

Quantum Metrology in the Finite-Sample Regime

Johannes Jakob Meyer^{1,*}, Sumeet Khatri,¹ Daniel Stilck França^{2,3}, Jens Eisert,^{1,4,5}
and Philippe Faist¹

¹*Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, 14195 Berlin, Germany*

²*Department of Mathematical Sciences, University of Copenhagen, 2100 København, Denmark*

³*Univ Lyon, ENS Lyon, UCBL, Inria, LIP, F-69342 Lyon Cedex 07, France*

⁴*Helmholtz-Zentrum Berlin für Materialien und Energie, 14109 Berlin, Germany*

⁵*Fraunhofer Heinrich Hertz Institute, 10587 Berlin, Germany*

 (Received 29 August 2023; revised 22 May 2025; accepted 8 July 2025; published 27 August 2025)

In quantum metrology, a major application of quantum technologies, the ultimate precision of estimating an unknown parameter is often stated in terms of the Cramér-Rao bound. Yet, the latter fails to completely characterize the distribution of estimates in the nonasymptotic regime. Optimizing metrology protocols with respect to the Cramér-Rao bound can therefore lead to surprisingly poor finite-sample performance. This can be avoided by quantifying the quality of a metrology protocol by the probability of obtaining an estimate with a given accuracy. Using this intrinsically single-shot quantity naturally accommodates the finite-sample regime. We show that the fundamental limits of this figure of merit can be quantified through a multihypothesis testing problem between quantum states. Building on this connection, we derive an analogue of the Cramér-Rao bound that contains explicit corrections relevant to the finite-sample regime. We apply the finite-sample metrology framework to the example task of phase estimation with an ensemble of spin-1/2 particles, giving unambiguous evidence that large quantum Fisher information does not guarantee that a metrology protocol has a good finite-sample performance. Overall, analyzing the probability of success allows the reliable study of quantum metrology in the finite-sample regime and opens up a plethora of new avenues for research at the interface of quantum information theory and quantum metrology.

DOI: [10.1103/qbn1-p6bq](https://doi.org/10.1103/qbn1-p6bq)

I. INTRODUCTION

Metrology, the scientific study of measurements, has naturally evolved to encompass the realm of quantum theory. Quantum metrology seeks to realize practical advantages by harnessing quantum effects. The growing quantum technologies sector, especially, holds high expectations for achieving unparalleled sensitivity with quantum sensors. Anticipated applications range from the calibration of atomic clocks over gravitational wave detection to potential medical uses [1–5].

It is crucial that the theory of quantum metrology accommodates the limitations of near-term quantum sensors. Usually, the number of quantum systems that can be controlled is restricted, for example, being in the dozens

for state-of-the-art sensors based on nitrogen-vacancy centers [6,7]. The slow operation speed of certain platforms furthermore constrains the number of experimental repetitions that can be performed in a reasonable time frame [8–10]. Another limiting factor is given by the inherent drift of system parameters, which limits the available time to conduct experiments before recalibration has to be performed [11]. Metrology in the near term therefore happens in the *finite-sample regime*, in which the number of experimental repetitions and simultaneously controllable quantum systems is limited. The finite-sample regime is also immediately relevant if the measurement time is very short, for example, in the context of spectator qubits that are used to perform a primitive form of error correction through real-time feedback [12,13].

The central goal of quantum metrology is to devise protocols that allow us to extract the parameter in question, which we denote as t , as precisely as possible with given resources. To determine an optimal protocol for a metrological task, we need a *figure of merit* that measures how well a given protocol estimates the parameter. The common approach in quantum metrology [1–3,5,14,15] is based on considering the mean-squared error of an estimate

*Contact author: johannes.meyer@fu-berlin.de

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

$\hat{t}$, i.e., $\mathbb{E}[(\hat{t} - t)^2]$, where the expectation value is relative to the randomness of quantum measurements as well as any inherent randomness of the protocol. The quantum Cramér-Rao bound [16] puts a fundamental lower bound on the achievable mean-squared error. For an unbiased estimator $\hat{t}$, i.e., satisfying $\mathbb{E}[\hat{t}] = t$, the quantum Cramér-Rao bound takes the form [5, 16–18]

$$\sqrt{\mathbb{E}[(\hat{t} - t)^2]} \geq \frac{1}{\sqrt{\mathcal{F}}}. \quad (1)$$

The quantity $\mathcal{F}$ is the *quantum Fisher information*, which measures how distinguishable the states of the family $\rho(t)$ are around the true value t . Much of the allure of the quantum Cramér-Rao bound comes from the fact that it can be saturated in many cases, for example, in the limit of many independent copies of the quantum system. One thus frequently resorts directly to the quantum Fisher information as a figure of merit. A significant part of the metrology literature revolves around finding protocols that exploit entanglement to achieve *Heisenberg scaling*, meaning that the quantum Fisher information scales as $\mathcal{F} = O(n^2)$ relative to the number n of copies of the quantum system, which contrasts with the classically achievable *standard quantum limit* $\mathcal{F} = O(n)$.

Optimizing protocols for large quantum Fisher information is a reasonable strategy in the asymptotic setting where a large number of experimental repetitions can be performed. The situation is considerably more involved in the practically important finite-sample regime. The reason for this is that the mean-squared error only constrains the second statistical moment of the distribution of the estimation error $\hat{t} - t$. Hence, it only completely characterizes the distribution when it is Gaussian. This is no problem asymptotically, due to the central limit theorem, but almost always leads to an incomplete characterization in the finite-sample regime, in which higher moments of the distribution cannot be neglected. This reasoning also clarifies that the quantum Cramér-Rao bound still constrains the achievable precision, even in the finite-sample regime, which means that large quantum Fisher information is *necessary* for good performance, but on its own is not *sufficient*—a phenomenon that we explicitly demonstrate in the setting of phase estimation using an ensemble of spins.

The existing literature contains a variety of approaches that provide alternatives to the Cramér-Rao bound. Our work is based on the notion of the *success probability* [19–23], which captures how likely it is that the estimate $\hat{t}$ is within a certain tolerance δ of the true value t . Formally, $\mathbb{P}[|\hat{t} - t| \leq \delta]$ replaces the mean-squared error $\mathbb{E}[(\hat{t} - t)^2]$ as the figure of merit. As we show, the success probability gives a figure of merit that is especially useful in the finite-sample regime.

Along these lines, it is worth noting that the success probability and the mean-squared error are not fully independent of each other. Indeed, Chebyshev’s inequality can always be used to put a lower bound on the success probability in terms of the mean-squared error of the protocol. If we consider a protocol that produces an unbiased estimate that saturates the Cramér-Rao bound of Eq. (1), we obtain the guarantee

$$\mathbb{P}[|\hat{t} - t| \leq \delta] \geq 1 - \frac{\mathcal{F}}{\delta^2}. \quad (2)$$

Chebyshev’s inequality predicts that the success probability approaches one polynomially fast when increasing δ . This is a weak guarantee, as we expect—and practically achieve—an exponential decrease in the error probability. In other words, guarantees based on Chebyshev’s inequality may vastly overestimate the resources necessary to achieve a desired success probability. In the context of the Cramér-Rao bound we can still get an exponential decrease if we aggregate multiple independent predictions through a sample mean, in which case Hoeffding’s inequality provides a tighter bound. As we showcase later for the specific case of phase estimation, we can significantly improve upon this strategy when using entangled probe systems.

Before we discuss our novel contributions in detail, we establish the notation and framework that we build on to reliably capture the performance of metrology protocols in the finite-sample regime.

II. FRAMEWORK FOR QUANTUM METROLOGY IN THE FINITE-SAMPLE REGIME

Combining and generalizing prior work on alternatives to the Cramér-Rao bound and global quantum metrology [19–26], this section is devoted to formally defining the *success probability* of a metrology protocol and to outlining how it can be used to define two other quantities, the *optimal tolerance* and the *sample complexity*. Together, these quantities form a holistic framework that is particularly suitable to exploring the finite-sample regime of quantum metrology.

Formally, we consider the task of estimating an unknown parameter that is encoded in a quantum state, as depicted in Fig. 1 (parametrized channels will be discussed in Sec. XI below). Let $t \mapsto \rho(t)$ be a one-parameter family of quantum states, where the parameter t belongs to some fixed real interval $t \in I \subseteq \mathbb{R}$. For technical convenience, we assume that the interval I is given as the domain of the function $t \mapsto \rho(t)$ and henceforth omit explicit mention of I , all while assuming that t belongs to the domain of $t \mapsto \rho(t)$. We note that $\rho(t)$ does not necessarily need to be a one-to-one function. However, if there are two values of t that correspond to the same state, distinguishing between these values will be impossible and hence limit the metrological performance accordingly.

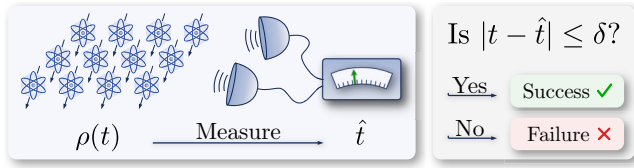


FIG. 1. The setup we consider for quantum metrology in the finite-sample regime consists of applying a measurement onto the quantum state $\rho(t)$ and inferring an estimate $\hat{t}$ of the value of the unknown parameter t from the outcome of the measurement. The estimation process is successful if the estimate $\hat{t}$ and the true parameter value t differ by at most some fixed error tolerance δ . The finite-sample approach quantifies the probability that the estimation procedure is successful. In contrast, the standard quantum Cramér-Rao bound quantifies the variance of outcome $\hat{t}$ for a given estimation procedure, fully characterizing the distribution of estimates only after collecting many outcomes.

An important particular example of the above setup is *phase estimation*, where an unknown phase shift imprinted on a quantum system by a Hamiltonian evolution is to be determined. It describes a wide range of tasks, including interferometers, magnetometers, and quantum clocks. We consider an ensemble of n spin- $\frac{1}{2}$ particles (qubits), which are prepared in a state ρ_0 . Every spin then independently undergoes an unknown phase shift of t , resulting in a state $\rho(t)$. Afterwards, a measurement is performed on the n spins, from which the value of the unknown phase shift is inferred. The combination of a *probe state* ρ_0 , a measurement, and the postprocessing of the measurement outcomes into predictions $\hat{t}$ for the value of t constitutes an example of a *metrological protocol*.

As we have motivated in the Introduction, the mean-squared error is a figure of merit for quantum metrology that may be misleading in the finite-sample regime. To circumvent this issue, we can approach the problem of estimating the parameter t from an operational angle. Imagine we want to estimate t up to a tolerance of δ , and if we managed to do so, we call our procedure *successful*. A very natural figure of merit for the quality of a metrological protocol is then the probability with which it produces such a successful estimate [19–22,25].

We consider the setup of Fig. 1, where a parameter is to be extracted from a parametrized quantum state $t \mapsto \rho(t)$ through a quantum measurement. Physically, this process is realized by performing a measurement and subsequently postprocessing its outcomes into a prediction for the underlying parameter, τ . Mathematically, we can formalize the metrological protocol as a positive operator-valued measure (POVM) $\tau \mapsto Q(\tau)$. For a fixed value t of the parameter, the probability of obtaining an estimate τ is then given by

$$\mathbb{P}[\tau | t] = \text{Tr}[\rho(t)Q(\tau)]. \quad (3)$$

We are left to mathematically model the fact that the value of the parameter t is unknown. In this work, we follow established strategies from the field of *global* quantum metrology [26] and consider two ways of doing so. The first takes the *Bayesian* point of view, where the true value of the parameter follows a prior distribution $t \sim \mu(t)$. The Bayesian approach is especially suitable if there is indeed existing knowledge about the underlying parameter, for example, from prior experiments. The second approach we analyze is one where we have no prior knowledge about the true value t . One might be tempted to model this situation using the Bayesian approach with a uniform prior distribution. This, however, still corresponds to an assumption about the underlying distribution. Instead, we take a more fundamental approach and consider the worst case over all possible values of t . This corresponds to a setting where the ground truth is selected by an adversary. We refer to this setting as the *minimax* setting, following standard terminology in statistics. The name stems from the two optimizations that are involved when we try to come up with a good protocol: our maximization of the success probability over possible protocols and the adversary’s minimization of it over the different choices of the true value t . For this point, see also Fig. 2. The minimax setting also gives guarantees that hold for all possible prior distributions of t and, as such, enables a rigorous treatment of the lack of any prior knowledge about the unknown parameter.

These thoughts lead to the following formal definitions. In the definitions,

$$w_\delta(t - \tau) := \begin{cases} 1 & \text{if } |t - \tau| \leq \delta, \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

denotes the rectangular *window function*, which we use to aggregate all possible predictions that are within our accepted estimation tolerance. The window function is a special case of a “deviation” or “dispersion” function as used in previous work [19,21,23] that allows for an interpretation as a probability.

Definition 1 (Success probability). For a given tolerance δ , a set of states $\rho(t)$, possibly with prior $\mu(t)$, and a measurement $Q(\tau)$, the Bayesian success probability is given by

$$\eta(\delta, \mu, \rho, Q) := \int dt \mu(t) d\tau w_\delta(t - \tau) \text{Tr}[\rho(t)Q(\tau)]. \quad (5)$$

The minimax success probability is given by

$$\bar{\eta}(\delta, \rho, Q) := \inf_t \int d\tau w_\delta(t - \tau) \text{Tr}[\rho(t)Q(\tau)]. \quad (6)$$

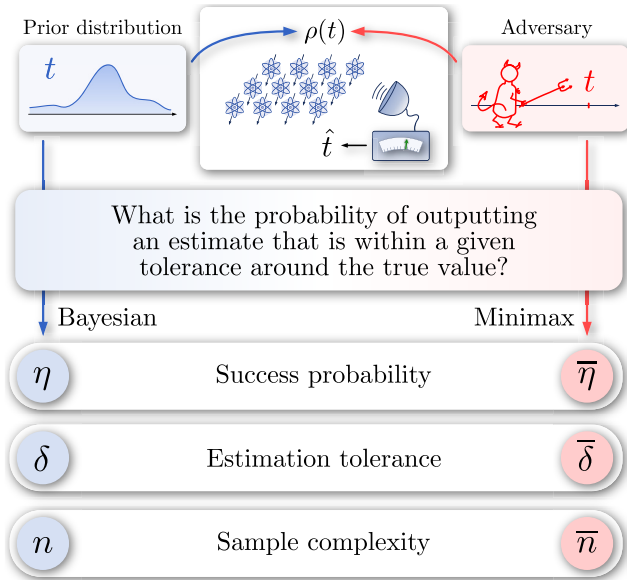


FIG. 2. The fundamental quantities forming the framework for finite-sample metrology used in this work. The success probability, defined intuitively through the statement given in the above figure, forms the cornerstone of the framework. The tolerance is obtained from it by answering the question: *what is the smallest tolerance that still guarantees a given success probability?* Similarly, the sample complexity is the answer to the question: *how many repetitions of an experiment do I need to perform to achieve a certain tolerance at a fixed success probability?* We rigorously define them in Definition 1 (success probability), Definition 2 (estimation tolerance), and Definition 3 (sample complexity).

We observe that the success probability can be more compactly written using the convolution notation

$$(w * \rho)(t) := \int d\tau w(t - \tau) \rho(\tau) \quad (7)$$

as

$$\begin{aligned} \eta(\delta, \mu, \rho, Q) &= \int dt \mu(t) \text{Tr}[\rho(t)(w_\delta * Q)(t)] \\ &= \int dt d\tau \text{Tr}[\mu(t) \rho(t) w_\delta(t - \tau) Q(\tau)] \\ &= \int dt d\tau \text{Tr}[\mu(t) \rho(t) w_\delta(\tau - t) Q(\tau)] \\ &= \int d\tau \text{Tr}[(w_\delta * \{\mu \cdot \rho\})(\tau) Q(\tau)]; \end{aligned} \quad (8)$$

we use this notation in the rest of this work.

The success probability of Definition 1 quantifies the probability that our metrology protocol outputs a correct estimate. This is very reminiscent of the strategy used to quantify the performance of learning algorithms pioneered by Valiant [27], which was coined *probably approximately correct learning*.

As we show in Sec. IV of the Supplemental Material [28], both the Bayesian and minimax success probabilities have basic continuity properties in all their arguments and allow for intuitive monotonicity relations.

It is equally operationally relevant to require that our metrology protocol should achieve a certain success probability and then to quantify the smallest possible estimation tolerance for which that success probability is still achieved. This leads to the following rigorous definition.

Definition 2 (Estimation tolerance). For a given success probability η , a set of states $\rho(t)$, possibly with prior $\mu(t)$, and a measurement $Q(\tau)$, the Bayesian estimation tolerance is given by

$$\delta(\eta, \mu, \rho, Q) := \inf \{\delta' \geq 0 \mid \eta(\delta', \mu, \rho, Q) \geq \eta\}. \quad (9)$$

The minimax estimation tolerance is given by

$$\bar{\delta}(\eta, \rho, Q) := \inf \{\delta' \geq 0 \mid \bar{\eta}(\delta', \rho, Q) \geq \eta\}. \quad (10)$$

Looking at the estimation tolerance as a figure of merit instead of the success probability allows for a simpler comparison with the standard bounds encountered in quantum metrology, as the tolerance has comparable meaning to the standard deviation of an estimator quantified by the quantum Cramér-Rao bound.

Both the success probability and the tolerance introduced in Definitions 1 and 2 are truly *single-shot* quantities, in that they consider a single outcome of the quantum measurement. Usually, however, the desired performance can only be achieved by performing multiple repetitions of the same experiment. We model multiple repetitions of an experiment by having access to n copies of the parametrized state, i.e., $t \mapsto \rho^{\otimes n}(t)$. The measurement $Q(\tau)$ is then performed collectively on the n copies of the state.

The multicopy scenario leads us to a third operationally relevant figure of merit, the *sample complexity*. It quantifies the number of copies necessary to achieve guarantees on *both* the success probability and the estimation tolerance.

Definition 3 (Sample complexity). For a given success probability η and tolerance δ , a set of states $\rho(t)$, possibly with prior $\mu(t)$, and a sequence of measurements $\{Q^{(n)}(\tau)\}$, the Bayesian sample complexity is given by

$$n(\eta, \delta, \mu, \rho, \{Q^{(n)}\}) := \inf \{n' \in \mathbb{N} \mid \eta(\delta, \mu, \rho^{\otimes n'}, Q^{(n')}) \geq \eta\}. \quad (11)$$

The minimax sample complexity is given by

$$\bar{n}(\eta, \delta, \rho, \{Q^{(n)}\}) := \inf \{n' \mid \bar{\eta}(\delta, \rho^{\otimes n'}, Q^{(n')}) \geq \eta\}. \quad (12)$$

Results in the context of more general metrology tasks, like state tomography or Hamiltonian learning, are therefore usually phrased in terms of the sample complexity. We explore this connection and the multivariate generalization of this framework in Sec. XIV below.

