

Complexity of Local Quantum Circuits under Nonunital Noise

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It is widely accepted that noisy quantum devices are limited to logarithmic depth circuits unless midcircuit measurements and error correction are employed. However, this conclusion holds only for unital error channels, such as depolarizing noise. Building on the idea of the “quantum refrigerator” Ben-Or *et al.* [Quantum refrigerator, arXiv1301.1995 (2013)], we improve upon previous results and show that geometrically local circuits in the presence of nonunital noise, in any dimension $d \geq 1$, can correct errors without midcircuit measurements and extend computation to any depth, with only polylogarithmic overhead in the depth and the number of qubits. This implies that local quantum dynamics subjected to sufficiently weak nonunital noise are computationally universal and nearly as hard to simulate as noiseless dynamics. Additionally, we quantify the contraction property of local random circuits in the presence of nonunital noise.

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I. INTRODUCTION

Noise affects the performance of quantum algorithms [1] and remains one of the most critical challenges in current quantum experiments [2]. Despite these challenges, recent advances in improving noisy quantum devices have enabled computations with simulation complexity that tests the limits of classical simulation techniques [3,4]. An interesting open question is whether noisy systems can provide an advantage as coherence times keep improving and the number of qubits scales up, before quantum error correction becomes available. Interestingly, the resolution of this question seems to depend on the nature of the noise being considered.

Unital noise models, such as depolarizing noise, are widely used in the field due to their simplicity, as they can be simulated by summing stochastic unitary trajectories. Their simple representation has not only made it easy to analyze noisy quantum circuits analytically [5–9] but also enabled the successful implementation of error-mitigation techniques on current noisy quantum hardware [3,10]. However, unital models rarely present a comprehensive description of the rich physics of quantum devices

[11,12]. Furthermore, this type of noise generally leads to an irreversible increase in the entropy of the system until it reaches the trivial maximally mixed state. As a result, the output of such noisy random circuits approaches a uniform distribution exponentially with increasing circuit depth [5,6,13,14], rendering circuits with superlogarithmic depth classically simulable.

While unital-noise models have been extensively characterized, the impact of general *nonunital* noise remains comparatively under-explored. In Ref. [15], it has been shown that, similar to unital noise, the output distribution of any circuit subject to general noise becomes independent on the input state at logarithmic depths—but only for noise-strength values exceeding a certain constant value. For sufficiently weak noise, however, this convergence result does not hold.

The seminal work in Ref. [16] has shown that circuits with all-to-all connectivity, when subjected to strictly nonunital noise, can efficiently simulate noiseless circuits with only a polylogarithmic overhead in depth and number of qubits. This makes the difference between nonunital and unital noise crucial from the perspective of classical simulation complexity. Here, we generalize the result from Ref. [16] to arbitrary local quantum circuits in dimensions $d \geq 1$ with similar overhead.

Circuits exhibiting such complexity must be carefully designed to exploit the properties of noise [9]. A natural question is whether a similar result holds for *random* circuits. As shown in Ref. [17], anticoncentration—commonly observed in the noiseless case [18]—does not occur in the presence of nonunital noise.

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This implies that current efficient classical algorithms for sampling from noisy circuits do not directly apply to this setting [6]. Nevertheless, as the circuit depth increases, the distance between any two input states inevitably collapses exponentially with the number of qubits after a certain depth, even under nonunital noise [15]. In this work, we demonstrate that for general nonunital noise, the decay rate of this distance is at most constant. This result implies that noisy local random circuits remain meaningful up to at least logarithmic depth. In contrast, for global n -qubit Haar-random operations, this distance becomes exponentially small even at constant depth, which makes considering deeper circuits senseless.

In our work, we address two main questions. First, we ask: does the result of Ref. [16] generalize to arbitrary architectures (including one-dimensional architectures) with the same polylogarithmic overhead? We answer this question positively in Theorem 1 below. We show that this is possible if the noise strength is below a certain critical value that depends on the introduced measure of noise nonunitality and the dimension of the lattice. This result establishes that the complexity of nonunital noise is essentially independent of architecture and can be applied to all existing hardware architectures.

To prove this result, we employ the same quantum refrigerator construction as in Ref. [16] but we incorporate the locality structure of the circuit and account for errors occurring during qubit swaps. We also derive how the resulting noise threshold depends on the lattice dimension. In the limit of all-to-all connectivity, we restore the original result from Ref. [16].

In Sec. IV, we examine random circuits affected by general types of noise. In Theorem 2, we demonstrate that the distance between two initially orthogonal product states in the computational basis contracts at a rate bounded above by a constant, for local circuits. This result extends the results of Ref. [7] to nonunital noise models and complements the results of Ref. [15]. Moreover, in Theorem 3, we provide an example of a noisy all-to-all Haar-random circuit, in which the distance between two states becomes exponentially small in the number of qubits already at a constant depth.

A. Circuits with worst-case complexity

We define a d -dimensional local noisy quantum circuit with noise described by the map $\mathcal{N}$ as a sequence of unitary operations applied between neighboring sites of a d -dimensional square lattice, followed by single-qubit completely positive trace-preserving (CPTP) maps $\mathcal{N}$ acting independently on the same qubits. In this work, we focus on general nonunital noise. Following the standard definition, a noise channel is classified as nonunital if it fails to preserve the identity operator.

Definition 1. We define a channel $\mathcal{N}$ to be nonunital if $\|\mathcal{N}(I) - I\|_1 > 0$, where I is the identity operator.

Generally, the noise strength of a channel $\mathcal{N}$ is defined as

$$\kappa = \|\mathcal{N} - \mathcal{I}\|_\diamond, \quad (1)$$

where $\mathcal{I}$ denotes the identity channel.

Also, we assume that for a typical channel, the distance between any pair of states contracts at a uniform rate. Let $\mathbf{S}_N = \{\rho : \rho \geq 0, \text{Tr } \rho = 1, \dim(\rho) = N\}$ denote the set of density matrices on an N -dimensional Hilbert space. Then, the contraction condition for $\mathcal{N}$ can be formulated as follows.

Definition 2. A channel $\mathcal{N}$ is said to be contracting if there exists $0 \leq \lambda < 1$ such that any pair of states $\rho, \sigma \in \mathbf{S}_2$ satisfies

$$\|\mathcal{N}(\rho - \sigma)\|_1 \leq \lambda \|\rho - \sigma\|_1. \quad (2)$$

We call a family of channels uniformly contracting if there exists $\Delta > 0$ such that $\lambda \leq 1 - \Delta\kappa$ for any channel $\mathcal{N}$ from the family.

This condition applies to most relevant families of parametrized nonunital quantum channels, such as the general damping channel and the replacement channel (see Appendix A 1) but it can be violated in some special cases; for an example, see Appendix A 2.

Finally, let $\sigma^* \in \mathbf{S}_2$ be a fixed point of the channel $\mathcal{N}$, defined by

$$\mathcal{N}(\sigma^*) = \sigma^*. \quad (3)$$

The fixed point can also be characterized by the purity parameter, defined as

$$\eta := \sqrt{2\text{Tr}[(\sigma^*)^2] - 1}. \quad (4)$$

From Definition 2, it follows that $\mathcal{N}$ must have a unique fixed point. Moreover, the parameter η for the fixed point of a nonunital channel must satisfy $\eta > 0$, since $\eta = 0$ implies a maximally mixed state, which is excluded by Definition 1. Thus, the parameter η serves as an alternative measure of the nonunitality of a channel.

Our goal is to show that for any family of nonunital uniformly contracting channels that satisfy Definitions 1 and 2, for any $\eta > 0$ there exists a threshold $\kappa_0(\eta)$ such that for any $\kappa \leq \kappa_0(\eta)$, simulating the resulting noisy dynamics is nearly as hard as simulating the noiseless dynamics. In particular, we prove the following result.

Theorem 1 (Informal). Consider d -dimensional quantum circuits with noise generated by a contracting noise

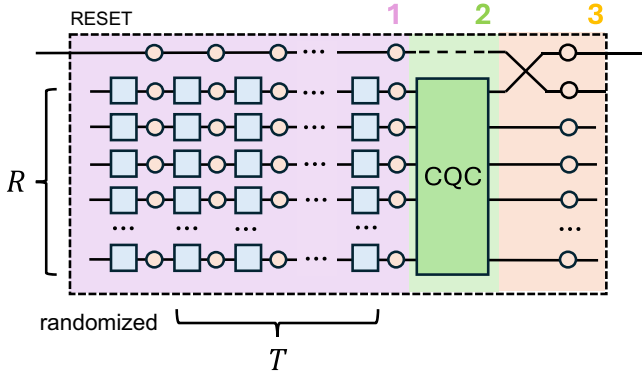


FIG. 1. The RESET circuit architecture. The circuit comprises three stages of the RESET protocol, each highlighted with a distinct color. Single-qubit gates are represented by blue boxes, while noise channels are denoted by circles labeled $\mathcal{N}(\cdot)$. In stage 1, the qubits undergo a passive-cooling protocol to achieve nonzero polarization. The green box indicates the noisy compound quantum compressor (CQC) subcircuit applied into stage 2, which facilitates coherent compression of polarization into the top qubit. Finally, the top qubit is swapped with the target qubit in stage 3.

channel $\mathcal{N}$ with strength κ and a unique fixed point σ^* with purity parameter η . Then, for an arbitrary $\eta > 0$, there exists nonzero $\kappa_0 = O(\eta^{\mu/d})$ with $\mu > \mu_0 = \log 3 / \log(3/2)$, such that for all $\kappa \leq \kappa_0$, noisy circuits allow for the simulation of noiseless unitary circuits with a logarithmic overhead.

The formal statement and its proof are provided in Sec. III E. Below, we outline the key steps of the proof.

First, we demonstrate that by utilizing nonunitary noise and ancilla qubits, it is possible to implement an approximate RESET operation that prepares the target qubit in the state $|0\rangle$. To simulate RESETS, we employ a modified “quantum refrigerator” algorithm, originally proposed in Ref. [16]. Our algorithm comprises three independent stages (see Fig. 1).

Stage 1 focuses on preparing the ancilla in an imperfect but predictable state with nonzero polarization. First, we “erase” the information stored in the ancilla qubits by applying random single-qubit unitary gates. These gates should be selected independently and at random, e.g., from the Haar distribution, for each run of the circuit. As a result, statistically, the ancilla qubits are left in a maximally mixed state. This step prevents correlated errors when the ancilla are reused for quantum error correction. Next, the qubits go through the process of passive cooling, where we apply a sequence of specially chosen noisy single-qubit gates to ensure that each ancilla qubit attains some nonzero polarization. Specifically, we construct a single-qubit noisy unitary circuit of depth T that transforms a maximally mixed state of a qubit into an output state ρ

satisfying polarization z scaling as (see Lemma 2)

$$z := \text{Tr}(Z\rho) = \Omega(\kappa\eta T), \quad (5)$$

for small κ and η , where Z is the Pauli-Z matrix and $\Omega(\cdot)$ denotes big-Omega asymptotic notation.

In stage 2, or the algorithmic cooling stage, the ancilla qubits are processed by a circuit that we call the compound quantum compressor (CQC), which concentrates the polarization into fewer qubits [19]. We show below that for any fixed $g > 0$, one can design a CQC that transforms R ancilla qubits of input polarization z_{in} into a single qubit of output polarization z_{out} , with

$$R \leq C \frac{[-\ln(3[1 - z_{\text{out}} + O(g)])]^\alpha}{z_{\text{in}}^{\mu + O(g)}}, \quad (6)$$

if the noise rate κ satisfies weakness condition $\kappa R^{1/d} \leq g$, where $\mu = \log 3 / \log(3/2) + \delta$ for any $\delta > 0$ and C is a constant that depends on δ (see Lemma 6). At the same time, the depth of the CQC circuit can be estimated as

$$T_{\text{CQC}} = O(R^{1/d} \log R). \quad (7)$$

In stage 3, we exchange the output of the compression cycle with the target qubit. This completes the RESET procedure.

The weakness condition on noise for CQC can be satisfied for arbitrary small g if we choose the passive-cooling depth T as

$$T \propto (\kappa\eta)^{-\mu/(\mu+d)}. \quad (8)$$

In this case, z_{in} given by the expression in Eq. (5) is sufficiently large to keep $\kappa R^{1/d} \leq g$.

Then, combining the results in Eqs. (5) and (6), we obtain the required number of ancillas per RESET operation to be (see Lemma 7)

$$R = O\left(\frac{\log^\alpha(\kappa\eta^{-\mu/d})}{(\kappa\eta)^{\mu d/(\mu+d)}}\right). \quad (9)$$

Since executing a RESET requires an additional $T_R = T + T_{\text{CQC}} + 1$ layers, all other qubits must wait for T_R steps in the worst-case scenario, thereby accumulating idling errors. These errors can be incorporated into an effective noise channel $\mathcal{N}_{\text{diag}} = \mathcal{N}^{T_R}$. As established in Corollary 1, under our choice of passive-cooling time in Eq. (8), the new effective noise strength $\kappa' = \|\mathcal{N}_{\text{diag}} - \mathcal{I}\|_\diamond$ satisfies

$$\kappa' = \tilde{O}((\kappa\eta^{-\mu/d})^{d/(\mu+d)}). \quad (10)$$

The RESET error itself is also comparable to κ' .

Although the idling-induced noise strength κ' in Eq. (10) exceeds the original noise strength κ , the inclusion of

RESET operations enables the ultimate reduction in the noise rate via quantum error correction. In particular, once κ' still falls below the threshold (see Theorem 4), the new circuit construction supports measurement-free fault-tolerant quantum computing with only logarithmic overhead in system size [20–23]. Crucially, this is possible only when $\kappa < \kappa_0 = O(\eta^{\mu/d})$; otherwise, the parameter κ' in Eq. (10) cannot be made small by reducing κ . The total qubit and depth overhead factorizes into the constant cost of implementing the RESETs and the polylogarithmic overhead required for quantum error correction.

Let us provide some simple estimates on the required threshold. To ensure that the effective noise rate satisfies $\kappa' \leq p_0$, where p_0 denotes the threshold for measurement-free quantum error correction, the asymptotic form in Eq. (10) must yield

$$\kappa_0 \propto p_0(p_0\eta)^{\mu/d}. \quad (11)$$

For a two-dimensional architecture ($d = 2$) with $\eta = 0.5$ and $p_0 \sim 10^{-2}$, as suggested by intuition from Ref. [24], this implies a noise threshold of $\kappa_0 \sim 10^{-5}$, assuming that the proportionality constant in Eq. (11) is close to one. At our operating point, the argument of the big- O scaling in Eq. (6) evaluates to $R \sim 10^7$, giving an order-of-magnitude estimate for the RESET qubit overhead (for constant factors of order unity), ignoring the quantum error-correction (QEC) overhead, which can be estimated separately and that depends on the code. It is important to note that both the threshold estimate and the overhead bound are specific to our theoretical approach; practical schemes may significantly reduce these values, potentially by several orders of magnitude.

The dependence of the threshold $\kappa_0 = O(\eta^{\mu/d})$ on both the purity parameter η of the noise fixed point and the lattice dimensionality d can be understood heuristically as follows. When η is small, passive cooling of stage 1 prepares a state with polarization $z \sim \eta$. This requires $R \sim z^{-\mu}$ algorithmic cooling rounds during stage 2 [see Eq. (6)]. Each round involves at most $R^{1/d} \sim z^{-\mu/d} \sim \eta^{-\mu/d}$ swaps connecting logical qubits across different ancilla blocks forming d -dimensional hypercubes. To keep the total swap error $\kappa R^{1/d} \sim \kappa \eta^{-\mu/d}$ below a small constant, one arrives at the condition $\kappa_0 = O(\eta^{\mu/d})$.

Notably, the scheme operates even for $d = 1$, in contrast to the local protocol of Ref. [16], which incurs polynomial overhead and applies only for $d \geq 2$. Also, in the limit $d \rightarrow \infty$, we recover the η -independent result of Ref. [16] for all-to-all connectivity.

B. Random circuits

In the previous section, we have demonstrated that local circuits subjected to nonunitary noise remain as powerful as noiseless circuits, even when the degree of nonunitarity is arbitrarily small. The goal of this section is to explore a

scenario in which the gates are generated randomly. This result can be seen as a generalization of Ref. [7] applied to unital noise and complements Ref. [15], which establishes similar upper bounds.

To define noisy parallel random circuits, we first consider an architecture that is defined as an arrangement of gates without specifying the exact gate operations. We focus on architectures that are parallel, meaning that the architecture can be divided into parallel layers such that each qubit is involved in exactly one gate per layer. The number of such layers is referred to as the parallel depth, denoted by D . In a noisy architecture, it is further assumed that noise is homogeneous, i.e., each gate is followed by the same two-qubit quantum noise channel. Unlike in the previous section, we allow the noise to be a more general form of a two-qubit channel that is symmetric under the swapping of qubits. Our results can be generalized to the case in which the noise is neither symmetric nor homogeneous.

A random circuit is defined as a collection of circuits with a specified noisy architecture in which the gate unitaries are distributed according to the Haar measure. We now state the following result: if starting from distinct pure computational basis states, these states remain efficiently distinguishable for logarithmically long times.

Theorem 2. Consider a noisy random unitary circuit consisting of D parallel layers of two-qubit gates. For any bit strings $\mathbf{z}, \mathbf{z}' \in \{0, 1\}^n$, with $\mathbf{z} \neq \mathbf{z}'$, the corresponding initial quantum states $\rho = |\mathbf{z}\rangle\langle\mathbf{z}|$ and $\sigma = |\mathbf{z}'\rangle\langle\mathbf{z}'|$ satisfy

$$\frac{1}{2} \mathbb{E}_{\mathcal{B}} \|\mathcal{C}(\rho - \sigma)\|_1 \geq e^{-\Gamma D}, \quad (12)$$

where $\Gamma > 0$ is a constant, $\mathcal{C}$ is a noisy random circuit channel, and $\mathbb{E}_{\mathcal{B}}$ denotes the expectation over the random two-qubit unitaries.

The proof of this theorem can be found in Sec. IV C. This result relies on the inherent locality of the gates and does not apply to more general nonlocal quantum gates. For example, consider an all-to-all architecture in which the system is subjected to a single all-to-all Haar-random unitary gate per layer. For such a circuit, we demonstrate the following result.

Theorem 3. Consider an n -qubit all-to-all random circuit of depth $D > 1$ subject to noise generated by a replacement channel with the rate parameter $0 < \gamma \leq 1$. Then,

$$\frac{1}{2} \mathbb{E} \|\mathcal{C}(\rho - \sigma)\|_1 \leq O\left(2^{n/2} \left(1 - \frac{\gamma}{2}\right)^{n(D-1)/2}\right), \quad (13)$$

where $\mathcal{C}$ denotes the channel corresponding to the noisy random circuit and $\mathbb{E}$ denotes the expectation over the gate unitaries.

The definition of the replacement channel is given in Appendix A 1 and the proof of the theorem is presented in Sec. IVE. This result demonstrates that for all-to-all circuits, the output depends only on the last *constant* number of layers, independent of the noise model or its strength. This result exhibits a tighter scaling than existing bounds for nonunital noise in local circuits [15], although it is derived specifically for the replacement channel. We show that this result can also be extended to the general amplitude damping.

C. Discussion

In this work, we have proved that quantum circuits subjected to nonunital noise are as hard to simulate classically as noiseless dynamics. This result is based on a construction that allows implementing RESET operations using ancilla qubits while maintaining coherence in the remaining qubits. Our results go beyond those in Ref. [16], which required either all-to-all connectivity or a polynomial overhead for local circuits. Moreover, our results can be extended to continuous-time dynamics using techniques developed in Ref. [25].

These results carry several implications. First, we present an explicit construction that harnesses arbitrary nonunital noise to perform substitute measurements in conventional syndrome-extraction protocols for quantum error correction, thereby extending existing measurement-free schemes [23]. Second, we introduce a versatile toolkit for establishing the computational complexity of multiple instances of local open-system quantum dynamics: dissipation-assisted quantum computing [26], variational algorithms augmented by engineered nonunitary operations, such as randomized RESET protocols [27], and the preparation of thermal states via quasilocal dissipative evolution [28]. Finally, we demonstrate that even in strictly one-dimensional architectures, the local dynamics under nonunital channels remain as intractable to simulate classically as their purely unitary counterparts, a result not previously shown.

An implication of these results is the new evidence suggesting that a universal classical algorithm is unlikely to simulate natural dissipative systems interacting with their environment (e.g., a thermal bath). This underscores the importance of quantum computing for studying states such as Gibbs states, which characterize thermal equilibrium in many-body physics.

