

Poor Man's Majorana Tetron

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The Majorana tetron is a prototypical topological qubit stemming from the ground-state degeneracy of a superconducting island hosting four Majorana modes. This degeneracy manifests as an effective non-local spin degree of freedom, the most paradigmatic signature of which is the topological Kondo effect. Degeneracies of states with different fermionic parities also characterize minimal Kitaev chains, which have lately emerged as a platform to realize and study unprotected versions of Majorana modes, dubbed poor man's Majorana modes. Here, we introduce the "poor man's Majorana tetron," comprising four quantum dots coupled via a floating superconducting island. Its charging energy yields nontrivial correlations among the dots, although, unlike a standard tetron, it is not directly determined by the fermionic parity of the Majorana modes. The poor man's tetron displays parameter regions with a twofold-degenerate ground state with odd fermionic parity, which gives rise to an effective Anderson impurity model when coupled to external leads. We show that this system can approach a regime featuring the topological Kondo effect under a suitable tuning of experimental parameters. Therefore, the poor man's tetron is a promising device to observe the nonlocality of Majorana modes and their related fractional conductance.

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I. INTRODUCTION

The past decade has witnessed steady progress in the fabrication techniques of semiconductor-superconductor hybrid systems, driven, in particular, by the research on Majorana zero-energy modes and the engineering of artificial topological superconductors [1,2]. These advances have allowed for the combination of quantum dots and

superconducting islands in the same controllable architecture, with nanowire devices coated by aluminum shells constituting the paradigmatic platform to engineer tunable superconducting devices. These systems take advantage of the interplay between the intrinsic spin-orbit coupling of the semiconductor nanowires and the superconducting pairing induced by their proximity to aluminum. Additionally, they offer the possibility of creating superconducting subgap states that can be controlled through electrostatic gates, the rich physics of which can be investigated via transport measurements.

These general features have allowed for the realization of efficient Cooper-pair splitters [3,4] and minimal Kitaev chains [5–9], which constitute the basic platform to study the so-called poor man's Majorana modes [10]. These are non-Abelian zero-energy modes that can be obtained through a fine tuning of the parameters of the system [10,11] and, for this reason, they are not protected

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against external perturbations, as in the case of a two-site Kitaev chain. Topological protection can be recovered by extending these systems toward longer quantum dot chains, thus progressively approaching the topological phase of the Kitaev model [12–14]. Compared with the conventional microscopic realization of Majorana modes in nanowires [15,16], this mesoscopic approach offers additional control to overcome the limitations imposed by disorder. Therefore, it constitutes a promising alternative for observing properties of Majorana modes that have so far been elusive.

In minimal devices with two quantum dots coupled through an intermediate superconducting island via weak-tunneling amplitudes, two main interdot interactions emerge. The first is the crossed Andreev reflection (CAR), corresponding to a process in which two electrons, one for each dot, couple to form a Cooper pair. The second is the elastic cotunneling of electrons from one dot to the other. Poor man’s Majorana modes can be realized when the amplitudes of these processes match. Such a condition can be achieved by mediating these processes through controllable Andreev bound states [5,11] in the intermediate superconductor. The onset of poor man’s Majorana modes, however, does not necessarily require these additional states and alternative techniques for their realization have been proposed based on the control of magnetic fluxes and superconducting phases [13,14,17].

Experimental investigations of minimal Kitaev chains [6–9] have been realized in setups with grounded superconducting islands, such that the charging energy of the electrons in the superconductors can be neglected [18]. The study of Majorana modes, however, has demonstrated that devices with floating superconducting islands display additional transport effects that can be used for their characterization [19–25]. Furthermore, electrostatic interactions constitute a key ingredient in the engineering of one of the most crucial building blocks for the design of Majorana platforms for quantum information processing: the tetron [26]. A tetron is a floating superconducting island, or Cooper-pair box, hosting four Majorana modes and it represents one of the paradigmatic examples of a topological qubit [27–30]. It is therefore a fundamental question to assess whether systems combining minimal Kitaev chains and charging-energy effects can be used to define possible qubit platforms and explore the nonlocal properties of Majorana modes.

In this work, we analyze the extent to which the physics of tetrons can be recovered with poor man’s Majorana modes. In particular, we investigate the fate of these modes in devices with four quantum dots coupled with a single floating superconducting island (see Fig. 1).

It is known that intradot interactions can be beneficial for the formation of poor man’s Majorana modes [31] and weak interdot interactions are not detrimental [17,32]. Furthermore, in two-dot setups, Majorana sweet spots can also

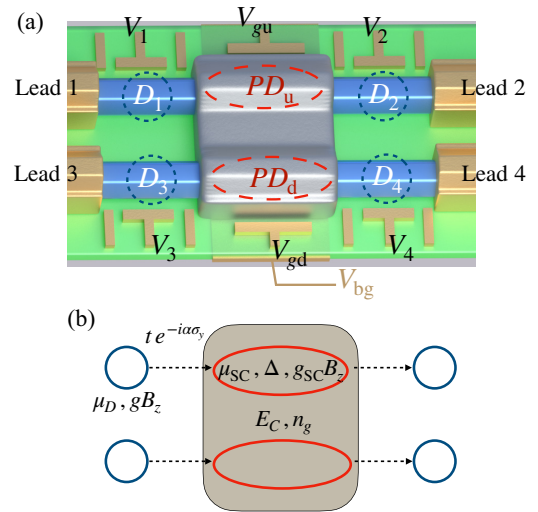


FIG. 1. (a) A sketch of the poor man’s tetron: two semiconducting nanowires (blue) are connected by a shared superconducting (SC) floating island (silver) and a set of electrostatic gates is used to define a quantum dot D_i on each side of each nanowire (dashed circles). The superconducting segments of the nanowires are treated as proximitized dots $PD_{u,d}$ (orange ellipses). Gate voltages are used to control the external dot energy levels μ_D (V_i voltages) and the induced charge n_g (V_{gr} voltages). The voltage V_{bg} of a back gate is used to tune the potential μ_{SC} in the proximitized dots. The device is then connected with four external leads (lateral gold contacts) to explore the emerging Kondo physics. (b) A schematic representation of the Hamiltonian terms of the poor man’s tetron. The arrows indicate the sign of the spin-orbit coupling in the tunneling term.

be identified in the presence of a sizeable charging energy in the intermediate superconducting island [33]. We will show instead that the situation is different in devices with four dots. Here, in general, the charging energy of the superconductor destroys the poor man’s Majorana modes. However, when suitably tuning the dot potentials, regimes with degenerate ground states can still be obtained. Therefore, we label such four-dot systems *poor man’s tetrons*. We envision that these devices can be fabricated by doubling the minimal Kitaev chain analyzed in Ref. [6] with the use of double nanowires [34,35] connected through a shared aluminum shell. Alternative devices fabricated in two-dimensional (2D) systems with gate-defined dots can be adopted as well, to obtain analogous controllable platforms [9,36].

One of the key characteristics of the topologically protected tetron is the emergence of a nonlocal effective spin-1/2 degree of freedom that couples in a nontrivial way with external leads in multiterminal transport experiments. In particular, the interaction of the Majorana modes of the tetron with $M = 3$ or 4 leads is predicted to generate the so-called topological Kondo effect [37–39]: in the low-temperature limit, the tetron displays a fractional nonlocal conductance $(2/M)e^2/h$ corresponding to the onset of a

non-Fermi-liquid fixed point in the limit of strong coupling between the leads and the Majorana modes. These distinctive transport features, however, have never been observed so far: engineering a physical platform to study the topological Kondo effect is still considered a crucial milestone in the study of Majorana modes and, in general, of non-Abelian anyons. Indeed, recent theoretical works have shed light on the relationship between the non-Fermi-liquid fixed point in this kind of quantum impurity problems and the emergence of non-Abelian topological objects [40–42].

In the following, we investigate whether the correlated ground states that characterize a poor man's tetron can reproduce the exotic transport features identifying the topological Kondo effect and whether other kinds of Kondo effects may emerge in this system.

In Sec. II, we present the general model for a poor man's tetron composed by four tunable dots, the interactions of which are mediated by a single floating superconducting island. In Sec. III, we analyze these devices in the perturbative regime with weak interdot tunneling and a strong superconducting gap. We show in Sec. IV that the many-body states of the poor man's tetron display non-local correlations reminiscent of the topological tetron. In Sec. V, we discuss how these correlations reflect in multiterminal transport experiments, and we show that, upon a suitable tuning of the dot potentials and subgap-state energy levels, it is possible to reach an effective topological Kondo point [37–39], characterized by fractional nonlocal conductance. In Sec. VI, we analyze the low-energy physics of poor man's tetron beyond perturbation theory through exact diagonalization. We investigate, in particular, the extension in parameter space of the regime in which the poor man's tetron behaves as a two-level system. In Sec. VII, we present estimates for the system parameters based on the possible realization of the poor man's Majorana tetron in double nanowires. Measurement schemes for the fermionic parities of the system and further applications are also discussed.

The appendices provide technical details on the perturbative analysis of the poor man's tetron and the definition of the related Kondo problems.

II. THE POOR MAN'S TETRON

Andreev bound states constitute a convenient tool to mediate the interactions between two quantum dots in a controllable way and realize poor man's Majorana modes. In recent experiments [6], this scenario has been realized by fabricating InSb nanowires, which present a central segment covered by an aluminum superconductor and a pair of quantum dots, one on each side of the superconducting segment, defined in the nanowire through suitable electrostatic gates. Analogous results have also been obtained in a regime of strong hybridization between the dots and

the central superconducting region, which leads to the formation of Yu-Shiba-Rusinov states mediating the interdot interactions [7].

In the limit of small quantum dots, the low-energy physics of each dot can be captured by a single energy level, the energy μ_D of which can be tuned via suitable electrostatic gates. Additionally, the superconducting segment of the nanowire can also be represented in its simplest approximation by a single-level central dot, subject to an s -wave superconducting pairing induced by proximity to the aluminum [11,31].

The construction of the poor man's tetron (see Fig. 1) relies on doubling this system. We consider, in particular, two parallel nanowires that share the same floating superconducting shell [34,35]. Each nanowire, labeled by $\tau = u, d$, hosts two external single-level dots and an intermediate superconducting segment, characterized by a spin-1/2 doublet of Andreev subgap states. The Andreev states in each nanowire are tunnel coupled with both quantum dots in the same wire; however, we assume that the Andreev states in different nanowires are independent, which corresponds to the situation in which the wires are separated and connected exclusively through the superconductor. As a consequence, the two nanowires interact only through the shared charging energy of the common superconducting shell, without any single-electron tunneling or crossed superconducting pairing between them. This scenario has been considered in, e.g., Ref. [43] in the context of multiterminal transport spectroscopy across hybrid semiconductor-superconductor double nanowires.

We decompose the Hamiltonian of the poor man's tetron in the form $H_{\text{sys}} = H_{\text{sp}} + H_c$, where H_{sp} is the quadratic Hamiltonian describing the single-particle dynamics of the electrons in the two nanowires; H_c defines instead the charging energy of the superconducting island and may also include the Coulomb interactions of electrons within the same dot.

We approximate the single-particle Hamiltonian in the following way:

$$\begin{aligned}
 H_{\text{sp}} = & \sum_{\substack{a=L,R \\ s=\uparrow,\downarrow \\ \tau=u,d}} \mu_D d_{a\tau}^\dagger d_{a\tau} + gB_z \sum_{\substack{s,s'=\uparrow,\downarrow \\ a=L,R \\ \tau=u,d}} d_{a\tau}^\dagger \sigma_{ss'}^z d_{as'\tau} \\
 & + g_{\text{SC}} B_z \sum_{\tau, s, s'=\uparrow,\downarrow} c_{s\tau}^\dagger \sigma_{ss'}^z c_{s'\tau} + \sum_{\substack{s=\uparrow,\downarrow \\ \tau=u,d}} \mu_{\text{SC}} c_{s\tau}^\dagger c_{s\tau} \\
 & + \Delta \sum_{\tau=u,d} \left(c_{\uparrow\tau}^\dagger c_{\downarrow\tau}^\dagger e^{i\varphi} + \text{H.c.} \right) \\
 & - \sum_{\substack{s,s'=\uparrow,\downarrow \\ a=L,R \\ \tau=u,d}} t_{a\tau} \left[c_{s\tau}^\dagger (e^{-ia\alpha_\tau \sigma_y})_{ss'} d_{as'\tau} + \text{H.c.} \right]. \quad (1)
 \end{aligned}$$

In this expression, we use fermionic creation and annihilation operators $d_{a\tau}^\dagger$ and $d_{a\tau}$ to represent the degrees of freedom of the external dots [indicated for simplicity by $D_{1,\dots,4}$ in Fig. 1(a)]. The label $a = L, R$ distinguishes the dots on the left and right sides of the central superconducting island, and we consider $a = +1, -1$, respectively, in the definition of the tunneling terms in the last line of Eq. (1). The label s refers to the spin of the electrons. The energy level of each dot can in principle be tuned independently; however, hereafter, we consider symmetric configurations where all the dots are equivalent and described by the same chemical potential μ_D . Additionally, we also include an out-of-plane Zeeman splitting gB_z , which is instrumental in splitting the spin degeneracies in the dots and obtaining poor man's Majorana modes in the noninteracting limit. We assume that the Zeeman splitting also affects the proximitized segments of the nanowires [labeled by PD_τ in Fig. 1(a)], with a different Landé factor g_{SC} . The electronic states in the central nanowire segments are described by the ladder operators $c_{s\tau}^\dagger$ and $c_{s\tau}$, and are subject to a proximity-induced s -wave pairing Δ associated with the superconducting phase operator $e^{i\varphi}$, which decreases the number of Cooper pairs N_{CP} by one: $[N_{CP}, e^{i\varphi}] = -e^{i\varphi}$. μ_{SC} is an effective chemical potential for these electrons and we assume that it can be tuned by a suitable back gate (see Fig. 1).

The superconducting segments in each wire are tunnel coupled with the external dots with amplitudes $t_{a\tau}$, which can be controlled by cutter gates. These tunneling processes are characterized by a spin rotation determined by the Rashba spin-orbit coupling of the wires. We consider a rotation around the y axis by effective angles $\pm\alpha_\tau$, depending on the direction of the tunneling. α_τ is proportional to the spin-orbit momentum of nanowire τ .

With regard to the interactions H_c , we focus on a scenario in which the Zeeman term gB_z is strong enough to suppress the population of the spin- $\uparrow$ states in the external dots, such that doubly occupied states are also prevented in the low-energy sector of the model. Therefore, we preliminarily disregard the effects of the on-site interactions of the four external dots and consider exclusively the charging energy of the intermediate superconducting island:

$$H_c = E_C (N - n_g)^2, \quad (2)$$

where the operator $N = 2N_{CP} + \sum_{s\tau} c_{s\tau}^\dagger c_{s\tau}$ describes the excess charge of the superconducting island, including the contribution given by the number of Cooper pairs N_{CP} and the occupations $c_{s\tau}^\dagger c_{s\tau}$ of the proximitized segments of the nanowires. The operator N , therefore, increases by 1 when an electron hops from one of the external dots into the central superconducting (SC) region. However, N is left unchanged by the Δ pairing term in the Hamiltonian given in Eq. (1), since this term annihilates one

Cooper pair to populate two electron states in the proximitized segments. The parameter $n_g = \sum_i C_{gi} V_{gi}/e$ defines the induced charge of the superconducting island and we assume that it can be tuned by suitable electrostatic side gates at voltage V_{gi} (see Fig. 1), while the capacitances C_{gi} are fixed by the geometry.

The choice in Eq. (2) provides an approximate description of the electrostatic interactions of the poor man's tetron. A more refined modeling of the interactions can be obtained by introducing a capacitance matrix that also accounts for the crossed repulsion between electrons in the dots and central island, thus considering additional interaction terms proportional to $N d_{a\tau}^\dagger d_{a\tau}$ (see the related analysis for two-dot systems in Ref. [33]). These interactions do not yield qualitative changes in the following analysis as long as they do not introduce a strong anisotropy between the four dots. We also observe that the physics of the topological tetron is recovered in the limit of strong cross capacitance among all the dots and between the dots and the superconducting islands. This situation is hardly achievable in hybrid semiconductor-superconductor systems, where the dot charges are easily screened by the environment and the central island; however, it would correspond to the limit in which the charging energy in Eq. (2) includes not only the charge of the superconducting island but also the charge of the four dots. In this respect, the introduction of a general capacitance matrix of the device allows for the interpolation between the poor man's tetron, where electrostatic interactions are limited to the central island, and the tetron, where the charge distribution between dots and superconductor is irrelevant.

The total Hamiltonian H of the poor man's tetron separately preserves the fermionic parities of the upper and lower wires:

$$P_\tau = (-1)^{\sum_{a,s} d_{a\tau}^\dagger d_{a\tau} + \sum_s c_{s\tau}^\dagger c_{s\tau}}. \quad (3)$$

As a consequence, these two conserved quantities define four independent symmetry sectors. Additionally, the total number $N_t = N + \sum_{a,s,\tau} d_{a\tau}^\dagger d_{a\tau}$ of electrons in the system is conserved as well and $(-1)^{N_t} = P_u P_d$. In the following, we will adopt these quantum numbers to label the related sectors of the states of the system.

III. PERTURBATIVE REGIME: CROSSED ANDREEV REFLECTION AND ELASTIC COTUNNELING

To gain a qualitative understanding of the poor man's tetron and its low-energy spectrum, it is useful to analyze the perturbative-tunneling regime in which, despite the presence of the charging energy, it is still possible to define elastic cotunneling processes and CARs between the external dots on the same nanowire.

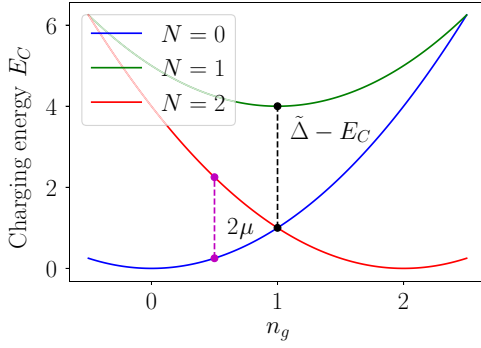


FIG. 2. The energy of the states of the superconducting island for $N = 0, 1, 2$ excess electrons as a function of n_g . The parabolas describe the charging energy in Eq. (2). For odd particle numbers, the energy is lifted by the SC gap $\tilde{\Delta}$. The energy difference between $N = 2$ and $N = 0$ states must be counterbalanced by the dot potential for optimal CAR processes (dashed purple line).

To this end, we extend to the poor man's tetron the analysis presented in Ref. [33] for two dots interacting via a floating superconducting island. In particular, we consider a regime in which the tunneling amplitudes t represent the smallest energy scale in the poor man's tetron Hamiltonian $H_{\text{sys}} = H_{\text{sp}} + H_c$. These amplitudes must be compared with the energies of the states of the superconducting island, labeled by the particle number N , as a function of the induced charge n_g (Fig. 2).

For the sake of simplicity, we consider a physical setup in which the superconductor screens the magnetic field in the proximitized segments of the nanowires, such that we impose $g_{\text{SC}} = 0$ in the Hamiltonian given in Eq. (1). The general case with $g_{\text{SC}} \neq 0$ is discussed in Appendix A. For $g_{\text{SC}} = 0$, the energy of the Bogoliubov quasiparticles in the proximitized region of the nanowires is given by

$$\tilde{\Delta} = \sqrt{\Delta^2 + \mu_{\text{SC}}^2}. \quad (4)$$

We focus on the regime $E_C < \tilde{\Delta}$, where the low-energy states of the superconducting island are all characterized by an even number of electrons N . In this situation, the energy cost for occupying any Bogoliubov mode is indeed larger than the difference in charging energy between even and odd states, even for the extreme case with n_g being an odd integer (see Fig. 2). When the tunneling amplitudes t are considerably smaller than $\tilde{\Delta} - E_C$, the dynamics of the corresponding strong-coupling regime can be described in terms of elastic cotunneling and CAR processes. Both processes emerge at second order in perturbation theory; they are mediated by high-energy virtual states with an odd number N of electrons in the superconducting island and their amplitudes are of order $t^2 / (\tilde{\Delta} - E_C)$ (for details, see Appendix A).

The Hamiltonian of the poor man's tetron given in Eq. (1) implies that the elastic cotunneling processes amount to the coherent tunneling of an electron from one external dot to the other in the same nanowire, mediated by a Bogoliubov state. In this process, the state of the central superconducting island is left untouched. The CAR, instead, corresponds to a process in which a pair of electrons coming from opposite dots in the same nanowire enter the superconducting island by forming a Cooper pair.

