

From Linear Differential Equations to Unitaries: A Moment-Matching Dilation Framework with Near-Optimal Quantum Algorithms

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We introduce a universal dilation framework to simulate arbitrary non-Hermitian dynamics $\dot{\mathbf{x}} = L(t)\mathbf{x}$ on unitary quantum processors. By imposing algebraic moment-matching conditions on an ancilla, our framework generates diverse families of dilation schemes that unify existing methods and broaden the design space for hardware-aware implementations. A tight-binding dilation that maps the ancilla to a 1D lattice with nearest-neighbor hopping is derived, overcoming the connectivity bottlenecks of dilation schemes. This construction handles general dynamics, including gain, without *a priori* rescaling and achieves near-optimal complexity, as demonstrated by benchmarks on dissipative wave propagations.

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I. INTRODUCTION

Nonunitary dynamics are ubiquitous in scientific computing and in the modeling of partial differential equations (PDEs). In physics, they arise in settings ranging from the Fokker-Planck description of stochastic dynamics to the breakdown of bulk-boundary correspondence in non-Hermitian skin effects [1]. From cooling many-body systems via imaginary-time evolution [2] to modeling driven-dissipative chemical dynamics [3], such nonunitary phenomena call for scalable simulation tools that go beyond the native unitary evolution of quantum processors. Classically, the cost of resolving these dynamics is often dominated by the growth of the discretized state dimension, which rapidly becomes prohibitive for realistic models.

To bring nonunitary dynamics within the scope of the well-developed machinery of quantum Hamiltonian simulation [4–6], one must first embed them into a strictly unitary dilation. While constructions such as Schrödingerization [7,8] and linear combination of Hamiltonian simulations (LCHS) [9] provide exact dilations and achieve near-optimal scaling [10,11], they rely on specific ancilla couplings, the implementation of which may require all-to-all qubit connectivity. Moreover, a general theory for constructing an exact dilation remains elusive.

The recent flurry of quantum algorithms for nonunitary dynamics, including Schrödingerization- and LCHS-type approaches as well as methods for systems with gain and loss, naturally raises three questions: (1) Is there a unifying framework that encompasses existing dilations while exposing the freedom in the ancillary construction? (2) Are there constructive ancillary algebraic constraints that guarantee an exact dilation? (3) Can these criteria be leveraged to generate new families of dilations with different structural properties, potentially useful for future hardware-aware implementations?

Main contribution A: *A framework of moment-matching dilations.* This paper gives constructive answers to the first two questions and exhibits a broad design space for the third. We present a universal, *moment-fulfilling* dilation built from an ancilla triple $(F, |r\rangle, (I))$, effectively a structured bath—obeying simple moment identities. These conditions (i) recover Schrödingerization and LCHS as special cases, (ii) establish the algebraic sufficiency conditions for exact dilation, and (iii) expose continuously parameterized families of new dilations. As summarized in Table I, these range from tight-binding models with only nearest-neighbor hoppings, ideal for digital qubit arrays, to continuous-variable realizations suited for photonic platforms. This enlarged design space may facilitate future hardware-aware implementations of nonunitary simulations.

At an intuitive level, the proposed algorithm follows an encode-evolve-evaluate paradigm. Starting from the initial condition $\mathbf{x}(0)$, one first embeds the state into a larger dilation space through the ancillary function/vector $|r\rangle$. In this picture, the ancilla acts as an additional transport direction: the nonunitary gain or loss in the original dynamics is converted into coherent amplitude propagation in the

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enlarged unitary system. For the differential-type construction studied below, this ancilla takes the form of a tight-binding lattice, so that the truncation error can be interpreted through finite-speed propagation and a light-cone mechanism. One then evolves the enlarged system under a unitary dynamics generated by the dilated Hamiltonian. Finally, the target solution is recovered by applying the ancillary evaluation functional $\langle l|$, i.e., by projecting/postselecting onto the appropriate ancilla component. From this perspective, the overall efficiency is governed by three coupled ingredients: the norm of the dilated Hamiltonian, which controls the simulation time scale; the cost of block-encoding the corresponding sparse/unitary generator; and the success probability of the final recovery step, which depends on the chosen encode-evolve-evaluate triple $(F, |r\rangle, \langle l|)$. Beyond query complexity, the practical cost of the algorithm also depends on several implementation overheads: the ancilla dimension required by the chosen dilation, the cost of encoding the ancillary generator into qubits, the locality/connectivity overhead induced by binary versus unary ancilla encodings, and the postselection or amplitude-amplification cost associated with the final recovery step. These overheads depend on the choice of the dilation triple $(F, |r\rangle, \langle l|)$ and will therefore differ across dilation families. In the present paper, these issues are analyzed in detail for the differential-type family.

Main contribution B: *A differential-type dilation with near-optimal complexity.* Among the exact moment-matching families generated by the framework, we analyze in detail the differential-type dilation, since its SBP finite-dimensional approximation yields a sparse tight-binding generator and hence a Hermitian dilation that is amenable to block-encoding.

Here, by “near-optimal” we mean the standard block-encoding benchmark for homogeneous linear differential equations [12]

$$\mathbf{x}'(t) = L(t)\mathbf{x}(t) :$$

query complexity linear in the evolution scale $\alpha_A T$, with

$$\alpha_A := \max_{0 \leq t \leq T} \|L(t)\|,$$

and only logarithmic dependence on $1/\epsilon$, up to the standard polylogarithmic overheads of Hamiltonian simulation. In the dissipative setting, the dependence on the solution enters through the unavoidable postselection factor $\|\mathbf{x}(0)\|/\|\mathbf{x}(T)\|$. A notable feature of the present framework is that we do not assume the Hermitian part

$$K(t) = \frac{1}{2} (L(t) + L(t)^\dagger)$$

to be negative semidefinite, and hence we do not rely on an *a priori* exponential rescaling to reduce the problem to a dissipative one.

This complexity behavior is established in two distinct regimes. In the weakly nonunitary regime $K_{\max} T = \mathcal{O}(1)$, where

$$K_{\max} := \max_{t \in [0, T]} \|K(t)\|,$$

Theorem 2 shows that the differential dilation, after a finite-difference approximation, yields an ϵ -accurate simulation with query complexity

$$\tilde{\mathcal{O}}\left(\left(TH_{\max} + \log \frac{X(T)}{\epsilon}\right) \frac{\|\mathbf{x}(0)\|}{\|\mathbf{x}(T)\|}\right),$$

where $X(T) := \sup_{t \in [0, T]} \|\mathbf{x}(t)\|$. In particular, when L is autonomous, one can invoke optimal time-independent Hamiltonian simulation, so that the precision dependence becomes an additive $\mathcal{O}(\log(1/\epsilon))$ term. In this sense, the precision dependence is more favorable than that of LCHS [10, 11], where the precision overhead remains multiplicative.

In the regime $K_{\max} T \gg 1$, Theorem 3 yields query complexity whose bulk scaling is linear in $(H_{\max} + K_{\max})T$, matching the corresponding bulk dependence in the improved LCHS framework [10] and Schrödingerization [11]. Aside from this leading term, the remaining distinction lies in the precision dependence: our graded-lattice construction has logarithmic dependence on $1/\epsilon$, whereas the current time-dependent LCHS analysis has precision factor $(\log(1/\epsilon))^{1+1/\beta}$ for $0 < \beta < 1$ [10]. A later work of Low and Somma [13] further improved the logarithmic dependence in the dissipative setting for autonomous ODEs.

Notation and conventions

Throughout the paper, we use the standard Dirac notation $|\cdot\rangle$ and $\langle\cdot|$ for normalized quantum states and their duals. For the exact dilation, however, the encoding and evaluation objects may lie outside the natural Hilbert space associated with the operator F . To distinguish them from physical quantum states, we use the round-bracket notation $|r\rangle$ and $\langle l|$. More precisely, $|r\rangle$ denotes a generally unnormalized function (or generalized vector) in the domain $\mathcal{X}$ of F , which in general does not belong to the Hilbert space on which F is Hermitian, while $\langle l| : \mathcal{X} \rightarrow \mathbb{R}$ denotes the corresponding linear functional, which plays the role of the evaluation/postselection map. On other hand, when the exact dilation is approximated on a finite-dimensional ancilla space, these objects are represented by standard normalized states, and the formulas reduce to the standard bra-ket expressions up to an overall scaling factor. In addition, we use $\mathcal{O}(\cdot)$ to denote asymptotic upper bounds and $\tilde{\mathcal{O}}(\cdot)$ to suppress logarithmic factors. Throughout the paper, $\|\cdot\|$ denotes the Euclidean norm for vectors and the spectral norm for matrices.

II. PROBLEM SETUP

We consider the dynamics

$$\dot{\mathbf{x}}(t) = L(t) \mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{C}^N, \quad (1)$$

with $L(t) \in \mathbb{C}^{N \times N}$. By linearity, we set $\|\mathbf{x}_0\| = 1$. Our task is to approximate $\mathbf{x}(t) = \mathcal{T} \exp\left(\int_0^t L(s) ds\right) \mathbf{x}_0$.

Following [7–9] we split

$$L(t) = -iH(t) + K(t), \quad (2)$$

with $H(t) = H(t)^\dagger$ and $K(t) = K(t)^\dagger$. The Hermitian part $K(t)$ in $L(t)$ makes Eq. (1) nonunitary on its natural Hilbert space $\mathcal{H}$, so standard Hamiltonian simulation techniques do not apply directly.

Our strategy is to embed Eq. (1) into a unitary evolution on an enlarged Hilbert space via an **E-E-E** pipeline (Encode $\rightarrow$ Evolve $\rightarrow$ Evaluate). We introduce an ancillary Hilbert space $\mathcal{H}_A$ and an ancilla triple $(F, |r\rangle, \langle l|)$:

(i) **Encode:** $|\Psi(0)\rangle = |r\rangle \otimes |\mathbf{x}_0\rangle$, where $|r\rangle$ is a generalised vector that may lie outside $\mathcal{H}_A$.

(ii) **Evolve:** $|\Psi(t)\rangle = \mathcal{T} \exp\left(-i \int_0^t \tilde{H}(s) ds\right) |\Psi(0)\rangle$, under the dilated Hamiltonian

$$\tilde{H}(t) := I_A \otimes H(t) + iF \otimes K(t), \quad (3)$$

where F is a skew-Hermitian operator on $\mathcal{H}_A$, and thus $\tilde{H}(t)$ is Hermitian on $\mathcal{H}_A \otimes \mathcal{H}$.

(iii) **Evaluate:** $\mathbf{x}(t) = (\langle l| \otimes I) |\Psi(t)\rangle$, for a suitable readout operator $\langle l|$ on the ancilla.

Altogether, we aim to design $(F, |r\rangle, \langle l|)$ such that

$$\mathcal{T} e^{\int_0^t L(s) ds} = (\langle l| \otimes I) \mathcal{T} e^{-i \int_0^t \tilde{H}(s) ds} (|r\rangle \otimes I), \quad \forall t \geq 0. \quad (4)$$

For exact dilations, we set up the ancilla in a function space $\mathcal{X}$ containing $|r\rangle$, with a dense Hilbert subspace $\mathcal{H}_A \subset \mathcal{X}$ where F is skew-Hermitian.

Theorem 1 (Moment-fulfilling dilation). Let $L(t)$ be split as in (2). If $(F, |r\rangle, \langle l|)$ satisfy *moment identities*,

$$\langle l| F^k |r\rangle = 1, \quad \forall k \geq 0, \quad (5)$$

then the dilated evolution and recovery Eq. (4) reproduces the physical evolution $\mathbf{x}(t)$ *exactly* for all $t \geq 0$.

We leave the proof in Appendix A.

A convenient sufficient condition for Eq. (5) is $F|r\rangle = |r\rangle$, $\langle l|r\rangle = 1$. Since F is skew-Hermitian on $\mathcal{H}_A$, $|r\rangle$ must lie in the larger function space $\mathcal{X} \supset \mathcal{H}_A$ to correspond to a real eigenvalue. Approximating $|r\rangle$ from $\mathcal{H}_A$ then yields a universal approximation property for the trajectories of Eq. (1) for any fixed T and ϵ .

A. New moment-fulfilling families

Despite the abstract framework, there are many specific dilation families.

Example 1 (Schrödingerization via warped-phase transformation [7]). The Schrödingerisation approach can be associated with an ancillary Hilbert space,

$$\mathcal{H}_A = \{f, f' \in L^2((0, \infty), dp) \mid f(0) = 0\},$$

with inner product $\langle f, g \rangle := \int_0^\infty f(p)^* g(p) dp$. In addition, the triple is selected as follows:

$$F := -\frac{d}{dp}, \quad |r\rangle = e^{-p}, \quad (\langle l|f\rangle) : f(p) \mapsto e^{p^*} f(p^*). \quad (6)$$

Here $p^* > 0$ is arbitrary. Another choice of the evaluation is $(\langle l|f\rangle) = \int_0^{+\infty} f(p) dp$ [14]. It is clear that $|r\rangle \notin \mathcal{H}_A$, due to the violation of the boundary condition. In this case, the function space $\mathcal{X}$ can be chosen to be $C_0(0, \infty)$ to contain $|r\rangle$. In addition, simple calculations verify that the exact moment conditions $(\langle l|F^k|r\rangle) = 1, k = 0, 1, 2, \dots$ are fulfilled.