Our work further shows that the claim that noisy quantum computers are only useful for logarithmic depth [2] may be misleading, as it relies on the intuition derived from depolarizing and other types of unital noise. One can envision a broad class of circuits in which nonunital dissipation processes are integrated into the design to provide a near-term quantum advantage. However, it remains unclear whether these circuits can be efficiently constructed.

It is worth noting that the noise threshold obtained in our work using a concatenated error-correction scheme is quite low. Thus, an open question remains as to whether it is possible to construct circuits with higher tolerance to nonunital noise. Nonetheless, our result reveals an intriguing behavior: even noise that is η -close to depolarizing noise allows fault-tolerant quantum computation with local circuits without measurements and feedback, provided that η is nonzero.

II. NOTATION

In this section, we provide a comprehensive list of the notation used throughout the paper.

A. Quantum states and operators

$\mathcal{M}_N$: Set of all operators O acting on an N -dimensional Hilbert space, i.e., $\mathcal{M}_N = \{O : \dim(O) = N\}$.

$\mathcal{S}_N$: Set of all density operators ρ on an N -dimensional Hilbert space, defined as $\mathcal{S}_N = \{\rho : \rho \geq 0, \text{Tr}(\rho) = 1, \dim(\rho) = N\}$.

$\Lambda_{d,n}$: Finite d -dimensional square lattice containing n qubits.

$\sigma^* \in \mathcal{M}_2$: Fixed point of channel $\mathcal{N}$; defined in Eq. (3).

$\eta \in [0, 1]$: Purity parameter of σ^* ; introduced in Eq. (4).

$\lambda \in [0, 1]$: Contraction coefficient for channel $\mathcal{N}$; defined in Lemma 2.

$\chi \in [0, 1]$: Polarization contraction coefficient; see Lemma 1.

$\gamma \in \mathbb{R}_+$: Noise-rate parameter other than the diamond norm; see noise channels in Appendix A 1.

$\rho_n^{(0)}$: Maximally mixed state on n qubits,

$$\rho_n^{(0)} := \frac{1}{2^n} I^{\otimes n}. \quad (14)$$

B. Quantum channels

$\mathcal{N} : \mathcal{M}_2 \rightarrow \mathcal{M}_2$: Target noise channel that describes uniform single-qubit noise; see Sec. IA.

$\mathcal{N}_{\text{diag}} : \mathcal{M}_2 \rightarrow \mathcal{M}_2$: Diagonalized form of $\mathcal{N}$; introduced in Definition 3.

$\mathcal{A}_k : \mathcal{M}_2 \rightarrow \mathcal{M}_2$: Noiseless CQC process that implements algorithmic cooling on $m = 3^k$ qubits in a mixed state, outputting a single qubit of enhanced purity; see Eq. (37).

$\mathcal{U} : \mathcal{M}_2 \rightarrow \mathcal{M}_2$: Unitary channel that diagonalizes the fixed point of noise channel σ^* ; see Definition 3.

$\mathcal{Q}_T : \mathcal{M}_2 \rightarrow \mathcal{M}_2$: Passive-cooling operation corresponding to a depth- T circuit; see Eq. (31).

$\mathcal{D} : \mathcal{M}_2 \rightarrow \mathcal{M}_2$: Diagonalization procedure introduced in Eq. (15).

$\tilde{\mathcal{U}}_C$: Noisy-circuit channel for elementary three-qubit CQCs; see Eq. (60).

C. Algorithms

$T_R \in \mathbb{N}$: Total dissipation-assisted RESET depth.
 $T_{PC} \equiv T \in \mathbb{N}$: Depth of the passive-cooling process.
 $T_{AC} = D_d(m) \in \mathbb{N}$: Depth of the algorithmic cooling process that takes place on the d -dimensional lattice and involves m ancilla qubits.
 $N \in \mathbb{N}$: Total number of qubits in the circuit.
 $D \in \mathbb{N}$: Parallel-circuit depth: the number of nonoverlapping gate layers.
 $m \in \mathbb{N}$: Number of ancilla qubits used to generate the reset map $\mathcal{R}$.
 $\mathcal{A}_k : \mathbf{M}_2 \rightarrow \mathbf{M}_2$: Compound algorithmic cooling process applied by CQC, which consists of k cycles.
 $\mathcal{R} : \mathbf{M}_2 \rightarrow \mathbf{M}_2$: RESET process generated by utilizing the noise.
 $\mu_0 = \log 3 / \log(3/2) \approx 2.7105$: Constant part of the exponent of the input polarization scaling for CQC; see Lemmas 3 and 6.
 $\alpha = \log_2 3 \approx 1.585$: Exponent of the polylogarithmic output scaling for CQC; see Lemmas 3 and 6.
 $\mu = \mu_0 + \delta$: Scaling exponent in the ansatz introduced in Eq. (118) for arbitrarily small $\delta > 0$.

D. Quantum error correction

L : Number of concatenation layers in the quantum code; see the introduction to Sec. III D.
 $\mathcal{N}_i$: Noise channel acting on physical qubit i .
 n : Number of logical qubits in the elementary quantum code; see the introduction to Sec. III D.
 n_A : Number of ancillas necessary to implement measurement-free correction on n logical qubits.
 $n_B = n + n_A$: Size of the elementary block of QEC; see the introduction to Sec. III D.
 $\mathcal{E}_q^{(m)}$: m -level encoding operation for a logical zero state of (logical) qubit q , introduced in Eq. (142).
 $P_\alpha^m : M_N \rightarrow M_N$: m -level logical Pauli; see Eq. (133).
 $P_\alpha^{m+1} : M_{Q_m} \rightarrow M_{Q_m}$: m -level logical Pauli; see Eq. (134), where $Q_m = n_B^m$.
 $T_{\alpha\beta}$: Pauli transfer matrix; see Eq. (135).
 $\mathcal{N}^{(m,\kappa)}$: m -local noise channel with rate κ ; see Notation 2.
 κ_m : Effective noise rate at concatenation level m ; see Eq. (144).
 $\mathcal{P}_m$: Projector onto the code space at level m .
 D_m : Depth of the circuit at concatenation level m ; see Eq. (142).
 $\mathcal{O}_k^{(0)}$: Physical gates; see Eq. (142).
 $\mathcal{O}_k^{(m)}$: Logical gates, being the m -level extension of the physical gates $\mathcal{O}_k^{(0)}$; see Eq. (142).
 M_r^m : List of parallel m -level logical-gate indices for layer r of the logical quantum gates.
 $d_{mr} = |M_r^m|$: Number of gates in logical layer r at the m th concatenation level.

$\tilde{\rho}_{\text{out}}^{(m)}$: Output of the m th-level noisy logical circuit; see Eq. (143).
 N_m : Number of logical qubits at the m th level of concatenation.
 N_O : Largest number of gates in any logical operation or error-correcting gadget for Steane's $[[7, 1, 3]]$ code.
 $\mathcal{F}^{(m)}$: m th-level extension of a channel $\mathcal{F}$; see Eq. (136).
 $\tilde{\mathcal{C}}^{(m)}$: Noisy QEC gadget; see Definition 5.

E. Random-circuit analysis

q : Local dimension of the qudit; $q = 2$ corresponds to a qubit.
 $T_{G,k}$: Marginal trace distance for subset G ; see Eq. (168).
 $S_{G,k}$: Marginal Hilbert-Schmidt (HS) distance for subset G ; see Eq. (169).
 K_k^i : Kraus operator for the noise channel, $\mathcal{N}_k$; see Eq. (166).
 $K_{\alpha\alpha,\beta\beta}^i$: Kraus matrix elements; see Eq. (176).
 $\mathcal{U}_k$: Noiseless unitary operation for gate k ; see Eq. (166).
 $\tilde{\mathcal{U}}_k$: Noisy unitary operation for gate k ; see Eq. (166).
 u, v : Noise parameters; see Eq. (173).
 $U_{\alpha\alpha,\beta\beta}$: Unitary matrix elements; see Eq. (176).
 N_G : Dimension of the Hilbert space for subset G .
 Ω : Subset on which the gate acts; see Proposition 2.
 c_1, c_2 : HS transformation coefficients; see Eq. (171).
 a, b : Noise parameters for two-qubit gates; defined in Eq. (183).
 A, B : Noise parameters for two-qubit gates; defined in Eq. (184).
 α, r : HS transformation parameters; defined in Eq. (186).
 β, μ : HS transformation parameters; defined in Eq. (188).

F. Diagonalization

For any Hermitian operator $O \in \mathbf{M}_2$, we define its eigenvalues $\lambda_0[O]$ and $\lambda_1[O]$ such that $\lambda_0[O] \geq \lambda_1[O]$. We also define the *diagonalization procedure* as

$$\mathcal{D}(O) := \lambda_0[O] \cdot |0\rangle\langle 0| + \lambda_1[O] \cdot |1\rangle\langle 1|. \quad (15)$$

Any operator that satisfies $\mathcal{D}(O) = O$ is called a *diagonal* operator. Note that the diagonalization procedure is *not* a linear operator.

G. In-out description

For any process $\mathcal{S} : \mathbf{M}_2 \rightarrow \mathbf{M}_2$, we generally denote its input state as $\rho_{\text{in}}[\mathcal{S}]$ and its output state as $\rho_{\text{out}}[\mathcal{S}] := \mathcal{S}(\rho_{\text{in}}[\mathcal{S}])$. The *polarizations* of these states are

defined as

$$\begin{aligned} z_{\text{in}}[S] &:= \text{Tr}(Z\rho_{\text{in}}), \\ z_{\text{out}}[S] &:= \text{Tr}(Z\rho_{\text{out}}). \end{aligned} \quad (16)$$

These definitions are used to describe general properties of the channel $\mathcal{S}$.

III. CONSTRUCTING CIRCUITS WITH WORST-CASE COMPLEXITY

In this section, we present a step-by-step procedure for constructing a local *noisy* unitary circuit that can simulate any local *noiseless* unitary circuit with only polylogarithmic overhead, under the assumption that the noise is strictly nonunital. We begin by introducing a quantum refrigeration scheme—a subcircuit that takes $R \gg 1$ qubits in a fully mixed state and outputs one qubit in an *almost* pure state by applying a sequence of algorithmic cooling procedures [19]. Building on this construction, we demonstrate that noisy unitary circuits can simulate other noisy unitary circuits augmented with RESET operations. Finally, we show how to relate the output of such RESET-enabled noisy circuits to that of noiseless circuits by invoking the threshold theorem [20,29].

We define a noise-enabled RESET operation as an algorithm that resets a target qubit from a state ρ to the zero state in the computational basis, using R ancilla qubits initialized in the maximally mixed state [see Eq. (14)] and noisy unitary operations $\tilde{\mathcal{U}}$ composed of local gates followed by noisy channels, i.e.,

$$\text{Tr}_{\text{anc}}(\tilde{\mathcal{U}}(\rho \otimes \rho_R^{(0)})) \approx |0\rangle\langle 0|. \quad (17)$$

This algorithm operates in three stages: (1) a *passive-cooling* stage, in which the ancilla qubits are randomized into a maximally mixed state and subsequently exposed to nonunital noise for some circuit depth $T > 0$; (2) an *algorithmic cooling* stage, which takes $R \gg 1$ qubits and returns one qubit in a (nearly) pure state by concentrating the entropy into the remaining ancillas; and (3) a *swap operation*, in which the target qubit is swapped with the output qubit from the algorithmic cooling stage.

A. Stage 1: Passive cooling

In the passive-cooling stage, each ancilla qubit undergoes an individual cooling process that involves unitary operations followed by a noise channel. In this process, the qubits do not interact and can therefore be treated independently. Consequently, all results in this section concern single-qubit processes.

We begin by describing the cooling effect of the noise channel. To do so, we first introduce the diagonalized form of the noise channel.

Definition 3. Consider a channel $\mathcal{N}$ with unique fixed point σ^* . Then, we define its diagonalized form as

$$\mathcal{N}_{\text{diag}} = \mathcal{U}\mathcal{N}\mathcal{U}^\dagger, \quad (18)$$

where $\mathcal{U}$ is a unitary operator chosen such that $\mathcal{U}(\sigma^*)$ is diagonal.

The definition of diagonal operator can be found in Sec. II F. Note that σ_{diag}^* , being the fixed point of the diagonalized channel,

$$\mathcal{N}_{\text{diag}}(\sigma_{\text{diag}}^*) = \sigma_{\text{diag}}^*, \quad (19)$$

has the same purity parameter η as the state σ^* . However, note that diagonalized channels do not necessarily output a diagonal density matrix, even when acting on a diagonal input state [30].

The diagonalized form of a nonunital noise channel $\mathcal{N}_{\text{diag}}$ leads to rapid convergence of the output polarization to the value η , as formalized in the following lemma. Throughout this discussion, we adopt the notation introduced in Sec. II G to describe the relationship between the input polarization $z_{\text{in}}[\mathcal{A}]$ and the output polarization $z_{\text{out}}[\mathcal{A}]$ of a quantum channel $\mathcal{A}$ for arbitrary input states, unless otherwise specified.

Lemma 1. Consider a channel $\mathcal{N}$ with a unique fixed point with $\eta > 0$ and let $\mathcal{N}_{\text{diag}}$ denote the diagonalized form of $\mathcal{N}$. Then, for any diagonal input state, we have

$$z_{\text{out}}[\mathcal{N}_{\text{diag}}] = \eta + \chi(z_{\text{in}}[\mathcal{N}_{\text{diag}}] - \eta), \quad (20)$$

for some $\chi < 1$.

Proof. Consider ρ_{in} and ρ_{out} as the input and output states of the diagonalized channel $\mathcal{N}_{\text{diag}}$, with corresponding polarizations z_{in} and z_{out} , respectively. Assuming that ρ_{in} is diagonal (as defined in Sec. II F), it must take the form

$$\rho_{\text{in}} = \frac{1}{2}(I + z_{\text{in}}Z), \quad (21)$$

where $Z \in \mathbf{M}_2$ is the Pauli-Z operator.

For this input state, the output of the quantum channel is

$$\rho_{\text{out}} \equiv \mathcal{N}_{\text{diag}}(\rho_{\text{in}}) = \frac{1}{2}(\mathcal{N}_{\text{diag}}(I) + z_{\text{in}}\mathcal{N}_{\text{diag}}(Z)). \quad (22)$$

Next, we express the fixed point of the channel $\mathcal{N}_{\text{diag}}$, defined by Eq. (19), as

$$\sigma_{\text{diag}}^* = \frac{1}{2}(I + \eta Z). \quad (23)$$

This follows from the fact that $\tilde{\sigma}^*$ has the same purity parameter η as the state σ^* and is diagonal. Using the

linearity of quantum maps, and by solving for

$$\mathcal{N}_{\text{diag}}\left(\frac{1}{2}(I + \eta Z)\right) = \frac{1}{2}(I + \eta Z), \quad (24)$$

we derive the following expression:

$$\mathcal{N}_{\text{diag}}(I) = I + \eta Z - \eta \mathcal{N}_{\text{diag}}(Z). \quad (25)$$

Substituting $\mathcal{N}_{\text{diag}}(I)$ from this relation into Eq. (22), we obtain

$$\rho_{\text{out}} = \frac{1}{2}(I + \eta Z + (z_{\text{in}} - \eta)\mathcal{N}_{\text{diag}}(Z)). \quad (26)$$

By applying the definition $z_{\text{out}} = \text{Tr}(\rho_{\text{out}}Z)$, we obtain the expression

$$z_{\text{out}} = \eta + \chi(z_{\text{in}} - \eta), \quad (27)$$

where

$$\chi := \frac{1}{2}\text{Tr}(Z\mathcal{N}_{\text{diag}}(Z)). \quad (28)$$

The final step is to show that $\chi < 1$. To prove this, we assume the contrary, i.e., $\chi = 1$, and demonstrate that it leads to a contradiction. Since $\mathcal{N}_{\text{diag}}$ is a CPTP map, we can express

$$\mathcal{N}_{\text{diag}}(Z) = \sum_{P_i \in \{X, Y, Z\}} n_i P_i, \quad (29)$$

where

$$n_i := \frac{1}{2}\text{Tr}[P_i \mathcal{N}_{\text{diag}}(Z)], \quad \sum_i n_i^2 \leq 1. \quad (30)$$

Suppose, for contradiction, that $\chi \equiv n_3 = 1$. Then, $\mathcal{N}_{\text{diag}}(Z) = Z$ and, by Eq. (25), also $\mathcal{N}_{\text{diag}}(I) = I$. But this means that $\mathcal{N}_{\text{diag}}$ is unital, in direct conflict with Definition 1. Hence we must have $\chi < 1$. Substituting this bound into Eq. (27) immediately yields the claimed monotonic contraction in Eq. (20). ■

A direct corollary of Lemma 1 is that it is possible to design a passive-cooling process that guarantees an output polarization approaching the value of η . To do so, consider the quantum map

$$\mathcal{Q}_T = (\mathcal{N}_{\text{diag}}\mathcal{D})^T, \quad (31)$$

where $\mathcal{D}$ is the diagonalization procedure outlined in Sec. II G

Lemma 2 (Passive-cooling process). The channel $\mathcal{Q}$ in Eq. (31) for $z_{\text{in}}[\mathcal{Q}] = 0$ satisfies

$$z_{\text{out}}[\mathcal{Q}_T] \geq \eta \Delta \kappa T + O(\eta \kappa^2 T^2). \quad (32)$$

Proof. Consider $|\psi_0\rangle$ and $|\psi_1\rangle$ to be the eigenstates of the fixed point σ^* of the channel $\mathcal{N}$ corresponding to the eigenvalues $\lambda_0[\sigma^*]$ and $\lambda_1[\sigma^*]$. Then, $\mathcal{U}^\dagger(Z) = |\psi_0\rangle\langle\psi_0| - |\psi_1\rangle\langle\psi_1|$. Following Definition 2, the parameter in Eq. (27) satisfies

$$\begin{aligned} \chi &:= \frac{1}{2}\text{Tr}(Z\mathcal{N}_{\text{diag}}(Z)) = \frac{1}{2}\text{Tr}(Z\mathcal{U}\mathcal{N}\mathcal{U}^\dagger(Z)) \\ &= \frac{1}{2}\text{Tr}\left(\mathcal{U}^\dagger(Z)\mathcal{N}(|\psi_0\rangle\langle\psi_0| - |\psi_1\rangle\langle\psi_1|)\right) \\ &\leq \frac{1}{2}\|\mathcal{N}(|\psi_0\rangle\langle\psi_0| - |\psi_1\rangle\langle\psi_1|)\|_1 \\ &\leq \frac{1}{2}\lambda\| |\psi_0\rangle\langle\psi_0| - |\psi_1\rangle\langle\psi_1| \|_1 \leq \lambda, \end{aligned} \quad (33)$$

where, as before, $\kappa = \|\mathcal{N} - \mathcal{I}\|_\diamond$. Next, by Lemma 1, for any $T \geq 0$, we have

$$z_{\text{out}}[\mathcal{Q}_T] \geq \chi^T z_{\text{in}}[\mathcal{Q}_T] + \eta(1 - \chi^T), \quad (34)$$

where we have used the fact that the diagonalization procedure $\mathcal{D}$ does not decrease the polarization. Assuming that the product κT is small and using Definition 2 by setting $\lambda \leq 1 - \kappa \Delta$, the state of the ancilla after T passive-cooling steps is given by

$$z_{\text{out}}[\mathcal{Q}_T] \geq \eta \Delta \kappa T + O(\eta \kappa^2 T^2), \quad (35)$$

which brings us to the statement of the lemma. ■

How can one implement the channel $\mathcal{Q}_T$ in practice? The approach is as follows. We consider a sequence of T unitary operations, each followed by the application of the noise channel $\mathcal{N}$, as dictated by the noisy architecture. For the first layer, we set the unitary to be $\mathcal{U}_1 = \mathcal{U}^\dagger$. For subsequent layers, $2 \leq k \leq T$, we define $\mathcal{U}_k = \mathcal{U}^\dagger \mathcal{G}_k \mathcal{U}$, where each $\mathcal{G}_k$ is a unitary chosen to diagonalize its input state and acting as $\mathcal{D}$. This construction is feasible because the input to each layer is known, allowing us to explicitly determine the required $\mathcal{G}_k$.

This leads to the effective channel

$$\mathcal{Q}'_T = (\mathcal{N}\mathcal{U}^\dagger\mathcal{D}\mathcal{U})^T = \mathcal{U}^\dagger(\mathcal{N}_{\text{diag}}\mathcal{D})^T\mathcal{U} = \mathcal{U}^\dagger\mathcal{Q}_T\mathcal{U}. \quad (36)$$

The first unitary $\mathcal{U}$ can be ignored, since we are initially acting on the maximally mixed state. The final unitary $\mathcal{U}^\dagger$ can be effectively canceled by adapting the subsequent unitary operations. Thus, one can use $\mathcal{Q}'_T$ to fully emulate the effect of $\mathcal{Q}_T$.

