

Measurement-Based Quantum Computation in Symmetry-Enriched Topological Phases

Paul Herringer^{1,2,3,*}, Vir B. Bulchandani^{4,5}, Younes Javanmard^{6,7,†}, David T. Stephen^{6,7,†} and Robert Raussendorf^{1,3}

¹*Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany*

²*Department of Physics and Astronomy, University of British Columbia, 6224 Agricultural Road, Vancouver, British Columbia V6T 1Z1, Canada*

³*Stewart Blusson Quantum Matter Institute, University of British Columbia, Vancouver, British Columbia V6T 1Z4, Canada*

⁴*Department of Physics and Astronomy, Rice University, 6100 Main Street, Houston, Texas 77005, USA*

⁵*Department of Physics, National University of Singapore, 117542, Singapore*

⁶*Department of Physics and Center for Theory of Quantum Matter, University of Colorado Boulder, Boulder, Colorado 80309, USA*

⁷*Department of Physics, California Institute of Technology, Pasadena, California 91125, USA*

 (Received 19 April 2025; revised 11 July 2025; accepted 13 August 2025; published 21 October 2025)

We present the first examples of topological phases of matter with uniform power for measurement-based quantum computation (MBQC). This is possible due to a new framework for analyzing the computational properties of phases of matter that is more general than previous constructions, which have been limited to short-range entangled phases in one dimension. We show that ground states of the toric code in an anisotropic magnetic field yield a natural, albeit noncomputationally universal, application of our framework. We then present a new model with topological order the ground states of which are universal resources for MBQC. Both topological models are enriched by subsystem symmetries and these symmetries protect their computational power. Our framework greatly expands the range of physical models that can be analyzed from the computational perspective.

DOI: [10.1103/j2z3-s6d6](https://doi.org/10.1103/j2z3-s6d6)

I. INTRODUCTION

Measurement-based quantum computation (MBQC) is a computational scheme that uses local measurements on an entangled many-body quantum state to perform quantum computations. In MBQC, the initial state determines the computational power and is thus a computational resource. It is known that computationally universal resource states, such as cluster states [1,2], Affleck, Kennedy, Lieb, and Tasaki (AKLT) states [3,4], and certain hypergraph states [5–8], exist. On these states, any quantum circuit can be efficiently simulated by local adaptive measurement. Yet, to date, no complete classification of MBQC resource

states exists and the precise nature of the computational resource leveraged by MBQC remains unclear.

One potential route to understanding MBQC resource states is by connecting them to “physical order,” by which we mean the condensed-matter classification of many-body quantum states. For gapped quantum phases, this classification encompasses topologically ordered states with long-range entanglement and topologically trivial states with short-range entanglement [9]. It is further refined by the addition of symmetries, which split the topologically trivial states into symmetry-protected topological (SPT) phases [10–13] and the topologically ordered states into symmetry-enriched topological (SET) phases [14–18]. The symmetries that define SPT and SET phases can be global, i.e., the action of the symmetry is the same at every site, or subsystem symmetries that act by a tensor product of local operators on a rigid subset of sites [19,20]. An example of the latter would be a linelike symmetry that acts on horizontal or vertical lines of sites.

When suitable symmetries are imposed, it has been shown that there exist phases of matter with uniform computational power for MBQC [21–30]. Specifically, the Lie

*Contact author: paul.herringer@itp.uni-hannover.de

†Present Address: Quantinuum, 303 S Technology Ct, Broomfield, Colorado 80021, USA

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

group of logical gates that can be realized through MBQC is the same for every state in the phase. Thus, we can speak of “computational order” in addition to physical order, and similarly of computational phases of matter [21–23]. The existence of such phases prompts us to ask: *What is the relationship between computational order and physical order in many-body quantum states?*

Until now, computational phases have only been identified in systems with SPT order [22–30]. One might thus conjecture that computational order is simply SPT order viewed through the lens of MBQC. However, as we show in this paper, this is not the case. Specifically, we develop a formalism based on nonlocal order parameters that characterizes computational power for a much broader class of two-dimensional (2D) quantum states, including states with symmetry-enriched topological order (SET). This opens up the possibility of identifying computational phases in topologically ordered systems, which we illustrate via two examples.

Our first example is the familiar toric code perturbed by an anisotropic magnetic field. It is known that the toric code state has limited but nontrivial computational power in MBQC [31] and we show that this power extends to a symmetry-enriched phase around it. This proceeds by establishing a duality between order parameters that detect the computational phase and Fredenhagen-Marcu-like string order parameters that detect the SET phase.

Our second example yields a computationally universal SET phase, demonstrating that computational universality is compatible with SET order as well as SPT order. While a complete picture of the relationship between computational order and physical order remains an open question, here we show that the phenomenology of computational phases is richer than previously known.

A. Overview of results

Our main result, Theorem 3, is a generalization of the existing one-dimensional (1D) computational order formalism [25] to 2D and to states that may have topological order. We apply this theorem to show that both the toric code and our new stabilizer model exhibit nontrivial computational order, in a manner that is inextricably related to their unusual (symmetry-enriched) topological order. For comparison, we first state a counterpart of Theorem 3 for SPT order in 1D. It is the informal (as well as restricted and partial) version of Ref. [25, Corollary 1]. The main idea is that computational order is characterized by the same string order parameters σ_k that detect SPT order in 1D.

Theorem 1 (1D computational power, informal [25]). Consider a short-range entangled state $|\Phi\rangle$ of a spin-1/2 chain, symmetric under a group $G = (\mathbb{Z}_2)^m$. Furthermore, assume that the string order parameters σ_k are nonzero for all sites k . Then, the group of realizable gates is

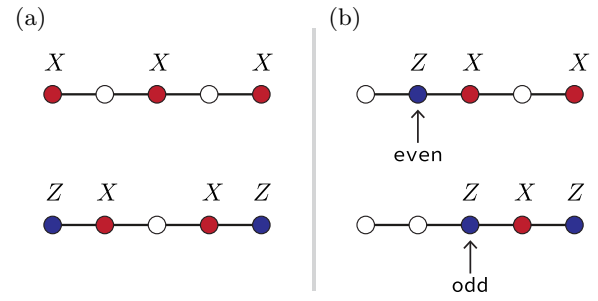


FIG. 1. Symmetry action and order parameters in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ cluster phase in one dimension (1D). (a) The action of the symmetry on the physical spins: top, a representation of the group element $(01) \in \mathbb{Z}_2 \times \mathbb{Z}_2$; bottom, a representation for the group element (10) . (b) The order parameters ending in even and odd sites, respectively.

$\mathcal{L} = \exp(-i\mathcal{A})$, where $\mathcal{A}$ is the Lie algebra generated by $\{T(g_k)\}$, with $T(G)$ a projective representation of the symmetry group G , and $g_k \in G$ is a special group element associated with the k th spin through the construction of the order parameter σ_k .

For illustration, let us consider the example of the cluster phase [25], with symmetry group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. The symmetries and operators yielding the order parameters are depicted in Fig. 1. In this case, the projective representation $T(G)$ is the maximally noncommutative one— $T(00) = I$, $T(01) = X$, $T(10) = Z$, $T(11) = Y$. The special group elements associated with the individual sites are (01) for all even and (10) for all odd sites. The Lie algebra $\mathcal{A}$ is thus generated by the Pauli operators Z and X ; hence the resulting Lie group of realizable circuits is $\mathcal{L} = SU(2)$.

1. Main theorem

We extend Theorem 1 as follows.

Theorem 2 (2D computational power, informal). Consider a state $|\Phi\rangle$ of a 2D spin-1/2 system, such that:

- (a) $|\Phi\rangle$ is symmetric under a group $G \cong \mathbb{Z}_2^m$.
- (b) $|\Phi\rangle$ is related to a suitable stabilizer state by a local finite-depth, symmetric unitary.
- (c) The order parameters σ_k are nonzero for every site k .

Then, the Lie group of quantum gates realizable by MBQC with $|\Phi\rangle$ as a resource state is $\mathcal{L} = \exp\{-i\mathcal{A}\}$, where $\mathcal{A}$ is the Lie algebra generated by $\{T(g_k)\}$, with $T(G)$ being a projective representation of G and g_k being a special group element associated with site k through the construction of the order parameter σ_k .

We emphasize condition (b) of Theorem 2 because it allows us to address certain phases of matter that do not

contain product states, thereby unlocking the study of computational phases of matter beyond invertible short-range entangled phases. In particular, this includes phases of matter with topological order.

Some comments on Theorem 2 are in order. First, the presence of a symmetry group G means that when we want to apply the Theorem to states with topological order, we necessarily enter the realm of symmetry-enriched topological (SET) phases of matter. Then, the theorem says that certain SET phases of matter have uniform computational power, akin to the previous results for SPT phases [23–25].

Second, as in the case of short-range entangled phases, we find that the most interesting SET phases from the perspective of MBQC have subsystem symmetries [26–29], which are symmetries that act on rigid lower-dimensional submanifolds. For the examples that we consider, these subsystem symmetries are linelike and the order parameters σ_k appearing in Theorem 2 are string order parameters that descend from the linelike symmetries. Topological phases enriched by subsystem symmetries, dubbed subsystem SET (SSET) phases, have recently been studied in Refs. [17, 18]. While these phases of matter may appear to be relatively obscure, a simple realization of this physics arises upon perturbing the toric code by a bond-anisotropic magnetic field, as we now discuss.

2. First example: Toric code in an anisotropic field

As an illustrative example of a model that falls under the assumptions of Theorem 2, we consider the toric code [33]

$$H_0 = - \sum_{(x,y) \in \Lambda} A_{(x,y)} - \sum_{(x,y) \in \Lambda} B_{(x,y)} \quad (1)$$

on a periodic square lattice Λ of L_x by L_y sites, where (x, y) labels the Cartesian coordinates of each vertex with $x = 1, 2, \dots, L_x$ and $y = 1, 2, \dots, L_y$, and $A_{(x,y)}$ and $B_{(x,y)}$ are the usual star and plaquette operators (see Fig. 2). We perturb H_0 by a bond-anisotropic magnetic field,

$$\delta H = -h_X \sum_{(x,y) \in \Lambda} X_{(x,y+\frac{1}{2})} - h_Z \sum_{(x,y) \in \Lambda} Z_{(x+\frac{1}{2},y)} \quad (2)$$

where we use half-integer coordinates to label edges and $h_X, h_Z \geq 0$. This yields the perturbed toric code Hamiltonian

$$H = H_0 + \delta H. \quad (3)$$

The phase diagram of this model is shown in Fig. 2. In contrast to more standard magnetic field perturbations without bond anisotropy [32], the anisotropic form of the perturbation given in Eq. (2) guarantees that the parity of the number of anyons [excitations of $A_{(x,y)}$ or $B_{(x,y)}$] is

conserved along every row (see Fig. 2). These conservation laws can be expressed as subsystem symmetries,

$$U(g_y^e) = \prod_{x=1}^{L_x} A_{(x,y)}, \quad U(g_y^m) = \prod_{x=1}^{L_x} B_{(x,y)}, \quad (4)$$

which are equal to products of X operators or products of Z operators, respectively, along two parallel rows (see Fig. 3). The phase diagram in Fig. 2 should be thought of as a deformation of the toric code that preserves these subsystem symmetries. In fact, we will find that the unperturbed toric code state lies in a nontrivial SSET phase of matter protected by these symmetries.

Using these symmetries, we can define string order parameters,

$$\begin{aligned} S_{[a,b]}(g_y^e) &= \prod_{x=a}^b A_{(x,y)} \\ &= X_{(a-\frac{1}{2},y)} \left(\prod_{x=a}^b X_{(x,y+\frac{1}{2})} X_{(x,y-\frac{1}{2})} \right) X_{(b+\frac{1}{2},y)}, \\ S_{[a,b]}(g_y^m) &= \prod_{x=a}^b B_{(x,y)} \\ &= Z_{(a,y+\frac{1}{2})} \left(\prod_{x=a}^b Z_{(x+\frac{1}{2},y)} Z_{(x+\frac{1}{2},y+1)} \right) Z_{(b+1,y+\frac{1}{2})}, \end{aligned} \quad (5)$$

as illustrated in Fig. 3. All ground states of H_0 have perfect string order, $\langle S_{[a,b]}(g_y^{e/m}) \rangle = 1$. When we perturb away from this point, the string order parameters are no longer perfect but, crucially, their expectation values do not decay with their length $|a-b|$. These string order parameters (or rather their appropriate generalizations for open boundary conditions) are precisely the computational order parameters that appear in Theorem 2. By mapping Eq. (3) to a stack of 1D transverse-field Ising models (for related results, see, e.g., Refs. [34] and [35]), we prove that these order parameters are nonzero everywhere in the symmetry-enriched topological phase $h_X, h_Z < 1$, so that every ground state in this phase is an MBQC resource for the same family of computations described by Theorem 2. This family of computations turns out to be nonuniversal, since the Lie group of gates that can be implemented in MBQC maps onto classically simulable free-fermionic circuits. This shortcoming is remedied by our second example.

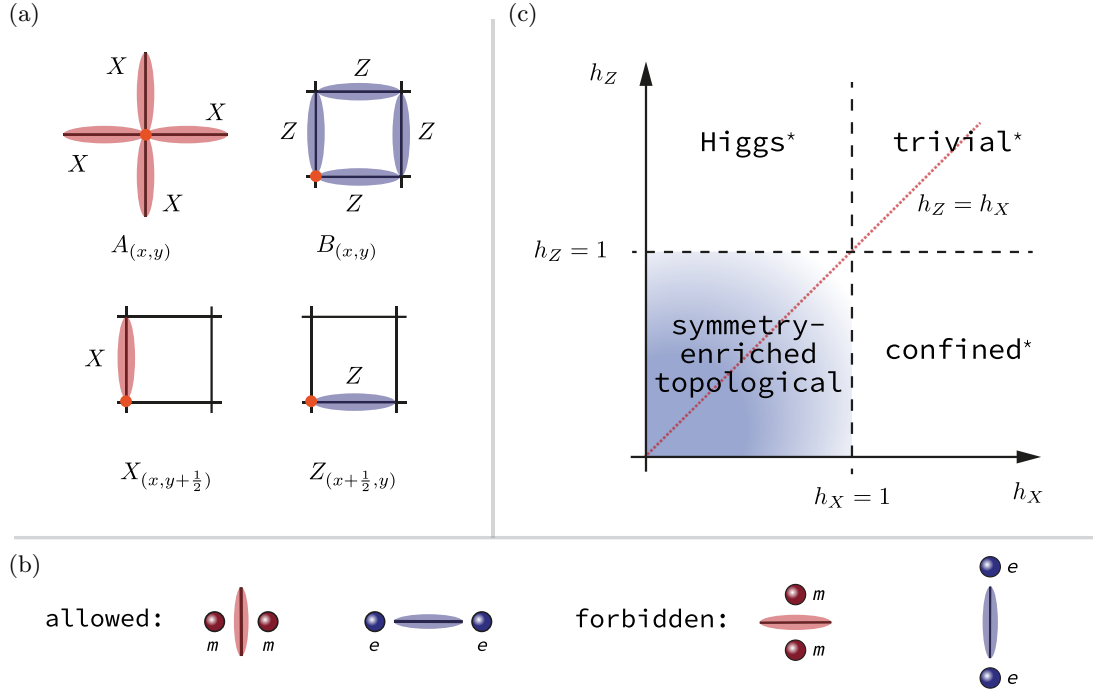


FIG. 2. (a) Our labeling conventions for the terms in the anisotropically perturbed toric code Hamiltonian given in Eq. (3). Orange dots indicate the lattice point (x, y) and throughout this paper, red and blue lobes will correspond to X and Z Pauli operators, respectively. (b) Unlike the usual magnetic field perturbation of the toric code, the anisotropic magnetic field perturbation given in Eq. (2) only creates perfectly horizontal pairs of electric or magnetic anyons. Thus only a subset of the usual pair-creation processes are allowed. (c) The ground-state phase diagram of the resulting Hamiltonian. Asterisks indicate that charges are only confined in the vertical direction or, equivalently, the presence of subsystem symmetry, as discussed in Sec. III B 4. We follow the standard terminology [32] borrowed from $U(1)$ gauge theory, according to which the e -condensed phase $h_Z > 1$ yields an m -confined or “Higgs” phase and the m -condensed phase $h_X > 1$ yields an e -confined or simply “confined” phase. The line $h_Z = h_X$ is indicated in red.

3. Second example: A computationally universal topological phase

We propose a new stabilizer model, the XZ -star model, that yields universal resource states for MBQC:

$$H_* = - \sum_x \sum_{y \text{ even}} C_{x,y} - \sum_x \sum_{y \text{ odd}} D_{x,y}, \quad (6)$$

where

$$\begin{aligned} C_{x,y} &= X_{x-1,y} X_{x,y} X_{x+1,y} X_{x,y-1} X_{x,y+1}, \\ D_{x,y} &= Z_{x-1,y} Z_{x,y} Z_{x+1,y} Z_{x,y-1} Z_{x,y+1}, \end{aligned} \quad (7)$$

and (x, y) index the sites of a square 2D lattice. This model again has topological order along with a set of subsystem symmetries and corresponding order parameters that satisfy the conditions of Theorem 2, and we find that the

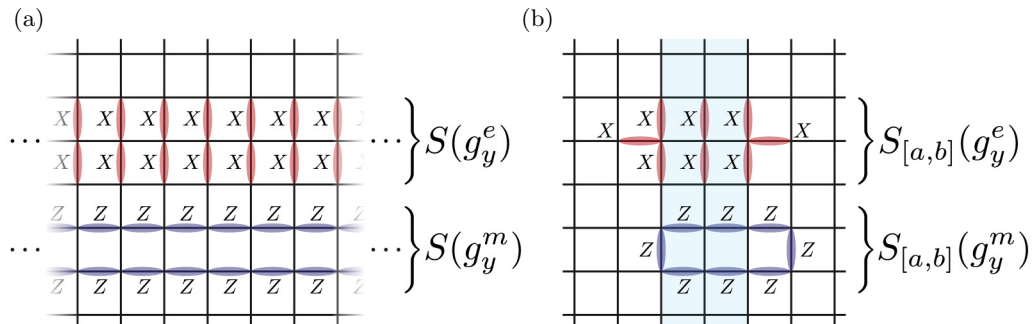


FIG. 3. (a) The subsystem symmetries of the toric code in an anisotropic magnetic field. (b) The corresponding string order parameters. The highlighted region corresponds to the interval $x \in [a, b]$.

corresponding Lie group of quantum gates is universal. This model and its symmetries therefore define a computationally universal SSET phase of matter. A computationally universal $\mathbb{Z}_2$ string-net condensate has previously been identified in Ref. [36]; however, to the best of our knowledge, the XZ -star model is the first example where computational universality is shown to extend to an entire SSET phase.

Interestingly, we also show that this model has the same topological order and symmetry fractionalization as a stack of two toric codes, which do not enable universal MBQC. This apparent tension is resolved by the fact that the two models have differently structured subsystem symmetries, highlighting the importance of the microscopic symmetry representation for MBQC.