Practical settings in quantum metrology often involve estimating an unknown parameter present in the dynamics of a system, rather than directly encoded into the state itself. We discuss this setting of *parametrized quantum channels* in Sec. XI below.

With the preceding definitions, we possess a set of figures of merit that capture the performance of finite-sample quantum metrology protocols. Here, we discuss the optimal values these quantities can take when we optimize over all possible metrological protocols. This brings us to the following definitions of the *optimal* counterparts of Definitions 1, 2, and 3.

Definition 4 (Optimal Bayesian quantities). The *optimal Bayesian success probability* is obtained by optimizing the Bayesian success probability over all possible POVMs:

$$\eta^*(\delta, \mu, \rho) := \sup_{Q(\tau)} \eta(\delta, \mu, \rho, Q). \quad (13)$$

We use it to define the *optimal Bayesian tolerance*

$$\delta^*(\eta, \mu, \rho) := \inf \{\delta' \geq 0 \mid \eta^*(\delta', \mu, \rho) \geq \eta\} \quad (14)$$

and *optimal Bayesian sample complexity*

$$n^*(\eta, \delta, \mu, \rho) := \inf \{n' \in \mathbb{N} \mid \eta^*(\delta, \mu, \rho^{\otimes n'}) \geq \eta\}. \quad (15)$$

The optimal minimax quantities are defined analogously.

Definition 5 (Optimal minimax quantities). The *optimal minimax success probability* is obtained by optimizing the minimax success probability over all possible POVMs:

$$\bar{\eta}^*(\delta, \rho) := \sup_{Q(\tau)} \bar{\eta}(\delta, \rho, Q). \quad (16)$$

We use it to define the *optimal minimax estimation tolerance*

$$\bar{\delta}^*(\eta, \rho) := \inf \{\delta' \geq 0 \mid \bar{\eta}^*(\delta', \rho) \geq \eta\} \quad (17)$$

and *optimal minimax sample complexity*

$$\bar{n}^*(\eta, \delta, \rho) := \inf \{n' \in \mathbb{N} \mid \bar{\eta}^*(\delta, \rho^{\otimes n'}) \geq \eta\}. \quad (18)$$

The optimal Bayesian and minimax success probabilities have desirable properties such as convexity in their arguments and data processing under noise channels, a point we elaborate on in Sec. IV of the Supplemental Material [28].

III. OVERVIEW OF RESULTS

In this work, we establish the following main results.

Computing optimal protocols. Optimizing protocols for a high success probability gives strong operational guarantees in the finite-sample regime. As we show in Sec. V, the optimization of protocols can be performed using semi-infinite programming [29,30], a continuous analogue of semidefinite programming [31,32]. In the specific case when the measurement is fixed, we analytically give the optimal way to postprocess the outcomes into estimates of the underlying parameter.

Fundamental limits via hypothesis testing. In Sec. VI, we discuss the relationship between the metrological success probability and quantum hypothesis testing. In Theorem 1, we rigorously establish that the task of estimating a parameter encoded in a state $\rho(t)$ is at least as hard as performing a quantum multihypothesis test between parametrized states $\{\rho(t_i)\}_{i=1}^M$ associated with parameter values that are separated by at least twice the tolerance δ , i.e., $|t_i - t_j| > 2\delta$ for $i \neq j$ (see Fig. 3). This provides us with an upper bound on the success probability for our parameter estimation task in terms of the success probability of the associated multihypothesis testing task. We make use of this theorem to derive a simple relation between the success probability and the fidelity of states in Corollary 2 that forms the basis of our further results on asymptotic rates. Our theorem bears resemblance to a result of Berry *et al.* [23] that also relates the performance of metrology protocols to hypothesis testing.

Asymptotic rates of quantum metrology. In Sec. VII, we exploit the hypothesis testing bound of Theorem 1 to study the rate with which the success probability approaches one when using increasingly more copies of the same quantum state $\rho(t)$ while maintaining a fixed tolerance. In particular, we provide upper and lower bounds on error rates in Theorems 2 and 3, and we present the exact asymptotic rate for commuting (i.e., classical) metrology problems in Corollary 3.

A finite-sample Cramér-Rao bound. The metrological tolerance, which quantifies the smallest deviation of metrological estimates that still guarantees a given success probability, fulfills a role similar in spirit to the standard deviation in the asymptotic framework of quantum metrology. We exemplify this similarity in Sec. VIII by giving a bound in Theorem 4 that resembles the Cramér-Rao bound, but is valid in the single-shot setting. This showcases the fact that, while not sufficient for good finite-sample performance, large Fisher information is still a necessary prerequisite.

A finite-sample analysis of phase estimation. We devote Secs. XII and XIII to a minimax analysis of the problem of estimating a phase imprinted on an ensemble of spin- $\frac{1}{2}$ particles by a Hamiltonian evolution. For the general case of a covariant Hamiltonian evolution, we give the measurement

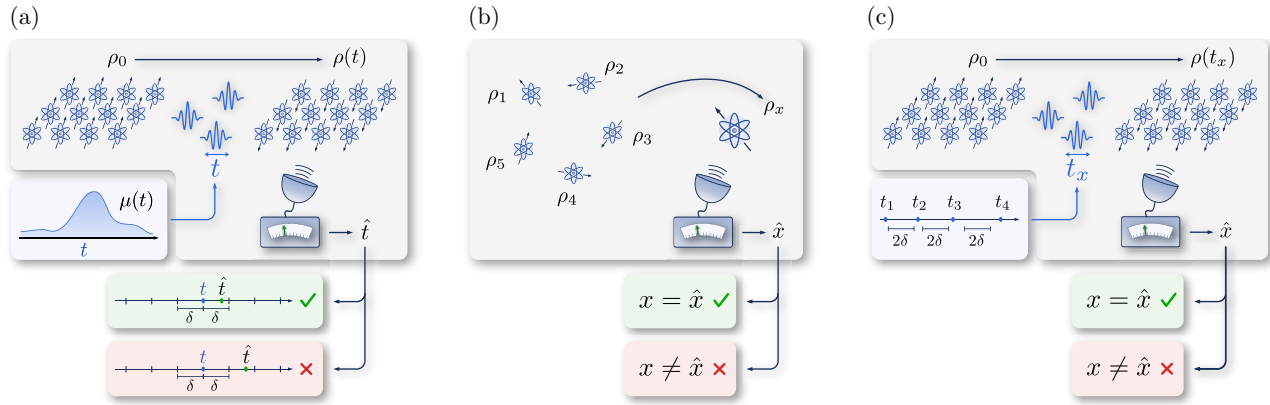


FIG. 3. Metrological tasks consist of estimating some physical properties of a quantum system. A basic protocol involves preparing a probe state ρ_0 , letting it interact with the quantum system of interest, and then measuring the output state. We describe the most general protocols in Sec. XI. (a) In quantum parameter estimation, the parameter $t \in \mathbb{R}$ indicates the physical property of interest, which may be governed by an underlying random process with probability density function μ . In the finite-sample approach, we identify the success of the estimation procedure by correctly identifying parameter t up to a given tolerance δ . (b) In quantum hypothesis testing, a set of quantum states with labels $x \in \mathcal{X}$ is given and the task is to design a measurement that maximizes the probability of correctly identifying the label x . (c) The core contribution of our work is a rigorous connection between quantum hypothesis testing and quantum parameter estimation. Specifically, we show that the parameter estimation problem of (a) is at least as hard as distinguishing states that are mutually at least 2δ apart in the hypothesis testing setting of (b); see Theorem 1.

achieving the optimal success probability and provide a formula in closed form in Theorem 5. This allows us to find the optimal probe state in the phase estimation scenario and to establish the optimal asymptotic rate of the error probability in Theorem 6.

An exhaustive numerical comparison of different probe states shows without a doubt that the quantum Fisher information fails to reliably predict the finite-sample performance of a probe state. A clear example of this is given by the Holland-Burnett state, which has nearly optimal performance according to the Cramér-Rao bound, but whose finite-sample performance is very poor. We also observe that the optimal probe states for finite-sample phase estimation are notably different from typically considered ones.

Further results. We use the hypothesis testing bounds and the finite-sample Cramér-Rao bound to derive corresponding bounds on the sample complexity of quantum metrology in Sec. IX. This work takes significant inspiration from *single-shot quantum information theory* [33–36], in particular in its mindset. Correspondingly, we show in Sec. X how the optimal success probability can be viewed in terms of one-shot entropic quantities.

In Sec. XI, we show that the framework we use and the main results we derive readily generalize to the setting in which unknown parameter t is accessed through the use of a parameter-dependent quantum channel $t \mapsto \mathcal{N}(t)$, where richer estimation strategies, like adaptive strategies, are available. We elucidate connections to other fields in Sec. XIV and discuss a variety of open problems in Sec. XV.

IV. RELATION TO PRIOR ART

Here we provide a brief overview of the relation of our work to prior works, and we refer the reader to Sec. I of the Supplemental Material [28] for a more detailed account.

Prior works focusing on metrology with finite repetitions [20,37–43] in both the classical and quantum cases usually take a “large deviation” perspective, such that the number of samples is understood to be finite but still large. Also, Hayashi and Watanabe [43] provided general and meaningful criteria for judging finite-length bounds for the task of channel coding. The two criteria are efficient computational complexity and asymptotic optimality. These criteria are also meaningful within our framework of finite-sample quantum metrology, and it is notable that the optimal success probability that we define satisfies both criteria.

The notion of the success probability that we define has already appeared in Refs. [20,22,24]. Hayashi [20] also defined a notion of the success probability with a given tolerance similar to Definition 1. He studied the properties of *consistent* protocols that ensure that the success probability, asymptotically, approaches unity for all nonzero tolerances. For those protocols, a large-deviation analysis where the number of samples is finite but still assumed large was performed. This allowed the author to bound the local curvature of the asymptotic rate defined in Eq. (32) below. In our work, we go beyond this result by establishing bounds that hold in the single-shot setting and that we then use to establish bounds on the asymptotic rate that hold for fixed tolerance δ and that at the same time give a

tighter constraint on its local curvature (see Sec. VI H of the Supplemental Material [28]).

The authors of Refs. [20,44] have quantified asymptotic properties related to the achievable precision under statistical assumptions on the estimators, something that can be tightened with our results, as we explain in the Supplemental Material [28]. References [41,42,45] have extended this approach to quantum state tomography. The problem of optimizing the metrological tolerance with a given guarantee on the success probability has been treated in Refs. [22,24,25,46,47], where some bounds have been given. The existing body of literature on confidence region tomography can also be seen as an instance of learning a state with a given tolerance [46,48–51]. In a spirit similar to our work, connections between quantum metrology and hypothesis testing have been used to obtain precision bounds in the standard approach to quantum metrology [24,46,52]. Berry *et al.* [23] gave a quantum Bell-Ziv-Zakai bound that is highly versatile. While the rectangular window function we use in our work does not fit into the framework of Ref. [23], we believe that a modification is possible that would yield a result that is similar to the Bayesian version of our hypothesis testing bound on the success probability. We detail this reasoning in Sec. I of the Supplemental Material [28]. The reverse direction of using metrology bounds to quantify channel discrimination has also been explored [53]. Our analysis of the phase estimation problem [19,54–59] generalizes a result by Holevo [19] and has overlaps with Ima and Hayashi [60] and Hayashi [61], who also used the Fourier-analytic approach. Alternative methods for multiparameter estimation are reviewed in Ref. [62]. Salmon *et al.* [63] studied the case of a fixed measurement with a focus on the admissibility of said measurement.

V. COMPUTING OPTIMAL MEASUREMENTS

Of the three quantities that form the basic framework treated in this work, the success probability is the most amenable to optimization. It is linear in the chosen measurement, and we can show that the optimization over the measurement $Q(\tau)$ can be cast into the form of a convex problem [32]. Specifically, it assumes the form of a *semi-infinite problem* [29,30], so a *semidefinite problem* with infinitely many objective variables and finitely many constraints, or the other way around. For the theory of semidefinite programming to largely take over, the involved functions must be analytically defined, smooth, and convex [30], which can be safely assumed here. In what follows, we refer to such infinite readings of semidefinite problems as convex problems. This formulation of the optimization problem does not only provide a path to more easily compute them, but is also a reliable tool to prove upper and lower bounds.

Proposition 1 (POVM optimization). For a given tolerance δ and a set of states $\rho(t)$ with prior distribution $\mu(t)$, the optimal success probability $\eta^*(\delta, \mu, \rho)$ can be computed using the convex program

$$\max_{Q(t) \geq 0} \left\{ \int dt \operatorname{Tr}[(w_\delta * \{\mu \cdot \rho\})(t)Q(t)] \mid \int dt Q(t) = \mathbb{I} \right\}. \quad (19)$$

The optimal success probability is equally characterized by the convex program

$$\min_{X \geq 0} \{ \operatorname{Tr}[X] \mid X \geq (w_\delta * \{\mu \cdot \rho\})(t) \text{ for all } t \}. \quad (20)$$

The minimax success probability can be computed in a similar way by additionally optimizing over all priors.

The second convex program in Eq. (20) is derived through the notion of duality in semi-infinite programming. The proof of the above proposition, along with a more detailed statement is provided in Proposition 2 within the Supplemental Material [28], where we also explicitly give the convex formulation of the minimax success probability.

From a practical standpoint, it is also important that the above semi-infinite program can actually be implemented numerically while having guarantees on the quality of the approximation.

Proposition 2 (Discretization). There exists a discretization of the convex program of Proposition 1 that yields a semidefinite program that can be solved with standard tools. If $\rho(t)$ is Lipschitz with respect to the trace distance, the error of the discretization can be made arbitrarily small by choosing a suitable scale of the discretization.

We give detailed statements of the above in Sec. II F of the Supplemental Material [28]. The optimal tolerance and sample complexity can in principle be computed by combining the above semidefinite program with binary search [64]. The influence of discretization in quantum metrology was also studied previously in Ref. [65].

The general structure of the solutions to the convex problem of Proposition 1 is unclear *a priori*. Consequently, the optimal measurement for a given metrological problem might be either impractical or impossible to implement in an experiment. A particular way to model experimental restrictions is to consider the measurement performed on the quantum system to be fixed. In the following, we study how to best turn the outcomes of that measurement into a prediction of the encoded parameter.

As shown in Fig. 4, we consider an estimation strategy that begins by applying a fixed quantum measurement described by a POVM $\{M(\lambda)\}_\lambda$. For instance, the POVM might represent a projective measurement in a fixed basis

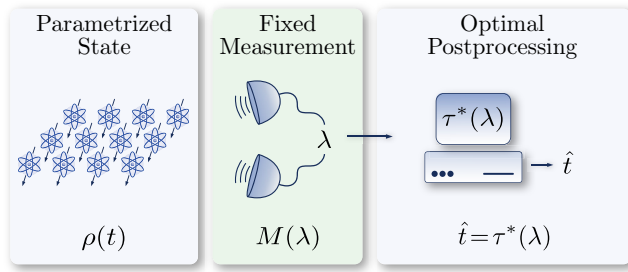


FIG. 4. Metrology setting with fixed measurement as discussed in Sec. V. A parametrized state $\rho(t)$ is measured with a POVM $\{M(\lambda)\}_\lambda$ fixed by, e.g., experimental feasibility restrictions. We show that the optimal way of postprocessing the measurement outcome λ into predictions of the underlying parameter t is given by a smoothed analogue of maximum *a posteriori* estimation in the Bayesian case.

or a POVM with continuous outcomes, like homodyne detection. Subsequently, the procedure infers from the outcome λ an estimate $\tau^*(\lambda)$ for the value t that is encoded in the measured state.

In classical statistics, an essential technique to infer the value of parameters is maximum-likelihood estimation. When combined with Bayesian priors, it gives rise to *maximum a posteriori estimation* (MAP). In our case, we can model the joint distribution of the measurement outcomes λ and the ground truth t as

$$(\lambda, t) \sim \mathbb{P}[\lambda]\mathbb{P}[t | \lambda], \quad (21)$$

where $\mathbb{P}[\lambda]$ is the distribution of measurement outcomes relative to the prior distribution $\mu(t)$ and the given POVM $\{M(\lambda)\}_\lambda$, and $\mathbb{P}[t | \lambda]$ is the posterior distribution of the ground truth given the observed measurement outcome, which can in practice be obtained using Bayes rule. In our setting, it is natural to predict the value for our ground truth that maximizes the amount of posterior probability within its tolerance window:

$$\tau_{\text{SMAP}}^*(\lambda) := \arg \max_{\tau} (w_\delta * \mathbb{P}[\cdot | \lambda])(\tau). \quad (22)$$

This *smoothed maximum a posteriori* (SMAP) estimate is indeed optimal.

Proposition 3 (Optimal postprocessing of a fixed measurement in the Bayesian setting). For a given tolerance δ , state set $\rho(t)$ with prior $\mu(t)$, and a fixed POVM $\{M(\lambda)\}_\lambda$, the optimal Bayesian success probability of an estimation strategy that measures $\{M(\lambda)\}$ and postprocesses the result is achieved by the smoothed maximum *a posteriori* estimator:

$$\sup_{\tau^*} \eta(\delta, \mu, \rho, Q_{M, \tau^*}) = \eta(\delta, \mu, \rho, Q_{M, \tau_{\text{SMAP}}^*}). \quad (23)$$

Here, Q_{M, τ^*} formally denotes the POVM obtained by postprocessing the outcome of measuring M using τ^* .

We present a proof of this fact in Sec. V of the Supplemental Material [28] and present explicit formulations of the achieved success probability. We note that the situation is more nuanced in the minimax setting. The strategy equivalent to SMAP is not necessarily optimal in this case, but yields a practical upper bound.

VI. ULTIMATE LIMITS OF FINITE-SAMPLE METROLOGY THROUGH HYPOTHESIS TESTING

Intuitively, determining a parameter encoded in a quantum system should be intimately related to the task of distinguishing states for different values of the parameter. In this section, we make this intuition rigorous and prove that the success probability of a quantum metrology protocol is fundamentally limited by the distinguishability of states associated with parameters spaced at the order of the desired tolerance.