As a result, this procedure yields ancilla qubits with small polarizations on the order of approximately $\eta\kappa T$. The next step is to design a circuit that compresses this ensemble of weakly polarized ancilla qubits into a single ancilla qubit with higher purity.

B. Stage 2: Algorithmic cooling

We implement the algorithmic cooling stage using a circuit that we refer to as the compound quantum compressor (CQC). In the ideal (noiseless) case, the CQC is represented by a linear transformation $\mathcal{A}_k$ and implements a unitary operation U_C composed of k layers acting on $R = R(k)$ identical input qubits, each prepared in the state ρ_{in} . After applying U_C , the compressed output is obtained by tracing out the $R - 1$ ancilla qubits:

$$\mathcal{A}_k(\rho) := \text{Tr}_{R-1} \mathcal{U}_C(\rho^{\otimes R}) \equiv \text{Tr}_{R-1}(U_C \rho^{\otimes R} U_C^\dagger), \quad (37)$$

where $R = R(k)$ is the total number of input qubits, which depends on the number of layers k , and Tr_{R-1} denotes the partial trace over the first $R - 1$ qubits.

We focus on a class of CQCs characterized by a linearized polarization gain

$$z_{\text{out}}[\mathcal{A}_k] = y z_{\text{in}}[\mathcal{A}_k] + O(z_{\text{in}}[\mathcal{A}_k]^2), \quad (38)$$

where $z_{\text{in}}[\mathcal{A}_k]$ and $z_{\text{out}}[\mathcal{A}_k]$ denote the input and output polarizations of the map $\mathcal{A}_k$ [cf. Eq. (16)] and $y > 1$ is the gain parameter quantifying the purity enhancement. Entropic considerations impose the bound $y^2 \leq R$ [19]. For fixed polarization values, our objective is to design a QCp that maximizes y .

More concretely, a QCp for three input qubits can be realized by implementing a controlled-NOT (CNOT) operation on qubit 2 controlled by qubit 3 and then a controlled-SWAP (CSWAP) on qubits 1 and 3 controlled by qubit 2 (here, qubits 1 and 3 are swapped if value of qubit 2 is |0⟩) [19]. This procedure is described by the unitary

$$\mathcal{A}_1 : U_C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \quad (39)$$

For the three-qubit QCp, when ρ_{in} is diagonal in the computational basis, Eq. (38) takes the form

$$z_{\text{out}}[\mathcal{A}_1] = \frac{3}{2} z_{\text{in}}[\mathcal{A}_1] - \frac{1}{2} (z_{\text{in}}[\mathcal{A}_1])^3. \quad (40)$$

A larger compressor $\mathcal{A}_k$ for $k > 1$ can be built by concatenating multiple three-qubit modules into k successive

cycles. In each cycle, the outputs of the previous stage are partitioned into groups of three qubits, each compressed by the elementary CQC before proceeding to the next cycle. The overall number of qubits is $R = 3^k$, which grows exponentially with the number of cycles k . We refer to this architecture as a *compound quantum compressor* (CQC).

1. CQC depth

The minimum depth required for a compound quantum compressor depends on the qubit connectivity of the available circuit architecture. In the all-to-all limit (equivalent to an infinite-dimensional lattice, $d \rightarrow \infty$), interactions are unconstrained, so the total depth, denoted $D_{d \rightarrow \infty}(R)$, is simply the sum of the depths of each three-qubit compression cycle,

$$D_{d \rightarrow \infty}(R) = D_C \log_3 R, \quad (41)$$

where D_C is the depth of the unitary U_C in Eq. (39), applied across $k = \log_3 R$ cycles.

On a spatially local lattice of finite dimension d , SWAP gates are required to bring distant qubits together. Embedding R qubits in a d -dimensional hypercube of side length $\Theta(R^{1/d})$ means that each qubit incurs at most $\Theta(R^{1/d})$ SWAPS per compression cycle. Hence the asymptotic depth of the spatially local CQC becomes

$$\begin{aligned} D_d(m) &= D_{d \rightarrow \infty}(R) + \Theta(R^{1/d} \log R) \\ &= \Theta(R^{1/d} \log R). \end{aligned} \quad (42)$$

Here, the $\Theta(R^{1/d} \log R)$ term reflects the $\Theta(R^{1/d})$ swaps required in each of the $k = \log_3 R$ cycles. We use the fact that the natural logarithm $\log$ and $\log_3$ are related by a constant factor, which can be omitted in big- O notation.

Minimizing the depth of compressor plays an important role, as the remaining qubits decohere while the compression takes place.

2. Input-output ratio for ideal CQCs

Next, we focus on establishing an upper bound on m , given the polarization of the input state $z_{\text{in}}[\mathcal{A}_k]$ and the target polarization of the output state $z_{\text{out}}[\mathcal{A}_k]$. We first consider the case of noiseless CQC. In the following section, we extend this result to account for the presence of noise. Additionally, we only consider the compound compressors built from the three-qubit QCps discussed earlier. Such a noiseless CQC is characterized by the following lemma.

Lemma 3 (Noiseless CQC). There exists a constant $C > 0$ such that, for any $0 < z_{\text{in}}[\mathcal{A}_k] < z_{\text{out}}[\mathcal{A}_k]$, $\frac{2}{3} < z_{\text{out}}[\mathcal{A}_k] < 1$, and $\delta > 0$ there exists a noiseless compressor

$\mathcal{A}_k$ that utilizes

$$R = 3^k \leq C \frac{[-\log(3(1 - z_{\text{out}}[\mathcal{A}_k]))]^\alpha}{(z_{\text{in}}[\mathcal{A}_k])^{\mu_0 + \delta}} \quad (43)$$

qubits, where $\alpha = \log_2 3 \approx 1.59$, $\mu_0 = (\alpha/(\alpha - 1)) \approx 2.7105$.

Proof. Consider the CQC described in Sec. III B, which consists of k cycles of individual three-qubit QCp layers. Let the output of the QCp in the l th cycle serve as the input for the $(l + 1)$ th cycle. Then, we have

$$z_{\text{out}}[\mathcal{A}_{l+1}] = \frac{3}{2}z_{\text{out}}[\mathcal{A}_l] - \frac{1}{2}(z_{\text{out}}[\mathcal{A}_l])^3. \quad (44)$$

This inequality implies that $z_{\text{out}}[\mathcal{A}_{l+1}] \geq z_{\text{out}}[\mathcal{A}_l]$, indicating nondecreasing polarization across cycles. Thus, the output of the $(l + 1)$ th cycle can be expressed as

$$\begin{aligned} z_{\text{out}}[\mathcal{A}_{l+1}] &= \left(\frac{3}{2} - \frac{(z_{\text{out}}[\mathcal{A}_l])^2}{2} \right) z_{\text{out}}[\mathcal{A}_l] \\ &\geq \left(\frac{3}{2} - \frac{(z_{\text{out}}[\mathcal{A}_k])^2}{2} \right) z_{\text{out}}[\mathcal{A}_l]. \end{aligned} \quad (45)$$

By applying Eq. (40) for $l = 1$ and propagating the bound recursively for all $l = 2, \dots, k$, the k -cycle CQC satisfies

$$z_{\text{out}}[\mathcal{A}_k] \geq \left(\frac{3}{2} - \frac{(z_{\text{out}}[\mathcal{A}_k])^2}{2} \right)^k z_{\text{in}}[\mathcal{A}_k], \quad (46)$$

where we have used the fact that all processes start from the same initial state, i.e., $z_{\text{in}}[\mathcal{A}_k] = z_{\text{in}}[\mathcal{A}_1]$.

From this expression, the minimum number of cycles k required to reach a target output polarization satisfies the inequality

$$k \leq \frac{\log(z_{\text{out}}[\mathcal{A}_k]/z_{\text{in}}[\mathcal{A}_k])}{\log\left(\frac{3}{2} - \frac{1}{2}(z_{\text{out}}[\mathcal{A}_k])^2\right)}. \quad (47)$$

For reference, a comparison between this bound and the exact numerical solution is shown in Fig. 2(a). As can be seen from this plot, the above bound is tight when $z_{\text{in}}[\mathcal{A}_k]$ is small and $z_{\text{out}}[\mathcal{A}_k]$ is not too close to unity.

Using Eq. (47), we obtain the following upper bound on the number of qubits required:

$$R = 3^k \leq \left(\frac{z_{\text{out}}[\mathcal{A}_k]}{z_{\text{in}}[\mathcal{A}_k]} \right)^{\mu_0 + f}, \quad (48)$$

where the exponent is given by

$$\begin{aligned} \mu_0 &= \frac{\log 3}{\log(3/2)} \approx 2.7095, \\ f &:= \frac{\log 3}{\log(3/2 - z_{\text{out}}[\mathcal{A}_k]^2/2)} - \frac{\log 3}{\log(3/2)} \\ &= O(z_{\text{out}}[\mathcal{A}_k]^2). \end{aligned} \quad (49)$$

We are halfway toward proving our result. To target the regime in which $1 - z_{\text{out}}[\mathcal{A}_k]$ is small, we first introduce the depolarization parameter, defined as

$$\begin{aligned} p_{\text{in}}[\mathcal{A}_l] &:= 1 - z_{\text{in}}[\mathcal{A}_l], \\ p_{\text{out}}[\mathcal{A}_l] &:= 1 - z_{\text{out}}[\mathcal{A}_l]. \end{aligned} \quad (50)$$

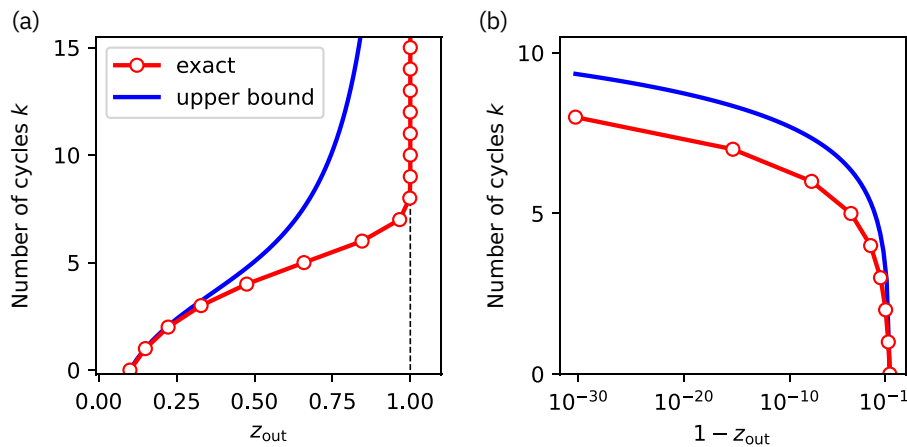


FIG. 2. Bounds on the number of cycles. (a) A comparison between the exact number of algorithmic compression cycles (represented by red dots) required to achieve the target polarization z_{out} starting from the initial value $z_{\text{in}} = 0.1$ and the upper bound given in Eq. (47) (dashed line). The bound shows a strong agreement for small values of z_{out} . (b) A similar comparison for the exact number of cycles required to achieve $p_{\text{out}} = 1 - z_{\text{out}}$ starting from the initial value $p_{\text{in}} = 0.3$, alongside the upper bound from Eq. (53). This bound effectively captures the rapid double-exponential convergence of p_{out} for $p_{\text{in}} > 1/3$ as the number of cycles increases.

We can then rewrite Eq. (40) as the update rule

$$p_{\text{out}}[\mathcal{A}_1] = (p_{\text{in}}[\mathcal{A}_1])^2 (3 - 2p_{\text{in}}[\mathcal{A}_1]). \quad (51)$$

Using the same method as for polarization, taking into account that the output state of the l th layer is an input for the $(l+1)$ th layer, we obtain

$$\begin{aligned} p_{\text{out}}[\mathcal{A}_{l+1}] &= (p_{\text{out}}[\mathcal{A}_l])^2 (3 - 2p_{\text{out}}[\mathcal{A}_l]) \\ &\leq 3(p_{\text{out}}[\mathcal{A}_l])^2. \end{aligned} \quad (52)$$

The output of the k th cycle satisfies the inequality

$$\begin{aligned} p_{\text{out}}[\mathcal{A}_k] &\leq \frac{1}{3}(3p_{\text{out}}[\mathcal{A}_2])^{2^{k-1}} \\ &\leq \frac{1}{3}(3p_{\text{in}}[\mathcal{A}_1])^{2^k} = \frac{1}{3}(3p_{\text{in}}[\mathcal{A}_k])^{2^k}. \end{aligned} \quad (53)$$

The bound on depolarization decreases monotonically, as long as $p_{\text{in}}[\mathcal{A}_k] < \frac{1}{3}$. Note that, as we describe below, we will apply this bound only to the few last layers of the compressor; therefore, we do not require this condition separately in the statement of the lemma.

The number of steps required to achieve the target value of depolarization is bounded as

$$k \leq \log_2 \left(\frac{\log 3 p_{\text{out}}[\mathcal{A}_k]}{\log 3 p_{\text{in}}[\mathcal{A}_k]} \right). \quad (54)$$

This bound is also illustrated in Fig. 2(b), alongside the exact solution. Going back to polarizations, we obtain the following bound on the number of qubits:

$$R = 3^k \leq \left(\frac{\log 3 (1 - z_{\text{out}}[\mathcal{A}_k])}{\log 3 (1 - z_{\text{in}}[\mathcal{A}_k])} \right)^\alpha, \quad (55)$$

where we have used the notation

$$\alpha = \log_2 3 \approx 1.585. \quad (56)$$

Note the logarithmic scaling with $p_{\text{out}}[\mathcal{A}_k] = 1 - z_{\text{out}}[\mathcal{A}_k]$.

We can now combine the two results to derive a hybrid bound on m . We divide the entire CQC into three sequences. The first sequence includes k_1 layers and operates with a small input polarization $z_{\text{in}}[\mathcal{A}_k] \ll 1$, producing $m/3^{k_1}$ qubits with a constant output polarization of $z_{\text{out}}[\mathcal{A}_{k_1}] = z_1$. The next k_2 layers continue this process as an input and output $m/3^{k_1+k_2}$ qubits with polarization $z_{\text{out}}[\mathcal{A}_{k_1+k_2}] = z_2 > 2/3$. The third sequence contains $k_3 = k - k_1 - k_2$ layers, processes qubits with polarization z_2 , and produces a single qubit with polarization of

$z_{\text{out}}[\mathcal{A}_k] = z_1 \approx 1$. For k_1 and k_2 , we use Eq. (47), and for k_3 , we use Eq. (54):

$$\begin{aligned} 3^{k_1} &\leq \left(\frac{z_1}{z_{\text{in}}[\mathcal{A}_k]} \right)^{\mu_0 + \delta}, \\ 3^{k_2} &\leq \left(\frac{z_2}{z_1} \right)^{\mu_0 + \delta'}, \\ 3^{k_3} &\leq \left(\frac{\log 3 (1 - z_{\text{out}}[\mathcal{A}_k])}{\log 3 (1 - z_2)} \right)^\alpha. \end{aligned} \quad (57)$$

Note that k_2 depends only on z_1 and z_2 , which can be chosen as constants, and, therefore, can be treated as a constant. Consequently, the resulting bound on the number of qubits is

$$R = 3^{k_1+k_2+k_3} \leq C \frac{(-\log 3 (1 - z_{\text{out}}[\mathcal{A}_k]))^\alpha}{(z_{\text{in}}[\mathcal{A}_k])^{\mu_0 + \delta}}. \quad (58)$$

Here, $C = z_1^{\delta - \delta'} z_2^{\mu_0 + \delta'} (-\log 3 (1 - z_2))^{-\alpha}$, $\delta = O(z_1^2)$, and $\delta' = O(z_2^2)$ are independent of both z_{in} and z_{out} , and z_2 is a constant. By assigning z_1 an arbitrarily small constant value, we can achieve arbitrarily small δ [see Eq. (49)], though with an increased value of C . This expression completes our proof. ■

The inequality in Eq. (43) underscores a key feature of CQCs: the input-output ratio scales polynomially with the input polarization, yet only logarithmically with the output depolarization. As we demonstrate in the following section, this logarithmic dependence enables the RESET error, given by $1 - z_{\text{out}}[\mathcal{A}_k]$, to be made arbitrarily small. Crucially, this can be achieved while keeping the swapping cost, $\kappa m^{1/d}$, negligible, provided that the noise strength κ remains sufficiently small.

3. Input-output ratio for noisy CQCs

Our next step is to generalize the results from the previous section to the case of a noisy CQC. We define the noisy quantum process in close analogy with the noiseless scenario in Eq. (37),

$$\tilde{\mathcal{A}}_k(\rho) := \text{Tr}_{\text{anc}}[\tilde{\mathcal{U}}_C(\rho^{\otimes m})], \quad (59)$$

where $\tilde{\mathcal{U}}_C$ denotes the actual (noisy) circuit intended to implement the ideal unitary U_C . As before, we model noise by single-qubit channels $\mathcal{N}$ acting immediately after each gate. Consequently, the overall noisy evolution can be written as

$$\tilde{\mathcal{U}}_C = \prod_{g=1}^G \left(\prod_{j \in \Omega_g} \mathcal{N}_j \right) \mathcal{U}_g, \quad (60)$$

where G is the number of gates in the circuit compilation of U_C , $\mathcal{U}_g$ is the ideal unitary channel for gate g , Ω_g is the set

of qubits acted upon by gate g , and $\mathcal{N}_j = \mathcal{I}_{/j} \otimes \mathcal{N}$ denotes the single-qubit noise channel applied to qubit j (tensored with the identity on all other qubits).

We start by establishing a noisy compression relation for a single noisy CQC.

Lemma 4 (Noisy compression for polarization). Noisy CQC map $\tilde{\mathcal{A}}_k$ in Eq. (59) for $k = 1$ satisfies

$$z_{\text{out}}[\tilde{\mathcal{A}}_1] = \left(\frac{3}{2} + O(\kappa)\right) z_{\text{in}}[\tilde{\mathcal{A}}_1] - \frac{1}{2} (z_{\text{in}}[\tilde{\mathcal{A}}_1])^3 + O(\kappa\eta). \quad (61)$$

Proof. We start from the expression for the elementary CQC that is given by the formula

$$\tilde{\mathcal{A}}_1(\rho) := \text{Tr}_{\text{anc}}[\tilde{\mathcal{U}}_C(\rho^{\otimes 3})], \quad (62)$$

where we assume that ρ is taken in the Z basis. Next, we decompose the input state of the noisy circuit as

$$\rho^{\otimes 3} = \rho_3^{(0)} + z_{\text{in}}[\mathcal{A}_1] \delta O, \quad (63)$$

where $\rho_3^{(0)}$ is the maximally mixed state and δO is a certain Hermitian operator. We define the perturbation

$$\delta \mathcal{N}_j := \mathcal{N}_j - \mathcal{I}, \quad (64)$$

which captures the nonideal contribution from noise at location j . As formally presented in Eq. (60), the circuit is composed of G ideal unitary gates $\mathcal{U}_k$ acting on subsets of qubits indexed by Ω_k , each preceded by a set of noisy channels. Then, the overall noisy-circuit channel can be written as

$$\tilde{\mathcal{U}}_C = \prod_{k=1}^G \left(\prod_{j \in \Omega_k} (\mathcal{I} + \delta \mathcal{N}_j) \right) \mathcal{U}_k. \quad (65)$$

Partially expanding the product in $\delta \mathcal{N}_j$, the noisy-circuit channel becomes

$$\tilde{\mathcal{U}}_C = \mathcal{U}_C + \delta \mathcal{U}_C, \quad (66)$$

where

$$\delta \mathcal{U}_C = \sum_{k'=1}^G \sum_{j \in \Omega_{k'}} \mathcal{E}_{k'} \delta \mathcal{N}_j \mathcal{U}_{k'} \prod_{k=1}^{k'-1} \mathcal{U}_k, \quad (67)$$

and each $\mathcal{E}_{k'} : \mathbf{M}_8 \rightarrow \mathbf{M}_8$ is a linear superoperator:

$$\mathcal{E}_{k'} = \prod_{k=k'+1}^G \left(\prod_{j \in \Omega_k} (\mathcal{I} + \delta \mathcal{N}_j) \right) \mathcal{U}_k. \quad (68)$$

Let Z_0 denote the Pauli- Z operator acting on the system qubit that is not discarded:

$$z_{\text{out}}[\mathcal{A}_1] = \text{Tr} [Z_0 \tilde{\mathcal{U}}_C(\rho^{\otimes 3})]. \quad (69)$$

Using Eqs. (63) and (66), we arrive at

$$z_{\text{out}}[\mathcal{A}_1] = \text{Tr} [Z_0 \mathcal{U}_C(\rho^{\otimes 3})] + \text{Tr} [Z_0 \delta \mathcal{U}_C(\rho_3^{(0)})] + z_{\text{in}}[\mathcal{A}_1] \text{Tr} [Z_0 \delta \mathcal{U}_C(\delta O)]. \quad (70)$$

Since $\mathcal{U}_C$ is a noiseless unitary channel, the first term yields the ideal expression

$$\text{Tr} [Z_0 \mathcal{U}_C(\rho^{\otimes 3})] = \frac{3}{2} z_{\text{in}}[\mathcal{A}_1] - \frac{1}{2} (z_{\text{in}}[\mathcal{A}_1])^3. \quad (71)$$

Next, we consider the contribution of the second term, which captures the deviation due to noise for the maximally mixed component. We can simplify

$$\begin{aligned} \delta \mathcal{U}_C(\rho_3^{(0)}) &= \sum_{k'=1}^G \sum_{j \in \Omega_{k'}} \mathcal{E}_{k'} \delta \mathcal{N}_j \mathcal{U}_{k'} \prod_{k=1}^{k'} \mathcal{U}_k (\rho_3^{(0)}) \\ &= \sum_{k'=1}^G \sum_{j \in \Omega_{k'}} \mathcal{E}_{k'} \delta \mathcal{N}_j (\rho_3^{(0)}), \end{aligned} \quad (72)$$

where we have used that $\rho_3^{(0)}$ is the maximally mixed three-qubit state and is thus invariant under any unitary transformation.