B. Structure of paper

The rest of the paper is structured as follows. In Sec. II, we discuss our MBQC scheme for 2D systems of qubits and formally state Theorem 2. Section III is devoted to the toric code example. In Sec. III A, we apply Theorem 2 to the toric code model with an anisotropic magnetic field and demonstrate that this model has computational order. We then discuss the topological characteristics and subsystem symmetries of the model in Sec. III B. We conclude our study of the toric code example by relating computational order and topological (string) order in Sec. III C.

In Sec. IV, we introduce the XZ -star model, discuss its properties from the perspective of topological order and subsystem symmetries, and prove that its ground states are computationally universal for MBQC. We conclude with a general discussion of our results in Sec. VI.

II. COMPUTATIONAL ORDER

In this section, we characterize the computational power of 2D systems of qubits with symmetries. Specifically, we consider states that are symmetric under a group $G \cong \mathbb{Z}_2^k$, which applies to local stabilizer states, including over 2600 examples known as stabilizer projected entangled-pair states (PEPS) [37]. It also applies to states in the surrounding phases, among which are SPT and SET phases.

The computational power of a state refers to the number of logical qubits available for computation and the Lie group of logical gates that can be implemented via MBQC. Here, we show that for symmetric quantum states, these computational properties are characterized by a set of computational order parameters (COPs). Our order parameters and our MBQC scheme are a generalization of the scheme defined in Ref. [25], which uses 1D string order parameters and applies to short-range entangled states. The technical reason for requiring short-range entangled states is that the proofs require expectation values of local observables

to decorrelate, $\langle PQ \rangle = \langle P \rangle \langle Q \rangle$, when their supports are separated by a finite distance.

In Lemma 1, we prove a less restrictive condition, namely, that decorrelation also occurs if the state in question is related to an appropriate stabilizer state by a local finite-depth unitary circuit. Notably, the allowed stabilizer states include states with topological order, such as our toric code model. After the statement and proof of Lemma 1, we define the COPs and specify how they characterize computational power for MBQC with Theorem 3.

A. Decorrelation

Consider a region Ω of a 2D lattice of qubits. We require a way to expand Ω by a fixed amount, so we introduce the k -neighborhood of Ω , which is Ω plus a ribbon of width k around its perimeter (Fig. 4). Formally, we say that the k -neighborhood of Ω is the set of all points (x, y) such that a sphere $\mathcal{S}_k(x, y)$ of radius k centered at (x, y) touches Ω :

$$\mathcal{N}_k(\Omega) = \{(x, y) : \mathcal{S}_k(x, y) \cap \Omega \neq \emptyset\}. \quad (8)$$

For an operator P , we use the shorthand $\mathcal{N}_k(P)$ to denote $\mathcal{N}_k(\text{supp}(P))$.

Definition 1. Let W be a quantum circuit acting on a 2D lattice of spins and let Ω be a region of the lattice. The spread $\Delta(W)$ is the smallest non-negative integer such that, for any choice of Ω and any linear operator P , the support of which is contained within Ω ,

$$\text{supp}(W^\dagger P W) \subset \mathcal{N}_\Delta(\Omega). \quad (9)$$

In other words, if W has spread Δ , then $\text{supp}(W^\dagger P W)$ lies within $\text{supp}(P)$ plus a ribbon of width Δ .

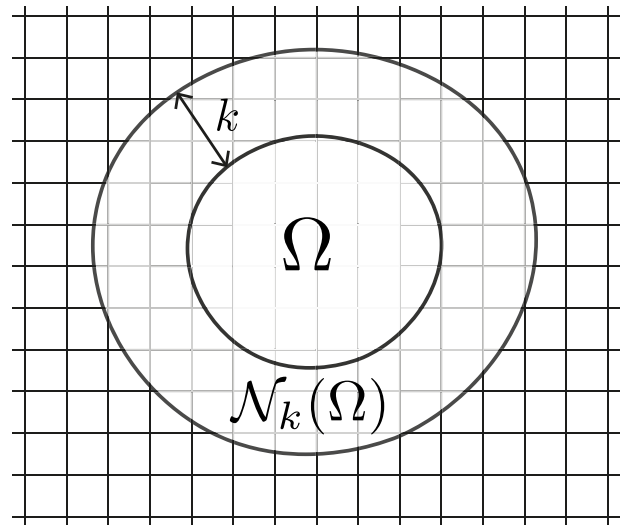


FIG. 4. Expanding a region Ω of the lattice by a ribbon of width k to obtain the k -neighborhood $\mathcal{N}_k(\Omega)$.

Let S_{Φ_0} be the stabilizer group of $|\Phi_0\rangle$. Given a region Ω , we can write a generating set for S_{Φ_0} as follows:

$$S_{\Phi_0} = \langle S(\Omega), \bar{S}(\Omega), \partial S(\Omega) \rangle, \quad (10)$$

where $S(\Omega)$ is a set of generators supported within Ω , $\bar{S}(\Omega)$ is a set of generators supported within the complement of Ω , and $\partial S(\Omega)$ is a set of generators that cross the boundary of Ω .

Lemma 1. Consider a state $|\Phi\rangle$ created from a stabilizer state $|\Phi_0\rangle$ by a quantum circuit W with spread Δ . Let P and Q be two linear operators and suppose that we can choose $\partial S(\mathcal{N}_\Delta(P))$ such that none of its elements overlap with the Δ -neighborhood of Q : $\text{supp}(s) \cap \mathcal{N}_\Delta(Q) = \emptyset$ for all $s \in \partial S(\mathcal{N}_\Delta(P))$. Then,

$$\langle \Phi | PQ | \Phi \rangle = \langle \Phi | P | \Phi \rangle \langle \Phi | Q | \Phi \rangle. \quad (11)$$

Proof. Letting $\Pi[S]$ be the projectors onto a set of stabilizer generators, we can write

$$|\Phi_0\rangle \propto \Pi[S(\Omega), \bar{S}(\Omega), \partial S(\Omega)] |\Phi_0\rangle. \quad (12)$$

Now consider $W^\dagger PW$, which has support contained in $\mathcal{N}_\Delta(P)$. We can pull a copy of $\Pi[\bar{S}(\mathcal{N}_\Delta(P))]$ past $W^\dagger PW$ because their supports do not overlap:

$$\begin{aligned} (W^\dagger PW) \Pi[S_{\Phi_0}] \\ = \Pi[\bar{S}(\mathcal{N}_\Delta(P))](W^\dagger AW) \Pi[S_{\Phi_0}]. \end{aligned} \quad (13)$$

With the assumption of the lemma, we also have

$$\begin{aligned} (W^\dagger QW) \Pi[S_{\Phi_0}] \\ = \Pi[S(\mathcal{N}_\Delta(P)), \partial S(\mathcal{N}_\Delta(P))](W^\dagger PW) \Pi[S_{\Phi_0}]. \end{aligned} \quad (14)$$

Combining the results of Eqs. (13) and (14), we have

$$\begin{aligned} \langle \Phi | PQ | \Phi \rangle &= \langle \Phi_0 | \Pi[S_{\Phi_0}](W^\dagger PW)(W^\dagger QW) \Pi[S_{\Phi_0}] | \Phi_0 \rangle \\ &= \langle \Phi_0 | \Pi[S_{\Phi_0}](W^\dagger PW) \Pi[\bar{S}(\mathcal{N}_\Delta(P), S(\mathcal{N}_\Delta(P)), \partial S(\mathcal{N}_\Delta(P))](W^\dagger QW) \Pi[S_{\Phi_0}] | \Phi_0 \rangle \\ &= \langle \Phi_0 | \Pi[S_{\Phi_0}](W^\dagger PW) \Pi[S_{\Phi_0}](W^\dagger QW) \Pi[S_{\Phi_0}] | \Phi_0 \rangle \\ &= \langle \Phi_0 | \Pi[S_{\Phi_0}](W^\dagger PW) \Pi[S_{\Phi_0}] | \Phi_0 \rangle \langle \Phi_0 | \Pi[S_{\Phi_0}](W^\dagger QW) \Pi[S_{\Phi_0}] | \Phi_0 \rangle \\ &= \langle \Phi | P | \Phi \rangle \langle \Phi | Q | \Phi \rangle, \end{aligned} \quad (15)$$

which completes the proof. $\blacksquare$

As an example, let us apply Lemma 1 to the toric code ground state on a lattice with smooth open boundaries. For simplicity, let $\text{supp}(P)$ and $\text{supp}(Q)$ be rectangular regions with either rough or smooth boundaries in the bulk. We then have three cases, as follows:

- (a) *Both smooth boundaries.* A rectangular region with a smooth boundary is defined by the edges around its perimeter. The toric code stabilizers that cross this boundary are vertex operators (Fig. 5). If we have a circuit W with spread Δ , then $\mathcal{N}_\Delta(P)$ and $\mathcal{N}_\Delta(Q)$ are rectangular regions with length and width both increased by 2Δ . The set $\partial S(\mathcal{N}_\Delta(P))$ consists of vertex stabilizers that cross the boundary of $\mathcal{N}_\Delta(P)$, and for Lemma 1 to be satisfied, we require that none of these overlap with $\mathcal{N}_\Delta(Q)$. This will be the case if there is at least one plaquette separating $\mathcal{N}_\Delta(P)$ from $\mathcal{N}_\Delta(Q)$ or, in other words, $\text{supp}(P)$ is separated from $\text{supp}(Q)$ by at least $2\Delta + 1$ unit cells (Fig. 5).

- (b) *Both rough boundaries.* A rectangular region with a rough boundary is defined by the plaquettes around its perimeter. The toric code stabilizers that cross this boundary are plaquette operators (Fig. 5). This situation is analogous to case (i) with vertex and plaquette operators swapped, so again we find that Lemma 1 is satisfied if $\text{supp}(P)$ is separated from $\text{supp}(Q)$ by at least $2\Delta + 1$ unit cells.
- (c) *One smooth and one rough boundary.* When $\text{supp}(P)$ has a smooth boundary and $\text{supp}(Q)$ has a rough boundary, the stabilizers that cross the boundary of $\mathcal{N}_\Delta(P)$ are vertex operators and the stabilizers that cross the boundary of $\mathcal{N}_\Delta(Q)$ are plaquette operators. As shown in Fig. 5, Lemma 1 is satisfied if $\mathcal{N}_\Delta(P)$ and $\mathcal{N}_\Delta(Q)$ are separated by at least 1.5 unit cells, which requires $\text{supp}(P)$ and $\text{supp}(Q)$ to be separated by at least $2\Delta + 1.5$ unit cells.

B. Symmetry representations

Our setting for MBQC is a 2D lattice of qubits with open boundaries. We define three subsets of the lattice: the left

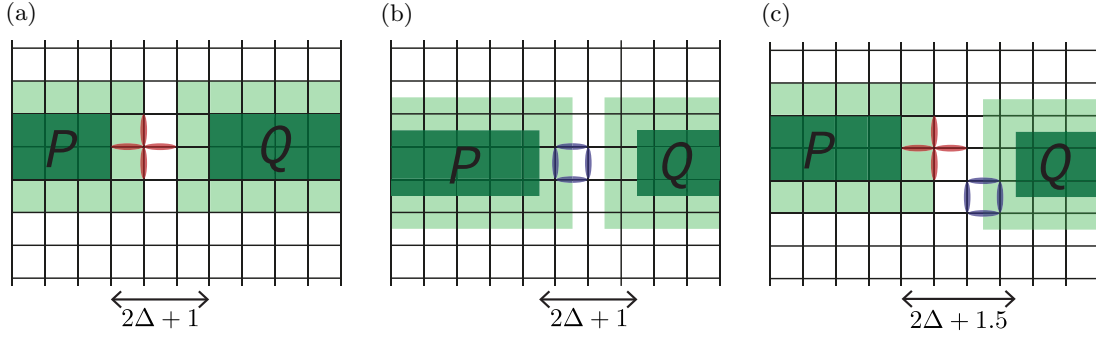


FIG. 5. The separation between operators P and Q required to satisfy Lemma 1 in states related to the toric code state by a local finite-depth unitary with spread Δ . The darker-green regions represent supports of P and Q , respectively, while the lighter-green regions represent the neighborhoods $N_\Delta(P)$ and $N_\Delta(Q)$. (a) If the supports of P and Q have smooth boundaries, we require vertex stabilizers that cross the boundary of $N_\Delta(P)$ not to cross the boundary of $N_\Delta(Q)$, so the separation must be $2\Delta + 1$. (b) The case in which P and Q are supported on regions with rough boundaries. (c) The case in which $\text{supp}(P)$ has a smooth boundary and $\text{supp}(Q)$ has a rough boundary.

boundary L , the bulk B , and the right boundary R . Let $|\Phi\rangle$ be a quantum state of the lattice that is symmetric under a linear representation $U(G)$ of a group $G \cong \mathbb{Z}_2^m$:

$$U(g) |\Phi\rangle = (-1)^{\chi(g)} |\Phi\rangle \quad g \in G, \quad (16)$$

where $\chi(g) \in \{0, 1\}$. We assume that $U(G)$ is a tensor product of single-qubit Pauli operators I, X, Y , or Z that commute in the bulk but may anticommute at the boundaries. In other words, $U(G)$ is a product of site-local representations that are linear in the bulk and projective at the boundaries. Specifically, we have:

- (1) Linear representations in the bulk: $\hat{u}_i(g) \in \{X_i, Y_i, Z_i, I_i\}, i \in B$.
- (2) Projective representations on the left boundary: $\hat{v}_i^L(g) \in \{X_i, Y_i, Z_i, I_i\}, i \in L$.
- (3) Projective representations on the right boundary: $\hat{v}_i^R(g) \in \{X_i, Y_i, Z_i, I_i\}, i \in R$.

We will often refer to symmetry representations acting on the entire left boundary, right boundary, or bulk, which we denote by

$$\begin{aligned} V_L(g) &= \bigotimes_{i \in L} \hat{v}_i^L(g), \\ V_R(g) &= \bigotimes_{i \in R} \hat{v}_i^R(g), \\ U_B(g) &= \bigotimes_{i \in B} \hat{u}_i(g). \end{aligned} \quad (17)$$

With this notation, we can expand $U(g)$ as

$$U(g) = V_L(g) \otimes U_B(g) \otimes V_R(g). \quad (18)$$

For $U(g)$ to be linear, Eq. (18) requires the commutation relations of $V_L(g)$ to match those of $V_R(g)$. We formalize

this by defining a function $\kappa : G \times G \rightarrow \mathbb{Z}_2$ via

$$V_L(g)V_L(g') = (-1)^{\kappa(g,g')} V_L(g')V_L(g) \quad (19)$$

and demand that $V_R(g)$ satisfy the same equation, i.e.,

$$V_R(g)V_R(g') = (-1)^{\kappa(g,g')} V_R(g')V_R(g). \quad (20)$$

C. Computational order parameters

Let k be a site in the bulk and denote by $\{> k\}$ the set of all sites for which the horizontal distance to the right boundary is less than k . In other words, $\{> k\}$ is the rectangular region starting immediately to the right of site k and running all the way up to and including the right boundary. We denote the restriction of $U(g)$ to $\{> k\}$ by $U_{>k}(g)$. This operator may fail to commute with $U(G)$ because the projective representation $V_R(G)$ at the right boundary is no longer counterbalanced by $V_L(g)$ at the left boundary. However, we can search for an operator $\hat{\beta}_k(g)$ acting on site k such that

$$R_k(g) := \hat{\beta}_k(g) \otimes U_{>k}(g) \quad (21)$$

commutes with all $U(G)$. When such an operator exists, we say that $U(g)$ is localizable to site k and that $\hat{\beta}_k(g)$ localizes $U(g)$. The condition for $U(g)$ to be localizable at site k is that there exists $\hat{\beta}_k(g)$ such that

$$\hat{u}_k(g')\hat{\beta}_k(g) = (-1)^{\kappa(g,g')}\hat{\beta}_k(g)\hat{u}_k(g') \quad (22)$$

for all $g' \in G$. At each site, we denote by $\mathcal{G}_k \subset G$ the subgroup of G for which there is a $\hat{\beta}_k(g)$ satisfying Eq. (22). The set of all elements of g that can be localized to at least one site is denoted $\mathcal{G} = \bigcup_i \mathcal{G}_i$.

Lemma 2. If Eq. (22) has a solution for site k and group element $g_k \in \mathcal{G}_k$, then without loss of generality, $\hat{\beta}_k(g_k)$ may be chosen to be a Pauli operator.

Proof. Without loss of generality, assume that $\hat{u}_k(g) \in \{X_k, I_k\}$ for all $g \in G$. Then, $\hat{\beta}_k(g_k)$ must anticommute with X_k . The space of single-qubit operators that anticommute with X is the complex vector space spanned by $\{Y_k, Z_k\}$, so we can make $\hat{\beta}_k(g_k)$ a Pauli operator by choosing $\hat{\beta}_k(g_k)$ to be Y_k or Z_k . ■

The reason for demanding that $R_k(g_k)$ commutes with all $U(G)$ is to allow for nonzero expectation values $\sigma_k(g_k) = \langle R_k(g_k) \rangle_\Phi$ on symmetric states $|\Phi\rangle$. These expectation values will turn out to be COPs and we will show that MBQC is possible whenever these order parameters are nonzero. Note that $\sigma_k(g_k)$ only exists for $g_k \in \mathcal{G}_k$; i.e., only localizable elements of G give rise to a COP.

D. Logical observables

To implement MBQC on symmetric states, we need to specify how information will be encoded and recovered from such states. To this end, we define the logical observables

$$T(g) = \begin{cases} V_L(g) \otimes U_B(g) \otimes V_R(g), & g \in H, \\ I_L \otimes U_B(g) \otimes V_R(g), & g \in G \setminus H, \end{cases} \quad (23)$$

where $I_L = \bigotimes_{i \in L} I_i$ is the identity on the left boundary and H is a maximal subgroup of G such that $V_R(H)$ is Abelian. Overall, $T(G)$ forms a projective representation of G . However, because the $T(H)$ commute, they can be measured simultaneously and they will be the basis states for our MBQC readout. They also define the initial state of the logical system, as shown in the following lemma.

Lemma 3. For all states $|\Phi\rangle$ satisfying Eq. (16),

$$\langle T(g) \rangle_\Phi = \begin{cases} (-1)^{\chi(g)}, & g \in H, \\ 0, & g \in G \setminus H. \end{cases} \quad (24)$$

Proof. For $g \in H$, we have $T(g) = U(g)$ and the result follows from Eq. (16). By assumption, $T(H)$ is a maximal Abelian subgroup of $T(G)$; thus for every $g \in G \setminus H$, there exists $h \in H$ such that $T(h)$ anticommutes with $T(g)$. Then, $\langle \Phi | T(g) | \Phi \rangle = \langle \Phi | T(h)^\dagger T(g) T(h) | \Phi \rangle = -\langle \Phi | T(g) | \Phi \rangle$, which implies $\langle \Phi | T(g) | \Phi \rangle = 0$. ■

E. Computational power

With $R_k(g_k)$, $\sigma_k(g_k)$, and $T(G)$ defined, we can state the following theorem regarding the computational power of symmetric states.