In the following, we assume for simplicity that the spin-orbit coupling α is the same in both nanowires and that all tunneling amplitudes $t_{a,\tau}$ have the same value t , as depicted in Fig. 1(b). This corresponds to devices that are symmetric under the exchange of the two nanowires, such that cotunneling and CAR amplitudes do not depend on the τ index.

The description of cotunneling and CAR processes is further simplified in a regime of strong Zeeman splitting $gB_z > t$, which offers a double-nanowire counterpart to the hybrid poor man's Majorana devices in Ref. [33]. In this spin-polarized limit, the low-energy behavior of the poor man's tetron can be described by the effective Hamiltonian

$$H_{\text{eff}} = - \left[t_{\text{COT}}(N) \sum_{\tau} d_{R\tau}^{\dagger} d_{L\tau} + \Delta_{\text{CAR}} e^{-i\varphi} \sum_{\tau} d_{R\tau} d_{L\tau} + \text{H.c.} \right] + \left[\mu_D - \frac{t_{\text{COT}}(N)}{\cos 2\alpha} - gB_z \right] \sum_{a,\tau} d_{a\tau}^{\dagger} d_{a\tau} + E_C (N - n_g)^2, \quad (5)$$

where we have dropped the $\downarrow$ index for the spin degree of freedom. Here, we have introduced the elastic cotunneling amplitudes t_{COT} , which, in general, depend on the charge N of the superconducting island, and the CAR amplitude Δ_{CAR} . Moreover, in Eq. (5), the bare potential μ_D in Eq. (1) is shifted by both the Zeeman energy and a second-order contribution $-t_{\text{COT}}(N)/\cos 2\alpha$ (see Appendix A). For the sake of simplicity, in the following, we will neglect the N dependence of this correction, as it does not qualitatively affect our results. Therefore, we simply label by

$$\mu = \mu_D - gB_z - \frac{\bar{t}_{\text{COT}}}{\cos 2\alpha} \quad (6)$$

an effective average energy level of the dots given by a suitable average cotunneling amplitude $\bar{t}_{\text{COT}}$.

Hereafter, we also restrict our analysis to induced charges $0 \leq n_g \leq 2$. For different values of n_g , the physical properties repeat periodically by adjusting the number of Cooper pairs N_{CP} . For $0 \leq n_g \leq 2$, the two lowest-energy states of the superconducting island have charge $N = 0$ and $N = 2$ and correspond to the two lowest parabolas in Fig. 2. CAR processes between the dots and the island

allow for transitions between these states and, to significantly affect the properties of the system, they require these states to be approximately degenerate. Hence, their charging-energy difference must be balanced by a suitable energy μ of the external dots. This corresponds to the condition $H_c(N=0) + 2\mu = H_c(N=2)$, depicted in Fig. 2 and yielding

$$\mu = 2E_C(1 - n_g). \quad (7)$$

This equation defines the tuning of μ as a function of n_g required to obtain optimal CAR processes in the system, even in the presence of a strong charging energy. The condition given in Eq. (7) matches an analogous constraint for the definition of poor man's Majorana modes in two-dot devices with an intermediate floating superconducting island [33]. The introduction of additional cross-capacitance terms in the Hamiltonian would simply result in a redefinition of H_c and Eq. (7), as long as all the dots are equivalent to each other.

The general derivation of the cotunneling and CAR amplitudes is presented in Appendix A. For spin-polarized dots tuned based on the condition given in Eq. (7), we obtain

$$t_{\text{COT}}(N = 1 \pm 1) = t^2 \cos(2\alpha) \frac{\mu_{\text{SC}} \tilde{\Delta} + \mu_{\text{SC}} E_C \mp 2E_C \tilde{\Delta}}{\tilde{\Delta} (\tilde{\Delta} + 3E_C) (\tilde{\Delta} - E_C)} \quad (8)$$

and

$$\Delta_{\text{CAR}} = \frac{t^2 \sin(2\alpha) \sqrt{\tilde{\Delta}^2 - \mu_{\text{SC}}^2}}{\tilde{\Delta} (\tilde{\Delta} - E_C)}. \quad (9)$$

We observe, in particular, that the dependence of the cotunneling amplitude on the occupation of the superconducting island is approximately proportional to $t^2 E_C \cos(2\alpha) / \tilde{\Delta}^2$ and thus vanishing for small $E_C / \tilde{\Delta}$. The CAR amplitude, on the other hand, is unaffected by N .

In the following sections, we will primarily consider setups in which the tuning condition given in Eq. (7) is fulfilled. We stress, however that deviations from this optimal chemical potential that are small compared to the CAR

and cotunneling amplitudes do not yield major effects on the following analysis.

IV. LOW-ENERGY STATES IN THE WEAK-TUNNELING REGIME

In the absence of coupling to external leads, the total number of electrons in the poor man's tetron is preserved. For $n_g \in [0, 2]$ and a sizeable E_C (comparable with CAR and cotunneling amplitudes), the states with the lowest energy display a total electron number $N_t = N + \sum_{a,\tau} d_{a\tau}^\dagger d_{a\tau} \in [0, 6]$. Therefore, in the following, we will focus on states with these total particle numbers.

Given the total number N_t of electrons, and thereby the total parity of the system $P_u P_d$, there are eight possible states that can be distinguished by the occupation numbers $n_{a\tau} = d_{a\tau}^\dagger d_{a\tau} = 0, 1$. These eight states are, in turn, subdivided into two sets of four states with $P_d = \pm 1$.

Consequently, the Hamiltonian H_{eff} can be decomposed into 4×4 sub-blocks, with two sub-blocks for each total electron number N_t . Their diagonalization allows us to determine the structure of the low-energy states of the poor man's tetron.

The 4×4 Hamiltonian blocks acquire, in general, four possible forms determined by the parities P_u and P_d , which establish whether the dynamics of the related sector in each wire are dictated by CAR or cotunneling processes. Therefore, in general, it will be convenient to label the sectors with the notation (N_t, P_d) , where the total number of particles N_t determines the total parity of the system $P_u P_d$, and P_d further specifies which of the four (P_u, P_d) sector is addressed.

In the following, we analyze the ground state of each parity block to understand the low-energy behavior of the system. In Table I, we summarize the main assumptions for the validity of the following perturbative results and their effect; in Appendix B, we present the effective Hamiltonians in each sector and the related ground-state energies.

A. The odd-parity states

The odd-parity sectors $(N_t = 2m + 1, P_d = +)$ and $(N_t = 2m + 1, P_d = -)$ are affected by both CAR and cotunneling processes. When the low-energy Hamiltonian given in Eq. (5) is invariant under the exchange of the

TABLE I. The assumptions adopted for the perturbative analysis of the poor man's tetron.

Assumption	Effect
$\tilde{\Delta} - g_{\text{SC}} B_z > E_C$	CAR processes are well defined at low energy
$\tilde{\Delta} - g_{\text{SC}} B_z - E_C \gg t \gg t_{\text{COT}} , \Delta_{\text{CAR}} $	Perturbative-tunneling regime
$g B_z \gg t_{\text{COT}} , \Delta_{\text{CAR}} $	Polarization of the dots
$E_C \gtrsim t_{\text{COT}} , \Delta_{\text{CAR}} $	The lowest-energy states display $N_t \in [0, 6]$
$\mu \sim 2E_C(1 - n_g)$	Degeneracy of the $N = 0$ and $N = 2$ charge states

two nanowires, $u \leftrightarrow d$, these two subspaces are equivalent. This emerging symmetry does not require a perfect physical equivalence of the two nanowires; it corresponds instead to a tuning of the poor man's tetron such that the differences of the CAR and cotunneling amplitudes between the two wires are negligible. Even if the parameters in the microscopic Hamiltonian H_{sp} (e.g., Δ or g_{SC}) are different for the two nanowires in realistic devices, the control of the surrounding electrostatic gates enables several strategies to compensate these discrepancies at the level of H_{eff} (see Appendix A).

If we impose the tuning condition given in Eq. (7), the odd sectors with $N_t = 3$ are the only subspaces in which all the CAR processes connect degenerate charge states (see Appendix B). This implies that the eigenstates in $(N_t = 3, P_d)$ display only a weak dependence on the charging energy, inherited by the CAR and cotunneling amplitudes. This scenario is close to the noninteracting limit described by poor man's Majorana modes and, for this reason, we focus our analysis on setups that favor the onset of global ground states in these sectors.

We first address the $(3, +)$ sector. By ordering the dot occupation numbers $n_i = d_i^\dagger d_i$ with $i = 1, \dots, 4$ as indicated in Fig. 1, we can express the four states in this subspace in the basis $|n_1 n_2 n_3 n_4, N\rangle$. They result in $|0100, 2\rangle$, $|1000, 2\rangle$, $|0111, 0\rangle$, and $|1011, 0\rangle$.

We consider a region of parameter space in which $t_{\text{COT}}(2) < t_{\text{COT}}(0) < 0$ and $\Delta_{\text{CAR}} > 0$; this corresponds to small spin-orbit coupling ($0 < \alpha < \pi/4$) and negative $\mu_{\text{SC}} < -2E_C \tilde{\Delta}/(\tilde{\Delta} + E_C)$. In this case, the ground state assumes the form

$$|3+\rangle = \frac{1}{\sqrt{2}} [\cos \theta (|0100, 2\rangle - |1000, 2\rangle) + \sin \theta (|0111, 0\rangle - |1011, 0\rangle)]. \quad (10)$$

Given the symmetry between the two nanowires, the sector $(3, -)$ displays a ground state with the same energy and the form

$$|3-\rangle = \frac{1}{\sqrt{2}} [\cos \theta (|0001, 2\rangle - |0010, 2\rangle) + \sin \theta (|1101, 0\rangle - |1110, 0\rangle)]. \quad (11)$$

The angle θ is a parameter that can be experimentally controlled through the energy level μ_D of the dots. When imposing the degeneracy condition given in Eq. (7), it reads

$$\theta = \frac{1}{2} \arctan \frac{2\Delta_{\text{CAR}}}{t_{\text{COT}}(0) - t_{\text{COT}}(2)}. \quad (12)$$

In the general case, on the other hand, the dot potential μ_D determines the denominator in Eq. (12).

In the limit $E_C \rightarrow 0$, the two cotunneling amplitudes $t_{\text{COT}}(0)$ and $t_{\text{COT}}(2)$ become equal [see Eq. (8)]. Therefore, based on Eq. (12), θ tends to $\pi/4$ and the odd states $|3, \pm\rangle$ become a superposition with equal amplitudes of their four components, hence recovering the noninteracting result expected for two two-site Kitaev chains with vanishing on-site potential.

The odd-parity sectors with $N_t \neq 3$ present ground states with the same form but their states display different charging energies (see Appendix B), which yield an explicit dependence of the parameter θ on E_C .

We finally mention that there is a special line in parameter space, obtained by imposing $\mu_{\text{SC}} = 0$ and fulfilling the degeneracy condition given in Eq. (7), such that $t_{\text{COT}}(2) = -t_{\text{COT}}(0)$ and the ground state of each sector $(3, P_d)$ becomes twofold degenerate. We emphasize, however, that our perturbative results indicate that the global ground state of the tetron does not belong to the $N_t = 3$ sectors in the limit $\mu_{\text{SC}} \rightarrow 0$.

B. The even-parity states

The low-energy sectors with even fermionic parity that compete with the odd states to be the global ground states of the poor man's tetron are characterized by $N_t = 2$ or $N_t = 4$ total electrons for $0 \leq n_g \leq 2$.

The even-parity sectors $(N_t = 2m, P_d = +)$ and $(N_t = 2m, P_d = -)$ are inequivalent since the former has an effective Hamiltonian that depends on CAR processes only, whereas the energies of the latter depend only on elastic cotunneling processes. Their Hamiltonians are presented in Appendix B.

The sector $(2m, +)$ is the only sector involving states with three different values of the charge in the superconducting island. For $N_t = 2, 4$, the tuning of the potential of the external dots allows us to set two of these three charge states to be degenerate. The third acquires a higher energy. As a result, the ground state of $(2m, +)$ displays, in general, a form that detaches considerably from the noninteracting limit $E_C \rightarrow 0$.

In the strongly interacting case $E_C \gg \Delta_{\text{CAR}}$, the related ground states result in

$$|4+\rangle \xrightarrow{E_C \gg \Delta_{\text{CAR}}} \frac{1}{2} |0011, 2\rangle + \frac{1}{2} |1100, 2\rangle + \frac{1}{\sqrt{2}} |1111, 0\rangle, \quad (13)$$

$$|2+\rangle \xrightarrow{E_C \gg \Delta_{\text{CAR}}} \frac{1}{2} |0011, 0\rangle + \frac{1}{2} |1100, 0\rangle + \frac{1}{\sqrt{2}} |0000, 2\rangle. \quad (14)$$

The sector $(2m, -)$, on the other hand, has eigenstates determined exclusively by the cotunneling processes and all its states display a charge $N_t - 2$ in the superconducting

island. For these reasons, its ground state does not depend on E_C and reads

$$|2m-\rangle = \frac{1}{2} [|0101, N_t - 2\rangle - |0110, N_t - 2\rangle - |1001, N_t - 2\rangle + |1010, N_t - 2\rangle]. \quad (15)$$

The choice of the signs in Eq. (15) corresponds to the case $t_{\text{COT}}(N_t - 2) < 0$. The case $t_{\text{COT}}(N_t - 2) > 0$ does not present any qualitative difference and the general scenario is considered in Appendices B and C.

C. Effects of the charging energy

The fermionic parities P_τ are symmetries of the Hamiltonian of the poor man's tetron irrespective of whether the superconducting island is floating or grounded. Thus, the previous sectors also have a clear counterpart in the noninteracting limit $E_C, \mu \rightarrow 0$, even when the total electron number is not conserved.

Therefore, in the noninteracting case also, we can distinguish four low-energy ground states which, for $t_{\text{COT}} < 0$ and $\Delta_{\text{CAR}} > 0$, have wave functions

$$\begin{aligned} |e+\rangle &= \frac{1}{2} (|00\rangle + |11\rangle)_u \otimes (|00\rangle + |11\rangle)_d, \\ |e-\rangle &= \frac{1}{2} (|01\rangle - |10\rangle)_u \otimes (|01\rangle - |10\rangle)_d, \\ |o+\rangle &= \frac{1}{2} (|01\rangle - |10\rangle)_u \otimes (|00\rangle + |11\rangle)_d, \\ |o-\rangle &= \frac{1}{2} (|00\rangle + |11\rangle)_u \otimes (|01\rangle - |10\rangle)_d. \end{aligned}$$

Their energies are as follows:

$$\begin{aligned} E^{(e+)} &= -|\Delta_{\text{CAR},u}| - |\Delta_{\text{CAR},d}|, \\ E^{(e-)} &= -|t_{\text{COT},u}| - |t_{\text{COT},d}|, \\ E^{(o+)} &= -|t_{\text{COT},u}| - |\Delta_{\text{CAR},d}|, \\ E^{(o-)} &= -|\Delta_{\text{CAR},u}| - |t_{\text{COT},d}|. \end{aligned}$$

For systems that are symmetric under $u \leftrightarrow d$, the odd ground states are degenerate and their energy, $-|t_{\text{COT}}| - |\Delta_{\text{CAR}}|$, lies in between the energies of the even ground states.

Additionally, a fourfold degeneracy of the system is obtained at the sweet spot corresponding to $|\Delta_{\text{CAR},\tau}| = |t_{\text{COT},\tau}|$ for both nanowires. This corresponds to the formation of four poor man's Majorana modes [11].

When the particle number is conserved, the four sectors are further split based on the total number of electrons N_t . Subsectors of the same kind, however, remain degenerate for $E_C \rightarrow 0$.

In floating devices with $E_C > 0$, only the state $|e-\rangle$ maintains its noninteracting form. When imposing the

degeneracy of states with charge $N = 0$ and $N = 2$ in the superconducting island via the tuning given in Eq. (7), the odd states with $N_t = 3$ acquire a correction in the coefficients of their wave functions that is represented by the displacement of θ from $\pi/4$ in Eqs. (10) and (11). This correction, at leading order, is linear in $[t_{\text{COT}}(0) - t_{\text{COT}}(2)]/\Delta_{\text{CAR}}$, thus in E_C/Δ [see Eqs. (8), (9)].

The states in odd sectors with $N_t \neq 3$, as well as the states in the $(2m, +)$ sectors, on the other hand, are more affected by E_C given the different energy costs of their charge configurations. With regard to the states $|4+\rangle$ and $|2+\rangle$, this can be easily seen by comparing Eqs. (13), (14), which display only three components, with the noninteracting form $|e+\rangle$, which, instead, is the equal superposition of four terms (for further details, see Appendix B). For this reason, the four poor man's Majorana modes are not stable, in general, under the introduction of the charging energy.

Despite this, however, it is still possible to identify regions in the parameter space of the poor man's tetron in which this device displays degenerate ground states and may approach the physics of the topological tetron. For this purpose, we need to identify regimes of the parameters in which the ground states of the odd sectors with $N_t = 3$ constitute the lowest-energy states of the poor man's tetron and the emerging nanowire symmetry $u \leftrightarrow d$ is achieved. Importantly, the appearance of global ground states with an odd fermionic parity requires a nonvanishing charging energy because, in the noninteracting limit, the energy of the odd ground states lies in between the energies of the even states.

The even sectors with $N_t = 2, 4$ are the main competitors to establish the global ground state for $n_g \sim 1$. This can be seen in the example reported in Fig. 3, where we illustrate which sector expresses the global ground state as a function of n_g and μ for a choice of parameters such that at $\mu = 0$ and $n_g = 1$, there is a degenerate odd ground state belonging to the sectors $(3, \pm)$. Ground states with even fermionic parity surround the central region with the odd ground states at $N_t = 3$.

To gain a qualitative intuition of the different behavior of the ground-state energies of even and odd sectors let us first observe that, due to the different electron numbers, the even states with $N_t = 2, 4$ are affected in a different way by the dot energy level and, in general, they gain a different energy contribution $\pm\mu$ with respect to the odd $N_t = 3$ ground states. In Fig. 4, we display the low-energy spectrum of the poor man's tetron close to $n_g = 1$, when the potential of the external dots fulfills the constraint given in Eq. (7). The chosen parameters correspond to the dashed cut in Fig. 3 and the ground state displays $N_t = 3$ electrons at $n_g = 1$ (blue line in Fig. 4). Equation (7) implies that the $\pm\mu$ energy shift of the even states is translated into a different slope of the even-ground-state energies with respect to n_g (compare the blue line with the dashed and dotted lines).

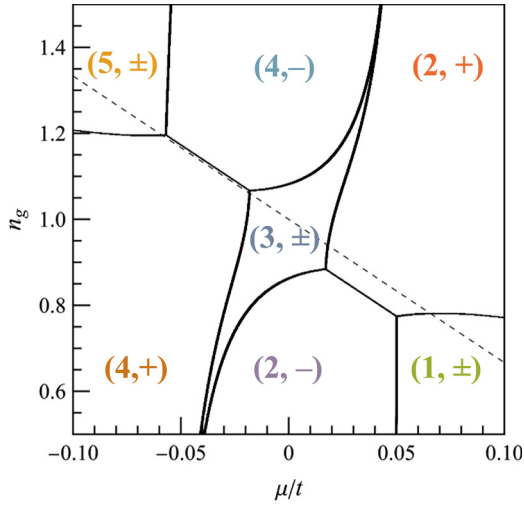


FIG. 3. The alternation of the parity sectors that express the global ground state of the poor man's tetron as a function of n_g and μ . The parameters are $\Delta = 4t$, $\mu_{SC} = -5t$, $E_C = 0.15t$, and $\alpha = \pi/6$. The dashed line corresponds to the constraint given in Eq. (7) and the energies in Fig. 4. The labels indicate the particle number N_i and the parity P_d .

Furthermore, the sign of μ changes at $n_g = 1$. Therefore even states with $N_i = 2$ (dashed lines in Fig. 4) are favored for lower values of n_g and $\mu > 0$, whereas even states with $N_i = 4$ (dotted lines) are favored for higher values of n_g and $\mu < 0$.