Introducing the notation $O_{k'} = \mathcal{E}_{k'}^\dagger(Z_0)$ and defining $\delta \mathcal{N} = \mathcal{N} - \mathcal{I}$ as the deviation from the identity channel for a single qubit, we obtain

$$\text{Tr} [Z_0 \delta \mathcal{U}_C(\rho_3^{(0)})] = \sum_{k'=1}^G \sum_{j \in \Omega_{k'}} \text{Tr} [O_{k'} \delta \mathcal{N}(I) \otimes \rho_2^{(0)}]. \quad (73)$$

This allows us to derive the following upper bound using the trace norm and operator norm:

$$\left| \text{Tr} [Z_0 \delta \mathcal{U}_C(\rho_3^{(0)})] \right| \leq \frac{1}{2} \|O_{k'}\| \cdot \|\delta \mathcal{N}(I)\|_1. \quad (74)$$

Now, we utilize the property that the noisy channel $\mathcal{N}$ has a fixed point σ^* characterized by a purity parameter η [see Eqs. (3) and (4)]. That is,

$$\begin{aligned} \mathcal{N}(\sigma^*) &= \sigma^*, \\ \sigma^* &= \frac{1}{2} I + \frac{\eta}{2} (|\psi_0\rangle\langle\psi_0| - |\psi_1\rangle\langle\psi_1|), \end{aligned} \quad (75)$$

where $|\psi_0\rangle$ and $|\psi_1\rangle$ are the eigenstates of σ^* corresponding to eigenvalues ordered in decreasing magnitude.

This leads to the following expression for the deviation of the identity under $\mathcal{N}$:

$$\begin{aligned} \mathcal{N}(I) - I &= -\eta (\mathcal{N}(|\psi_0\rangle\langle\psi_0|) - |\psi_0\rangle\langle\psi_0|) \\ &\quad + \eta (\mathcal{N}(|\psi_1\rangle\langle\psi_1|) - |\psi_1\rangle\langle\psi_1|). \end{aligned} \quad (76)$$

Applying the triangle inequality, we obtain a bound on the trace norm,

$$\|\mathcal{N}_j(I) - I\|_1 \leq 2\eta \|\mathcal{N}_j - \mathcal{I}\|_\diamond = 2\eta\kappa, \quad (77)$$

where κ denotes the noise strength defined in Eq. (1).

Finally, noting that the operator norm satisfies $\|O_{k'}\| = O(1)$ with respect to κ and η , we conclude that

$$\text{Tr} [Z_0 \delta\mathcal{U}_C(\rho_3^{(0)})] = O(\kappa\eta). \quad (78)$$

Similarly, we analyze the effect of noise on the operator δO and obtain

$$\delta\mathcal{U}_C(\delta O) = \sum_{k'=1}^G \sum_{j \in \Omega_{k'}} \mathcal{E}_{k'} \delta\mathcal{N}_j \mathcal{U}_{k'} \prod_{k=1}^{k'} \mathcal{U}_k(\delta O), \quad (79)$$

which describes how noise accumulates as δO propagates through the circuit under imperfect channels.

This leads to the following expression for the expectation value:

$$\begin{aligned} |\text{Tr} [Z_0 \delta\mathcal{U}_C(\delta O)]| &= \left| \sum_{k'=1}^G \sum_{j \in \Omega_{k'}} \text{Tr} [O_{k'} \delta\mathcal{N}_j(\delta O)] \right| \\ &\leq \frac{1}{2} \|O_{k'}\| \cdot \|\delta\mathcal{N}\|_\diamond \cdot \|\delta O\|_1 \\ &= O(\kappa), \end{aligned} \quad (80)$$

where we have used the duality between the diamond norm and the trace norm, and again the fact that $\|O_{k'}\| = O(1)$ and $\|\delta\mathcal{N}\|_\diamond = \kappa$.

Thus, we arrive at the following bound:

$$\text{Tr} [Z_0 \delta\mathcal{U}_C(\delta O)] = O(\kappa). \quad (81)$$

Finally, combining the intermediate results obtained in Eqs. (71), (78), and (81), and substituting them into Eq. (70), we obtain the desired statement of the lemma. This concludes the proof. $\blacksquare$

Lemma 5 (Noisy compression relation for depolarization). The noisy CQC map $\tilde{\mathcal{A}}_k$ in Eq. (59) for $k = 1$ satisfies

$$p_{\text{out}}[\tilde{\mathcal{A}}_1] \leq (p_{\text{in}}[\tilde{\mathcal{A}}_1])^2 (3 - 2p_{\text{in}}[\tilde{\mathcal{A}}_1]) + O(\kappa). \quad (82)$$

Proof. Following the approach from the previous lemma, let Z_0 denote the Pauli-Z operator acting on the system qubit that is not discarded. Define

$$P_0 := I - Z_0, \quad (83)$$

where I is the identity operator on the three-qubit system. Using Eq. (66), we can rewrite the output probability as

$$\begin{aligned} p_{\text{out}}[\tilde{\mathcal{A}}_1] &= \text{Tr} [P_0 \tilde{\mathcal{U}}_C(\rho^{\otimes 3})] \\ &= \text{Tr} [P_0 \mathcal{U}_C(\rho^{\otimes 3})] + \text{Tr} [P_0 \delta\mathcal{U}_C(\rho^{\otimes 3})]. \end{aligned} \quad (84)$$

The noiseless contribution is bounded by Eq. (51):

$$\text{Tr} [P_0 \mathcal{U}_C(\rho^{\otimes 3})] \leq (p_{\text{in}}[\tilde{\mathcal{A}}_1])^2 (3 - 2p_{\text{in}}[\tilde{\mathcal{A}}_1]). \quad (85)$$

We now introduce the following notation:

$$\begin{aligned} \rho_{k'} &:= \mathcal{U}_{k'} \prod_{k=1}^{k'} \mathcal{U}_k(\rho^{\otimes 3}) \in \mathbf{D}_8, \\ P_{k'} &:= \mathcal{E}_{k'}^\dagger(P_0), \end{aligned} \quad (86)$$

which allows us to express the noise contribution as

$$\begin{aligned} |\text{Tr} [P_0 \delta\mathcal{U}_C(\rho^{\otimes 3})]| &= \left| \sum_{k'=1}^G \sum_{j \in \Omega_{k'}} \text{Tr} [P_{k'} \delta\mathcal{N}_j(\rho_{k'})] \right| \\ &\leq \frac{1}{2} \|P_{k'}\| \cdot \|\delta\mathcal{N}\|_\diamond \\ &= O(\kappa). \end{aligned} \quad (87)$$

Here, we have used that, similar to in the previous proof, $\|P_{k'}\|$ is a constant, as the operator acts on a finite number of qubits. Combining the bounds for the noiseless and noisy contributions yields the claimed result. $\blacksquare$

Next, we use this result to find a bound on the input-output ratio for the compound compressor.

Lemma 6 (Noisy CQC). For any $\delta > 0$, there exist absolute constants $g_0, C > 0$ (depending only on the dimension d) such that the following holds. Let $\tilde{\mathcal{A}}_k$ denote the noisy CQC operation with $k \geq 1$ layers and $m = 3^k$ qubits. Suppose that

$$z_{\text{in}}[\tilde{\mathcal{A}}_k] = \Omega(\kappa\eta) \quad (88)$$

and fix any target tolerance g satisfying $0 < g \leq g_0$. Assume that the local noise strength κ satisfies

$$\kappa R^{1/d} \leq g. \quad (89)$$

Then, to achieve any desired output fidelity within the range

$$\frac{2}{3} + O(g) < z_{\text{out}}[\tilde{\mathcal{A}}_k] < 1 - O(g), \quad (90)$$

it suffices to use

$$R \leq C \frac{[-\log(1 - z_{\text{out}} - O(g))]^\alpha}{z_{\text{in}}^{\mu_0 + \delta + O(g)}} \quad (91)$$

qubits, where $\alpha = \log_2 3$ and $\mu_0 = \log 3 / \log(3/2)$.

Proof. We start by rewriting the expression of the k -layer CQC in terms of a three-qubit CQC process. We can formally write $\tilde{\mathcal{A}}_k$ as

$$\tilde{\mathcal{A}}_k = \prod_{l=1}^k (\mathcal{N}^{M_l} \tilde{\mathcal{A}}_1), \quad (92)$$

where M_l is the number of swaps required between layers l and $l+1$. The number of swaps never exceeds the side of the hypercube that contains all m ancilla qubits, i.e., $M_l \leq m^{1/d}$:

$$\begin{aligned} z_{\text{out}}[\mathcal{N}^{M_l}] &\geq (1 - \kappa)^{M_l} z_{\text{in}}[\mathcal{N}^{M_l}] \\ &\geq (1 - \kappa M_l) z_{\text{in}}[\mathcal{N}^{M_l}] \\ &\geq (1 - \kappa m^{1/d}) z_{\text{in}}[\mathcal{N}^{M_l}] \\ &\geq (1 - g) z_{\text{in}}[\mathcal{N}^{M_l}]. \end{aligned} \quad (93)$$

Next, we use Lemma 4 to establish that

$$\begin{aligned} z_{\text{out}}[\tilde{\mathcal{A}}_{l+1}] &\geq \left(\frac{3}{2} + O(\kappa) \right) z_{\text{out}}[\tilde{\mathcal{A}}_l] \\ &\quad - \frac{1}{2} (z_{\text{out}}[\tilde{\mathcal{A}}_l])^3 + O(\kappa \eta). \end{aligned} \quad (94)$$

According to the statement of the lemma, for all $l \geq 1$, we have

$$z_{\text{in}}[\tilde{\mathcal{A}}_l] = \Omega(\kappa \eta) \quad (95)$$

and we can obtain the following recurrence relation:

$$z_{\text{out}}[\tilde{\mathcal{A}}_{l+1}] \geq (1 - g) \left(\frac{3}{2} + \gamma - \frac{1}{2} (z_{\text{out}}[\tilde{\mathcal{A}}_k])^2 \right) z_{\text{out}}[\tilde{\mathcal{A}}_l], \quad (96)$$

where $\gamma = o(1)$ is subconstant and $\gamma \rightarrow 0$ when $\kappa \eta \rightarrow 0$.

From this expression, the minimum number of cycles k required to reach a target output polarization satisfies the inequality

$$k \leq \frac{\log(z_{\text{out}}[\tilde{\mathcal{A}}_k]/z_{\text{in}}[\tilde{\mathcal{A}}_k])}{\log((1 - g)(\frac{3}{2} + \gamma - \frac{1}{2}(z_{\text{out}}[\tilde{\mathcal{A}}_k])^2))}. \quad (97)$$

This leads us to the condition

$$R = 3^k \leq \left(\frac{z_{\text{out}}[\tilde{\mathcal{A}}_k]}{z_{\text{in}}[\tilde{\mathcal{A}}_k]} \right)^{\mu_0 + f'}, \quad (98)$$

where $\mu_0 = \log 3 / \log(3/2) \approx 2.7095$ as previously and

$$\begin{aligned} f' &:= \gamma + \frac{\log 3}{\log(3/2 - z_{\text{out}}[\mathcal{A}_k]^2/2)} - \frac{\log 3}{\log(3/2)} \\ &= \gamma + O(z_{\text{out}}[\mathcal{A}_k]^2). \end{aligned} \quad (99)$$

Similarly to Lemma 3, we can write a bound for the depolarization improvement over compression rounds using Lemma 5, as

$$p_{\text{out}}[\tilde{\mathcal{A}}_1] \leq 3(p_{\text{in}}[\tilde{\mathcal{A}}_1])^2 + O(\kappa). \quad (100)$$

To account for the error during swaps, we can rewrite Eq. (93) as

$$p_{\text{out}}[\mathcal{N}^{M_l}] \leq (1 - g)p_{\text{in}}[\mathcal{N}^{M_l}] + g \leq p_{\text{in}}[\mathcal{N}^{M_l}] + g. \quad (101)$$

This allows us to write down the recurrence condition for the output depolarization for two consecutive rounds of compression as

$$p_{\text{out}}[\tilde{\mathcal{A}}_{l+1}] \leq 3(p_{\text{out}}[\tilde{\mathcal{A}}_l] + g)^2 + O(\kappa). \quad (102)$$

Since $g \sim \kappa$, there always exists a constant $a > 0$ that depends only on the dimension of the lattice, such that

$$p_{\text{out}}[\tilde{\mathcal{A}}_{l+1}] \leq 3(p_{\text{out}}[\tilde{\mathcal{A}}_l] + g)^2 + ag. \quad (103)$$

To facilitate analysis, let us introduce a new variable, defined as

$$\pi_l := p_{\text{out}}[\tilde{\mathcal{A}}_l] + bg. \quad (104)$$

This leads to a recurrence relation for π_l of the form

$$\pi_{l+1} = 3\pi_l^2 + (a + b)g - 6\pi_l(b - 1)g + 3(b - 1)^2g^2. \quad (105)$$

To ensure that the recurrence does not increase, we require the following inequality to hold:

$$(a + b)g - 6\pi_l(b - 1)g + 3(b - 1)^2g^2 \leq 0. \quad (106)$$

Solving this inequality gives us a lower bound on π_l , as

$$\pi_l \geq \frac{1}{6} \frac{b+a}{b-1} + \frac{b-1}{2} g. \quad (107)$$

Substituting back into the original probability variable, we obtain a required lower bound on $p_{\text{out}}[\tilde{\mathcal{A}}_l]$,

$$p_{\text{out}}[\tilde{\mathcal{A}}_l] \geq \frac{1}{6} \frac{b+a}{b-1} + \frac{b-3}{2} g. \quad (108)$$

Putting $b = \max(2, a+1)$ to optimize the bound, we find that

$$p_{\text{out}}[\tilde{\mathcal{A}}_l] \geq \frac{2}{3} + \frac{b-3}{2} g. \quad (109)$$

Finally, we can express the depth k needed to reach a target output probability in terms of the input and output probabilities:

$$k \leq \log_2 \left(\frac{\log 3 (p_{\text{out}} + bg)}{\log 3 (p_{\text{in}} + bg)} \right). \quad (110)$$

Then, the input-output ratio satisfies

$$m = 3^k \leq \left(\frac{\log 3 (p_{\text{out}} + bg)}{\log 3 (p_{\text{in}} + bg)} \right)^\alpha$$

$$\text{if } p_{\text{in}} \leq \frac{2}{3} + \frac{b-3}{2} g, \quad (111)$$

where the exponent $\alpha = \log_2 3$.

Finally, by combining the two bounds, we obtain a unified estimate for m . We model our compressor as a sequence of three consecutive modules. The initialization module, of depth k_1 , takes the input polarization

$$z_{\text{in}}[\tilde{\mathcal{A}}_k] = \Omega(\eta\kappa) \quad (112)$$

and raises it to some intermediate value z_1 . The amplification module, of depth k_2 , then boosts this polarization to

$$z_2 > \frac{2}{3} + cg, \quad (113)$$

where the constant c depends only on the structural properties of the circuit and k_2 remains fixed, independent of the input and output polarizations. Finally, the compression module, of depth k_3 , increases the polarization from z_2 to the final output value $z_{\text{out}}[\tilde{\mathcal{A}}_k]$. Using the same reasoning that led to Eq. (58), we bound k_1 using Eq. (97) and k_3 using Eq. (110), while k_2 contributes only a constant overhead. Hence,

$$R = 3^{k_1+k_2+k_3} \leq C \frac{(-\log_3 [1 - z_{\text{out}} - O(g)])^\alpha}{z_{\text{in}}^{\mu_0+\delta+O(g)}}, \quad (114)$$

where C and $\delta = \gamma + O(z_1^2)$ depend only on our choices of z_1 and z_2 . In particular, by selecting z_1 sufficiently close

to its lower bound, one can make δ arbitrarily small, at the cost of increasing C . This completes the proof. ■

C. Cooling procedure and RESET process

Once we have figured out the compression process, we need to find the cooling procedure so that the resulting depth and number of ancillas are compatible with the rest of the circuit. In particular, we want the implementation depth T_R to satisfy $\kappa T_R \ll 1$; otherwise, the rest of the qubits will decohere before the RESET operation is complete. We also require that $\kappa m^{1/d} \ll 1$, to ensure that one can perform the circuit operations by navigating the target qubits around the ancillas using SWAP operations.

The possibility of such a cooling schedule and the entire RESET process can be established using the following lemma. Here, we employ standard big-Theta notation, $\Theta(x)$, along with big- O notation, $\tilde{O}(x)$, which omits logarithmic factors.

Lemma 7. Consider a d -dimensional noisy circuit composed of unitary operations followed by single-qubit noise described by a general contracting channel $\mathcal{N}$ as in Eq. (2), with a fixed-point purity $\eta > 0$ and a strength $\|\mathcal{N} - \mathcal{I}\|_\diamond = \kappa > 0$ satisfying $\kappa = O(\eta^{\mu/d})$, where $\mu = \log 3 / \log(3/2) + \delta$ for any $\delta > 0$. Such a circuit can implement a noisy RESET operation that outputs a single qubit with the following polarization:

$$z_{\text{out}}[\mathcal{R}] = 1 - \tilde{O}\left((\kappa\eta^{-\mu/d})^{d/(\mu+d)}\right), \quad (115)$$

using m ancilla qubits and depth T_R satisfying

$$R = O\left(\frac{\log^\alpha(\kappa\eta^{-\mu/d})}{(\kappa\eta)^{\mu d/(\mu+d)}}\right),$$

$$\kappa T_R = \tilde{O}\left((\kappa\eta^{-\mu/d})^{d/(\mu+d)}\right), \quad (116)$$

for $\alpha = \log_2 3$.

Proof. We start by describing the passive-cooling process. Consider maximally mixed ancilla states that are subjected to $T \gg 1$ layers of circuit evolution. Using the expression that we derived in Lemma 2 from the uniform contracting properties of the noise channel $\mathcal{N}$, we obtain

$$z_{\text{in}}[\mathcal{A}_k] = \Omega(\kappa\eta T). \quad (117)$$

It is convenient to use the following ansatz for the passive-cooling time:

$$T \propto (\kappa\eta)^{-\mu/(\mu+d)}, \quad (118)$$

where the symbol $\propto$ denotes a relation up to an arbitrary multiplicative factor and $\mu > 0$ is a positive exponent. Then, the total depth of the cooling part can be

described by

$$\kappa T \propto (\kappa \eta^{-\mu/d})^{d/(\mu+d)}. \quad (119)$$

Therefore, the polarization from Eq. (117) takes the form

$$z_{\text{in}}[\mathcal{A}_k] = \Omega((\kappa \eta)^{d/(\mu+d)}). \quad (120)$$

Next, we aim to bound the total number of ancillas required for the CQC. We consider a CQC that implements the algorithmic cooling process $\mathcal{A}_k$ for $m = 3^k$ ancilla qubits. Substituting $z_{\text{in}}[\mathcal{A}_k]$ from Eq. (120) and setting $z_{\text{out}}[\mathcal{A}_k] = z_{\text{out}}[\mathcal{R}]$ in Eq. (115) into the statement of Lemma 6, we obtain

$$R = O\left(\frac{\log^\alpha(\kappa \eta^{-\mu/d})}{(\kappa \eta)^{(\mu_0 + O(g))d/(\mu+d)}}\right). \quad (121)$$

Let us show that there exists a proper choice of μ such that all the conditions of Lemma 6 are satisfied; in particular,

$$\begin{aligned} z_{\text{in}}[\tilde{\mathcal{A}}_k] &= \Omega(\kappa \eta), \\ \kappa R^{1/d} &\leq g, \end{aligned} \quad (122)$$

for sufficiently small $g \leq g_0$.