Theorem 3 (Computational power). Consider a state $|\Phi\rangle$ of a 2D spin-1/2 system satisfying the following conditions:

- (a) $|\Phi\rangle$ is related to a stabilizer state $|\Phi_0\rangle$ by a local finite-depth unitary that commutes with $U(G)$ and has finite spread Δ .
- (b) There exists a finite d such that Lemma 1 holds for any two regions of the lattice separated by at least $2\Delta + d$.
- (c) $|\Phi\rangle$ is symmetric under $U(G)$ as defined in Eq. (16).
- (d) The COPs $\sigma_k(g_k)$ are nonzero for all k .

Then, MBQC with $|\Phi\rangle$ as a resource state can implement the following:

- (i) initialization and readout of $\text{rank}(H)$ logical qubits, where H is a maximal subgroup of G such that $V_R(H)$ is Abelian.
- (ii) logical gates from the Lie group $\mathcal{L} = \exp(-i\mathcal{A})$, where $\mathcal{A}$ is the Lie algebra generated by $T(\mathcal{G})$ under $i[\cdot, \cdot]$ and linear combinations

We prove Theorem 3 in Appendix A. The key takeaway is that every symmetric state such that the order parameters $\sigma_k(g_k)$ are nonzero has equivalent computational power, in the sense that they enable the same Lie group of logical gates for MBQC. Conditions (a) and (b) above generalize the notion of a short-range entangled state to allow for states with topological order and the quantity d depends on the details of the stabilizer state $|\Phi_0\rangle$; e.g., the toric code stabilizers as shown in Fig. 5.

As background to the theorem, we sketch the process of MBQC on symmetric states below:

- (a) *Initialization.* The initial state of the logical system is an eigenstate of $T(H)$ with eigenvalues given by Lemma 3.
- (b) *Readout.* After measuring all sites on the left boundary and in the bulk, the measurement statistics for $T(g)$ can be reproduced by measuring $V_R(g)$ on the right boundary. $V_R(H)$ can be inferred simultaneously from the single-qubit measurement outcomes of $\hat{v}_i^R(H)$.
- (c) *Logical rotations.* For every site k in the bulk, there is a symmetry-respecting measurement basis, namely, the eigenbasis of the nonidentity $\hat{u}_k(G)$. Recall that the $\hat{u}_k(G)$ are Paulis and commute, so there must be a unique $\hat{\alpha}_k \in \{X_k, Y_k, Z_k\}$ the eigenbasis of which is the symmetry-respecting basis. If $\hat{\beta}_k(g_k)$, $R_k(g_k)$, and $\sigma_k(g_k)$ exist for some $g_k \in \mathcal{G}$, then measurement in the rotated eigenbasis of

$$O_k(g_k, \theta_k) = e^{-i\frac{\theta_k}{2}\hat{\beta}_k(g_k)}\hat{\alpha}_k e^{i\frac{\theta_k}{2}\hat{\beta}_k(g_k)} \quad (25)$$

implements the logical completely positive trace-preserving (CPTP) map

$$\mathcal{M}_k(g_k, \theta_k) = \frac{1 + \sigma_k(g_k)}{2} \left[\exp\left(-i\frac{\theta_k}{2}T(g_k)\right) \right] + \frac{1 - \sigma_k(g_k)}{2} \left[\exp\left(i\frac{\theta_k}{2}T(g_k)\right) \right], \quad (26)$$

where the square brackets denote superoperators. This map acts on the logical observables, i.e., it takes

$$T(g) \rightarrow T'(g) = \mathcal{M}_k(g_k, \theta_k)(T(g)) \quad (27)$$

for $g_k \in \mathcal{G}_k$ and $g \in G$. When $\sigma_k(g_k) = 1$, $\mathcal{M}_k$ simplifies to a unitary rotation

$$M_{\log,k}(g_k, \theta_k) = \exp\left(-i\frac{\theta_k}{2}T(g_k)\right), \quad (28)$$

and when $0 \leq \sigma_k(g_k) < 1$, we obtain a linear combination of $M_{\log,k}(g_k, \theta_k)$ and $M_{\log,k}(g_k, -\theta_k)$.

The full proof that measurement in the basis given in Eq. (25) leads to the CPTP map Eq. (26) is long and technical and is given in Appendix A. Below, we sketch part of the proof to provide some intuition for how MBQC works in our formalism. The computational output of MBQC consists of the expectation values of logical observables $\langle \Phi | T(g) | \Phi \rangle$, which can be determined by repeated rounds

of measuring the bulk in the symmetry-respecting basis and the boundary in the eigenbasis of the projective representation of $T(g)$. Now, consider the expectation value

$$\langle \Phi | e^{i\frac{\theta_k}{2}L_k(g_k)} T(g) e^{-i\frac{\theta_k}{2}L_k(g_k)} | \Phi \rangle, \quad (29)$$

where $L_k(g_k) = T(g_k)R_k(g_k)$. L_k looks like $T(g_k)$ to the left of site k , has $\hat{\beta}_k$ at site k , and is the identity to the right of site k . In other words, L_k is a product of symmetry-respecting single-qubit operators, except at site k . Thus, when measuring in the symmetry-respecting basis, we can replace all operators but $\hat{\beta}_k$ with eigenvalues, $L_k(g_k) \rightarrow (\prod_{i < k} (-1)^{s_i}) \hat{\beta}_k$ where $s_i \in \{0, 1\}$. Applying this to Eq. (29), we obtain

$$\langle \Phi | e^{i\frac{\tilde{\theta}_k}{2}\hat{\beta}_k} T(g) e^{-i\frac{\tilde{\theta}_k}{2}\hat{\beta}_k} | \Phi \rangle, \quad (30)$$

where $\tilde{\theta}_k = (\prod_{i < k} (-1)^{s_i}) \theta_k$. This expectation value—and therefore also Eq. (29)—can be sampled by measuring all sites except k in the symmetry-respecting basis and site k in the eigenbasis of $O_k(g_k, \tilde{\theta}_k)$. The dependence of the rotation angle $\tilde{\theta}_k$ on previous measurement outcomes is a critical part of MBQC; it allows us to construct a deterministic computation from the inherently probabilistic process of quantum measurement.

Having established that we can recover Eq. (29) from local measurements, let us now show that it amounts to a logical rotation in the case in which $|\Phi\rangle$ is the reference stabilizer state $|\Phi_0\rangle$ and $L_k(g_k)$ anticommutes with $T(g)$. We have

$$\begin{aligned} \langle \Phi_0 | e^{i\frac{\theta_k}{2}L_k(g_k)} T(g) e^{-i\frac{\theta_k}{2}L_k(g_k)} | \Phi_0 \rangle &= \cos(\theta_k) \langle \Phi_0 | T(g) | \Phi_0 \rangle - i \sin(\theta_k) \langle \Phi_0 | T(g) L_k(g_k) | \Phi_0 \rangle \\ &= \cos(\theta_k) \langle \Phi_0 | T(g) | \Phi_0 \rangle - i \sin(\theta_k) \langle \Phi_0 | T(g) T(g_k) R_k(g_k) | \Phi_0 \rangle \\ &= \cos(\theta_k) \langle \Phi_0 | T(g) | \Phi_0 \rangle - i \sin(\theta_k) \langle \Phi_0 | T(g) T(g_k) | \Phi_0 \rangle \\ &= \langle \Phi_0 | e^{i\frac{\theta_k}{2}T(g_k)} T(g) e^{-i\frac{\theta_k}{2}T(g_k)} | \Phi_0 \rangle \\ &= \langle \Phi_0 | M_{\log,k}^\dagger(g_k, \theta_k) T(g) M_{\log,k}(g_k, \theta_k) | \Phi_0 \rangle, \end{aligned} \quad (31)$$

where from the second to the third line we use that R_k is a stabilizer of $|\Phi_0\rangle$.

When $|\Phi\rangle \neq |\Phi_0\rangle$, we show in Appendix A that $\langle \Phi | T(g) T(g_k) R_k(g_k) | \Phi \rangle = \sigma_k(g_k) \langle \Phi | T(g) T(g_k) | \Phi \rangle$. In combination with the case in which $L_k(g_k)$ and $T(g)$ commute, this leads to the evolution of the logical observables being described by the logical CPTP map given in Eq. (26) instead of a logical unitary. We also show that

any number of logical CPTP maps can be performed in sequence, as long as the sites being measured away from the symmetry-respecting basis are separated by a large enough distance.

To approximately recover a unitary gate, we can split up the desired rotation by θ_k into N consecutive rotations by θ_k/N . This approximates $M_{\log,k}(g_k, \theta_k)$ up to an error ϵ under the diamond norm with the properties stated in

Lemma 12 of Ref. [38]. The error is given by

$$\epsilon = \frac{\theta_k^2}{N} \frac{1 - \sigma_k(g_k)^2}{\sigma_k(g_k)^2}. \quad (32)$$

Thus, while every state with nonzero order parameters has equivalent computational power, they vary in terms of computational efficiency. The most efficient states for computation are those for which all the order parameters are equal to one.

Finally, given the ability to implement the logical rotations $M_{\log,k}(g_k, \theta_k)$ for all $g_k \in \mathcal{G}_k$, we can also implement rotations about their commutators via

$$\begin{aligned} & \exp\left(-i\frac{d\theta}{2}T(g_k)\right) \exp\left(-i\frac{d\theta}{2}T(g_{k'})\right) \\ & \approx \exp\left(-\frac{(d\theta)^2}{4}[T(g_k), T(g_{k'})]\right). \end{aligned} \quad (33)$$

This gives us access to all the unitaries in $\mathcal{L} = \exp(-i\mathcal{A})$, where $\mathcal{A}$ is the Lie algebra generated by $T(\mathcal{G})$.

III. COMPUTATIONAL VERSUS PHYSICAL ORDER IN THE TORIC CODE

A. Computational order

1. Boundary conditions for MBQC

The MBQC scheme described in Sec. II requires a lattice with open boundary conditions, so we modify the toric code Hamiltonian given in Eq. (3) to have smooth boundaries [39] on all sides. The plaquette stabilizers $B_{(x,y)}$ are unchanged but we modify the vertex stabilizers $A_{(x,y)}$ by cutting off any tensor factors that fall outside the lattice (Fig. 6). We set the magnetic field strengths $h_X = h_Z = \alpha$ and we also remove the $X_{(x,y+\frac{1}{2})}$ terms at the left and right boundaries so that the system is invariant under a global

linear representation $U(G)$ with projective representations at the boundaries, as required by Theorem 3.

Putting all these changes together yields the Hamiltonian

$$\begin{aligned} H_{\text{MBQC}} = & - \sum_{y=1}^{L_y} \sum_{x=1}^{L_x} A_{(x,y)} - \sum_{y=1}^{L_y-1} \sum_{x=1}^{L_x-1} B_{(x,y)} \\ & - \alpha \sum_{y=1}^{L_y-1} \sum_{x=2}^{L_x-1} X_{(x,y+\frac{1}{2})} - \alpha \sum_{y=1}^{L_y} \sum_{x=1}^{L_x-1} Z_{(x+\frac{1}{2},y)}. \end{aligned} \quad (34)$$

Note that H_{MBQC} has a unique ground state $|\psi_\alpha\rangle$ for all finite values of α . These ground states will be our resource states for MBQC.

2. Symmetries and order parameters

a. Symmetries. H_{MBQC} and its ground states $|\psi_\alpha\rangle$ are invariant under the same subsystem symmetries defined in Eq. (4), with modifications near the smooth boundary to ensure that they commute with the boundary terms. Specifically, we have

$$U(g_y^e) = \prod_{x=1}^{L_x} A_{(x,y)}, \quad U(g_y^m) = \prod_{x=1}^{L_x-1} B_{(x,y)}. \quad (35)$$

The smooth boundary conditions create one more row of vertices than plaquettes, yielding the symmetry group $G = \langle g_y^e, g_y^m \rangle \cong \mathbb{Z}_2^{2L_y-1}$.

At the right boundary, we can find the projective representations by looking at how $U(g_y^e)$ and $U(g_y^m)$ terminate

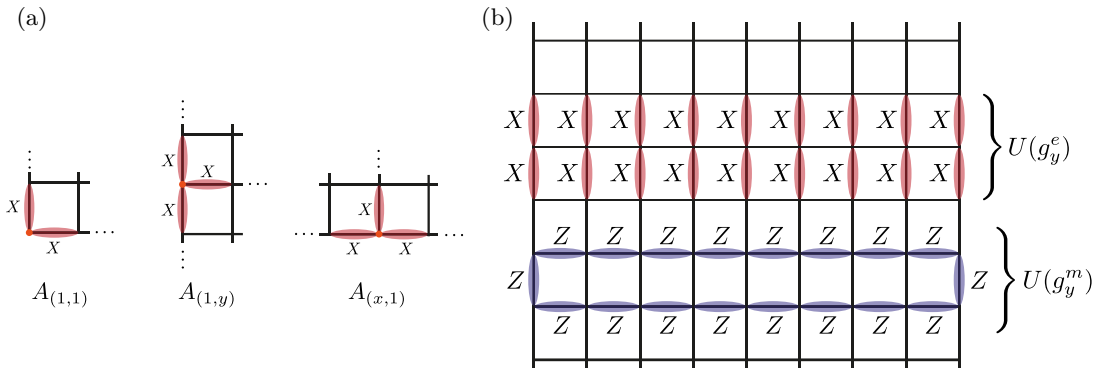


FIG. 6. The toric code with smooth open boundary conditions. (a) We modify the vertex stabilizers by cutting them off at the edges. (b) The action of the subsystem symmetries $U(G)$.

(Fig. 6). This yields

$$V_R(g_y^e) = \begin{cases} X_{(L_x, \frac{3}{2})}, & y = 1, \\ X_{(L_x, y - \frac{1}{2})} X_{(L_x, y + \frac{1}{2})}, & 1 < y < L_y, \\ X_{(L_x, L_y - \frac{1}{2})}, & y = L_y, \end{cases} \quad (36)$$

$$V_R(g_y^m) = Z_{(L_x, y + \frac{1}{2})}.$$

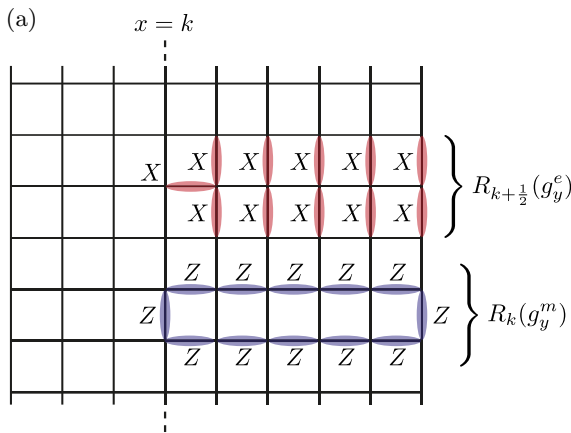
We can see that this is indeed a projective representation because $V_R(g_y^e)$ anticommutes with $V_R(g_y^m)$ and $V_R(g_{y-1}^m)$, i.e., $\kappa(g_y^e, g_y^m) = \kappa(g_y^e, g_{y-1}^m) = 1$. The projective representations on the left boundary are the same.

b. Computational order parameters. Now let us localize one of the linear representations $U(g_y^e)$ to create a COP. First, we restrict $U(g_y^e)$ to the region right of site $(k + \frac{1}{2}, y)$ with $1 < k < L_x$, yielding

$$U_{>k+\frac{1}{2}}(g_y^e) = X_{(k+\frac{1}{2}, y)} \prod_{x=k+1}^{L_x} A_{(x, y)}. \quad (37)$$

$U_{>k+\frac{1}{2}}(g_y^e)$ anticommutes with $U(g_y^m)$ and $U(g_{y-1}^m)$ due to the projective representations at the right boundary, but Eq. (22) says that we may be able to “fix up” this anticommutation by adding an operator $\hat{\beta}_{k+\frac{1}{2}, y}(g_y^e)$ at site $(k + \frac{1}{2}, y)$. On this site, $U(g_y^m)$ and $U(g_{y-1}^m)$ act by Z , so we can choose $\hat{\beta}_{k+\frac{1}{2}, y}(g_y^e) = X_{(k+\frac{1}{2}, y)}$. Then, the product

$$R_{k+\frac{1}{2}}(g_y^e) = X_{(k+\frac{1}{2}, y)} \otimes U_{>k+\frac{1}{2}}(g_y^e) = \prod_{x=k+1}^{L_x} A_{(x, y)} \quad (38)$$



commutes with all $U(G)$. This gives us an “electric” COP $\sigma_{k+\frac{1}{2}}(g_y^e) = \langle R_{k+\frac{1}{2}}(g_y^e) \rangle$. Note that there are no other $U(g)$ that can be localized at site $(k + \frac{1}{2}, y)$; thus $\mathcal{G}_{k+\frac{1}{2}, y} = \{g_y^e\}$.

Similarly, we can localize $U(g_y^m)$ to any site $(k, y + \frac{1}{2})$, $2 \leq k \leq L_x - 1$ and obtain

$$R_k(g_y^m) = \prod_{x=k}^{L_x-1} B_{(x, y)}, \quad (39)$$

from which we obtain a “magnetic” computational order parameter $\sigma_k(g_y^m) = \langle R_k(g_y^m) \rangle$. Once again, $U(g_y^m)$ is the only symmetry that can be localized to site $(k, y + \frac{1}{2})$, so we have $\mathcal{G}_{k, y+\frac{1}{2}} = \{g_y^m\}$. Taking the union $\bigcup_{x, y} \mathcal{G}_{x, y}$, we have $\mathcal{G} = \{g_y^e, g_y^m : y = 1, \dots, L_y\}$. The operators $R_{k+\frac{1}{2}}(g_y^e)$ and $R_k(g_y^m)$ are shown in Fig. 7.

c. Logical observables. Recall that we have one more row of vertices than we do of plaquettes; therefore, a maximal subgroup $H \subset G$ such that $V_R(H)$ is Abelian is given by $H = \langle g_y^e : i = 1, \dots, m+1 \rangle$. With respect to H , we can define the logical observables

$$T(g_y^e) = U(g_y^e) = \prod_{x=1}^{L_x} A_{(x, y)}, \quad (40)$$

$$T(g_y^m) = Z_{(1, y+\frac{1}{2})} \prod_{x=1}^{L_x-1} B_{(x, y)}.$$

The observables $T(g_y^e)$ all commute, so we use them as L_y logical qubits for initialization and readout.

