

Anyon-Trions in Atomically Thin Semiconductor Heterostructures

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Topologically ordered quantum systems give rise to anyonic quasiparticles, whose controlled braiding operations form the foundation of topological quantum computation. Traditionally, studies of anyons have relied on edge-state interferometry, leaving the direct detection and manipulation of anyons in the bulk a major experimental challenge. Here, we propose and theoretically investigate a pathway toward this goal by demonstrating that a long-lived, optically generated interlayer exciton can bind to a quasihole in a fractional quantum Hall state, forming a novel composite excitation: the anyon-trion. Using exact diagonalization techniques, we reveal that anyon-trions exhibit millielectronvolt-scale binding energies and a linear dependence on the fractional charge of the quasihole. This scaling offers a powerful means to optically extract the quasihole's fractional charge through measurable shifts in exciton resonances. We outline a feasible experimental implementation via photoluminescence spectroscopy in a quantum twisting microscope setup, providing a promising route for the direct optical observation of anyon-trions within the bulk.

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I. INTRODUCTION

The discovery of the integer quantum Hall (IQH) and fractional quantum Hall (FQH) effects in two-dimensional electron gases under strong magnetic fields revolutionized condensed matter physics by revealing the crucial role of topology in understanding quantum phases of matter [1–5]. A key milestone in the FQH regime was the identification of fractionally charged quasiparticles obeying anyonic statistics [6,7], providing the first experimental realization of Abelian anyons—quasiparticles existing in two dimensions and described by braid group representations. Their exotic properties and potential use in topological quantum computing [8] continue to drive efforts toward their control and detection both in solids and synthetic quantum simulators.

Experimental probes of anyons include shot noise measurements to detect fractional charge [9–12], Fabry-Pérot interferometry to observe conductance phase slips [13,14], collider setups [15], and time-domain measurements of anyon braiding [16]. Although these experimental approaches have provided ground-breaking results that reveal fractional charge and braiding phase of anyons, they lack spatial resolution [17] and bulk control that are crucial for applications such as topological quantum computing.

Recently, van der Waals heterostructures based on transition metal dichalcogenides (TMDs) [18] have emerged as powerful platforms for exploring strongly correlated electronic states via optical spectroscopy [19–21], revealing phenomena such as generalized Wigner crystal formation [22–27], as well as integer [28] and fractional [29–31] Chern insulator states. A rapidly growing direction involves optically detecting and controlling exotic quantum Hall phases. This includes optical sensing of FQH states [32], manipulation using vortex light [33], attractive polaron spectroscopy in quantum anomalous Hall states [34], and magnetic field-dependent trion photoluminescence as a probe of Chern number and composite Fermi liquid states [35]. Complementary theoretical work has

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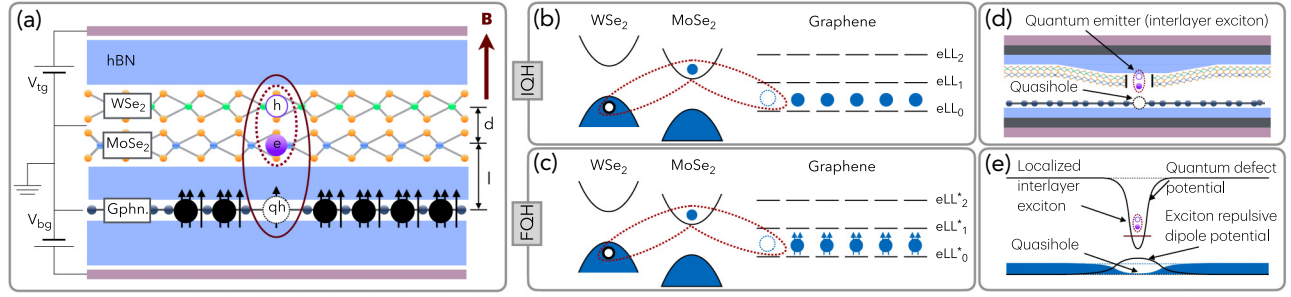