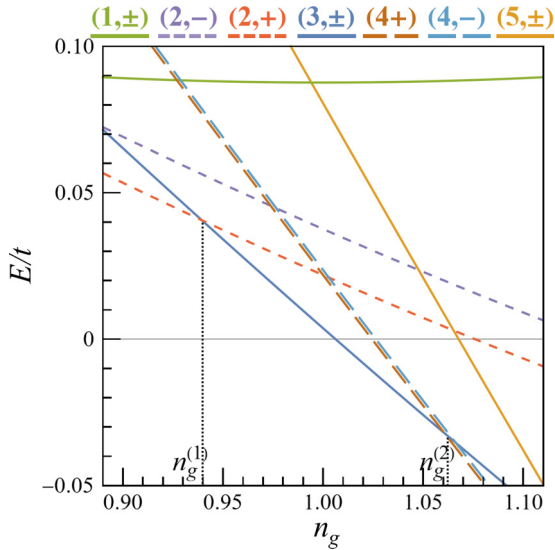


FIG. 4. The energies of the ground states of the parity sectors of the poor man's tetron with $N_i = 1, \dots, 5$ in the perturbative regime as a function of the induced charge n_g . The labels indicate the particle number N_i and the parity P_d . The chemical potential $\mu(n_g)$ is chosen to fulfill Eq. (7). The other parameters are $\Delta = 4t$, $\mu_{SC} = -5t$, $E_C = 0.15t$, and $\alpha = \pi/6$. For this choice of parameters, $\Delta_{CAR} \approx 0.087t$, $t_{COT}(0) \approx -0.056t$, and $t_{COT}(2) \approx -0.063t$.

To better understand the spectrum close to $n_g = 1$, let us consider the regime with $\tilde{\Delta} \gg E_C \gg |\Delta_{CAR}|$. In this case, the energy differences between the ground states of the sectors with $N_i = 2, 4$ (at energies $E^{(N_i=2,4,\pm)}$) and the odd sectors $N_i = 3$ (at energies $E^{(3)}$) can be roughly approximated as

$$\delta E_+^{(N_i=3\pm 1)} \equiv E^{(N_i=3\pm 1,+)} - E^{(3)} \approx \pm\mu - (\sqrt{2} - 1)\Delta_{CAR} + \bar{t}_{COT}, \quad (16)$$

$$\delta E_-^{(N_i=3\pm 1)} \equiv E^{(3\pm 1,-)} - E^{(3)} \approx \pm\mu - \bar{t}_{COT} + \Delta_{CAR}, \quad (17)$$

where the $\pm$ subscripts indicate the parity P_d of the related even sectors. For simplicity, in the previous equations we have replaced all the cotunneling amplitudes with the average value

$$\bar{t}_{COT} \equiv \frac{|t_{COT}(2) + t_{COT}(0)|}{2}. \quad (18)$$

By setting the dot energies to the value $\mu(n_g)$ in Eq. (7), the energy gaps in Eqs. (16) and (17) determine whether there exists a range of induced charges n_g such that the degenerate odd states $|3+\rangle$ and $|3-\rangle$ become the ground states of the poor man's tetron. The perturbative analysis suggests that these gaps are positive around $n_g = 1$ for intermediate values of the charging energies. On the one hand, E_C must be sufficiently larger than Δ_{CAR} such that the modulus of the coefficient of Δ_{CAR} in Eq. (16) is sufficiently smaller than 1. Equation (16) indeed corresponds to the large- E_C limit in Eqs. (13), (14) such that this coefficient acquires its minimum value $\sqrt{2} - 1$. On the other, E_C must be sufficiently small to avoid a large difference between the cotunneling amplitudes $t_{COT}(0)$ and $t_{COT}(2)$. Indeed, in Eqs. (16) and (17), we have neglected this difference which, however, linearly displaces the two ranges of the parameter μ that, respectively, fulfill $\delta E_+^{(N_i=2,4)} > 0$ and $\delta E_-^{(N_i=2,4)} > 0$. Therefore, to satisfy all conditions $\delta E_{\pm}^{(N_i=2,4)} > 0$, we require

$$|t_{COT}(0)| - |t_{COT}(2)| \ll \Delta_{CAR} - \bar{t}_{COT}. \quad (19)$$

Under these assumptions, the onset of ground states with odd fermionic parity is favored at $\Delta_{CAR} \lesssim \sqrt{2}\bar{t}_{COT}$. We emphasize that these conditions are derived based on perturbation theory. In Sec. VI, we will consider the system beyond the perturbative regime and show that the poor man's tetron can display degenerate odd ground states even if these conditions are violated when considering systems with stronger tunneling amplitudes t .

With regard to the choice of parameters in Fig. 4, the CAR amplitude is approximately $\Delta_{CAR} \approx 1.45\bar{t}_{COT}$ and the degenerate states $|3+\rangle$ and $|3-\rangle$ are ground states of the

system in a range of induced charges $n_g^{(1)} < n_g < n_g^{(2)}$. This is the regime that we will consider in the next section to investigate the Kondo effects emerging in the poor man's tetron.

At the points $n_g = n_g^{(1,2)}$, the ground state becomes threefold degenerate, with one of the even states crossing the odd ground states. At $n_g = n_g^{(1)}$ in particular, $E^{(2,+)} = E^{(3)}$: this condition corresponds to a situation in which μ counterbalances the difference between CAR and cotunneling amplitudes and matches the sweet spot discussed in Ref. [33] for a single-nanowire device with a floating superconducting island. For this specific value of the induced charge, the state $|2-\rangle$ is an excited state lifted in energy from the three ground states by approximately $\sqrt{2}\Delta_{\text{CAR}} - 2|t_{\text{COT}}(0)|$ in the limit $E_C \gg \Delta_{\text{CAR}}$.

The structure of the eigenstates close to $n_g^{(1)}$ in the strong- E_C limit is reminiscent of the threefold-degenerate spinful systems studied in Ref. [44], where the nanowire degree of freedom of the spin-polarized poor man's tetron plays a role similar to spin in those two-dot setups. This regime with three low-lying ground states well separated in energy from $|2-\rangle$ is the situation that we address in Sec. VC by mapping the poor man's tetron onto an Anderson impurity model.

In experimental setups, two-terminal conductance measurements performed with external leads connected to the dots on the same wire would allow for the determination of these spectral features: the gaps $\delta E_{\pm}$ can be measured by finite-bias spectroscopy and the charges $n_g^{(i)}$ can be determined by observing the zero-bias peak of the system. Finally, we emphasize that the observations presented so far refer only to the ground states of each of the four sectors. With regard to the excited states within each sector, however, the related gaps are approximately given by $2\Delta_{\text{CAR}}$ or $2t_{\text{COT}}(N)$ and are larger than the gaps $\delta E_{\pm}^{(N_i=2,4)}$ in our regimes of interest.

V. THE THREE-LEAD KONDO MODEL

The most straightforward experimental investigations that can be performed in a hybrid quantum dot–superconductor system such as the poor man's tetron are based on its transport properties. Therefore, in the following, we extend our analysis of this device to include external leads. In particular, we focus on a multiterminal scenario in which the zero-temperature limit of the conductance matrix is intimately related to the onset of Kondo-like correlations between the poor man's tetron and these external electrodes, such that transport can reveal important signatures about the spectrum of the system and its low-energy states.

In the case of topologically protected tetrons, their coupling to external leads results in the topological Kondo

effect [37–39], as the Majorana modes of the tetron contribute to define a nonlocal spin-1/2 impurity. In the following, we investigate the extent to which the peculiar forms and correlations of the ground states discussed in the previous section may allow us to retrieve the topological Kondo effect through a tuning of the poor man's tetron, even without topological Majorana modes.

To this end, we extend our analysis to the following Hamiltonian:

$$H = H_{\text{sys}} + H_{\text{leads}} - \sum_{\alpha} J_{\alpha} l_{\alpha}^{\dagger} d_{\alpha} + \text{H.c.} \quad (20)$$

The index α labels the leads [Fig. 1(a)], each of which is tunnel coupled with one of the dots with amplitude J_{α} . As in the previous section, we assume here that the Zeeman term in the dots is strong, such that only spin-down electrons are involved in the low-energy physics of this model and, therefore, the leads can be also approximated as polarized systems. Hence, the operator $l_{\alpha}^{\dagger}$ constitutes the creation operators of an electron at the extremity of lead α . In Sec. VI, we will present results on the more general spinful scenario.

H can be interpreted as the Hamiltonian of an interacting impurity, corresponding to the poor man's tetron, coupled with external leads as in an Anderson problem. Specifically, we are interested in the regime in which the two odd ground states of the tetron [Eqs. (10) and (11)] constitute two degenerate states of an impurity which, as a consequence, can be treated as an effective spin-1/2 system. In the following, we will indeed consider setups in which the two nanowires are equivalent at low energies, such that the odd ground states $|3+\rangle$ and $|3-\rangle$ are degenerate [see Appendix B]. A violation of this condition would correspond to the introduction of an effective Zeeman term in the analogous Anderson model.

We can obtain insight into the Kondo problem emerging from the coupling of this effective spin-1/2 degree of freedom with the leads by considering a perturbative regime in which the parameters J_{α} are the smallest energy scale in the system, such that $|J_{\alpha}| \ll \delta E_{\pm}^{(N_i)}$ for all even sectors $(2m, \pm)$. In particular, a Kondo model emerges at second order in perturbation theory and can be described by the following Hamiltonian:

$$H_{\text{Kondo}} = \sum_{\alpha, \beta} J_{\alpha} J_{\beta}^{*} (l_{\alpha}^{\dagger} l_{\beta} \Upsilon^{\alpha\beta} + l_{\beta} l_{\alpha}^{\dagger} \Xi^{\alpha\beta}). \quad (21)$$

Here, α and β run over the external leads and the operators $\Upsilon^{\alpha\beta}$ and $\Xi^{\alpha\beta}$ are 2×2 matrices acting on the effective spin-1/2 degree of freedom defined by the ground states $\{|3+\rangle, |3-\rangle\}$. By considering the low-energy subspace

described in the previous section, these matrices read

$$\Upsilon^{\alpha\beta} = \sum_{n, a, b, c=\pm} \frac{-1}{E_n^{(4,c)} - E^{(3)}} |3a\rangle \langle 3a| d_\alpha |4, c, n\rangle \langle 4, c, n| d_\beta^\dagger |3b\rangle \langle 3b|, \quad (22)$$

$$\Xi^{\alpha\beta} = \sum_{n, a, b, c=\pm} \frac{-1}{E_n^{(2,c)} - E^{(3)}} |3a\rangle \langle 3a| d_\beta^\dagger |2, c, n\rangle \langle 2, c, n| d_\alpha |3b\rangle \langle 3b|. \quad (23)$$

In these expressions, the index n runs over the four eigenstates in each even sector of the Hamiltonian H_{eff} in Eq. (5). The energy $E_n^{(N_l, c)}$ corresponds to the eigenenergy of the n th eigenstate of the even sector (N_l, c), labeled by $|N_l, c, n\rangle$ (see Appendix B). $E^{(3)}$ is the energy of the odd ground states with $N_l = 3$ electrons [Eq. (B4)]. The sums in the definition of Υ and Ξ run over the even states with $N_l = 4$ and $N_l = 2$ electrons, respectively.

For the sake of simplicity, in the following subsections, we will consider only the ground states $n = 1$ of the four involved even sectors. States with higher n indeed have higher energies and do not qualitatively affect the formulation of the Kondo problem emerging from the Hamiltonian given in Eq. (21). The quantitative effect of the excited states with $n > 1$ is outlined in Appendix C and it amounts to a variation of the order of approximately 4% in the definition of the Υ and Σ matrices for the parameters considered in Fig. 4.

When restricting to the ground states ($n = 1$) and considering the limit $E_C \gg \Delta_{\text{CAR}}$ in Eq. (13), the matrix

elements of the operator $d_\beta^\dagger$ in Eq. (22) read

$$\sum_{b, c=\pm} \langle 4, c, 1| d_\beta^\dagger |3b\rangle |4, c, 1\rangle \langle 3b| = \frac{\cos \theta}{\sqrt{2}} \begin{pmatrix} \xi_4(\delta_{\beta,1} + \delta_{\beta,2}) & \xi_4(\delta_{\beta,3} + \delta_{\beta,4}) \\ \delta_{\beta,3} - \delta_{\beta,4} & -\delta_{\beta,1} + \delta_{\beta,2} \end{pmatrix}. \quad (24)$$

Here, we have introduced the numerical parameter ξ_4 for the sector $N_l = 4$:

$$\xi_4 = \frac{1 + \sqrt{2} \tan \theta}{2}. \quad (25)$$

This parameter codifies the ratio of the transitions from the odd ground states to the even ground states of the sectors $(4, c)$. The angle θ is defined in Eq. (12). We observe that ξ is related to the deviations of the four many-body ground states from their noninteracting limits: E_C indeed dictates the angle θ and the factor $\sqrt{2}$ is obtained in the strong-interaction limit [Eq. (13)]. In the noninteracting case ($E_C = 0$), $\xi_4 = 1$.

In an analogous way, the matrix elements of d_α in Eq. (23) read

$$\sum_{b, c=\pm} \langle 2, c, 1| d_\alpha |3b\rangle |2, c, 1\rangle \langle 3b| = \frac{\sin \theta}{\sqrt{2}} \begin{pmatrix} \xi_2(\delta_{\alpha,2} - \delta_{\alpha,1}) & \xi_2(\delta_{\alpha,4} - \delta_{\alpha,3}) \\ -\delta_{\alpha,3} - \delta_{\alpha,4} & \delta_{\alpha,1} + \delta_{\alpha,2} \end{pmatrix}, \quad (26)$$

with

$$\xi_2 = \frac{1 + \sqrt{2} \cot \theta}{2}. \quad (27)$$

Based on the previous equations, we obtain

$$\Upsilon^{\alpha\beta} = -\frac{\cos^2 \theta}{2} \begin{pmatrix} \frac{\xi_4^2(\delta_{\alpha,1} + \delta_{\alpha,2})(\delta_{\beta,1} + \delta_{\beta,2})}{\delta E_+^{(4)}} + \frac{(\delta_{\alpha,3} - \delta_{\alpha,4})(\delta_{\beta,3} - \delta_{\beta,4})}{\delta E_-^{(4)}} & \frac{\xi_4^2(\delta_{\alpha,3} + \delta_{\alpha,4})(\delta_{\beta,1} + \delta_{\beta,2})}{\delta E_+^{(4)}} - \frac{(\delta_{\alpha,1} - \delta_{\alpha,2})(\delta_{\beta,3} - \delta_{\beta,4})}{\delta E_-^{(4)}} \\ \frac{\xi_4^2(\delta_{\alpha,1} + \delta_{\alpha,2})(\delta_{\beta,3} + \delta_{\beta,4})}{\delta E_+^{(4)}} - \frac{(\delta_{\alpha,3} - \delta_{\alpha,4})(\delta_{\beta,1} - \delta_{\beta,2})}{\delta E_-^{(4)}} & \frac{\xi_4^2(\delta_{\alpha,3} + \delta_{\alpha,4})(\delta_{\beta,3} + \delta_{\beta,4})}{\delta E_+^{(4)}} + \frac{(\delta_{\alpha,1} - \delta_{\alpha,2})(\delta_{\beta,1} - \delta_{\beta,2})}{\delta E_-^{(4)}} \end{pmatrix}, \quad (28)$$

$$\Xi^{\alpha\beta} = -\frac{\sin^2 \theta}{2} \begin{pmatrix} \frac{\xi_2^2(\delta_{\alpha,1} - \delta_{\alpha,2})(\delta_{\beta,1} - \delta_{\beta,2})}{\delta E_+^{(2)}} + \frac{(\delta_{\alpha,3} + \delta_{\alpha,4})(\delta_{\beta,3} + \delta_{\beta,4})}{\delta E_-^{(2)}} & \frac{\xi_2^2(\delta_{\alpha,3} - \delta_{\alpha,4})(\delta_{\beta,1} - \delta_{\beta,2})}{\delta E_+^{(2)}} - \frac{(\delta_{\alpha,1} + \delta_{\alpha,2})(\delta_{\beta,3} + \delta_{\beta,4})}{\delta E_-^{(2)}} \\ \frac{\xi_2^2(\delta_{\alpha,1} - \delta_{\alpha,2})(\delta_{\beta,3} - \delta_{\beta,4})}{\delta E_+^{(2)}} - \frac{(\delta_{\alpha,3} + \delta_{\alpha,4})(\delta_{\beta,1} + \delta_{\beta,2})}{\delta E_-^{(2)}} & \frac{\xi_2^2(\delta_{\alpha,3} - \delta_{\alpha,4})(\delta_{\beta,3} - \delta_{\beta,4})}{\delta E_+^{(2)}} + \frac{(\delta_{\alpha,1} + \delta_{\alpha,2})(\delta_{\beta,1} + \delta_{\beta,2})}{\delta E_-^{(2)}} \end{pmatrix}, \quad (29)$$

where these 2×2 matrices act on the effective spin-1/2 degree of freedom constituted by the odd ground states.

The Hamiltonian given in Eq. (21) with the previous matrices allows us to define the Kondo problem associated with the poor man's tetron as a function of the physical parameters of the system. For the sake of simplicity, in the following subsections, we will restrict to the three-lead problem by setting $J_4 = 0$.

A. The poor man's topological Kondo point

Topologically protected tetrons with four Majorana modes are expected to display a topological Kondo effect characterized by a fractional off-diagonal conductance $2e^2/Mh$, where $M = 3, 4$ is the number of external leads coupled to the zero-energy modes [37–39]. This conductance is the main signature of the topological Kondo fixed point, which is stable against anisotropies of the couplings with the leads [37] but it is unstable under the Majorana overlaps, which constitute an effective relevant Zeeman term in the related Kondo problem [45,46]. Moreover, renormalization-group analysis and numerical simulations of the dynamics of tetrons show that the topological Kondo effect is magnified in the proximity of the charge-degeneracy point of these devices [47,48].

In the following, we investigate under which tuning conditions the poor man's tetron displays an analogous topological Kondo point that can be detected by a fractionalized conductance at low voltage bias.

As shown by the definition of the effective Hamiltonian in Eq. (21) and the matrices Υ and Ξ , the Kondo problem emerging from the poor man's tetron depends on the energies of the eigenstates of the Hamiltonian given in Eq. (5), which, in turn, are affected by the charging energy E_C . This is the main difference between the poor man's tetron and the topologically protected tetron and it originates from their different geometry entailing that, as a first approximation, the charging energy of the poor man's tetron does not depend on the occupation of the four external dots but only on the charge N of the superconducting island. This situation is different in the standard tetron, in which the fermionic parity of the Majorana modes directly affects its charging energy.

To build the analogy between the Hamiltonian H_{Kondo} and the topological Kondo effect, we rewrite it in the following form:

$$H_{\text{Kondo}} = \sum_{\alpha, \beta} J_{\alpha} J_{\beta}^* I_{\alpha}^{\dagger} I_{\beta} (\Upsilon^{\alpha\beta} - \Xi^{\alpha\beta}) + \sum_{\alpha} |J_{\alpha}|^2 \Xi^{\alpha\alpha}. \quad (30)$$

In this formulation, the leads are coupled to the two odd ground states via the matrices $\Upsilon^{\alpha\beta} - \Xi^{\alpha\beta}$ and the matrices $\Xi^{\alpha\alpha}$ may cause an additional Zeeman splitting of the effective spin-1/2 degree of freedom. In the case with three leads, by following the construction in Refs. [49,50], we

can define a set of spin-1 lead operators in the form

$$I_{\gamma} = i \sum_{\alpha, \beta=1}^3 \epsilon_{\gamma\beta\alpha} I_{\alpha}^{\dagger} I_{\beta}, \quad (31)$$

where ϵ is the Levi-Civita antisymmetric tensor. These operators correspond to generators of the $SO(3)$ group that underlies the topological Kondo effect.

A topological Kondo point is indeed achieved when H_{Kondo} acquires the ideal form [37,49]:

$$H_{\text{TK}} = \sum_{\gamma} \lambda_{\gamma\gamma} I_{\gamma} \sigma_{\gamma}, \quad (32)$$

which constitutes an over-screened Kondo problem with the spin-1 lead operators I_{α} coupled to the effective spin-1/2 Pauli matrices σ_{γ} . Additional Zeeman terms affecting the spin-1/2 degree of freedom are relevant perturbations that must vanish to retrieve the ideal topological Kondo effect.