Example 2 (Linear combination Hamiltonian simulations (LCHS) [9]). LCHS expresses the solution of the ODE as an integral with the Lorentzian kernel. To put it into the framework of moment conditions, we can choose the ancillary Hilbert space to be the real Schwartz class endowed with the usual L^2 inner product,

$$\mathcal{H}_A = L^2(\mathbb{R}, dk), \quad \langle f, g \rangle = \int_{-\infty}^{\infty} f(k)^* g(k) dk.$$

The Schwartz space $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$ is used as a dense invariant test domain, rather than as the Hilbert space itself. Thus, it is regarded as a construction through a rigged Hilbert space. Multiplication by ik leaves $\mathcal{H}_A$ invariant, hence we may take the skew-Hermitian ancillary generator $F = -ik$. In addition, we set

$$|r\rangle(k) = \frac{-i}{k - i}, \quad (\langle l|f\rangle) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{f(k)}{k + i} dk,$$

thus giving the LCHS formula,

$$\mathcal{T} e^{\int_0^t L(s) ds} = \frac{1}{\pi} \int_{\mathbb{R}} \mathcal{T} e^{-i \int_0^t H(s) + kK(s) ds} \frac{1}{k^2 + 1} dk.$$

Although $|r\rangle \notin \mathcal{S}(\mathbb{R})$, it is naturally paired with Schwartz test functions and may be interpreted in the dual space $\mathcal{S}'(\mathbb{R})$. The moment identities are then understood through this test-function/dual pairing, or equivalently through the corresponding contour-integral representation in the complex k -plane. The moment condition Eq. (5) is then interpreted via contour integrals in the complex k -plane (with

TABLE I. Examples of one-parameter families of exact moment-matching dilations. Here $\mathcal{X}$ denotes the ancillary function space, r is the corresponding encoding function (or generalized vector), and $(l|f)$ denotes the action of the evaluation functional on a general element $f \in \mathcal{X}$. Detailed constructions for all listed examples, including the Bargmann case, are provided in Appendix B.

Dilation type	Ancilla space $\mathcal{X}$	Generator F	Representative r	Evaluation $(l f)$
Differential	$H^1(0, 1)$	$\theta(p\partial_p + \frac{1}{2})$	p^β $\beta = \frac{1}{\theta} - \frac{1}{2}$	$2^\beta f\left(\frac{1}{2}\right)$
Integral	$L^2(\mathbb{R})$	$[Ff](p) = \int_{\mathbb{R}} p e^{-\theta p-q } f(q) dq$	$e^{a(\theta)p}$	$f(0)$
Pseudo-diff.	$L^2(\mathbb{R})$	$-i(-\Delta)^\theta$	$e^{i\xi_0 x}$, $\xi_0 = e^{i\pi/(4\theta)}$	$f(0)$
Bargmann	$\mathcal{B}$	$\theta(a^\dagger - a)$	$e^{\frac{z^2}{2} - \frac{z}{\theta}}$	$f(0)$
Difference	ℓ^2	$[Ff]_n = \theta(f_n - f_{n-1})$	$\{\lambda_\theta^n\}_{n \geq 0}$, $\lambda_\theta = \frac{\theta}{1 + \theta}$	f_0

contributions from the simple poles at $k = i$) and can be verified using Cauchy's residue theorem. LCHS has been improved to near-optimal query complexity [10,13,15,16] and extended to infinite dimensions [17].

The LCHS has an interesting connection to the Schrödingerization method in the previous example. Let $\mathcal{F}$ denote Fourier transform, $\hat{f}(k) = \mathcal{F}f(p) = \int_{\mathbb{R}} f(p) e^{-ikp} dp$, one can apply the inverse Fourier transform to the triple, to maintain the moment condition under a unitary similarity transformation:

$$1 = (l|F^k|r) = (l|\circ \mathcal{F}(\mathcal{F}^\dagger F \mathcal{F})^k \mathcal{F}^\dagger|r).$$

By restricting to the half-line $p > 0$: $(\mathcal{F}^\dagger|r)(p) = e^{-p} 1_{(0,\infty)}(p)$. Similarly, one can show that $\mathcal{F}^{-1} \circ F \circ \mathcal{F} = -d/dp$ and $((l|\circ \mathcal{F})f) = 2 \int_0^\infty e^{-p} f(p) dp$. Thus in the dual p -representation the triple coincides with the warped-phase Schrödingerization example, with an equivalent integral evaluation functional.

Table I lists five one-parameter families (differential, integral, pseudo-differential, Bargmann-Fock, finite-difference), with a tuning parameter θ that deforms F and $|r\rangle$ to balance norm growth and hardware constraints. See the Appendix A for the specific constructions.

B. Related works

Quantum algorithms for linear differential equations were first developed in the QLSE-based framework, where the evolution problem is reduced to the solution of a structured linear system [18,19]. These works established the basic complexity-theoretic foundation for the subject.

A different line of work is based on *time marching*, in which the solution is propagated over short time intervals instead of being encoded into a single global linear system [20,21]. This approach avoids storing the full history

of the solution and instead focuses on short-time propagation together with the control of success probability and amplification overhead.

There are also related embedding-based approaches, including recent Lindbladian formulations [22], as well as recent refinements of LCHS-type methods [9,23]. These works are closely related to our setting, but they pursue different algorithmic objectives and use different forms of dilation or embedding.

The present work is closest in spirit to Schrödingerization- and LCHS-based constructions [9,10,24]. Our aim, however, is not to propose only a single simulation scheme, but rather to develop a general moment-matching dilation framework that both unifies several existing approaches and generates new dilation families with different structural and implementation properties. In addition, LCHS-type constructions are formulated first for dissipative systems and then extended through an exponential rescaling, whereas the present framework does not require a sign-definiteness assumption such as $K(t) \leq 0$.

III. DILATION VIA DIFFERENTIAL OPERATORS

We instantiate the differential family of Table I tailored to digital quantum computers, by deriving a tight-binding operator with nearest hopping elements. We take the ancilla space

$$\mathcal{H}_A = H_0^1(0, 1) = \{f \in L^2(0, 1) : f' \in L^2(0, 1), f(1) = 0\},$$

with inner product $\langle f, g \rangle := \int_0^1 f(p)^* g(p) dp$, and

$$F_\theta := \theta F, \quad F := p\partial_p + \frac{1}{2}. \quad (7)$$

We prove in Appendix C that the moment identities Eq. (5) are fulfilled by

$$|r\rangle = p^\beta, \quad (l|f = p_*^{-\beta} f(p_*), \quad p_* > 0, \quad \beta = \frac{1}{\theta} - \frac{1}{2}, \quad (8)$$

which reproduces the first row of Table I.

To render the ancilla compatible with digital architectures, we map the continuous differential generator onto a discrete 1D lattice with lattice points $\{p_j\}_{j=0}^M$. Employing summation-by-parts (SBP) operators [25,26] (or, equivalently, a skew-symmetric subspace projection [27]; see the Appendix C for details), we obtain an $(M+1)$ -dimensional ancilla space in which F_h has only nearest-neighbor hopping terms and the inner product is defined by a diagonal overlap matrix W .

The resulting dilated Hamiltonian is

$$\tilde{H}(t) := I_A \otimes H(t) + i\theta F_h \otimes K(t), \quad (9)$$

which is Hermitian on $\mathbb{C}^{M+1} \otimes \mathcal{H}$. Crucially, unlike Fourier-based dilations that imply all-to-all connectivity, our constructed F_h maps directly to a tight-binding chain with hopping amplitudes $f_j = (F_h)_{jj+1}$.

For the particularly simple choice of lattice spacing $h = 1/M$, $\theta = 2$ [i.e., $\beta = 0$ and $|r_h\rangle \propto \mathcal{W}^{1/2} \mathbf{1}$ in Eq. (8)], the bulk lattice preserves the state, $\theta F_h |r_h\rangle = |r_h\rangle$ on all interior sites, with deviations confined to the right boundary. Physically, this implies the error is localized to the edge. Consequently, by choosing $\langle l_h| \propto \langle j_*|$, one finds $\langle l_h| (\theta F_h)^k |r_h\rangle = 1, \forall k < M - j_*$, meaning the discrete triple $(F_h, |r_h\rangle, \langle l_h|)$ fulfills the first $M - j_*$ moments exactly.

To enforce the moment identities at *all* orders with a finite ancilla, we introduce a boundary impurity potential at the edge of the chain:

$$\hat{F}_h := F_h + \alpha |M\rangle \langle M|, \quad \alpha := \frac{1}{\theta} - \frac{\langle M| F_h |r_h\rangle}{\langle M| |r_h\rangle}. \quad (10)$$

This Moment-Locking Closure (MLC) tunes the boundary defect such that $\theta \hat{F}_h |r_h\rangle = |r_h\rangle$ is an exact eigenstate. Thus, $\langle l_h| (\theta \hat{F}_h)^k |r_h\rangle = 1, \forall k \geq 0$. Theorem 1 implies that this finite-dimensional non-Hermitian triple $(\hat{F}_h, |r_h\rangle, \langle l_h|)$ yields an *exact* dilation of (1).

In the implementation, we simulate the Hermitian dilated Hamiltonian $\tilde{H}(t)$ built from F_h and view $\hat{H}(t) := I \otimes H(t) + i\theta \hat{F}_h \otimes K(t)$ as a reference that propagates the exact dilated solution $|\Phi(t)\rangle = |r_h\rangle \otimes |\mathbf{x}(t)\rangle$. Writing $\chi(t) := \Phi(t) - \Psi(t)$ as the error, one finds that $\chi(0) = 0$ and $i(d/dt)\chi(t) = \tilde{H}(t)\chi(t) + i\theta\alpha(|M\rangle \langle M| \otimes K(t))\Phi(t)$. Crucially, because the tight-binding operator F_h only couples nearest neighbors, this boundary disturbance propagates into the bulk with a finite speed. Since F_h approximates the first-order differential operator Eq. (7), which

generates a hyperbolic PDE, this finite-speed transport is closely related to the classical Courant-Friedrichs-Lewy (CFL) principle in numerical PDEs [28]: information can only travel a finite distance over a finite time interval. In the present lattice setting, this phenomenon is expressed through a Lieb-Robinson-type light-cone bound [29],

$$\varrho := \frac{2e\theta K_{\max} T}{(M - j_*)h} < \frac{1}{2}, \quad K_{\max} := \max_{0 \leq t \leq T} \|K(t)\|. \quad (11)$$

We prove that the error at the readout site j_* decays *geometrically*: $\|(\langle j_*| \otimes I)\chi(T)\| \leq C 2^{-(M-j_*)}$, where $C = \alpha K_{\max} T X(T)$ with $X(T) := \sup_{t \in [0, T]} \|\mathbf{x}(t)\|$. More specifically, for times

$$t \in \left[0, \frac{(M - j_*)h}{2e\theta K_{\max}}\right],$$

the influence of the boundary perturbation at site j_* on the bulk is exponentially small. In this sense, $K_{\max}$ plays the role of an effective propagation speed. Thus choosing $M - j_* = \Omega(\log(K_{\max} T X(T)/\epsilon))$ suffices to suppress the error at j_* under ϵ . When $K_{\max} T X(T) = \mathcal{O}(1)$, the required ancilla dimension grows only polylogarithmically in $1/\epsilon$. Hence, with a binary encoding of the ancilla lattice, the ancilla register requires only

$$n_A = \mathcal{O}(\log \log(1/\epsilon))$$

qubits. However, under such a binary encoding, the resulting qubit Hamiltonian generally no longer inherits the geometric locality of F_h , and additional compilation overhead may arise on hardware without all-to-all connectivity. By contrast, a unary encoding preserves the one-dimensional nearest-neighbor structure more directly and is therefore preferable when hardware connectivity is limited, at the expense of a larger ancilla register, $n_A = \mathcal{O}(\log(1/\epsilon))$. We postpone the proof to Lemma D1 in Appendix D.

Combining this with the efficient simulation of the sparse lattice Hamiltonian [5] yields our first complexity bound. The proof is in Appendix D.