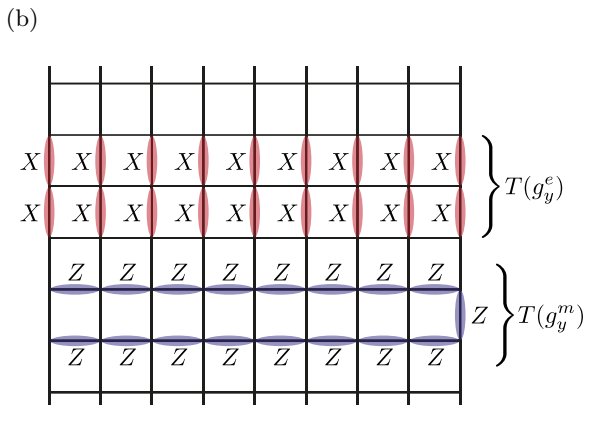


FIG. 7. (a) The operators $R_{k+\frac{1}{2}}(g_y^e)$ and $R_k(g_y^m)$ that give rise to computational order parameters (COPs) in the toric code. (b) The logical observables $T(g_y^e)$ and $T(g_y^m)$.

d. Logical gates. The logical rotation associated with the electric COPs is $\exp(i(\theta/2)T(g_y^e))$ and the logical rotation for the magnetic COPs is $\exp(i(\theta/2)T(g_y^m))$. These are all the logical rotations that exist, because we have established that $\mathcal{G} = \{g_y^e, g_y^m : y = 1, \dots, L_y\}$. The anti-commutation relations of $T(g_y^e)$ and $T(g_y^m)$ are completely determined by the projective representations $V_R(g_y^e)$ and $V_R(g_y^m)$ at the right boundary, so that the Lie algebra $\mathcal{A}$ generated by $T(\mathcal{G})$ is isomorphic to the Lie algebra generated by $V_R(\mathcal{G})$. If we think of the right boundary as a 1D system of $L_y - 1$ qubits, then Eq. (36) tells us that $V_R(\mathcal{G})$ acts as

$$\{X_1, X_1 X_2, \dots, X_{L_y-2} X_{L_y-1}, X_{L_y-1}, \\ Z_1, \dots, Z_{L_y-1}\}. \quad (41)$$

These operators can be mapped via a Jordan-Wigner transformation to the creation operators for a system of L_y free Majorana fermions, which generate the special orthogonal Lie algebra $\mathfrak{so}(2L_y)$. Therefore, we have $\mathcal{A} \cong \mathfrak{so}(2L_y)$ and the Lie group of gates available for MBQC is $\mathcal{L} = \text{SO}(2L_y)$.

e. Computational power. Having derived the global symmetries and COPs, we can check that the conditions of Theorem 3 are satisfied for the ground states $|\psi_\alpha\rangle$:

- (a) The $|\psi_\alpha\rangle$ are gapped ground states of the Hamiltonian given in Eq. (34). In the thermodynamic limit, any given $|\psi_\alpha\rangle$ is in the same phase as the toric code state $|\psi_{TC}\rangle$ or a product state, both of which are stabilizer states. It is known that two states are in the same topological phase if and only if they are related by a local finite-depth unitary [40]; thus, for any α , there exists a local finite-depth unitary connecting it to a stabilizer state. Such a unitary necessarily has finite spread.
- (b) As discussed in Sec. II A, we have $d = 1.5$ such that regions with any combination of rough or smooth boundaries will satisfy Lemma 1 if they are separated by at least $2\Delta + d$.
- (c) $|\psi_\alpha\rangle$ are symmetric under the linear representation $U(G)$ as defined in Eq. (35).

With conditions (a)–(c) fulfilled, Theorem 3 tells us that any $|\psi_\alpha\rangle$ such that the computational order parameters are nonzero can be used as a resource state for MBQC, with L_y logical qubits and gates drawn from $\text{SO}(2L_y)$. In the next section, we show that every COP is dual to a string order parameter as defined in Eq. (48) and therefore that the ground-state topological phase of H_{MBQC} including the toric code corresponds to a computational phase of matter [26].

B. Topological order

1. Derivation of the phase diagram

Having understood the ground-state computational power of the Hamiltonian given in Eq. (3), we now investigate its ground-state topological order with a view to showing how these two properties are related. To understand the ground-state physics of the Hamiltonian given in Eq. (3), it is simplest to borrow the language of lattice gauge theory [33,41] and split H into “electric” and “magnetic” contributions,

$$H = H^e + H^m, \\ H^e = - \sum_{(x,y) \in \Lambda} A_{(x,y)} - h_Z \sum_{(x,y) \in \Lambda} Z_{(x+\frac{1}{2},y)}, \\ H^m = - \sum_{(x,y) \in \Lambda} B_{(x,y)} - h_X \sum_{(x,y) \in \Lambda} X_{(x,y+\frac{1}{2})}. \quad (42)$$

A distinctive feature of our anisotropic perturbing magnetic field in Eq. (2) compared to the standard [32] isotropic perturbing magnetic field of the toric code is that the electric and magnetic terms in Eq. (42) commute with one another, $[H^e, H^m] = 0$. In fact, these Hamiltonians are completely decoupled from one another. This is intuitively clear in the language of anyonic excitations of H_0 [33,42], according to which the operators $A_{(x,y)}$ ($B_{(x,y)}$) count the number of electric charges (magnetic vortices) at their centers, while the operators $Z_{(x+\frac{1}{2},y)}$ ($X_{(x,y+\frac{1}{2})}$) create perfectly horizontal pairs of electric charges (magnetic vortices) on either side of the bonds on which they act. Thus, unlike the standard perturbed toric code, our Hamiltonian H is equivalent to L_y mutually decoupled pairs of Ising models acting on electric and magnetic anyons, respectively [see Fig. 2(b)]. Let us now make this statement mathematically precise.

We can formalize the qualitative mapping described above by taking $L_x \rightarrow \infty$ and mapping electric or magnetic anyons to Ising domain walls defined in terms of new Pauli spin variables σ (on horizontal bonds of Λ) and τ (on sites of Λ), respectively, as

$$A_{(x,y)} \mapsto \sigma_{(x-\frac{1}{2},y)}^z \sigma_{(x+\frac{1}{2},y)}^z, \\ Z_{(x+\frac{1}{2},y)} \mapsto \sigma_{(x+\frac{1}{2},y)}^x \quad (43)$$

in the electric sector and

$$B_{(x,y)} \mapsto \tau_{(x,y)}^z \tau_{(x+1,y)}^z, \\ X_{(x,y+\frac{1}{2})} \mapsto \tau_{(x,y)}^x \quad (44)$$

in the magnetic sector [43]. In terms of these effective spin variables, we find that

$$H^e = \sum_{y=1}^{L_y} H_y^e, \quad H^m = \sum_{y=1}^{L_y} H_y^m, \quad (45)$$

where $[H_y^e, H_{y'}^e] = [H_y^e, H_{y'}^m] = [H_y^m, H_{y'}^m] = 0$ for all $y, y' = 1, 2, \dots, L$, with

$$H_y^e = - \sum_{x \in \mathbb{Z}} \sigma_{(x-\frac{1}{2}, y)}^z \sigma_{(x+\frac{1}{2}, y)}^z - h_Z \sum_{x \in \mathbb{Z}} \sigma_{(x+\frac{1}{2}, y)}^x \quad (46)$$

and

$$H_y^m = - \sum_{x \in \mathbb{Z}} \tau_{(x, y)}^z \tau_{(x+1, y)}^z - h_X \sum_{x \in \mathbb{Z}} \tau_{(x, y)}^x. \quad (47)$$

We have thus mapped the Hamiltonian H to a stack of decoupled transverse-field Ising chains. Similar Hamiltonians and mapping to decoupled Ising chains have previously been obtained for the Wen plaquette model [34] and for the bond-anisotropic Kitaev model [35]. It follows that the confinement transitions out of the topological ground-state phase of H as h_X and h_Z are increased from zero are equivalent to the ferromagnetic-to-paramagnetic ground-state phase transition of the exactly solvable [44] 1D transverse-field Ising model. We deduce that for all values of L_y , electric charges condense (and magnetic vortices are confined) when $h_Z > 1$ and magnetic vortices condense (and electric charges are confined) when $h_X > 1$, leading to the simple phase diagram shown in Fig. 2(c) as $L_y \rightarrow \infty$. Note that in our discussion of measurement-based quantum computing above, we have set $h_X = h_Z = \alpha$ so that the confinement transition out of the topological phase occurs at the (trivially) bicritical point $\alpha = 1$ (see Fig. 2).

2. Topological order parameters

We have shown that the Hamiltonian H exhibits a topological ground-state phase in the thermodynamic limit for $|h_X|, |h_Z| < 1$. Let us now briefly recall how topological phases are diagnosed in practice, e.g., for the usual magnetic field perturbation of the toric code [32,45]. The key idea is “string order” [46–49] that motivates the construction of Fredenhagen-Marcu order parameters [47–49] in lattice gauge theory. Such order parameters have proved theoretically [50,51], experimentally [52], and numerically [45] valuable for diagnosing topological order in condensed-matter systems.

For concreteness, let $L_x = L_y = L$ and let $|\psi_{\text{TC}}\rangle$ denote the ground state of the unperturbed toric code Hamiltonian H_0 . For any (closed or open) string Γ^e on the bonds of the

direct lattice or Γ^m on the bonds of the dual lattice, define string operators or Wilson lines

$$W[\Gamma^e] = \prod_{b \in \Gamma^e} Z_b, \quad W[\Gamma^m] = \prod_{b \in \Gamma^m} X_b. \quad (48)$$

When the strings in question are open, the corresponding string operators act on $|\psi_{\text{TC}}\rangle$ to create pairs of anyons at their end points, so that the resulting state is in a different $\mathbb{Z}_2$ -charge sector from the ground state and $\langle \psi_{\text{TC}} | W[\Gamma^{e/m}] | \psi_{\text{TC}} \rangle = 0$. For open strings with length $\Omega(L)$, one might similarly expect that $\langle \psi | W[\Gamma^{e/m}] | \psi \rangle \rightarrow 0$ in the thermodynamic limit whenever the relevant anyons are deconfined, i.e., $|\psi\rangle$ is in a “topological” phase, while $\langle \psi | W[\Gamma^{e/m}] | \psi \rangle > 0$ in the thermodynamic limit whenever the relevant anyons are confined, i.e., $|\psi\rangle$ is in a “trivial” phase. This motivates the following definition. Let $\Gamma_o^{e/m}$ be open strings of length $\Omega(L)$. We then define “string order parameters”

$$O[\Gamma_o^{e/m}] = \lim_{L \rightarrow \infty} \sqrt{|\langle \psi | W[\Gamma_o^{e/m}] | \psi \rangle|}. \quad (49)$$

However, for the usual magnetic field perturbation of the toric code, the string order parameter has the fatal flaw that it is susceptible to “perimeter-law” scaling at generic points of the phase diagram and therefore decays too fast to be distinguishable from zero in practice, even in the trivial phase. This is usually remedied by the prescription due to Fredenhagen and Marcu [47–49], according to which additional closed loops $\Gamma_c^{e/m}$ with length $|\Gamma_c^{e/m}| = 2|\Gamma_o^{e/m}|$ are introduced, yielding the definition

$$\tilde{O}[\Gamma_o^{e/m}] = \lim_{L \rightarrow \infty} \sqrt{\frac{|\langle \psi | W[\Gamma_o^{e/m}] | \psi \rangle|}{\sqrt{|\langle \psi | W[\Gamma_c^{e/m}] | \psi \rangle|}}}. \quad (50)$$

This rescaling of the string order parameter cancels out the decay due to the perimeter law, yielding a practically useful tool for diagnosing topological order [45,51,52].

Note that the phase diagram in Fig. 2 differs from this conventional understanding in the important respect that its perimeter-law scaling is highly anisotropic: only vertical bonds of the direct lattice contribute to perimeter-law scaling of $O(\Gamma_o^e)$, while only vertical bonds of the dual lattice contribute to perimeter-law scaling of $O(\Gamma_o^m)$. What this means in practice is that the string order parameters $O(\Gamma_o^{e/m})$ suffice to diagnose the various ground-state phases of H without Fredenhagen-Marcu-type rescaling, provided that the long open strings $\Gamma_o^{e/m}$ are taken to be perfectly horizontal. In the remainder of this paper, “string operators” and “string order parameters” will always refer to this special case of perfectly horizontal strings.

3. Kramers-Wannier duality

We have shown above that the Hamiltonian H could be solved exactly by mapping the bare toric code operators to new Pauli operators σ and τ via the map

$$\begin{aligned} M(A_{(x,y)}) &= \sigma_{(x-\frac{1}{2},y)}^z \sigma_{(x+\frac{1}{2},y)}^z, \\ M\left(Z_{(x+\frac{1}{2},y)}\right) &= \sigma_{(x+\frac{1}{2},y)}^x, \\ M(B_{(x,y)}) &= \tau_{(x,y)}^z \tau_{(x+1,y)}^z, \\ M\left(X_{(x,y+\frac{1}{2})}\right) &= \tau_{(x,y)}^x. \end{aligned} \quad (51)$$

However, the image of this map famously exhibits Kramers-Wannier duality [53], implying two distinct ways of mapping to Pauli operators for each row of Λ and for each sector (electric or magnetic) within a given row, yielding 2^{2L_y} distinct ways of mapping H to a stack of transverse-field Ising models. If we demand that the same choice is made in every row and between electric and magnetic sectors, this vast freedom reduces to two possible choices, namely, the mapping Eq. (51) and its Kramers-Wannier dual map $\tilde{M}$, involving Pauli spin variables $\tilde{\sigma}$ (on sites of Λ) and $\tilde{\tau}$ (on horizontal bonds of Λ) such that

$$\begin{aligned} \tilde{M}(A_{(x,y)}) &= \tilde{\sigma}_{(x,y)}^x, \\ \tilde{M}\left(Z_{(x+\frac{1}{2},y)}\right) &= \tilde{\sigma}_{(x,y)}^z \tilde{\sigma}_{(x+1,y)}^z, \\ \tilde{M}(B_{(x,y)}) &= \tilde{\tau}_{(x+\frac{1}{2},y)}^x, \\ \tilde{M}\left(X_{(x,y+\frac{1}{2})}\right) &= \tilde{\tau}_{(x-\frac{1}{2},y)}^z \tilde{\tau}_{(x+\frac{1}{2},y)}^z. \end{aligned} \quad (52)$$

Under this dual map, we find that the decomposition into a stack of independent transverse-field Ising models given in Eq. (45) continues to hold up to an exchange of coupling constants, i.e.,

$$H_y^e = -h_Z \sum_{x \in \mathbb{Z}} \tilde{\sigma}_{(x,y)}^z \tilde{\sigma}_{(x+1,y)}^z - \sum_{x \in \mathbb{Z}} \tilde{\sigma}_{(x,y)}^x \quad (53)$$

and

$$H_y^m = -h_X \sum_{x \in \mathbb{Z}} \tilde{\tau}_{(x-\frac{1}{2},y)}^z \tilde{\tau}_{(x+\frac{1}{2},y)}^z - \sum_{x \in \mathbb{Z}} \tilde{\tau}_{(x+\frac{1}{2},y)}^x. \quad (54)$$

Comparison with Eqs. (46) and (47) reveals that if there is a unique phase transition along the “self-dual line” $h_Z = h_X = \alpha > 0$ in Fig. 2, it must occur at $\alpha = 1$, which provides a nonperturbative way of understanding the location of this phase transition.

In terms of the Kramers-Wannier duality mapping D , which can be defined explicitly as

$$\begin{aligned} D\left(\sigma_{(x-\frac{1}{2},y)}^z \sigma_{(x+\frac{1}{2},y)}^z\right) &= \tilde{\sigma}_{(x,y)}^x, \\ D\left(\sigma_{(x+\frac{1}{2},y)}^x\right) &= \tilde{\sigma}_{(x,y)}^z \tilde{\sigma}_{(x+1,y)}^z, \\ D(\tau_{(x,y)}^z \tau_{(x+1,y)}^z) &= \tilde{\tau}_{(x+\frac{1}{2},y)}^x, \\ D(\tau_{(x,y)}^x) &= \tilde{\tau}_{(x-\frac{1}{2},y)}^z \tilde{\tau}_{(x+\frac{1}{2},y)}^z, \end{aligned} \quad (55)$$

we have

$$\tilde{M} = D \circ M. \quad (56)$$

4. Subsystem symmetry and SSET order

As discussed in Sec. I, the Hamiltonian H can also be understood as a model with subsystem symmetry. Namely, because of the anisotropy of the magnetic field, H commutes with all Wilson loops Γ_c^e (Γ_c^m), where c is a perfectly horizontal closed loop on the direct (dual) lattice. There are two such operators for each row $y = 1, 2, \dots, L_y$. Thus the Hamiltonian H enjoys a much higher degree of symmetry than more generic [32] magnetic field perturbations of the toric code.

However, we do not consider all of these Wilson loops to be symmetries. This is because a single Wilson loop, when truncated, creates anyons at its end points. These anyons cannot be removed by acting only near the end points of the Wilson loop. Therefore, a single Wilson loop cannot be “localized” or used to define a string order parameter. This is due to the fact that the 1-form symmetry generated by the Wilson loops is spontaneously broken. A pair of adjacent Wilson loops, however, is preserved since it creates a pair of anyons at its end points, and this pair can be annihilated by acting with a local operator. Therefore, we define the subsystem symmetries of H to be generated by all pairs of adjacent horizontal Wilson loops. This matches the structure of the symmetries derived in Sec. III A. Furthermore, under the duality mapping in Eq. (52), such symmetries correspond to the $\mathbb{Z}_2$ symmetries of the individual Ising chains, which provides a second intuitive picture of the origin of the subsystem symmetries (alongside the conservation law along rows discussed in Sec. I A).