The optimal probability of success for distinguishing a set of N quantum states $\{\rho_i\}_{i=1}^N$ with prior probabilities $\{p_i\}_{i=1}^N$ —a test between N quantum hypotheses—is given by [66]

$$P_s^*(\{\rho_i\}_{i=1}^N) := \sup_{\substack{\text{POVMs} \\ \{Q_i\}_{i=1}^N}} \sum_{i=1}^N p_i \text{Tr}[\rho_i Q_i]. \quad (24)$$

Similarly, the optimal minimax success probability is defined as

$$\bar{P}_s^*(\{\rho_i\}_{i=1}^N) := \sup_{\substack{\text{POVMs} \\ \{Q_i\}_{i=1}^N}} \min_{1 \leq i \leq N} \text{Tr}[\rho_i Q_i]. \quad (25)$$

Already, Yuen *et al.* [67] have studied an extension of this definition with an additional *cost matrix* C_{ij} and the associated *optimal expected cost*

$$\inf_{\substack{\text{POVMs} \\ \{Q_i\}_{i=1}^N}} \sum_{i=1}^N \sum_{j=1}^N p_i C_{ij} \text{Tr}[\rho_i Q_j], \quad (26)$$

and have shown that this constitutes a semidefinite problem. A look at Proposition 1 shows that our notion of the Bayesian success probability can be understood as a continuous version of this definition in which the complement of the window function, $1 - w_\delta$, takes the role of the cost function.

Because of this close resemblance of metrological problems to generalized multihypothesis testing problems, it is unsurprising that every quantum multihypothesis testing problem can be written as a particular metrology problem

with suitably chosen prior distribution and tolerance (see Sec. VI A of the Supplemental Material [28]).

We also establish a result in the other direction and bound the optimal Bayesian and minimax success probabilities for a metrological problem through multihypothesis testing. To gather some intuition, let us consider the minimax setting. The minimax optimal POVM Q guarantees a success probability $\bar{\eta}^*$ for our metrological task of estimating the parameter t up to a precision of δ . If we are now tasked to distinguish between two states $\rho(t)$ and $\rho(t')$, we can approach the problem by measuring Q and then predicting the value t or t' that is closest to the measurement outcome. If the tolerance regions around t and t' do not overlap then, by the definition of the minimax success probability, this approach also guarantees a probability of $\bar{\eta}^*$ for successfully distinguishing between $\rho(t)$ and $\rho(t')$. This in turn means that any upper bound on the success probability for the distinguishing task is also an upper bound on the success probability of the metrological task.

The theorem below expands this reasoning to the Bayesian setting and to the possibility of trying to distinguish multiple states along the trajectory.

Theorem 1 (Hypothesis testing bound). For a given tolerance δ , fix any set $\mathcal{S} = \{(v, s)\}$ of prior probabilities $v \in [0, 1]$ and shifts $s \in \mathbb{R}$ such that, for all distinct $s, s' \in \mathcal{S}$, we have $|s - s'| > 2\delta$ and $\sum_{v \in \mathcal{S}} v = 1$. Then, for a state set $\rho(t)$ with prior $\mu(t)$, we have the upper bound

$$\eta^*(\delta, \mu, \rho) \leq \int dt P_s^* (\{v \mu(t+s) \rho(t+s)\}_{(v,s) \in \mathcal{S}}). \quad (27)$$

Optimizing over the prior probabilities v then yields the analogous upper bound in the minimax setting:

$$\bar{\eta}^*(\delta, \rho) \leq \inf_t \bar{P}_s^* (\{\rho(t+s)\}_{s \in \mathcal{S}}). \quad (28)$$

The proof of the theorem is presented in Sec. VI B of the Supplemental Material [28]. It also generalizes to parametrized quantum channels, as discussed in Sec. XI below. This theorem is related to a result by Berry *et al.* [23], which established a connection between hypothesis testing and quantum metrology as quantified by a “distortion function” $D(x)$ that measures how “bad” an estimate is. Our setting corresponds to the choice of distortion function $D(x) = 1 - w_\delta(x)$. While technically requiring the distortion function to be differentiable, we believe that the results of Ref. [23] imply a theorem that is similar to Theorem 1. We detail this relation in Sec. I of the Supplemental Material [28]. The upper bound on η^* obtained in this way differs from Eq. (27), but would also be suitable in our analysis of the asymptotic rate of the success probability that we carry out below.

As the general formulation of Theorem 1 might appear unwieldy, we present some more intuitive corollaries. We first note that application of Theorem 1 is especially useful analytically when restricting to two shifts, because then the analytic expression for the success probability of binary hypothesis testing due to Helstrom [68] can be used. Looking at the minimax case with two shifts, we immediately obtain what can be considered an analogue of Le Cam’s two-point method [69].

Corollary 1 (Two-point method). For a given tolerance δ and state set $\rho(t)$, we have the following upper bound on the minimax success probability:

$$\bar{\eta}^*(\delta, \rho) \leq \inf_{|t-t'| > 2\delta} \bar{P}_s^*(\rho(t), \rho(t')). \quad (29)$$

The above result, together with the Fuchs–van de Graaf inequalities, allows us to further deduce a relation to the quantum fidelity

$$F(\rho, \sigma) := \text{Tr}[(\sigma^{1/2} \rho \sigma^{1/2})^{1/2}], \quad (30)$$

which we use later to give a concise sample complexity bound, as shown in Sec. VI C of the Supplemental Material [28].

Corollary 2. For a given tolerance δ and state set $\rho(t)$, we have the minimax error probability lower bound

$$1 - \bar{\eta}^*(\delta, \rho) \geq \frac{1}{4} \sup_{|t-t'| > 2\delta} F(\rho(t), \rho(t'))^2. \quad (31)$$

We can use similar reasoning as in the derivation of Theorem 1 to obtain a lower bound that makes use of *asymmetric* hypothesis testing. The resulting bounds are presented in Sec. VI D of the Supplemental Material [28].

We explore other directions that yield metrology bounds of different forms in Sec. VI F of the Supplemental Material [28]. Along the way, we present a bound for quantum multihypothesis testing in Sec. VI E of the Supplemental Material [28].

VII. ASYMPTOTIC BEHAVIOR OF THE SUCCESS PROBABILITY IN THE INDEPENDENT AND IDENTICALLY DISTRIBUTED REGIME

In this section, we consider our framework in the limit where many independent and identically distributed (IID) copies of the state are available. We aim to understand this asymptotic limit of the finite-sample framework in order to connect it to standard approaches in metrology and information theory where many copies of the state are assumed to be available. We exploit Theorem 1 to compute an upper bound on the asymptotic rate with which both the

Bayesian and minimax success probabilities approach one. Formally, we define the asymptotic error rate as

$$R^*(\delta, \mu, \rho) := \lim_{n \rightarrow \infty} -\frac{1}{n} \log[1 - \eta^*(\delta, \mu, \rho^{\otimes n})], \quad (32)$$

when the limit exists. The minimax rate $\bar{R}^*(\delta, \rho)$ is defined analogously.

We relate the asymptotic rate of a metrological problem to the asymptotic rate of hypothesis testing [70,71], which is given in terms of the *Chernoff divergence*

$$C(\rho, \sigma) := - \inf_{0 \leq s \leq 1} \log \text{Tr}[\rho^s \sigma^{1-s}]. \quad (33)$$

In analogy to multihypothesis testing, where the asymptotic rate is given by the smallest Chernoff divergence between two states that are to be discriminated, we obtain the following upper bound.

Theorem 2 (Upper bound on the asymptotic rate). For a given tolerance δ and a set of states $\rho(t)$, possibly with prior $\mu(t)$, the Bayesian and minimax rates obey the upper bounds

$$\bar{R}^*(\delta, \rho) \leq R^*(\delta, \mu, \rho) \leq \inf_{|t-t'| > 2\delta} C(\rho(t), \rho(t')), \quad (34)$$

where the optimization is over time values that have nonvanishing support in the possible priors.

We emphasize that, for the above statement and those that follow in this section, we implicitly assumed that the support of μ encompasses the whole admissible set of values for parameter t that is used to compute the minimax success probability. Restrictions to smaller admissible sets of parameters are possible and the results carry over straightforwardly. The above theorem establishes that a metrological problem is asymptotically at most as hard as the hardest binary hypothesis testing problem of two states whose associated time values are at least 2δ apart. The proof of the theorem uses Theorem 1 together with Laplace's principle and is presented in Sec. VI G of the Supplemental Material [28].

In quantum multihypothesis testing, the smallest pairwise rate is actually achievable [71] and, because of the results below on the commuting case, we believe this to also be true in the case of quantum metrology, at least under suitable regularity assumptions. While we were not able to prove this general statement, we succeeded in establishing a lower bound that guarantees an asymptotic rate equal to the best pairwise hypothesis testing rate for a fixed measurement sequence. Formally, we consider a fixed sequence of measurements $\{M^{(n)}\}$ for $n \in \mathbb{N}$ with outcomes λ (cf. Sec. V), and denote the channel that maps

states to their output distributions over λ as

$$\mathcal{M}^{(n)}[\rho] = \int d\lambda |\lambda\rangle\langle\lambda| \text{Tr}[\rho M^{(n)}(\lambda)]. \quad (35)$$

This sequence achieves the following rate for binary state discrimination:

$$R(\rho, \sigma, \{M^{(n)}\}_{n \in \mathbb{N}}) := \lim_{n \rightarrow \infty} -\frac{1}{n} \log\{1 - P_s^*(\mathcal{M}^{(n)}[\rho^{\otimes n}], \mathcal{M}^{(n)}[\sigma^{\otimes n}])\}. \quad (36)$$

With this notation in place, we have the following theorem, which adds the additional assumption that the set of states is continuous.

Theorem 3 (Asymptotic rate with a fixed measurement scheme). For a given tolerance $\delta > 0$, a continuous set of states $\rho(t)$, possibly with prior $\mu(t)$ of full support, and any fixed measurement sequence $\{M^{(n)}\}$ for $n \in \mathbb{N}$, we have

$$R^*(\delta, \mu, \rho) \geq \bar{R}^*(\delta, \rho) \geq \inf_{|t-t'| > 2\delta} R(\rho(t), \rho(t'), \{M^{(n)}\}). \quad (37)$$

In the above theorem, the possible values of t are constrained to the domain of $\rho(t)$ and priors that have full support on said domain. Priors with restricted support fulfill the theorem with t and t' contained in the restricted domain. The proof uses a discretization argument together with the smoothed maximum *a posteriori* estimation technique introduced in Sec. V and is given in Sec. VI A of the Supplemental Material [28]. We note that Hayashi [20] presented some more explicit achievability statements in the large deviation regime $\delta \rightarrow 0$.

In the case where an optimal measurement sequence is known, however, this lower bound already achieves the rate. This is particularly true in the commuting (i.e., classical) case.

Corollary 3 (Asymptotic rate for commuting states). For a given tolerance $\delta > 0$ and a continuous set of states $\rho(t)$ such that $[\rho(t), \rho(t')] = 0$ for all t, t' , possibly with prior $\mu(t)$, we have

$$R^*(\delta, \mu, \rho) = \bar{R}^*(\delta, \rho) = \inf_{|t-t'| > 2\delta} C(\rho(t), \rho(t')), \quad (38)$$

where the optimization is over time values that have nonvanishing support in the possible priors.

We believe that the strong connection we established here to hypothesis testing between multiple quantum states serves as a motivation to further explore the connection between quantum metrology and quantum information tasks. This should be a fruitful endeavor that allows researchers with a background in quantum metrology as well as in quantum information processing to make an impact on quantum metrology.

VIII. A FINITE-SAMPLE CRAMÉR-RAO BOUND

In the asymptotic approach to quantum metrology, the most important tool is the *quantum Cramér-Rao bound*, which gives a lower bound on the standard deviation of any locally unbiased estimate, relating it to the inverse square root of the quantum Fisher information. We already argued that the quantum Cramér-Rao bound also constrains the precision in the finite-sample regime, as a good value of the bound is a necessary but not a sufficient condition for successful estimation.

In this section, we derive a lower bound on the minimax estimation tolerance of Definition 2 that makes this fact explicit. We note that lower bounds on the estimation tolerance sufficient to establish that the tolerance can at most decrease with Heisenberg scaling have been derived in Refs. [24,25,46]. These bounds, however, do not resemble the Cramér-Rao bound.

We derive our bound from Corollary 2, which we rephrase in terms of the sandwiched Rényi relative entropies. They are defined in terms of a constant $\alpha \in (0, 1) \cup (1, \infty)$ as [36]

$$\tilde{D}_\alpha(\rho \parallel \sigma) := \frac{1}{\alpha - 1} \log \text{Tr}[(\sigma^{(1-\alpha)/2\alpha} \rho \sigma^{(1-\alpha)/2\alpha})^\alpha]. \quad (39)$$

In our case, the value $\alpha = \frac{1}{2}$ is crucial, as it connects to the fidelity of quantum states defined in Eq. (30):

$$\begin{aligned} \tilde{D}_{1/2}(\rho \parallel \sigma) &= -\frac{1}{2} \log \text{Tr}[(\sigma^{1/2} \rho \sigma^{1/2})^{1/2}] \\ &= -\frac{1}{2} \log F(\rho, \sigma). \end{aligned} \quad (40)$$

With this notation at hand, Corollary 2 can be rephrased as

$$\log \left(\frac{1}{4[1 - \bar{\eta}^*(\delta, \rho)]} \right) \leq 4 \inf_{|t-t'| > 2\delta} \tilde{D}_{1/2}(\rho(t), \rho(t')). \quad (41)$$

Before we come to the formal statement of our theorem, let us outline how this bound behaves in the special case of IID copies. We can make the choice $t' = t + 2\delta$ in the above bound. For sufficiently small δ , we can perform a Taylor expansion. We denote it as

$$\begin{aligned} \tilde{D}_{1/2}(\rho(t) \parallel \rho(t + \tau)) &= \frac{1}{2} f_2(t) \tau^2 + \frac{1}{3!} f_3(t) \tau^3 \\ &\quad + \frac{1}{4!} f_4(t) \tau^4 + \dots, \end{aligned} \quad (42)$$

where we envision choosing $\tau = 2\delta$. The values $f_k(t)$ are the Taylor coefficients of the function $\tau \mapsto \tilde{D}_{1/2}[\rho(t) \parallel \rho(t + \tau)]$ at $\tau = 0$; an explicit formula is given in Eq. (46) below. Because of the relationship between the sandwiched Rényi relative entropy and the fidelity, the

coefficient $f_2(t)$ is a constant multiple of the quantum Fisher information $f_2(t) = \frac{1}{8} \mathcal{F}(t)$; see Sec. VII B of the Supplemental Material [28].

Let us now for simplicity drop the explicit time dependence. If we could ignore the higher-order terms in the Taylor expansion, we could simply choose $2\delta = \tau \propto 1/\sqrt{f_2}$ to render the error probability lower bound constant, which would give us the desired scaling of the bound. We will, however, need to work a bit harder to get something analytically meaningful. Let us choose $\tau = \gamma/\sqrt{f_2}$ in the above expansion. Then, as long as $2\delta = \tau$ is within the radius of convergence of the Taylor series, we have

$$\begin{aligned} \tilde{D}_{1/2}(\rho(t) \parallel \rho(t + \tau)) \\ = \frac{1}{2} \gamma^2 + \frac{1}{3!} \frac{f_3}{f_2^{3/2}} \gamma^3 + \frac{1}{4!} \frac{f_4}{f_2^2} \gamma^4 + \dots \end{aligned} \quad (43)$$

We immediately observe that the quality of the second-order approximation depends on the ratios $f_p/f_2^{p/2}$. To get some intuition for these ratios, it is instructive to look at the case of IID copies, i.e., $\rho(t) \rightarrow \rho^{\otimes n}(t)$. In this case, the additivity of the sandwiched Rényi relative entropy implies that

$$\tilde{D}_{1/2}(\rho^{\otimes n}(t) \parallel \rho^{\otimes n}(t + \tau)) = n \tilde{D}_{1/2}(\rho(t), \rho(t + \tau)). \quad (44)$$

This in turn means that all coefficients f_p grow linearly in n , i.e., $f_p \rightarrow n f_p$, which implies that

$$\frac{f_p}{f_2^{p/2}} \rightarrow \frac{1}{n^{(p-2)/2}} \frac{f_p}{f_2^{p/2}}, \quad (45)$$

meaning that higher-order ratios decay more quickly, justifying the second-order approximation in the limit of large n . Note that this is the same regime in which the standard Cramér-Rao bound is attainable.

With the intuition we gained in the IID setting, we can now make sense of the following theorem that applies in the single-shot setting.

Theorem 4 (Finite-sample Cramér-Rao bound). For a given smooth set of states $\rho(t)$, we define

$$f_k(t) := -\frac{1}{2} \frac{\partial^k}{\partial \tau^k} \log F(\rho(t), \rho(t + \tau)) \Big|_{\tau=0} \quad (46)$$

and the coefficient

$$q := \sup_t \sup_{3 \leq p \in \mathbb{N}} \left| \frac{f_p(t)}{f_2^{p/2}(t)} \right|^{1/(p-2)}. \quad (47)$$

We then have, for any desired minimax success probability $\bar{\eta} > 3/4$,

$$\bar{\delta}^*(\bar{\eta}, \rho) \geq \frac{\Gamma}{\sqrt{\inf_t \mathcal{F}(t)}}, \quad (48)$$

where

$$\frac{q}{6\sqrt{2}} \log \frac{1}{4(1-\bar{\eta})} > \sqrt{\log \frac{1}{4(1-\bar{\eta})}} - \Gamma > 0. \quad (49)$$

The bound holds as long as $\Gamma/\sqrt{\inf_t \mathcal{F}(t)}$ does not exceed the smallest convergence radius of the Taylor expansion of the sandwiched Rényi relative entropy.

We note that the coefficient q is a measure of closeness to a Gaussian shape of the fidelity curve $\tau \mapsto F(\rho(t), \rho(t+\tau))$ and again emphasize that $f_2(t)$ relates to the quantum Fisher information as $f_2(t) = \frac{1}{8}\mathcal{F}(t)$. The proof is presented in Sec. VII B of the Supplemental Material [28]. As we discussed above, in the case of IID copies $\rho(t) \rightarrow \rho^{\otimes n}(t)$, we have $q = O(1/\sqrt{n})$, giving a bound that is asymptotically constant.

The above gives a lower bound for the optimal minimax tolerance that, asymptotically, has the expected scaling both in the quantum Fisher information and the desired logarithmic dependence on the success probability. It furthermore is valid in the single-shot setting, where the finite-size corrections depend on the higher-order derivatives of the sandwiched Rényi relative entropy. We find it conceivable that the ratio of the third to second derivative could give the factual second-order asymptotics, but conclude that it would be necessary to find a matching upper bound (i.e., a protocol) with similar performance guarantees.