Theorem 2 (Simulation by finite-difference dilation). Assume block encodings of $H(t)$ and $K(t)$ with operator norms $H_{\max}$ and $K_{\max}$, respectively, and that $K_{\max} T = \mathcal{O}(1)$. The above dilation scheme using Eq. (9) produces an ϵ -accurate approximation of $|\mathbf{x}(T)\rangle$ with success probability $\Omega(1)$ using

$$\tilde{\mathcal{O}}\left(\left(TH_{\max} + \log \frac{X(T)}{\epsilon}\right) \frac{\|\mathbf{x}(0)\|}{\|\mathbf{x}(T)\|}\right)$$

block-encoding queries to $H(t)$ and $K(t)$, where $\tilde{\mathcal{O}}$ hides the $(\log TH_{\max}/\epsilon)/(\log \log TH_{\max}/\epsilon)$ factor.

For autonomous dynamics where L is time-independent, one can invoke the optimal time-independent Hamiltonian simulation [4,6], improving the logarithmic dependence on ϵ into an additive $\mathcal{O}(\log(1/\epsilon))$ term in the complexity. For larger values of $K_{\max}T$, one extends this short-time result by combining time segmentation with intermediate postselection in the spirit of [30].

IV. NEAR-OPTIMAL COMPLEXITY VIA GRADED LATTICES

To extend the simulation to longer times T , we introduce a graded lattice (geometric grid) where the lattice spacing is spatially modulated with a grading parameter $\delta > 0$:

$$p_j = \exp[-\delta(M-j)], \quad \delta = \Theta(1). \quad (12)$$

This exponential grading suppresses the hopping amplitudes near the boundary, drastically slowing its propagation speed (see the Appendix D). Furthermore, a direct computation shows that the hopping elements $f_j := (F_h)_{j,j+1}$ are $\Theta(1)$ and that $\|F_h\| = \Theta(1)$. We again choose the discrete right vector $|r_h\rangle \propto W^{1/2}\mathbf{1}$ and apply the same MLC Eq. (10), finding that $\alpha = \Theta(1)$.

To saturate the fundamental query complexity bounds for simulating nonunitary dynamics, we apply oblivious amplitude amplification after each time segment. Specifically, consider a partition $[0, T] = \bigcup_k [t_k, t_{k+1}]$, where $t_k := k\tau$, with τ selected as follows:

$$\tau \leq \frac{1}{e\theta K_{\max}}. \quad (13)$$

The overall workflow is depicted in Fig. 1.

To show that OAA can be efficiently applied, we first select δ , j_* , and M such that $\|F_h\| = \Theta(1)$, the light cone condition holds, and $|r_h\rangle$ has a finite lumped weight in the window $[0, j_*]$: $P_{\text{win}} := \sum_{j \in \text{win}} |\langle j | r_h \rangle|^2 > \Delta > 0$ (see the Appendix E for the proof). By using a reflection $2\Pi_{\text{win}} - I$, where $\Pi_{\text{win}} := \left(\sum_{j \in \text{win}} |j\rangle\langle j|\right) \otimes I$, and

$2(|r_h\rangle\langle r_h| \otimes I) - I$, we can invoke OAA to restore the ancilla back to $|r_h\rangle$ to continue to the next time segment. In each segment, the raw success probability factors as $p_k \approx P_{\text{win}}(j_*) \|\mathbf{x}(t_{k+1})\|^2 / \|\mathbf{x}(t_k)\|^2$, so the OAA iteration count scales like $1/\sqrt{p_k}$.

Theorem 3. Assume block encodings of $H(t)$ and $K(t)$, choose $\delta \geq \delta_0 > 0$ so that $\|F_h\| = \Theta(1)$. Let τ satisfy Eq. (13), and let $N = \lceil T/\tau \rceil$. Choose (M, j_*) so that $P_{\text{win}}(j_*) \geq \Delta = \Omega(1)$ and the light-cone leakage per segment and Hamiltonian simulation error is $\epsilon_{\text{seg}} \leq \Theta(\epsilon/N)$.

(i) Nonautonomous dynamics. There is a segmented algorithm based on Dyson-series time-dependent simulation of $\mathcal{U}_k$ and OAA restoration such that the total number of block-encoding queries is

$$\mathcal{O}\left((H_{\max}T + K_{\max}T) \frac{\log(\Lambda_{\tau} K_{\max} T / \epsilon)}{\log \log(\Lambda_{\tau} K_{\max} T / \epsilon)} \Gamma\right), \quad (14)$$

where $\Lambda_{\tau} := H_{\max}\tau + \mathcal{O}(1)$, and Γ is defined as

$$\Gamma := \prod_{k=0}^{N-1} \max \left\{ 1, \frac{\|\mathbf{x}(t_k)\|}{\|\mathbf{x}(t_{k+1})\|} \right\}. \quad (15)$$

In particular, if $K(t) \leq 0$ for all t then $\Gamma = \|\mathbf{x}(0)\|/\|\mathbf{x}(T)\|$.

(ii) Autonomous dynamics. If $H(t) \equiv H$ and $K(t) \equiv K$, then using qubitization/QSVT per segment yields queries

$$\mathcal{O}((\|H\| + \|K\|)T + \|K\|T \log(\|KT/\epsilon\|) \Gamma). \quad (16)$$

Here we work in the standard block-encoding model: a single query applies a controlled unitary that block-encodes $H(t)$ or $K(t)$, while preparation of the normalized initial state $|\mathbf{x}_0\rangle$ is counted separately. Theorem 3 shows that the number of queries scales linearly in $\alpha_A T$ with $\alpha_A := \max_{0 \leq t \leq T} \|L(t)\|$ and only logarithmically in $1/\epsilon$, with the dependence on the solution entering through the factor $\|\mathbf{x}(0)\|/\|\mathbf{x}(T)\|$ due to postselection. Compared to the lower bounds [12], our algorithm attains near-optimal

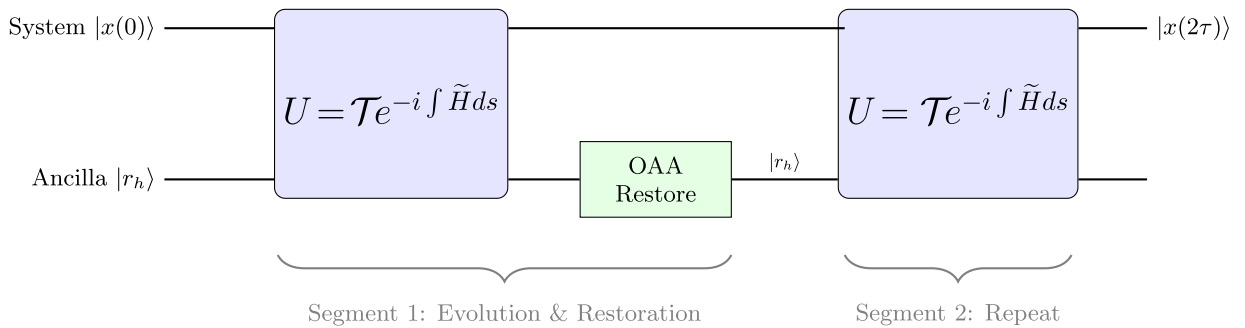


FIG. 1. Circuit diagram for the segmented simulation. The algorithm proceeds in time steps of duration τ . In each segment, the joint evolution $U_k = \mathcal{T} \exp(-i \int \tilde{H} ds)$ is applied. The system state evolves from $|\mathbf{x}(t)\rangle$ to $|\mathbf{x}(t + \tau)\rangle$ (conditioned on success), while the ancilla is restored to the reference state $|r_h\rangle$ via Oblivious Amplitude Amplification (OAA) before the next segment begins.

TABLE II. Ancillary structures in representative dilation-based approaches. The present differential construction provides a native tight-binding topology, while Schrödingerization and LCHS use spectral or selector-register ancillas.

Method	Ancillary structure	Connectivity/implementation feature
Schrödingerization [8,24]	Auxiliary continuous coordinate, typically truncated/discretized and represented in a Fourier or momentum basis.	Fourier diagonalization leads to momentum-indexed controlled Hamiltonian evolutions. The native structure is spectral rather than a nearest-neighbor ancilla chain.
LCHS [9,10]	LCU selector register over Hamiltonian simulations generated by $H + \omega K$.	The main overhead is in PREPARE/SELECT operations, quadrature weights, and controlled block encodings. The ancilla is a selector register rather than a transport lattice.
Differential/tight-binding dilation (this work)	One-dimensional ancilla lattice generated directly by the tight-binding operator F_h .	The ancilla has native nearest-neighbor transport. Unary encoding preserves this two-local chain structure; binary encoding uses $O(\log M)$ qubits but generally loses geometric locality.

scaling in all parameters, including T , ϵ , and the operator norms of $H(t)$ and $K(t)$. Compared to prior works that assumed $K(t)$ to be negative definite or rely on a known stabilizing rescaling, we handle arbitrary $K(t)$ without such an *a priori* rescaling. We show in the Appendix H that the dilation $\tilde{H}$ also admits a linear-combination-of-unitaries (LCU) representation closely analogous to the implementation of LCHS [9]. Finally, in the unitary limit $K \rightarrow 0$, the dilation becomes unnecessary and one recovers the optimal Hamiltonian-simulation scaling for e^{-iHT} (up to the usual additive $\log(1/\epsilon)$ term).

Connectivity and implementation comparison. In Table II, we briefly compare the ancillary structure of the present construction with Schrödingerization and LCHS. This comparison is not meant as a device-level benchmark, since the final cost depends on the chosen block encodings of $H(t)$ and $K(t)$, but it clarifies the topology of the ancillary degrees of freedom, among these three different approaches.

From the complexity viewpoint, the bulk scaling of the graded-lattice construction in Theorem 3 is linear in $(H_{\max} + K_{\max})T$, matching the leading evolution-time dependence of the improved LCHS framework and recent Schrödingerization variants. The distinction is therefore not only asymptotic query scaling, but also the structure of the dilation: the present method realizes the ancillary dynamics as local transport on a one-dimensional chain, which enables the light-cone analysis and gives a direct nearest-neighbor implementation route under unary encoding.

V. NUMERICAL EXAMPLES

We first demonstrate the method on a canonical model of dissipative wave propagation: a 2D viscoelastic solid with

a Maxwell-type constitutive law coupling stress σ , strain ϵ , and an internal variable γ , $\sigma = \kappa_1(\epsilon - \gamma)$ and $\eta\dot{\gamma} = \sigma - \kappa_2\gamma$, with series modulus κ_1 , parallel modulus κ_2 , viscosity η , and density ρ [31]. Introducing $\mathbf{u} = (u_1, \mathbf{u}_2, u_3)^T$ via $u_1 = \sqrt{\kappa_1}(\epsilon - \gamma)$, $\mathbf{u}_2 = \rho^{-1/2}\mathbf{p}$ with momentum $\mathbf{p} = (p_x, p_y)^T$, and $u_3 = \sqrt{\kappa_2}\gamma$, the differential equations can be written in first-order form

$$\frac{d}{dt} \begin{pmatrix} u_1 \\ \mathbf{u}_2 \\ u_3 \end{pmatrix} = (-iH + K) \begin{pmatrix} u_1 \\ \mathbf{u}_2 \\ u_3 \end{pmatrix}. \quad (17)$$

We apply standard finite difference schemes on a 64×64 periodic grid on $[0, 2] \times [0, 2]$, yielding an $N = 16384$ -dimensional system of the form Eq. (1). As initial data, we take two localized Gaussian compressions in the volumetric strain and set $\mathbf{p} = \gamma = 0$, so that two cylindrical waves emanate from the bumps, wrap around the domain periodically, and attenuate due to viscoelastic relaxation, as shown in the direct numerical simulations in Fig. 2.

To benchmark the tight-binding dilation, we couple this system to an ancilla using F_h on a geometric grid Eq. (12). For fixed M and δ , we vary the postselection point p_* in the ancilla. Figure 3 compares the pressure signal $p(t)$ at $(x, y) = (1/4, 1/4)$ for several choices of p_* . For small p_* the dilated trajectory is visually indistinguishable from the reference solution over the entire time interval, while for larger p_* deviations appear earlier. In the Appendix D we show that this behavior confirms the causal structure of the lattice embedding: where the boundary-induced error is exponentially suppressed in $2eK_{\max}T/(M - j_*) \sinh(\delta/2)$ with j_* being the readout site. The simulations are performed classically with a Runge-Kutta time integrator and a small time step so that the time discretization error is negligible. Because the dilated Hamiltonian requires

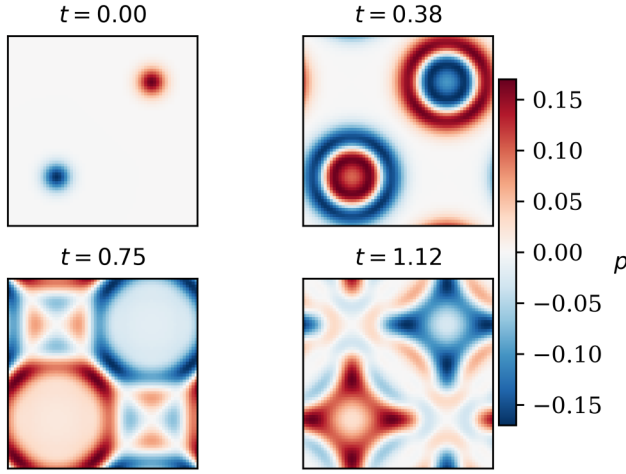


FIG. 2. Snapshots of the pressure field $p = -K_{\text{bulk}}(\varepsilon - \gamma)$.

only nearest-neighbor interactions in the ancilla, we expect these dynamics to be implementable on near-term digital quantum devices without complex connectivity overhead. The code used to generate these results is available at Ref. [32].