The first condition in Eq. (122) is satisfied by the input polarization in Eq. (120) for any $\mu \geq 0$. To satisfy the second condition, we limit our choice to a subconstant value

$$g = \tilde{O}((\kappa \eta^{-\mu/d})^{d/(\mu+d)}), \quad (123)$$

where $\tilde{O}$ ignores logarithmic corrections.

Then, choosing $\mu = \mu_0 + \delta$ for any $\delta > 0$, we automatically satisfy the second condition in Eq. (122) for small enough κ and η , provided that $\kappa = O(\eta^{\mu/d})$. Indeed, we can rewrite

$$\kappa m^{1/d} = A \cdot B, \quad (124)$$

where the terms A and B take the form

$$\begin{aligned} A &:= \tilde{O}((\kappa \eta^{-\mu/d})^{d/(\mu+d)}), \\ B &:= (\kappa \eta)^{(\delta + O(g))/(\mu+d)}. \end{aligned} \quad (125)$$

For any $\delta > 0$, the value of B is polynomially small in $\kappa \eta$. Therefore, since $A \sim g$ are the same up to a polylogarithmic factor in $\kappa \eta^{-\mu/d}$, the first condition in Eq. (122) is always satisfied for sufficiently small κ and η . Also, since $1 - z_{\text{out}}[\mathcal{A}_k] \sim g$, it also satisfies the conditions of this theorem.

Since g is subconstant, we can write that Eq. (121) finally yields

$$R = O\left(\frac{\log^\alpha(\kappa \eta^{-\mu/d})}{(\kappa \eta)^{\mu d/(\mu+d)}}\right). \quad (126)$$

What remains is the total depth of the reset operation, which is

$$T_R = T_{\text{PC}} + T_{\text{AC}} = T + O(m^{1/d} \log m), \quad (127)$$

where T_R is the total depth of the RESET operation, $T_{\text{PC}} = T$ is the depth required for passive cooling, and T_{AC} is the time required for algorithmic cooling. Then, we arrive at the expression

$$\kappa T_R = \tilde{O}((\kappa \eta^{-\mu/d})^{d/(\mu+d)}). \quad (128)$$

This expression concludes our proof. $\blacksquare$

Corollary 1. Consider an N -qubit d -dimensional noisy unitary circuit of depth D , where each qubit is subject to single-qubit noise described by a channel $\mathcal{N}$ with a fixed-point purity $\eta > 0$ and a strength $\|\mathcal{N} - \mathcal{I}\|_\diamond = \kappa > 0$ satisfying $\kappa = O(\eta^{\mu/d})$, where $\mu = \log 3 / \log(3/2) + \delta$ for any $\delta > 0$. Then, such a circuit can simulate an N' -qubit d -dimensional noisy circuit of depth D' , incorporating unitary gates and RESET operations subject to a single-qubit noise $\mathcal{N}_{\text{diag}}$ with a strength

$$\kappa' := \|\mathcal{N}_{\text{diag}} - \mathcal{I}\|_\diamond = \tilde{O}((\kappa \eta^{-\mu/d})^{d/(\mu+d)}), \quad (129)$$

with overhead

$$\begin{aligned} N/N' &= O((\kappa \eta)^{-\mu d/(\mu+d)} \text{polylog}(\kappa \eta^{-\mu/d})), \\ D/D' &= O((\kappa \eta)^{-\mu/(\mu+d)}). \end{aligned} \quad (130)$$

This result directly follows from the fact that RESET produces qubits with polarization in Eq. (115) and the fact that the rest of the qubits accumulate the noise

$$\kappa' = \kappa T_R + O(\kappa m^{1/d}) = \tilde{O}((\kappa \eta^{-\mu/d})^{d/(\mu+d)}). \quad (131)$$

The overhead can be computed as $N'/N = m$ because each circuit qubit must have m auxiliary qubits to implement RESET operations. The depth overhead can be computed as $D'/D \leq T_R O(\kappa m^{1/d})$ since, in the worst case, each layer may contain a RESET requiring depth T_R .

D. Connection between RESETs and fault tolerance

In this section, we prove the threshold theorem that quantifies the overhead in simulating a noiseless unitary circuit of depth D' up to some error ε using a D -depth noisy

circuit with RESETs when the noise strength is below some certain constant value.

Theorem 4 (Threshold theorem). There exists a constant $\kappa_0 > 0$ such that for all $\kappa := \|\mathcal{N} - \mathcal{I}\|_\diamond < \kappa_0$, a N -qubit d -dimensional noisy unitary circuit with RESETs of depth D can simulate an N' -qubit d -dimensional noiseless unitary circuit of depth D' up to error ϵ with overhead

$$D/D' \sim N/N' = O(\log(N'D'/\epsilon)). \quad (132)$$

Before proving this theorem, we begin by introducing some definitions. First, consider the standard notation of a $[[n, 1, 3]]$ quantum code, which encodes a single logical qubit into n physical qubits and is capable of tolerating any single-qubit error [31]. We use this code to build a concatenated code. For each encoded subset, we assign n_A ancilla qubits, which are necessary for performing error correction and implementing fault-tolerant gates. Consequently, the code and ancillas together form a block consisting of $n_B = n + n_A$ qubits. Thus, given a fixed number of qubits N' in the target noiseless circuit, we consider the noisy circuit containing $N = N'n_B^L$ qubits, where $L \geq 0$ represents the total number of concatenation levels.

Next, we define the logical operators for each concatenation level. First, we set $P_0^0 := I$, $P_1^0 := X$, $P_2^0 := Y$ and $P_3^0 := Z$ as the single-qubit Pauli matrices. For each concatenation level $L \geq m \geq 0$, we then define the level- m logical Pauli operators P_α^m ,

$$P_\alpha^m := \bigotimes_{i=1}^{N_m} P_{\alpha_i}^m, \quad (133)$$

where $\alpha \in \mathbb{Z}_4^{N_m}$ is an integer-valued vector with components $\alpha_i \in \{0, 1, 2, 3\}$, $N_m = N'n_B^{L-m}$ is the number of logical qubits at m th level, and we define level- m Pauli operators for $m \geq 0$ through the recurrence relation

$$P_\alpha^{m+1} := (P_\alpha^m)^{\otimes n} \otimes (P_0^m)^{\otimes n_A}, \quad (134)$$

where we have chosen the first n qubits as encoding qubits and the remaining n_A qubits as ancillas.

After defining the logical operators, we define the logical superoperators that describe quantum channels. First, for any k -qubit channel $\mathcal{E}(\cdot)$ acting on a physical-qubit subset Ω and any pair of vectors $\alpha, \beta \in \mathbb{Z}_4^k$, we define $T_{\alpha\beta}$ as the coefficients of the (Pauli) transfer matrix of the channel such that

$$\begin{aligned} \forall \alpha \in \mathbb{Z}_4^k, O \in \mathcal{O}_{F \setminus \Omega} : \\ \mathcal{F}^\dagger(P_\alpha^0 \otimes O) = \sum_{\beta \in \mathbb{Z}_4^k} T_{\alpha\beta} P_\beta^0 \otimes O, \end{aligned} \quad (135)$$

where the O belong to the set of operators $\mathcal{O}_{F \setminus \Omega}$ acting on the subset $F \setminus \Omega$ complementary to Ω . The transfer matrix

uniquely defines the quantum channel acting on physical qubits. Then, we define a quantum channel $\mathcal{F}^{(m)}(\cdot)$ to be a level- m logical extension of $\mathcal{F}(\cdot)$ if its action follows $T_{\alpha\beta}$ such that

$$\begin{aligned} \forall \alpha \in \mathbb{Z}_4^k, O \in \mathcal{O}_{F \setminus \Omega} : \\ \mathcal{F}^{(m)\dagger}(P_\alpha^m \otimes O) = \sum_{\beta \in \mathbb{Z}_4^k} T_{\alpha\beta} P_\beta^m \otimes O. \end{aligned} \quad (136)$$

There may be multiple extensions of a given Pauli channel, distinguished by their action on Pauli operators other than P_α^m .

Finally, let us define the logical gates. Before that, to simplify the expressions, we extend the traditional product notation to the ordered product of superoperators as follows.

Notation 1. We denote the multiplication operator according to an ordered list of operations as

$$\prod_{l \in (l_1, \dots, l_N)} \mathcal{O}_l := \mathcal{O}_{l_N} \dots \mathcal{O}_{l_1}. \quad (137)$$

We also use a notation for an unspecified single-qubit noise model of given strength.

Notation 2. By $\mathcal{N}^{(m, \kappa)}$, we denote an unspecified CPTP map that satisfies two conditions:

- (1) $\mathcal{N}^{(m, \kappa)} = \mathcal{N}_1^{(m)} \otimes \dots \otimes \mathcal{N}_{N_m}^{(m)}$, where $\mathcal{N}_i^{(m)}$ is an m -level extension of a single-qubit channel $\mathcal{N}_i$.
- (2) $\|\mathcal{N}_i - \mathcal{I}\|_\diamond \leq \kappa$.

Using these definitions, we now outline the properties of the gates used in the evolution. By the Solovay-Kitaev theorem, one can argue that any circuit can be represented with a limited set of quantum gates up to an arbitrarily small error. Therefore, without loss of generality, we assume that the target circuit is already composed of such gates and disregard compilation errors.

The logical extension of the standard gates must satisfy a notion of fault tolerance, which can be formulated as follows. Note that the operations must also include the RESET.

Definition 4 (Fault-tolerant logical operation). We call a subcircuit represented by a list of gate indices O a fault-tolerant logical operation if the noisy and noiseless operations, defined as

$$\begin{aligned} \mathcal{O}^{(m)} = \prod_{i \in O} \mathcal{O}_i^{(m-1)}, \quad \tilde{\mathcal{O}}^{(m)} = \prod_{i \in O} \mathcal{N}_i^{(m-1, \kappa)} \mathcal{O}_i^{(m-1)} \end{aligned} \quad (138)$$

for any $\kappa, \kappa' \geq 0$, satisfy the condition

$$\tilde{\mathcal{O}}^{(m)} \mathcal{N}^{(m-1, \kappa')} = \mathcal{N}^{(m-1, \kappa' + \kappa|O|)} \mathcal{O}^{(m)}, \quad (139)$$

where $|O|$ is the length of the list O (i.e., the number of operations in O).

In addition to logical operations, the logical circuit requires operations that mitigate the effect of errors. We formulate the main property of such a QEC gadget as follows.

Definition 5 (QEC gadget). We call a subcircuit represented by a list of gate indices C a QEC gadget if, for the operation defined as

$$\tilde{\mathcal{C}}^{(m)} := \prod_{i \in C} \mathcal{N}^{(m-1, \kappa)} \mathcal{O}_i^{(m-1)}, \quad (140)$$

there exists $c > 0$ such that for all $\kappa, \kappa' \geq 0$, the condition

$$\tilde{\mathcal{C}}^{(m)} \mathcal{N}^{(m-1, \kappa')} \mathcal{P}^m = \mathcal{N}^{(m-1, \kappa|C|)} \mathcal{N}^{(m, c(\kappa'^2 + \kappa^2))} \mathcal{P}^m \quad (141)$$

is satisfied, where $|C|$ is the length of the list C , $\mathcal{P}^m = \sum_{v=1}^{2^{N_m}} |v, m\rangle \langle v, m|$ is the projector on the m -level logical space, and $|v, m\rangle$ are m -level logical code words.

The construction of logical gates and QEC gadgets using only unitary gates and RESET operations is well established in the literature. Here, we summarize the results with the following proposition.

Proposition 1. For the $[[7, 1, 3]]$ Steane code, there exists a complete set of fault-tolerant operations, according to Definition 4, that includes RESET and a computationally universal set of gates, as well as a QEC gadget according to Definition 5.

Proof. Consider a universal set of gates consisting of the S , Hadamard, CNOT, T , and Toffoli gates. For the $[[7, 1, 3]]$ Steane code, the Hadamard, S , and CNOT gates are transversal, i.e., they can be implemented as a product of single-qubit or pairwise two-qubit operations [32]. As a result, Eq. (139) is satisfied automatically. A scheme for devising a fault-tolerant T gate and Toffoli gate is provided in Ref. [22]. The measurement-free RESET and QEC gadget are presented in Ref. [23]. ■

Using the notation and definitions introduced above, we can proceed to the proof of Theorem 4, which constitutes the main result of this section. The proof employs similar ideas to those in previous works [20, 21] but explicitly avoids using midcircuit measurements.

Proof. Consider the $[[7, 1, 3]]$ Steane code and a family of noiseless concatenated logical circuit $\mathcal{U}$ consisting of

two parts: (i) initialization of each L -level logical qubit q in $|0\rangle$ state using L -level encoding operation $\mathcal{E}_q^L$; and (ii) a sequence of L -level fault-tolerant logical operations $\mathcal{O}_k^L$ including logical unitary gates and logical RESETs. Part (ii) can be divided into $N_L \geq 0$ layers. The initialization and each operation in turn is represented as a combination of $(L-1)$ -level logical operations forming N_{L-1} layers, and so on. As a result, for any level of concatenation, there exists a circuit representation that implements the target unitary operation,

$$\begin{aligned} \forall m \in [L], \quad \exists D_m \geq 0, d_{mr} \geq 1, M_r^m \in \mathbb{Z}_S^{d_{mr}} : \\ \rho_{\text{out}} := \mathcal{U}(|0\rangle\langle 0|^{\otimes N}) \\ \equiv \prod_{r=1}^{D_m} \left(\prod_{k \in M_r^m} \mathcal{O}_k^{(m)} \right) \prod_{q \in Q_m} \mathcal{E}_q^{(m)}(|0\rangle\langle 0|^{\otimes N}), \end{aligned} \quad (142)$$

where $Q_m = [N' n_B^{L-m}]$ is the list of m -level logical qubits, $\mathcal{O}_k^{(m)}$ are logical gates, being m -level extension of physical gates $\mathcal{O}_k^{(0)}$ that satisfy the recurrence relation in Eq. (138), $\mathcal{E}_q^{(m)}$ is an m -level encoding operation for a logical zero state, M_r^m is the list of parallel m -level logical-gate indices for layer r , D_m is the number of logical-gate layers, d_{mr} is the number of gates in layer r , and S is the total set of fault-tolerant operations including RESET. We assume that M_r^L exactly matches the decomposition of the target noiseless circuit that we want to simulate with the noisy circuit.

Next, consider a family of noisy implementations of the process in Eq. (142). For m -level implementation, we consider the output and the process

$$\begin{aligned} \tilde{\rho}_{\text{out}}^{(m)} := \mathcal{U}_m(|0\rangle\langle 0|^{\otimes N}) \\ \equiv \prod_{r=1}^{D_m} \left(\prod_{q \in Q_m} \tilde{\mathcal{C}}_q^{(m)} \prod_{k \in \tilde{M}_r^m} \tilde{\mathcal{O}}_k^{(m)} \right) \prod_{q \in Q_m} \tilde{\mathcal{E}}_q^{(m)}(|0\rangle\langle 0|^{\otimes N}), \end{aligned} \quad (143)$$

where the $\tilde{\mathcal{C}}_q^{(m)}$ are noisy QEC gadgets (see Definition 5) and the $\tilde{\mathcal{O}}_k^{(m)}$ are noisy m -level logical operations that satisfy the recurrence relation

$$\begin{aligned} \tilde{\mathcal{O}}_j^{(m)} &:= \prod_{i \in O_j} \mathcal{N}^{(m-1, \kappa_m)} \mathcal{O}_i^{(m-1)}, \\ \mathcal{O}_j^m &:= \prod_{i \in O_j} \mathcal{O}_i^{(m-1)}, \end{aligned} \quad (144)$$

where $\mathcal{N}^{(m, \kappa_m)}$ denotes an unspecified noise channel (see Notation 2) of m -dependent noise strength $\kappa_m \geq 0$. Our goal is to find proper strength values κ_m and sets $\tilde{M}_r^m$ such that we can relate $\rho_{\text{out}}^{(m)}$ to $\rho_{\text{out}}^{(0)}$ in Eq. (143) and to the noiseless-circuit output ρ_{out} in Eq. (142).

Let us start our analysis at concatenation level $m = 1$, where we set $\kappa_1 \equiv \kappa$ to be the physical noise strength. The product of noisy fault-tolerant operations in Eq. (144) can be represented as a combination of noisy physical operations and satisfies

$$\begin{aligned}\tilde{\mathcal{O}}_j^1 \mathcal{P}^1 &= \left(\prod_{i \in O_j} \mathcal{N}^{(0,\kappa)} \mathcal{O}_i^{(0)} \right) \mathcal{P}^1 \\ &= \mathcal{N}^{(0,N_O\kappa)} \left(\prod_{i \in O_j} \mathcal{O}_i^{(0)} \right) \mathcal{P}^1 \\ &= \mathcal{N}^{(0,N_O\kappa)} \mathcal{O}_j^1 \mathcal{P}^1 = \mathcal{N}^{(0,N_O\kappa)} \mathcal{P}^1 \mathcal{O}_j^1, \quad (145)\end{aligned}$$

where $\mathcal{O}_j^1 \equiv \prod_{i \in O_j} \mathcal{O}_i^{(0)}$ is the noiseless one-level logical operation, O_j is the list of all gates in the logical operation $\mathcal{O}_j^1$, and $N_O := \sup_j |O_j|$ is the largest number of gates in any such logical operation. Here, in the second transformation, we have used the property of operation fault-tolerance (see Definition 4). A similar statement can be made about the noisy RESET operations:

$$\begin{aligned}\prod_{q \in Q_1} \tilde{\mathcal{E}}_q^{(1)}(|0\rangle\langle 0|^{\otimes N}) &:= \prod_{q \in Q_1} \prod_{i \in E_q} \mathcal{N}^{(0,\kappa)} \mathcal{O}_i^{(0)}(|0\rangle\langle 0|^{\otimes N}) \\ &= \mathcal{N}^{(0,N_O\kappa)} \prod_{q \in Q_1} \prod_{i \in E_q} \mathcal{O}_i^{(0)}(|0\rangle\langle 0|^{\otimes N}) \\ &= \mathcal{N}^{(0,N_O\kappa)} \mathcal{P}^1 \prod_{q \in Q_1} \mathcal{E}_q^{(1)}(|0\rangle\langle 0|^{\otimes N}), \quad (146)\end{aligned}$$

where $\mathcal{E}_q^{(1)}$ is the noiseless one-level encoding operation and E_q is the list of all gates in $\mathcal{E}_q^{(1)}$. The last transition is justified by the fact that noiseless RESET operation $\mathcal{E}_q^{(1)}$

is an m -level pure logical state $|0\rangle\langle 0|_q$ satisfying $|0\rangle\langle 0|_q = \mathcal{P}^1(|0\rangle\langle 0|_q)$. Next, the corrected m -level logical operations satisfy

$$\begin{aligned}\prod_{q \in Q_1} \tilde{\mathcal{C}}_q^1 \prod_{j \in \tilde{M}_r^1} \tilde{\mathcal{O}}_j^1 \mathcal{N}^{(0,N_O\kappa)} \mathcal{P}^1 &= \prod_{q \in Q_1} \tilde{\mathcal{C}}_q^1 \mathcal{N}^{(0,2N_O\kappa)} \mathcal{P}^1 \prod_{k \in \tilde{M}_r^1} \mathcal{O}_k^{(1)} \\ &= \mathcal{N}^{(0,N_O\kappa)} \mathcal{P}^1 \mathcal{N}^{(1,c(4N_O^2+1)\kappa^2)} \prod_{k \in \tilde{M}_r^1} \mathcal{O}_k^{(1)}. \quad (147)\end{aligned}$$

This expression leads to the expression for the noisy-circuit channel:

$$\begin{aligned}\tilde{\rho}_{\text{out}}^2 &= \mathcal{N}^{(0,N_O\kappa)} \prod_{r=1}^{N_1} \left(\prod_{k \in \tilde{M}_r^1} \mathcal{N}^{(1,c(4N_O^2+1)\kappa^2)} \mathcal{O}_k^{(1)} \right) \\ &\times \prod_{q \in Q_1} \mathcal{E}_q^{(1)}(|0\rangle\langle 0|^{\otimes N}). \quad (148)\end{aligned}$$