It is also interesting to gather some intuition about the workings of the above bound. The right-hand side involves the quantum Fisher information, which captures how much states change under infinitesimal, and hence *asymptotically small*, perturbations. This quantity yet puts a bound on $\bar{\delta}$, which is a nonasymptotic quantity. This is because we effectively reduce to a setting where the quantum Fisher information captures the dominant contributions to the sandwiched Rényi relative entropy even at noninfinitesimal distances—the case when it is dominated by the second-order expansion. The coefficient q measures how close we are to this setting.

The proof of the above theorem hinges on the convergence of the Taylor expansion of the sandwiched Rényi relative entropy. As the convergence radius is the distance to the closest pole of the function, we can conclude that the smallest radius of convergence is

$$r := \min\{|\tau| \mid F(\rho(t), \rho(t+\tau)) = 0 \text{ for some } t\}. \quad (50)$$

This shows that in the case of IID copies, for example, the radius of convergence is independent of the number of copies. We thus do not expect the radius of convergence to be an issue in practically relevant scenarios.

We further observe that the expansion we use is very reminiscent of the Edgeworth–Gram–Charlier series

expansion method in statistics [72], with the ratios $f_p/f_2^{p/2}$ in analogy to the normalized p th cumulants. It is an important direction of research to further our understanding of the higher derivatives of the sandwiched Rényi relative entropy.

IX. BOUNDS ON THE OPTIMAL SAMPLE COMPLEXITY

The sample complexity of Definition 3 captures the number of copies of a quantum system needed to achieve a target tolerance with a guaranteed success probability. It is a quantity for which bounds follow in a relatively straightforward manner from our previous results. This is because any bound that relates the tolerance and the success probability to each other can be used to establish a sample complexity bound.

We can use Corollary 2 in the form of Eq. (41) to obtain the following concise minimax sample complexity lower bound that involves the sandwiched Rényi relative entropy of order 1/2 introduced in Eq. (40).

Corollary 4 (Two-point sample complexity bound). For a given tolerance δ and state set $\rho(t)$, we have the following lower bound on the minimax sample complexity:

$$\bar{n}^*(\eta, \delta, \rho) \geq \frac{1}{4 \inf_{|t-t'| > 2\delta} \tilde{D}_{1/2}(\rho(t) \parallel \rho(t'))} \log \frac{1}{4(1-\eta)}. \quad (51)$$

When talking about sample complexities, we usually present them in big- O notation, where the relevant limits are $\delta \rightarrow 0$ and $\eta \rightarrow 1$. In the limit $\delta \rightarrow 0$, the above bound is dominated by close values $t' = t + 2\delta$, motivating a Taylor expansion. As we have discussed in detail in Sec. VIII, we have

$$\tilde{D}_{1/2}(\rho(t) \parallel \rho(t+2\delta)) = \frac{1}{4}\mathcal{F}(t)\delta^2 + O(\delta^3), \quad (52)$$

where $\mathcal{F}(t)$ is the quantum Fisher information at time t . This means that Corollary 4 immediately implies an IID sample complexity lower bound of

$$\bar{n}^*(\eta, \delta, \rho) \geq O\left(\frac{1}{\min_t \mathcal{F}(t)} \frac{1}{\delta^2} \log \frac{1}{1-\eta}\right). \quad (53)$$

The above result applies to IID states. It is, however, equally important to have a sample complexity bound that does not rely on the IID structure of the underlying state. This can, for example, happen when quantum metrology with quantum channels is performed. In that case, we deal with a parametrized family of states $\rho^{(n)}(t)$ for $n \in \mathbb{N}$. To get results about this case, we build on our finite-sample Cramér–Rao bound of Theorem 4. It relies on two quantities: the smallest quantum Fisher information $\inf_t \mathcal{F}(t)$ and

the coefficient q that quantifies the validity of a quadratic approximation to the sandwiched Rényi relative entropy.

Many results in the literature on quantum metrology analyze the asymptotic scaling of the quantum Fisher information, which is usually $\mathcal{F} = O(n)$ in the standard quantum limit and $\mathcal{F} = O(n^2)$ in the Heisenberg limit [73]. We can “import” results of this type in the form of the following corollary.

Corollary 5 (Sample complexity scaling bound). In the setting of Theorem 4, where $\rho(t)$ is replaced with a parametrized family of states $\{\rho^{(n)}(t)\}_{n \in \mathbb{N}}$, assume that we have the following asymptotic scalings in n :

$$\inf_t \mathcal{F}^{(n)}(t) = O(n^\alpha), \quad q^{(n)} = o(1), \quad (54)$$

for $\alpha > 0$. Then we have

$$\bar{n}(\eta, \delta) \geq O\left(\left[\frac{\log[1/(1-\eta)]}{\delta^2}\right]^{1/\alpha}\right). \quad (55)$$

The proof of the above corollary is given in Sec. VIII of the Supplemental Material [28]. As was shown in Sec. VIII, in the case of IID copies, we have $\alpha = 1$ and $q = O(n^{-1/2}) = o(1)$, reproducing the scaling of Corollary 4 up to a worse dependence on the inverse error probability.

X. RELATION TO ENTROPIC QUANTITIES

In the single-shot approach to quantum information theory [35,36], many quantum information processing tasks can be quantified through generalized notions of *entropy*. In this section, we explore how our definitions relate to some of these concepts.

We start by giving an alternative definition of the optimal Bayesian success probability. To do so, we define the *conditional min entropy* of a bipartite and positive semidefinite classical-quantum operator [74]

$$H_{\min}(T | S)_X := -\inf_{\sigma_S} D_{\max}(X_{TS} \| \mathbb{I}_T \otimes \sigma_S), \quad (56)$$

where the *max-relative entropy* [75] of two positive semidefinite operators X and Y is given by

$$D_{\max}(X \| Y) := \log \| Y^{-1/2} X Y^{-1/2} \|_{\infty}. \quad (57)$$

We then have the following corollary of the dual formulation of the optimal success probability in Proposition 1.

Corollary 6 (Relation to the conditional min entropy). For a given tolerance δ and a set of states $\rho(t)$ with prior

distribution $\mu(t)$, define the classical-quantum operator

$$X_{TS} := \int dt |t\rangle\langle t|_T \otimes (w_\delta * [\mu \cdot \rho])(t)_S, \quad (58)$$

where T is the time register and S the system register. Then

$$-\log \eta^*(\delta, \mu, \rho) = H_{\min}(T | S)_{X_S}. \quad (59)$$

Another angle on the same fact is given by defining the *max-relative entropy radius* [76] of a set of positive semidefinite operators $\mathcal{X}$ as

$$r_{\max}(\mathcal{X}) := \inf_{\sigma} \max_{x \in \mathcal{X}} D_{\max}(x \| \sigma). \quad (60)$$

In this case, the optimal success probability can be understood as the max-relative entropy radius of the set of states “smoothed” using the window function

$$\log \eta^*(\delta, \mu, \rho) = r_{\max}(\{(w_\delta * [\mu \cdot \rho])(t)\}). \quad (61)$$

Both relations can be thought of as continuous analogues of the known connection between the success probability of multihypothesis testing and the conditional min entropy of classical-quantum states [74,76]. In addition to the various known operational meanings of the conditional min entropy [33,74,77–79], the relation of Corollary 6 endows it with yet another operational meaning, this time in the context of quantum metrology. Notably, operator X_{TS} of Corollary 6 is not normalized, unlike in the case of multihypothesis testing, due to the window function w_δ . Another instance in which the conditional min entropy of a nonnormalized operator has an operational meaning is in the context of channel simulation [78,79]. We provide the proofs of both relations in Sec. III of the Supplemental Material [28].

Finally, we exploit the connection between quantum metrology and hypothesis testing derived in Ref. [46] to give an alternative lower bound on the optimal minimax success probability $\bar{\eta}^*$. The lower bound is expressed in terms of the hypothesis testing relative entropy [36,80–84]

$$D_h^\eta(\rho \| \sigma) := -\log \beta_h^\eta(\rho \| \sigma), \quad (62)$$

where the asymmetric hypothesis testing error β_h^η is defined as

$$\beta_h^\eta(\rho \| \sigma) := \inf_{0 \leq M \leq \mathbb{I}} \{\text{Tr}[M\sigma] \mid \text{Tr}[M\rho] \geq \eta\}. \quad (63)$$

We have the following proposition, reminiscent of a comparable result in Ref. [46].

Proposition 4 (Lower bound on the estimation tolerance in terms of the hypothesis testing relative entropy). For a

set of states $\rho(t)$ and any $0 \leq \eta \leq \bar{\eta}^*(\delta, \rho)$, the minimax tolerance satisfies

$$\bar{\delta}(\eta, \rho) \geq \frac{1}{2} \int dt \exp(-D_h^\eta(\rho(t) \parallel \sigma)) \quad (64)$$

for any state σ .

The proof exploits the semidefinite program dual formulation of the hypothesis testing relative entropy and is given next to the more general statement of Theorem 29 within the Supplemental Material [28]. We provide a proof that is independent of Ref. [46] for completeness.

XI. METROLOGY OF QUANTUM CHANNELS IN THE FINITE-SAMPLE REGIME

So far, we have considered the task of estimating a parameter t encoded in a set of states $\rho(t)$. In practice, as alluded to in Fig. 3, parametrized quantum states arise from the interaction of a probe system with some physical process whose properties in the form of parameter t we wish to determine. In this case, the primary object of interest is not a set of parametrized states, but a set of parametrized *quantum channels* $t \mapsto \mathcal{N}(t)$, which represent the parameter-dependent evolution of the probe system.

A. Access models

As quantum channels represent evolutions of quantum systems, the possible ways of interacting with multiple copies of them to extract the underlying parameter are much richer than in the case of a parametrized quantum state where the only way of interacting is to choose a suitable POVM. As we show in Fig. 5, a variety of *access models* for multiple copies of quantum channels can be distinguished. In the simplest case, the same single-shot protocol is repeated multiple times and the individual outcomes are processed classically, effectively reducing to the case of the parametrized quantum state—we therefore refer to it as the *IID case*. In the *parallel* case, we use n copies of the channel in parallel, but possibly with entangled inputs and measurements. If *adaptive* processing is allowed then every use of the channel can be followed by a round of adaptive quantum processing before the next channel use. Such causally ordered strategies are modeled by *quantum combs* [85,86]. Finally, the most general conceivable access to n copies of a quantum channel allows for *indefinite causal order* of the channel uses, e.g., through the use of a *quantum switch* [87]. If such strategies can be physically realized is still debated [88–90], but in theory they lead to an asymptotic quadratic advantage over the Heisenberg limit for infinite-dimensional systems [73,91], and there exists a strict performance hierarchy in the finite-dimensional case [92].

If we use a particular access model to interact with n copies of a parametrized quantum channel and subsequently measure a POVM that predicts the underlying parameter, we refer to this as a *strategy*. We can give a formal description of access models and strategies via the Choi representation formalism. In this framework, every *strategy* within an access model for n copies of the channel, $\mathbf{S}_n$, is a function $\tau \mapsto P_n(\tau)$, $P_n \in \mathbf{S}_n$, that maps possible predictions to positive semidefinite operators. With this, we can extend the definition of the success probability (Definition 1) as follows.

Definition 6 (Success probability (channels)). For a given tolerance δ , a set of channels $\mathcal{N}(t)$ of which we can access n copies, possibly with prior $\mu(t)$, and a strategy $P_n(\tau)$, the Bayesian success probability is given by

$$\eta(\delta, \mu, \mathcal{N}, P_n) := \int dt \mu(t) d\tau w_\delta(t - \tau) \times \text{Tr}\{C[\mathcal{N}(t)]^{\otimes n} P_n(\tau)\}, \quad (65)$$

where $C[\mathcal{N}(t)]$ is the Choi representation of the quantum channel $\mathcal{N}(t)$. The minimax success probability is given by

$$\bar{\eta}(\delta, \mathcal{N}, P_n) := \inf_t \int d\tau w_\delta(t - \tau) \text{Tr}\{C[\mathcal{N}(t)]^{\otimes n} P_n(\tau)\}. \quad (66)$$

Naturally, the *optimal* (Bayesian) success probability relative to an access model is defined as

$$\eta^*(\delta, \mu, \mathcal{N}, \mathbf{S}_n) := \sup_{P_n \in \mathbf{S}_n} \eta(\delta, \mu, \mathcal{N}, P_n). \quad (67)$$

The other quantities of the framework are then obtained from the success probabilities similar to Definitions 2 and 3.

In Sec. V, we showed that the optimal Bayesian and minimax success probabilities can be computed by solving a convex optimization problem without duality gap. As evidenced in Fig. 5, protocols that involve a parametrized quantum channel have more moving parts that can and need to be optimized. Let us, for example, take the simplest case of only one use of the parametrized quantum channel. In this case we have to optimize over both the probe state ρ_0 that is fed into the quantum channel and measurement $Q(\tau)$. Naively, the objective is then a nonlinear function of the arguments of the optimization ρ_0 and $Q(\tau)$ and we would not expect that this can be cast as a convex program. However, if we change our perspective and combine the preparation of the probe state and the measurement into a single object represented by a parametrized quantum comb, we can exploit the convexity of the set of quantum combs to again cast the computation of the optimal success probability as a semidefinite program. This reasoning

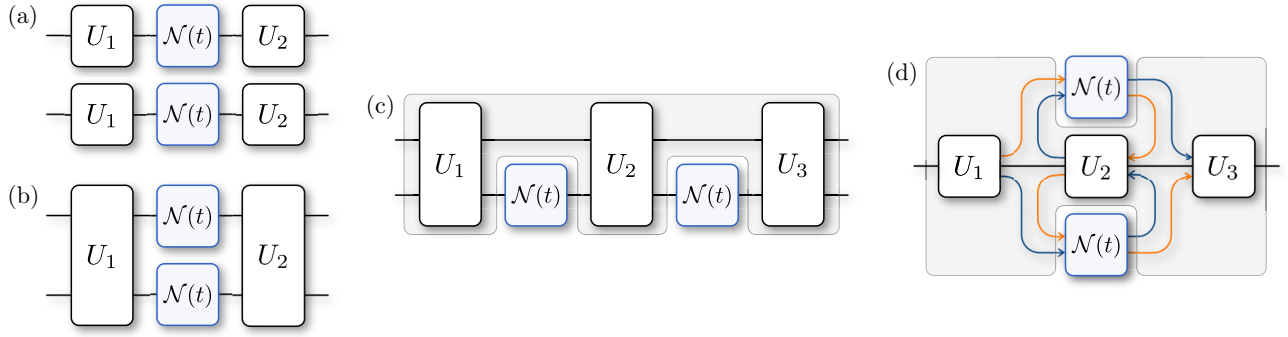


FIG. 5. The different types of protocols that can be realized when accessing multiple (in this case, two) copies of the same parametrized channel $\mathcal{N}(t)$. We distinguish (a) the *IID* case when the same single-shot protocol is repeated multiple times; (b) the *parallel* case where the probe state can be entangled by a suitable unitary transformation; (c) the *adaptive* case where an auxiliary system can be used as a memory to coherently adjust processing based on the outcomes of the first use of the channel (the gray shaded area represents an instance of a *quantum comb*); (d) the case of *indefinite causal order*, where a superposition of orders of invocations of the quantum channel can be used to boost the sensitivity. The blue and orange arrows represent the two orders of invocation. The solid black line in the middle indicates an auxiliary system. The gray shaded area represents a general strategy with indefinite causal order.

then immediately means that the case of adaptively interacting with n copies of the channel can also be efficiently optimized for. The same is true for strategies involving indefinite causal order, as we summarize in the following proposition.

Proposition 5 (Joint optimization). For a given tolerance δ and a set of channels $\mathcal{N}(t)$ of which we can access n copies, possibly with prior $\mu(t)$, the optimal success probability $\eta^*(\delta, \mu, \mathcal{N}, \mathbf{S}_n)$ and optimal minimax success probability $\bar{\eta}^*(\delta, \mathcal{N}, \mathbf{S}_n)$ can be computed using a semi-infinite program without duality gap for both adaptive $\mathbf{S}_n = \mathbf{S}_n^{\text{ada}}$ and indefinitely causally ordered $\mathbf{S}_n = \mathbf{S}_n^{\text{ico}}$ access.

The detailed statements of the convex programs and their duals are given in Sec. II of the Supplemental Material [28].

B. Connection to hypothesis testing

The core contribution of this work is the rigorous connection of success probability-based metrology with quantum hypothesis testing given in Sec. VI. There, we gave an upper bound on the success probability through the success probability of a corresponding multihypothesis testing problem between quantum states. As we show below, these results carry over to metrology with parametrized quantum channels. In this case, the reduction is to multihypothesis testing between channels under different access models. Formally, we define the optimal Bayesian success probability of testing n copies of the quantum channels $\{\mathcal{N}_i\}_{i=1}^N$ with prior probabilities $\{p_i\}_{i=1}^N$ under the access model $\mathbf{S}_n$

as

$$P_s^*(\{p_i \mathcal{N}_i\}_{i=1}^N, \mathbf{S}_n) := \sup_{\{P_i\}_{i=1}^N \subset \mathbf{S}_n} \sum_{i=1}^N p_i \text{Tr}(C[\mathcal{N}_i]^{\otimes n} P_i), \quad (68)$$

under the condition that the set $\{P_i\}_{i=1}^N$ corresponds to a valid combination of processing and POVM. The minimax statement is given analogously by choosing the prior probabilities adversarially for the chosen strategy. The fact that our results from the state case carry over to this more general case comes with little surprise when we realize that upon fixing the strategy $P(\tau)$ to the optimal strategy and executing it right until before the final measurement is performed, we obtain a parametrized set of states to which we can then apply Theorem 1, leading to the following result.

Corollary 7 (Hypothesis testing bound (channels)). For a given tolerance δ , fix any set $\mathcal{S} = \{(\lambda, s)\}$ of prior probabilities $\lambda \in [0, 1]$ and shifts $s \in \mathbb{R}$ such that, for all distinct $s, s' \in \mathcal{S}$, we have $|s - s'| > 2\delta$ and $\sum_{\lambda \in \mathcal{S}} \lambda = 1$. Then, for a set of channels $\mathcal{N}(t)$ of which we can access n copies with prior $\mu(t)$ and a fixed access model $\mathbf{S}_n$, we have the upper bound

$$\eta^*(\delta, \mu, \mathcal{N}, \mathbf{S}_n) \leq \int dt P_s^*(\{\lambda \mu(t+s) \mathcal{N}(t+s)\}_{(\lambda, s) \in \mathcal{S}}, \mathbf{S}_n). \quad (69)$$

Optimizing over the prior probabilities λ then yields the analogous upper bound in the minimax setting

$$\bar{\eta}^*(\delta, \mathcal{N}, \mathbf{S}_n) \leq \inf_t \bar{P}_s^*(\{\mathcal{N}(t+s)\}_{s \in \mathcal{S}}, \mathbf{S}_n). \quad (70)$$

Because of the richer structure embodied by different access models for multiple copies of the same parametrized channel $\mathcal{N}(t)$, we can define multiple types of asymptotic rates to generalize the analysis carried out in Sec. VII. Of particular interest to us are the rates corresponding to IID strategies R_{iid}^* [Fig. 5(a)], because there we can make use of our results on the asymptotics of the state case, the rates corresponding to parallel strategies R_{par}^* [Fig. 5(b)], and the rates corresponding to adaptive strategies R_{ada}^* [Fig. 5(c)]. Starting from Corollary 7 and following the same logic that led to Theorem 2 shows that the asymptotic rates cannot exceed the corresponding rates of distinguishing quantum channels, for which at least some results are known [93–96].