We now demonstrate that the toric code belongs to a nontrivial SSET phase protected by these symmetries. For now, we return to the thermodynamic limit and define the symmetries more precisely as

$$S(g_y^e) = \prod_{x \in \mathbb{Z}} A_{(x,y)}, \quad S(g_y^m) = \prod_{x \in \mathbb{Z}} B_{(x,y)}, \quad (57)$$

which are equal to products of X and Z , respectively, on neighboring rows, as in Fig. 3. These symmetries generate an infinite group. However, the symmetries are not all independent, as they satisfy a pair of *global relations*

$$\begin{aligned} \prod_{y \in \mathbb{Z}} S(g_y^e) &= \prod_{x, y \in \mathbb{Z} \times \mathbb{Z}} A_{(x, y)} = 1, \\ \prod_{y \in \mathbb{Z}} S(g_y^m) &= \prod_{x, y \in \mathbb{Z} \times \mathbb{Z}} B_{(x, y)} = 1. \end{aligned} \quad (58)$$

According to the procedure described in Ref. [17], identifying global relations between symmetry generators is the first step in determining to which SSET phase a model belongs. The next step is to truncate these global relations to finite regions. We first define the truncation of the symmetry generators to finite intervals $\mathcal{I}_x$,

$$\begin{aligned} S_{\mathcal{I}_x}(g_y^e) &= \prod_{x \in \mathcal{I}_x} A_{(x, y)}, \\ S_{\mathcal{I}_x}(g_y^m) &= \prod_{x \in \mathcal{I}_x} B_{(x, y)}. \end{aligned} \quad (59)$$

These truncations are defined such that they act in the same way as the untruncated generators inside the region $\mathcal{I}_x$ while still commuting with the Hamiltonian. They are in fact identical to the string order parameters illustrated in Fig. 3. Now, we truncate the global relation by restricting the product in Eq. (58) to a finite interval $\mathcal{I}_y$,

$$\begin{aligned} \prod_{y \in \mathcal{I}_y} S_{\mathcal{I}_x}(g_y^e) &= \prod_{x, y \in \mathcal{I}_x \times \mathcal{I}_y} A_{(x, y)} \equiv W[\Gamma_{\partial \mathcal{R}}^m], \\ \prod_{y \in \mathcal{I}_y} S_{\mathcal{I}_x}(g_y^m) &= \prod_{x, y \in \mathcal{I}_x \times \mathcal{I}_y} B_{(x, y)} \equiv W[\Gamma_{\partial \mathcal{R}}^e], \end{aligned} \quad (60)$$

where $\mathcal{R} = \mathcal{I}_x \times \mathcal{I}_y$ is a finite rectangular patch on the lattice and $\partial \mathcal{R}$ is its boundary. We see that restricting the global relation to a finite patch $\mathcal{R}$ is equivalent to the action of a Wilson loop around the boundary of $\mathcal{R}$, as in Fig. 8. More specifically, we have the following correspondence:

$$\begin{aligned} \phi \left(\prod_{y \in \mathbb{Z}} S(g_y^e) \right) &= m, \\ \phi \left(\prod_{y \in \mathbb{Z}} S(g_y^m) \right) &= e, \end{aligned} \quad (61)$$

where the function ϕ takes in a global relation and returns the label of the superselection sector (i.e., anyon type) corresponding to the Wilson loop that appears at the boundary of the truncated relation. It is shown in Ref. [17] that the function ϕ is invariant under symmetry-preserving perturbations and that it completely characterizes the pattern of

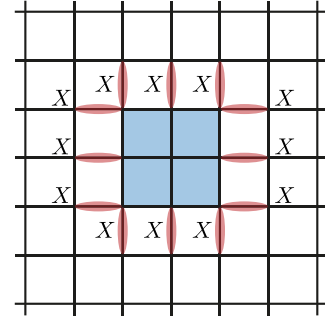


FIG. 8. Truncating the global relation in the e -sector to a finite region $\mathcal{R}$ (shaded region) is equal to a Wilson loop around the boundary of that region. The truncation for the m -sector is similar.

subsystem symmetry fractionalization, and therefore the SSET phase of the model. The fact that ϕ is a nontrivial map shows that the subsystem symmetries have a nontrivial interplay with the anyons. In the present model, this is not so surprising since the symmetries themselves are products of Wilson loops. In more general cases, this interplay is not obvious but can nonetheless be revealed by the above truncation procedure [17].

The nontrivial subsystem-symmetry fractionalization in this model is related to the restricted mobility of its anyons. As we have seen in Fig. 2, the symmetry restrictions only allow anyons to be created in pairs belonging to the same row. Equivalently, symmetry-preserving operations can only move a single anyon horizontally. This is, in fact, a necessary consequence of subsystem-symmetry fractionalization, as shown in Ref. [17].

Finally, we observe that the above calculations have been performed entirely in a finite region of the bulk of the lattice. Therefore, even if we consider a finite lattice with arbitrary boundary conditions, the above procedure can still be performed in the bulk to recover the invariants in Eq. (61). This is even true in the case in which the global relations are violated near the boundary of the lattice. This is akin to the observation that conventional SPT phases are still well defined even when the bulk symmetry needs to be modified near the boundary to commute with the Hamiltonian.

C. Duality between string and computational order

We would now like to relate the notion of computational order, as discussed above, to ground-state topological order of the Hamiltonian H_{MBQC} . This immediately leads us to the following tension: MBQC pertains to finite systems by construction, while phases of matter are only sharply defined in the thermodynamic limit.

In order to make a meaningful comparison between the two concepts, we must therefore take a suitable limit of MBQC. The simplest procedure is to return to the

thermodynamic limit of the toric code Hamiltonian H in Eq. (3) and consider the analog of the “loop operators” given in Eqs. (38) and (39), which, in this setting, correspond to horizontal contractible Wilson loops,

$$R_y^e = \prod_{x=0}^{\ell} A_{(x,y)}, \quad R_y^m = \prod_{x=0}^{\ell-1} B(x,y). \quad (62)$$

We would like to compare these objects to purely horizontal string operators

$$\begin{aligned} W[\Gamma_o^e] &= \prod_{x=0}^{\ell-1} Z_{(x+\frac{1}{2},y)}, \\ W[\Gamma_o^m] &= \prod_{x=0}^{\ell} X_{(x,y+\frac{1}{2})}. \end{aligned} \quad (63)$$

Under the mapping M , we find that the loop operators detect ferromagnetic correlations in the Ising model,

$$\begin{aligned} M(R_y^e) &= \sigma_{(-\frac{1}{2},y)}^z \sigma_{(\ell+\frac{1}{2},y)}^z, \\ M(R_y^m) &= \tau_{(0,y)}^z \tau_{(\ell,y)}^z, \end{aligned} \quad (64)$$

while the string operators that diagnose the topological order map to Ising string operators of length order ℓ ,

$$\begin{aligned} M(W[\Gamma_o^e]) &= \prod_{x'=0}^{\ell-1} \sigma_{(x'+\frac{1}{2},y)}^x, \\ M(W[\Gamma_o^m]) &= \prod_{x'=0}^{\ell} \tau_{(x',y)}^x. \end{aligned} \quad (65)$$

This might naively suggest that the computational and topological order parameters are probing distinct kinds of order in this model. However, under Kramers-Wannier duality these two kinds of order are interchanged. Specifically, under the mapping $\tilde{M}$, we find that loop operators now map to Ising-string-type operators,

$$\begin{aligned} \tilde{M}(R_y^e) &= \prod_{x'=0}^{\ell} \tilde{\sigma}_{(x',y)}^x, \\ \tilde{M}(R_y^m) &= \prod_{x'=0}^{\ell-1} \tilde{\tau}_{(x'+\frac{1}{2},y)}^x, \end{aligned} \quad (66)$$

while string operators now detect ferromagnetic correlations,

$$\begin{aligned} \tilde{M}(W[\Gamma_o^e]) &= \tilde{\sigma}_{(0,y)}^z \tilde{\sigma}_{(\ell,y)}^z, \\ \tilde{M}(W[\Gamma_o^m]) &= \tilde{\tau}_{(-\frac{1}{2},y)}^z \tilde{\tau}_{(\ell+\frac{1}{2},y)}^z. \end{aligned} \quad (67)$$

Let us now take stock of what we have shown. Taking $\ell \rightarrow \infty$, Eq. (67) implies that ground-state topological

order of H , as detected by string order, is equivalent to paramagnetic order in the Kramers-Wannier dual Ising models given in Eqs. (53) and (54), the effective transverse-field strengths of which are $\tilde{h}_X = h_X^{-1}$ and $\tilde{h}_Z = h_Z^{-1}$, respectively. Meanwhile, Eq. (64) implies that the ground-state computational order of H , as detected by “loop order,” is equivalent to the ferromagnetic order in the Ising models given in Eqs. (46) and (47), with effective transverse-field strengths h_X and h_Z , respectively. It follows that in the thermodynamic limit, the computational and topological phases of H are identical and arise for $|h_X|, |h_Z| < 1$, as depicted in Fig. 2.

More abstractly, this discussion reveals that for the anisotropic toric code Hamiltonian H , computational and topological order are “two sides of the same coin”: their respective loop and string order parameters both probe ferromagnetic order but for small perturbations $|h_X|, |h_Z| \ll 1$, this ferromagnetic order (or lack thereof) arises in very different models, weakly and strongly coupled Ising chains, respectively, which have qualitatively distinct physics but are nevertheless related by a Kramers-Wannier duality.

It is instructive to check numerically that our conclusions about the computational ground-state phase of H , which have been based on reasoning in the thermodynamic limit, are borne out in finite systems, which are required for MBQC. To this end, we simulate the true finite-size MBQC Hamiltonian given in Eq. (34) and its associated loop operators Eqs. (38) and (39). We compare the behavior of these operators to the finite-system string operators

$$\begin{aligned} W[\Gamma_o^e] &= \prod_{x=\frac{L_X}{2}}^{L_X-1} Z_{(x+\frac{1}{2},y)}, \\ W[\Gamma_o^m] &= \prod_{x=\frac{L_X}{2}}^{L_X-1} X_{(x,y+\frac{1}{2})}. \end{aligned} \quad (68)$$

By the above discussion, we expect finite-size loop and string operators to probe the same phase transition at $h_X = h_Z = \alpha = 1$ and to have nonzero ground-state expectation values on opposite sides of this transition. These expectations are confirmed by a density matrix renormalization group (DMRG) simulation of the ground state of H_{MBQC} , as shown in Fig. 9 (for technical details, see Appendix B).

IV. A UNIVERSAL MBQC RESOURCE STATE WITH SET ORDER

The toric code model described above is well understood from the perspective of topological order and condensed-matter physics, but from the perspective of MBQC it has only limited computational power. As discussed in Sec. III A 1, the Lie group of gates available

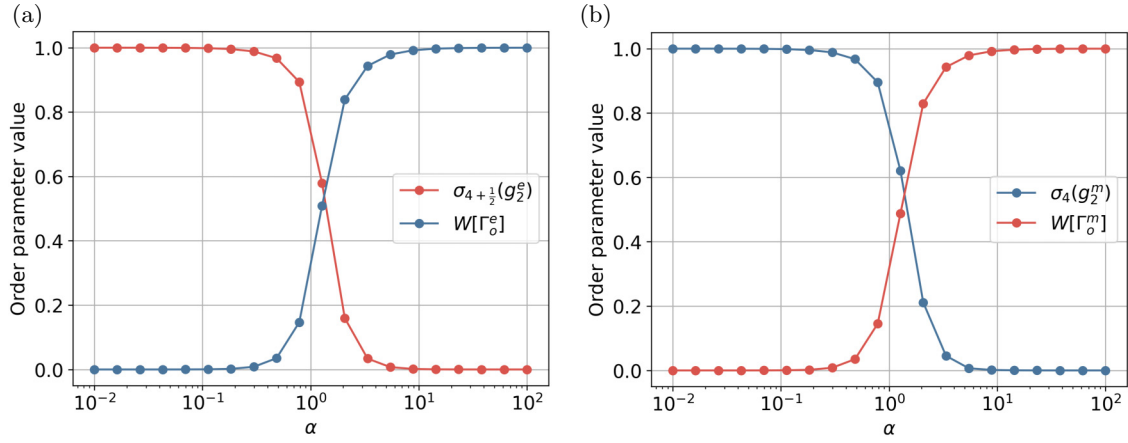


FIG. 9. A density matrix renormalization group (DMRG) simulation of (a) electric and (b) magnetic order parameters in the ground state of H_{MBQC} , with $L_x = 7, L_y = 3$. We observe that the COPs σ_k and string order parameters $W[\Gamma_o^{e/m}]$ are nonzero on opposite sides of the expected phase transition at $\alpha = 1$. For details of the simulation techniques, see Appendix B.

for MBQC with the toric code ground state is $\text{SO}(2L_y)$, which is not a universal group of gates—in fact, quantum computation restricted to operations from $\text{SO}(2L_y)$ is efficiently classically simulable [54,55]. In other words, while the computational power of the toric code is nontrivial, it is far from optimal. Given this, one might ask whether topologically ordered states limit the computational power available for MBQC.

We answer this question in the negative by presenting a state that is computationally universal for MBQC and also topologically ordered. Our example is the ground state of a stabilizer Hamiltonian on a square 2D lattice with qubits on the vertices and dimensions $L_x \times L_y$. We call this model the XZ -star model, because the Hamiltonian is a sum of star-shaped operators

$$H_* = - \sum_x \sum_{y \text{ even}} C_{x,y} - \sum_x \sum_{y \text{ odd}} D_{x,y}, \quad (69)$$

where

$$\begin{aligned} C_{x,y} &= X_{x-1,y} X_{x,y} X_{x+1,y} X_{x,y-1} X_{x,y+1}, \\ D_{x,y} &= Z_{x-1,y} Z_{x,y} Z_{x+1,y} Z_{x,y-1} Z_{x,y+1}. \end{aligned} \quad (70)$$

A. Topological order

The elementary excitations of H_* come in pairs and are created by string operators $W[\Gamma^e]$ and $W[\Gamma^o]$ acting along strings Γ^e (Γ^o) that start and end on sites with even (odd) y coordinate. Given a string Γ^e starting at site (x, y) , $W[\Gamma^e]$ acts in a three-periodic pattern $Z_{x,y} Z_{x+1,y} I_{x+2,y} \dots$ when moving horizontally and on every second site, i.e., $Z_{x,y} I_{x,y+1} \dots$ when moving vertically. $W[\Gamma^o]$ acts in the same pattern but with X instead of Z . Thus the excitations

can only move by hopping three sites horizontally or two sites vertically.

There are four independent types of excitations in the model: e ($\bar{e}$) created in pairs by $W[\Gamma^e]$ starting at $x = 0$ ($x = 1 \bmod 3$), respectively, and m ($\bar{m}$) created in pairs by $W[\Gamma^o]$ starting at $x = 1$ ($x = 0 \bmod 3$). The strings starting at $x = 2 \bmod 3$ create the composite particles $e \times \bar{e}$ and $m \times \bar{m}$. From the commutation relations of the string operators, we can deduce that $\{e, \bar{e}, m, \bar{m}\}$ reproduce the anyon theory of two copies of the toric code, implying that there is a local finite-depth quantum circuit that maps the XZ -star model onto two copies of the toric code [56]. An example of such a circuit is given in Fig. 10. Therein, every third column of XZ -star stabilizer generators $C_{k,y}$ and $D_{k,y}$ for $k = 2 \bmod 3$ are mapped to single X or Z operators, respectively. Thus, every third column becomes a column of ancilla qubits in a product state. The products of $C_{k,y} C_{k+1,y}$ and $D_{k,y} D_{k+1,y}$ for $k = 1, 2 \bmod 3$ are mapped to the vertex and plaquette stabilizers of two copies of the toric code. Together, these terms generate the stabilizer group of the XZ -star model, which completes the mapping.

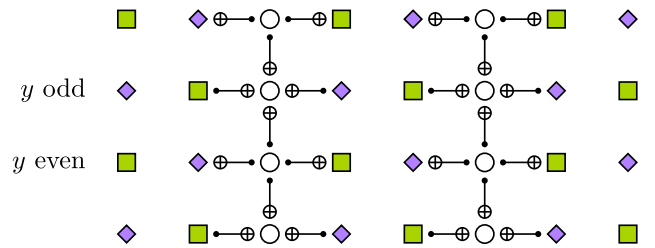


FIG. 10. A quantum circuit that maps the XZ -star model to two independent copies of the toric code (one on the green squares and the other on the purple diamonds) plus ancillas in a product state (white circles).

B. Subsystem symmetries and SSET order

The XZ -star model admits a set of subsystem symmetries that are similar to those in our toric code model:

$$\begin{aligned}
 U(g_{110}^y) &= \prod_{x=0,1,\text{mod } 3} C_{x,y}, \\
 U(g_{011}^y) &= \prod_{x=1,2,\text{mod } 3} C_{x,y}, \\
 U(g_{101}^y) &= \prod_{x=0,2,\text{mod } 3} C_{x,y}, \\
 U(\bar{g}_{110}^y) &= \prod_{x=0,1,\text{mod } 3} D_{x,y}, \\
 U(\bar{g}_{011}^y) &= \prod_{x=1,2,\text{mod } 3} D_{x,y}, \\
 U(\bar{g}_{101}^y) &= \prod_{x=0,2,\text{mod } 3} D_{x,y}.
 \end{aligned} \tag{71}$$

These are parallel lines of X or Z in a three-periodic pattern. $U(g)$ acts on odd rows and $U(\bar{g})$ acts on even rows. Since we have yet to perturb the model, H_* actually has a much larger set of symmetries. Here, we have chosen a minimal set of symmetries that are needed to protect the universal computational power of the model, as we demonstrate shortly.

These symmetries admit a set of global relations, namely,

$$\begin{aligned}
 \prod_{y \text{ even}} U(g_{110}^y) &= 1, & \prod_{y \text{ even}} U(g_{011}^y) &= 1, \\
 \prod_{y \text{ odd}} U(\bar{g}_{110}^y) &= 1, & \prod_{y \text{ odd}} U(\bar{g}_{011}^y) &= 1.
 \end{aligned} \tag{72}$$

As for the toric code, we determine the SSET phase to which H_* belongs by calculating the function ϕ that identifies the anyon type of the boundary Wilson loop that arises upon restricting these relations to rectangular

regions. Skipping the details, we find that

$$\begin{aligned}
 \phi \left(\prod_{y \text{ even}} U(g_{110}^y) \right) &= m, & \phi \left(\prod_{y \text{ even}} U(g_{011}^y) \right) &= \bar{m}, \\
 \phi \left(\prod_{y \text{ odd}} U(\bar{g}_{110}^y) \right) &= e, & \phi \left(\prod_{y \text{ odd}} U(\bar{g}_{011}^y) \right) &= \bar{e}.
 \end{aligned} \tag{73}$$

This is the same fractionalization pattern as two decoupled toric codes, with each individual toric code equipped with the subsystem symmetry described in Sec. III A 1. Such a decoupled stack of toric codes is not universal for MBQC, since we have shown that a single toric code is not universal and that single-site measurements cannot couple the two states. Thus, even though H_* and two toric codes share the same topological order and the same kind of SSET order, they do not have the same computational power. This matches similar observations that have been made for SPT phases [27]. The explanation lies in the symmetry representation, which differs between H_* and two decoupled toric codes. In the context of MBQC, the microscopic representation of the symmetry of the lattice is an important ingredient due to the restriction to single-site measurements. In the context of phases of matter, the representation often plays a secondary role since it can change under real-space renormalization or local quantum circuits. We conclude that computational universality of the ground states of H_* lies not in their topological or SSET order but, rather, in the particular way in which the two constituent toric codes have been intertwined.