For any desirable set of layer instructions $\{\tilde{M}_r^2\}$, there exist a number of parallel layers $D_2 < D_1$ and a set of instructions $\{\tilde{M}_r^1\}$ such that

$$\begin{aligned}\prod_{r=1}^{D_1} \left(\prod_{k \in \tilde{M}_r^1} \mathcal{O}_k^{(1)} \right) \prod_{q \in Q_1} \mathcal{E}_q^{(1)} &= \prod_{r=1}^{D_2} \left(\prod_{q \in Q_2} \mathcal{C}_q^{(2)} \prod_{k \in \tilde{M}_r^2} \mathcal{O}_k^{(2)} \right) \prod_{q \in Q_2} \mathcal{E}_q^{(2)}. \quad (149)\end{aligned}$$

To see this, we can also just decompose $\mathcal{E}_q^{(2)} = \left(\prod_{i \in E_q} \mathcal{O}_i^{(1)} \right) \prod_{q \in Q_1} \mathcal{E}_q^{(1)}$ and $\mathcal{O}_j^{(2)} = \sum_{i \in O_j} \mathcal{O}_i^{(1)}$ and combine all gates in the new set of instructions. Using equivalence in Eq. (149), the expression in Eq. (148) becomes

$$\tilde{\mathcal{U}}_1(|0\rangle\langle 0|^{\otimes N}) = \mathcal{N}^{(0,N_O\kappa)} \prod_{r=1}^{N_2} \left(\prod_{q \in Q_2} \tilde{\mathcal{C}}_q^{(2)} \prod_{k \in \tilde{M}_r^2} \tilde{\mathcal{O}}_k^{(2)} \right) \prod_{q \in Q_2} \tilde{\mathcal{R}}_q^{(2)}(|0\rangle\langle 0|^{\otimes N}) = \mathcal{N}^{(0,N_O\kappa)} \tilde{\mathcal{U}}_2(|0\rangle\langle 0|^{\otimes N}), \quad (150)$$

where $\tilde{\mathcal{O}}_k^2$ satisfies Eq. (144) with $\kappa_2 = c(4N_O^2 + 1)\kappa^2$. Repeating the steps in Eqs. (145)–(150), replacing $1 \rightarrow m$ and $2 \rightarrow m + 1$, we obtain the recurrence relation

$$\begin{aligned}\rho_{\text{out}}^{(m)} &:= \tilde{\mathcal{U}}_m(|0\rangle\langle 0|^{\otimes N}) = \mathcal{N}^{(m-1,\kappa_m)} \tilde{\mathcal{U}}_{m+1}(|0\rangle\langle 0|^{\otimes N}) \\ &= \mathcal{N}^{(m-1,\kappa_m)} \rho_{\text{out}}^{(m+1)}, \quad (151)\end{aligned}$$

for $\kappa_m := N_O \mu^{-1} (\mu \kappa)^{2^{m-1}}$ for $m > 1$ and $\mu = c(4N_O^2 + 1)$. This recurrence relation has a solution:

$$\tilde{\mathcal{U}}_1(|0\rangle\langle 0|^{\otimes N}) = \left(\prod_{m \in [L]} \mathcal{N}^{(m-1,\kappa_m)} \right) \tilde{\mathcal{U}}_{L-1}(|0\rangle\langle 0|^{\otimes N}). \quad (152)$$

Next, we choose the set $\{\tilde{M}_r^{L-1}\}$ such that

$$\begin{aligned} & \prod_{r=1}^{N_{L-1}} \left(\prod_{k \in \tilde{M}_r^{L-1}} \mathcal{O}_k^{(L-1)} \right) \prod_{q \in Q_L} \mathcal{E}_q^{(1)} \\ &= \prod_{r=1}^D \left(\prod_{k \in M_r^L} \mathcal{O}_k^{(L)} \right) \prod_{q \in Q_L} \mathcal{E}_q^L. \end{aligned} \quad (153)$$

Using the fault tolerance of the operations, we obtain

$$\tilde{\mathcal{U}}_L = \mathcal{N}^{(L, G_{KL})} \mathcal{U}, \quad (154)$$

where $G = N'D' + 1$ is the total number of gates in the operations in the L -level representation of the circuit $\mathcal{U}$, including RESETs and excluding QEC gadgets (since they do not affect L -level logic errors):

$$\tilde{\rho}_{\text{out}}^1 = \left(\prod_{m \in [L-1]} \mathcal{N}^{(m, \kappa_m)} \right) \mathcal{N}^{(L, G_{KL})} \rho_{\text{out}}, \quad (155)$$

where $G = N'D'$ is the total number of gates in the operations in the L -level representation of the circuit. This expression connects the output physically implementable circuit $\tilde{\rho}_{\text{out}}^1$ in Eq. (143) and the output of the logical circuit ρ_{out} in Eq. (142). Without loss of generality, we consider individual qubit measurements in the z basis, i.e.,

$$z_q = \text{Tr}(Z_q^{(L)} \tilde{\rho}_{\text{out}}^1), \quad (156)$$

where $Z^{(L)}$ is the L -level Pauli- Z operator applied to qubit q .

For the concatenated Steane code, the errors for levels $m \leq L-1$ can be mitigated by taking a majority vote over the logical block for each logical qubit. Thus, the probability that the channel $\mathcal{N}^{(m, \kappa_m)}$ will affect the value of a logical level- L qubit measurement is given by

$$\begin{aligned} \epsilon_m &= O\left(\kappa_m^{\frac{1}{2}(n^{L-m}+1)}\right) \\ &= O\left([N_O \mu^{-1} (\mu \kappa)^{2^{m-1}}]^{\frac{1}{2}(n^{L-m}+1)}\right) \\ &= O\left((\mu \kappa)^{2^{L-2}}\right). \end{aligned} \quad (157)$$

In the case $\kappa < \kappa_0 := \mu^{-1}$, all these errors are marginal compared to the logical error from the L th level of concatenation, which can be expressed as

$$\epsilon = O(N'D' \kappa_L) = O(N'D' (\mu \kappa)^{2^L}). \quad (158)$$

Thus, the number of concatenation levels to achieve the target error ϵ is

$$L = O(\log \log N'D'/\epsilon). \quad (159)$$

Taking into account that each block requires n_B qubits and that each gadget in Proposition 1 requires no more than Q layers of gates, we obtain

$$\begin{aligned} N/N' &= n_B^L = O(\log(D'N'/\epsilon)), \\ D/D' &= Q^L = O(\log(D'N'/\epsilon)). \end{aligned} \quad (160)$$

This expression concludes our proof. $\blacksquare$

E. Main result

In this section, we prove our main results by combining results from previous sections.

Theorem 1 (Formal). Consider quantum circuits with noise generated by a general contracting channel, $\mathcal{N}$, characterized by $\kappa = \|\mathcal{N} - \mathcal{I}\|_\diamond$ and a fixed point σ^* with purity parameter $\eta = \sqrt{2\text{Tr}[(\sigma^*)^2]} - 1$. Then, for any $\eta > 0$, $\epsilon > 0$, and integer $d \geq 1$, there exists $\kappa = O(\eta^{\mu/d})$ for any $\mu > \mu_0 = \log 3 / \log(3/2)$, such that an N -qubit d -dimensional noisy circuit of depth D can simulate the output of any N' -qubit d -dimensional noiseless circuit of depth D' with an error of at most ϵ and overhead

$$\begin{aligned} \frac{N}{N'} &= O((\kappa \eta)^{-\mu d/(\mu+d)} \text{polylog}(\kappa \eta^{-\mu/d}) \log(N'D'/\epsilon)), \\ \frac{D}{D'} &= O((\kappa \eta)^{-\mu/(\mu+d)} \log(N'D'/\epsilon)). \end{aligned} \quad (161)$$

Proof. The proof of this theorem follows directly by combining Corollary 1 and Theorem 4. Corollary 1 implies that there exists a local noisy unitary circuit with noise strength $\kappa > 0$ that can emulate any given local RESET-enabled noisy unitary circuit with a noise strength $\kappa' > 0$ in Eq. (129). Then, by invoking Theorem 4, we demonstrate that this RESET-enabled circuit can be employed to simulate the target unitary circuit up to error ϵ with an overhead of $O(\log(D'N'/\epsilon))$, provided that κ' is below a fixed constant threshold. According to the expression in Eq. (129), this condition can be satisfied for sufficiently small values of $\kappa = O(\eta^{\mu/d})$. Then, the total overheads in terms of the qubit count N/N' and circuit depth D/D' are obtained by taking the product of the respective overheads for error correction, given in Eq. (132), and for the implementation of RESET, as defined in Eq. (130). This yields the statement of this theorem. $\blacksquare$

If the error correction has a threshold p_0 , this means that we must set $\kappa' \leq p_0$, which, using Corollary 1, leads to a

threshold in the physical noise rate:

$$\kappa = \tilde{O}(p_0(\eta p_0)^{\mu/d}). \quad (162)$$

As an additional use case, let us consider a scenario in which passive cooling of the ancilla can be achieved via engineered noise (e.g., amplitude damping) at a rate faster than the intrinsic noise, i.e., $\tilde{\kappa} = \alpha\kappa$. If α is large enough, one can use only passive qubits to implement RESETs. However, this is only feasible when $\alpha = \tilde{O}(p_0^{-1})$; otherwise, the time required to reset qubits would introduce errors on the idling qubits that exceed the threshold.

Our results show that fault tolerance is achievable in this case for any value of α if we use the full RESET construction using CQC. To see this, note that in this case we must replace Eq. (117) with

$$z_{\text{in}}[\mathcal{A}_k] = \Omega(\alpha\kappa\eta T). \quad (163)$$

With the substitution effectively replacing $\eta \rightarrow \eta\alpha$, the rest of the analysis remains unchanged. Therefore, we arrive at the effective noise rate

$$\kappa' = O\left((\kappa(\eta\alpha)^{-\mu/d})^{\frac{d}{\mu+d}}\right) \quad (164)$$

and the corresponding threshold condition,

$$\kappa = \tilde{O}(p_0(\alpha\eta p_0)^{\mu/d}), \quad (165)$$

which, of course, is valid only if αp_0 is small. We emphasize that this result constitutes a universal asymptotic estimate. Identifying a concrete scheme tailored to a specific noise channel remains an important and promising direction for future research.

IV. ANALYSIS OF RANDOM CIRCUITS

In this section, we prove our results corresponding to random quantum circuits in the presence of noise.

A. Algebra of subset distances

We represent a noisy Haar-random circuit as a sequence of gate operations, where each gate is described by a super-operator denoted as $\tilde{\mathcal{U}}_k(\cdot)$, which acts on a specific subset

of qudits Ω_k and can be expressed in the following form:

$$\begin{aligned} \tilde{\mathcal{U}}_k &= \mathcal{N}_k \circ \mathcal{U}_k, \\ \mathcal{U}_k(\rho) &:= U_k \rho U_k^\dagger, \\ \mathcal{N}_k(\rho) &:= \sum_{i=1}^{q^{2|\Omega_k|-1}} K_k^i \rho K_k^{i\dagger}, \end{aligned} \quad (166)$$

where $U_k \sim \mathcal{B}$ is a unitary transformation acting on qudits in Ω_k , sampled from the Haar distribution $\mathcal{B}$, the K_k^i are Kraus operators satisfying $\sum_i K_k^{i\dagger} K_k^i = I$, and $|\Omega_k|$ represents the number of qudits in the set Ω_k . Following standard approximations, we assume that the operators K_k^i are independent of the gates U_k .

Then, the state of the qubits after applying k gates takes the form

$$\rho_k := \mathcal{C}_k \circ \cdots \circ \mathcal{C}_1(\rho_{k=0}), \quad (167)$$

where $\rho_{k=0}$ denotes the input state.

Given an arbitrary pair of states ρ_k and σ_k as defined in Eq. (167), we focus on their *subset distances*. First, we define the marginal trace distance for a subset G as

$$\begin{aligned} T_{G,k} &:= \frac{1}{2} \mathbb{E}_{\mathcal{B}} \|\text{Tr}_{F \setminus G}(\rho_k - \sigma_k)\|_1 \\ &\equiv \frac{1}{2} \mathbb{E}_{\mathcal{B}} \text{Tr}_G |\text{Tr}_{F \setminus G}(\rho_k - \sigma_k)|, \end{aligned} \quad (168)$$

where F denotes the set of all qubits, Tr_A represents the partial trace over the subset of qubits A , $\mathbb{E}_{\mathcal{B}}$ denotes the gate average over the Haar distribution, $|O| := \sqrt{O^\dagger O}$ is the absolute value of the operator, and $\|M\|_1 := \text{Tr} \sqrt{M^\dagger M}$ is the trace norm. The set-theoretic notation is illustrated in Fig. 3.

We also define the marginal Hilbert-Schmidt (HS) distance as

$$\begin{aligned} S_{G,k} &:= \frac{1}{2} \mathbb{E}_{\mathcal{B}} \|\text{Tr}_{F \setminus G}(\rho_k - \sigma_k)\|_2^2 \\ &\equiv \frac{1}{2} \mathbb{E}_{\mathcal{B}} \text{Tr}_G [\text{Tr}_{F \setminus G}(\rho_k - \sigma_k)^2], \end{aligned} \quad (169)$$

where $\|M\|_2 = \sqrt{\text{Tr} M^\dagger M}$ is the Frobenius norm. The trace distance and the HS distance are related by the

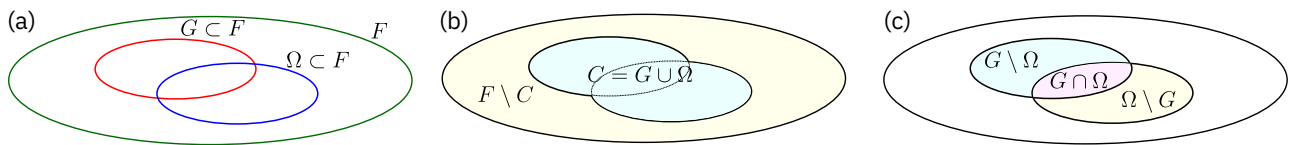


FIG. 3. The set-theory notation. Venn diagrams illustrating different subsets and their intersections. (a) The complete set F and the subsets $G \subset F$ and $\Omega \subset F$. (b) The union of two sets $C = G \cup \Omega$ (blue) and the difference of two sets $F \setminus C$ (yellow). (c) Along with the set differences $G \setminus \Omega$ (blue) and $\Omega \setminus G$ (yellow), we also illustrate the intersection $G \cap \Omega$ of the subsets G and Ω (red).

inequalities [33]

$$S_{G,k} \leq T_{G,k}, \quad T_{G,k}^2 \leq N_G S_{G,k}, \quad (170)$$

where $N_G := q^{|G|}$ is the Hilbert-space dimension of the subset G . The first inequality follows from the monotonicity of the Schatten norms [34], while the second inequality results from the Cauchy-Schwarz inequality. By default, $S_{\emptyset,k} = T_{\emptyset,k} = 0$, where $\emptyset$ is the empty set of qubits.

In the derivations, we use the HS distance as a convenient measure that can be averaged over the Haar distribution using the well-known expression for the fourth moments of random matrix elements. One can then identify simple linear transformations that describe the evolution of these distances. We summarize these transformations in the following proposition.

Proposition 2 (Evolution of HS distances). Suppose that the d th gate is a noisy Haar-random gate in the form of Eq. (166) and applied to a subset $\Omega_k \equiv \Omega$. For any subset $G \subset F$, the HS distance in Eq. (169) satisfies

$$S_{G,k} = c_1 S_{G \setminus \Omega, k-1} + c_2 S_{G \cup \Omega, k-1}, \quad (171)$$

where the coefficients are

$$c_1 = \frac{N_{G \cap \Omega} (N_{\Omega \setminus G}^2 u - v)}{N_{\Omega}^2 - 1}, \quad c_2 = \frac{N_{\Omega \setminus G} (N_{G \cap \Omega}^2 v - u)}{N_{\Omega}^2 - 1}, \quad (172)$$

and the noise parameters are

$$u = \frac{1}{N_{\Omega} N_{\Omega \setminus G}} \text{Tr}_{G \cap \Omega} \left| \text{Tr}_{\Omega \setminus G} \sum_i K_k^i K_k^{i\dagger} \right|^2, \quad (173)$$

$$v = \frac{1}{N_{\Omega} N_{G \cap \Omega}} \sum_{ij} \text{Tr}_{\Omega \setminus G} \left| \text{Tr}_{G \cap \Omega} K_k^i K_k^{j\dagger} \right|^2.$$

Proof. Since the entire proof concerns the effect of a single gate k , we can simplify the notation by omitting the index k , i.e., setting $K_k^i \equiv K^i$, $U_k \equiv U$, and $\Omega_k \equiv \Omega$. Combining the definition of the HS distance in Eq. (169), the explicit form of the gate channel in Eq. (166), and the set-theoretic notation (see Fig. 3), we write

$$S_{G,k} = \mathbb{E}_U \text{Tr}_G \left(\text{Tr}_{F \setminus G} K^i U \delta \rho_{k-1} U^\dagger K^{i\dagger} \right)^2, \quad (174)$$

where $\delta \rho_k := \rho_k - \sigma_k$, $\mathbb{E}_U$ denotes the average over the gate unitary $U \sim \mathcal{B}_\Omega$, sampled from the Haar distribution over the subset Ω . The sum is taken over $i = 1, \dots, 4^{|\Omega|} - 1$, as in Eq. (166). Here and for the rest of the proof, we also adopt the “silent-sum” convention: if an index appears twice in a single term, it implies summation over that index without explicitly writing the summation symbol.

Next, let us divide the entire set into four complementary parts, $F \setminus (G \cup \Omega)$, $G \setminus \Omega$, $G \cap \Omega$, and $\Omega \setminus G$ (for illustration, see Fig. 3). Using this division, we can introduce two basis states defined on the full qubit set F and the subset G , respectively,

$$|\xi A \alpha a\rangle := |\xi\rangle_{F \setminus (G \cup \Omega)} \otimes |A\rangle_{G \setminus \Omega} \otimes |\alpha\rangle_{G \cap \Omega} \otimes |a\rangle_{\Omega \setminus G},$$

$$|\alpha a\rangle := |\alpha\rangle_{G \cap \Omega} \otimes |a\rangle_{\Omega \setminus G}. \quad (175)$$

In these expressions, $|\xi\rangle_{F \setminus (G \cup \Omega)}$, $|A\rangle_{G \setminus \Omega}$, $|\alpha\rangle_{G \cap \Omega}$, and $|a\rangle_{\Omega \setminus G}$ represent arbitrary basis states of qubits within the sets $F \setminus (G \cup \Omega)$, $G \setminus \Omega$, $G \cap \Omega$, and $\Omega \setminus G$. Using these basis states, we define the following three matrix elements:

$$[\delta \rho]_{\xi A \alpha a, \xi' B \beta b} := \langle \xi A \alpha a | \delta \rho_{d-1} | \xi' B \beta b \rangle,$$

$$K_{\alpha a, \beta b}^i := \langle \alpha a | K^i | \beta b \rangle,$$

$$U_{\alpha a, \beta b} := \langle \alpha a | U | \beta b \rangle. \quad (176)$$

This notation allows us to explicitly rewrite Eq. (174) as

$$S_{G,k} = \mathbb{E}_U [U_{\beta' b', \beta b} U_{\alpha' a', \alpha a}^* U_{\mu' m', \mu m} U_{\nu' n', \nu n}^* K_{\gamma' g', \beta' b'}^i K_{\gamma g, \alpha' a'}^{i*} K_{\gamma g', \mu' m'}^j K_{\gamma' g', \nu' n'}^{j*} [\delta \rho]_{\xi B \beta b, \xi A \alpha a} [\delta \rho]_{\xi' A \mu m, \xi' B \nu n}], \quad (177)$$

where we have used the “silent-sum” convention.