To compare the Hamiltonians given in Eqs. (30) and (32), we define the matrices

$$\Sigma^{\alpha\beta} \equiv \Upsilon^{\alpha\beta} - \Xi^{\alpha\beta} \quad (33)$$

and we consider their symmetrization and antisymmetrization under the exchange $\alpha \leftrightarrow \beta$:

$$\Sigma_s^{\alpha\beta} \equiv \frac{\Sigma^{\alpha\beta} + \Sigma^{\beta\alpha}}{2}, \quad \Sigma_{\text{as}}^{\alpha\beta} \equiv \frac{\Sigma^{\alpha\beta} - \Sigma^{\beta\alpha}}{2}. \quad (34)$$

Introducing the constants

$$A_{\pm} = \frac{\cos^2 \theta \xi_4^2}{2\delta E_{+}^{(4)}} \pm \frac{\sin^2 \theta \xi_2^2}{2\delta E_{+}^{(2)}}, \quad (35)$$

$$B_{\pm} = \frac{\cos^2 \theta}{2\delta E_{-}^{(4)}} \pm \frac{\sin^2 \theta}{2\delta E_{-}^{(2)}}, \quad (36)$$

these may be expressed as

$$\Sigma_s^{12} = -\sigma_3 \frac{A_{+} + B_{+}}{2} + \sigma_0 \frac{B_{+} - A_{+}}{2}, \quad \Sigma_{\text{as}}^{12} = 0, \quad (37)$$

$$\Sigma_s^{23} = -\sigma_1 \frac{A_{+} + B_{+}}{2}, \quad \Sigma_{\text{as}}^{23} = i\sigma_2 \frac{A_{+} - B_{+}}{2}, \quad (38)$$

$$\Sigma_s^{13} = \sigma_1 \frac{B_{-} - A_{-}}{2}, \quad \Sigma_{\text{as}}^{13} = i\sigma_2 \frac{A_{-} + B_{-}}{2}. \quad (39)$$

Without loss of generality, we consider a gauge choice for the degrees of freedom of the leads such that $J_2 = J$ is real and $J_1 = J_3 = iJ$ are imaginary. Based on these

definitions, the Hamiltonian given in Eq. (30) can be rewritten as

$$\begin{aligned} \frac{H_{\text{Kondo}}}{J^2} = & -I_3 \Sigma_s^{12} - I_1 \Sigma_s^{23} - iI_2 \Sigma_{\text{as}}^{13} \\ & + iF_1 \Sigma_{\text{as}}^{23} + F_2 \Sigma_s^{13} \\ & + uZ\sigma_3 + \sum_{\alpha} \Xi_{\alpha\alpha} - w \sum_{\alpha} l_{\alpha}^{\dagger} l_{\alpha}, \end{aligned} \quad (40)$$

where $u = (B_- - A_-)/2$, $w = (A_- + B_-)/2$, and we have introduced the following (symmetric) lead operators:

$$F_{\gamma} = \sum_{\alpha, \beta=1}^3 |\epsilon_{\gamma\alpha\beta}| l_{\alpha}^{\dagger} l_{\beta}, \quad (41)$$

$$Z = l_1^{\dagger} l_1 + l_2^{\dagger} l_2 - l_3^{\dagger} l_3. \quad (42)$$

The operators F correspond to generators of the $SU(3)$ group, whereas the operator Z is not traceless and can be decomposed in terms of generators of $U(3)$.

In Eq. (40), the last term plays the role of a diagonal potential scattering and it can simply be considered as a shift of the voltage bias of the leads; therefore we disregard it.

The most straightforward approach to a topological Kondo point is to set the constraint $A_+ = B_+$:

$$\frac{\cos^2 \theta \xi_4^2}{2\delta E_+^{(4)}} + \frac{\sin^2 \theta \xi_2^2}{2\delta E_+^{(2)}} = \frac{\cos^2 \theta}{2\delta E_-^{(4)}} + \frac{\sin^2 \theta}{2\delta E_-^{(2)}}. \quad (43)$$

Under this assumption, we indeed obtain

$$\begin{aligned} H_{\text{Kondo}} = & J^2 \left(A_+ I_1 \sigma_1 + \frac{A_- + B_-}{2} I_2 \sigma_2 + A_+ I_3 \sigma_3 \right) \\ & + J^2 \frac{B_- - A_-}{2} \left(F_2 \sigma_1 + Z \sigma_3 - \frac{\sigma_3}{2} \right). \end{aligned} \quad (44)$$

Here, the first line matches the description of the three-lead topological Kondo Hamiltonian given in Eq. (32), with an anisotropic coupling along the y direction of the spins; the topological Kondo effect, however, is robust against this kind of anisotropy [37]. The second line of Eq. (44), on the other hand, corresponds to perturbations away from the topological Kondo fixed point, which share the same coupling,

$$\varepsilon \equiv J^2 \frac{B_- - A_-}{2} = \frac{J^2 \cos^2 \theta}{2} \left(\frac{1}{\delta E_-^{(4)}} - \frac{\xi_4^2}{\delta E_+^{(4)}} \right), \quad (45)$$

at the fine-tuned point provided by Eq. (43). We observe that $\varepsilon \rightarrow 0$ in the limit of small charging energies.

For the parameters adopted in Fig. 4, the condition given in Eq. (43) is fulfilled for $n_g \approx 0.997$, where the ratio

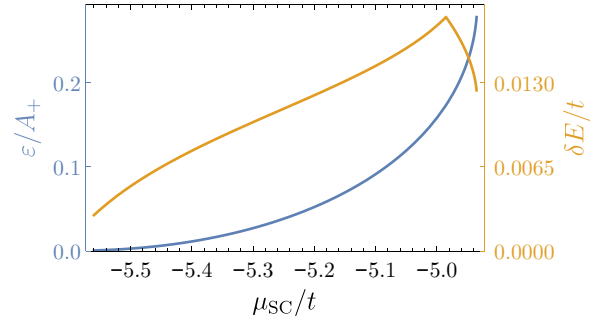


FIG. 5. The relative error ε/A_+ (blue) and energy gap δE (orange) between the even excited states and the odd ground states as a function of μ_{SC} . The data are calculated by considering the optimal induced charge n_g fulfilling the constraint given in Eq. (43). The other system parameters are the same as for Fig. 4.

between the perturbation given in Eq. (45) and the coefficient λ_{11} of the topological Kondo Hamiltonian becomes $\varepsilon/A_+ \approx 0.15$. In Fig. 5, we observe, however, that by varying the parameter μ_{SC} , it is possible to achieve considerably lower relative perturbations ε/A_+ at the fine-tuned points fulfilling Eq. (43) in correspondence with the sizeable gaps between the odd ground states and the even excited states.

Requiring that both the tuning given in Eq. (43) is fulfilled and the perturbation ε vanishes corresponds to setting

$$\xi_{N_t}^2 = \frac{\delta E_+^{(N_t)}}{\delta E_-^{(N_t)}} \quad (46)$$

for both $N_t = 2, 4$. This double condition cannot, in general, be achieved by tuning only n_g but it requires a simultaneous control of at least two parameters (e.g., n_g and μ_{SC}).

Importantly, the constraint given in Eq. (43) does not depend on the amplitudes of the couplings $|J_{\alpha}|$. Anisotropic couplings J_{α} would simply modify the different terms in Eqs. (40) and (44) through multiplicative constants. Since the topological Kondo effect is robust against the anisotropy of the coefficients λ in Eq. (32), small differences in the parameters J_{α} do not hinder the possibility of approaching the topological Kondo point.

In the following, we show that the ε perturbations in the second line of the Hamiltonian given in Eq. (44) are relevant in the renormalization-group sense and we discuss their effect.

B. Effects of perturbations

The Kondo Hamiltonian in Eq. (44) differs from the topological Kondo Hamiltonian in Eq. (32) by the additional exchange interactions $F_2 \sigma_1$ and $Z \sigma_3$, besides a Zeeman term along σ_3 . Furthermore, in realistic scenarios,

we must consider several additional effects that contribute to perturb the topological Kondo point identified by the optimal tuning given in Eq. (43).

The first is determined by the contribution of the excited states of the even sectors ($N_i, -$) with $N_i = 2, 4$ in the perturbative definition of the matrices Υ and Ξ (see Appendix C). These high-energy states cause a shift of the optimal point [as described in Eq. (C21)], modify the Zeeman field along σ_3 , and introduce additional potential scattering terms proportional to $l_2^\dagger l_1 \sigma_0 + \text{H.c.}$

The second important perturbation is caused by a possible detuning away from the optimal condition in Eq. (43) [or, analogously, Eq. (C21)], which determines the onset of the term $F_1 \sigma_2$ in Eq. (40).

Therefore, in general, we will consider the following Hamiltonian for a three-lead setup:

$$H = H_{\text{TK}} + \delta H, \quad (47)$$

where the topological Kondo Hamiltonian H_{TK} is defined in Eq. (32) and the perturbations away from it acquire the general form

$$\begin{aligned} \delta H = & h_3 \sigma_3 + \varepsilon_2 (l_1^\dagger l_3 \sigma_1 + \text{H.c.}) + \varepsilon_1 (l_2^\dagger l_3 \sigma_2 + \text{H.c.}) \\ & + \varepsilon_2 (l_1^\dagger l_1 + l_2^\dagger l_2 - l_3^\dagger l_3) \sigma_3 + \sum_{\alpha, \beta} w_{\alpha\beta} l_\alpha^\dagger l_\beta, \end{aligned} \quad (48)$$

where $\varepsilon_2 \propto J^2(B_- - A_-)$ and $\varepsilon_1 \propto J^2(B_+ - A_+)$ correspond to the detuning away from the condition given in Eq. (43).

In the absence of the corrections δH , a device described by H_{TK} displays the topological Kondo effect below the Kondo temperature $T_K \propto \rho \lambda^2 \exp[-1/(\rho \lambda)]$ that depends on the lead density of states ρ and the coupling constants $\lambda_{\gamma\gamma}$ ($\lambda = \lambda_{\gamma\gamma}$ in the isotropic limit [49]). T_K dictates the crossover between this strong-coupling regime and weak-coupling regimes that can typically be addressed with resonant-level models [47,48]. The strongly coupled topological Kondo regime, in particular, is characterized by the onset of the fractional nonlocal conductance $2e^2/Mh$, the corrections of which with the voltage bias scale with exponents reflecting its non-Fermi-liquid behavior [51,52] (for the related thermal-transport properties, see also Ref. [53]).

When including the perturbations described by δH , a general analysis of their effects can be obtained by renormalization-group considerations. The last term in Eq. (48) describes a potential scattering that does not involve the effective spin-1/2 degree of freedom. This term is marginal in the renormalization-group analysis of the topological Kondo problem determined by H_{TK} . As previously mentioned, the components with $\alpha = \beta$ can be corrected by a suitable shift of the voltage bias of the leads. The off-diagonal terms with $\alpha \neq \beta$ do not qualitatively

affect the non-Fermi-liquid properties of the topological Kondo effect [49].

The Zeeman term $h_3 \sigma_3$ is relevant and drives the system away from the topological Kondo point if $h_3 \gtrsim T_K$. A small Zeeman term introduces an additional energy scale $T_h \sim T_K (h_3/T_K)^{3/2}$ [45,46]. T_h sets the crossover between the topological Kondo effect, with $M = 3$ leads for $T_h \ll T \ll T_K$ and a trivial regime for $T \ll T_h$ with vanishing zero-bias conductance. Hence, in order to observe the physics of the topological Kondo effect, the Zeeman term h_3 , as estimated in Eq. (45), must be sufficiently smaller than T_K .

To investigate the role of the perturbations ε_i in three-lead devices, we follow the construction in Ref. [49] and introduce a set of rotated creation and annihilation lead operators of the form

$$\ell_{+1} = \frac{l_1 + il_2}{\sqrt{2}}, \quad \ell_{-1} = \frac{-l_1 + il_2}{\sqrt{2}}, \quad \ell_0 = l_3. \quad (49)$$

The Hamiltonian $H'_{\text{Kondo}} = H_{\text{TK}} + \delta H'$ can be rewritten as

$$H'_{\text{Kondo}} = \frac{\lambda_{11} + \lambda_{22} + \varepsilon_2 - \varepsilon_1}{\sqrt{2}} (\ell_0^\dagger \ell_{+1} \sigma^+ + \ell_{+1}^\dagger \ell_0 \sigma^-) \quad (50a)$$

$$+ \frac{\lambda_{11} + \lambda_{22} - \varepsilon_2 + \varepsilon_1}{\sqrt{2}} (\ell_0^\dagger \ell_{-1} \sigma^- + \ell_{-1}^\dagger \ell_0 \sigma^+) \quad (50b)$$

$$+ \frac{\lambda_{11} - \lambda_{22} + \varepsilon_2 + \varepsilon_1}{\sqrt{2}} (\ell_0^\dagger \ell_{+1} \sigma^- + \ell_{+1}^\dagger \ell_0 \sigma^+) \quad (50c)$$

$$+ \frac{\lambda_{11} - \lambda_{22} - \varepsilon_2 - \varepsilon_1}{\sqrt{2}} (\ell_0^\dagger \ell_{-1} \sigma^+ + \ell_{-1}^\dagger \ell_0 \sigma^-) \quad (50d)$$

$$+ (\lambda_{33} + \varepsilon_2) (\ell_{+1}^\dagger \ell_{+1} - \ell_0^\dagger \ell_0) \sigma_3 \quad (50e)$$

$$+ (\lambda_{33} - \varepsilon_2) (\ell_0^\dagger \ell_0 - \ell_{-1}^\dagger \ell_{-1}) \sigma_3 \quad (50f)$$

$$+ \varepsilon_2 \ell_0^\dagger \ell_0 \sigma_3. \quad (50g)$$

This form makes it easy to understand the distinctiveness of the topological Kondo point and the role of the ε_i perturbations. In the unperturbed axial case defined by $\lambda_{11} = \lambda_{22} = \lambda$ and $\varepsilon_1 = \varepsilon_2 = 0$, the three-channel topological Kondo problem in Eq. (32) decomposes into two standard spin-1/2 Kondo problems with spin-1/2 conduction electrons. These two models, however, share the same impurity and one of their conduction channels, namely, that associated with the lead operators $\ell_0^\dagger$ and ℓ_0 [49,50]. These two Kondo problems have their off-diagonal components defined by the first two lines in Eq. (50). For $\varepsilon_1 = \varepsilon_2$, the two related amplitudes are equal and their competition is crucial to generate the non-Fermi-liquid topological

Kondo fixed point that, indeed, corresponds to the over-screened Kondo problem with spin-1 conduction electrons. If $\varepsilon_2 - \varepsilon_1 \neq 0$, however, one of the two Kondo problems dominates over the other. For instance, when $\varepsilon_2 > \varepsilon_1$, the first coupling constant in Eq. (50) grows faster than the second under the renormalization group and the system flows toward an isotropic strong-coupling fixed point at which the channels 0 and +1 screen the effective spin-1/2 impurity [49]. The perturbation corresponding to $\varepsilon_2 - \varepsilon_1$ is thus relevant and drives the system from the topological Kondo point to a standard spin-1/2 Kondo problem.

The terms given in lines (50c) and (50d), on the other hand, correspond to the anisotropy of the couplings $\lambda_{\gamma\gamma}$ of the topological Kondo problem. This anisotropy is irrelevant [37], which also suggests that the role of the perturbation associated with $\varepsilon_2 + \varepsilon_1$ does not affect the features of the topological Kondo point.

The terms given in lines (50e) and (50f) constitute the diagonal terms of the two spin-1/2 Anderson impurity models. Analogously to the related off-diagonal terms given in lines (50a) and (50b), the perturbation ε_2 favors either of them, such that, in this case also, ε_2 appears as a relevant perturbation that strengthens the renormalization flow toward a fixed point at which the effective impurity is screened by two of the lead degrees of freedom represented by Eq. (49) and either the +1 or -1 channel remains decoupled.

Finally, the last term, given in line (50g) may strengthen the role of the Zeeman term but does not lead to an asymmetry between the two standard Anderson models.

We therefore conclude that both perturbations, ε_2 and ε_1 , taken singularly, will drive the system away from the topological Kondo fixed point. In particular, given the analogy to the axial anisotropy studied in Refs. [49,50], these perturbations behave in a way similar to the Zeeman terms and define an additional energy scale:

$$T_\varepsilon \sim T_K [\rho |\varepsilon_2 - \varepsilon_1|]^{3/2}. \quad (51)$$

The fractional conductance and non-Fermi-liquid features typical of the topological Kondo point are expected to emerge for bias voltages and temperatures in the interval $T_\varepsilon \ll T, eV_b \ll T_K$. If, instead, the perturbations ε_i become comparable to the Kondo temperature T_K , the topological Kondo features will not emerge in the poor man's tetron.

We end the analysis of the perturbations away from the topological Kondo point by considering the differences between poor man's tetrans coupled with $M = 3$ and $M = 4$ leads. In general, the Zeeman term in Eq. (40) originates from the sum over the lead index of the operators $\Xi^{\alpha\alpha}$. In the four-lead case, when all the couplings J_α have the same modulus, this term simply returns a trivial contribution proportional to the 2×2 identity in Eq. (44). This makes it easier to observe the onset of the topological Kondo effect.

The topological Kondo problem stemming from the four-lead devices can be analyzed by generalizing the lead operators I_γ in Eq. (31) to the antisymmetric lead generators of the $SO(4)$ group (see, e.g., Ref. [54]). Its perturbations can be expressed in terms of symmetric lead generators of $SU(4)$ that generalize the F_γ operators in Eq. (41). Furthermore, analogously to the Zeeman term, the Z term in Eq. (40) is replaced by additional $SU(4)$ generators in four-lead systems with equal couplings such that, differently from the three-lead case, no perturbations driving the system toward a $U(4)$ Kondo model appear.

The terms stemming from symmetric generators are also relevant perturbations for $M = 4$ but the related crossover energy between trivial and topological Kondo regimes is expected to scale as ε^2/T_K . Therefore, we envisage that the poor man's topological Kondo regime is more easily observable in four-lead devices. A perturbative renormalization-group analysis of this problem is deferred to future works.

C. The Anderson-impurity-model regime

The decomposition of the perturbed topological Kondo Hamiltonian in the form given in Eq. (50) suggests that driving the system away from the topological Kondo point defined by the constraint given in Eq. (43) modifies the low-temperature behavior of the poor man's tetron from the non-Fermi-liquid topological Kondo fixed point to a strongly coupled Kondo regime corresponding to an effective spin-1/2 impurity screened by two species of conduction electrons.

This picture can be further confirmed by the analysis of the Kondo problem in proximity to the degeneracy points $n_g = n_g^{(1/2)}$ (see Fig. 4). At these points, one of the even ground states becomes degenerate with the odd ground states. In the following, we show that such a regime can be mapped onto a simple spin-1/2 Anderson impurity model with infinite interaction, which, indeed, shares the low-temperature behavior of the standard Kondo problem.

We focus in particular on the neighborhood of $n_g \approx n_g^{(1)}$. The behavior at $n_g^{(2)}$ is analogous through a suitable particle-hole transformation.

Close to $n_g^{(1)}$, the lowest-energy states of the poor man's tetron are the almost-degenerate $|3+\rangle$, $|3-\rangle$, and $|2+\rangle$. The lowest excitation is then provided by $|2-\rangle$, which is separated in energy by $\delta E_-^{(2)} \sim \sqrt{2}\Delta_{\text{CAR}} - 2|t_{\text{COT}}(0)|$. For different parameter regimes with respect to the one depicted in Fig. 4, the roles of $|2+\rangle$ and $|2-\rangle$ may be exchanged.

We consider a regime in which the J_α couplings with the leads in Eq. (20) are small compared to the energy gaps of the tetron: $|J_\alpha| \ll \delta E_-^{(2)}$. Therefore, we can approximate the low-energy dynamics of the system by considering only the three ground states. Given the fact that $|2+\rangle$ has one electron less than the odd states, the resulting effective

Hamiltonian acquires the form

$$\begin{aligned} H(n_g^{(1)}) &= - \sum_{\alpha,b} J_{\alpha} l_{\alpha}^{\dagger} (2 + |d_{\alpha}|3b) |2+\rangle \langle 3b| + \text{H.c.} \\ &= \frac{\xi_2 \sin \theta}{\sqrt{2}} \left[(J_2 l_2^{\dagger} - J_1 l_1^{\dagger}) |2+\rangle \langle 3+| + \right. \\ &\quad \left. + (J_4 l_4^{\dagger} - J_3 l_3^{\dagger}) |2+\rangle \langle 3-| \right] + \text{H.c.}, \quad (52) \end{aligned}$$

where we have used Eq. (26). From the previous equation, it is easy to observe that only two linear combinations of the lead operators are involved in the Hamiltonian. In particular, we relabel as follows:

$$\ell_u \propto J_2^* l_2 - J_1^* l_1, \quad \ell_d \propto J_4^* l_4 - J_3^* l_3. \quad (53)$$

The Hamiltonian $H(n_g^{(1)})$ becomes equivalent to the spin-1/2 Anderson impurity model,

$$H_{\text{AIM}} = (J_u \ell_u^{\dagger} d_u + J_d \ell_d^{\dagger} d_d + \text{H.c.}) + U d_u^{\dagger} d_u d_d^{\dagger} d_d, \quad (54)$$

in the limit of infinite interactions $U \rightarrow \infty$. This mapping corresponds to considering the even state $|2-\rangle$ as the vacuum state of the impurity problem and the odd states as the doublet defined by the creation operators $d_u^{\dagger}$ and $d_d^{\dagger}$.