XII. PHASE ESTIMATION OF A PURE STATE HAMILTONIAN EVOLUTION

In this section, we analyze one of the most prototypical scenarios of quantum metrology, namely, phase estimation with pure states. We analyze the minimax success probability, as it represents the most stringent achievable guarantees. It is further important to emphasize that our analysis takes the perspective of *global* estimation, contrary to the *local* estimation routinely seen in the literature.

As a first step, we establish a general result on the minimax success probability in the $U(1)$ -group-covariant setting, which applies beyond phase estimation. We consider a set of states $|\psi(t)\rangle$ generated by unitary evolution of a pure initial probe state $|\psi\rangle$ under a Hamiltonian H , reflecting the evolution of a closed quantum system

$$|\psi(t)\rangle = e^{-itH}|\psi\rangle = U(t)|\psi\rangle. \quad (71)$$

To ensure that t can be understood as a “phase,” H is assumed to have all differences between eigenvalues integer valued, in which case the recurrence time of the Hamiltonian is guaranteed to be 2π . Let H have a spectral decomposition $H = \sum_{\lambda} \lambda \Pi_{\lambda}$, where the λ are the different eigenvalues and the Π_{λ} are the projectors onto the possibly degenerate eigenspaces. Then, we can expand

$$|\psi\rangle = \sum_{\lambda} \psi_{\lambda} |\psi_{\lambda}\rangle, \quad (72)$$

where we defined the normalized projections of $|\psi\rangle$ onto the eigenspaces of H such that $\Pi_{\lambda}|\psi\rangle = \psi_{\lambda}|\psi_{\lambda}\rangle$.

For such a covariant set of states $\psi(t) = |\psi(t)\rangle\langle\psi(t)|$, the following theorem establishes that the *pretty good measurement* [97–102] is minimax optimal, and we obtain a closed-form solution for the optimal minimax success probability. Our result generalizes an optimality result by Holevo [19, Theorem 3], which included a restriction on the window function that the rectangular window w_{δ} does not fulfill. It is well in line with known results on the

optimality of the pretty good measurement in other covariant state discrimination and parameter estimation tasks [19,21,56,65,103–106].

Theorem 5 (Minimax optimal measurement). For a state set $\psi(t)$ given by a pure initial probe state $\rho_0 = |\psi\rangle\langle\psi|$ evolving under a Hamiltonian with integer eigenvalue differences for time $t \in [0, 2\pi]$, the pretty good measurement

$$Q_{\text{PGM}}(t) = R^{-1/2} \psi(t) R^{-1/2}, \quad \text{where } R = \int dt \psi(t), \quad (73)$$

achieves the optimal minimax success probability, equal to

$$\bar{\eta}^*(\delta, \psi) = \sum_{\lambda, \lambda'} |\psi_{\lambda}| |\psi_{\lambda'}| \hat{w}_{\delta}(\lambda - \lambda'), \quad (74)$$

where $\hat{w}_{\delta}(\omega) = \sin(\delta\omega)/(\pi\omega)$ is the Fourier transform of the rectangular window w_{δ} at frequency ω .

The above theorem establishes a direct relation between the amplitudes of the probe state $|\psi\rangle$ and the minimax success probability. It especially shows that only the spectrum of the Hamiltonian and the *absolute values* of the amplitudes matter. The proof exploits strong duality and complementary slackness to establish a formula for the optimal dual variable and is presented as Theorem 35 within the Supplemental Material [28].

One important consequence of Theorem 5 is that it greatly simplifies the search for a minimax optimal probe state. If we arrange the absolute values of the amplitudes in a vector $\boldsymbol{\psi} := (|\psi_{\lambda}|)_{\lambda}$ and construct the matrix associated with the Fourier transform, $W_{\lambda, \lambda'} := \hat{w}_{\delta}(\lambda - \lambda')$, then the optimal minimax success probability is given by the quadratic form

$$\bar{\eta}^*(\delta, \psi) = \langle \boldsymbol{\psi}, W \boldsymbol{\psi} \rangle. \quad (75)$$

The optimal probe state is hence obtained by solving the optimization problem

$$\boldsymbol{\psi}^* := \arg \max_{\boldsymbol{\psi}} \{ \langle \boldsymbol{\psi}, W \boldsymbol{\psi} \rangle \mid \psi_{\lambda} \geq 0 \text{ for all } \lambda, \|\boldsymbol{\psi}\|_2 = 1 \}. \quad (76)$$

Because of the positivity constraint on the entries of the vector, this is in general an NP-hard optimization problem [107] in worst-case complexity.

XIII. PHASE ESTIMATION WITH AN ENSEMBLE OF SPIN- $\frac{1}{2}$ PARTICLES

We now turn our attention to the special case of phase estimation on a spin chain [54,108], which we introduced in the Introduction. This is a covariant problem in the

sense of Sec. XII, with the single-spin Hamiltonian given by $H = \text{diag}(0, 1)$. If we have n spins separately evolving under this Hamiltonian, the effective Hamiltonian is given by summing up the local terms on the individual copies

$$H_n = \sum_{i=1}^n H_i, \quad H_i = \mathbb{I}^{\otimes i-1} \otimes H \otimes \mathbb{I}^{\otimes n-i}. \quad (77)$$

As we have seen in Theorem 5, only the spectrum of the Hamiltonian matters. For the Hamiltonian H_n , it is given by $\text{spec}(H_n) = \{0, 1, \dots, n\}$, which grows linearly in n . We can therefore treat the equivalent problem of a Hamiltonian with spectral decomposition $H = \sum_{k=0}^n k|k\rangle\langle k|$, where the eigenvectors $|k\rangle$ are understood to be any eigenstate of the Hamiltonian H_n with energy k , e.g., $|3\rangle$ could be $|0, 0, 1, 1, 0, 1\rangle$ for $n = 6$.

A. Comparison of probe states

Applying Theorem 5 allows us to compute the optimal minimax success probability (Fig. 6) and the optimal minimax tolerance (Fig. 7) for different kinds of probe states.

Our first and most obvious candidate for a probe state is a generalized *Greenberger-Horne-Zeilinger* (GHZ) state

$$|\text{GHZ}_n\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |n\rangle), \quad (78)$$

which is optimal in the standard approach to quantum metrology [109–111]. However, in the minimax setting, it fails spectacularly—with a minimax success probability amounting to random guessing and a similarly high tolerance (see Figs. 6 and 7). This is an immediate consequence of the fact that the standard approach to quantum metrology is concerned with *local* estimation. In our case, however, the probe needs to be able to perform well in a task of *global* estimation, i.e., the probe state should allow us to discern values in the whole interval $[0, 2\pi]$. The recurrence time of about $2\pi/n$ of the generalized GHZ state means that it can very well resolve small differences in values, but not larger ones.

Our next contender is the *Holland-Burnett* (HB) state [112]

$$|\text{HB}_n\rangle := \frac{1}{\sqrt{n+1}} \sum_{k=0}^n |k\rangle, \quad (79)$$

which consists of an equal superposition of all energy eigenstates. The associated success probability in the context of phase estimation was also studied in Refs. [58,60]. As shown in Fig. 6, this probe state achieves almost optimal success probability in the regime where $n \ll 1/\delta$. This behavior can be explained by expanding the Fourier transform of the window function, $\hat{w}_\delta(\omega) = \delta \text{sinc}(\delta\omega)/\pi$,

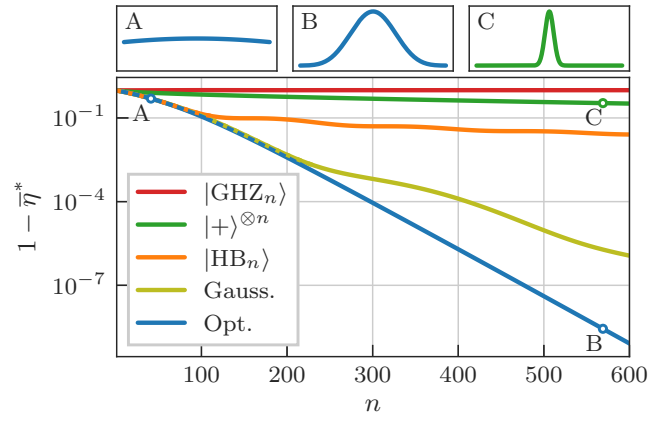


FIG. 6. Optimal error probability of quantum metrology that can be guaranteed for any prior distribution for different probes in the phase estimation scenario for $\bar{\delta} = 0.04$. We compare a generalized GHZ state [Eq. (78), red], a tensor power of plus states (green), the Holland-Burnett state [Eq. (79), orange], the Gaussian state [Eq. (82), yellow], and the optimal state [Eq. (76), blue]. The generalized GHZ state never performs well because it cannot resolve the time globally. For $n \ll 1/\bar{\delta}$, the Holland-Burnett state performs almost optimally, but has comparable asymptotic performance to the tensor power probe. The Gaussian probe performs almost optimally in a larger regime than the Holland-Burnett state and has an intermediary asymptotic. The overset plots show the energy profiles of the optimal probe for $n = 41$ (A), the optimal probe for $n = 561$ (B), and the tensor power probe for $n = 561$ (C). The asymptotic rate in the IID case is consistent with $R \approx \bar{\delta}^2$, whereas the optimal entangled rate is consistent with $R \approx \bar{\delta}$. We present additional numerical results for different values of $\bar{\delta}$ in Sec. XI C of the Supplemental Material [28].

around $\omega = 0$, because the largest frequency scales as $O(n)$:

$$\hat{w}_\delta(\omega) = \frac{\delta}{\pi} [1 - O(\delta^2 \omega^2)]. \quad (80)$$

The zeroth-order contribution to this term is given by

$$\begin{aligned} \bar{\eta}^*(\delta, \psi) &= \frac{\delta}{\pi} \left(\sum_{\lambda, \lambda'} |\psi_\lambda| |\psi_{\lambda'}| + O(\delta^2 \omega^2) \right) \\ &= \frac{\delta}{\pi} [\|\psi\|_1^2 + O(\delta^2 \omega^2)]. \end{aligned} \quad (81)$$

In this limit, a probe that maximizes the one norm of the amplitude vector is clearly optimal, which corresponds to the Holland-Burnett state. This state was also identified in previous work as a suitable probe state for phase estimation with other cost functions [58,60,113]. In the case of the optimal tolerance (see Fig. 7), we observe the inverse of this behavior: for small n , the performance is not satisfactory. Asymptotically, however, the Holland-Burnett state achieves the same Heisenberg scaling $\bar{\delta}^* = O(1/n)$ as the

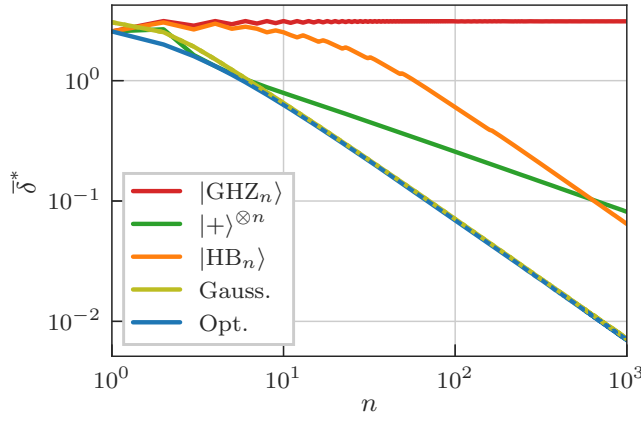


FIG. 7. Optimal tolerance of quantum metrology that can be guaranteed for any prior distribution for different probes in the phase estimation scenario for fixed success probability $\bar{\eta} = 0.99$. We compare a generalized GHZ state [Eq. (78), red], a tensor power of plus states (green), the Holland-Burnett state [Eq. (79), orange], the Gaussian state [Eq. (82), yellow], and the optimal state [Eq. (76), blue]. The generalized GHZ state never performs well because it cannot resolve the time globally. The tensor power of plus states performs adequately for small n , but only achieves standard quantum limited scaling $O(1/\sqrt{n})$ asymptotically. For small n , the Holland-Burnett state does not perform satisfactorily, but achieves the optimal Heisenberg scaling $O(1/n)$ in the asymptotic limit. The Gaussian probe performs almost optimally except for very small n . We present additional numerical results for different values of $\bar{\eta}$ in Sec. XI C of the Supplemental Material [28].

optimal probe. As we show in additional numerics presented in Sec. XI C of the Supplemental Material [28], the critical value of n at which the Holland-Burnett state starts to enter the Heisenberg-scaling regime increases with increasing success probability. This behavior also contrasts with the fact that, with the $\sin^2$ cost function, Heisenberg-limited scaling cannot be achieved with the Holland-Burnett state [58].

Next, we have analyzed the performance of a separable probe state. We chose IID copies of the optimal single-spin probe state, the $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ state. We observe in Fig. 6 that the success probability achieved with this state reaches towards unity much more slowly than the optimal probe state, with an asymptotic rate quadratically smaller. When looking at the optimal tolerance in Fig. 7, we observe the expected asymptotic scaling of the standard quantum limit $\bar{\delta}^* = O(1/\sqrt{n})$.

We additionally compare the aforementioned probe states with a *Gaussian probe* whose amplitudes have a Gaussian shape:

$$\psi_\lambda \propto \exp \left[-\frac{1}{2} \frac{2\delta}{n+1} \left[\lambda - \frac{n}{2} \right]^2 \right]. \quad (82)$$

Choosing the standard deviation, $\sqrt{(n+1)/2\delta}$ optimizes the trade-off between the tails of the Gaussian in real and frequency space, as becomes evident from the proof of Theorem 7 in Sec. XI of the Supplemental Material [28]. Regarding the success probability, we observe in Fig. 6 that the Gaussian probe performs close to optimally in a larger regime than the Holland-Burnett state and achieves better asymptotics, but also does not match the optimal probe. In the case of the tolerance, we do, however, observe in Fig. 7 that it nearly reproduces the optimal probe state. As we show in additional numerics presented in Sec. XI C of the Supplemental Material [28], a gap in tolerance opens between the optimal and Gaussian probes when the target success probability is increased, but the Gaussian probe preserves Heisenberg scaling and a good performance.

Last but not least, we study the optimal probe state. Normally, we would need to solve the optimization problem of Eq. (76). However, in the special case we encounter here, in which W is defined through the Fourier transform of a rectangular window function and the eigenvalue spectrum has no gaps, we can build on prior work studying a similar problem in the context of classical signal processing [114], where W is referred to as the *prolate matrix*. The continuous version of these tools [115] has already been used by Imai and Hayashi to solve this problem in the asymptotic setting [60]. Slepian [114] established that the largest eigenvector of matrix W is given by the so-called *discrete prolate spheroidal sequence* of zeroth order. While he studied the problem without the positivity constraint on the eigenvector, we can build on a different result of Slepian to show that the largest eigenvector is always nonnegative. In the case of phase estimation, we can therefore compute the optimal probe by finding the eigenvector associated with the largest eigenvalue of W .

In our numerical investigations, we observe in Fig. 6 that the success probability tends towards unity with an asymptotic rate quadratically greater than what is possible with the separable probe and also outperforms the Gaussian probe significantly. It further achieves a clear Heisenberg scaling $\bar{\delta}^* = O(1/n)$ for the tolerance, as evident in Fig. 7, in accordance with the asymptotic results of Ref. [60]. To get a feeling for the amplitude distributions of the different probes, we plot in Fig. 6 the amplitude distribution over the eigenvalues of the optimal probe states for $n = 41$ and $n = 561$. These plots clearly show that the optimal probe state for small n has a flat spectrum, whereas, asymptotically, a moderately concentrated shape is optimal. We compare this with the corresponding shape of the tensor power probe at $n = 561$, which is much more concentrated, explaining its inferior performance.

Our numerical results make it quite clear that the setting of optimizing the success probability for a fixed tolerance and of optimizing the tolerance for a fixed success probability are qualitatively different. We especially see

that probes that perform well in one setting do not necessarily perform well in the other.

B. Comparison with the Cramér-Rao bound

Finally, we also want to shine a light on the relation of the minimax tolerance with the quantum Cramér-Rao bound. To compare the optimal estimation tolerance, which is fundamentally dependent on the targeted success probability, and the Cramér-Rao bound on an equal footing, we need a sensible way to assign a success probability to the latter. In general, as the Cramér-Rao bound (CRB) describes an achievable variance, we can model the distribution of the error $\hat{t} - t$ as a normal distribution with standard deviation $\sigma = 1/\sqrt{\mathcal{F}}$ and then compute the probability of obtaining an error within the target confidence interval $[-\delta, \delta]$. This yields a function

$$\eta_{\text{CRB}}(\delta) = \mathbb{P}\{\tau \in [-\delta, \delta] \mid \tau \sim \mathcal{N}(0, 1/\sqrt{\mathcal{F}})\} \quad (83)$$

that we can invert to obtain $\delta_{\text{CRB}}(\eta)$. While this is arguably the most sensible way of adding a success probability dependence to the Cramér-Rao bound, it also clearly shows the difference between the asymptotic and finite-sample regimes where other moments of the distribution cannot be neglected. As the domain of parameter t in phase estimation is equivalent to a circle, we make our comparison using a von Mises distribution [116] (a circular analogue of the normal distribution) with concentration parameter $\kappa = \mathcal{F}$.

In Fig. 8, we plot the achievable tolerance for a subset of the states presented above together with the interpolated quantum Cramér-Rao bound. We observe that, for the IID state and the optimal state, the Cramér-Rao bound faithfully predicts the achievable minimax tolerance after $n \approx 10$ repetitions. This is, however, not the case for all states. The achievable tolerance for the GHZ state essentially amounts to that associated with random guessing because of the global nature of the problem, while it achieves the maximum possible quantum Fisher information.