Now, we can use the standard expression for the fourth moment of matrix elements in a Haar ensemble:

$$\mathbb{E}_U [U_{\beta' b', \beta b} U_{\alpha' a', \alpha a}^* U_{\mu' m', \mu m} U_{\nu' n', \nu n}^*] = \frac{1}{N_{\Omega}^2 - 1} \left(\delta_{\alpha \beta} \delta_{ab} \delta_{\mu \nu} \delta_{mn} \delta_{\alpha' \beta'} \delta_{a' b'} \delta_{\mu' \nu'} \delta_{m' n'} + \delta_{\alpha \mu} \delta_{am} \delta_{\beta \nu} \delta_{bn} \delta_{\beta' \nu'} \delta_{b' n'} \delta_{\alpha' \mu'} \delta_{a' m'} \right. \\ \left. - \frac{1}{N_{\Omega}} \delta_{\alpha \beta} \delta_{ab} \delta_{\mu \nu} \delta_{mn} \delta_{\beta' \nu'} \delta_{b' n'} \delta_{\alpha' \mu'} \delta_{a' m'} \right. \\ \left. - \frac{1}{N_{\Omega}} \delta_{\alpha \mu} \delta_{am} \delta_{\beta \nu} \delta_{bn} \delta_{\alpha' \beta'} \delta_{a' b'} \delta_{\mu' \nu'} \delta_{m' n'} \right), \quad (178)$$

where δ_{ab} is Kronecker delta and $N_\Omega := 2^{|\Omega|}$ is the Hilbert-space dimension of the subset Ω . Note that since each symbol is mentioned once in each term, the silent sum does not apply to this expression. Inserting Eq. (178) into Eq. (177), we obtain

$$S_{G,k} = \frac{1}{N_\Omega^2 - 1} \left(K_{\gamma'g,\alpha'a'}^i K_{\gamma g,\alpha'a'}^{i*} K_{\gamma'g',\mu'm'}^j K_{\gamma'g',\mu'm'}^{j*} [\delta\rho]_{\xi B\alpha a,\xi A\alpha a} [\delta\rho]_{\xi' A\mu m,\xi' B\mu m} \right. \\ + K_{\gamma'g,\beta'b'}^i K_{\gamma'g',\beta'b'}^{j*} K_{\gamma g',\alpha'a'}^j K_{\gamma g,\alpha'a'}^{i*} [\delta\rho]_{\xi B\beta b,\xi A\alpha a} [\delta\rho]_{\xi' A\alpha a,\xi' B\beta b} \\ - \frac{1}{N_\Omega} K_{\gamma'g,\beta'b'}^i K_{\gamma'g',\beta'b'}^{j*} K_{\gamma g',\alpha'a'}^j K_{\gamma g,\alpha'a'}^{i*} [\delta\rho]_{\xi B\alpha a,\xi A\alpha a} [\delta\rho]_{\xi' A\mu m,\xi' B\mu m} \\ \left. - \frac{1}{N_\Omega} K_{\gamma'g,\alpha'a'}^i K_{\gamma g,\alpha'a'}^{i*} K_{\gamma'g',\mu'm'}^j K_{\gamma'g',\mu'm'}^{j*} [\delta\rho]_{\xi B\beta b,\xi A\alpha a} [\delta\rho]_{\xi' A\alpha a,\xi' B\beta b} \right). \quad (179)$$

These products of matrix elements can be represented as follows:

$$S_{G,k+1} = \frac{1}{N_\Omega^2 - 1} \left(\text{Tr}_{G \cap \Omega} \left| \sum_\alpha \text{Tr}_{\Omega \setminus G} K^i K^{j\dagger} \right|^2 S_{G \setminus \Omega,k} + \sum_{ij} \text{Tr}_{\Omega \setminus G} |\text{Tr}_{\Omega \cap G} K^i K^{j\dagger}|^2 S_{G \cup \Omega,k} \right. \\ \left. - \frac{1}{N_\Omega} \sum_{ij} \text{Tr}_{\Omega \setminus G} |\text{Tr}_{\Omega \cap G} K^i K^{j\dagger}|^2 S_{G \setminus \Omega,k} - \frac{1}{N_\Omega} \text{Tr}_{G \cap \Omega} \left| \sum_\alpha \text{Tr}_{\Omega \setminus G} K^i K^{j\dagger} \right|^2 S_{G \cup \Omega,k} \right). \quad (180)$$

Next, we use the definitions of the parameters u and v given in Eq. (173) to obtain

$$S_{G,k+1} = \frac{1}{N_\Omega^2 - 1} \left([N_\Omega N_{\Omega \setminus G} u - N_{G \cap \Omega} v] S_{G \setminus \Omega,k} + [N_\Omega N_{G \cap \Omega} v - N_{\Omega \setminus G} u] S_{G \cup \Omega,k} \right). \quad (181)$$

Finally, we consider that $\Omega \equiv (\Omega \setminus G) \cup (G \cap \Omega)$. Therefore, the dimension of the Hilbert space for the set Ω is given by

$$N_\Omega = N_{\Omega \setminus G} N_{G \cap \Omega}. \quad (182)$$

Inserting this expression into Eq. (181), we obtain the results in Eqs. (171) and (172). This last step completes our proof. $\blacksquare$

B. Two-qudit gates case, uniform symmetric noise

Consider the case of a circuit that consists of two-qudit gates subject to a noise model that does not depend on the gate (i.e., uniform) and symmetric in respect of swapping the qubits. We introduce the following two sets of parameters. The first set is

$$a = \frac{1}{q^4} \sum_{ij} |\text{Tr}_\Omega K^i K^{j\dagger}|^2, \quad b = \frac{1}{q^2} \text{Tr}_\Omega \left| \sum_i K^i K^{i\dagger} \right|^2. \quad (183)$$

Let $\Omega_k = \{m, m'\}$, where m and $m' \neq m$ are indices of two qubits that constitute the set Ω_k . Then, we define

$$A = \frac{1}{q^2} \sum_{ij} \text{Tr}_m |\text{Tr}_{m'} K^i K^{j\dagger}|^2, \quad (184) \\ B = \frac{1}{q^2} \text{Tr}_m \left| \text{Tr}_{m'} \sum_i K^i K^{i\dagger} \right|^2,$$

where Tr_m and $\text{Tr}_{m'}$ indicate partial traces taken over qudit m and m' of the set $\{m, m'\}$, respectively. Note that A and B are independent of the order of the qubits. Using these parameters, we analyze in detail the rules in Eq. (171) for three different cases of alignment between the two-qudit gate subspace Ω and the subset G .

1. Case 1: The gate applies within the subset

Consider $\Omega \in G$, where the gate is applied within the qudit subset G . Then, Eq. (171) leads to

$$S_{G,k} = (\alpha - r) S_{G \setminus \Omega,k-1} + r S_{G,k-1}, \quad (185)$$

where we define the parameters

$$\alpha := \frac{q^2 a + b}{q^2 + 1}, \quad r := \frac{q^4 a - 1}{q^4 - 1}. \quad (186)$$

Note that for a unitary noise channel, we have $a = b = 1$ and, therefore, $S_{G,k} = S_{G,k-1}$.

2. Case 2: The gate intersects with the subset

Consider $\Omega_k \cap G \neq \{\emptyset\}$ and $\Omega_k \setminus G \neq \{\emptyset\}$ and let there be two qudits m and m' such that $G \cap \Omega_k = \{m\}$ and $G \setminus \Omega_k = \{m'\}$. Then, Eq. (171) leads to

$$S_{G,k} = \frac{1}{2}(\beta + \mu)S_{G \setminus \Omega_k, k-1} + \frac{1}{2}(\beta - \mu)S_{G \cup \Omega_k, k-1}, \quad (187)$$

where we have defined

$$\beta := \frac{q}{q^2 + 1}(A + B), \quad \mu := \frac{q}{q^2 - 1}(B - A). \quad (188)$$

Again, for a unitary noise channel, $A_{mm'} = B_{m'm} = 1$, we obtain

$$S_{G,k} = \frac{q^2}{q^2 + 1}(S_{G \setminus \Omega_k, k-1} + S_{G \cup \Omega_k, k-1}). \quad (189)$$

This expression is exactly the same as for a noiseless circuit, because multiplying a Haar-random unitary by any unitary does not change the distribution.

3. Case 3: The gate applies outside the subset

Finally, consider the case $\Omega \cap G = \{\emptyset\}$, i.e., the gate applies to the complementary subset $F \setminus G$. Then, Eq. (171) leads to a simple relation

$$S_{G,k} = S_{G,k-1}. \quad (190)$$

C. Logarithmic depth lower bound for local circuits

Consider the case of qubits $q = 2$. We recall that the *parallel layer* is a part of the circuit where each qubit participates in one and only one gate.

Theorem 2 (Restatement). Consider a noisy random unitary circuit consisting of D parallel layers of two-qubit gates. For any bit strings $\mathbf{z}, \mathbf{z}' \in \{0, 1\}^n$, with $\mathbf{z} \neq \mathbf{z}'$, the corresponding initial quantum states $\rho = |\mathbf{z}\rangle\langle\mathbf{z}|$ and $\sigma = |\mathbf{z}'\rangle\langle\mathbf{z}'|$ satisfy

$$\frac{1}{2}\mathbb{E}_{\mathcal{B}}\|\mathcal{C}(\rho - \sigma)\|_1 \geq e^{-\Gamma D}, \quad (191)$$

where $\Gamma > 0$ is a constant, $\mathcal{C}$ is a noisy random circuit channel, and $\mathbb{E}_{\mathcal{B}}$ denotes the expectation over the random two-qubit unitaries.

Proof. Consider the marginal HS distance $S_{\{i,j\},l}$ between states ρ and σ after $0 \leq l \leq D - 1$ parallel layers of the circuit for any pair of qubits i and j . Let us define the quantity

$$S_l^{(2)} := \max_{\{i,j\} \in C_{l+1}} S_{\{i,j\},l}, \quad (192)$$

where C_l (layer connectivity) is the set of all two-qubit pairs on which gates at the layer l act. Let us now consider the evolution of $S_{\{i,j\},k}$. For the parallel circuit, depending on the choice of the qubit group and the layer, there are two possible scenarios.

In the first scenario, qubits i and j are subject to the same two-qubit gate. Then, we can use Eq. (185) and rewrite

$$S_{\{i,j\},l} = rS_{\{i,j\},l-1}, \quad (193)$$

where we have taken into account that the HS distance for any empty subset is zero by default, and r is taken from Eq. (186).

In the second scenario, the qubits i and j are each subjected to different gates, coupled to qubits i' and j' , respectively. First, consider the action of the gate between qubits i and i' . For this, we introduce the “midlayer” values of the HS distance $S'_{\Omega,l}$ for the state of the qubits before the gate between qubits i and i' is applied. Using Eq. (187), we obtain

$$S_{\{i,j\},l} = \frac{1}{2}(\beta + \mu)S_{\{j\},l} + \frac{1}{2}(\beta - \mu)S'_{\{i,i'\},l}, \quad (194)$$

where

$$\beta = \frac{q}{q^2 + 1}(A_{ii'} + B_{i'i}), \quad \mu = \frac{q}{q^2 - 1}(B_{i'i} - A_{ii'}). \quad (195)$$

Next, we take into account the effect of the gate between j and j' , which leads us to

$$\begin{aligned} S'_{\{j\},l} &= \frac{1}{2}(\beta - \mu)S_{\{j,j'\},l-1} \\ S'_{\{i,i'\},l} &= \frac{1}{2}(\beta + \mu)S_{\{i,i'\},l-1} \\ &\quad + \frac{1}{2}(\beta - \mu)S_{\{i,i',j,j'\},l-1}, \end{aligned} \quad (196)$$

where

$$\beta' = \frac{q}{q^2 + 1}(A + B), \quad \mu' = \frac{q}{q^2 - 1}(B - A). \quad (197)$$

Combining the expressions in Eqs. (194) and (196), we obtain

$$\begin{aligned} S_{\{i,j\},l} &= \frac{1}{4}(\beta^2 - \mu^2) \left(S_{\{i,i'\},l} + S_{\{j,j'\},l-1} \right) \\ &\quad + \frac{1}{4}(\beta - \mu)^2 S_{\{i,i',j,j'\},l-1} \\ &\geq \frac{1}{4}(\beta^2 - \mu^2) (S_{\{i,i'\},l-1} + S_{\{j,j'\},l-1}), \end{aligned} \quad (198)$$

where we have used the fact that the HS distance is non-negative. We observe that after applying either Eq. (193) or Eq. (198), depending on the specific case, to the HS of all qubit pairs in C_{l+1} , at least one pair will be expressed in terms of the most recent HS from C_l . Thus,

$$\begin{aligned} S_l^{(2)} &\geq e^{-\Gamma} S_{l-1}^{(2)}, \\ \Gamma &= -\log \left(\min \left\{ r, \frac{1}{4}(\beta^2 - \mu^2) \right\} \right). \end{aligned} \quad (199)$$

This expression allows us to derive the full solution

$$\max_{i \neq j} S_{\{i,j\},D} \geq e^{-\Gamma} S_{D-1}^{(2)} \geq S_0^{(2)} e^{-\Gamma D} = e^{-\Gamma D}, \quad (200)$$

where we have taken into account that for two distinct bit strings, $S_0^{(2)} = 1$. Consider qubit indices $\{a, b\}$ such that $S_{\{a,b\},D} = \max_{i \neq j} S_{\{i,j\},D}$. Then, using the inequality in Eq. (170), we obtain

$$T_{F,D} := \frac{1}{2} \mathbb{E} \|\mathcal{C}(\rho - \sigma)\|_1 \geq T_{\{a,b\},D} \geq S_{\{a,b\},D} \geq e^{-\Gamma D}, \quad (201)$$

which completes the proof. $\blacksquare$

D. Linear-depth upper bound for local circuits

Theorem 5. Consider an n -qubit noisy random unitary circuit consisting of D parallel layers of two-qubit gates. For any pair of input states $\rho, \sigma \in \mathbb{S}_{2^n}$, we have

$$\frac{1}{2} \mathbb{E}_B \|\mathcal{C}(\rho - \sigma)\|_1 \leq 2^{n/2} e^{-\Gamma' D}, \quad (202)$$

where $\Gamma' > 0$ is a constant, $\mathcal{C}$ is a noisy random circuit channel, and $\mathbb{E}$ denotes the expectation over the random two-qubit unitaries.

Proof. Let us introduce a new set of parameters $\{S'_{G,k}\}$ in the following way. First, we set that if the d th gate applies

within set G , i.e., $\Omega \subset G$, but $\Omega \neq G$, we have

$$S'_{G,k} = S_{G,k} = \alpha \left[\left(1 - \frac{r}{\alpha}\right) S_{G \setminus \Omega, k-1} + \frac{r}{\alpha} S_{G, k-1} \right], \quad (203)$$

where α is defined in Eq. (186). The exception to this is when $G = \Omega$. Then,

$$S'_{\Omega, k} = \alpha S_{\Omega, k-1}. \quad (204)$$

In this case, $S_{\Omega, k} = r S_{\Omega, k-1} \leq S'_{\Omega, k}$ because $r \leq \alpha$. Next, if Ω and G overlap but $\Omega \setminus G \neq \{\emptyset\}$, we obtain

$$\begin{aligned} S'_{G,k} &= S_{G,k} \\ &= \beta \left[\left(1 + \frac{\mu}{2\beta}\right) S_{G \setminus \Omega, k-1} + \left(1 - \frac{\mu}{2\beta}\right) S_{G, k-1} \right]. \end{aligned} \quad (205)$$

Finally, if the gate applies outside G , i.e., $G \cap \Omega = \{\emptyset\}$, we obtain

$$S'_{G,k} = S_{G,k} = S_{G, k-1}. \quad (206)$$

To summarize, the quantities $S'_{G,k} = S_{G,k}$ for all G except when $G = \Omega$, in which case they are given by Eq. (204). Therefore, we set that $S'_{G,k} \leq S_{G,k}$.

Using Eqs. (203)–(206), we get that there exist integers N_1 and N_2 and coefficients $1 \geq p(Q, G) \geq 0$ corresponding to each subset Q, G in $\mathcal{F}$ excluding the empty set (denoted as $\mathcal{F} \setminus \{\emptyset\}$) and satisfying $\sum_{G \in \mathcal{F} \setminus \{\emptyset\}} p(Q, G) = 1$ such that

$$\begin{aligned} \forall G' \in \mathcal{F} \setminus \{\emptyset\}: \\ S'_{Q,k} &= \alpha^{N_1} \beta^{N_2} \sum_{G \in \mathcal{F} \setminus \{\emptyset\}} p(Q, G) S_{G, k-ND/2}. \end{aligned} \quad (207)$$

Next, because every qudit participates at least at each gate and because Q cannot be empty, we have

$$2 \leq N_1 + N_2 \leq ND/2. \quad (208)$$

Then, we can transform

$$\begin{aligned} S_{F,k} &\leq S'_{F,k} \leq e^{-2\gamma D} \sum_{G \in \mathcal{F} \setminus \{\emptyset\}} p(Q, G) S_{G,0} \\ &\leq e^{-2\gamma D} \sum_{G \in \mathcal{F} \setminus \{\emptyset\}} p(Q, G) \\ &\leq e^{-2\Gamma' D}, \end{aligned} \quad (209)$$

where $\gamma = -\log \max(\alpha, \beta)$ as in the condition of the theorem. Using Eq. (170), we then obtain

$$T_{F,k} \leq \sqrt{N_F S_{F,k}} = 2^{n/2} e^{-\Gamma' D}, \quad (210)$$

which completes the proof. $\blacksquare$

E. Constant-depth upper bound for all-to-all random circuits

Consider a model in which each layer consists of a global Haar-random n -qubit gate, followed by a layer of single-qubit noise channels. We then consider the operator

$$X_d := \mathcal{C}_d(\rho - \sigma), \quad (211)$$

where we assume that ρ and σ are two orthogonal pure states. Here, $\mathcal{C}_d$ represents the noisy unitary circuit channel,

$$\mathcal{C}_d := \mathcal{N}^{\otimes n} \circ \mathcal{U}_d \cdots \mathcal{N}^{\otimes n} \circ \mathcal{U}_1, \quad (212)$$

where $\mathcal{U}_k$ is the unitary transformation through the k th Haar-random unitary gate, and $\mathcal{N}$ is the single-qubit noise channel,

$$\mathcal{N}(\rho) = \sum_{i=1}^q K_i \rho K_i^\dagger, \quad (213)$$

which results in the total channel

$$\mathcal{N}^{\otimes n}(\rho) := \sum_{\mathbf{s} \in C_n} \hat{K}_{\mathbf{s}} \rho \hat{K}_{\mathbf{s}}^\dagger, \quad \hat{K}_{\mathbf{s}} = K_{s_1} \circ K_{s_2} \circ \cdots \circ K_{s_n}, \quad (214)$$

where $K_1, \dots, K_q$ are the Kraus operators for a single-qubit channel and C_n is the set of all combinations of n numbers taking values $s_i \in \{1, \dots, q\}$. Our goal is to find the expectation of the trace distance between the initially orthogonal states ρ and σ , i.e.,

$$\frac{1}{2} \mathbb{E}_{\mathcal{U}} \|X_d\|_1 \equiv \frac{1}{2} \mathbb{E}_{\mathcal{U}} \text{Tr} \sqrt{X_d^2}, \quad (215)$$

where $\mathbb{E}_{\mathcal{U}}$ is the expectation over Haar-random gates. Then, we prove that for the replacement channel (see Definition A1), the following result holds.

Theorem 3 (Restatement). Consider an n -qubit all-to-all random circuit of depth $D > 1$ subject to noise generated by a replacement channel with the rate parameter $0 < \gamma \leq 1$. Then,

$$\frac{1}{2} \mathbb{E} \|\mathcal{C}(\rho - \sigma)\|_1 \leq O\left(2^{n/2} \left(1 - \frac{\gamma}{2}\right)^{n(D-1)/2}\right), \quad (216)$$

where $\mathcal{C}$ denotes the channel corresponding to the noisy random circuit and $\mathbb{E}$ denotes the expectation over the gate unitaries.