In this scenario, there is no term in Hamiltonian given in Eq. (54) that allows for a flip of the τ degrees of freedom of the electrons, such that any differential conductance between the top and bottom wires vanishes at zero bias. This constitutes the main distinction in the transport features between the topological Kondo regime discussed in Sec. V A and the Anderson-model regime with strong ε perturbations. Other differences between these two regimes affect the transport from the left to the right leads: we observe that the model in Eq. (52) displays strong similarities with the infinite interaction limit of a two-channel impurity model with spin-polarized leads [55,56], with the τ degree of freedom playing the role of the spin. For anisotropic amplitudes J_{α} that break the symmetry between the two nanowires, it is known that the Kondo peak of this model is split by a different renormalization of the doublet energy levels. Consequently, the differential conductance of the poor man's tetron between left and

right leads within the same nanowire is expected to exhibit a splitting of the Kondo peak away from zero bias when the ε perturbations dominate.

VI. NUMERICAL RESULTS BEYOND THE PERTURBATIVE REGIME

The analysis performed in the previous sections has relied on two main simplifying assumptions. (i) We have assumed that the dots are weakly coupled to the superconducting islands, namely, that $t \ll \tilde{\Delta} - E_C$, to describe the dynamics in terms of CAR and coherent cotunneling processes. (ii) We have considered spin-polarized systems by assuming that the electrons in the dots are affected by a strong Zeeman term ($gB_z \gg \tilde{t}_{\text{COT}}, \Delta_{\text{CAR}}$) and possibly by strong intradot repulsions.

The poor man's tetron, however, can be analyzed in more general regimes by performing exact diagonalization on the full Hamiltonian $H_{\text{sys}} = H_{\text{sp}} + H_c$ defined by Eqs. (1) and (2), which we implement numerically using the SNEG software [57,58]. In this section, we observe that the onset of ground states with odd fermionic parity can actually be favored when increasing the tunneling amplitude t beyond the validity of our previous perturbative analysis.

Additionally, since the energy splitting between the ground states of the various parity sectors mostly scales with the tunneling amplitude t , the nonperturbative regimes with a stronger hybridization between dots and the superconducting island offers larger energy gaps. This is analogous to the enhanced stability of the poor man's Majorana modes experimentally observed in the strong-coupling regime [7].

In Fig. 6, we show how the parameter region with degenerate $N_t = 3$ ground state in the μ_D, n_g plane evolves for different values of the potential μ_{SC} in the weak-tunneling regime. The related energy gap δE is calculated in units of t and the numerical data correspond to a strongly polarized scenario in which Zeeman term $gB_z = 1.5\Delta = 6t$. Figure 6(b) corresponds to the parameters adopted in Fig. 3. To make the comparison easier, we have shifted μ_D in Fig. 6 by gB_z [see also Fig. 8(a)]. With respect to Fig. 3 we observe that μ_D and μ are displaced by a quantity roughly corresponding the average contribution $-\tilde{t}_{\text{COT}}/\cos 2\alpha = -2\tilde{t}_{\text{COT}}$ in Eq. (5). Besides this, the

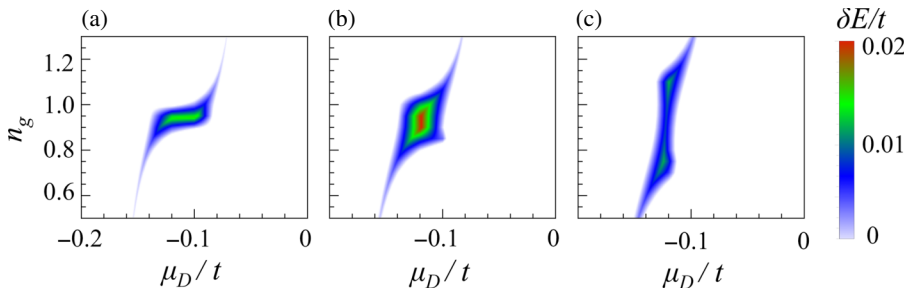


FIG. 6. The excitation energy $\delta E/t = \min(\delta E_{\pm}^{(2)}, \delta E_{\pm}^{(4)})/t$ versus μ_D and n_g for (a) $\mu_{\text{SC}}/t = -6$, (b) $\mu_{\text{SC}}/t = -5$, and (c) $\mu_{\text{SC}}/t = -4$, respectively. Only the region with the odd $N_t = 3$ ground state is shown. The other parameters are $\Delta = 4t$, $\alpha = \pi/6$, $E_C = 0.15t$, and $E_Z = 1.5\Delta$.

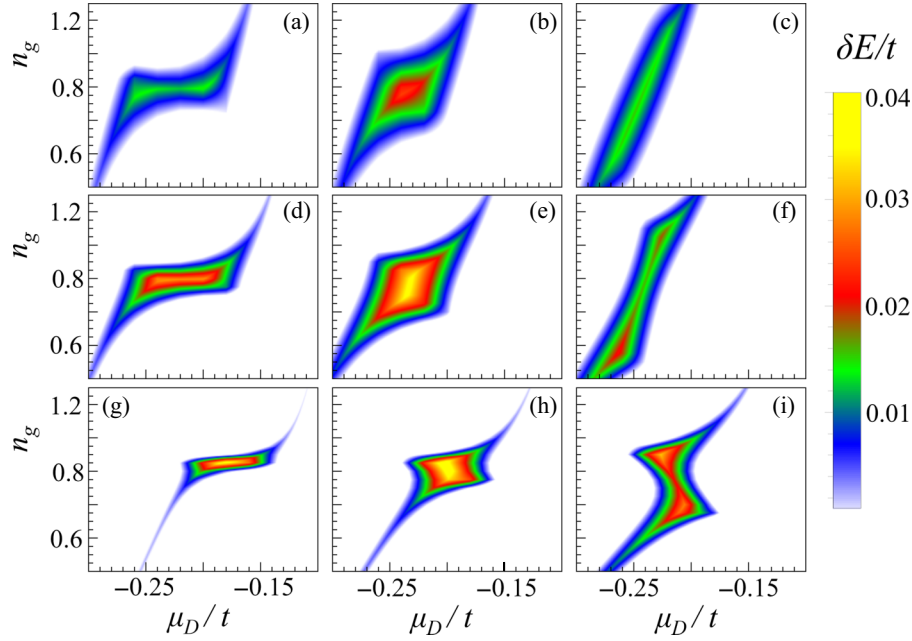


FIG. 7. The excitation energy $\delta E/t = \min(\delta E_{\pm}^{(2)}, \delta E_{\pm}^{(4)})/t$ versus μ_D and n_g in the nonperturbative regime at $\Delta = 2t$ for various μ_{SC} and charging-energy values: (a) $E_C = 0.05t$, $\mu_{SC} = -3t$; (b) $E_C = 0.05t$, $\mu_{SC} = -2.7t$; (c) $E_C = 0.05t$, $\mu_{SC} = -2.4t$; (d) $E_C = 0.15t$, $\mu_{SC} = -3t$; (e) $E_C = 0.15t$, $\mu_{SC} = -2.5t$; (f) $E_C = 0.15t$, $\mu_{SC} = -2t$; (g) $E_C = 0.5t$, $\mu_{SC} = -3.8t$; (h) $E_C = 0.5t$, $\mu_{SC} = -3t$; (i) $E_C = 0.5t$, $\mu_{SC} = -2.2t$. Only the region with the odd $N_t = 3$ ground state is shown. The other parameters are $\alpha = \pi/6$ and $E_Z = 1.5\Delta$. The color scale and the ranges of the parameter n_g are the same as in Fig. 6.

exact-diagonalization results in Fig. 6 confirm the general validity of the perturbative results in Sec. IV.

In Fig. 7, we show, instead, the results in a regime with a stronger hybridization between the dots and the superconducting island. In particular, we have considered $\Delta = 2t$ and we illustrate the extension of the region with degenerate $N_t = 3$ ground states in the μ_D, n_g plane for different values of μ_{SC} . In this scenario, the odd ground states dominate over a considerably broader region in parameter space and the related gaps grow with the tunneling amplitude t . The comparison between Figs. 7 and 6 suggests that the Kondo physics discussed in Sec. V is more easily observable for large tunnel coupling t , where the tuning of the other parameters to obtain degenerate ground states is easier to achieve. The three panels in Fig. 7 additionally display the extension of the region with $N_t = 3$ degenerate ground states for different values of the charging energy, ranging from a case in which E_C is less than half of the corresponding CAR and cotunneling amplitudes to a strong- E_C example. We conclude that the degeneracy of the ground states of the poor man's tetron can be achieved for a broad range of charging energies.

Finally, we estimate the deviations of the exact odd ground states from the perturbative form presented in Eqs. (10) and (11). The perturbative analysis in Sec. V is based on the assumption that the Bogoliubov states of the proximitized dots at energy $\tilde{\Delta}$ are not occupied. In this case, the total fermionic parity matches with the fermionic

parity of the external dots. Away from the perturbative regime, however, this assumption is not fulfilled in general, such that ground states with odd N_t also present components with an even fermionic parity of the total occupation $\sum_{a,s,\tau} d_{a\tau}^\dagger d_{a\tau}$. These components do not have a counterpart in the description given in Eqs. (10) and (11) and may cause significant deviations from the Kondo problem analyzed in Sec. V. In Fig. 8, we depict the probability of observing a total odd fermionic parity of the external dots in the exact ground states with $N_t = 3$. In the perturbative regime [Fig. 8(a)], this probability is always above 0.9, such that we do not expect significant nonperturbative errors in the analysis of Sec. V. When increasing t [Figs. 8(b)–8(d)], this probability may decrease for certain choices of the parameters. However, even at $t/\Delta = 0.5$, we typically observe that this probability is beyond 0.7, such that the exact ground states do not deviate excessively from the analyzed $|3+\rangle$ and $|3-\rangle$ and we do not expect major deviations from the Kondo models analyzed in Sec. V.

VII. PHYSICAL IMPLEMENTATION OF THE POOR MAN'S MAJORANA TETRON AND APPLICATIONS

A. Experimental architectures and parameters

In the following, we discuss the experimental accessibility to the low-energy physics of the poor man's tetron. Systems of quantum dots hybridized by superconducting

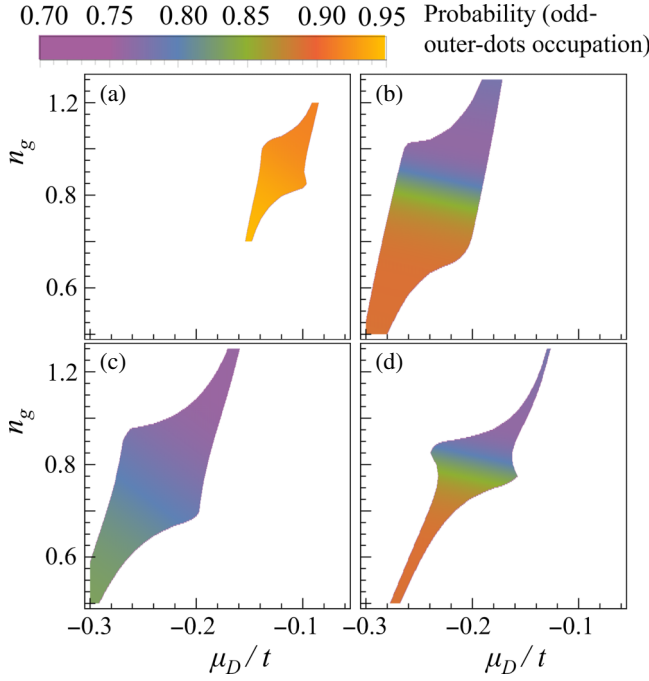


FIG. 8. The probability of odd occupation of the four quantum dots for the data displayed in Fig. 6(b) and Figs. 7(b), 7(e), and 7(h): (a) $\Delta = 4t$, $E_C = 0.15t$, $\mu_{SC} = -5t$; (b) $\Delta = 2t$, $E_C = 0.05t$, $\mu_{SC} = -2.7t$; (c) $\Delta = 2t$, $E_C = 0.15t$, $\mu_{SC} = -2.5t$; (d) $\Delta = 2t$, $E_C = 0.5t$, $\mu_{SC} = -3t$. The other parameters are $\alpha = \pi/6$ and $E_Z = 1.5\Delta$.

islands can be experimentally fabricated by appropriately patterning and locally gating semiconductor-superconductor heterostructures. Two main architectures have recently been developed for this purpose: semiconductor nanowires with strong spin-orbit coupling, usually InSb or InAs [6,7], and 2D electron gases, such as InSbAs planar geometries [9,36].

We consider first architectures based on proximitized double nanowires. The Al film coat of these hybrid systems typically induces in the nanowires a superconducting gap $\Delta \simeq (100\text{--}210) \mu\text{eV}$ [7,34], depending on the semiconductor material, geometry, and voltage gates [59]. In experiments, the charging energy E_C strongly depends on the geometry of the superconducting island. In order to have well-defined CAR processes, we need to set $\tilde{\Delta} > E_C$ and thus consider sufficiently large mesoscopic islands. Geometries studied in Ref. [34] suggest the range $E_C = (50\text{--}100) \mu\text{eV}$ as feasible.

The tunneling amplitudes t can be tuned via external electrostatic gates and single-nanowire experiments display both weak [6] and strong [7] hybridization regimes, with $t_{\text{COT}} \simeq (24\text{--}60) \mu\text{eV}$. Theoretical modeling and numerical fits in Ref. [7] are compatible with values of the spin-orbit parameter such that $\tan \alpha \sim 1/2$.

To ensure that the low-energy physics of the poor man's tetron remains observable, the gaps in Eqs. (16)–(17)

between the $|3\pm\rangle$ states and first excited states should exceed the typical temperatures of the system $T = (20\text{--}30) \text{ mK} = (1.7\text{--}2.6) \mu\text{eV}$ [6,7,34]. Based on these experimental parameters, we consider in the following a regime that satisfies the assumptions of the perturbative analysis of the previous sections.

The analytical results in Sec. IV yield a maximal gap $\delta E_{\pm}^{(N_f=3\pm 1)} \simeq 0.024\Delta \simeq 4.8 \mu\text{eV}$ that can be reached for $\Delta = 200 \mu\text{eV}$ and $E_C = \Delta/4$, in the proximity of the charge-degeneracy point $n_g = 1.0$. In this regime, the cotunneling amplitudes given in Eq. (8) span a range of values $|t_{\text{COT}}| \simeq (17\text{--}25) \mu\text{eV}$, which fall in with the observations in single-nanowire experiments. In experimental setups, this requires a large spin-orbit angle $\alpha = \pi/5$ that may not be easily accessible. For lower values of the spin-orbit parameter, achieving a maximal gap $\delta E_{\pm}^{(N_f=3\pm 1)} \simeq 0.013\Delta$ necessitates a smaller charging energy $E_C \simeq 0.05\Delta$. These results are based on the assumption that the system operates within the perturbative regime. However, these estimates are also compatible with the numerical data reported in Fig. 7 that display gaps reaching the values $\delta E_{\pm}^{(N_f=3\pm 1)} \simeq 0.02\Delta$ in the nonperturbative tunneling regime.

These findings suggest that the poor man's tetron can exhibit an experimentally accessible energy gap of around 50–60 mK, reinforcing the feasibility of observing its relevant low-energy features in state-of-the-art nanowire-based platforms.

Regarding the topological Kondo effect, we distinguish two regimes for its physical implementation.

The first scenario has previously been discussed in Sec. VA and corresponds to the *Coulomb valley* topological Kondo effect [37,49]: two degenerate ground states $|3\pm\rangle$ are separated in energy from the excited states. The Kondo Hamiltonian given in Eq. (32) gives rise to the usual perturbative Kondo renormalization-group flow, which in turn sets the Kondo temperature $T_K \sim \delta E \exp(-D\delta E/J^2)$, where D is the bandwidth of the electronic leads. The small energy gaps $\delta E_{\pm}^{N_f} \lesssim 0.025\Delta \simeq 50 \text{ mK}$ (see Fig. 7) limit T_K to small values, making the observation of the topological Kondo effect challenging. In the strong-coupling limit $J^2/D\delta E \sim 1$, the Kondo temperature indeed increases to the value of the gap $\delta E \lesssim 0.025\Delta$, which represents the high-energy cutoff of the quantum impurity problem [47, 48,51]. Increasing the induced gap Δ is therefore beneficial for observing the topological Kondo effect. For this purpose, one can consider materials alternative to Al: the epitaxial growth of Pb films ($\Delta \simeq 1.2 \text{ meV}$) on semiconducting nanowires has recently been achieved in experimental platforms [60] and a realization of the poor man's Majorana tetron with superconducting Pb may yield gaps 6 times larger than that for aluminum.

The second scenario involves the tuning of the impurity in a regime with four almost degenerate ground states

corresponding to the *charge-degeneracy point* in a standard topological Kondo effect [47,61,62]. Importantly, the total charge fluctuations of the impurity give rise to a crossover Kondo temperature T_K that is parametrically higher than in the Coulomb-valley regime [61]. Nonequilibrium numerics with tensor-network techniques have recently confirmed this result [48]. The perturbative low-energy spectrum of the poor man's tetron in Fig. 4 reveals that the two odd ground states $|3\pm\rangle$ become nearly degenerate with the even states at $N_t = 4$ for suitable values of the induced charge close to $n_g^{(2)}$ (for different parameters, similar results are achieved for $N_t = 2$). This allows for a choice of the lead couplings such that $\delta E_{\pm}^{(4)} \ll J_{\alpha} < \delta E_{\pm}^{(2)} \sim 0.048\Delta$, which opens up the path to accessing the topological Kondo effect at the charge-degeneracy point [47,61], thus enhancing the related Kondo temperature and the accessibility of its observations.

Similar parameter ranges are expected for the realization of quantum dots in 2D electron gases of InSb or InAs proximitized with Al. Furthermore, the recent realization of superconducting quantum dots in Ge systems [63] promises to extend the pool of materials that can be used for the study of poor man's Majorana modes and could be adopted for the fabrication of the poor man's Majorana tetron.

B. Readout of the poor man's Majorana tetron and applications

The construction of the poor man's tetron with double nanowires is well suited for quantum transport experiments and the exploration of the Kondo effects.

The low-energy properties of the poor man's tetron, however, can also be investigated through different approaches and, in the following, we focus on setups in which the poor man's tetron is embedded in a planar geometry. Indeed, its realization in 2D heterostructures would open up the path toward combining multiple poor man's tetrans for both the study of adiabatic protocols to explore the properties of poor man's Majorana modes (see, e.g., Refs. [64–66]) and the engineering of more complex phases of matter, in analogy with their topological counterparts [27–29].

The characterization of the states of the poor man's tetron relies on the conservation of the fermionic parities P_{τ} in Eq. (3). When operating in the regime with, e.g., the twofold-degenerate ground states $|3+\rangle$ and $|3-\rangle$, the state of the system can be determined through a measurement of either fermionic parity P_{τ} .

Therefore, in order to verify the degeneracy of the system ground states and distinguish them, it is possible to apply readout procedures based on parity to charge conversion [67,68], which offer an alternative to the Kondo features discussed in Sec. V that does not rely on transport through the poor man's tetron.

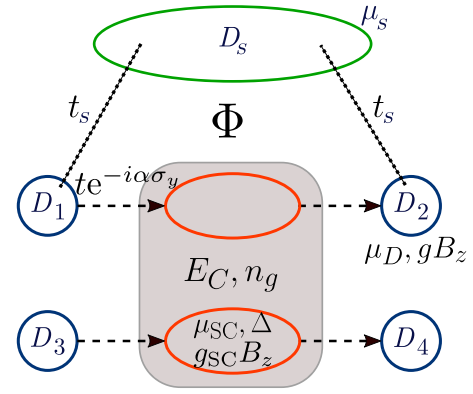


FIG. 9. A schematic setup to integrate the poor man's Majorana tetron with a sensor quantum dot D_s in order to measure the fermionic parity P_u . The coupling of the dots D_1 and D_2 to D_s is defined in Eq. (55). The sensor quantum dot D_s can, in turn, be coupled to a reflectometry setup or other sensing devices.

In particular, the poor man's tetron can be integrated with several of the setups proposed for the readout of tetrans [67–69], Majorana modes in nanowires [70], and other devices with poor man's Majorana modes [64,65]. These readout schemes can be applied to monitor either the distribution of charge on the external quantum dots of the poor man's tetron or its fermionic parities P_{τ} .