To demonstrate the versatility of our framework beyond digital architectures, we turn to the Bargmann-Fock family (Table I, row 4). Unlike the tight-binding ancilla, this construction exploits the native continuous degrees of freedom in continuous-variable (CV) photonic hardware. We consider a canonical testbed for non-Hermitian topology: a four-site $\mathcal{PT}$ -symmetric SSH dimer [1], governed by

$$H = \begin{pmatrix} \delta & J_1 & 0 & 0 \\ J_1 & 0 & J_2 & 0 \\ 0 & J_2 & \delta & J_1 \\ 0 & 0 & J_1 & 0 \end{pmatrix}, \quad K = \gamma \text{diag}(1, 0, 1, 0), \quad (18)$$

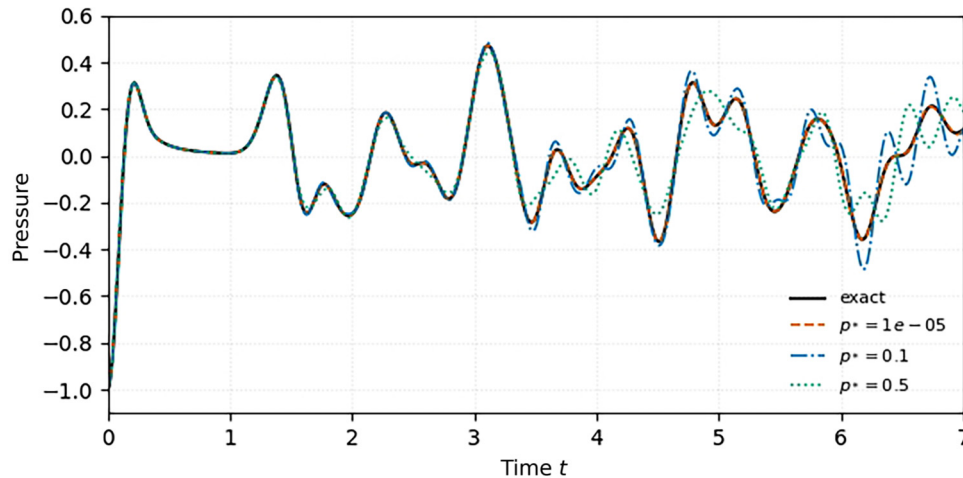


FIG. 3. Pressure time series at $(x, y) = (1/4, 1/4)$ for the Maxwell viscoelastic wave. Colored curves show the dilated dynamics for varying readout points p_* . Smaller p_* tracks the reference for longer times, while larger p_* leads to earlier deviations, illustrating the boundary light-cone effect.

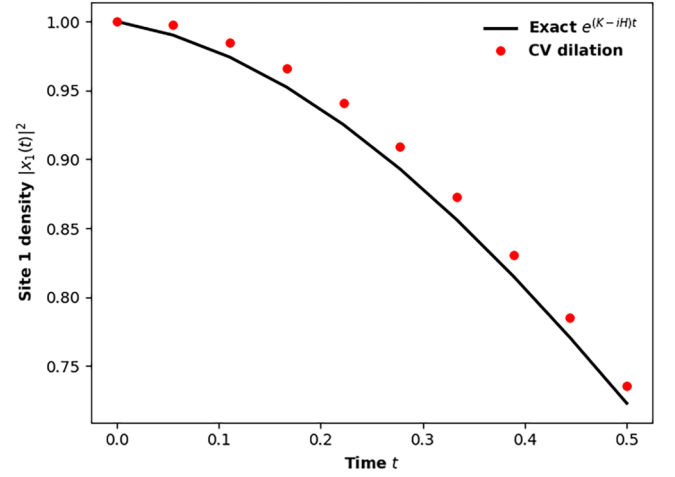


FIG. 4. Edge-density dynamics for the $\mathcal{PT}$ -SSH dimer: Bargmann-Fock CV dilation (markers) vs. exact evolution from matrix exponentiation (solid line).

with parameters $(J_1, J_2, \delta, \gamma) = (1.0, 0.6, 0.3, -1/16)$ and final time $T = 0.5$. Here, the dilation $\tilde{H}$ acts on a hybrid space: discrete modes for the system and a single continuous bosonic mode for the ancilla.

We simulate the dilated evolution using the STRAWBERRY FIELDS library [33]. Our moment-matching state $|r_\theta\rangle$ maps directly to a native Gaussian resource state prepared via squeezing and displacement gates, while the evolution utilizes standard beam-splitters and optical loss channels. Figure 4 shows that the CV-dilated dynamics of the topological edge population $\rho_{\text{edge}}(t) = |x_1(t)|^2$ reproduce the exact evolution with high fidelity. While the system size is limited here by the exponential cost of classically simulating the CV ancilla's Fock basis, the quantum algorithm scales efficiently, confirming that the Bargmann-Fock dilation offers a pathway to simulate non-Hermitian

topological matter on near-term photonic chips without artificial discretization.

VI. DISCUSSION

By formulating the dilation of non-Hermitian dynamics in terms of algebraic moment conditions on an ancilla, we expose a design space that extends well beyond existing methods and naturally organizes into new families of dilations. This flexibility lets the mathematical embedding mirror the hardware's native architecture—whether through the tight-binding lattices or continuous-variable modes demonstrated here.

In the present paper, we have focused the detailed complexity analysis on the differential-type family of moment-matching dilations. The reason is that, because of its differential structure, the error introduced by finite-dimensional truncation can be interpreted through a finite-propagation (light-cone) mechanism, which in turn leads to explicit complexity estimates. The other exact dilation families shown in Table I demonstrate the wide variety of dilations. On the other hand, the truncation error is expected to be governed by different mechanisms. A systematic analysis of their approximation error, algorithmic complexity, and compatibility with different hardware architectures is left for future work.

Future works will also extend these structured constructions to emerging non-Hermitian dynamics in condensed matter physics and to nonlinear systems via quantum Carleman linearization, further establishing this framework as a versatile bridge between nonunitary phenomenology and unitary quantum dynamics.

ACKNOWLEDGMENT

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DATA AVAILABILITY

The data that support the findings of this article are openly available [32], at github.com/xxl12/quantum-dilation-algorithms, embargo periods may apply.

APPENDIX A: THE PROOF OF EXACT DILATION VIA THE MOMENT MATCHING CONDITIONS

Theorem A1 (Moment-fulfilling and Exact Dilation). Let $L(t) = -iH(t) + K(t)$ be split so that H and K are both Hermitian. Then the dilated evolution and recovery *exactly* reproduces the physical solution of

$$\frac{d}{dt}\mathbf{x}(t) = L(t)\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{C}^N, \quad (\text{A1})$$

provided that the following moment conditions are fulfilled,

$$(l|F^k|r) = 1, \quad \forall k \geq 0. \quad (\text{A2})$$

Proof. For autonomous ODEs, L is time-independent. We can express the exponential as a Taylor series, each term can be expanded with powers: $(I_A \otimes H + iF \otimes K)^k$. When taking the moments of each term, $(l|$ and $|r)$ will only act on F , which then is reduced to the Taylor expansion of the matrix exponential of $-iH + K$ and thus completes the proof.

To prove the time-dependent case, we can resort to the time-ordered evolution operator, which involves the time-ordered integrals of the following form:

$$(l|(I_A \otimes H(t_1) + iF \otimes K(t_1))(I_A \otimes H(t_2) + iF \otimes K(t_2)) \cdots (I_A \otimes H(t_k) + iF \otimes K(t_k))|r).$$

Multiplying the operators and using the fact that $(l|I_A|r) = 1$ and the properties in Eq. (5), we find that the product is reduced to

$$(H(t_1) + iK(t_1))(H(t_2) + iK(t_2)) \cdots (H(t_k) + iK(t_k)),$$

leading exactly to the Dyson series expansion for the time-ordered evolution operator associated with $L(t)$. ■

APPENDIX B: FURTHER EXAMPLES OF MOMENT-MATCHING DILATIONS

Another natural choice for constructing a Hilbert space is by using integral operators.

Example B1 (Integral-kernel dilation). For a scale $\theta > 0$ define the odd kernel $K_\theta(p) = pe^{-\theta|p|}$. Its convolution $(F_\theta f)(p) = \int_{\mathbb{R}} K_\theta(p - q)f(q)dq$ is skew-Hermitian on $\mathcal{H}_A = L^2(\mathbb{R})$. Meanwhile, for $a \in (-\theta, 0)$ one finds $F_\theta e^{ax} = \lambda_\theta(a)e^{ax}$, $\lambda_\theta(a) = -4\theta a/(\theta^2 - a^2)^2$. Choosing a so that $\lambda_\theta(a) = 1$ (e.g., $\theta = 1 \Rightarrow a \approx -0.2253$) gives

$$|r\rangle(x) = e^{ax}, \quad F_\theta|r\rangle = |r\rangle.$$

By taking $(l|f) = f(0)$, the moment identities are exactly satisfied.

Although $|r\rangle \notin \mathcal{H}_A$, it lies in the weighted space $\mathcal{X} = L^2(e^{-2\gamma|x|}dx)$ for any $\gamma \in (-a, \theta)$, where $\mathcal{H}_A$ is dense. Discretizing F via Nyström quadrature yields a narrow-band matrix that might be amenable to block-encoding.

Another convenient way to generate skew operators is through pseudodifferential symbols $a(x, \xi)$:

$$(\text{Op}(a)f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(x-y)\xi} a(x, \xi) f(y) dy d\xi.$$

Example B2 (Dilation via a pseudodifferential generator). Let the ancillary Hilbert space be $\mathcal{H}_A = L^2(\mathbb{R}, dx)$. Fix an order $0 < \theta \leq 1$ and define the pseudodifferential generator

$$F := -i(-\Delta)^\theta, \quad \left[\widehat{(-\Delta)^\theta f} \right](\xi) = (\xi^2)^\theta \hat{f}(\xi).$$

Because $(-\Delta)^\theta$ is self-adjoint and positive, the factor $-i$ makes F skew-Hermitian on $\mathcal{H}_A$. A natural choice for $|r\rangle$ is the plane wave $|r\rangle(x) = e^{i\xi_0 x}$, with a complex frequency ξ_0 so that $|r\rangle \notin \mathcal{H}_A$ and the evaluation functional $\langle l|f\rangle = f(0)$. By choosing ξ_0 such that $\xi_0^{2\theta} = i$, we have $F|r\rangle = -i\xi_0^{2\theta}|r\rangle = |r\rangle$. Thus all the moment conditions in Eq. (A2) are satisfied. Selecting $\arg(\xi_0^2) = \pi/2$ then forces $(\xi_0^2)^\theta = e^{i\pi/2} = i$, i.e., $\xi_0 = e^{i\pi/(4\theta)}$ from the principal branch.

Example B3 (Bargmann-Fock dilation). The Bargmann-Fock construction gives a continuous-variable realization of the moment-matching framework and corresponds to the fourth row of Table I. Let the ancillary Hilbert space be the Bargmann-Fock space $\mathcal{B}$ of entire functions, equipped with the inner product

$$\langle f, g \rangle_{\mathcal{B}} = \frac{1}{\pi} \int_{\mathbb{C}} f(z)^* g(z) e^{-|z|^2} d^2 z.$$

In this representation, the bosonic annihilation and creation operators act as

$$a = \partial_z, \quad a^\dagger = z.$$

Thus $a^\dagger - a = z - \partial_z$ is skew-Hermitian on its natural domain in $\mathcal{B}$. For any $\theta > 0$, define

$$F_\theta = \theta(a^\dagger - a) = \theta(z - \partial_z),$$

$$|r_\theta\rangle = \exp\left(\frac{z^2}{2} - \frac{z}{\theta}\right), \quad \langle l|f\rangle = f(0).$$

A direct calculation gives

$$(z - \partial_z) \exp\left(\frac{z^2}{2} - \mu z\right) = \mu \exp\left(\frac{z^2}{2} - \mu z\right).$$

Taking $\mu = 1/\theta$, we obtain

$$F_\theta |r_\theta\rangle = |r_\theta\rangle.$$

Moreover, since $r_\theta(0) = 1$, we have

$$\langle l|r_\theta = r_\theta(0) = 1.$$

Therefore,

$$\langle l|F_\theta^k|r_\theta\rangle = 1, \quad k = 0, 1, 2, \dots,$$

and the moment identities in Eq. (5) are satisfied exactly.