More interesting is the behavior for the Holland-Burnett state. We see that, for the moderate success probability $\bar{\eta} = 0.9$, the performance is practically on par with the optimal state and close to what the Cramér-Rao bound would predict. For a larger target success probability of $\bar{\eta} = 0.999$, however, a substantial gap opens where the Holland-Burnett state performs as bad as the GHZ state before it enters the Heisenberg scaling regime. As we observe from the right-hand side of Fig. 8, this gap cannot be explained by the Cramér-Rao bound that still predicts a near-optimal performance. This behavior starkly underscores the fact that the Cramér-Rao bound—and with it the quantum Fisher information—cannot serve as a figure of merit to judge the performance of metrological protocols in the finite-sample regime.

While the quantum Fisher information should not be used as a figure of merit to optimize probe states, our numerical results suggest that the *optimal* achievable quantum Fisher information over all possible probe states is still a good proxy for the *optimal* achievable tolerance over all possible probe states. Care has to be taken, however, because the states that achieve the optimal achievable tolerance and the optimal achievable quantum Fisher information are very different. This observation is in agreement with the results of Refs. [117,118], which showed that the expected ratio between the square root of the optimal achievable quantum Fisher information and the achievable root-mean-squared error in a Bayesian [118] or minimax [117] sense is π . The ratios between the optimal minimax tolerance and the optimal Cramér-Rao bound in our numerical experiments were a bit smaller than π , but increase with $\bar{\eta}$. We suspect that the value π is approached in the limit $\bar{\eta} \rightarrow 1$.

Another way of quantifying the optimal achievable precision in phase estimation is given by the *Bayesian Cramér-Rao bound* [108]. It generalizes the Cramér-Rao bound by replacing the mean-squared error with the sine-squared cost, a periodic surrogate, given as $C(t - \tau) = 4 \sin^2[(t - \tau)/2]$, and then quantifying the best achievable value of this quantity in expectation over a prior distribution of parameter t . This lifts the Cramér-Rao bound to the global estimation setting. We repeat the comparison described above for the Bayesian Cramér-Rao bound in Sec. XI B of the Supplemental Material [28].

We observe that the Bayesian Cramér-Rao bound faithfully predicts that the GHZ state is a poor probe state for global estimation. It does, however, show pathological behavior that we attribute to the choice of the sine-squared cost function: the minimax optimal probe state, as well as the Holland-Burnett state, only achieve standard quantum limited scaling in terms of the Bayesian Cramér-Rao bound; see also Ref. [58] for a detailed discussion. As we believe to have made a compelling case that the estimation tolerance is an appropriate figure of merit to quantify precision in metrology, we attribute this mismatch to a shortcoming of the Bayesian Cramér-Rao bound caused by the choice of the sine-squared cost function. As such, we conclude that the Bayesian Cramér-Rao bound in this form is a questionable figure of merit for quantum metrology.

C. Analytical results

The numerical observations presented in Fig. 6 motivate an analytical study of the asymptotics of the minimax error probability, extending the results of Sec. VI. There, we established results on the asymptotic rate for IID copies of the same state, which corresponds to the case of the tensor power probe discussed above. We observe that, when entangled probe states are allowed—corresponding to the parallel setting—the asymptotic rate is much improved, as

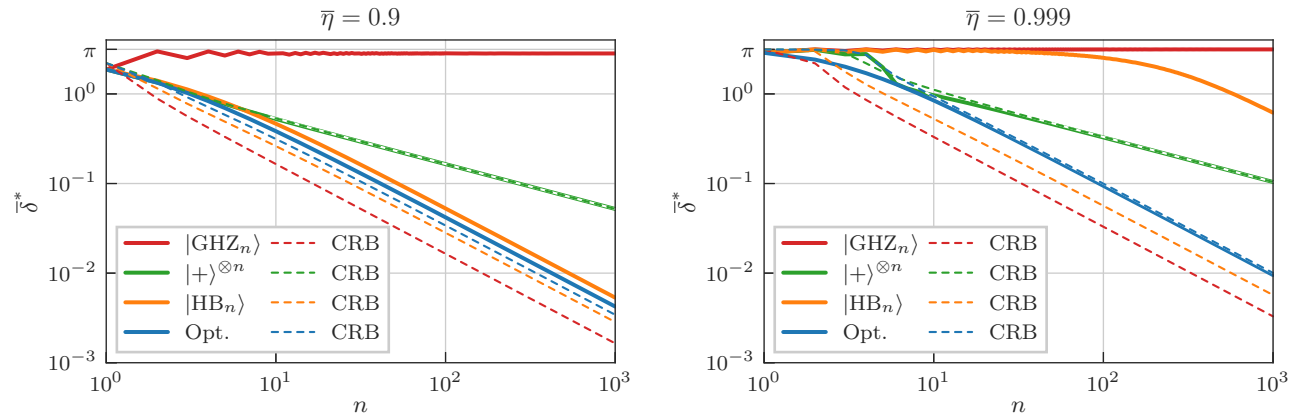


FIG. 8. Optimal tolerance of quantum metrology that can be guaranteed for any prior distribution for different probes in the phase estimation scenario compared to the quantum Cramér-Rao bound interpolated with a von-Mises distribution. We compare a generalized GHZ state [Eq. (78), red], a tensor power of plus states (green), the Holland-Burnett state [Eq. (79), orange], and the optimal state [Eq. (76), blue]. The associated quantum Cramér-Rao bound is shown with dashed lines. As expected, the quantum Cramér-Rao bound predicts the asymptotically achievable precision quite well. The probe states achieving the optimal precision, however, are notably different. The GHZ state optimal for the Cramér-Rao bound performs badly in the finite-sample setting due to the global nature of the estimation problem. The Holland-Burnett state achieves Heisenberg scaling asymptotically, but we observe that a gap between the performance predicted by the CRB and the actual finite-sample performance opens up depending on the desired success probability that underscores the fact that the Cramér-Rao bound alone cannot be used as a figure of merit in the finite-sample regime.

the rates we observe numerically are consistent with

$$\bar{R}_{\text{iid}}^*(\bar{\delta}) \approx \bar{\delta}^2, \quad \bar{R}_{\text{par}}^*(\bar{\delta}) \approx \bar{\delta}, \quad (84)$$

where we recall the definitions of rates associated with different strategies explained at the end of Sec. XI. In the following, we make this observation rigorous.

As we already argued above, the optimal probe for phase estimation on a spin chain is given by the *discrete prolate spheroidal sequence* (DPSS) of zeroth order [114]. We obtain the optimal rate in the parallel case.

Theorem 6 (Optimal minimax rate). For a given minimax tolerance $0 < \bar{\delta} < \pi/2$, the parallel minimax error rate is given by

$$\bar{R}_{\text{par}}^*(\bar{\delta}) = \log \left(\frac{1 + \sin(\bar{\delta}/2)}{1 - \sin(\bar{\delta}/2)} \right) = \bar{\delta} + O(\bar{\delta}^3).$$

The proof combines two results from Ref. [114] and exploits the Perron-Frobenius theorem to establish positivity of the DPSS of zeroth order and is given as Theorem 37 within the Supplemental Material [28]. We note that this result is not implied by a comparable result of Ref. [60] due to the different setup. While the DPSS has no closed form, there exist efficient approximations involving the modified Bessel function of the first kind of zeroth order I_0 [119]. In our case, the optimal probe can thus be approximated by

choosing

$$\psi_\lambda \propto I_0 \left[\frac{\bar{\delta} n}{2} \sqrt{1 - \left(\frac{2\lambda + 1}{n + 1} - 1 \right)^2} \right] \quad (85)$$

and subsequently normalizing.

We can use the result on the rate for IID probes of Theorem 2 to calculate

$$\bar{R}_{\text{iid}}^*(\bar{\delta}) \leq -\log \cos^2(\bar{\delta}) \approx \bar{\delta}^2 \quad (86)$$

for small $\bar{\delta}$; see Theorem 40 within the Supplemental Material [28]. Combined with the Theorem 6 this implies that entangled strategies have a quadratic advantage in the asymptotic minimax rate, which can be understood as the rate analogue of the dichotomy between the standard quantum and Heisenberg limits.

We further perform a theoretical analysis of the optimal standard deviation for the Gaussian probe. We can use tail bound estimates for the Gaussian distribution to prove that the optimal choice of standard deviation [see Eq. (82)] achieves half the optimal rate.

Theorem 7 (Minimax rate for Gaussian probes). For a given minimax tolerance $\bar{\delta} > 0$, the Gaussian probe achieves the minimax error rate of

$$\bar{R}_{\text{Gauss}}(\bar{\delta}) = \frac{\bar{\delta}}{2}. \quad (87)$$

The proof is presented as Theorem 38 within the Supplemental Material [28]. The structure of the proof also

allows us to determine the scaling of the optimal asymptotic tolerance, as detailed in Observation 39 within the Supplemental Material [28].

XIV. EXTENSIONS AND CONNECTIONS TO OTHER FIELDS

This section is dedicated to exploring the various connections and possible generalizations of our definitions and results to other areas of quantum metrology and quantum information theory.

A. Multiparameter quantum metrology

Measuring multiple parameters at the same time [120, 121] creates additional challenges, such as having to reconcile measurements [122, 123] and probe states [53] that are optimal for each parameter, but might be incompatible. It furthermore offers a framework to study networks of quantum sensors [124] and the optimal estimation of functions of multiple parameters [125].

We can capture arbitrary instances of multivariate quantum metrology by replacing the parameter space $\mathbb{R}$ with an arbitrary set $\mathcal{X}$ equipped with a distance measure $d(x, y)$. In this case, the success probability for a given set of states $x \mapsto \rho(x)$, prior distribution $\mu(x)$, and POVM $y \mapsto Q(y)$ is

$$\eta(\delta, \mu, \rho, Q) = \int_{\mathcal{X}} dx \mu(x) \int_{\mathcal{X}} dy w_{\delta}(d(x, y)) \text{Tr}[\rho(x)Q(y)]. \quad (88)$$

The definitions for the tolerance and sample complexity, the corresponding minimax quantities, and the optimal quantities follow analogously as in the previous sections. In Sec. IX of the Supplemental Material [28], we give a generalization of the hypothesis testing bound of Theorem 1 to the multivariate case and use it to derive a multiparameter analogue of Corollary 1.

A particularly well-studied variant of multivariate quantum metrology concerns the task of *state tomography*, in which the parameter space $\mathcal{X}$ is taken to be the quantum state space itself and the distance is, for instance, taken to be the trace distance. In this setting, our approach connects with a series of works in quantum tomography on establishing confidence regions in state space given measurement data [48–51, 126]. For pure states, the optimally achievable error probability was quantified in Ref. [104].

In the same spirit, other tomography tasks, e.g., the tomography of quantum channels and non-Markovian processes in the form of quantum combs, can be considered in the framework presented in Sec. II.

A similar situation is encountered when seeking to estimate some property x , e.g., the expectation value of an observable, of a general unknown state ρ . The fact that

multiple states can share the same property value x prevents us from applying our tools directly, as we assumed that one parameter value corresponds to one state.

This can be resolved by introducing so-called “nuisance” parameters, additional parameters that ensure that each quantum state is associated with a distinct set of parameter values. Taking the space of quantum states as the parameter space introduces the maximal possible number of nuisance parameters. Using the distance function

$$d_x(\rho, \sigma) := |x(\rho) - x(\sigma)|, \quad (89)$$

then allows us to reduce the property estimation problem to a multiparameter estimation problem as described above. A particular instance of this is shadow tomography [127, 128], where the distance between quantum states is evaluated by comparing the expectation values on a given set of observables $\mathcal{O} = \{O_i\}_{i=1}^M$:

$$d_{\mathcal{O}}(\rho, \sigma) := \sup_{O \in \mathcal{O}} |\text{Tr}[O(\rho - \sigma)]|. \quad (90)$$

B. Cryptography and adversarial parameter estimation

There is an emerging subfield of quantum metrology concerned with its intersection with cryptography [129–131]. As an example, we might seek a metrological protocol where the precision with which an eavesdropper might estimate a parameter should be as low as possible. This regime corresponds to a regime of small success probability, $\eta \rightarrow 0$.

We make use of the fact that the success probability and tolerance have a functional relationship that corresponds to an inversion of the success probability seen as a function of the tolerance. This allows us to analyze the limit of small success probability for smooth, i.e., arbitrarily often differentiable, measurements $Q(\tau)$.

Proposition 6. For a given set of states $\rho(t)$, possibly with prior $\mu(t)$, a smooth measurement $Q(\tau)$, and a small success probability η , the Bayesian tolerance is given by

$$\delta(\eta, \mu, \rho, Q) = \frac{\eta}{2} \left(\int dt \mu(t) \text{Tr}[\rho(t)Q(t)] \right)^{-1} + O(\eta^2), \quad (91)$$

whereas the minimax tolerance is given by

$$\bar{\delta}(\eta, \rho, Q) = \frac{\eta}{2} \left(\inf_t \text{Tr}[\rho(t)Q(t)] \right)^{-1} + O(\eta^2). \quad (92)$$

Both statements hold conditioned on the inverted quantity being nonzero.

The proof is shown in Sec. VII of the Supplemental Material [28].

Let us now assume that Alice performs a quantum metrology protocol and obtains a quantum state $\rho_A(t)$ that is communicated to Bob via a quantum channel $\mathcal{N}$, such that he receives the state $\rho_B(t) = \mathcal{N}[\rho_A(t)]$. At the same time, an eavesdropper Eve tries to obtain as much information as possible about the transmitted state. The state Eve can obtain in the worst case is modeled by the *complementary channel* $\mathcal{N}_c$ [132], which describes the information the environment can obtain when viewing $\mathcal{N}$ as part of a larger, unitary evolution. As such, we assume that Eve holds the state $\rho_E(t) = \mathcal{N}_c[\rho_A(t)]$. The above proposition tells us that, if we want to limit the precision with which Eve can estimate parameter t from state $\rho_E(t)$, we need to choose the initial state of Alice such that the quantity

$$\begin{aligned} & \sup_{Q(t)} \int dt \mu(t) \text{Tr}[\rho_E(t)Q(t)] \\ &= \sup_{Q(t)} \int dt \mu(t) \text{Tr}\{\rho_A(t)\mathcal{N}_c^\dagger[Q(t)]\} \end{aligned} \quad (93)$$

is as small as possible.

C. Optimization over constrained sets of measurements

In the formulation of the success probability as a convex problem of Proposition 1, we optimize over all possible quantum measurements of the system. We already obtained the optimal postprocessing for the practically important case of a fixed measurement. Another possible way of including practical constraints is to only optimize over POVMs from a set $\mathcal{Q}$ that represents the implementable set of measurements.

In this case, we obtain a restricted optimal success probability $\eta_{\mathcal{Q}}^*(\delta, \mu, \rho)$ quantified as

$$\begin{aligned} & \eta_{\mathcal{Q}}^*(\delta, \mu, \rho) \\ &:= \max \left\{ \int dt \text{Tr}[(w_\delta * \{\mu \cdot \rho\})(t)Q(t)] \mid Q \in \mathcal{Q} \right\}. \end{aligned} \quad (94)$$

The minimax variant, as well as the corresponding tolerance and sample complexity are then defined analogously to their nonrestricted counterparts. A particular case of interest appears when $\mathcal{Q}$ is a convex set, turning the computation into a convex optimization problem. Such a situation occurs, for instance, if we assume that Alice prepares a state $\rho(t)$ and sends it through a noisy channel $\mathcal{E}$ to Bob, who attempts to estimate t using any possible POVM on his system. Bob's optimal POVM $Q(t)$ can be mapped to the POVM $\mathcal{E}^\dagger(Q(t))$ on Alice's system. Consequently, Bob's optimization over any POVM can be equivalently expressed as Alice optimizing over all POVMs in the image of $\mathcal{E}^\dagger$. The set $\mathcal{Q} = \{\mathcal{E}^\dagger[Q] \mid \int dt Q(t) = \mathbb{I}, Q(t)$

$\geq 0\}$ is a convex set and Bob's optimal success probability can be computed efficiently.

XV. FUTURE DIRECTIONS

In the standard approach to quantum metrology built on top of the quantum Cramér-Rao bound, many fundamental problems have already been resolved. In the single-shot setting, on the contrary, a large number of problems, both fundamental and practical in nature, are still unsolved. In this section, we highlight some problems of particular interest. We also note that, for most of these problems, classical counterparts also exist, extending to the task of uncovering what features of finite-sample metrology are purely quantum.

Open problem 1: developing measurement schemes with performance guarantees. Arguably the most important open problem is the development of metrology protocols that have good guarantees on their performance in the finite-sample regime. The development of such protocols is necessary to judge in what regimes the bounds we developed in this work are tight and would also guide further endeavors to improve them. First steps in this direction have already been performed [20,58,104,118], but much work still remains to be done, especially in the limit of very small sample size.

The formulation of the success probability as a convex problem provides an efficient way to compute the optimal POVM associated with a discretized version of the estimation problem. These solutions, however, are not likely to provide additional insight on measurement schemes that are perhaps suboptimal, but far more convenient to implement than the optimal measurement.

Natural candidates for practical measurement schemes are adaptive protocols [55,59], while on the theoretical side the pretty good measurement that constitutes the optimal measurement in the pure and covariant settings of Theorem 5 could give generic achievability statements. Promising candidates might furthermore be constructed using the class of measurements studied in Sec. V in conjunction with a particularly suitable basis for the fixed measurement.

Open problem 2: improving the finite-sample Cramér-Rao bound. The quantum Cramér-Rao bound (1) is the fundamental cornerstone of the standard approach to quantum metrology. Beyond its simplicity, much of its appeal comes from the fact that there exists a measurement that attains the bound. The situation is more complicated when considering multiple parameters [62,121] or limitations on our prior knowledge about the unknown parameter [117] or the allowed measurement strategies. Nevertheless, it is undisputed that in the asymptotic realm, the quantum Cramér-Rao bound, and with it the quantum Fisher information, carries detailed information about the attainable precision. Understanding how quickly the convergence to

the asymptotic regime happens is also of critical importance. We believe that a look into the classical literature might offer some guidance [133].

As the estimation tolerance of Definition 2 fulfills a comparable role to the standard deviation, establishing *achievable* fundamental lower bounds is of key importance. We already argued in the Introduction that the Cramér-Rao bound still constrains the precision in the finite-sample regime. Our finite-sample Cramér-Rao bound of Theorem 4 is an important first step to rigorously quantify this fact. It is, however, still an open problem as to whether this bound is achievable. As the asymptotic nature of the Cramér-Rao bound eats away many complications, we expect that the exact way the achievable fundamental limits of the estimation tolerance are quantified is much more subtle. As such, continuing to develop finite-sample Cramér-Rao bounds is of paramount importance to our understanding of the finite-sample regime and the properties of the underlying set of states that influence the achievable tolerance.