As an example, we consider the setup schematically depicted in Fig. 9, where the quantum dots D_1 and D_2 of the upper nanowire are coupled through tunneling to an additional sensor dot D_s , which can then be coupled to radio-frequency resonators to apply dispersive gate sensing techniques [70–72] and measure the related quantum capacitance of the system.

The Hamiltonian of the setup in Fig. 9 is described by $H_{\text{ro}} = H_{\text{sys}} + H_s$, where we have introduced the Hamiltonian of the sensor dot:

$$H_s = \mu_s d_s^\dagger d_s - t_s \left(d_{Lu}^\dagger d_s + d_s^\dagger d_{Lu} \right) - t_s \left(e^{-i\Phi} d_{Ru}^\dagger d_s + e^{i\Phi} d_s^\dagger d_{Ru} \right). \quad (55)$$

Analogously to other parity to charge-conversion schemes [67], the role of the sensor dot is to introduce an interferometric loop involving the two dots of the upper nanowire. The loop can be characterized by a magnetic flux such that Φ is the corresponding Aharonov-Bohm phase. However, differently from the case of noninteracting zero-energy Majorana modes, the insertion of a flux is not necessary for readout procedures in the poor man's tetron and one can operate at generic values of the flux. The Hamiltonian given in Eq. (55) corresponds to a regime in which the quantum dots are spin polarized, analogously to Eq. (5). The amplitudes t_s characterize the tunneling of electrons between D_s and the quantum dots of the upper nanowire.

The parameter μ_s is the effective potential of the sensor dot and we assume that it can be varied in time.

Instead of preserving the parity P_u , H_{ro} preserves the parity $P_{ro} = P_u(-1)^{d_s^\dagger d_s}$, which also includes the fermionic parity $(-1)^{d_s^\dagger d_s}$ of the sensor dot. This implies that by initializing the sensor dot in its empty state, the time evolution defined by $H_{ro} = H_{sys} + H_s$ mixes the poor man's tetron states in the sector (N_t, P_d) with the states in $(N_t - 1, P_d)$ as an effect of the tunneling of an electron into the sensor dot. However, if the system is initialized in the state $|3P_d\rangle$ with $\langle d_s^\dagger d_s \rangle = 0$, the parity P_{ro} corresponds to $P_u = -P_d$.

When the sensor dot is capacitively coupled to a resonator, the system relaxes to a steady state characterized by $N_t = 3$ and $P_{ro} = -P_d$. This steady state is close to the ground state for low coupling with the measurement apparatus and low temperature and is characterized by an average energy E_{ro} that depends on P_{ro} .

The coupling with the resonator allows for the dispersive measurement of the inverse quantum capacitance of this steady state, corresponding to $C_q^{-1} = d^2 E_{ro} / d\mu_s^2$, which crucially depends on $P_{ro} = -P_d$. In particular, by taking $\mu_s = 0$ and $\Phi = 0$, C_q^{-1} vanishes for the ground state of H_{ro} if $P_{ro} = -1$ (corresponding to three electrons in total and $P_d = +1$) and is different from zero if $P_{ro} = +1$ (three electrons in total and $P_d = -1$).

Therefore, by initializing the sensor dot in the empty state (with large μ_s) and then abruptly lowering μ_s to zero, reflectometry measurements applied to the sensor dot allow us to distinguish the initial states $|3+\rangle$ and $|3-\rangle$.

Furthermore, the state of the system can be continuously monitored and the observation of telegraph noise would provide a direct measurement of quasiparticle poisoning in the system.

Recent experiments based on these techniques in hybrid superconducting systems have revealed a timescale for quasiparticle poisoning of the order of 1 ms [70] and we expect a similar poisoning timescale for state-of-the-art implementations of the poor man's tetron.

The limiting factor to the use of the poor man's tetron as a qubit, however, relies on its expected low dephasing time: analogously to other poor man's Majorana devices, the poor man's tetron in the degenerate ground-state regime behaves similarly to a charge qubit: noise in the voltage gates and electrostatic environment causes the dephasing of the superpositions of its ground states over timescales predicted to be of the order of 10 ns [7,8].

This distinguishes the poor man's tetron from the standard tetron, which, on the other hand, displays topological protection against voltage noise in its ideal forms.

We emphasize, however, that this susceptibility to charge noise does not affect the use of the poor man's tetron for transport experiments out of equilibrium, as long as the tunneling rates between the leads and the device are sufficiently larger than the dephasing rate. This is expected

to be the case for the analysis of the transport properties outlined in the previous section.

Furthermore, the construction of poor man's Majorana modes [11] is beneficial in avoiding the onset of additional unwanted low-energy states: the transport spectroscopy results presented in, e.g., Refs. [6,7] show that poor man's Majorana systems with two quantum dots can be tuned in such a way that the only subgap states involved in transport are the ones emerging from the effective Hamiltonian given in Eq. (5) (for $E_C = 0$). This suggests that a poor man's Majorana implementation of the topological Kondo effect can controllably avoid the quasiparticle poisoning due to "dangerous" subgap states analyzed in Ref. [73].

Finally, the poor man's tetron shares with the topological tetron the property of being easily connected to other quantum devices. Besides its connection to external leads analyzed in Sec. V, its quantum dots can be easily coupled with either tunneling or capacitive interactions to other quantum dots. This offers the possibility of using it as a building block for the engineering of artificial quantum matter and, e.g., the realizations of many-body systems displaying topological order, in a way analogous to other Majorana architectures [27–29].

VIII. CONCLUSIONS

The experimental realization of minimal Kitaev chains in single-nanowire devices [5–8] provides a platform to study key properties of Majorana modes by trading off their topological protection for an improved controllability of these systems.

The poor man's Majorana tetron that we have presented constitutes a fundamental element to extend the study of the interplay between elastic cotunneling and crossed Andreev reflection processes to interacting hybrid systems.

We have proposed a blueprint for this device, which is composed by four quantum dots coupled via a floating superconducting island, and we have characterized its low-energy features. The tuning of its physical parameters enables the observation of the topological Kondo effect emerging in multiterminal conductance experiments when the poor man's tetron is coupled to external leads.

The physics of the related four poor man's Majorana modes is heavily altered by the introduction of the charging energy in the poor man's tetron: this device is characterized by local electrostatic interactions that, in general, separately affect the dots and the superconducting island mediating the interactions among them. These interactions are therefore qualitatively different from the topologically protected tetrans [26,28–30], in which the fermionic parity encoded in the Majorana modes is directly linked with the charging energy of the Majorana–Cooper-pair box. A similar distinction also holds for the comparison with hybrid quantum dot–superconducting island platforms for the observation of the topological symplectic Kondo effect [74].

Importantly, this difference in the form of the electrostatic interactions affects both the structure of the many-body low-energy states of the poor man's tetron and their degeneracies, which in general are completely split by the charging energy even when tuning the poor man's Majorana modes to their sweet spot. From this point of view, the poor man's tetron is also qualitatively different from its single-nanowire counterpart which, instead, allows for a tuning of its parameters that enables the onset of zero-energy poor man's Majorana modes [33].

Although poor man's Majorana modes are unstable against the charging energy in the poor man's tetron, we have shown that by carefully tuning the dot energy levels, the induced charge on the superconducting island, and the Andreev energy levels of the hybridized nanowires, extended regions of parameter space can be found in which the ground state exhibits a twofold degeneracy and behaves as an effective nonlocal spin-1/2 degree of freedom. This degeneracy is obtained if the system parameters are tuned such that a symmetry between the two nanowires emerges at low energies; in particular, the respective crossed Andreev reflection and cotunneling amplitudes must match and their effective spin-orbit coupling must be sufficiently strong. When a degenerate configuration of the poor man's tetron is reached, the transport properties of this device become nontrivial and are characterized by the onset of an effective spin-1/2 Kondo impurity problem stemming from the different nanowire fermionic parities that distinguish the degenerate ground states.

We have studied the resulting Kondo effect based on perturbation theory and shown that under a suitable tuning of the system parameters, it approaches the topological Kondo regime that characterizes topologically protected tetrans. We have examined the main perturbations that can drive the poor man's tetron away from the topological Kondo point in realistic devices and concluded that there exists a regime of parameters such that the fractional conductance of the topological Kondo point can be observed for intermediate temperatures and voltage biases. Since these perturbations are relevant in the renormalization-group sense, however, the poor man's tetron in the low-temperature and -bias regime flows toward trivial fixed points corresponding to a suitably defined spin-1/2 Anderson impurity model. In this respect, we have confirmed that the topological Kondo effect [37–39] in the limit of zero temperature and voltage bias is a genuine effect that reflects the topological protection of its constituent Majorana modes and it can be obtained by a poor man's Majorana device only through fine tuning.

We have also discussed how our perturbative findings evolve in a nonperturbative regime through exact diagonalization and shown that a strong coupling between the dots and the superconducting island is beneficial to extend the parameter region in which the poor man's tetron behaves as a two-level system.

We emphasize that in our analysis we have modeled the coupling between the superconductor and the nanowires by considering simple intermediate dots with an effective mean-field proximity-induced superconducting pairing. A more accurate modeling of the induced superconductivity can be performed by the use of surrogate models [75–77] to include the role of the Bogoliubov quasiparticles in the system more realistically.

The poor man's tetron is a device with a rich structure of eigenstates, the hierarchy of which can be modified by tuning its external parameters. Its realization is within reach both in double-nanowire devices [34] and in 2D electron gases with gate-defined quantum dots. The latter scenario can, e.g., be achieved by doubling the architecture realized in Ref. [9] for the case of a two-site Kitaev chain and it offers the possibility of integrating the poor man's tetron with several readout setups [64,65,67–70].

The electric tunability of this device will allow for extending the study of its multiterminal transport features beyond the analysis presented in this work, which has primarily been aimed at identifying regimes in which the three-lead topological Kondo effect emerges as a recognizable phenomenon.

In particular, our analysis has focused on setups with twofold-degenerate ground states with odd fermionic parity. We remark, however, that the regime we have studied is not the only region in the physical parameters that displays degenerate ground states. Other configurations allow for the emergence of different effective spin degrees of freedom expected to favor the onset of a variety of Kondo effects and non-Fermi liquids. For instance, the poor man's tetron is a suitable platform to explore the physics of the charge Kondo effect emerging from transport phenomena mediated by Andreev processes and the related multichannel Kondo effects [78].

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: PERTURBATIVE DEFINITION OF ELASTIC COTUNNELING AND CROSSED ANDREEV REFLECTION

In this appendix, we present the details of the second-order perturbative calculation that defines elastic cotunneling and CAR processes in the poor man's tetron. The following analysis extends the analogous description of devices with a single nanowire in Ref. [33]. It applies when the tunneling amplitudes t from the external dots to the central SC segments of the nanowires can be considered small with respect to the other energy scales in the setup. The unperturbed Hamiltonian accounts for the charging energy, the chemical potential of the dots, and the energy of the Andreev subgap states in the SC segments. In particular, the single-particle Hamiltonian describing these subgap states corresponds to

$$H_{SC} = \sum_{s,\tau} \tilde{\Delta}_s f_{s\tau}^\dagger f_{s\tau}, \quad (A1)$$

with energies

$$\tilde{\Delta}_{\uparrow/\downarrow} = \sqrt{\mu_{SC}^2 + \Delta^2} \pm g_{SC} B_z. \quad (A2)$$

The Bogoliubov modes are defined by

$$f_{\uparrow\tau} = uc_{\uparrow\tau} e^{-i\varphi/2} + vc_{\downarrow\tau}^\dagger e^{i\varphi/2}, \quad (A3)$$

$$f_{\downarrow\tau} = uc_{\downarrow\tau} e^{-i\varphi/2} - vc_{\uparrow\tau}^\dagger e^{i\varphi/2}, \quad (A4)$$

with

$$u = \sqrt{\frac{1}{2} + \frac{\mu_{SC}}{2\tilde{\Delta}}}, \quad v = \sqrt{\frac{1}{2} - \frac{\mu_{SC}}{2\tilde{\Delta}}}, \quad (A5)$$

where $\tilde{\Delta} = \sqrt{\mu_{SC}^2 + \Delta^2}$. By inverting Eqs. (A3) and (A4), we obtain

$$c_{s\tau}^\dagger = (uf_{s\tau}^\dagger - svf_{s\tau}) e^{-i\varphi/2}. \quad (A6)$$

In the last equation, we consider $s = \pm 1$ for spin up and down, respectively; $\bar{s}$ is a spin aligned opposite to s . Expressed in terms of the Bogoliubov modes f_s , the tunneling Hamiltonian reads

$$H_t = -t \sum_{\tau,a,s,s'} [(uf_{s\tau}^\dagger - svf_{s\tau}) (e^{-ia\alpha\sigma_y})_{ss'} d_{\tau s' a} e^{-i\varphi/2} + \text{H.c.}] \quad (A7)$$

Here, we have considered the same tunneling amplitude t for all the dots. The operator $e^{-i\varphi/2}$ suitably increases the charge of the SC box. This operator is conjugate to N and obeys the commutation relation $[N, e^{-i\varphi/2}] = e^{-i\varphi/2}$.

In the following, we also assume that the magnetic field B_z is pointing up and that it is strong enough to neglect the external dot states with the polarization $s = \uparrow$ ($gB_z \gg t$). Additionally, we assume that all the chemical potentials of the external dots are the same, such that we consider only a single energy level μ for all the dot electrons with $s = \downarrow$ spin. Finally, we consider a regime of strong induced pairing, such that $E_C < \tilde{\Delta} - g_{SC} B_z$. Under these assumptions, the elastic cotunneling at second order results in

$$H_{COT} = \sum_{\tau,a',a'',s} -d_{a''\downarrow\tau}^\dagger (e^{ia'\alpha\sigma_y})_{\downarrow s} \frac{t^2 u^2}{\tilde{\Delta}_s + H_c(N_I + 1) - H_c(N_I) - \mu} (e^{-ia'\alpha\sigma_y})_{s\downarrow} d_{a'\downarrow\tau} - d_{a''\downarrow\tau} (e^{-ia'\alpha\sigma_y})_{s\downarrow} \frac{t^2 v^2}{\tilde{\Delta}_{\bar{s}} + H_c(N_I - 1) - H_c(N_I) + \mu} (e^{ia'\alpha\sigma_y})_{\downarrow s} d_{a'\downarrow\tau}^\dagger, \quad (A8)$$

where we have labeled by $N = N_I$ the total charge of the superconducting island in the initial state. In the limit $E_C = 0$, we recover the results in Ref. [11].

In Sec. IV, we have considered systems with $n_g \in [0, 2]$ and, in this case, the lowest-energy states of the unperturbed Hamiltonian H_0 correspond to the ones with charge $N = 0$ and $N = 2$ in the superconducting island. In particular, to have significant CAR processes, it is necessary

to enforce the condition in Eq. (7) that guarantees the degeneracy between these states. Under this condition, the energy differences appearing in the denominators of Eq. (A8) become

$$H_c(N_I + 1) - H_c(N_I) - \mu = E_C (2N_I - 1), \quad (A9)$$

$$H_c(N_I - 1) - H_c(N_I) - \mu = E_C (3 - 2N_I). \quad (A10)$$

The calculation of the relevant energy differences for different values of N_I can be carried out straightforwardly. The cotunneling Hamiltonian acquires the general form

$$H_{\text{COT}}(N_I) = - \sum_{\tau, a', a'', s} d_{a'' \downarrow \tau}^\dagger \left(e^{ia'' \alpha \sigma_y} \right)_{\downarrow s} \tilde{t}_s(N_I) \left(e^{-ia' \alpha \sigma_y} \right)_{s \downarrow} d_{a' \downarrow \tau}, \quad (\text{A11})$$

with the following values of the amplitudes in the most relevant sectors:

$$\tilde{t}_s(2) = t^2 \frac{\tilde{\Delta}_s u^2 - \tilde{\Delta}_s v^2 - E_C (u^2 + 3v^2)}{(\tilde{\Delta}_s + 3E_C)(\tilde{\Delta}_s - E_C)}, \quad (\text{A12})$$

$$\tilde{t}_s(0) = t^2 \frac{\tilde{\Delta}_s u^2 - \tilde{\Delta}_s v^2 + E_C (3u^2 + v^2)}{(\tilde{\Delta}_s + 3E_C)(\tilde{\Delta}_s - E_C)}. \quad (\text{A13})$$

When considering $g_{\text{SC}} = 0$ and imposing $a' \neq a''$, one obtains the cotunneling amplitudes in Eq. (8). The terms with $a' = a''$, on the other hand, correspond to the shift of the energy level μ that we included in Eq. (5).

The calculation of the CAR processes is performed in an analogous way. When considering transitions between states with N_I and $N_I + 2$ electrons in the superconducting island, we obtain an effective p -wave pairing between the dot degrees of freedom of the form

$$H_{\text{CAR}}(N_I) = t^2 e^{-i\varphi} \sum_{\tau, a', a'', s} d_{a'' \downarrow \tau} \left(e^{-ia'' \alpha \sigma_y} \right)_{\bar{s} \downarrow} \frac{svu}{\tilde{\Delta}_s + H_c(N_I + 1) - [H_c(N_I) + H_c(N_I + 2)]/2} \left(e^{-ia' \alpha \sigma_y} \right)_{s \downarrow} d_{a' \downarrow \tau} + \text{H.c.}, \quad (\text{A14})$$

where $s = \pm 1$ labels the spin of the virtual Bogoliubov state involved in the process. Only processes between configurations with $N_I = 0$ and $N_I = 2$ connect degenerate states when we impose the condition given in Eq. (7). However, the former expression for CAR processes provides an approximate value of the amplitudes when $\tilde{\Delta}_s \gg t \gtrsim E_C$.

Analogously to the cotunneling, we rewrite the CAR processes in the form

$$H_{\text{CAR}}(N_I) = e^{-i\varphi} \sum_{\tau, a', a'', s} d_{a'' \downarrow \tau} \left(-i\sigma_y e^{ia'' \alpha \sigma_y} \right)_{\downarrow s} \tilde{\Delta}_{\text{CAR},s}(N_I) \left(e^{-ia' \alpha \sigma_y} \right)_{s \downarrow} d_{a' \downarrow \tau} + \text{H.c.} \quad (\text{A15})$$

The denominators in Eq. (A14) depend only on the curvature of the parabolas $H_c(N_I)$; hence the CAR amplitudes do not depend on the number of electrons N_I . We have

$$\tilde{\Delta}_{\text{CAR},s} = \frac{t^2 \sqrt{\tilde{\Delta}^2 - \mu_{\text{SC}}^2}}{2\tilde{\Delta}(\tilde{\Delta}_s - E_C)}. \quad (\text{A16})$$

For $g_{\text{SC}} = 0$, and thus $\tilde{\Delta}_s = \tilde{\Delta}$, we obtain the result in Eqs. (5) and (9).

These calculations have been performed under the simplifying assumption that the two nanowires are equivalent at the microscopic level. In realistic devices, however, we expect that the proximity-induced pairing Δ and the effective Zeeman coupling g_{SC} acquire a dependence on τ . Additionally, the spin-orbit parameter α is also subject to differences among the four dots.

In the general scenario, therefore, the effective cotunneling and CAR amplitudes are different for the two nanowires. This results, in turn, in a further dependence on τ of the amplitudes in Eqs. (5), (8), and (9).

In order to implement an emergent nanowire symmetry $u \leftrightarrow d$ in the effective Hamiltonian given in Eq. (5) for any value of n_g , we need therefore to be able to separately control the ratios of the four amplitudes $t_{\text{COT},\tau}$ and $\Delta_{\text{CAR},\tau}$. The general ratio between amplitudes in the top and bottom nanowire can be tuned by varying the tunneling amplitudes $t_{a\tau}$ with suitable cutter gates. The remaining two ratios $t_{\text{COT},\tau}/\Delta_{\text{CAR},\tau}$, on the other hand, require two additional controls to be suitably set.

In our previous analysis, we have assumed that the parameter μ_{SC} is the same for both wires. However, by considering two separate side gates close to the superconducting island [labeled by V_{gu} and V_{gd} in Fig. 1(a)], this potential also acquires a dependence on τ such that the independent control of $\mu_{\text{SC},u}$ and $\mu_{\text{SC},d}$ guarantees a sufficient tunability of the system to achieve the desired nanowire symmetry. Alternative strategies to tune the device can also be envisioned by considering that the cutter gates controlling the tunneling between the external dots and the central segments of the nanowires also affect the dot spin-orbit parameters.

APPENDIX B: EFFECTIVE HAMILTONIANS IN THE FOUR PARITY SECTORS

In this appendix, we report the 4×4 blocks of the effective Hamiltonian given in Eq. (5) under the conditions presented in Table I.