As in the other exact dilation examples, the right vector $|r_\theta\rangle$ is a generalized vector rather than a normalizable Bargmann-Fock state. In finite-dimensional or truncated continuous-variable implementations, $|r_\theta\rangle$ is replaced by a normalized approximation in a finite Fock subspace. This is the construction used in the continuous-variable numerical example in Sec. V. In quadrature notation, $iF_\theta = i\theta(a^\dagger - a)$ is proportional to the momentum quadrature, so the coupling $iF_\theta \otimes K(t)$ is naturally interpreted as a quadrature-mediated ancilla-system interaction.

Example B4 (Dilations using difference operators). Consider a class of difference operators, applied to the space of infinite sequences $\mathcal{H}_A = \ell^2$, with each $f \in \ell^2$ being $f = (0, f_1, \dots)$ equipped with the inner product: $\langle f, g \rangle = \sum_{n \geq 1} f_n^* g_n$. We can define a difference operator θF as $(\theta F f)_0 = 0$ and $(\theta F f)_n = \theta(f_n - f_{n-1})$ when $n \geq 1$. A telescopic (summation-by-parts) calculation gives $F^\dagger = -F$ on $\mathcal{H}_A$. In addition, one can verify that with $|r\rangle = \{\lambda_\theta^n\}_{n \geq 0}$, $\lambda_\theta = (\theta/(1+\theta)) \in (0, 1)$, and $\langle l|f\rangle = \lambda_\theta^{-1} f_1$ the moment conditions Eq. (A2) are satisfied.

These examples show the overwhelming possibilities of achieving the exact dilations, in addition to the dilation scheme created by a similarity transformation among $\langle l|$, F , and $|r\rangle$. This wide variety of such dilation methods forms a large platform where different quantum algorithms can be invoked. For example, the dilated Hamiltonian in Example B2 and Example B1 might be efficiently implemented using the block-encoding techniques in Refs. [34,35].

APPENDIX C: SUMMATION-BY-PARTS (SBP) DISCRETIZATION ON UNIFORM GRID

Here we consider the dilation using

$$\mathcal{H}_A := H_0^1(0, 1)$$

$$= \{f \in L^2(0, 1) \mid f' \in L^2(0, 1), f(1) = 0\},$$

$$\langle f, g \rangle = \int_0^1 f(p)^* g(p) dp.$$

Here we consider the Sobolev space,

$$H^1(0, 1) = \int_0^1 f(x)^2 + f'(x)^2 dx < +\infty\}.$$

Namely both the function and its derivative are square integrable. The subscript 0 indicates the homogeneous boundary condition at $x = 1$.

Define the generator as the differential operator $F_\theta = \theta F$, and

$$F := p \partial_p + \frac{1}{2}. \quad (\text{C1})$$

Lemma C1. For every $\theta > 0$ the operator θF is skew-Hermitian on $\mathcal{H}_A$. Moreover, the moment conditions of Eq. (A2) are satisfied with

$$|r\rangle = p^\beta, \quad \langle l|f = p_*^{-\beta} f(p_*),$$

for some $p_* \in (0, 1)$, where the exponent β is

$$\beta = \frac{1}{\theta} - \frac{1}{2}. \quad (\text{C2})$$

Proof. Skewness: for $f, g \in \mathcal{H}_A$, we examine the inner product denoted by $\langle, \rangle$,

$$\langle f, Fg \rangle = \int_0^1 f^* \left(pg' + \frac{1}{2}g \right) dp = pf^*g \Big|_0^1 - \langle Ff, g \rangle, \quad (\text{C3})$$

and the boundary term vanishes since $p = 0$ at the left endpoint and $f(1) = g(1) = 0$ at $p = 1$. Hence F is skew-Hermitian and so is θF . For the moments, $F(p^\beta) = (\beta + \frac{1}{2})p^\beta$; with Eq. (C2) we get $F_\theta |r\rangle = \theta(\beta + \frac{1}{2}) |r\rangle = |r\rangle$. Also, $\langle l| |r\rangle = p_*^{-\beta} r(p_*) = p_*^{-\beta} \cdot p_*^\beta = 1$. ■

Note $|r\rangle \notin H_0^1$ only because it fails the condition $r(1) = 0$, but $r \in C^1(0, 1)$ and $r \in H^1(0, 1)$ since $\beta > 1/2$ when $\theta \in (0, 1)$, so the dilation remains well-posed. Specifically, to ensure $|r\rangle \in H^1(0, 1)$, we require its derivative to be square integrable. Since

$$r'(p) = \beta p^{\beta-1},$$

this is equivalent to

$$\int_0^1 p^{2\beta-2} dp < \infty,$$

namely $\beta > \frac{1}{2}$. For positive θ , this is exactly the condition $\theta \in (0, 1)$.

For a qubit implementation, we discretize $\{F, \langle l|, |r\rangle\}$ as follows. Partition $[0, 1]$ into M intervals with grid points $\{p_i\}_{i=0}^M$ and set $v_i = v(p_i)$. The integration by parts that ensured the skew Hermitian property can be extended to the discrete level using summation by parts (SBP). Following [25,26], we let $h_j := p_{j+1} - p_j$ and define the SBP trapezoid weights

$$W = \text{diag}(w_0, \dots, w_M), \quad w_0 = \frac{1}{2}h_0, \\ w_j = \frac{1}{2}(h_{j-1} + h_j) \quad (1 \leq j \leq M-1), \quad w_M = \frac{1}{2}h_{M-1}. \quad (\text{C4})$$

Let Q be the tridiagonal matrix for a centered difference operator with

$$(Q)_{j,j+1} = \frac{1}{2}, \quad (Q)_{j+1,j} = -\frac{1}{2}, \quad Q_{00} = -\frac{1}{2}, \quad Q_{MM} = +\frac{1}{2} \quad (\text{C5})$$

and set $D := W^{-1}Q$. Then the diagonal-norm SBP identity holds

$$WD + D^\dagger W = Q + Q^\dagger = B := \text{diag}(-1, 0, \dots, 0, 1).$$

This implies that for all grid functions $\mathbf{u}, \mathbf{v}$, $\langle \mathbf{u}, D\mathbf{v} \rangle_W + \langle D\mathbf{u}, \mathbf{v} \rangle_W = u_M v_M - u_0 v_0$ with weighted inner product $\langle \mathbf{x}, \mathbf{y} \rangle_W := \mathbf{x}^\dagger W \mathbf{y}$, thus mimicking the integration by parts property in Eq. (C3) and thus automatically maintain the skew property after the discretization. Specifically, let $P := \text{diag}(p_0, \dots, p_M)$ and define the (Hamiltonian) split form

$$F_w := \frac{1}{2}(PD + DP) - \frac{1}{2}W^{-1}BP, \quad F_h := W^{1/2}F_w W^{-1/2}. \quad (\text{C6})$$

The SBP property automatically guarantees that F_h is skew-Hermitian and tridiagonal, while F_w is skew with respect to the W -inner product. On a uniform grid one has $h_i \equiv h := 1/M$, $p_i = ih$, $i = 0, \dots, M$, and $w_0 = w_M = h/2$, $w_i = h, \forall 1 \leq i \leq M-1$. The second-order SBP approximation of F in Eq. (C1) reads

$$(F_h v)_0 = \frac{1}{2\sqrt{2}}v_1, \\ (F_h v)_1 = \frac{3}{4}v_2 - \frac{1}{2\sqrt{2}}v_0, \\ (F_h v)_i = \frac{p_{i+1} + p_i}{4h}v_{i+1} - \frac{p_i + p_{i-1}}{4h}v_{i-1}, \quad 2 \leq i \leq M-1, \\ (F_h v)_M = -\frac{p_{M-1} + p_M}{4h}v_{M-1}. \quad (\text{C7})$$

The skew-Hermitian property is apparent.

With the approximation of F by $F_h \in \mathbb{C}^{(M+1) \times (M+1)}$, we obtain a dilated Hamiltonian,

$$\tilde{H}(t) := I \otimes H(t) + i\theta F_h \otimes K(t). \quad (\text{C8})$$

The implementation of $\tilde{H}(t)$ requires an appropriate representation of F_h on quantum devices. The SBP split form can be written as $F_h = a - a^\dagger$, where $a := \sum_{j=0}^{M-1} f_j |j\rangle\langle j+1|$ is a weighted lowering operator. In this sense iF_h plays the role of a discrete momentum/current operator on a tight-binding chain (a discrete derivative up to constants). With $|r_h\rangle = W^{1/2}\mathbf{1}$, the right vector is the mass-weighted flat mode: for $\theta = 2$, the SBP closure gives $\theta F_h |r_h\rangle = |r_h\rangle$ on all interior sites.

We can also associate F_h with Pauli strings, which provide more alternatives to the implementation on digital quantum computers. Using a unary (one-hot) encoding of the ancilla with computational basis $|j\rangle_{j=0}^M$, define $\sigma_j^\pm = (X_j \mp iY_j)/2$. For the off-diagonal couplings f_j , the split-form chain satisfies $(F_h)_{j,j+1} = f_j$, $(F_h)_{j+1,j} = -f_j$, hence

$(iF_h)_{j,j+1} = if_j$, $(iF_h)_{j+1,j} = -if_j$. It admits the two-local Pauli expansion

$$F_h = \sum_{j=0}^{M-1} f_j \left(\sigma_j^+ \sigma_{j+1}^- - \sigma_j^- \sigma_{j+1}^+ \right),$$

$$\implies iF_h = -\frac{1}{2} \sum_{j=0}^{M-1} f_j (X_j Y_{j+1} - Y_j X_{j+1}).$$

Meanwhile, for the initial state $|r\rangle$ (an eigenvector of F), we take the direct discretization with normalizing factor Z_β ,

$$|r_h\rangle = |r_h\rangle = Z_\beta^{-1} \sum_{j=0}^M p_j^\beta w_j^{1/2} |j\rangle, \quad (\text{C9})$$

$$Z_\beta = \left(\sum_{j=0}^M w_j p_j^{2\beta} \right)^{1/2} = \mathcal{O} \left(\frac{1}{\sqrt{2\beta+1}} \right). \quad (\text{C10})$$

Take $(l_h| = (1/\langle j_*| r_h) \langle j_*|$, for some $0 \leq j_* < M$. Then $(l_h|r_h) = 1$ by construction.

These elements constitute a simple procedure to simulate the ODEs Eq. (A1): Starting with $|r_h\rangle \otimes |\mathbf{x}_0\rangle$, one applies Hamiltonian simulations with the dilated Hamiltonian Eq. (C8) to time T , and then postselect at j_* on the ancilla. In the following sections, we will analyze the error due to the discretization and the success probability from the postselection.

APPENDIX D: LIGHT-CONE PROPERTIES AND EXACT DILATION

Considering the simple choice $\beta = 0$, i.e., $\theta = 2$, then $|r_h\rangle = \sqrt{h/2} |0\rangle + \sqrt{h} |1\rangle + \dots + \sqrt{h} |M-1\rangle + \sqrt{h/2} |M\rangle$. A direct calculation shows that $\langle j| \theta F_h |r_h\rangle = \langle j| r_h\rangle$ for all $j < M$, which is guaranteed by the SBP construction. Similarly, $\langle j| (\theta F_h)^2 |r_h\rangle = \langle j| r_h\rangle$ for all $j < M-1$. Therefore,

$$(l_h|(\theta F_h)^k |r_h\rangle = 1, \quad \forall k < M - j_*, \quad (\text{D1})$$

which is an indication of the approximate property via a finite moment matching of Eq. (A2).

One remarkable observation is that by adding one diagonal entry to F_h ,

$$\hat{F}_h = F_h + \alpha |M\rangle \langle M|, \quad \alpha = \frac{1}{\theta} - \frac{\langle M| F_h |r_h\rangle}{\langle M| r_h\rangle} = \mathcal{O}(M). \quad (\text{D2})$$

we obtain $\theta \hat{F}_h |r_h\rangle = |r_h\rangle$, implying that the triple $\langle l_h|, \hat{F}_h, |r_h\rangle$ satisfies moment conditions of *all orders* with a finite-dimensional, almost skew-Hermitian operator

$\hat{F}_h$. In light of Theorem A1, the dilation using a finite-dimensional non-Hermitian $\hat{F}_h$ recovers the exact solution of the ODEs Eq. (A1). We will refer to such a correction as moment-locking closure (MLC).

The benefit of MLC is two-fold. First, using $\hat{F}_h$ provides a simple route to quantify the error by the dilated Hamiltonian without going to infinite dimensions. Second, compared to the infinite-dimensional formulae, such as Schrödingerization [7] and LCHS [9], it provides an easier starting point to design successively improved approximations.