C. Computational order

For MBQC, we will consider open boundary conditions, with $x = 0, 1, \dots, L_x$, $y = 1, 2, \dots, L_y$, and L_x divisible by 3. As in the toric code example, we modify the stabilizers at the boundary by cutting them off. This causes the subsystem symmetries to act projectively at the boundaries. Specifically, we have the projective representations

$$\begin{aligned}
 V_L(g_{110}^y) &= X_{0,y-1} X_{0,y+1}, & V_R(g_{110}^y) &= X_{L_x,y-1} X_{L_x,y} X_{L_x,y+1}, \\
 V_L(\bar{g}_{110}^y) &= Z_{0,y-1} Z_{0,y+1}, & V_R(\bar{g}_{110}^y) &= Z_{L_x,y-1} Z_{L_x,y} Z_{L_x,y+1}, \\
 V_L(g_{011}^y) &= X_{0,y}, & V_R(g_{011}^y) &= X_{L_x,y}, \\
 V_L(\bar{g}_{011}^y) &= Z_{0,y}, & V_R(\bar{g}_{011}^y) &= Z_{L_x,y}, \\
 V_L(g_{101}^y) &= X_{0,y-1} X_{0,y} X_{0,y+1}, & V_R(g_{101}^y) &= X_{L_x,y-1} X_{L_x,y} X_{L_x,y+1}, \\
 V_L(\bar{g}_{101}^y) &= Z_{0,y-1} Z_{0,y} Z_{0,y+1}, & V_R(\bar{g}_{101}^y) &= Z_{L_x,y-1} Z_{L_x,y} Z_{L_x,y+1}.
 \end{aligned} \tag{74}$$

A maximal Abelian subgroup of $V_R(G)$ is generated by $\{V_R(\bar{g}_{011}^y) : y = 1, 2, \dots, L_y\}$. Therefore, we define $H = \langle \bar{g}_{011}^y : y = 1, 2, \dots, L_y \rangle$ and choose the logical observables $T(G)$ according to Eq. (23). The rank of H is L_y ; therefore, we have L_y logical qubits.

To determine the Lie group of gates for MBQC, we need to construct the operators $R_k(g)$. As in the toric code example, we do this by solving Eq. (22) for every site (k, y) in the bulk. This yields

$$\begin{aligned} k = 0 \bmod 3 : \quad R_k(g_{011}^y) &= \prod_{x=1,2,\bmod 3} C_{k+x,y}, & R_k(\bar{g}_{011}^y) &= \prod_{x=1,2,\bmod 3} D_{k+x,y}, \\ k = 1 \bmod 3 : \quad R_k(g_{101}^y) &= \prod_{x=1,2,\bmod 3} C_{k+x,y}, & R_k(\bar{g}_{101}^y) &= \prod_{x=1,2,\bmod 3} D_{k+x,y}, \\ k = 2 \bmod 3 : \quad R_k(g_{110}^y) &= \prod_{x=1,2,\bmod 3} C_{k+x,y}, & R_k(\bar{g}_{110}^y) &= \prod_{x=1,2,\bmod 3} D_{k+x,y}, \end{aligned} \quad (75)$$

as shown in Fig. 11. Therefore, we have

$$\mathcal{G} = \{g_{110}^y, \bar{g}_{110}^y, g_{011}^y, \bar{g}_{011}^y, g_{101}^y, \bar{g}_{101}^y\}, \quad (76)$$

where $y = 1, \dots, L_y$.

The Lie algebra of MBQC gates is generated by the projective representations $V_R(g), g \in \mathcal{G}$, which are precisely the representations given in Eq. (74). To determine which Lie algebra is generated, we note that any two sets of n -qubit Pauli operators with the same commutation relations generate the same Lie algebra. Thus we need only find a set of Paulis with the same commutation relations as in Eq. (74) and such that the Lie algebra that they generate is known. One set satisfying these conditions is the set of terms from the Heisenberg-chain Hamiltonian: $\{X_i X_{i+1}, Y_i Y_{i+1}, Z_i Z_{i+1} : i = 1, \dots, L_y - 1\}$. The algebra generated by this set—and therefore by $V_R(\mathcal{G})$ —is

shown in Ref. [57, Theorem IV.1] to be

$$\mathcal{A} = \begin{cases} \text{su}(2^{L_y-1}), & L_y \text{ odd}, \\ \text{su}(2^{L_y-2})^{\oplus 4}, & L_y \text{ even}. \end{cases} \quad (77)$$

Thus the Lie group $\mathcal{L}$ of MBQC gates is $\text{SU}(2^{L_y-1})$ for L_y odd and $\text{SU}(2^{L_y-2})$ for L_y even, which are universal gate sets for $L_y - 1$ and $L_y - 2$ qubits, respectively.

V. EFFECTS OF TOPOLOGICAL ORDER ON MBQC

A. Topological obstruction to symmetry localization

Symmetry localization is the process that takes symmetry elements to COPs and thus to MBQC executable gates. Not every symmetry can be localized to form an order parameter, which is why the gate set for MBQC may be smaller than the set of symmetries. In the case of the toric

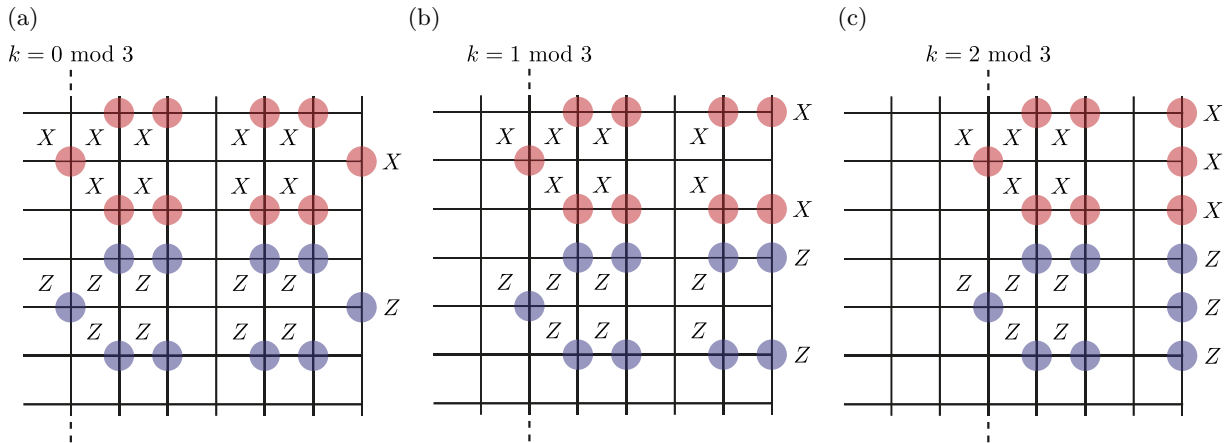


FIG. 11. The operators $R_k(g)$ that yield COPs in the XZ -star model. The operators in (a) correspond to $\bar{m}$ and e anyons, those in (b) to m and $\bar{e}$, and those in (c) to $m \times \bar{m}$ and $e \times \bar{e}$, as discussed in Sec. VB. (a) Top row, $R_k(g_{011}^y)$; bottom row, $R_k(\bar{g}_{011}^y)$. (b) Top row, $R_k(g_{101}^y)$; bottom row, $R_k(\bar{g}_{101}^y)$. (c) Top row, $R_k(g_{110}^y)$; bottom row, $R_k(\bar{g}_{110}^y)$.

code, this failure to localize can be explained by a topological obstruction to localization, indicating that topological properties have a direct effect on MBQC.

For example, consider the restriction of an electric symmetry $U(g_y^e)$ to the region right of site $(k + \frac{1}{2}, y)$, which consists of two strings of X operators on neighboring rows of vertical edges. Applying this restricted symmetry operator to the toric code ground state creates a pair of magnetic charges at $(k + \frac{1}{2}, y \pm \frac{1}{2})$, which can be “cleaned up” by the local operator $X_{k+\frac{1}{2},y}$ to give the localized symmetry $R_{k+\frac{1}{2}}(g_y^e)$ and the corresponding COP $\sigma_{k+\frac{1}{2}}(g_y^e)$. In contrast, consider a single horizontal string of X operators, which is also a symmetry of the Hamiltonian Eq. (34). Truncating this single string to the right of site $(k, y + \frac{1}{2})$ gives a string operator as in Eq. (68) and creates a single magnetic charge at $(k + \frac{1}{2}, y + \frac{1}{2})$. Due to the topological nature of the charges in the toric code, a single charge cannot be annihilated by any local operator, preventing the localization of a single string operator.

B. Relationship between anyons and logical rotations

As described above, the construction of computational order parameters by symmetry truncation associates each order parameter with a particular type of anyon, and can be thought of as an operator that moves a pair of anyons into the bulk from the right boundary and then annihilates them. For two different order parameters, the braiding relation of the corresponding anyons determines the anticommutation of their projective representations at the right boundary and thus whether the MBQC gates associated with those order parameters commute or anticommute. For example, in the toric code, the X -type COPs are associated with m particles and the Z -type COPs are associated with e particles. The corresponding $X_i X_{i+1}$ and Z_i rotations anticommute and this anticommutation can be traced back to the nontrivial braiding of the e and m particles in the toric code. Any set of Pauli rotations with the same anticommutation relations generates the same Lie group; thus the braiding of anyons also determines the Lie group of gates for MBQC.

This relationship between anyons and MBQC gates offers some insight into why the XZ -star model is computationally universal even though it is equivalent to two copies of the toric code. The COPs $R_k(g)$ in this model can be associated with anyons depending on the column k in which the tip is located, modulo 3 (see Fig. 11). If $k = 0 \bmod 3$, then $R_k(g)$ (y even) corresponds to an $\bar{m}$ anyon, while $R_k(\bar{g})$ (y odd) corresponds to an e anyon. For $k = 1 \bmod 3$, the order parameters correspond to m and $\bar{e}$ anyons, respectively, and for $k = 2 \bmod 3$, they correspond to the composite anyons $m \times \bar{m}$ and $e \times \bar{e}$. This last column is the key to unlocking computational universality, because the anyons $m \times \bar{m}$ and $e \times \bar{e}$ braid nontrivially with anyons from *both* copies of the toric code present in the XZ -star

model. Thus, the corresponding MBQC gates anticommute with gates from both toric codes, allowing the Lie group to be larger than it would be for two decoupled toric codes.

C. Constants of motion

Another mechanism by which topological order appears to affect MBQC is “constants of motion.” Specifically, in both the toric code and the XZ -star model, we observe that there exist operators on the right boundary that commute with all MBQC gates (excluding those at the corners). Such operators must be conserved by any MBQC circuit—hence the name. In the toric code, there is one constant of motion, $\prod_y Z_{L_x, y+\frac{1}{2}}$. In the XZ -star model, there are two: $\prod_{y \text{ even}} X_{L_x, y}$ and $\prod_{y \text{ odd}} Z_{L_x, y}$.

We conjecture that constants of motion are a generic feature of MBQC in topologically ordered systems for the following reason. In 2D, topologically ordered systems have anyons created by string operators that commute with every stabilizer generator (i.e., every term in the Hamiltonian) except at their end points. A string operator spanning the entire right boundary must therefore commute with the projective representation $V_R(G)$ that generates the MBQC gates (with the possible exception of those at the corners of a system with open boundaries). For a cylindrical system, a string operator spanning the right boundary commutes with all of $V_R(G)$ and is therefore a constant of motion.

VI. DISCUSSION AND CONCLUSIONS

We have presented a formalism based on nonlocal order parameters that characterizes MBQC in 2D systems, including for the first time systems with topological order. Our formalism allows us to describe the first examples of computational phases of matter with topological order: the toric code in an anisotropic field and the computationally universal XZ -star model. These examples take computational phases beyond SPT ordered systems, which have previously been the only systems known to host such phases. They also illustrate that computational order fails to correspond neatly to SET order or SSET order, in the sense that the topological data and symmetry fractionalization pattern of a given phase are not sufficient to determine its computational power. Instead, there appears to be a more complicated interplay between computational power and the symmetries that protect it. It seems to us that understanding the precise nature of this relationship between symmetry, computational order, and physical order is an important direction for future work.

A. Subsystem symmetry and entanglement

Subsystem symmetry is particularly significant from this perspective because it appears in nearly all known examples of nontrivial computational order, including those

presented here. Now that we can characterize the computational power of long-range entangled systems, it would be interesting to study the models described in Ref. [16,17] from the perspective of MBQC. For example, for the models described in Ref. [17], there are simple external fields that drive the model to a trivial SSET phase without destroying the topological order. By studying this transition, can we distinguish the roles of topological order and subsystem-symmetry enrichment in MBQC more clearly?

In addition to subsystem symmetry, entanglement plays an important role in MBQC. In particular, MBQC is efficiently classically simulable on states where the Schmidt-rank width grows at most logarithmically with the system size [58], implying that there is a minimum amount of entanglement required for universal MBQC. Yet one can also have too much entanglement; in fact, the majority of states in the Hilbert space of a many-qubit system are too entangled to be useful resource states [59]. From this perspective, one might expect long-range entanglement to be detrimental for MBQC, but our XZ -star model shows that long-range entanglement and computational universality can coexist. Can understanding the connection between MBQC and physical order help to clarify the role of entanglement in MBQC?

B. Practical considerations

All states in the phases that we study here have equivalent power for MBQC, which promises a larger set of target states to aim for when preparing resource states in the laboratory. This affords some robustness but does not make our MBQC scheme fault tolerant against arbitrary environmental noise, because we require the subsystem symmetries to remain unbroken.

However, in three or higher dimensions, the story may be different. It is already known that MBQC with the three-dimensional (3D) cluster state is both computationally universal and fault tolerant [60]. This fault tolerance can be understood via foliation, a duality between MBQC with the 3D cluster state and repeated rounds of quantum error correction on the 2D surface code. In fact, it is possible to embed any 2D stabilizer code into a 3D resource state via foliation [61–63], and there are even fault-tolerant MBQC schemes that go beyond foliation [64].

We expect a version of Theorem 3 to hold in three and higher dimensions, so there may exist computationally universal and fault-tolerant SSET phases in 3D. It would be interesting to find examples of such phases and to understand whether the combination of symmetry, topological excitations, and MBQC offers any advantages for fault-tolerant quantum computation. Furthermore, it has been shown that 3D SPT phases can be used as self-correcting quantum memories [65], which suggests that symmetry protection and active quantum error correction can be unified in 3D models. We expect that MBQC with 3D SSET

states will be a useful setting in which this unification can be further explored.

Returning to 2D, our results have implications for the efficiency of MBQC with different algorithms. For any computationally ordered state characterized by Theorem 3, there is an elementary gate set, namely, rotations about the Pauli axes given by the projective symmetry representations on the boundary of the system. Even if this gate set is universal, there may be significant differences in the number of measurements and qubits required to implement an arbitrary gate, because arbitrary gates must be composed as sequences of commutators of the elementary gates. Thus, depending which gates appear most frequently in the desired quantum algorithm, one resource state may have a practical advantage over another. Showing that MBQC can be universal in SSET ordered states allows us to have a larger pool of elementary gate sets to draw from and may enable better tailoring of resource states to quantum algorithms.

C. Nonlocal games

The generalization of utility for MBQC from SPT phases to topological phases mirrors a similar generalization that has been obtained recently in the setting of quantum nonlocal games [66]. The first works on playing nonlocal games with ground states of local Hamiltonians found that a robust quantum advantage for such games could be obtained in SPT [67,68] and conventional symmetry-breaking [68] phases of matter, but an apparent lack of robustness in topological phases, e.g., for nonlocal games played in the topological phase of the toric code perturbed by bond-isotropic magnetic fields [68,69]. This lack of robustness has been remedied by a game that can be played with local [70], rather than nonlocal [69], stabilizers of the toric code. Thus, quantum advantage for nonlocal games appears to be less demanding than fully fledged MBQC (no subsystem symmetry is required). What properties must a quantum phase of matter possess in order to yield a robust quantum advantage for nonlocal games (in the sense of, e.g., Ref. [71])?

D. Conclusions

In summary, our results highlight the significant overlap between objects of interest for MBQC—such as symmetries, order parameters, and logical observables—and objects of interest to condensed-matter physics. For the examples studied in this paper, especially the anisotropic toric code, we have been able to make these connections quantitatively precise, but there appears to be plenty of scope for developing broader and more systematic connections between computational power for MBQC and the kinds of quantum states that arise naturally in condensed-matter physics.

ACKNOWLEDGMENTS

We thank Arnab Adhikary, Ehud Altman, Michael Knap, Tobias J. Osborne, and Frank Pollmann for helpful discussions. P.H. is funded by the National Science and Engineering Research Council of Canada (NSERC). This work was done in part while V.B.B. was visiting the Simons Institute for the Theory of Computing, supported by U.S. Department of Energy (DOE) Quantum Systems Accelerator (QSA) Grant No. #FP00010905. Y.J. acknowledges funding by the Ministry of Science and Culture

of Lower Saxony through *Quantum Valley Lower Saxony Q1 (QVLS-Q1)*. D.T.S. acknowledges support from the Simons Collaboration on Ultra-Quantum Matter, which is a grant from the Simons Foundation (Grant No. 651440). R.R. is funded by the Humboldt Foundation.

DATA AVAILABILITY

The data that support the findings of this paper are openly available [72].

APPENDIX A: PROOF OF THEOREM 3

Theorem 3 (Computational power). Consider a state $|\Phi\rangle$ of a 2D spin-1/2 system satisfying the following conditions:

- (a) $|\Phi\rangle$ is related to a stabilizer state $|\Phi_0\rangle$ by a local finite-depth unitary that commutes with $U(G)$ and has finite spread Δ .
- (b) There exists a finite d such that Lemma 1 holds for any two regions of the lattice separated by at least $2\Delta + d$.
- (c) $|\Phi\rangle$ is symmetric under $U(G)$ as defined in Eq. (16).
- (d) The COPs $\sigma_k(g_k)$ are nonzero for all k .

Then, MBQC with $|\Phi\rangle$ as a resource state can implement the following:

- (i) Initialization and readout of $\text{rank}(H)$ logical qubits, where H is a maximal subgroup of G such that $V_R(H)$ is Abelian.
- (ii) Logical gates from the Lie group $\mathcal{L} = \exp(-i\mathcal{A})$, where $\mathcal{A}$ is the Lie algebra generated by $T(\mathcal{G})$ under $i[\cdot, \cdot]$ and linear combinations.

This proof is a generalization of the proof given in Ref. [25] to 2D systems. We will make use of the operators

$$L_k(g) = T(g)R_k(g), \quad (\text{A1})$$

which act nontrivially on the region to the left of site k . Before we continue, it will be useful to establish some commutation relations between the various operators defined so far.