1. Odd sector ($N_t = 2m + 1, P_d = +$)

This is the sector with parities $P_u = -1, P_d = 1$.

Its basis is $\{|0100, N_t - 1\rangle, |1000, N_t - 1\rangle, |0111, N_t - 3\rangle, |1011, N_t - 3\rangle\}$. The effective Hamiltonian for this sector reads

$$H_{\text{eff}}^{\text{odd}+} = \begin{pmatrix} \mu + H_c(N_t - 1) & -t_{\text{COT}}(N_t - 1) & -\Delta_{\text{CAR}} & 0 \\ -t_{\text{COT}}(N_t - 1) & \mu + H_c(N_t - 1) & 0 & -\Delta_{\text{CAR}} \\ -\Delta_{\text{CAR}} & 0 & 3\mu + H_c(N_t - 3) & -t_{\text{COT}}(N_t - 3) \\ 0 & -\Delta_{\text{CAR}} & -t_{\text{COT}}(N_t - 3) & 3\mu + H_c(N_t - 3) \end{pmatrix}. \quad (\text{B1})$$

When $N_t = 3$, the diagonal elements become equal to E_C at the charge-degeneracy point $n_g = 1, \mu = 0$. When the potential μ is tuned to achieve the degeneracy of the diagonal terms [Eq. (7)], the ground state of this sector results in

$$|3+\rangle = \frac{1}{\sqrt{2}} [\cos \theta |0100, N_t - 1\rangle + \mathbf{s} \cos \theta |1000, N_t - 1\rangle + \sin \theta |0111, N_t - 3\rangle + \mathbf{s} \sin \theta |1011, N_t - 3\rangle], \quad (\text{B2})$$

where $\mathbf{s} \equiv \text{sign}[t_{\text{COT}}(0) + t_{\text{COT}}(2)]$. In this case, the parameter θ is defined as

$$\theta = \frac{1}{2} \text{Arg}[t_{\text{COT}}(N_t - 3) - t_{\text{COT}}(N_t - 1) + i2\Delta_{\text{CAR}}]. \quad (\text{B3})$$

Equation (10) corresponds to the ground state given in Eq. (B2) when $t_{\text{COT}}(2) < t_{\text{COT}}(0) < 0$; thus $\mathbf{s} = -1$ and

$\Delta_{\text{CAR}} > 0$ ($0 < \alpha < \pi/4$ and $\mu_{\text{SC}} < -2E_C\tilde{\Delta}/(\tilde{\Delta} + E_C)$). In this regime, considered in the main text, the energy of the ground states $|3+\rangle$ and $|3-\rangle$ results in

$$E^{(N_t=3)} = E_C \left[6 - 6n_g + n_g^2 \right] - \bar{t}_{\text{COT}} - \sqrt{\left(\frac{t_{\text{COT}}(0) - t_{\text{COT}}(2)}{2} \right)^2 + \Delta_{\text{CAR}}^2}, \quad (\text{B4})$$

where we have imposed the degeneracy condition given in Eq. (7).

For generic μ , the ground state maintains the form given in Eq. (B2) but the angle θ acquires a dependence on $\mu - 2E_C(1 - n_g)$ such that, in general, θ can be tuned through a suitable choice of μ . For general $N_t \neq 3$ and μ , the diagonal elements in $H_{\text{eff}}^{\text{odd}+}$ are indeed not degenerate and the ground-state energy acquires the form

$$E^{(N_t=2m+1)} = 2\mu + \frac{H_c(N_t - 1) + H_c(N_t - 3)}{2} + \mathbf{s} \frac{(t_{\text{COT}}(N_t - 1) + t_{\text{COT}}(N_t - 3))}{2} - \sqrt{\frac{[2\mu + 4E_C(2 + n_g - N_t) + \mathbf{s}(t_{\text{COT}}(N_t - 1) - t_{\text{COT}}(N_t - 3))]^2}{4} + \Delta_{\text{CAR}}^2}, \quad (\text{B5})$$

with the signs now set to minimize the energy. The ground state maintains the same form as given in Eq. (B2) with the angle θ suitably determined and the same dependence on the sign $\mathbf{s}$. More refined calculations can be performed by explicitly taking into account the dependence of μ from the cotunneling amplitudes in Eq. (5).

We conclude by observing that in the special case in which $\mu_{\text{SC}} = 0$, the cotunneling amplitudes in Eq. (8) become opposite to each other. Therefore the sign $\mathbf{s}$ is not well defined. In this situation, we obtain a twofold degeneracy of the ground state of this symmetry sector when the degeneracy condition given in Eq. (7) is fulfilled.

2. Odd sector ($N_t = 2m + 1, P_d = -$)

This sector corresponds to the parities $P_u = 1, P_d = -1$ and is analogous to the previous sector under the exchange of the two nanowires $u \leftrightarrow d$. The basis is $\{|0001, N_t - 1\rangle, |0010, N_t - 1\rangle, |1101, N_t - 3\rangle, |1110, N_t - 3\rangle\}$:

$$H_{\text{eff}}^{\text{odd}-} = \begin{pmatrix} \mu + H_c(N_t - 1) & -t_{\text{COT}}(N_t - 1) & -\Delta_{\text{CAR}} & 0 \\ -t_{\text{COT}}(N_t - 1) & \mu + H_c(N_t - 1) & 0 & -\Delta_{\text{CAR}} \\ -\Delta_{\text{CAR}} & 0 & 3\mu + H_c(N_t - 3) & -t_{\text{COT}}(N_t - 3) \\ 0 & -\Delta_{\text{CAR}} & -t_{\text{COT}}(N_t - 3) & 3\mu + H_c(N_t - 3) \end{pmatrix}. \quad (\text{B6})$$

Analogously to the previous sector, for $N_t = 3$ under the tuning condition given in Eq. (7), the ground state of $(3, -)$ results in

$$|3-\rangle = \frac{1}{\sqrt{2}} [\cos \theta |0001, N_t - 1\rangle + \mathbf{s} \cos \theta |0010, N_t - 1\rangle + \sin \theta |1101, N_t - 3\rangle + \mathbf{s} \sin \theta |1110, N_t - 3\rangle], \quad (\text{B7})$$

where θ is defined in Eq. (B3) and $\mathbf{s}$ is the sign of the cotunneling amplitude $t_{\text{COT}}(N_{\text{max}})$ with the largest modulus.

In the general case, the energy of the ground state matches Eq. (B5).

3. Even sector ($N_t = 2m, P_d = +$)

This sector corresponds to the parities $P_u = 1, P_d = 1$.

Its basis is $\{|0000, N_t\rangle, |0011, N_t - 2\rangle, |1100, N_t - 2\rangle, |1111, N_t - 4\rangle\}$:

$$H_{\text{eff}}^{\text{even}+} = \begin{pmatrix} H_c(N_t) & -\Delta_{\text{CAR}} & -\Delta_{\text{CAR}} & 0 \\ -\Delta_{\text{CAR}} & 2\mu + H_c(N_t - 2) & 0 & -\Delta_{\text{CAR}} \\ -\Delta_{\text{CAR}} & 0 & 2\mu + H_c(N_t - 2) & -\Delta_{\text{CAR}} \\ 0 & -\Delta_{\text{CAR}} & -\Delta_{\text{CAR}} & 4\mu + H_c(N_t - 4) \end{pmatrix}. \quad (\text{B8})$$

This is the only sector in which three values of the charge of the superconducting island are involved. As a consequence, only three of the four diagonal entries can be tuned to be degenerate. To study the role of the excited states in the definition of the Kondo problem, we represent the general form of the eigenstates of this Hamiltonian as

$$|N_t, +, n\rangle = A_{1,1}^{(n)}(N_t) |0000, N_t\rangle + A_{1,2}^{(n)}(N_t) |0011, N_t - 2\rangle + A_{1,3}^{(n)}(N_t) |1100, N_t - 2\rangle + A_{1,4}^{(n)}(N_t) |1111, N_t - 4\rangle, \quad (\text{B9})$$

where we have introduced the amplitudes $A_{c=1,j}^{(n)}$. Here, the index $c = 1$ labels the Hilbert sector ($P_{\uparrow} = 1, P_{\downarrow} = 1$), j indicates the states in the basis of this sector, and $n = 1 \dots 4$ refers to the four eigenstates of this sector ordered by increasing eigenenergies. Therefore, $|4, +, 1\rangle$

corresponds to the ground state $|4+\rangle$ that we have analyzed in the main text.

The sectors with total particle number $N_t = 2, 4$ present three equal diagonal terms when considering the tuning condition given in Eq. (7).

We observe, in general, that the state $(|0011, N_t - 2\rangle - |1100, N_t - 2\rangle) / \sqrt{2}$ is an eigenstate of $H_{\text{eff}}^{\text{even}+}$ with eigenenergy $2\mu + H_c(N_t - 2)$. For $N_t = 2, 4$, this corresponds to the second energy state $n = 2$. In particular, we have $A_{1,3}^{(n)} = A_{1,2}^{(n)}$ for $n = 1, 3, 4$ and $A_{1,3}^{(n)} = -A_{1,2}^{(n)} = 1/\sqrt{2}$ for $n = 2$.

For $N_t = 4$, at the charge-degeneracy point $n_g = 1$, $\mu = 0$, the diagonal entries of $H_{\text{eff}}^{\text{even}+}$ become $9E_C, E_C, E_C$, and E_C . In general, when $E_C \gg |\Delta_{\text{CAR}}|$ and $\mu < 2E_C$, the state $|0000, 4\rangle$ acquires a considerably higher energy than the others and can effectively be removed from the low-energy basis. The Hamiltonian can then be approximated as

$$H_{\text{eff}}^{\text{even}+}(N_t = 4) \xrightarrow{E_C \gg \Delta_{\text{CAR}}} \begin{pmatrix} 2\mu + H_c(2) & 0 & -\Delta_{\text{CAR}} \\ 0 & 2\mu + H_c(2) & -\Delta_{\text{CAR}} \\ -\Delta_{\text{CAR}} & -\Delta_{\text{CAR}} & 4\mu + H_c(0) \end{pmatrix}. \quad (\text{B10})$$

Equation (13) corresponds to the ground state of this Hamiltonian. For $N_t = 2$, the situation is analogous: in this case, the state $|1111, -2\rangle$ acquires a higher energy and becomes decoupled for large E_C . In the limit of large E_C , the energies of the ground states $|N_t = 2, 4, +\rangle$ can thus be

approximated as

$$E^{(N_t=2,4,+)} \xrightarrow{E_C \gg \Delta_{\text{CAR}}} E_C \left[(N_t - 2 - n_g)^2 + 4(1 - n_g) \right] - \sqrt{2}\Delta_{\text{CAR}}. \quad (\text{B11})$$

4. Even sector ($N_t = 2m, P_d = -$)

This sector corresponds to ($P_u = -1, P_d = -1$) and its basis displays only states having the same charge $N_t - 2$ in the superconducting island: $\{|0101, N_t - 2\rangle, |0110, N_t - 2\rangle, |1001, N_t - 2\rangle, |1010, N_t - 2\rangle\}$:

$$H_{\text{eff}}^{\text{even-}} = \begin{pmatrix} 2\mu + H_c(N_t - 2) & -t_{\text{COT}}(N_t - 2) & -t_{\text{COT}}(N_t - 2) & 0 \\ -t_{\text{COT}}(N_t - 2) & 2\mu + H_c(N_t - 2) & 0 & -t_{\text{COT}}(N_t - 2) \\ -t_{\text{COT}}(N_t - 2) & 0 & 2\mu + H_c(N_t - 2) & -t_{\text{COT}}(N_t - 2) \\ 0 & -t_{\text{COT}}(N_t - 2) & -t_{\text{COT}}(N_t - 2) & 2\mu + H_c(N_t - 2) \end{pmatrix}. \quad (\text{B12})$$

Analogously to the previous sector, we introduce the following notation for the eigenstates of this sector:

$$|N_t, -, n\rangle = A_{2,1}^{(n)}|0101, N_t - 2\rangle + A_{2,2}^{(n)}|0110, N_t - 2\rangle + A_{2,3}^{(n)}|1001, N_t - 2\rangle + A_{2,4}^{(n)}|1010, N_t - 2\rangle. \quad (\text{B13})$$

Given the structure of the Hamiltonian $H_{\text{eff}}^{\text{even-}}$, all the coefficients $A_{2,j}^{(n)}$ have modulus $1/2$ and signs that depend on the sign of $t_{\text{COT}}(N_t - 2)$.

The ground-state energy of this sector results in

$$E^{(N_t=2m,-)} = E_C (N_t - 2 - n_g)^2 + 2\mu - 2|t_{\text{COT}}(N_t - 2)|. \quad (\text{B14})$$

APPENDIX C: THE ROLE OF THE EVEN EXCITED STATES IN THE DEFINITION OF THE KONDO PROBLEM

In the main text, we have truncated the summation over the indices n in Eqs. (22) and (23) to the even ground states ($n = 1$) in Eqs. (13), (14), and (15). Here, we consider the full summation in the definition of these operators to show how the topological Kondo point is modified by the inclusion of the excited states. In particular, we will obtain that the condition given in Eq. (43) to approach the topological Kondo point is slightly modified and, in general, the excited states determine both the onset of additional marginal perturbations that do not affect its non-Fermi-liquid behavior and additional contributions to the effective Zeeman terms.

We consider the even states defined in Eqs. (B9) and (B13) for $N_t = 4$ and $N_t = 2$. With regard to the sector ($N_t = 4, +$), the matrix elements in Eq. (24) are generalized to the following expressions:

$$\langle 4, +, n | d_{\beta}^{\dagger} | 3+ \rangle = \frac{A_{1,3}^{(n)}(4) \cos \theta + A_{1,4}^{(n)}(4) \sin \theta}{\sqrt{2}} \delta_{\beta,1} - \mathbf{s} \frac{A_{1,3}^{(n)}(4) \cos \theta + A_{1,4}^{(n)}(4) \sin \theta}{\sqrt{2}} \delta_{\beta,2} \equiv D_1^{(n)} (\delta_{\beta,1} - \mathbf{s} \delta_{\beta,2}), \quad (\text{C1})$$

$$\langle 4, +, n | d_{\beta}^{\dagger} | 3- \rangle = \frac{A_{1,2}^{(n)}(4) \cos \theta + A_{1,4}^{(n)}(4) \sin \theta}{\sqrt{2}} \delta_{\beta,3} - \mathbf{s} \frac{A_{1,2}^{(n)}(4) \cos \theta + A_{1,4}^{(n)}(4) \sin \theta}{\sqrt{2}} \delta_{\beta,4} \equiv D_3^{(n)} (\delta_{\beta,3} - \mathbf{s} \delta_{\beta,4}). \quad (\text{C2})$$

Importantly, given the structure of the Hamiltonian given in Eq. (B8) and the related coefficients $A_{1,j}^{(n)}$, we obtain that $D_1^{(n)} = D_3^{(n)}$ for $n = 1, 2, 3$ and $D_1^{(2)} = -D_3^{(2)} = (\cos \theta)/2$. Regarding the even sector 2, the matrix elements result in

$$\langle 4, -, n | d_{\beta}^{\dagger} | 3+ \rangle = -\frac{\cos \theta}{\sqrt{2}} (A_{2,2}^{(n)}(4) + \mathbf{s} A_{2,4}^{(n)}(4)) \delta_{\beta,3} - \frac{\cos \theta}{\sqrt{2}} (A_{2,1}^{(n)}(4) + \mathbf{s} A_{2,3}^{(n)}(4)) \delta_{\beta,4} \equiv -C_3^{(n)} \delta_{\beta,3} - C_4^{(n)} \delta_{\beta,4}, \quad (\text{C3})$$

$$\langle 4, -, n | d_\beta^\dagger | 3 - \rangle = \frac{\cos \theta}{\sqrt{2}} \left(A_{2,3}^{(n)}(4) + \mathbf{s} A_{2,4}^{(n)}(4) \right) \delta_{\beta,1} + \frac{\cos \theta}{\sqrt{2}} \left(A_{2,1}^{(n)}(4) + \mathbf{s} A_{2,2}^{(n)}(4) \right) \delta_{\beta,2} \equiv C_1^{(n)} \delta_{\beta,1} + C_2^{(n)} \delta_{\beta,2}. \quad (\text{C4})$$

Based on these expressions, the terms entering the sum over n to define $\Upsilon = \sum_n \Upsilon^{(n)}$ in Eq. (22) result in

$$\Upsilon_{(1,1)}^{(n)} = \frac{\left(D_1^{(n)}\right)^2}{\delta E_+^{(4,n)}} (\delta_{\alpha,1} - \mathbf{s} \delta_{\alpha,2}) (\delta_{\beta,1} - \mathbf{s} \delta_{\beta,2}) + \frac{1}{\delta E_-^{(4,n)}} \left(C_3^{(n)} \delta_{\alpha,3} + C_4^{(n)} \delta_{\alpha,4}\right) \left(C_3^{(n)} \delta_{\beta,3} + C_4^{(n)} \delta_{\beta,4}\right), \quad (\text{C5})$$

$$\Upsilon_{(1,2)}^{(n)} = \frac{D_1^{(n)} D_3^{(n)}}{\delta E_+^{(4,n)}} (\delta_{\alpha,3} - \mathbf{s} \delta_{\alpha,4}) (\delta_{\beta,1} - \mathbf{s} \delta_{\beta,2}) - \frac{1}{\delta E_-^{(4,n)}} \left(C_1^{(n)} \delta_{\alpha,1} + C_2^{(n)} \delta_{\alpha,2}\right) \left(C_3^{(n)} \delta_{\beta,3} + C_4^{(n)} \delta_{\beta,4}\right), \quad (\text{C6})$$

$$\Upsilon_{(2,1)}^{(n)} = \frac{D_1^{(n)} D_3^{(n)}}{\delta E_+^{(4,n)}} (\delta_{\alpha,1} - \mathbf{s} \delta_{\alpha,2}) (\delta_{\beta,3} - \mathbf{s} \delta_{\beta,4}) - \frac{1}{\delta E_-^{(4,n)}} \left(C_3^{(n)} \delta_{\alpha,3} + C_4^{(n)} \delta_{\alpha,4}\right) \left(C_1^{(n)} \delta_{\beta,1} + C_2^{(n)} \delta_{\beta,2}\right), \quad (\text{C7})$$

$$\Upsilon_{(2,2)}^{(n)} = \frac{\left(D_3^{(n)}\right)^2}{\delta E_+^{(4,n)}} (\delta_{\alpha,3} - \mathbf{s} \delta_{\alpha,4}) (\delta_{\beta,3} - \mathbf{s} \delta_{\beta,4}) + \frac{1}{\delta E_-^{(4,n)}} \left(C_1^{(n)} \delta_{\alpha,1} + C_2^{(n)} \delta_{\alpha,2}\right) \left(C_1^{(n)} \delta_{\beta,1} + C_2^{(n)} \delta_{\beta,2}\right), \quad (\text{C8})$$

where the differences $\delta E_c^{(4,n)} = E_n^{(4,c)} - E^{(3)}$ match the denominators in the perturbative expression given in Eq. (22) and measure the energy gap between the n th eigenstate of the even sector (4, c) and the degenerate odd ground states.

To explicitly express the coefficients $C_\beta^{(n)}$, we introduce the sign $\mathbf{t} = \text{sign}[t_{\text{COT}}(2)]$. If $|t_{\text{COT}}(2)| > |t_{\text{COT}}(0)|$, $\mathbf{s} = \mathbf{t}$ (this corresponds to $\mu_{\text{SC}} < -2E_C \tilde{\Delta}/(\tilde{\Delta} + E_C)$ and $0 < \alpha < \pi/4$ for $g_{\text{SC}} = 0$). Otherwise, $\mathbf{s} = -\mathbf{t}$. In the following, we will address these two situations.

As a function of the signs $\mathbf{s}$ and $\mathbf{t}$, the $C_\beta^{(n)}$ can be read in Table II.