To elaborate on the first point, we also define $\hat{H}(t) := I \otimes H(t) + i\theta \hat{F}_h \otimes K(t)$, along with the dynamics they generate

$$i \frac{d}{dt} \Psi(t) = \tilde{H}(t) \Psi(t), \quad \Psi(0) = |r_h\rangle \otimes \mathbf{x}_0, \quad (\text{D3})$$

$$i \frac{d}{dt} \Phi(t) = \hat{H}(t) \Phi(t), \quad \Phi(0) = |r_h\rangle \otimes \mathbf{x}_0. \quad (\text{D4})$$

Since $\theta \hat{F}_h$ has $|r_h\rangle$ as an eigenvector with eigenvalue 1, $\Phi(t) = |r_h\rangle \otimes \mathbf{x}(t)$. Define the error $\chi(t) := \Phi(t) - \Psi(t)$. Subtracting the two evolutions yields

$$\begin{aligned} \frac{d}{dt} \chi(t) &= (-iI_A \otimes H(t) + \theta F_h \otimes K(t)) \chi(t) \\ &\quad + \alpha \theta \langle M| r_h\rangle |M\rangle \otimes K(t) \mathbf{x}(t), \quad \chi(0) = 0. \end{aligned} \quad (\text{D5})$$

In the error equation, the source of the error clearly concentrates at the right boundary only in the ancilla. Since F_h in $\tilde{H}$ only has nonzero entries at off-diagonals $(j, j+1)$ and $(j+1, j)$, the error propagation to the left, especially the arrival at j_* , where we postselect the solution, can be estimated by a light-cone type of approach [29], which in the context of the differential operator in Eq. (C1), is known as the CFL condition [28]. As a simple demonstration, consider the first-order linear PDE

$$u_t = -pu_p - \frac{1}{2}u, \quad 0 < p < 1, \quad t \geq 0.$$

Its behavior is completely transparent through the *method of characteristics*. One can find the solution by following a family of trajectories $(p(t), t)$, satisfying $dp/dt = -p$, and along these trajectories, the solution of the above PDE is reduced to the solution of the ODEs (by chain rule)

$$\frac{du}{dt} = -\frac{1}{2}u.$$

The first equation integrates to an exponential

$$p(t) = p_0 e^{-(t-t_0)} \implies t - t_0 = \ln \frac{p_0}{p(t)}.$$

Thus every “signal” injected at some point p_0 travels *inward* with speed proportional to the current p ; it needs

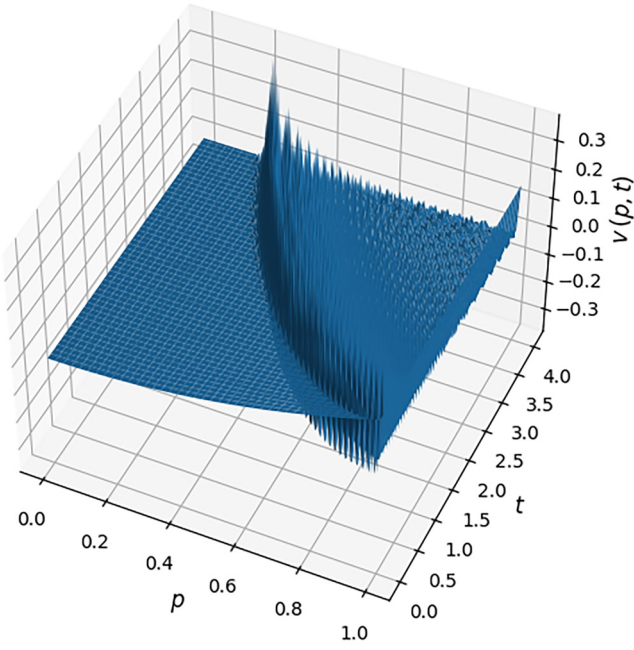


FIG. 5. Snapshots of the semidiscrete solution for $v' = -\theta F_h v$ with $v_j(0) \propto p_j^{1/\theta-1/2}$.

the finite time

$$T(p_0 \rightarrow p_*) = \ln \frac{p_0}{p_*}$$

to reach a location $p_* < p_0$.

Figure 5 shows a numerical demonstration of such back propagation of the boundary error using the ODE $(d/dt)v = -\theta \kappa(t) F_h v$, where one sees the boundary perturbation confined to the characteristic cone emanating from $p = 1$ with speed $-\theta p$; points outside that domain of dependence remain unchanged.

We now quantify such error propagation.

Lemma D1 (Finite-speed propagation bound). Let $\theta = 2$ and consider the uniform grid with grid size $h = 1/M$. Suppose that

$$\varrho := \frac{e\theta K_{\max} T}{(M - j_*)h} < \frac{1}{2}, \quad (\text{D6})$$

then the boundary-induced error arriving at $p_* = j_* h$ obeys

$$\|(\langle j_* | \otimes I) \chi(T)\| \leq 2\theta \alpha X(T) K_{\max} T \left(\frac{1}{2}\right)^{M-j_*},$$

$$X(T) := \sup_{t \in [0, T]} \|\mathbf{x}(t)\|. \quad (\text{D7})$$

Consequently, to guarantee $\|(\langle j_* | \otimes I) \chi(T)\| \leq \epsilon$, it suffices to choose

$$M - j_* \geq \log_2 \frac{4\alpha X(T) K_{\max} T}{\epsilon}. \quad (\text{D8})$$

Proof. We first eliminate the $I_A \otimes H$ term in Eq. (D5) by working in the interaction picture. Let $U_H(t)$ solve $U'_H(t) = -iH(t)U_H(t)$ with $U_H(0) = I$ and define the interaction-picture quantities

$$K_I(t) := U_H(t)^\dagger K(t) U_H(t), \quad \mathbf{x}_I(t) := U_H(t)^\dagger \mathbf{x}(t),$$

$$\chi_I(t) := (I_A \otimes U_H(t)^\dagger) \chi(t).$$

Since $U_H(t)$ is unitary, $\|K_I(t)\| = \|K(t)\|$ and $\|\mathbf{x}_I(t)\| = \|\mathbf{x}(t)\|$.

Then Eq. (D5) becomes the inhomogeneous linear ODE

$$\frac{d}{dt} \chi_I(t) = (\theta F_h \otimes K_I(t)) \chi_I(t)$$

$$+ \alpha \theta \langle M | r_h \rangle |M\rangle \otimes K_I(t) \mathbf{x}_I(t), \quad \chi_I(0) = 0. \quad (\text{D9})$$

Let $W(T, t)$ denote the fundamental (propagator) solution of the homogeneous part,

$$\partial_T W(T, t) = (\theta F_h \otimes K_I(T)) W(T, t), \quad W(t, t) = I.$$

Variation of constants gives

$$\chi_I(T) = \alpha \theta \langle M | r_h \rangle \int_0^T W(T, t) |M\rangle \otimes K_I(t) \mathbf{x}_I(t) dt. \quad (\text{D10})$$

Because F_h is time-independent, the Dyson (time-ordered) series for $W(T, t)$ reads

$$W(T, t) = I + \sum_{k=1}^{\infty} \theta^k \underbrace{\int_{t \leq t_1 \leq \dots \leq t_k \leq T} (F_h \otimes K_I(t_k)) \dots (F_h \otimes K_I(t_1)) dt_1 \dots dt_k}_{=: \mathcal{J}_k(T, t)}. \quad (\text{D11})$$

Since F_h commutes with itself,

$$\mathcal{J}_k(T, t) = F_h^k \otimes \left(\int_{t \leq t_1 \leq \dots \leq t_k \leq T} K_I(t_k) \dots K_I(t_1) dt_1 \dots dt_k \right) = F_h^k \otimes J_k(T, t),$$

where, with the convention $J_0(T, t) := I$,

$$J_k(T, t) := \int_{t \leq t_1 \leq \dots \leq t_k \leq T} K_I(t_k) \cdots K_I(t_1) dt_1 \cdots dt_k. \quad (\text{D12})$$

Thus,

$$W(T, t) = \sum_{k=0}^{\infty} \theta^k F_h^k \otimes J_k(T, t). \quad (\text{D13})$$

Insert Eq. (D13) into Eq. (D10):

$$\begin{aligned} \chi_I(T) &= \alpha \theta \langle M | r_h \rangle \sum_{k=0}^{\infty} \theta^k F_h^k |M\rangle \\ &\quad \otimes \left(\int_0^T J_k(T, t) K_I(t) \mathbf{x}_I(t) dt \right). \end{aligned}$$

Apply $(\langle i | \otimes I)$ and use the band structure of F_h :

$$\langle i | F_h^k | M \rangle = 0 \quad \text{for all } k < m, \quad m := M - i, \quad (\text{D14})$$

so only the terms with $k \geq m$ remain

$$\begin{aligned} (\langle i | \otimes I) \chi_I(T) &= \alpha \theta \langle M | r_h \rangle \sum_{k=m}^{\infty} \theta^k \langle i | F_h^k | M \rangle \\ &\quad \times \left(\int_0^T J_k(T, t) K_I(t) \mathbf{x}_I(t) dt \right). \quad (\text{D15}) \end{aligned}$$

Since each row of F_h has at most two nonzero entries of size $\leq 1/(2h)$,

$$\|F_h\| \leq \frac{1}{h} \implies \|F_h^k\| \leq h^{-k}.$$

Also, $\|K_I(t)\| \leq K_{\max}$ and $\|\mathbf{x}_I(t)\| \leq X(T)$. For the time-ordered integrals Eq. (D12),

$$\begin{aligned} \|J_k(T, t)\| &\leq \int_{t \leq t_1 \leq \dots \leq t_k \leq T} \|K_I(t_k) \cdots K_I(t_1)\| dt_1 \cdots dt_k \\ &\leq \frac{(K_{\max})^k}{k!} (T - t)^k. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left\| \int_0^T J_k(T, t) K_I(t) \mathbf{x}_I(t) dt \right\| \\ &\leq \int_0^T \frac{(K_{\max})^k}{k!} (T - t)^k K_{\max} X(T) dt \\ &= \frac{(K_{\max} T)^{k+1}}{(k+1)!} K_{\max} X(T). \end{aligned}$$

Combining with $\|\langle i | F_h^k | M \rangle\| \leq \|F_h^k\|_{\infty} \leq h^{-k}$, Eq. (D15) yields

$$\|(\langle i | \otimes I) \chi_I(T)\| \leq \alpha \theta K_{\max} X(T) T \sum_{k=m}^{\infty} \frac{(\theta K_{\max} T/h)^k}{(k+1)!}.$$

Use $(k+1)! \geq k!$ to get the simpler tail

$$\sum_{k=m}^{\infty} \frac{(\theta K_{\max} T/h)^k}{(k+1)!} \leq \sum_{k=m}^{\infty} \frac{(\theta K_{\max} T/h)^k}{k!}.$$

Now apply Stirling's bound $k! \geq (k/e)^k$:

$$\frac{(\theta K_{\max} T/h)^k}{k!} \leq \left(\frac{e \theta K_{\max} T}{hk} \right)^k.$$

For $k \geq m$ and $hm \geq 1 - p_*$ (since $p_i \leq p_*$), we obtain the uniform ratio

$$\left(\frac{e \theta K_{\max} T}{hk} \right)^k \leq \left(\frac{e \theta K_{\max} T}{1 - p_*} \right)^k =: \varrho^k, \quad \varrho = \frac{e \theta K_{\max} T}{1 - p_*} < 1$$

by the travel-time hypothesis Eq. (D6). Summing the geometric tail gives

$$\sum_{k=m}^{\infty} \frac{(\theta K_{\max} T/h)^k}{k!} \leq \frac{\varrho^m}{1 - \varrho}.$$

Inserting this into the previous bound yields the bound for $\|(\langle i | \otimes I) \chi_I(T)\|$, and since the physical and interaction pictures differ by a unitary on the second tensor factor, the same bound holds for $\|(\langle i | \otimes I) \chi(T)\|$. ■

The proof of Theorem 2.