It would not be surprising if a different geometric quantity than the quantum Fisher information would appear in such a bound. This is because the converse on hypothesis testing involving the fidelity of quantum states we use to derive Theorem 4 is known to not be tight in certain settings and that a proper converse should rely on the Chernoff divergence, which is in turn associated with a different geometrical quantity, namely, the Wigner-Yanase-Dyson information.

The achievability of the Cramér-Rao bound can also be understood as the statement that there exists a measurement such that the deviation of our estimate, $\hat{t} - t$, follows a distribution with variance given by the inverse quantum Fisher information. This yields a full characterization of the estimation performance, involving a way to assign success probabilities, if the distribution is completely specified by its variance—which means that it is a normal distribution. Because of the central limit theorem, this is exactly what happens in the IID limit. Conversely, understanding the higher moments and cumulants of the distribution of estimates should give us plenty of insight into the finite-sample regime.

As the quantum Fisher information quantifies the behavior of the fidelity $F(\rho(t), \rho(t + \tau))$ for small perturbations τ , we expect that other ways to quantify the distinguishability of states along the trajectory $\rho(t)$ that capture regimes beyond small perturbations should be useful in that regard. Higher-order derivatives of the curve $\rho(t)$ could serve as a starting point for this, which makes quantitatively connecting them to finite-sample quantities like the success probability and the estimation tolerance a crucial direction of research.

Another very important problem in the same spirit is to tackle the Bayesian case and see how our approach can be connected to the Bayesian Cramér-Rao bound [108]

on a more fundamental level and how the choice of cost function therein influences our conclusions.

Open problem 3: establishing the advantages from entanglement and adaptivity. Delineating the boundary of performance between different classes of protocols using entangled and nonentangled probe states, adaptive and nonadaptive processing, and coherent and incoherent measurements is one of the principal research directions in quantum metrology. Because of the more subtle nature of the finite-sample regime, we expect a richer theory to emerge in this regard. In particular, it is unclear whether optimal measurements in the finite-sample regime require the use of large amounts of coherence. When estimating a single parameter in the many-sample regime, the Heisenberg scaling can be achieved with the use of only entangled probe states and without the use of coherent measurements [134]; does a similar statement also hold in the nonasymptotic case? Also, can entanglement be replaced with adaptivity, as, e.g., has been done in Ref. [58]?

Similarly, the impact adaptivity can have in the finite-sample regime needs to be much better understood. While there is no scaling advantage asymptotically [73], we suspect that adaptive strategies could yield significant improvements in the finite-sample regime. Also, they could yield more practicable protocols. In the asymptotic setting, a single qubit memory is sufficient to estimate a quantum phase with Heisenberg scaling under adaptive access [58,59].

In our investigations of the phase estimation example, we observe an exact quadratic relation between the asymptotic rates for IID states and entangled states for small δ . This echoes the quadratic relation between the standard quantum and the Heisenberg limits in the standard approach to quantum metrology. Understanding under what conditions the relation

$$\lim_{\delta \rightarrow 0} \frac{[R_{\text{par}}^*(\delta, \mu, \mathcal{N})]^2}{R_{\text{iid}}^*(\delta, \mu, \mathcal{N})} = 1 \quad (95)$$

holds—especially for channels where Heisenberg-limited scaling is impossible [135]—would further deepen our understanding of the asymptotic rate for entangled inputs.

Open problem 4: establishing the effect of noise on estimation performance. While noise can be implicitly treated in our formalism by including it in the construction of the parametrized set of states $\rho(t)$ or channels $\mathcal{N}(t)$, a much deeper understanding of the influence of noise is desirable, especially with an eye towards practical applications. It would be instructive to see how much the influence of noise destroys advantages of entanglement in our setting [58,111,135,136]. As outlined in Sec. XIV, the minimax success probability also pertains to communication tasks, where the study of noisy channels is of utmost importance. Can we therefore relate the decrease in success probability

(or the increase in achievable tolerance) to properties of the noise channel?

A deeper understanding of noise could then be used to study the application of quantum error correction to finite-sample quantum metrology [131,137]. As the existing approaches to metrological error correction are intimately tied to the local picture of quantum metrology, we also expect a richer theory to emerge in the finite-sample case.

Open problem 5: understanding incompatibility in finite-sample multiparameter quantum metrology. As discussed in Sec. XIV A, our definitions extend in a natural way to multiparameter quantum metrology [138,139]. In this setting, challenges arise that are not present in the single-parameter setting, such as the incompatibility of measurements and probes that are optimal for different parameters [53,122]. These phenomena, particular to multiparameter quantum metrology, have been extensively studied in the asymptotic context. To understand incompatibility in the finite-sample regime, we first have to clarify *how* it would be best quantified using the framework of Sec. II. Any result in that direction will surely deepen the understanding of these effects and their severity in the finite-sample regime.

Open problem 6: quantifying generalized communication tasks. The minimax setting of quantum metrology has a further interesting interpretation, as it directly quantifies how well a sender Alice can communicate a scalar parameter $t \in \mathbb{R}$ to a receiver Bob when she encodes the parameter in a set of states $\rho(t)$. It is important to note here that the worst-case nature of the minimax setting is critical in quantifying the performance, as Alice wants to be able to communicate any value t with similar guarantees. In this sense, the minimax setting of quantum metrology quantifies a generalization of the commonly encountered task of communicating one of multiple discrete symbols [36]. In practical scenarios, Alice will usually send the state through a quantum channel $\mathcal{N}$ that degrades the message, giving an additional impetus to study quantum metrology with the noise model $\rho(t) \mapsto \mathcal{N}[\rho(t)]$ [131].

Open problem 7: resolving information-theoretic questions. In Sec. VII, we gave an upper bound on the asymptotic rate of quantum metrology, which is tight for commuting states. It is an open problem to determine under which general conditions on the set of states this bound is tight. Naturally, it is also interesting to quantify the second-order asymptotics of the success probability. Presumably, this would first require progress on the second-order asymptotics of the symmetric hypothesis testing error probability for quantum state discrimination in the general case. For pure quantum state tomography, explicit expressions are known and could serve as guidance for further improvement [104]. Another quite natural extension is the question of *mixed asymptotics*: if we fix a desired scaling of the tolerance, decreasing slower than the optimal

scaling, how fast can the success probability still attain unity?

As already outlined in Sec. XI, the setting of parametrized quantum channels offers much richer structure in its asymptotics, owing to the possibility of different strategies (separable use, parallel use, adaptive use, and indefinite causal order). Given that Corollary 7 explicitly relates the success probability of quantum metrology under different access modes with the success probability of a corresponding hypothesis testing problem with similar access modes, we expect that the asymptotics of quantum metrology with different kinds of strategies should relate to the asymptotics of the corresponding multihypothesis testing tasks.

XVI. CONCLUSION

Finite resources constrain the performance of quantum metrology. This is especially true in today's quantum sensors, where both the number of experimental repetitions that can be performed as well as the number of controllable systems are limited. Metrology with present-day hardware platforms thus happens in the *finite-sample regime*. Despite its fundamental importance, the dominant approach to quantum metrology, based on the Cramér-Rao bound, is inadequate when it comes to reliably characterizing metrology in the finite-sample regime. The reason for this is that the quantum Fisher information only characterizes the variance of the distribution of estimates, which can degrade the performance significantly if the estimates do not follow a Gaussian distribution (as they would in the asymptotic limit). Hence, optimizing metrology protocols for large Fisher information can lead to protocols with suboptimal performance in the finite-sample regime.

To get the most out of present-day hardware and to build the most reliable quantum sensors, we need a theory of quantum metrology that is reliable in the finite-sample regime. In our work, we show that we can build a complete theory of quantum metrology by shifting our perspective away from the variance (equivalently, the mean-squared error) as a figure of merit. Instead, we employ the probability that the estimate of the parameter lies within a fixed estimation tolerance of the true value. This quantity tells us something operationally relevant about the performance of the metrology protocol, even if only a single repetition is performed.

When considering the Cramér-Rao bound, the setting of local estimation emerges naturally. The estimated value will become increasingly accurate when employing more samples in the asymptotic limit, justifying the assumption that the value of the parameter is essentially known. This contrasts with the finite-sample regime. When the parameter is already known to lie within a small interval, we typically require a large number of samples to refine this knowledge to a more accurate estimate of the parameter.

If a finite number of samples are available, the limited accuracy to which the parameter can be estimated from those samples might be insufficient to provide any further meaningful information about the parameter. Therefore, parameter estimation in the finite-sample regime is more naturally associated with the global estimation setting. We adopt two ways of quantifying the fact that we do not have precise knowledge about the parameter value: one where we evaluate the worst case over the possible values of the parameter (minimax), and another where we evaluate the success probability relative to a prior distribution (Bayesian).

We use the success probability to establish a triad of figures of merit for finite-sample metrology. Next to the success probability, which quantifies the precision at fixed tolerance, we have the estimation tolerance, which quantifies the minimal tolerance at fixed success probability. The triad is completed by the sample complexity, which measures how many repetitions we have to perform to achieve guarantees on both the success probability and the estimation tolerance. These three quantities form a general and self-consistent framework for quantum metrology in the finite-sample regime.

The suitability of our framework for finite-sample metrology depends crucially on the fact that we can optimize metrology protocols for our figure of merit. We show that optimizing the success probability is a well-posed problem that can be solved through semi-infinite programming, a continuous generalization of semidefinite programming, even when the parameter is encoded in a parametrized quantum channel and adaptive protocols are considered.

The Cramér-Rao bound quantifies the fundamental limits to the achievable mean-squared error of a metrology protocol. The core contribution of our work is to similarly establish the fundamental limits of our triad of figures of merit. We show that the success probability of a metrology protocol is limited by the distinguishability of the quantum states associated with the different values of the parameter, more precisely by the success probability of a quantum multihypothesis testing problem between states spaced proportional to the tolerance. Our results identify a new bridge between the information-theoretic field of quantum hypothesis testing, in which statements are often of strong mathematical nature, and the significantly more practical setting of finite-sample quantum metrology. Building on this result, we establish a lower bound on the optimal estimation tolerance that takes the form of a finite-sample Cramér-Rao bound, again highlighting the fact that large quantum Fisher information is indeed a necessary condition for good finite-sample performance. Similarly, we establish bounds on the sample complexity that show that appropriate scaling of the quantum Fisher information is necessary to achieve Heisenberg-limited scaling asymptotically.

To illustrate our results, we analyze one of the most prototypical problems in quantum metrology: the estimation of an unknown phase shift imprinted on an ensemble of spin- $\frac{1}{2}$ particles through a Hamiltonian evolution. We make use of the semi-infinite programming formulation to establish the measurement achieving the optimal success probability. In doing so, we give a closed-form formula for the optimal success probability and use it to compare different families of probe states for phase estimation.

This comparison shines light on the intricacies of the finite-sample regime and shows that optimizing metrology protocols for large quantum Fisher information can lead to poor performance in the finite-sample regime, a finding consistent with previous works [57,117]. We observe that the optimal probe states have a distinctly different form from the probe states considered in the literature. Our numerics further exhibit clearly that the quantum Fisher information is not indicative of the finite-sample performance of a probe state. The Holland-Burnett state serves as a good example: while it has near-optimal Fisher information and achieves Heisenberg scaling asymptotically, the performance in the finite-sample regime degrades significantly compared to the optimally achievable one when the desired success probability is increased.

Our work reinforces the value of a change of perspective towards an operationally motivated figure of merit to study the practically important finite-sample regime in quantum metrology. We think that the theory of quantum metrology advanced in this work will provide a fertile ground for further investigation. While many fundamental questions are already answered in the traditional approach to quantum metrology, we observe the opposite picture in the finite-sample regime, as we detailed in the preceding section. It offers a plethora of intriguing questions, both about a better understanding of the fundamental connection to hypothesis testing, as well as to establishing new practical metrology protocols with superior performance in the finite-sample regime.

ACKNOWLEDGMENTS

The authors would like to thank Sergii Strelchuk, Wilfred Salmon, Francesco Albarelli, Jasminder Sidhu, and Maximilian Reichert for valuable feedback on earlier versions of this manuscript. This work has been supported by the DFG (CRC 183, FOR 2724), the BMBF (Hybrid, Hybrid++, DAQC), the BMWK (PlanQK, EniQmA), BMFTR (Hybrid++, QuSol), the Munich Quantum Valley (K-8), QuantERA (HQCC), Berlin Quantum, the Quantum Flagship (Millenion, PasQuans2), the European Research Council (DebuQC), and the Einstein Foundation (Einstein Research Unit on Quantum Devices). This work has also been funded by the DFG under Germany's Excellence Strategy – The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID: 390685689).

D.S.F. is supported by France 2030 under the French National Research Agency Award No. ANR-22-PNCQ-0002 and by the Novo Nordisk Foundation (Grant No. NNF20OC0059939, Quantum for Life).

J.J.M. led the project, derived a significant majority of the technical results, and wrote the manuscript draft. All authors contributed substantially to the technical results and to writing the paper.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

-
- [1] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, *Nat. Photonics* **5**, 222 (2011).
 - [2] M. G. A. Paris, Quantum estimation for quantum technology, *Int. J. Quantum Inf.* **07**, 125 (2011).
 - [3] C. L. Degen, F. Reinhard, and P. Cappellaro, Quantum sensing, *Rev. Mod. Phys.* **89**, 035002 (2017).
 - [4] L. Pezzè, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, Quantum metrology with nonclassical states of atomic ensembles, *Rev. Mod. Phys.* **90**, 035005 (2018).
 - [5] S. L. Braunstein, and C. M. Caves, Statistical distance and the geometry of quantum states, *Phys. Rev. Lett.* **72**, 3439 (1994).
 - [6] C. E. Bradley, J. Randall, M. H. Abobeih, R. C. Berrevoets, M. J. Degen, M. A. Bakker, M. Markham, D. J. Twitchen, and T. H. Taminiau, A ten-qubit solid-state spin register with quantum memory up to one minute, *Phys. Rev. X* **9**, 031045 (2019).
 - [7] M. H. Abobeih, J. Randall, C. E. Bradley, H. P. Bartling, M. A. Bakker, M. J. Degen, M. Markham, D. J. Twitchen, and T. H. Taminiau, Atomic-scale imaging of a 27-nuclear-spin cluster using a quantum sensor, *Nature* **576**, 411 (2019).
 - [8] A. Omran, M. Boll, T. A. Hilker, K. Kleinlein, G. Salomon, I. Bloch, and C. Gross, Microscopic observation of Pauli blocking in degenerate fermionic lattice gases, *Phys. Rev. Lett.* **115**, 263001 (2015).
 - [9] J. Franke, S. R. Muleady, R. Kaubruegger, F. Kranzl, R. Blatt, A. M. Rey, M. K. Joshi, and C. F. Roos, Quantum-enhanced sensing on optical transitions through finite-range interactions, *Nature* **621**, 740 (2023).
 - [10] D. Bluvstein *et al.*, Logical quantum processor based on reconfigurable atom arrays, *Nature* **626**, 58 (2024).
 - [11] T. Proctor, M. Reville, E. Nielsen, K. Rudinger, D. Lobser, P. Maunz, R. Blume-Kohout, and K. Young, Detecting and tracking drift in quantum information processors, *Nat. Commun.* **11**, 5396 (2020).
 - [12] S. Majumder, L. Andreta de Castro, and K. R. Brown, Real-time calibration with spectator qubits, *Npj Quantum Inf.* **6**, 1 (2020).
 - [13] K. Singh, C. E. Bradley, S. Anand, V. Ramesh, R. White, and H. Bernien, Mid-circuit correction of correlated phase errors using an array of spectator qubits, *Science* **380**, 1265 (2023).
 - [14] J. S. Sidhu, and P. Kok, Geometric perspective on quantum parameter estimation, *AVS Quantum Sci.* **2**, 014701 (2020).
 - [15] J. J. Meyer, Fisher information in noisy intermediate-scale quantum applications, *Quantum* **5**, 539 (2021).
 - [16] C. W. Helstrom, Minimum mean-squared error of estimates in quantum statistics, *Phys. Lett. A* **25**, 101 (1967).
 - [17] H. Cramér, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, 1946).
 - [18] C. R. Rao, Information and the accuracy attainable in the estimation of statistical parameters, *Bull. Calcutta Math. Soc.* **37**, 81 (1945).
 - [19] A. S. Holevo, Covariant measurements and uncertainty relations, *Rep. Math. Phys.* **16**, 385 (1979).
 - [20] M. Hayashi, Two quantum analogues of Fisher information from a large deviation viewpoint of quantum estimation, *J. Phys. A* **35**, 7689 (2002).
 - [21] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (Springer Science & Business Media, Pisa, 2011).
 - [22] T. Sugiyama, Precision-guaranteed quantum metrology, *Phys. Rev. A* **91**, 042126 (2015).
 - [23] D. W. Berry, M. Tsang, M. J. W. Hall, and H. M. Wiseman, Quantum Bell-Ziv-Zakai bounds and Heisenberg limits for waveform estimation, *Phys. Rev. X* **5**, 031018 (2015).
 - [24] M. Walter, and J. M. Renes, Lower bounds for quantum parameter estimation, *IEEE Trans. Inf. Theory* **60**, 8007 (2014).
 - [25] Y. Yang, G. Chiribella, and M. Hayashi, Quantum stopwatch: How to store time in a quantum memory, *Proc. Roy. Soc. A* **474**, 20170773 (2018).
 - [26] C. Mukhopadhyay, V. Montenegro, and A. Bayat, Current trends in global quantum metrology, *J. Phys. A: Math. Theor.* **58**, 063001 (2025).
 - [27] L. G. Valiant, A theory of the learnable, *Commun. ACM* **27**, 1134 (1984).
 - [28] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/qbn1-p6bq> for more details, omitted proofs, and additional figures.
 - [29] R. Hettich, and K. O. Kortanek, Semi-infinite programming: Theory, methods, and applications, *SIAM Rev.* **35**, 380 (1993).
 - [30] L. Vandenbergh, and S. Boyd, in *Semi-infinite programming*, edited by R. Reemtsen and J.-J. Rückmann (Springer US, Boston, MA, 1998), p. 277.
 - [31] L. Vandenbergh, and S. Boyd, Semidefinite programming, *SIAM Rev.* **38**, 49 (1996).
 - [32] S. P. Boyd, and L. Vandenbergh, *Convex Optimization* (Cambridge University Press, Cambridge, UK, 2004).
 - [33] R. Renner, Security of quantum key distribution, *Int. J. Quantum Inf.* **06**, 1 (2008).
 - [34] G. Bowen, and N. Datta, in *2006 IEEE International Symposium on Information Theory* (2006), pp. 451–455.
 - [35] M. Tomamichel, *Quantum Information Processing with Finite Resources* (Springer Cham, Cham, 2016).