With regard to the transitions mediated by states with $N_t = 2$ electrons, we generalize the matrix elements in Eq. (26) in the following form:

$$\langle 2, +, n | d_\alpha | 3 + \rangle = \mathbf{s} \frac{A_{1,1}^{(n)}(2) \cos \theta + A_{1,2}^{(n)}(2) \sin \theta}{\sqrt{2}} \delta_{\alpha,1} + \frac{A_{1,1}^{(n)}(2) \cos \theta + A_{1,2}^{(n)}(2) \sin \theta}{\sqrt{2}} \delta_{\alpha,2} \equiv Q_1^{(n)} (\mathbf{s} \delta_{\alpha,1} + \delta_{\alpha,2}), \quad (\text{C9})$$

$$\langle 2, +, n | d_\alpha | 3 - \rangle = \mathbf{s} \frac{A_{1,1}^{(n)}(2) \cos \theta + A_{1,3}^{(n)}(2) \sin \theta}{\sqrt{2}} \delta_{\alpha,3} + \frac{A_{1,1}^{(n)}(2) \cos \theta + A_{1,3}^{(n)}(2) \sin \theta}{\sqrt{2}} \delta_{\alpha,4} \equiv Q_3^{(n)} (\mathbf{s} \delta_{\alpha,3} + \delta_{\alpha,4}), \quad (\text{C10})$$

$$\langle 2, -, n | d_\alpha | 3 + \rangle = -\frac{\sin \theta}{\sqrt{2}} \left(A_{2,1}^{(n)}(2) + \mathbf{s} A_{2,3}^{(n)}(2) \right) \delta_{\alpha,3} + \frac{\sin \theta}{\sqrt{2}} \left(A_{2,2}^{(n)}(2) + \mathbf{s} A_{2,4}^{(n)}(2) \right) \delta_{\alpha,4} \equiv -P_3^{(n)} \delta_{\alpha,3} + P_4^{(n)} \delta_{\alpha,4}, \quad (\text{C11})$$

$$\langle 2, -, n | d_\alpha | 3 - \rangle = \frac{\sin \theta}{\sqrt{2}} \left(A_{2,1}^{(n)}(2) + \mathbf{s} A_{2,2}^{(n)}(2) \right) \delta_{\alpha,1} - \frac{\sin \theta}{\sqrt{2}} \left(A_{2,3}^{(n)}(2) + \mathbf{s} A_{2,4}^{(n)}(2) \right) \delta_{\alpha,2} \equiv P_1^{(n)} \delta_{\alpha,1} - P_2^{(n)} \delta_{\alpha,2}. \quad (\text{C12})$$

The $Q_\alpha^{(n)}$ coefficients behave analogously to the $D_\beta^{(n)}$ coefficients, such that $Q_1^{(n)} = Q_3^{(n)}$ for $n = 1, 3, 4$ and

$Q_1^{(2)} = -Q_3^{(2)}$. The P coefficients are instead summarized in Table III and depend on the sign of $t_{\text{COT}}(0)$.

TABLE II. A table of the coefficients $2\sqrt{2}C_\beta^{(n)}/\cos \theta$ in Eqs. (C3) and (C4) as a function of $\mathbf{t} = \text{sign}[t_{\text{COT}}(2)]$ and $\mathbf{s} = \text{sign}[t_{\text{COT}}(2) + t_{\text{COT}}(0)]$.

$\frac{2\sqrt{2}C_\beta^{(n)}}{\cos \theta}$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$\beta = 1$	$-\mathbf{s} + \mathbf{t}$	$-\mathbf{s} + \mathbf{t}$	$-\mathbf{s} - \mathbf{t}$	$\mathbf{s} - \mathbf{t}$
$\beta = 2$	$1 + \mathbf{s} \mathbf{t}$	$1 - \mathbf{s} \mathbf{t}$	$1 + \mathbf{s} \mathbf{t}$	$1 - \mathbf{s} \mathbf{t}$
$\beta = 3$	$\mathbf{s} + \mathbf{t}$	$-\mathbf{s} - \mathbf{t}$	$-\mathbf{s} + \mathbf{t}$	$\mathbf{s} - \mathbf{t}$
$\beta = 4$	$1 + \mathbf{s} \mathbf{t}$	$1 + \mathbf{s} \mathbf{t}$	$1 - \mathbf{s} \mathbf{t}$	$1 - \mathbf{s} \mathbf{t}$

TABLE III. A table of the coefficients $2\sqrt{2}P_\alpha^{(n)}/\sin \theta$ in Eqs. (C11) and (C12) as a function of $\mathbf{t} = \text{sign}[t_{\text{COT}}(0)]$ and $\mathbf{s} = \text{sign}[t_{\text{COT}}(2) + t_{\text{COT}}(0)]$.

$\frac{2\sqrt{2}P_\alpha^{(n)}}{\sin \theta}$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$\alpha = 1$	$1 + \mathbf{s} \mathbf{t}$	$1 + \mathbf{s} \mathbf{t}$	$1 - \mathbf{s} \mathbf{t}$	$1 - \mathbf{s} \mathbf{t}$
$\alpha = 2$	$\mathbf{s} + \mathbf{t}$	$-\mathbf{s} - \mathbf{t}$	$-\mathbf{s} + \mathbf{t}$	$\mathbf{s} - \mathbf{t}$
$\alpha = 3$	$1 + \mathbf{s} \mathbf{t}$	$1 - \mathbf{s} \mathbf{t}$	$1 + \mathbf{s} \mathbf{t}$	$1 - \mathbf{s} \mathbf{t}$
$\alpha = 4$	$\mathbf{s} + \mathbf{t}$	$-\mathbf{s} + \mathbf{t}$	$-\mathbf{s} - \mathbf{t}$	$\mathbf{s} - \mathbf{t}$

1. Case $t_{\text{COT}}(2)t_{\text{COT}}(0) > 0$

Let us focus first on a regime of parameters such that both the cotunneling amplitudes $t_{\text{COT}}(2)$ and $t_{\text{COT}}(0)$ share the same sign and therefore $\mathbf{s} = \mathbf{t}$ for both the states at $N_t = 2$ and $N_t = 4$. This situation occurs when $|\mu_{\text{SC}}| > 2E_C\tilde{\Delta}/(\tilde{\Delta} + E_C)$ (for $g_{\text{SC}} = 0$).

The regime with $t_{\text{COT}}(2) < t_{\text{COT}}(0) < 0$ corresponds to the analysis in Sec. V. The situation is analogous for $0 < t_{\text{COT}}(2) < t_{\text{COT}}(0)$. In these cases, the coefficients

$C_\beta^{(1)}$ and $P_\alpha^{(1)}$ do not vanish, such that the even ground states $|N_t, -, 1\rangle$ play a major role in the definition of the matrices Υ and Ξ , and thus on the Kondo problem. The excited states $|N_t, -, 4\rangle$, on the other hand, do not contribute to these operators, since $C_\beta^{(4)} = P_\alpha^{(4)} = 0$ for $\mathbf{t} = \mathbf{s}$.

In this regime, by considering the properties of the coefficients $D_\alpha^{(n)}$, we can rewrite the operators Υ in the form

$$-\sum_n \Upsilon_{(1,1)}^{(n)} = \sum_n \left[\frac{D_1^{(n)} D_3^{(n)}}{\delta E_+^{(4,n)}} (\delta_{\alpha,1} - \mathbf{s}\delta_{\alpha,2}) (\delta_{\beta,1} - \mathbf{s}\delta_{\beta,2}) + \frac{\cos^2 \theta}{2\delta E_-^{(4,1)}} (\delta_{\alpha,3} + \mathbf{s}\delta_{\alpha,4}) (\delta_{\beta,3} + \mathbf{s}\delta_{\beta,4}) \right. \\ \left. + \frac{\cos^2 \theta}{2\delta E_+^{(4,2)}} (\delta_{\alpha,1} - \mathbf{s}\delta_{\alpha,2}) (\delta_{\beta,1} - \mathbf{s}\delta_{\beta,2}) + \frac{\cos^2 \theta}{2\delta E_-^{(4,2)}} (\delta_{\alpha,3} - \mathbf{s}\delta_{\alpha,4}) (\delta_{\beta,3} - \mathbf{s}\delta_{\beta,4}) \right], \quad (\text{C13})$$

$$-\sum_n \Upsilon_{(1,2)}^{(n)} = \sum_n \left[\frac{D_1^{(n)} D_3^{(n)}}{\delta E_+^{(4,n)}} (\delta_{\alpha,3} - \mathbf{s}\delta_{\alpha,4}) (\delta_{\beta,1} - \mathbf{s}\delta_{\beta,2}) - \frac{\cos^2 \theta}{2\delta E_-^{(4,1)}} (\delta_{\alpha,1} + \mathbf{s}\delta_{\alpha,2}) (\delta_{\beta,3} + \mathbf{s}\delta_{\beta,4}) \right], \quad (\text{C14})$$

$$-\sum_n \Upsilon_{(2,1)}^{(n)} = \sum_n \left[\frac{D_1^{(n)} D_3^{(n)}}{\delta E_+^{(4,n)}} (\delta_{\alpha,1} - \mathbf{s}\delta_{\alpha,2}) (\delta_{\beta,3} - \mathbf{s}\delta_{\beta,4}) - \frac{\cos^2 \theta}{2\delta E_-^{(4,1)}} (\delta_{\alpha,3} + \mathbf{s}\delta_{\alpha,4}) (\delta_{\beta,1} + \mathbf{s}\delta_{\beta,2}) \right], \quad (\text{C15})$$

$$-\sum_n \Upsilon_{(2,2)}^{(n)} = \sum_n \left[\frac{D_1^{(n)} D_3^{(n)}}{\delta E_+^{(4,n)}} (\delta_{\alpha,3} - \mathbf{s}\delta_{\alpha,4}) (\delta_{\beta,3} - \mathbf{s}\delta_{\beta,4}) + \frac{\cos^2 \theta}{2\delta E_-^{(4,1)}} (\delta_{\alpha,1} + \mathbf{s}\delta_{\alpha,2}) (\delta_{\beta,1} + \mathbf{s}\delta_{\beta,2}) \right. \\ \left. + \frac{\cos^2 \theta}{2\delta E_+^{(4,2)}} (\delta_{\alpha,3} - \mathbf{s}\delta_{\alpha,4}) (\delta_{\beta,3} - \mathbf{s}\delta_{\beta,4}) + \frac{\cos^2 \theta}{2\delta E_-^{(4,2)}} (\delta_{\alpha,1} - \mathbf{s}\delta_{\alpha,2}) (\delta_{\beta,1} - \mathbf{s}\delta_{\beta,2}) \right]. \quad (\text{C16})$$

The operators Ξ result instead:

$$-\sum_n \Xi_{(1,1)}^{(n)} = \sum_n \left[\frac{\mathcal{Q}_1^{(n)} \mathcal{Q}_3^{(n)}}{\delta E_+^{(2,n)}} (\delta_{\alpha,1} + \mathbf{s}\delta_{\alpha,2}) (\delta_{\beta,1} + \mathbf{s}\delta_{\beta,2}) + \frac{\sin^2 \theta}{2\delta E_-^{(4,1)}} (\delta_{\alpha,3} - \mathbf{s}\delta_{\alpha,4}) (\delta_{\beta,3} - \mathbf{s}\delta_{\beta,4}) \right. \\ \left. + \frac{\sin^2 \theta}{2\delta E_+^{(2,2)}} (\delta_{\alpha,1} + \mathbf{s}\delta_{\alpha,2}) (\delta_{\beta,1} + \mathbf{s}\delta_{\beta,2}) + \frac{\sin^2 \theta}{2\delta E_-^{(2,2)}} (\delta_{\alpha,3} + \mathbf{s}\delta_{\alpha,4}) (\delta_{\beta,3} + \mathbf{s}\delta_{\beta,4}) \right], \quad (\text{C17})$$

$$-\sum_n \Xi_{(1,2)}^{(n)} = \sum_n \left[\frac{\mathcal{Q}_1^{(n)} \mathcal{Q}_3^{(n)}}{\delta E_+^{(2,n)}} (\delta_{\alpha,3} + \mathbf{s}\delta_{\alpha,4}) (\delta_{\beta,1} + \mathbf{s}\delta_{\beta,2}) - \frac{\sin^2 \theta}{2\delta E_-^{(2,1)}} (\delta_{\alpha,1} - \mathbf{s}\delta_{\alpha,2}) (\delta_{\beta,3} - \mathbf{s}\delta_{\beta,4}) \right], \quad (\text{C18})$$

$$-\sum_n \Xi_{(2,1)}^{(n)} = \sum_n \left[\frac{\mathcal{Q}_1^{(n)} \mathcal{Q}_3^{(n)}}{\delta E_+^{(2,n)}} (\delta_{\alpha,1} + \mathbf{s}\delta_{\alpha,2}) (\delta_{\beta,3} + \mathbf{s}\delta_{\beta,4}) - \frac{\sin^2 \theta}{2\delta E_-^{(2,1)}} (\delta_{\alpha,3} - \mathbf{s}\delta_{\alpha,4}) (\delta_{\beta,1} - \mathbf{s}\delta_{\beta,2}) \right], \quad (\text{C19})$$

$$-\sum_n \Xi_{(2,2)}^{(n)} = \sum_n \left[\frac{\mathcal{Q}_1^{(n)} \mathcal{Q}_3^{(n)}}{\delta E_+^{(2,n)}} (\delta_{\alpha,3} + \mathbf{s}\delta_{\alpha,4}) (\delta_{\beta,3} + \mathbf{s}\delta_{\beta,4}) + \frac{\sin^2 \theta}{2\delta E_-^{(2,1)}} (\delta_{\alpha,1} - \mathbf{s}\delta_{\alpha,2}) (\delta_{\beta,1} - \mathbf{s}\delta_{\beta,2}) \right. \\ \left. + \frac{\sin^2 \theta}{2\delta E_+^{(2,2)}} (\delta_{\alpha,3} + \mathbf{s}\delta_{\alpha,4}) (\delta_{\beta,3} + \mathbf{s}\delta_{\beta,4}) + \frac{\sin^2 \theta}{2\delta E_-^{(2,2)}} (\delta_{\alpha,1} + \mathbf{s}\delta_{\alpha,2}) (\delta_{\beta,1} + \mathbf{s}\delta_{\beta,2}) \right]. \quad (\text{C20})$$

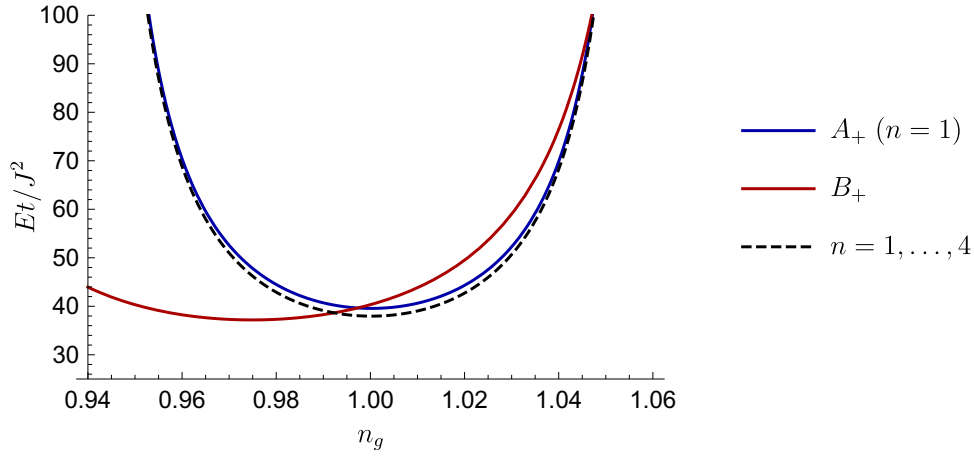


FIG. 10. A comparison between the contribution A_+ of the even ground states $|N_i, +\rangle$ (blue line) and the full expression, also including the excited states (dashed black line) in the definition of the Σ matrices, as a function of n_g . The topological Kondo conditions in Eqs. (43) and (C21) correspond to the crossings of these lines with the red line that depicts B_+ . The chosen parameters are $\Delta = 4t$, $\mu_{\text{SC}} = -5t$, $E_C = 0.15t$, and $\alpha = \pi/6$. These matrix elements are plotted in units of J^2/t .

When considering the matrices $\Sigma = \Upsilon - \Xi$, a possible fine tuning to retrieve a perturbed topological Kondo point is the following:

$$\sum_n \left[\frac{D_1^{(n)} D_3^{(n)}}{\delta E_+^{(4,n)}} + \frac{Q_1^{(n)} Q_3^{(n)}}{\delta E_+^{(2,n)}} \right] = \frac{\cos^2 \theta}{2\delta E_-^{(4,1)}} + \frac{\sin^2 \theta}{2\delta E_-^{(2,1)}}. \quad (\text{C21})$$

This tuning generalizes the condition given in Eq. (43) and discussed in Sec. V A by including additional contributions of the excited states to the left-hand side of the equation. The overall effect of the excited states amounts to a correction of a few percent in the definition of the Σ matrix elements, since the terms with $n = 2$ and $n = 3, 4$ contribute with opposite signs. A comparison between the term A_+ [Eq. (35)], which represents the $n = 1$ contribution, and the full expression given by the left hand side of Eq. (C21) is displayed in Fig. 10. The optimal topological Kondo point is met when the latter crosses B_+ [Eq. (36)], which constitutes the right-hand side of Eq. (C21). Such a constraint allows us to obtain a topological Kondo Hamiltonian in the form given in Eq. (30), with the addition of the terms in the second lines of Eqs. (C13), (C16), (C17), and (C20).

Let us consider these additional perturbations. For $\alpha = \beta$, these terms can be corrected through a suitable shift of the potentials in the leads and do not qualitatively affect the Kondo problem apart for a correction of the Zeeman term σ_3 entering Eq. (44) for the three-lead setups. For $\alpha = 1$ and $\beta = 2$ or vice versa, on the other hand, these terms provide a potential scattering contribution ($I_2^\dagger I_1 \sigma_0 + \text{H.c.}$), analogous to the ones discussed in Sec. V B, which does not renormalize and does not affect the non-Fermi-liquid

behavior of the topological Kondo point [38,49]. Symmetric terms are generated in the four-lead devices for $\alpha = 3$ and $\beta = 4$ or vice versa.

2. Case $t_{\text{COT}}(2)t_{\text{COT}}(0) < 0$

If $|\mu_{\text{SC}}| < 2E_C \tilde{\Delta}/(\tilde{\Delta} + E_C)$ (for $g_{\text{SC}} = 0$), the cotunneling amplitudes $t_{\text{COT}}(2)$ and $t_{\text{COT}}(0)$ acquire opposite signs. Depending on whether $|t_{\text{COT}}(0)| \geq |t_{\text{COT}}(2)|$ (for $\mu_{\text{SC}} \geq 0$, respectively), either the states $|4, -, 1\rangle$ or $|2, -, 1\rangle$ do not contribute to the definition of the Υ or Ξ matrices, respectively. This is because the related coefficients $C_\beta^{(1)}$ or $P_\alpha^{(1)}$ vanish. The role of the missing $n = 1$ state is replaced by the higher-energy state $n = 4$, the contribution of which, however, is reduced by the larger denominator $\delta E_-^{(N_i,4)}$. In this situation, it is easy to rederive the related Υ and Ξ matrices. The corresponding fine-tuning conditions to approach the topological Kondo point result in

$$\begin{aligned} \sum_n \left[\frac{D_1^{(n)} D_3^{(n)}}{\delta E_+^{(4,n)}} + \frac{Q_1^{(n)} Q_3^{(n)}}{\delta E_+^{(2,n)}} \right] \\ = \frac{\cos^2 \theta}{2\delta E_-^{(4,4)}} + \frac{\sin^2 \theta}{2\delta E_-^{(2,1)}}, \quad \text{for } |t_{\text{COT}}(0)| > |t_{\text{COT}}(2)|, \end{aligned} \quad (\text{C22})$$

$$\begin{aligned} \sum_n \left[\frac{D_1^{(n)} D_3^{(n)}}{\delta E_+^{(4,n)}} + \frac{Q_1^{(n)} Q_3^{(n)}}{\delta E_+^{(2,n)}} \right] \\ = \frac{\cos^2 \theta}{2\delta E_-^{(4,1)}} + \frac{\sin^2 \theta}{2\delta E_-^{(2,4)}}, \quad \text{for } |t_{\text{COT}}(0)| < |t_{\text{COT}}(2)|. \end{aligned} \quad (\text{C23})$$

Our analysis of the Kondo problems stemming from the poor man's tetron holds as long as the global ground

states are in the odd $N_t = 3$ sector and the system does not display the additional degeneracy at $\mu_{\text{SC}} = 0$. If, however, $t_{\text{COT}}(2) = -t_{\text{COT}}(0)$, the odd $N_t = 3$ sectors acquire an additional degeneracy when the tuning given in Eq. (7) is fulfilled. In this situation, the study of the related Kondo physics would require us to treat the system as an effective spin-3/2 impurity rather than a spin-1/2 impurity. Our perturbative results, however, suggest that the global ground state of the poor man's tetron does not belong to the degenerate $N_t = 3$ sectors for $\mu_{\text{SC}} = \mu = 0$ and $n_g = 1$, such that a fourfold degeneracy of the ground state is not realized, at least in the small- t regime.

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