Proof. With the geometrically decaying error in Lemma D1, in particular, under the condition in Eq. (D6), when $K_{\max} T = \mathcal{O}(1)$ we can make the geometric factor smaller than ϵ by taking $M = \Omega(\log(K_{\max} T X(T)/\epsilon))$ (Thus, the number of ancilla qubits has $\log \log(K_{\max} T X(T)/\epsilon)$ scaling). Since F_h is 2-sparse and $\|F_h\|_2 = \Theta(1/h) = \Theta(M)$, by using the matrix norm bounds for sparse matrices, the dilated Hamiltonian Eq. (C8) obeys

$$\begin{aligned} \|\tilde{H}(t)\| &= \mathcal{O}(H_{\max} + \theta K_{\max} M) \\ &= \mathcal{O}\left(H_{\max} + \theta K_{\max} \log \frac{K_{\max} T X(T)}{\epsilon}\right), \end{aligned}$$

where $H_{\max} := \max_{0 \leq t \leq T} \|H(t)\|$. Leveraging the time-dependent Hamiltonian simulations using Dyson series [5], we arrive at Theorem 2. ■

APPENDIX E: SBP DISCRETIZATION ON A GEOMETRICALLY SCALED GRID

The arrival-time estimate based on the PDEs generated by F suggests that the propagation of the boundary effect slows down when it is approaching the left boundary $-\log p_*$. This suggests that we mitigate a large $K_{\max}T$ by refining only near the left boundary while keeping the bulk step size moderate. We therefore adopt a *geometric* grid

$$p_j = \exp[-\delta(M-j)], \quad j = 0, 1, \dots, M, \quad (\text{E1})$$

with grading parameter $\delta > 0$ ($\delta = 1$ is enough for the discussions below). One can directly verify that the (trapezoidal) diagonal quadrature weights are

$$w_j = \begin{cases} \frac{1}{2}(e^\delta - 1)p_0, & j = 0, \\ \frac{p_{j+1} - p_{j-1}}{2} = p_j \sinh \delta, & 1 \leq j \leq M-1, \\ \frac{1}{2}(1 - e^{-\delta})p_M, & j = M, \end{cases}$$

and the split-form skew tridiagonal off-diagonals $f_j := (F_h)_{j,j+1}$ in Eq. (C6) are uniform in the interior, which makes it easier to analyze and implement. Specifically, we have

$$f_j = \frac{1}{4 \sinh(\delta/2)} \times \begin{cases} \sqrt{1 + e^{-\delta}}, & j = 0, \\ 1, & 1 \leq j \leq M-2, \\ \sqrt{1 + e^\delta}, & j = M-1. \end{cases} \quad (\text{E2})$$

We consider $\delta \geq 1$, which implies that

$$\|F_h\| \leq 2 \max_{0 \leq j \leq M-1} f_j = \frac{\sqrt{1 + e^\delta}}{2 \sinh(\delta/2)} = \Theta(1). \quad (\text{E3})$$

We now state the finite-propagation estimate on the geometric chain.

Lemma E1 (Finite-speed propagation on a geometric chain). Let F_h be the skew-Hermitian chain with off-diagonals f in Eq. (E2). Assume that $\delta \geq 1$. If

$$\varrho := \frac{e\theta K_{\max}T}{4(M-j_*) \sinh(\frac{\delta}{2})} < 1, \quad (\text{E4})$$

then the boundary influence decays geometrically,

$$\|\langle j_* | \otimes I \chi(T) \| \leq C(\delta) \theta \alpha X(T) K_{\max} T \frac{\varrho^{M-j_*}}{1 - \varrho^2}, \quad (\text{E5})$$

where C is a constant that bounds f_{M-1} in Eq. (E2).

The condition in Eq. (E4) is significantly better than Eq. (D6). Choosing $\delta = \Theta(1)$ and $M = \Theta(K_{\max}T)$ is

sufficient to make the base $\varrho < 1$, and an additional $\mathcal{O}(\log(1/\epsilon))$ points pushes the error below ϵ . Furthermore, one can choose a much larger δ to enhance this property, e.g., $\delta = \log(K_{\max}T)$. However, the downside of this approach is that the right vector $\langle j | r_h \propto W_j^{1/2}$ has exponentially small components near the left boundary, thus leading to a small success probability. On the other hand, by choosing a large postselect window $[0, p_{j_*}]$, the small probability can be improved to $\Omega(1)$, but in this case, Eq. (E4) is difficult to satisfy. In the next section, we present an approach to bypass Eq. (E4) to treat the stiff case when $K_{\max}T \gg 1$.

Proof sketch. Write the split-form SBP chain as

$$F_h = \sum_{j=0}^{M-1} f_j (|j\rangle \langle j+1| - |j+1\rangle \langle j|) = A - A^\dagger, \\ A := \sum_{j=0}^{M-1} f_j |j\rangle \langle j+1|.$$

Thus A lowers the site index by one (a left hop) with weight f_j , while $-A^\dagger$ raises the index by one (a right hop) with weight f_j and an extra minus sign. Fix $j_* \leq M$ and set $m := M - j_*$. For every integer $k \geq m$, $\langle j_* | F_h^k | M \rangle$ constitutes a sum over all length- k nearest-neighbour walks on the line $\{0, 1, \dots, M\}$ that start at M and end at j_* ; each appearance of an oriented edge $e = (\ell \leftrightarrow \ell + 1)$ contributes a factor f_j .

Expand $\chi(T)$ in Dyson series and consider again F_h^k . Only $k \geq m$ with $k \equiv m \pmod{2}$ contribute to $\langle j_* | F_h^k | M \rangle$. For any admissible walk of length k from M to j_* , its edge-weight monomial is a product of k factors drawn from $\{f_{\text{int}}, f_{M-1}, f_0\}$ with $f_{\text{int}} = 1/(4 \sinh(\delta/2))$. Let $C(\delta) := \sup_{\delta \geq 1} \sqrt{1 + e^\delta}/4 \sinh(\delta/2)$. Notice that $f_{M-1}, f_0 \leq C(\delta)$, and the path must cross the boundary edge M at least once. The number of such walks is at most $\binom{k}{(k+m)/2} \leq 2^k$, so

$$|\langle j_* | F_h^k | M \rangle| \leq C(\delta) f_{\text{int}}^k.$$

Hence

$$\left| \frac{(\theta K_{\max}T)^k}{k!} \langle j_* | F_h^k | M \rangle \right| \leq C(\delta) \frac{(\theta K_{\max}T f_{\text{int}})^k}{k!} \\ \leq C(\delta) \left(\frac{e\theta K_{\max}T f_{\text{int}}}{k} \right)^k,$$

using Stirling $k! \geq (k/e)^k$. The rest is the same as the previous Lemma. $\blacksquare$

APPENDIX F: SEGMENTATION-WISE SIMULATION AND OBLIVIOUS AMPLITUDE AMPLIFICATION

For long-time simulation where $K_{\max}T \gg 1$, we divide the simulation into segments, each with length $\tau \mathcal{O}(1/K_{\max})$. Upon the completion of the Hamiltonian evolution of each segment, one obtains an accurate solution in the window $[0, p_{j_*}]$, thanks to the light-cone condition in Eq. (E4). To transition to the next time segment, we simply apply oblivious amplitude amplifications (OAA) to restore the ancilla state back to $|r_h\rangle$.

Corollary F1 (Light-cone protected window with constant weight). Fix constants $\delta_0 > 0$ and $\Delta \in (0, 1/2)$ and assume $\delta \geq \delta_0$. Let $m := M - j_* \geq 1$ and choose

$$m \geq \max \left\{ \frac{1}{2 \sinh(\delta/2)}, \frac{1}{\delta} \log \left(\frac{1 + e^\delta}{2\Delta} \right) \right\}. \quad (\text{F1})$$

If the segment length τ satisfies

$$\tau \leq \frac{1}{e\theta K_{\max}}, \quad (\text{F2})$$

then for the readout site $j_* := M - m$ we have

(i) (Light-cone) the geometric-chain parameter obeys

$$\varrho = \frac{e\theta K_{\max} \tau}{4m \sinh(\delta/2)} \leq \frac{1}{2},$$

so Lemma E1 applies at j_* over time τ ;

(ii) the window probability satisfies

$$P_{\text{win}}(j_*) = \frac{\frac{1}{2}(p_{j_*} + p_{j_*+1}) - p_0}{1 - p_0} \geq \Delta - p_0,$$

$$p_0 = e^{-\delta M}.$$

Since p_0 is exponentially small with M , we can pick a safe lower bound $\Delta/2$ by choosing a sufficiently large M .

(iii) $\|F_h\| = \mathcal{O}(1)$.

Proof. (i) Using Eq. (13) gives

$$\varrho = \frac{e\theta K_{\max} \tau}{4m \sinh(\delta/2)} \leq \frac{1}{4m \sinh(\delta/2)} \leq \frac{1}{2},$$

by the first lower bound in Eq. (F1).

(ii) With $j_* = M - m$, we have $p_{j_*} = e^{-\delta m}$ and $p_{j_*+1} = e^{-\delta(m-1)}$, hence

$$P_{\text{win}}(j_*) = \frac{\frac{1}{2}(e^{-\delta m} + e^{-\delta(m-1)}) - p_0}{1 - p_0}$$

$$\geq \frac{1}{2}(e^{-\delta m} + e^{-\delta(m-1)}) - p_0 = \frac{1 + e^\delta}{2} e^{-\delta m} - p_0.$$

The second lower bound in Eq. (F1) ensures $((1 + e^\delta)/2) e^{-\delta m} \geq \Delta$, giving the claim.

(iii) This comes directly from the magnitude of f_j 's. ■

We describe a segmented algorithm for approximating the normalized solution. Specifically, consider a partition $[0, T] = \bigcup_k [t_k, t_{k+1}]$, where $t_k := k\tau$, with τ selected as follows:

$$\tau K_{\max} = \mathcal{O}(1). \quad (\text{F3})$$

We also let

$$\mathcal{U}_k := \mathcal{T} \exp \left(-i \int_{t_k}^{t_{k+1}} \tilde{H}(s) ds \right) \quad (\text{F4})$$

be the segment propagator on $[t_k, t_{k+1}]$. After each segment, we restore the ancilla back to $|r_h\rangle$ using OAA.

The light-cone property ensures that the prefront window $\text{win} = \{0, 1, \dots, j_*\}$ on the ancilla lattice is unaffected by the boundary error (up to an ϵ error) and retains the solution $\mathbf{x}(t)$ within the win. In particular, within the choice of Eq. (F3) for τ , $P_{\text{win}} := \sum_{j \in \text{win}} |\langle j | r_h \rangle|^2$ is a constant lower bound thanks to Corollary F1, and without loss of generality, we may set $P_{\text{win}} \geq 1/4$. Let us define the corresponding projector and the truncated ancilla mode, respectively,

$$\Pi_{\text{win}} := \left(\sum_{j \in \text{win}} |j\rangle \langle j| \right) \otimes I, \quad |r_{\text{win}}\rangle := \sum_{j \in \text{win}} |j\rangle \langle j| |r_h\rangle. \quad (\text{F5})$$

Write the (normalized) solution from the previous segment k as $|\Psi_k\rangle$. The postwindow state decomposes as

$$\mathcal{U}_k |\Psi_k\rangle = \underbrace{\Pi_{\text{win}} \mathcal{U}_k |\Psi_k\rangle}_{=: |G_k\rangle} + \underbrace{(I - \Pi_{\text{win}}) \mathcal{U}_k |\Psi_k\rangle}_{=: |B_k\rangle},$$

$$\langle G_k | B_k \rangle = 0. \quad (\text{F6})$$

The success probability of a direct window postselection is $p_k := \| |G_k\rangle \|^2$.

Since the boundary-induced component that reaches the window win during a single segment is exponentially

small, there exists an error vector $|e_k\rangle$ supported on win such that

$$\Pi_{\text{win}} \mathcal{U}_k |\Psi_k\rangle = |r_{\text{win}}\rangle \otimes |\mathbf{x}(t_{k+1})\rangle + |e_k\rangle, \quad \|e_k\| \leq \epsilon. \quad (\text{F7})$$

Equation (F7) formalizes an important observation that the unwanted boundary disturbance has *negligible overlap* with the “good” component supported on the window win, and one can invoke oblivious amplitude amplification (OAA). This requires reflections defined by Π_{win} and $|r_h\rangle$, the ancilla reference used at the beginning of each segment. Define the two reflections

$$R_{\text{win}} := I - 2\Pi_{\text{win}}, \quad R_r := 2(|r_h\rangle\langle r_h| \otimes I) - I. \quad (\text{F8})$$

Both are implementable with ancilla-only controls. The OAA applies iterations

$$\mathcal{Q}_k := -R_r \mathcal{U}_k^\dagger R_{\text{win}} \mathcal{U}_k. \quad (\text{F9})$$

In the two-dimensional invariant subspace spanned by $|G_k\rangle$ and $|B_k\rangle$, $\mathcal{Q}_k$ acts as a rotation that moves $|r_{\text{win}}\rangle$ to $|r_h\rangle$.

Operationally, in the subspace $\text{span}\{|G_k\rangle, |B_k\rangle\}$, $(\mathcal{Q}_k)$ performs a Grover rotation that amplifies the Π_{win} (“good”) component of $\mathcal{U}_k |\Psi_k\rangle$; an additional ancilla-only unitary then restores the amplified window state $|r_{\text{win}}\rangle / |r_{\text{win}}|$ to $|r_h\rangle$.