Lemma A1. The symmetry representations $U(G)$, $T(G)$, $R_k(\mathcal{G}_k)$, and $L_k(\mathcal{G}_k)$ have the following commutation relations:

$$[U(g)]R_k(g_k) = 0, \quad g \in G, g_k \in \mathcal{G}, k \in B, \quad (\text{A2})$$

$$[T(g)]R_k(g_k) = 0, \quad g \in G, g_k \in \mathcal{G}, k \in B, \quad (\text{A3})$$

$$T_k(g')T(g) = (-1)^{\kappa(g,g')}T(g)T_k(g'), \quad g, g' \in G, \quad (\text{A4})$$

$$L_k(g_k)T(g) = (-1)^{\kappa(g,g_k)}T(g)L_k(g_k), \quad g \in G, g_k \in \mathcal{G}, k \in B. \quad (\text{A5})$$

Proof. The operators $R_k(g)$ are already constructed such that Eq. (A2) holds. The $T(g)$ differ from the $U(g)$ only on the left boundary, so Eq. (A3) follows from Eq. (A2). To prove Eq. (A4), recall that the $T(g)$ are constructed such that they all commute on the left boundary, so they inherit their commutation relations from the right boundary, which are precisely the relations encoded by $\kappa(g, g')$. Finally, for Eq. (A5), we can apply the definition Eqs. (A1) and (A3) to show that

$$\begin{aligned} L_k(g_k)T(g) &= T(g_k)R_k(g_k)T(g) = (-1)^{\kappa(g,g_k)}T(g)T(g_k)R_k(g_k) \\ &= (-1)^{\kappa(g,g_k)}T(g)L_k(g_k). \end{aligned} \quad (\text{A6})$$

■

The rotation operators obtained by exponentiating $L_k(g_k)$ are

$$M_k(g_k, \theta_k) := \exp\left(-i\frac{\theta_k}{2}L_k(g_k)\right) \quad g_k \in \mathcal{G}_k. \quad (\text{A7})$$

For each rotation, there is a corresponding logical rotation

$$M_{\log,k}(g_k, \theta_k) = \exp\left(-i\frac{\theta_k}{2}T(g_k)\right) \quad (\text{A8})$$

and a logical CPTP map

$$\mathcal{M}_k(g_k, \theta_k) = \frac{1 + \sigma_k(g_k)}{2} [M_{\log,k}(\theta_k)] + \frac{1 - \sigma_k(g_k)}{2} [M_{\log,k}^\dagger(\theta_k)]. \quad (\text{A9})$$

In the following lemma, we will show that conjugation by M_k is equivalent to the logical CPTP map $\mathcal{M}_k$, in the sense that it reproduces the same measurement statistics for all logical observables.

Lemma A2. Consider a state $|\Phi\rangle$ satisfying conditions (a)–(c) of Theorem 3 and a sequence of sites $\{k_1, \dots, k_f\}$ such that the horizontal distance between site k_i and k_{i-1} is at least $2\Delta + d$ for all i . Then, the following equation holds:

$$\langle \Phi | M_{k_1}^\dagger \dots M_{k_f}^\dagger T(g) M_{k_f} \dots M_{k_1} | \Phi \rangle = \text{Tr} \left[T(g) \mathcal{M}_{k_f} \dots \mathcal{M}_{k_1} (|\Phi\rangle\langle\Phi|) \right]. \quad (\text{A10})$$

Proof. We will prove a single step of the evolution and the rest follows by induction. Let $|\Phi(k_i)\rangle = M_{k_i} \dots M_{k_1} |\Phi\rangle$ and consider the expectation value

$$\langle \Phi(k_{i-1}) | M_{k_i}^\dagger T(g) M_{k_i} | \Phi(k_{i-1}) \rangle. \quad (\text{A11})$$

If $T(g_{k_i})$ commutes with $T(g)$, then, by Lemma A1, so does $L_k(g_{k_i})$ and we have

$$\langle \Phi(k_{i-1}) | M_{k_i}^\dagger T(g) M_{k_i} | \Phi(k_{i-1}) \rangle = \langle \Phi(k_{i-1}) | T(g) | \Phi(k_{i-1}) \rangle. \quad (\text{A12})$$

If $T(g_{k_i})$ does not commute with $T(g)$, then it must anticommute, in which case

$$\langle \Phi(k_{i-1}) | M_{k_i}^\dagger T(g) M_{k_i} | \Phi(k_{i-1}) \rangle = \cos(\theta_{k_i}) \langle \Phi(k_{i-1}) | T(g) | \Phi(k_{i-1}) \rangle - i \sin(\theta_{k_i}) \langle \Phi(k_{i-1}) | T(g) L_{k_i}(g_{k_i}) | \Phi(k_{i-1}) \rangle. \quad (\text{A13})$$

For the second term, we have

$$\begin{aligned} \langle \Phi(k_{i-1}) | T(g) L_{k_i}(g_{k_i}) | \Phi(k_{i-1}) \rangle &= \langle \Phi(k_{i-1}) | T(g) T(g_{k_i}) R_{k_i}(g_{k_i}) | \Phi(k_{i-1}) \rangle \\ &= \langle \Phi | M_{k_1}^\dagger \dots M_{k_{i-1}}^\dagger T(g) T(g_{k_i}) M_{k_{i-1}} \dots M_{k_1} R_{k_i}(g_{k_i}) | \Phi \rangle \\ &= \langle \Phi | U^\dagger(gg_{k_i}) M_{k_1}^\dagger \dots M_{k_{i-1}}^\dagger T(g) T(g_{k_i}) M_{k_{i-1}} \dots M_{k_1} R_{k_i}(g_{k_i}) | \Phi \rangle \\ &= \langle \Phi | U^\dagger(gg_{k_i}) M_{k_1}^\dagger \dots M_{k_{i-1}}^\dagger T(g) T(g_{k_i}) M_{k_{i-1}} \dots M_{k_1} | \Phi \rangle \langle \Phi | R_{k_i}(g_{k_i}) | \Phi \rangle \\ &= \sigma_{k_i}(g_{k_i}) \langle \Phi(k_{i-1}) | T(g) T(g_{k_i}) | \Phi(k_{i-1}) \rangle. \end{aligned} \quad (\text{A14})$$

In the first line, we have used Eq. (A1) and in the second line, we have moved R_{k_i} past $M_{k_{i-1}} \dots M_{k_1}$ because they have disjoint support. In the third line, we pull out a global symmetry $U(gg_{k_i})$ and we note that

$$\text{supp}(U^\dagger(gg_{k_i}) M_{k_1}^\dagger \dots M_{k_{i-1}}^\dagger T(g) T(g_{k_i}) M_{k_{i-1}} \dots M_{k_1}) \subseteq \{\leq k_{i-1}\}, \quad (\text{A15})$$

while $\text{supp}(R_{k_i}(g_{k_i})) \subseteq \{\geq k_i\}$, which allows us to apply Lemma 1 in the fourth line.

Combining Eqs. (A12)–(A14), we have

$$\begin{aligned}
 \langle \Phi(k_{i-1}) | M_{k_i}^\dagger T(g) M_{k_i} | \Phi(k_{i-1}) \rangle &= \frac{1 + \sigma_{k_i}(g_{k_i})}{2} \langle \Phi(k_{i-1}) | M_{\log, k_i}^\dagger T(g) M_{\log, k_i} | \Phi(k_{i-1}) \rangle \\
 &\quad + \frac{1 - \sigma_{k_i}(g_{k_i})}{2} \langle \Phi(k_{i-1}) | M_{\log, k_i} T(g) M_{\log, k_i}^\dagger | \Phi(k_{i-1}) \rangle \\
 &= \langle \Phi(k_{i-1}) | \mathcal{M}_{k_i}^\dagger(T(g)) | \Phi(k_{i-1}) \rangle \\
 &= \text{Tr}[T(g) \mathcal{M}_{k_i}(|\Phi(k_{i-1})\rangle \langle \Phi(k_{i-1})|)].
 \end{aligned} \tag{A16}$$

■

Lemma A3. Consider a state $|\Phi\rangle$ and sequence of sites $\{k_1, \dots, k_f\}$ as in Lemma A2. The expectation value

$$\langle \Phi | M_{k_1}^\dagger \dots M_{k_d}^\dagger T(h) M_{k_d} \dots M_{k_1} | \Phi \rangle, \quad h \in H \tag{A17}$$

can be reproduced by adaptive single-qubit measurements on $|\Phi\rangle$ for any subgroup $H \subset G$ such that $V_R(H)$ is Abelian.

Proof. Denote

$$U_L(g) = \begin{cases} V_L(g), & g \in H, \\ I_L, & g \in G \setminus H, \end{cases} \tag{A18}$$

and let $U_{<k}^B(g)$ be the restriction of $U_B(g)$ to $\{<k\}$. Note that this is different than $U_{<k}(g) = V_L(g) \otimes U_{<k}^B(g)$.

Let $M_{\leq k_i} = M_{k_i} \dots M_{k_1}$ and define the intermediate logical observables

$$\begin{aligned}
 T_0(g) &= U_L(g), \\
 T_{\leq k_i}(g) &= M_{\leq k_i}^\dagger U_L(g) \otimes U_{\leq k_i}^B(g) M_{\leq k_i}.
 \end{aligned} \tag{A19}$$

Also, let $U_{k_i:k_j}^B(g) = \bigotimes_{x \in \{<k_j\} \setminus \{\leq k_i\}} \hat{u}_x(g)$ be the restriction of $U_B(g)$ to sites the horizontal position of which is between k_i and k_j and let $\partial\{<k\}$ be the set of sites on the boundary of $\{<k\}$. The intermediate observables satisfy a recursion relation

$$\begin{aligned}
 T_{\leq k_{i+1}}(g) &= M_{\leq k_{i+1}}^\dagger \left(U_L(g) \otimes U_{\leq k_{i+1}}^B(g) \right) M_{\leq k_{i+1}} \\
 &= M_{\leq k_{i+1}}^\dagger \left(U_L(g) \otimes U_{\leq k_i}^B(g) \otimes U_{k_i:k_{i+1}}^B(g) \otimes \left(\bigotimes_{j \in \partial\{<k_{i+1}\}} \hat{u}_j(g) \right) \right) M_{\leq k_{i+1}} \\
 &= \left(M_{\leq k_i}^\dagger (U_L(g) \otimes U_{\leq k_i}^B(g)) M_{\leq k_i} \otimes U_{k_i:k_{i+1}}^B(g) \otimes \left(\bigotimes_{j \in \partial\{<k_{i+1}\} \setminus k_{i+1}} \hat{u}_j(g) \right) \right) (M_{\leq k_{i+1}}^\dagger \hat{u}_{k_{i+1}}(g) M_{\leq k_{i+1}}) \\
 &= \left(T_{\leq k_i}(g) \otimes U_{k_i:k_{i+1}}^B(g) \otimes \left(\bigotimes_{j \in \partial\{<k_{i+1}\} \setminus k_{i+1}} \hat{u}_j(g) \right) \right) (M_{\leq k_i}^\dagger M_{k_{i+1}}^\dagger M_{\leq k_i} \hat{u}_{k_{i+1}}(g) M_{\leq k_i}^\dagger M_{k_{i+1}} M_{\leq k_i})
 \end{aligned} \tag{A20}$$

where in the third and fourth lines we use that $L_{k_i}(g_{k_i})$ —and therefore M_{k_i} —commutes with $\hat{u}_j(g)$ for all $j \neq k_i$ and $g \in G$. The product of rotations in the last line simplifies to

$$\begin{aligned}
 M_{\leq k_i}^\dagger M_{k_{i+1}}^\dagger M_{\leq k_i} &= M_{\leq k_i}^\dagger \exp \left(-i \frac{\theta_{k_{i+1}}}{2} L_{k_{i+1}}(g_{k_{i+1}}) \right) M_{\leq k_i} \\
 &= \exp \left(-i \frac{\theta_{k_{i+1}}}{2} M_{\leq k_i}^\dagger L_{k_{i+1}}(g_{k_{i+1}}) M_{\leq k_i} \right) \\
 &= \exp \left(-i \frac{\theta_{k_{i+1}}}{2} T_{\leq k_i}(g_{k_{i+1}}) \otimes \hat{\beta}_{k_{i+1}}(g_{k_{i+1}}) \right)
 \end{aligned} \tag{A21}$$

We use this to define the observables

$$O_{\leq k_i}(g, g_{k_i}, \theta_{k_i}) = \exp\left(i \frac{\theta_{k_i}}{2} T_{\leq k_{i-1}}(g_{k_i}) \otimes \hat{\beta}_{k_i}(g_{k_i})\right) \hat{u}_{k_i}(g) \exp\left(-i \frac{\theta_{k_i}}{2} T_{\leq k_{i-1}}(g_{k_i}) \otimes \hat{\beta}_{k_i}(g_{k_i})\right). \quad (\text{A22})$$

Then, the recursion relation in Eq. (A20) becomes

$$T_{\leq k_{i+1}}(g) = \left(T_{\leq k_i}(g) \otimes U_{k_i: k_{i+1}}^B(g) \otimes \left(\bigotimes_{j \in \partial\{\leq k_{i+1}\} \setminus k_{i+1}} \hat{u}_j(g) \right) \right) O_{\leq k_i}(g, g_{k_i}, \theta_{k_i}). \quad (\text{A23})$$

We now show by induction that the measurement outcome $\lambda_{\leq k_i}(g)$ of $T_{\leq k_i}(g)$ can be inferred from single-qubit measurements. For the first step, $T_0(g) = U_L(g)$, we have that $\lambda_0(g)$ can be simultaneously measured for all $g \in G$ because $U_L(G)$ are products of commuting single-qubit observables. Then, we can replace $T_0(g)$ by $\lambda_0(g)$ in Eqs. (A22) and (A23), yielding

$$T_{\leq k_1}(g) = \lambda_0(g) U_{< k_1}^B(g) \otimes \left(\bigotimes_{j \in \partial\{\leq k_1\} \setminus k_1} \hat{u}_j(g) \right) \otimes O_{k_1}(g, g_{k_1}, \theta_{k_1}), \quad (\text{A24})$$

where

$$O_{k_1}(g, g_{k_1}, \theta_{k_1}) = \exp\left(i \frac{\lambda_0(g) \theta_{k_1}}{2} \hat{\beta}_{k_1}(g_{k_1})\right) \hat{u}_{k_1}(g) \exp\left(-i \frac{\lambda_0(g) \theta_{k_1}}{2} \hat{\beta}_{k_1}(g_{k_1})\right). \quad (\text{A25})$$

The observables $U_{< k_1}^B(g) \otimes \left(\bigotimes_{j \in \partial\{\leq k_1\} \setminus k_1} \hat{u}_j(g) \right)$ are a product of commuting single-qubit observables, so they can be measured simultaneously for all $g \in G$. Then, note that the outcome of $O_{k_1}(g, g_{k_1}, \theta_{k_1})$ for all $g \in G$ can be inferred by measuring the qubit at site k_1 in a rotated basis

$$\exp\left(i \frac{\lambda_0(g) \theta_{k_1}}{2} \hat{\beta}_{k_1}(g_{k_1})\right) \hat{\alpha}_{k_1} \exp\left(-i \frac{\lambda_0(g) \theta_{k_1}}{2} \hat{\beta}_{k_1}(g_{k_1})\right), \quad (\text{A26})$$

where $\hat{\alpha}_{k_1} \in \{X_{k_1}, Y_{k_1}, Z_{k_1}\}$ is the Pauli operator at site k_1 that commutes with $\hat{u}_{k_1}(G)$. Therefore, we can infer $\lambda_{\leq k_1}(g)$ for all $g \in G$ from single-qubit measurements. Similarly, we can infer $\lambda_{\leq k_{i+1}}(g)$ from $\lambda_{\leq k_i}(g)$.

The induction continues until we reach $\lambda_{\leq k_f}(g)$. To obtain the measurement outcome of

$$M_{k_1}^\dagger \dots M_{k_d}^\dagger T(g) M_{k_d} \dots M_{k_1} = T_{\leq k_f}(g) \otimes U_{> k_d}^B(g) \otimes V_R(g), \quad (\text{A27})$$

we need to measure $U_{> k_d}^B(g)$ and $V_R(g)$. $U_{> k_d}^B(g)$ is a linear representation, so it can be simultaneously measured for all $g \in G$, but $V_R(g)$ is a projective representation; thus we can only simultaneously measure a subgroup $H \subset G$ such that $V_R(H)$ is Abelian. ■

Lemma A4. Given a state $|\Phi\rangle$ satisfying conditions (a)–(c) of Theorem 3, MBQC can implement the logical unitaries $M_{\log, k}(g_k, \theta_k)$ with error ϵ under the diamond norm with the properties stated in Ref. [38, Lemma 12], given by

$$\epsilon = \frac{\theta_k^2}{N} \frac{1 - \sigma_k(g_k)^2}{\sigma_k(g_k)^2}. \quad (\text{A28})$$

Proof. We refer to Ref. [25, Appendix B], where it is proven that a sequence of N logical CPTP maps $\mathcal{M}_k(g_k, \theta_k/N)$ approximate the unitary $M_{\log, k}(g_k, \theta_k)$ to error ϵ . ■

Proof of Theorem 3. With Lemma A2, we have shown that the expectation values of the logical observables $T(g)$ after conjugation by $M_{k_f} \dots M_{k_1}$ are equal to their expectation values after applying the logical CPTP maps $\mathcal{M}_{k_f} \dots \mathcal{M}_{k_1}$. With Lemma A3, we have shown that these expectation values can be inferred by local measurements. With Lemma A4, we have established that a sequence of N logical CPTP maps with rotation angle θ/N can approximate a logical unitary $M_{\log,k}(g_k, \theta_k)$ to arbitrary precision as long as the corresponding order parameter $\sigma_k(g_k)$ is nonzero.

The logical unitaries that are available for MBQC are rotations about axes defined by the logical observables $T(\mathcal{G})$. These logical observables derive their commutation relations from $V_R(\mathcal{G})$; thus the Lie algebra that they generate under commutators and linear combinations is isomorphic to the Lie algebra generated by $V_R(\mathcal{G})$. The same statement holds at the level of Lie groups after applying the exponential map. This establishes item (ii) of Theorem 3. To establish item (i), we note that the initial state of the logical observables is given by Lemma 3, and Lemma A3 tells us that we can read out $T(H)$ by local measurement. ■

APPENDIX B: DETAILS OF NUMERICAL SIMULATIONS

In this appendix, we summarize the numerical method used to generate Fig. 9. We use White's *density matrix renormalization group* (DMRG) [73,74], a classical numerical technique that provides a variational representation of a ground state $|\psi\rangle$. This ground state is expressed as a *matrix product state* (MPS), which is structured as follows [75,76]:

$$|\psi\rangle = \sum_{\substack{\sigma_1 \dots \sigma_N \\ \alpha_1, \dots, \alpha_{N-1}}} T_{\alpha_0, \alpha_1}^{\sigma_1} \dots T_{\alpha_{i-1}, \alpha_i}^{\sigma_i} \dots T_{\alpha_{N-1}, \alpha_N}^{\sigma_N} |\sigma_1 \dots \sigma_N\rangle, \quad (\text{B1})$$

where each $T_{\alpha_{i-1}, \alpha_i}^{\sigma_i}$ is a rank-3 tensor. The indices α_{i-1} and α_i span the *bond dimension* χ_i at site i , with the maximum bond dimension, which controls the accuracy of the MPS, denoted $\chi_{\max}$. Open boundary conditions are imposed by setting $\chi_0 = \chi_N = 1$.