Proof. To obtain an upper bound on this expectation, we use the invariance of the trace distance with respect to the

unitary transformation and the concavity of the square-root operation, which gives us the inequality

$$\begin{aligned} \mathbb{E} \text{Tr} \sqrt{X_d^2} &= \mathbb{E} \text{Tr} \sqrt{\mathcal{U}^\dagger [X_d^2]} \leq \frac{1}{2} \text{Tr} \sqrt{\mathbb{E} \mathcal{U}^\dagger [X_d^2]} \\ &= \text{Tr} \sqrt{\mathbb{E}_1 \mathcal{U}_1^\dagger \cdots \mathbb{E}_d \mathcal{U}_d^\dagger [X_d^2]}, \end{aligned} \quad (217)$$

where $\mathcal{U} := \prod_k \mathcal{U}_k$ is the noiseless-circuit representation. Next, for any X that satisfies $\text{Tr} X = 0$, we obtain [see Eq. (A17) in the Appendix A]

$$\begin{aligned} \mathbb{E}_k \mathcal{U}_k^\dagger [(\mathcal{N} \circ \mathcal{U}_k [X])^2] &= \mathbb{E} \sum_{ij} U_k^\dagger (\hat{K}_i U_k X U_k^\dagger \hat{K}_i^\dagger) (\hat{K}_j U_k X U_k^\dagger \hat{K}_j^\dagger) U_k \\ &= \alpha_n X^2 + \gamma_n \text{Tr}(X^2) I, \end{aligned} \quad (218)$$

where I is the n -qubit identity operator and α_n and γ_n are described by Eqs. (A18) and (A20) in the Appendix A, respectively. Also, we obtain [see Eq. (A21)]

$$\begin{aligned} \mathbb{E} \text{Tr} (\mathcal{N} \circ \mathcal{U}_k [X])^2 &= \mathbb{E}_U \sum_{ij} \text{Tr} [\hat{K}_j^\dagger \hat{K}_i U X U^\dagger \hat{K}_i^\dagger \hat{K}_j U X U^\dagger] \\ &= \delta_n \text{Tr}(X^2), \end{aligned} \quad (219)$$

where the expression for δ_n is shown in Eq. (A22) in the Appendix A. Based on this algebra, we obtain

$$\mathbb{E}_k \mathcal{U}_k^\dagger [X_k^2] = \alpha_n X_{k-1}^2 + \gamma_n \text{Tr}(X_{k-1}^2) I. \quad (220)$$

Using these formulas self-consistently, we derive that

$$\|\mathcal{C}_d(\rho - \sigma)\|_1 \leq \frac{1}{2} \sqrt{A X_0^2 + B \text{Tr}(X_0^2) I}, \quad (221)$$

where

$$X_0 \equiv \rho - \sigma, \quad (222)$$

and the coefficients A and B can be found as a solution of

$$\begin{pmatrix} A \\ B \end{pmatrix} = \mathcal{T}^d \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}, \quad (223)$$

where $\mathcal{T}$ is a simple 2×2 transfer matrix. The solution of Eq. (223) is

$$A = \alpha_n^d, \quad B = \gamma \frac{\alpha_n^d - \delta_n^d}{\alpha_n - \delta_n} = O(\gamma \max(\alpha, \delta)^{d-1}), \quad (224)$$

where in the last expression we take an asymptotic limit of large d . Using the fact the initial states are pure and

orthogonal, we obtain

$$\Delta(d) \leq \sqrt{A + 2B} + (N_H/2 - 1)\sqrt{2B}, \quad (225)$$

where $N_H = 2^n$ is the dimension of the Hilbert space.

Using the estimates of α_n and γ_n provided in Eq. (A26) of the Appendix A, we obtain that $\alpha_n = O((1 - 7\kappa/8)^n)$, $\gamma_n = O(2^{-n})$, and $\delta_n = O((1 - \gamma/2)^n)$. In this limit, $\alpha_n/\delta_n \rightarrow 0$ as $n \rightarrow \infty$; therefore, for large enough n we have $\max(\alpha_n, \delta_n) = \delta_n$. This means that

$$\begin{aligned} \Delta(d) &= O(\alpha_n^{d/2}) + O(N_H \gamma^{1/2} \delta_n^{d/2}) \\ &= O\left(2^{n/2} \left(1 - \frac{\gamma}{2}\right)^{n(d-1)/2}\right). \end{aligned} \quad (226)$$

This expression concludes our proof. $\blacksquare$

The direct corollary of the theorem is that for $d > N_0 = 1 - 1/\log_2(1 - \gamma/2)$, the distance decays exponentially in the total circuit volume $V = nd$. Thus all-to-all random circuits possess only constant memory.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A

1. Uniformly contracting quantum channels

Below, we provide definitions of two quantum channels that satisfy the conditions in Definition 2, namely, replacement and generalized-damping channels.

Definition A1 (Replacement channel). For any $0 < \gamma \leq 1$ and a fixed state $\sigma^* \in \mathcal{S}_2$, we define the replacement channel $\mathcal{N}_{\text{repl}}$ such that for any state $\rho \in \mathcal{S}_2$, the action of the channel is given by

$$\mathcal{N}_{\text{repl}}(\rho) := (1 - \gamma)\rho + \gamma\sigma^*. \quad (A1)$$

In this formulation, the channel mixes the input state ρ with the fixed state σ^* , where γ controls the degree of replacement. Consider the case in which the state σ^* has purity $\eta = \sqrt{2\text{Tr}((\sigma^*)^2) - 1}$. This channel can be written in the form

$$\mathcal{N}_{\text{repl}}(\rho) = \sum_{\mu=0}^4 K_{\mu} \rho K_{\mu}^{\dagger}, \quad (A2)$$

where the K_{μ} are the following Kraus operators:

$$\begin{aligned} K_0 &= \sqrt{1 - \gamma} I, \\ K_1 &= \sqrt{\gamma \frac{1 + \eta}{2}} |\psi_0\rangle\langle\psi_0|, \\ K_2 &= \sqrt{\gamma \frac{1 + \eta}{2}} |\psi_0\rangle\langle\psi_1|, \\ K_3 &= \sqrt{\gamma \frac{1 - \eta}{2}} |\psi_1\rangle\langle\psi_0|, \\ K_4 &= \sqrt{\gamma \frac{1 - \eta}{2}} |\psi_1\rangle\langle\psi_1|, \end{aligned} \quad (A3)$$

and the $\{|\psi_{\mu}\rangle\}$ are eigenstates of σ^* . The case $\eta = 0$ constitutes the case of a depolarizing channel. The replacement-channel family satisfies the uniform contractivity condition with $\Delta = 1/2$.

Definition A2 (Generalized-damping channel). For any $0 \leq \eta \leq 1$ and $0 < \gamma \leq 1$, we define the (generalized) damping channel as

$$\mathcal{N}_{\text{damp}}(\rho) := \sum_{\mu=1}^4 K_{\mu} \rho K_{\mu}^{\dagger}, \quad (A4)$$

where

$$\begin{aligned} K_1 &= \sqrt{\frac{1 + \eta}{2}} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \gamma} \end{pmatrix}, \\ K_2 &= \sqrt{\frac{1 - \eta}{2}} \begin{pmatrix} \sqrt{1 - \gamma} & 0 \\ 0 & 1 \end{pmatrix}, \\ K_3 &= \sqrt{\frac{1 + \eta}{2}} \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \\ K_4 &= \sqrt{\frac{1 - \eta}{2}} \begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}. \end{aligned} \quad (A5)$$

The parameters are chosen such that this channel also has the purity of the steady state σ^* equal to η . The case $\eta = 1$ corresponds to the commonly used notion of the amplitude damping channel. The generalized-damping family also satisfies the uniform contractivity condition with $\Delta = 1/2$.

2. Nonuniformly contracting channels

This section provides a counterexample to the uniform contractivity condition in Definition 2. For each $0 < \varepsilon < 1/\sqrt{2}$, define $\mathcal{N}_{\varepsilon} : \mathcal{M}_2 \rightarrow \mathcal{M}_2$ by its action on the Bloch

vector,

$$\begin{aligned}\rho &= \frac{1}{2}(I + \mathbf{r} \cdot \boldsymbol{\sigma}) \mapsto \\ \mathcal{N}_\varepsilon(\rho) &= \frac{1}{2}(I + (T_\varepsilon \mathbf{r} + \mathbf{t}_\varepsilon) \cdot \boldsymbol{\sigma}),\end{aligned}\quad (\text{A6})$$

where $\boldsymbol{\sigma} = (X, Y, Z)$ and

$$T_\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - \varepsilon^2 \end{pmatrix}, \quad \mathbf{t}_\varepsilon = \begin{pmatrix} 0 \\ 0 \\ \varepsilon \end{pmatrix}. \quad (\text{A7})$$

Equivalently,

$$\mathcal{N}_\varepsilon(\rho) = \frac{1}{2} \left[I + (r_x X + r_y Y + ((1 - \varepsilon^2)r_z + \varepsilon)Z) \right]. \quad (\text{A8})$$

One can check that for $\varepsilon < 1/\sqrt{2}$, the image of the Bloch ball remains within itself, so $\mathcal{N}_\varepsilon$ is CPTP. However,

$$\mathcal{N}_\varepsilon(I) = I + \varepsilon Z \neq I, \quad (\text{A9})$$

i.e., it is nonunital. On traceless inputs $\delta \sim (r_x, r_y, r_z) \cdot \boldsymbol{\sigma}$, the affine shift vanishes and

$$\mathcal{N}_\varepsilon(\delta) = (r_x, r_y, (1 - \varepsilon^2)r_z) \cdot \boldsymbol{\sigma}. \quad (\text{A10})$$

Therefore,

$$\lambda(\varepsilon) = \sup_{\text{Tr } \delta = 0} \frac{\|\mathcal{N}_\varepsilon(\delta)\|_1}{\|\delta\|_1} = 1 - \varepsilon^2. \quad (\text{A11})$$

Hence, $1 - \lambda(\varepsilon) = \varepsilon^2$. The worst-case input for $\|\mathcal{N}_\varepsilon - \mathcal{I}\|_\diamond$ is $I/2$, yielding

$$(\mathcal{N}_\varepsilon - \mathcal{I})(I/2) = \frac{1}{2}(I + \varepsilon Z) - \frac{1}{2}I = \frac{\varepsilon}{2}Z, \quad (\text{A12})$$

the trace norm of which is ε . One can verify that no other input yields a larger deviation; thus,

$$\kappa(\varepsilon) = \|\mathcal{N}_\varepsilon - \mathcal{I}\|_\diamond = \varepsilon. \quad (\text{A13})$$

Therefore, as $\varepsilon \rightarrow 0$, we obtain

$$1 - \lambda(\varepsilon) = \varepsilon^2 = \kappa(\varepsilon)^2 = O(\kappa^2), \quad (\text{A14})$$

so that $(1 - \lambda)/\kappa = \varepsilon \rightarrow 0$, and thus no positive Δ can satisfy $\|\mathcal{N}_\varepsilon(\rho - \sigma)\|_1 \leq (1 - \Delta \kappa)\|\rho - \sigma\|_1$ uniformly across this family.

3. Weingarten calculus

Consider $\mathcal{N}$ to be a single-qubit noise channel, represented as

$$\mathcal{N}(\rho) = \sum_{i=1}^q K_i \rho K_i^\dagger. \quad (\text{A15})$$

Also, consider its action on m qubits forming a subset Ω of the full set,

$$\mathcal{N}^{\otimes m}(\rho) := \sum_{\mathbf{s} \in C_n} \hat{K}_{\mathbf{s}} \rho \hat{K}_{\mathbf{s}}^\dagger, \quad \hat{K}_{\mathbf{s}} = K_{s_1} \circ K_{s_2} \circ \cdots \circ K_{s_m}, \quad (\text{A16})$$

where $K_1, \dots, K_q$ are the Kraus operators for a single-qubit channel and C_m is the set of all combinations of n numbers taking values $s_i \in \{1, \dots, q\}$.

Using the Weingarten calculus, we derive that

$$\begin{aligned}E_U \sum_{ij} U^\dagger \hat{K}_i U X U^\dagger \hat{K}_i^\dagger \hat{K}_j U X U^\dagger \hat{K}_j^\dagger U \\ = \alpha_m X^2 + \beta_m (\text{Tr}_\Omega X)^2 \otimes I_\Omega \\ + \omega_m \text{Tr}_\Omega (X^2) \otimes I_\Omega,\end{aligned}\quad (\text{A17})$$

where $\mathbb{E}_U$ is the expectation over the Haar distribution of m -qubit unitaries U and I_Ω is the identity operator acting on the subset Ω on which U acts, Tr_Ω is the partial trace over the subset Ω , and the coefficients are expressed by the Kraus operators of the noise channel, as

$$\begin{aligned}\alpha_m = \frac{1}{d(d^4 - 5d^2 + 4)} \sum_{ij} d^2 [\text{Tr}(K_i) \text{Tr}(K_i^\dagger K_j) \text{Tr}(K_j^\dagger)]^m + 4 [\text{Tr}(K_i^\dagger K_j K_j^\dagger K_i)]^m + d^2 [\text{Tr}(K_j^\dagger K_i^\dagger K_j K_i)]^m \\ - 2d [\text{Tr}(K_i) \text{Tr}(K_i^\dagger K_j K_j^\dagger)]^m + [\text{Tr}(K_i K_i^\dagger K_j) \text{Tr}(K_j^\dagger)]^m + [\text{Tr}(K_i^\dagger K_j) \text{Tr}(K_j^\dagger K_i)]^m, \quad (\text{A18})\end{aligned}$$

$$\begin{aligned}\beta_m = \frac{1}{d(d^4 - 5d^2 + 4)} \sum_{ij} 2 [\text{Tr}(K_i) \text{Tr}(K_j^\dagger) \text{Tr}(K_i^\dagger K_j)]^m + (d^2 - 2) [\text{Tr}(K_i^\dagger K_j K_j^\dagger K_i)]^m + 2 [\text{Tr}(K_j^\dagger K_i^\dagger K_j K_i)]^m \\ - d [\text{Tr}(K_i^\dagger K_j) \text{Tr}(K_j^\dagger K_i)]^m + [\text{Tr}(K_j^\dagger) \text{Tr}(K_i^\dagger K_j K_i)]^m + [\text{Tr}(K_i) \text{Tr}(K_j^\dagger K_i^\dagger K_j)]^m, \quad (\text{A19})\end{aligned}$$

$$\omega_m = \frac{1}{d(d^4 - 5d^2 + 4)} \sum_{ij} 2[\text{Tr}(K_i)\text{Tr}(K_j^\dagger K_i^\dagger K_j)]^m + 2[\text{Tr}(K_i^\dagger K_j K_i)\text{Tr}(K_j^\dagger)]^m + (d^2 - 2)[\text{Tr}(K_i^\dagger K_j)\text{Tr}(K_j^\dagger K_i)]^m - d \left[[\text{Tr}(K_i)\text{Tr}(K_i^\dagger K_j)\text{Tr}(K_j^\dagger)]^m + [\text{Tr}(K_i K_i^\dagger K_j K_j^\dagger)]^m + [\text{Tr}(K_i K_j^\dagger K_i^\dagger K_j)]^m \right], \quad (\text{A20})$$

where m is the number of qubits participating in the gate and $d := 2^m$. Also, we derive that for operators that satisfy $\text{Tr}_{\Omega} X = 0$, we obtain

$$\mathbb{E}_U \sum_{ij} \text{Tr} \left[\hat{K}_j^\dagger \hat{K}_i U X U^\dagger \hat{K}_i^\dagger \hat{K}_j U X U^\dagger \right] = \delta \text{Tr}(X^2), \quad (\text{A21})$$

where

$$\delta_m = \frac{1}{d(d^2 - 1)} \sum_{ij} d[\text{Tr}(K_i^\dagger K_j)\text{Tr}(K_j^\dagger K_i)]^m - [\text{Tr} K_i K_i^\dagger K_j K_j^\dagger]^m \quad (\text{A22})$$

is a real parameter.

a. Replacement channel

For a single-qubit replacement channel, we have

$$\alpha_m = \frac{1}{N_m} \left(N_m^2 E_1^m + 4E_2^m + N_m^2 E_3^m - 2N_m (E_4^m + E_5^m + E_6^m) \right), \quad (\text{A23})$$

$$\omega_m = \frac{1}{N_m} \left(2E_4^m + 2E_5^m + (N_m^2 - 2)E_6^m - N_m (E_1^m + E_2^m + E_3^m) \right),$$

where $N_m = 2^m$, $N_m = N_m(N_m^4 - 5N_m^2 + 4)$, and we have used the following notation:

$$\begin{aligned} E_1 &:= \sum_{ij} \text{Tr}(K_i)\text{Tr}(K_i^\dagger K_j)\text{Tr}(K_j^\dagger) = 8 - 12\kappa + \frac{1}{2}(9 + \epsilon^2)\kappa^2, \\ E_2 &:= \sum_{ij} \text{Tr}(K_i K_i^\dagger K_j K_j^\dagger) = 2(1 + \epsilon^2\kappa^2), \\ E_3 &:= \sum_{ij} \text{Tr}(K_j^\dagger K_i^\dagger K_j K_i) = 2 + \frac{1}{2}(\epsilon^2 - 3)\kappa^2, \\ E_4 &:= \sum_{ij} \text{Tr}(K_i)\text{Tr}(K_i^\dagger K_j K_j^\dagger) = 4 - 3\kappa + \epsilon^2\kappa^2, \\ E_5 &:= \sum_{ij} \text{Tr}(K_i K_i^\dagger K_j)\text{Tr}(K_j^\dagger) = 4 - 3\kappa + \epsilon^2\kappa^2, \\ E_6 &:= \sum_{ij} \text{Tr}(K_i^\dagger K_j)\text{Tr}(K_j^\dagger K_i) = 4 - 6\kappa + (3 + \epsilon^2)\kappa^2. \end{aligned} \quad (\text{A24})$$

To derive the expression in the asymptotic limit of large m , we first note that the normalization factor behaves as $N_m = \Theta(2^{5m})$. At the same time, we can bound each component of the larger formula as

$$\frac{1}{2} \leq E_1 \leq 8 \left(1 - \frac{7}{8}\kappa\right), \quad 2 \leq E_2 \leq 4 \quad \frac{1}{2} \leq E_3 \leq 2, \quad 1 \leq E_4, E_5 \leq 4 \quad 1 \leq E_6 \leq 4 \left(1 - \frac{1}{2}\kappa\right). \quad (\text{A25})$$

Most of the terms in the above expressions become marginal, which leaves us with

$$\alpha_m = O\left(\left(1 - \frac{7}{8}\kappa\right)^m\right) + O(2^{-2m}), \quad \gamma_m = O(2^{-m}), \quad \delta_m = O\left(\left(1 - \frac{1}{2}\kappa\right)^m\right). \quad (\text{A26})$$

b. Generalized-damping channel

For a generalized-damping channel, we have

$$\begin{aligned}
 E_1 &:= \sum_{ij} \text{Tr}(K_i) \text{Tr}(K_i^\dagger K_j) \text{Tr}(K_j^\dagger) = -2(\sqrt{1-\kappa} + 2)\kappa + 4(\sqrt{1-\kappa} + 1) + \frac{1}{2}\kappa^2(\epsilon^2 + 1), \\
 E_2 &:= \sum_{ij} \text{Tr}(K_i^\dagger K_j K_j^\dagger K_i) = 2(1 + \kappa^2 \epsilon^2), \\
 E_3 &:= \sum_{ij} \text{Tr}(K_j^\dagger K_i^\dagger K_j K_i) = 2(\sqrt{1-\kappa} - 1)\kappa + \frac{1}{2}\kappa^2(\epsilon^2 + 1) + 2, \\
 E_4 &:= \sum_{ij} \text{Tr}(K_i) \text{Tr}(K_i^\dagger K_j K_j^\dagger) = -\kappa + 2\sqrt{1-\kappa} + \kappa^2 \epsilon^2 + 2, \\
 E_5 &:= \sum_{ij} \text{Tr}(K_i K_i^\dagger K_j) \text{Tr}(K_j^\dagger) = -\kappa + 2\sqrt{1-\kappa} + \kappa^2 \epsilon^2 + 2, \\
 E_6 &:= \sum_{ij} \text{Tr}(K_i^\dagger K_j) \text{Tr}(K_j^\dagger K_i) = -4\kappa + \kappa^2(\epsilon^2 + 1) + 4,
 \end{aligned} \tag{A27}$$

$$\begin{aligned}
 \alpha &= \frac{1}{45}(\kappa - 1) \left(-10\kappa^2 + 3(4\sqrt{1-\kappa} + 9)\kappa - 3(8\sqrt{1-\kappa} + 7) \right), \\
 \gamma &= \frac{1}{180}(\kappa - 1) \left(-12(\sqrt{1-\kappa} - 2)\kappa + 24(\sqrt{1-\kappa} - 1) + \kappa^3(6\epsilon^2 + 3) - \kappa^2(18\epsilon^2 + 11) \right),
 \end{aligned} \tag{A28}$$

$$1 \leq E_1 \leq 8(1 - c\kappa), \quad 2 \leq E_2 \leq 4, \quad \frac{1}{2} \leq E_3 \leq 2, \quad 1 \leq E_4, E_5 \leq 4, \quad 1 \leq E_6 \leq 4(1 - \kappa/2), \tag{A29}$$

where $c \approx 0.8452 > 5/6$.

In the asymptotic limit of large m , we obtain

$$\alpha = O\left((1 - c_{\text{damp}}\kappa)^m + 2^{-2m}\right), \quad \gamma = O(2^{-m}). \tag{A30}$$

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