- [36] S. Khatri, and M. M. Wilde, Principles of quantum communication theory: A modern approach, [ArXiv:2011.04672](#).
- [37] R. R. Bahadur, Rates of convergence of estimates and test statistics, [Ann. Math. Stat. 38, 303 \(1967\)](#).
- [38] R. R. Bahadur, and J. C. Gupta, Distribution optimality and second-order efficiency of test procedures, [Lect. Notes-Monograph Ser. 8, 315 \(1986\)](#).
- [39] R. R. Bahadur, in *Some Limit Theorems in Statistics* (Society for Industrial and Applied Mathematics, Philadelphia, 1971), p. 1.
- [40] V. Spokoiny, Parametric estimation. Finite sample theory, [Ann. Stat. 40, 2877 \(2012\)](#).
- [41] T. Sugiyama, P. S. Turner, and M. Murao, Error probability analysis in quantum tomography: A tool for evaluating experiments, [Phys. Rev. A 83, 012105 \(2011\)](#).
- [42] T. Sugiyama, *Finite Sample Analysis in Quantum Estimation*, Springer Theses (Springer Tokyo, Tokyo, 2014).
- [43] M. Hayashi, and S. Watanabe, Finite-length analyses for source and channel coding on Markov chains, [Entropy 22, 460 \(2020\)](#).
- [44] Y. Yang, G. Chiribella, and M. Hayashi, Attaining the ultimate precision limit in quantum state estimation, [Commun. Math. Phys. 368, 223 \(2019\)](#).
- [45] T. Sugiyama, P. S. Turner, and M. Murao, Precision-guaranteed quantum tomography, [Phys. Rev. Lett. 111, 160406 \(2013\)](#).
- [46] M. Walter, and J. M. Renes, in *2014 IEEE Int. Symp. Inf. Th. (IEEE, Honolulu, 2014)*, p. 1126.
- [47] N. Kura, and M. Ueda, Finite-error metrological bounds on multiparameter Hamiltonian estimation, [Phys. Rev. A 97, 012101 \(2018\)](#).
- [48] R. Blume-Kohout, Robust error bars for quantum tomography, [ArXiv:1202.5270](#).
- [49] M. Christandl, and R. Renner, Reliable quantum state tomography, [Phys. Rev. Lett. 109, 120403 \(2012\)](#).
- [50] P. Faist, and R. Renner, Practical and reliable error bars in quantum tomography, [Phys. Rev. Lett. 117, 010404 \(2016\)](#).
- [51] J. Wang, V. B. Scholz, and R. Renner, Confidence polytopes in quantum state tomography, [Phys. Rev. Lett. 122, 190401 \(2019\)](#).
- [52] M. Tsang, Ziv-Zakai error bounds for quantum parameter estimation, [Phys. Rev. Lett. 108, 230401 \(2012\)](#).
- [53] F. Albarelli, and R. Demkowicz-Dobrzanski, Probe incompatibility in multiparameter noisy quantum metrology, [Phys. Rev. X 12, 011039 \(2022\)](#).
- [54] V. Bužek, R. Derka, and S. Massar, Optimal quantum clocks, [Phys. Rev. Lett. 82, 2207 \(1999\)](#).
- [55] D. W. Berry, and H. M. Wiseman, Optimal states and almost optimal adaptive measurements for quantum interferometry, [Phys. Rev. Lett. 85, 5098 \(2000\)](#).
- [56] G. Chiribella, G. M. D'Ariano, P. Perinotti, and M. F. Sacchi, Covariant quantum measurements that maximize the likelihood, [Phys. Rev. A 70, 062105 \(2004\)](#).
- [57] M. Hayashi, Parallel treatment of estimation of SU(2) and phase estimation, [Phys. Lett. A 354, 183 \(2006\)](#).
- [58] M. Hayashi, Z.-W. Liu, and H. Yuan, Global Heisenberg scaling in noisy and practical phase estimation, [Quantum Sci. Technol. 7, 025030 \(2022\)](#).
- [59] R. Cleve, A. Ekert, C. Macchiavello, and M. Mosca, Quantum algorithms revisited, [Proc. R. Soc. Lond. A 454, 339 \(1998\)](#).
- [60] H. Imai, and M. Hayashi, Fourier analytic approach to phase estimation in quantum systems, [New J. Phys. 11, 043034 \(2009\)](#).
- [61] M. Hayashi, Fourier analytic approach to quantum estimation of group action, [Commun. Math. Phys. 347, 3 \(2016\)](#).
- [62] R. Demkowicz-Dobrzanski, W. Gorecki, and M. Guta, Multi-parameter estimation beyond quantum Fisher information, [J. Phys. A 53, 363001 \(2020\)](#).
- [63] W. Salmon, S. Strelchuk, and D. Arvidsson-Shukur, Only classical parameterised states have optimal measurements under least squares loss, [Quantum 7, 998 \(2023\)](#).
- [64] I. F. D. Oliveira, and R. H. C. Takahashi, An enhancement of the bisection method average performance preserving minmax optimality, [ACM Trans. Math. Softw. 47, 1 \(2020\)](#).
- [65] W. Rządowski, and R. Demkowicz-Dobrzański, Discrete-to-continuous transition in quantum phase estimation, [Phys. Rev. A 96, 032319 \(2017\)](#).
- [66] This is the definition of *symmetric* hypothesis testing, where all states are treated equally and the average success probability is used as the quantifier. In *asymmetric* hypothesis testing, the states are treated independently.
- [67] H. Yuen, R. Kennedy, and M. Lax, Optimum testing of multiple hypotheses in quantum detection theory, [IEEE Trans. Inf. Theory 21, 125 \(1975\)](#).
- [68] C. W. Helstrom, Quantum detection and estimation theory, [J. Stat. Phys. 1, 231 \(1969\)](#).
- [69] B. Yu, in *Festschrift for Lucien Le Cam: Research papers in probability and statistics*, edited by D. Pollard, E. Torgersen, and G. L. Yang (Springer, New York, 1997), p. 423.
- [70] K. M. R. Audenaert, M. Nussbaum, A. Szkoła, and F. Verstraete, Asymptotic error rates in quantum hypothesis testing, [Commun. Math. Phys. 279, 251 \(2008\)](#).
- [71] K. Li, Discriminating quantum states: The multiple Chernoff distance, [Ann. Stat. 44, 1661 \(2016\)](#).
- [72] D. L. Wallace, Asymptotic approximations to distributions, [Ann. Math. Stat. 29, 635 \(1958\)](#).
- [73] S. Kurdziałek, W. Górecki, F. Albarelli, and R. Demkowicz-Dobrzański, Using adaptiveness and causal superpositions against noise in quantum metrology, [Phys. Rev. Lett. 131, 090801 \(2023\)](#).
- [74] R. König, R. Renner, and C. Schaffner, The operational meaning of min- and max-entropy, [IEEE Trans. Inf. Theory 55, 4337 \(2009\)](#).
- [75] N. Datta, Min- and max-relative entropies and a new entanglement monotone, [IEEE Trans. Inf. Theory 55, 2816 \(2009\)](#).
- [76] K. M. R. Audenaert, and M. Mosonyi, Upper bounds on the error probabilities and asymptotic error exponents in quantum multiple state discrimination, [J. Math. Phys. 55, 102201 \(2014\)](#).
- [77] F. Dupuis, M. Berta, J. Wullschleger, and R. Renner, One-shot decoupling, [Commun. Math. Phys. 328, 251 \(2014\)](#).
- [78] R. Duan, and A. Winter, No-signalling-assisted zero-error capacity of quantum channels and an information theoretic

- interpretation of the Lovász number, *IEEE Trans. Inf. Theory* **62**, 891 (2016).
- [79] K. Fang, X. Wang, M. Tomamichel, and M. Berta, Quantum channel simulation and the channel's smooth max-information, *IEEE Trans. Inf. Theory* **66**, 2129 (2020).
- [80] F. Hiai, and D. Petz, The proper formula for relative entropy and its asymptotics in quantum probability, *Commun. Math. Phys.* **143**, 99 (1991).
- [81] F. Dupuis, L. Kraemer, P. Faist, J. M. Renes, and R. Renner, in *XVIIth International Congress on Mathematical Physics* (World Scientific, Singapore, 2013), p. 134.
- [82] N. Datta, M. Mosonyi, M.-H. Hsieh, and F. G. S. L. Brandão, A smooth entropy approach to quantum hypothesis testing and the classical capacity of quantum channels, *IEEE Trans. Inf. Theory* **59**, 8014 (2013).
- [83] L. Wang, and R. Renner, One-shot classical-quantum capacity and hypothesis testing, *Phys. Rev. Lett.* **108**, 200501 (2012).
- [84] M. Tomamichel, and M. Hayashi, A hierarchy of information quantities for finite block length analysis of quantum tasks, *IEEE Trans. Inf. Theory* **59**, 7693 (2013).
- [85] G. Gutoski, and J. Watrous, in *Proceedings of the Thirty-Ninth Annual ACM Symposium on Theory of Computing, STOC '07* (Association for Computing Machinery, New York, NY, USA, 2007), p. 565.
- [86] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Theoretical framework for quantum networks, *Phys. Rev. A* **80**, 022339 (2009).
- [87] G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, Quantum computations without definite causal structure, *Phys. Rev. A* **88**, 022318 (2013).
- [88] N. Paunkovic, and M. Vojinovic, Causal orders, quantum circuits and spacetime: Distinguishing between definite and superposed causal orders, *Quantum* **4**, 275 (2020).
- [89] N. Ormrod, A. Vanrietvelde, and J. Barrett, Causal structure in the presence of sectorial constraints, with application to the quantum switch, *Quantum* **7**, 1028 (2023).
- [90] V. Vilasini, and R. Renner, Embedding cyclic information-theoretic structures in acyclic space-times: No-go results for indefinite causality, *Phys. Rev. A* **110**, 022227 (2024), publisher: American Physical Society.
- [91] X. Zhao, Y. Yang, and G. Chiribella, Quantum metrology with indefinite causal order, *Phys. Rev. Lett.* **124**, 190503 (2020).
- [92] Q. Liu, Z. Hu, H. Yuan, and Y. Yang, Optimal strategies of quantum metrology with a strict hierarchy, *Phys. Rev. Lett.* **130**, 070803 (2023).
- [93] M. Hayashi, Discrimination of two channels by adaptive methods and its application to quantum system, *IEEE Trans. Inf. Theory* **55**, 3807 (2009).
- [94] M. M. Wilde, M. Berta, C. Hirche, and E. Kaur, Amortized channel divergence for asymptotic quantum channel discrimination, *Lett. Math. Phys.* **110**, 2277 (2020).
- [95] F. Salek, M. Hayashi, and A. Winter, Usefulness of adaptive strategies in asymptotic quantum channel discrimination, *Phys. Rev. A* **105**, 022419 (2022).
- [96] B. Bergh, N. Datta, R. Salzmänn, and M. M. Wilde, Parallelization of adaptive quantum channel discrimination in the non-asymptotic regime, *IEEE Trans. Inf. Theory* **70**, 2617 (2024).
- [97] V. P. Belavkin, Optimal multiple quantum statistical hypothesis testing, *Stochastics* **1**, 315 (1975).
- [98] V. P. Belavkin, Optimum distinction of non-orthogonal quantum signals, *Radiotekh. Elektron.* **20**, 1177 (1975).
- [99] A. S. Holevo, On asymptotically optimal hypothesis testing in quantum statistics, *Theory Probab. Appl.* **23**, 411 (1979).
- [100] V. P. Belavkin, and V. P. Maslov, Design of optimal dynamic analyzers: Mathematical aspects of wave pattern recognition, Advances in Science and Technology in the USSR, *Arxiv:quant-ph/0412031*.
- [101] P. Hausladen, and W. K. Wootters, A 'pretty good' measurement for distinguishing quantum states, *J. Mod. Opt.* **41**, 2385 (1994).
- [102] L. P. Hughston, R. Jozsa, and W. K. Wootters, A complete classification of quantum ensembles having a given density matrix, *Phys. Lett. A* **183**, 14 (1993).
- [103] M. Ban, K. Kurokawa, R. Momose, and O. Hirota, Optimum measurements for discrimination among symmetric quantum states and parameter estimation, *Int. J. Theor. Phys.* **36**, 1269 (1997).
- [104] M. Hayashi, Asymptotic estimation theory for a finite-dimensional pure state model, *J. Phys. A* **31**, 4633 (1998).
- [105] G. Chiribella, and G. M. D'Ariano, Extremal covariant positive operator valued measures, *J. Math. Phys.* **45**, 4435 (2004).
- [106] G. Chiribella, and G. Mauro D'Ariano, Extremal covariant measurements, *J. Math. Phys.* **47**, 092107 (2006).
- [107] K. G. Murty, and S. N. Kabadi, Some NP-complete problems in quadratic and nonlinear programming, *Math. Prog.* **39**, 117 (1987).
- [108] R. Demkowicz-Dobrzański, Optimal phase estimation with arbitrary *a priori* knowledge, *Phys. Rev. A* **83**, 061802 (2011).
- [109] D. J. Wineland, J. J. Bollinger, W. M. Itano, F. L. Moore, and D. J. Heinzen, Spin squeezing and reduced quantum noise in spectroscopy, *Phys. Rev. A* **46**, R6797 (1992).
- [110] D. J. Wineland, J. J. Bollinger, W. M. Itano, and D. J. Heinzen, Squeezed atomic states and projection noise in spectroscopy, *Phys. Rev. A* **50**, 67 (1994).
- [111] S. F. Huelga, C. Macchiavello, T. Pellizzari, A. K. Ekert, M. B. Plenio, and J. I. Cirac, Improvement of frequency standards with quantum entanglement, *Phys. Rev. Lett.* **79**, 3865 (1997).
- [112] M. J. Holland, and K. Burnett, Interferometric detection of optical phase shifts at the Heisenberg limit, *Phys. Rev. Lett.* **71**, 1355 (1993).
- [113] G. Chesi, R. Rubboli, A. Riccardi, L. Maccone, and C. Macchiavello, Protocol for global multiphase estimation, *Phys. Rev. A* **108**, 012613 (2023).
- [114] D. Slepian, Prolate spheroidal wave functions, Fourier analysis, and uncertainty – V: The discrete case, *Bell Syst. Tech. J.* **57**, 1371 (1978).
- [115] D. Slepian, and H. O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty-I, *Bell System Tech. J.* **40**, 43 (1961).
- [116] N. I. Fisher, *Statistical Analysis of Circular Data* (Cambridge University Press, Cambridge, 1995).

- [117] M. Hayashi, Comparison between the Cramer-Rao and the mini-max approaches in quantum channel estimation, *Commun. Math. Phys.* **304**, 689 (2011).
- [118] W. Gorecki, R. Demkowicz-Dobrzanski, H. M. Wiseman, and D. W. Berry, π -corrected Heisenberg limit, *Phys. Rev. Lett.* **124**, 030501 (2020).
- [119] A. T. Walden, Accurate approximation of a 0th order discrete prolate spheroidal sequence for filtering and data tapering, *Sig. Proc.* **18**, 341 (1989).
- [120] J. Liu, H. Yuan, X.-M. Lu, and X. Wang, Quantum Fisher information matrix and multiparameter estimation, *J. Phys. A* **53**, 023001 (2020).
- [121] F. Albarelli, J. F. Friel, and A. Datta, Evaluating the Holevo Cramér-Rao bound for multiparameter quantum metrology, *Phys. Rev. Lett.* **123**, 200503 (2019).
- [122] F. Belliardo, and V. Giovannetti, Incompatibility in quantum parameter estimation, *New J. Phys.* **23**, 063055 (2021).
- [123] X.-M. Lu, and X. Wang, Incorporating Heisenberg's uncertainty principle into quantum multiparameter estimation, *Phys. Rev. Lett.* **126**, 120503 (2021).
- [124] T. J. Proctor, P. A. Knott, and J. A. Dunningham, Multiparameter estimation in networked quantum sensors, *Phys. Rev. Lett.* **120**, 080501 (2018).
- [125] K. Qian, Z. Eldredge, W. Ge, G. Pagano, C. Monroe, J. V. Porto, and A. V. Gorshkov, Heisenberg-scaling measurement protocol for analytic functions with quantum sensor networks, *Phys. Rev. A* **100**, 042304 (2019).
- [126] C. Ferrie, and R. Blume-Kohout, Minimax quantum tomography: Estimators and relative entropy bounds, *Phys. Rev. Lett.* **116**, 090407 (2016).
- [127] S. Aaronson, in *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2018* (Association for Computing Machinery, New York, 2018), p. 325.
- [128] H.-Y. Huang, R. Kueng, and J. Preskill, Predicting many properties of a quantum system from very few measurements, *Nat. Phys.* **16**, 1050 (2020).
- [129] N. Shettell, E. Kashefi, and D. Markham, Cryptographic approach to quantum metrology, *Phys. Rev. A* **105**, L010401 (2022).
- [130] N. Shettell, M. Hassani, and D. Markham, Private network parameter estimation with quantum sensors, *ArXiv:2207.14450*.
- [131] P. Faist, M. P. Woods, V. V. Albert, J. M. Renes, J. Eisert, and J. Preskill, Time-energy uncertainty relation for noisy quantum metrology, *PRX Quantum* **4**, 040336 (2023).
- [132] M. M. Wilde, *Quantum Information Theory* (Cambridge University Press, New York, 2017), 2nd ed.
- [133] S. L. Braunstein, How large a sample is needed for the maximum likelihood estimator to be approximately Gaussian? *J. Phys. A* **25**, 3813 (1992).
- [134] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum metrology, *Phys. Rev. Lett.* **96**, 010401 (2006).
- [135] R. Demkowicz-Dobrzański, J. Kołodyński, and M. Guţă, The elusive Heisenberg limit in quantum-enhanced metrology, *Nat. Commun.* **3**, 1063 (2012).
- [136] A. Smirne, J. Kołodyński, S. F. Huelga, and R. Demkowicz-Dobrzański, Ultimate precision limits for noisy frequency estimation, *Phys. Rev. Lett.* **116**, 120801 (2016).
- [137] S. Zhou, M. Zhang, J. Preskill, and L. Jiang, Achieving the Heisenberg limit in quantum metrology using quantum error correction, *Nat. Commun.* **9**, 78 (2018).
- [138] H. Yuen, and M. Lax, Multiple-parameter quantum estimation and measurement of nonselfadjoint observables, *IEEE Trans. Inf. Theory* **19**, 740 (1973).
- [139] M. Hayashi, *Asymptotic Theory of Quantum Statistical Inference: Selected Papers* (World Scientific, Singapore, 2005).