DMRG is a powerful algorithm that optimizes the variational parameters of the MPS, specifically the rank-3 tensors $T_{\alpha_{i-1}, \alpha_i}^{\sigma_i}$. The algorithm proceeds through a process of sequential quadratic variational optimizations:

$$E_{\text{gs}} = \min_{T_{\alpha_{i-1}, \alpha_i}^{\sigma_i}} \langle \psi | \hat{H} | \psi \rangle, \quad (\text{B2})$$

where the optimization is carried out for each site i in the system and continues until convergence.

In our simulations, we begin by expressing the Hamiltonian $\hat{H}_{\text{MBQC}}$ as a matrix product operator

$$\hat{H}_{\text{MBQC}} = \sum_{\substack{\sigma_1 \dots \sigma_N \\ \sigma'_1 \dots \sigma'_N}} M^{\sigma_1 \sigma'_1} \dots M^{\sigma_i \sigma'_i} \dots M^{\sigma_N \sigma'_N} |\sigma_1 \dots \sigma_N\rangle \langle \sigma'_1 \dots \sigma'_N|, \quad (\text{B3})$$

where each tensor $M^{\sigma_i \sigma'_i}$ is a $D \times D$ matrix, with D representing the bond dimension between sites i and $i+1$. The states $|\sigma_i\rangle$ and $|\sigma'_i\rangle$ refer to the local basis states at site i . To extend the MPS algorithms to two dimensions, one can label the sites of the lattice in a snakelike pattern that spans all the sites vertically or horizontally. In this work, we use the pattern depicted in Fig. 12, in which the direction of the arrow shows the labeling order. By using this convention, one can write the matrix-product-operator (MPO) representation of the Hamiltonian or any other observable.

To find the MPO representation of the Hamiltonian in the form given in Eq. (34), we use a matrix product diagram based on a finite state machine [77,78]. Depending on the structure of the Hamiltonian, different approaches are required to determine the MPO representation. When the Hamiltonian follows a regular structure, such as nearest-neighbor interactions, the MPO typically shows uniformity throughout the system. However, for Hamiltonians with arbitrary structures, a matrix product diagram-based representation is necessary [77–80].

The parameters of our DMRG simulation are as follows: maximum bond dimension 200, energy convergence threshold 10^{-5} , negligible singular-value cutoff 10^{-32} , and a maximum of 20 sweeps. As α becomes much larger than one, the ground state becomes exponentially close to degeneracy because of the lack of magnetic fields at the left and right boundaries. Therefore, instead of setting these fields to zero, we set them to 10^{-3} , which is always at least an order of magnitude smaller than any other term in the Hamiltonian. The raw data for Fig. 9 and code to reproduce our results are available from the authors upon reasonable request.

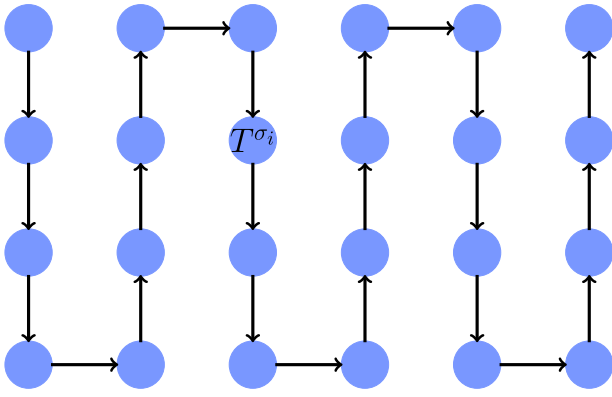


FIG. 12. Mapping a two-dimensional (2D) lattice to a matrix product state form by labeling each site in a snakelike path through the lattice and representing each site with its corresponding rank-3 tensor.

-
- [1] R. Raussendorf and H. J. Briegel, A one-way quantum computer, *Phys. Rev. Lett.* **86**, 5188 (2001).
- [2] R. Raussendorf, D. E. Browne, and H. J. Briegel, Measurement-based quantum computation on cluster states, *Phys. Rev. A* **68**, 022312 (2003).
- [3] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Rigorous results on valence-bond ground states in antiferromagnets, *Phys. Rev. Lett.* **59**, 799 (1987).
- [4] G. K. Brennen and A. Miyake, Measurement-based quantum computer in the gapped ground state of a two-body Hamiltonian, *Phys. Rev. Lett.* **101**, 010502 (2008).
- [5] M. Rossi, M. Huber, D. Bruß, and C. Macchiavello, Quantum hypergraph states, *New J. Phys.* **15**, 113022 (2013).
- [6] O. Gühne, M. Cuquet, F. E. S. Steinhoff, T. Moroder, M. Rossi, D. Bruß, B. Kraus, and C. Macchiavello, Entanglement and nonclassical properties of hypergraph states, *J. Phys. A: Math. Theor.* **47**, 335303 (2014).
- [7] J. Miller and A. Miyake, Hierarchy of universal entanglement in 2D measurement-based quantum computation, *npj Quantum Inf.* **2**, 16036 (2016).
- [8] J. Miller and A. Miyake, Latent computational complexity of symmetry-protected topological order with fractional symmetry, *Phys. Rev. Lett.* **120**, 170503 (2018).
- [9] X. Chen, Z.-C. Gu, and X.-G. Wen, Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order, *Phys. Rev. B* **82**, 155138 (2010).
- [10] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Entanglement spectrum of a topological phase in one dimension, *Phys. Rev. B* **81**, 064439 (2010).
- [11] X. Chen, Z.-C. Gu, and X.-G. Wen, Classification of gapped symmetric phases in one-dimensional spin systems, *Phys. Rev. B* **83**, 035107 (2011).
- [12] N. Schuch, D. Pérez-García, and I. Cirac, Classifying quantum phases using matrix product states and projected entangled pair states, *Phys. Rev. B* **84**, 165139 (2011).
- [13] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Symmetry protected topological orders and the group cohomology of their symmetry group, *Phys. Rev. B* **87**, 155114 (2013).
- [14] X. Chen, Symmetry fractionalization in two dimensional topological phases, *Rev. Phys.* **2**, 3 (2017).
- [15] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, Symmetry fractionalization, defects, and gauging of topological phases, *Phys. Rev. B* **100**, 115147 (2019).
- [16] D. T. Stephen, J. Garre-Rubio, A. Dua, and D. J. Williamson, Subsystem symmetry enriched topological order in three dimensions, *Phys. Rev. Res.* **2**, 033331 (2020).
- [17] D. T. Stephen, A. Dua, J. Garre-Rubio, D. J. Williamson, and M. Hermele, Fractionalization of subsystem symmetries in two dimensions, *Phys. Rev. B* **106**, 085104 (2022).
- [18] P.-S. Hsin, D. T. Stephen, A. Dua, and D. J. Williamson, Subsystem symmetry fractionalization and foliated field theory, *SciPost Phys.* **18**, 147 (2025).
- [19] Y. You, T. Devakul, F. J. Burnell, and S. L. Sondhi, Subsystem symmetry protected topological order, *Phys. Rev. B* **98**, 035112 (2018).
- [20] T. Devakul, Y. You, F. J. Burnell, and S. L. Sondhi, Fractal symmetric phases of matter, *SciPost Phys.* **6**, 007 (2019).
- [21] A. S. Darmawan, G. K. Brennen, and S. D. Bartlett, Measurement-based quantum computation in a 2D phase of matter, *New J. Phys.* **14**, 013023 (2012).
- [22] A. Miyake, Quantum computation on the edge of a symmetry-protected topological order, *Phys. Rev. Lett.* **105**, 040501 (2010).
- [23] D. V. Else, I. Schwarz, S. D. Bartlett, and A. C. Doherty, Symmetry-protected phases for measurement-based quantum computation, *Phys. Rev. Lett.* **108**, 240505 (2012).
- [24] D. T. Stephen, D.-S. Wang, A. Prakash, T.-C. Wei, and R. Raussendorf, Computational power of symmetry-protected topological phases, *Phys. Rev. Lett.* **119**, 010504 (2017).
- [25] R. Raussendorf, W. Yang, and A. Adhikary, Measurement-based quantum computation in finite one-dimensional systems: String order implies computational power, *Quantum* **7**, 1215 (2023).
- [26] R. Raussendorf, C. Okay, D.-S. Wang, D. T. Stephen, and H. P. Nautrup, Computationally universal phase of quantum matter, *Phys. Rev. Lett.* **122**, 090501 (2019).
- [27] D. T. Stephen, H. P. Nautrup, J. Bermejo-Vega, J. Eisert, and R. Raussendorf, Subsystem symmetries, quantum cellular automata, and computational phases of quantum matter, *Quantum* **3**, 142 (2019).
- [28] T. Devakul and D. J. Williamson, Universal quantum computation using fractal symmetry-protected cluster phases, *Phys. Rev. A* **98**, 022332 (2018).
- [29] A. K. Daniel, R. N. Alexander, and A. Miyake, Computational universality of symmetry-protected topologically ordered cluster phases on 2D Archimedean lattices, *Quantum* **4**, 228 (2020).
- [30] D. T. Stephen, W. W. Ho, T.-C. Wei, R. Raussendorf, and R. Verresen, Universal measurement-based quantum computation in a one-dimensional architecture enabled by dual-unitary circuits, *Phys. Rev. Lett.* **132**, 250601 (2024).
- [31] S. Bravyi and R. Raussendorf, Measurement-based quantum computation with the toric code states, *Phys. Rev. A* **76**, 022304 (2007).
- [32] I. S. Tupitsyn, A. Kitaev, N. V. Prokof'ev, and P. C. E. Stamp, Topological multicritical point in the phase diagram of the toric code model and three-dimensional lattice gauge Higgs model, *Phys. Rev. B* **82**, 085114 (2010).

- [33] A. Y. Kitaev, Fault-tolerant quantum computation by anyons, *Ann. Phys. (NY)* **303**, 2 (2003).
- [34] J. Yu, S.-P. Kou, and X.-G. Wen, Topological quantum phase transition in the transverse Wen-plaquette model, *Europhys. Lett.* **84**, 17004 (2008).
- [35] G. Wachtel and D. Orgad, Confinement transition in a Kitaev-like honeycomb model with bond anisotropy, *Phys. Rev. B* **99**, 115104 (2019).
- [36] T. Morimae, Quantum computational tensor network on string-net condensate, *Phys. Rev. A* **85**, 062328 (2012).
- [37] P. Herringer and R. Raussendorf, Classification of measurement-based quantum wire in stabilizer PEPS, *Quantum* **7**, 1041 (2023). [arXiv:2207.00616](https://arxiv.org/abs/2207.00616).
- [38] D. Aharonov, A. Kitaev, and N. Nisan, in *Proceedings of the Thirtieth Annual ACM Symposium on Theory of Computing* (Association for Computing Machinery, New York, NY, USA, 1998), pp. 20–30.
- [39] S. B. Bravyi and A. Y. Kitaev, Quantum codes on a lattice with boundary, [arXiv:quant-ph/9811052](https://arxiv.org/abs/quant-ph/9811052).
- [40] S. Bravyi, M. B. Hastings, and F. Verstraete, Lieb-Robinson bounds and the generation of correlations and topological quantum order, *Phys. Rev. Lett.* **97**, 050401 (2006).
- [41] J. B. Kogut, An introduction to lattice gauge theory and spin systems, *Rev. Mod. Phys.* **51**, 659 (1979).
- [42] A. Kitaev and C. Laumann, Topological phases and quantum computation, [arXiv:0904.2771](https://arxiv.org/abs/0904.2771).
- [43] Equivalently, one can define the σ and τ operators microscopically as $\sigma_{(x+\frac{1}{2},y)}^z = \prod_{x' \leq x} A_{(x',y)}$ and $\sigma_{(x+\frac{1}{2},y)}^x = Z_{(x+\frac{1}{2},y)}$ in the electric sector and $\tau_{(x,y)}^z = \prod_{x' < x} B_{(x',y)}$ and $\tau_{(x,y)}^x = X_{(x,y+\frac{1}{2})}$ in the magnetic sector.
- [44] P. Pfeuty, The one-dimensional Ising model with a transverse field, *Ann. Phys. (NY)* **57**, 79 (1970).
- [45] W.-T. Xu, F. Pollmann, and M. Knap, Critical behavior of Fredenhagen-Marcu string order parameters at topological phase transitions with emergent higher-form symmetries, *npj Quantum Inf.* **11**, 74 (2025).
- [46] J. Bricmont and J. Frölich, An order parameter distinguishing between different phases of lattice gauge theories with matter fields, *Phys. Lett. B* **122**, 73 (1983).
- [47] K. Fredenhagen and M. Marcu, Charged states in Z_2 gauge theories, *Commun. Math. Phys.* **92**, 81 (1983).
- [48] K. Fredenhagen and M. Marcu, Confinement criterion for QCD with dynamical quarks, *Phys. Rev. Lett.* **56**, 223 (1986).
- [49] M. Marcu, in *Lattice Gauge Theory '86*, edited by H. Satz, I. Harrity, and J. Potvin (Springer US, Boston, Massachusetts, 1987), pp. 223–233.
- [50] D. A. Huse and S. Leibler, Are sponge phases of membranes experimental gauge-Higgs systems? *Phys. Rev. Lett.* **66**, 437 (1991).
- [51] K. Gregor, D. A. Huse, R. Moessner, and S. L. Sondhi, Diagnosing deconfinement and topological order, *New J. Phys.* **13**, 025009 (2011).
- [52] G. Semeghini, H. Levine, A. Keesling, S. Ebadi, T. T. Wang, D. Bluvstein, R. Verresen, H. Pichler, M. Kalinowski, R. Samajdar, A. Omran, S. Sachdev, A. Vishwanath, M. Greiner, V. Vuletić, and M. D. Lukin, Probing topological spin liquids on a programmable quantum simulator, *Science* **374**, 1242 (2021).
- [53] H. A. Kramers and G. H. Wannier, Statistics of the two-dimensional ferromagnet. Part I, *Phys. Rev.* **60**, 252 (1941).
- [54] L. G. Valiant, in *Proceedings of the Thirty-Third Annual ACM Symposium on Theory of Computing* (Association for Computing Machinery, New York, NY, USA, 2001), pp. 114–123.
- [55] B. M. Terhal and D. P. DiVincenzo, Classical simulation of noninteracting-fermion quantum circuits, *Phys. Rev. A* **65**, 032325 (2002).
- [56] J. Haah, Classification of translation invariant topological Pauli stabilizer codes for prime dimensional qudits on two-dimensional lattices, *J. Math. Phys.* **62**, 012201 (2021).
- [57] R. Wiersema, E. Kökcü, A. F. Kemper, and B. N. Bakalov, Classification of dynamical Lie algebras of 2-local spin systems on linear, circular and fully connected topologies, *npj Quantum Inf.* **10**, 1 (2024).
- [58] M. Van den Nest, W. Dür, G. Vidal, and H. J. Briegel, Classical simulation versus universality in measurement-based quantum computation, *Phys. Rev. A* **75**, 012337 (2007).
- [59] D. Gross, S. T. Flammia, and J. Eisert, Most quantum states are too entangled to be useful as computational resources, *Phys. Rev. Lett.* **102**, 190501 (2009).
- [60] R. Raussendorf, J. Harrington, and K. Goyal, Topological fault-tolerance in cluster state quantum computation, *New J. Phys.* **9**, 199 (2007). [arXiv:quant-ph/0703143](https://arxiv.org/abs/quant-ph/0703143).
- [61] A. Bolt, G. Duclos-Cianci, D. Poulin, and T. M. Stace, Foliated quantum error-correcting codes, *Phys. Rev. Lett.* **117**, 070501 (2016).
- [62] B. J. Brown and S. Roberts, Universal fault-tolerant measurement-based quantum computation, *Phys. Rev. Res.* **2**, 033305 (2020).
- [63] M. Newman, L. A. de Castro, and K. R. Brown, Generating fault-tolerant cluster states from crystal structures, *Quantum* **4**, 295 (2020). [arXiv:1909.11817](https://arxiv.org/abs/1909.11817).
- [64] N. Nickerson and H. Bombin, Measurement based fault tolerance beyond foliation, [arXiv:1810.09621](https://arxiv.org/abs/1810.09621).
- [65] S. Roberts and S. D. Bartlett, Symmetry-protected self-correcting quantum memories, *Phys. Rev. X* **10**, 031041 (2020).
- [66] G. Brassard, A. Broadbent, and A. Tapp, Quantum pseudo-telepathy, *Found. Phys.* **35**, 1877 (2005).
- [67] A. K. Daniel and A. Miyake, Quantum computational advantage with string order parameters of one-dimensional symmetry-protected topological order, *Phys. Rev. Lett.* **126**, 090505 (2021).
- [68] V. B. Bulchandani, F. J. Burnell, and S. L. Sondhi, Playing nonlocal games with phases of quantum matter, *Phys. Rev. B* **107**, 045412 (2023).
- [69] V. B. Bulchandani, F. J. Burnell, and S. L. Sondhi, A multiplayer multiteam nonlocal game for the toric code, *Phys. Rev. B* **107**, 035409 (2023).
- [70] O. Hart, D. T. Stephen, D. J. Williamson, M. Foss-Feig, and R. Nandkishore, Playing nonlocal games across a topological phase transition on a quantum computer, *Phys. Rev. Lett.* **134**, 130602 (2025).

- [71] C. Lin, V. B. Bulchandani, and S. L. Sondhi, Quantum tasks assisted by quantum noise, [arXiv:2308.10969](https://arxiv.org/abs/2308.10969).
- [72] P. Herringer and Y. Javanmard, pherringer/setmbqc_paper: SETMBQC, Zenodo, v1.0.0, <https://doi.org/10.5281/zenodo.17175550> (2025).
- [73] S. R. White, Density matrix formulation for quantum renormalization groups, *Phys. Rev. Lett.* **69**, 2863 (1992).
- [74] U. Schollwöck, The density-matrix renormalization group in the age of matrix product states, *Ann. Phys.* **326**, 96 (2011).
- [75] M. Fannes, B. Nachtergaele, and R. F. Werner, Finitely correlated pure states, *J. Funct. Anal.* **120**, 511 (1994).
- [76] F. Verstraete and J. I. Cirac, Matrix product states represent ground states faithfully, *Phys. Rev. B* **73**, 094423 (2006).
- [77] G. M. Crosswhite and D. Bacon, Finite automata for caching in matrix product algorithms, *Phys. Rev. A* **78**, 012356 (2008).
- [78] S. Keller, M. Dolfi, M. Troyer, and M. Reiher, An efficient matrix product operator representation of the quantum chemical Hamiltonian, *J. Chem. Phys.* **143**, 244118 (2015).
- [79] I. P. McCulloch, From density-matrix renormalization group to matrix product states, *J. Stat. Mech.: Theory Exp.* **2007**, P10014 (2007).
- [80] J. Ren, W. Li, T. Jiang, and Z. Shuai, A general automatic method for optimal construction of matrix product operators using bipartite graph theory, *J. Chem. Phys.* **153**, 084118 (2020).