OAA requires $\Theta(1/\sqrt{p_k})$ rounds to increase $\sqrt{p_k}$ to 1. In each segment, the raw success probability factors as $p_k \approx P_{\text{win}}(j_*) \|x(t_{k+1})\|^2 / \|x(t_k)\|^2$ (up to the exponentially small light-cone leakage into win), so the OAA iteration count scales like $1/\sqrt{p_k}$. By choosing j_* (equivalently, the window) so that $P_{\text{win}}(j_*) \geq \Delta = \Omega(1)$, we ensure that the window overlap contributes only a constant amplification overhead. Thus, with a renormalization after each segment, the OAA overhead per segment scales like

$$R_k = \mathcal{O}\left(\max\left\{1, \frac{\|\mathbf{x}(t_k)\|}{\|\mathbf{x}(t_{k+1})\|}\right\}\right). \quad (\text{F10})$$

For monotone dissipative dynamics ($\|\mathbf{x}(t)\|$ nonincreasing), the cumulative amplification overhead is

$$\prod_k \|\mathbf{x}(t_k)\| / \|\mathbf{x}(t_{k+1})\| = \|\mathbf{x}(0)\| / \|\mathbf{x}(T)\|.$$

Another interesting scenario is when the solution goes through transient growth. If $\|\mathbf{x}(t)\|$ is *not* monotone and exhibits transient amplification before decaying, then Eq. (F10) shows that OAA is only costly on segments where the norm *decreases*. One may therefore omit amplification on growth segments (where p_k is larger) and apply OAA only after the trajectory passes a local maximum. For example, when $\|\mathbf{x}(t)\|$ experiences a transient growth followed by a stabilized path with a “hump” profile, the total

amplification overhead is controlled by the peak-to-final ratio,

$$\prod_k \max\left\{1, \frac{\|\mathbf{x}(t_k)\|}{\|\mathbf{x}(t_{k+1})\|}\right\} = \frac{\sup_{t \in [0, T]} \|\mathbf{x}(t)\|}{\|\mathbf{x}(T)\|}. \quad (\text{F11})$$

Define the cumulative amplification factor

$$\Gamma := \prod_{k=0}^{N-1} \max\left\{1, \frac{\|\mathbf{x}(t_k)\|}{\|\mathbf{x}(t_{k+1})\|}\right\}. \quad (\text{F12})$$

Theorem F1 (Query complexity via segmentation + OAA). Assume block encodings of $H(t)$ and $K(t)$, choose $\delta \geq \delta_0 > 0$ so that $\|F_h\| = \Theta(1)$. Let τ satisfy Eq. (F3), and let $N = \lceil T/\tau \rceil$. Choose (M, j_*) so that $P_{\text{win}}(j_*) \geq \Delta = \Omega(1)$ and the light-cone leakage per segment and Hamiltonian simulation error is $\epsilon_{\text{seg}} \leq \Theta(\epsilon/N)$.

(i) Nonautonomous dynamics. There is a segmented algorithm based on Dyson-series time-dependent simulation of $\mathcal{U}_k$ and OAA restoration such that the total number of block-encoding queries is

$$\mathcal{Q}^{(\text{td})} = \mathcal{O}\left((H_{\text{max}} T + K_{\text{max}} T) \frac{\log(\Lambda_\tau K_{\text{max}} T / \epsilon)}{\log \log(\Lambda_\tau K_{\text{max}} T / \epsilon)} \Gamma\right), \quad \Lambda_\tau := H_{\text{max}} \tau + \mathcal{O}(1), \quad (\text{F13})$$

where Γ is defined in Eq. (F12). In particular, if $K(t) \leq 0$ for all t then $\Gamma = \|\mathbf{x}(0)\| / \|\mathbf{x}(T)\|$.

(ii) Autonomous dynamics. If $H(t) \equiv H$ and $K(t) \equiv K$, then using qubitization/QSVT per segment yields

$$\mathcal{Q}^{(\text{ti})} = \mathcal{O}((\|H\| + \|K\|) T + \|K\| T \log(\|K\| T / \epsilon)) \Gamma. \quad (\text{F14})$$

Proof sketch. Fix $\epsilon_{\text{seg}} = \Theta(\epsilon/N) = \Theta(\epsilon/(K_{\text{max}} T))$ and apply Hamiltonian simulation algorithms to each segment. ■

Complexity perspective. From a resource-scaling viewpoint, the segmented tight-binding dilation inherits the near-optimal precision dependence of interaction-picture simulation: the overhead in ϵ enters only through the standard $\log(\cdot)/\log \log(\cdot)$ factor from truncated Dyson-series techniques. By contrast, LCHS-type LCU embeddings express the nonunitary propagator as a spectral integral over Hamiltonian evolutions; after truncation and quadrature this typically introduces an *additional* polylogarithmic overhead in the target accuracy through the required frequency cutoff (e.g., in LCHS the kernel has slow tail decay, which can force a large cutoff and correspondingly amplify the effective Hamiltonian scale). In the autonomous setting, qubitization/QSVT applied segment-wise yields an *additive* logarithmic dependence on $1/\epsilon$ governed by the dissipative scale (the $\log(1/\epsilon)$ term is weighted by $\|K\|T$,

not by $\|H\|T$. Finally, in the unitary limit $K \rightarrow 0$ the dilation becomes unnecessary and one recovers the optimal Hamiltonian-simulation scaling for e^{-iHT} (up to the usual additive $\log(1/\epsilon)$ term).

APPENDIX G: THE 2D MAXWELL VISCOELASTICITY MODEL

Here we present a numerical test to demonstrate the performance of the dilation scheme in the previous section. We consider wave propagation in a 2D solid with viscoelastic behavior. The model involves a stress-strain pair (σ, ϵ) , augmented by an internal variable γ . The variables

obey Maxwell-type constitutive relations:

$$\sigma = K_1(\epsilon - \gamma), \quad \eta \dot{\gamma} = \sigma - K_2 \gamma.$$

Here, K_1 is the series elastic modulus, K_2 is the parallel elastic modulus, and η is the viscosity [31]. To express the PDEs in the form of Eq. (1) with the decomposition $L = -iH + K$, we define a new state vector $\mathbf{u} = (u_1, \mathbf{u}_2, u_3)^T$ where

$$u_1 = \sqrt{K_1}(\epsilon - \gamma), \quad \mathbf{u}_2 = \sqrt{\rho^{-1}} \mathbf{p}, \quad u_3 = \sqrt{K_2} \gamma,$$

and we have denoted the momentum as $\mathbf{p} = (p_x, p_y)^T$.

The coupled PDEs for the transformed state vector $\mathbf{u}$ can be expressed in the first-order form

$$\frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ \mathbf{u}_2 \\ u_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \frac{\sqrt{K_1}}{\sqrt{\rho}} \nabla \cdot & 0 \\ \frac{\sqrt{K_1}}{\sqrt{\rho}} \nabla & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{-iH} + \underbrace{\begin{pmatrix} -K_1/\eta & 0 & \sqrt{K_1 K_2}/\eta \\ 0 & 0 & 0 \\ \sqrt{K_1 K_2}/\eta & 0 & -K_2/\eta \end{pmatrix}}_K \begin{pmatrix} u_1 \\ \mathbf{u}_2 \\ u_3 \end{pmatrix}. \quad (\text{G1})$$

To maintain the structure of these operators, we apply a direct finite-difference scheme to approximate the gradient and divergence on a 2D grid in the domain $[0, 2] \times [0, 2]$ with 64 grid points in each direction. The total number of unknowns is thus $N = 16\,384$.

APPENDIX H: IMPLEMENTATION BY LCU

Our dilation admits a linear-combination-of-unitaries (LCU) form that is closely related to the LCHS technique [9,10], but arises from a different spectral transform (Mellin rather than Fourier). Assume the interior generator F admits the spectral resolution

$$F = -i \int_{\mathbb{R}} \omega |\psi_\omega\rangle \langle \psi_\omega| d\mu(\omega), \quad (\text{H1})$$

so that $iF = \int \omega |\psi_\omega\rangle \langle \psi_\omega| d\mu$. Then the dilated Hamiltonian factorizes as

$$\begin{aligned} I_A \otimes H(t) + iF \otimes K(t) \\ = \int_{\mathbb{R}} |\psi_\omega\rangle \langle \psi_\omega| \otimes (H(t) + \omega K(t)) d\mu(\omega). \end{aligned}$$

Consequently, for any left-right test vectors $\langle l|$, $|r\rangle$ and initial state $|\mathbf{x}_0\rangle$,

$$\begin{aligned} (\langle l| \otimes I) \mathcal{T} \exp \left(-i \int_0^T [I_A \otimes H(s) + iF \otimes K(s)] ds \right) \\ \times (|r\rangle \otimes |\mathbf{x}_0\rangle) \\ = \int_{\mathbb{R}} [\langle l| \psi_\omega] [\langle \psi_\omega| |r\rangle] \mathcal{T} e^{-i \int_0^T (H(s) + \omega K(s)) ds} |\mathbf{x}_0\rangle d\mu(\omega). \end{aligned} \quad (\text{H2})$$

Equation (H2) is an LCHS-type integral representation: after quadrature in ω , one obtains a concrete LCU over propagators of the form $\mathcal{T} e^{-i \int (H + \omega K)}$.

For the differential operator F , the Mellin-wave basis [36]

$$\psi_\omega(p) = p^{-1/2-i\omega}, \quad \omega \in \mathbb{R},$$

satisfies $F\psi_\omega = -i\omega\psi_\omega$, and $d\mu(\omega) = d\omega/(2\pi)$, so Eq. (H1) is precisely the Mellin transform of F .

A fully discrete LCU form also appears after the discretization on the SBP stencil. Since the grid length $M = \Omega(\log(1/\epsilon))$ is modest, we can *classically* diagonalize the

interior matrix $F_h \in \mathbb{C}^{(M+1) \times (M+1)}$:

$$F_h = -i \sum_{j=0}^M \omega_j |\phi_j\rangle \langle \phi_j|, \\ \omega_j \in \mathbb{R}, \quad \{|\phi_j\rangle\}_{j=0}^M \text{ orthonormal.} \quad (\text{H3})$$

Then

$$I_A \otimes H(t) + iF_h \otimes K(t) \\ = \sum_{j=0}^M |\phi_j\rangle \langle \phi_j| \otimes (H(t) + \omega_j K(t)),$$

and the time-ordered propagator decomposes as

$$U(t) = \sum_{j=0}^M |\phi_j\rangle \langle \phi_j| \otimes \mathcal{T} e^{-i \int_0^t (H(s) + \omega_j K(s)) ds}.$$

Hence, for the dilation-consistent pair

$$|r_h\rangle = |r_h\rangle = \frac{1}{\sqrt{S}} \sum_{j=0}^M \sqrt{w_j} |j\rangle, \quad \langle l_h| = \sqrt{\frac{S}{S_W}} \langle l_h|, \\ \langle l_h| = \frac{1}{\sqrt{S_W}} \sum_{j=0}^{j_*} \sqrt{w_j} \langle j|, \quad (\text{H4})$$

with normalizing constants $S_W = \sum_{j=0}^{j_*} w_j$, $S = \sum_{j=0}^M w_j$ we obtain the explicit LCU expansion

$$|\mathbf{x}(t)\rangle = \sum_{j=0}^M \underbrace{\langle l_h | \phi_j \rangle \langle \phi_j | r_h \rangle}_{=: c_j} \mathcal{T} e^{-i \int_0^t (H(s) + \omega_j K(s)) ds} |\mathbf{x}_0\rangle \\ = \sum_j c_j U_j(t) |\mathbf{x}_0\rangle. \quad (\text{H5})$$

By Cauchy-Schwarz with the orthonormal basis $\{|\phi_j\rangle\}$,

$$\|c\|_1 \leq \| |l_h\rangle \| \| |r_h\rangle \| = \frac{\sqrt{S}}{\sqrt{S_W}} = \frac{1}{\sqrt{P_{\text{win}}(j_*)}}, \\ P_{\text{win}}(j_*) := \frac{S_W}{S}.$$

Thus a direct (single-shot) LCU implementation has success amplitude scaling like $\sqrt{P_{\text{win}}}$; if the window weight P_{win} is tiny, the LCU ℓ_1 -weight $\|c\|_1$ is large. A standard remedy is time segmentation: split $[0, T]$ into L segments of length $\tau = \Theta(1/K_{\text{max}})$, apply LCU on each segment (so that ℓ_1 -weights per segment are $\mathcal{O}(1)$), use robust oblivious amplitude amplification (OAA) per segment, and coherently *restore* the ancilla back to $|r_h\rangle$ at the end of

each segment. In this way the per-segment LCU ℓ_1 -weight L_{seg} can be kept ≤ 2 (cf. [5, Sec. 3]), yielding an overall near-optimal scaling as well.

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