FIG. 1. (a) Schematic of the device configuration to study anyon-trion formation. A $\text{MoSe}_2/\text{WSe}_2$ heterobilayer constitutes the optically active layer. At strong magnetic fields, the two-dimensional electron gas in the proximate graphene monolayer can form quantum Hall states. An interlayer exciton in the $\text{MoSe}_2/\text{WSe}_2$ bilayer with repulsive interaction with the electrons in graphene can bind a hole of an integer quantum Hall state (b) or a quasihole of a fractional quantum Hall state (c). In panels (a) and (c), a representative $\nu = 1/3$ fractional quantum Hall state in the composite fermion picture is illustrated as a $\nu^* = 1$ state of ^2CF composite fermions. (d) An interlayer exciton as a quantum emitter on the tip of a quantum twist microscope can detect quasiholes of a fractional quantum Hall state with nanometer-scale precision. (e) Concept of quasihole detection by a quantum twist microscope: an interlayer exciton localized in a quantum defect, exerts a repulsive potential to the electrons. At the position of a quasihole—where the negative charge is reduced—the exciton experiences weaker repulsion compared to the uniform background, leading to a redshift in its energy. This redshift serves as a direct optical signature of the quasihole.

proposed using quantum optics techniques to study and manipulate quantum Hall systems [36–39]. In TMDs, excitons behave as tightly bound particles, which can be modeled as mobile impurities and form a natural framework for realizing hybrid exciton-electron systems. Interestingly, the use of mobile impurities in FQH systems is regarded as a compelling approach for exploring their exotic properties [40–44].

In this work, we demonstrate that excitons in TMDs, interacting repulsively with electrons in a nearby quantum Hall system, can serve as sensitive probes for detecting positively charged excitations in the bulk of incompressible integer and fractional quantum Hall states. For repulsive exciton-electron interactions, the exciton resonance exhibits a local reduction in blueshift at the positions of (quasi)holes in (F)IQH states, indicating exciton binding to these excitations. Our proposed setup consists of a $\text{MoSe}_2/\text{WSe}_2$ bilayer placed near a graphene monolayer in a uniform magnetic field B [see Fig. 1(a)]. In this setup, interlayer excitons [45], generated via nonresonant optical pumping [46], act as quantum sensors for detecting localized charged excitations in graphene.

In an IQH state, we show that this free exciton can bind a single hole, forming a few-body composite object we refer to as a *magnetic trion* [see Fig. 1(b)]. We find that the magnetic trion consists primarily of a $1s$ exciton scattering off a hole in graphene’s lowest Landau level (LLL).

We then consider interlayer excitons interacting with $\nu = 1/3$ graphene FQH states. To capture the full many-body nature of the FQH state and its coupling to the exciton, we study small systems (up to eight electrons) on a sphere [47,48], with the exciton confined to the same spherical surface. We employ a unitary transformation—the Lee-Low-Pines (LLP) transformation

[49–51]—to exploit the full translational and rotational symmetry of the exciton-quantum Hall system. Combining this approach with exact diagonalization (ED), we compute the low-energy states of the exciton-quantum Hall system and find that the exciton can bind a quasihole, forming an *anyon-trion* [see Fig. 1(c)]. The binding energy of this state, which we estimate to be ~ 0.5 meV, signals the reduced charge of the quasihole. Although a trion binding energy of this magnitude is, in principle, experimentally accessible, the typical linewidth of mobile interlayer excitons—of the order of 1 meV in current samples—renders its direct detection challenging. In contrast, localized interlayer excitons offer a far more promising route: their linewidths can be as narrow as 25 μeV [52], enabling substantially improved energy resolution.

Motivated by this, we also study the case of a localized interlayer exciton positioned at the tip of a quantum twist microscope (QTM) [53]. In the quantum optical implementation of the QTM, such an exciton serves as a quantum emitter that provides a sensitive, minimally invasive probe of the quasihole’s fractional charge with nanometer-scale spatial resolution [see Figs. 1(d) and 1(e)]. We find that the anyon-trion binding energy—arising from the binding of a quasihole to a localized exciton at the QTM tip—can reach values as large as ~ 0.9 meV. This places the direct experimental observation of anyon-trions within reach of current experiments.

II. EXCITONS IN IQH STATES: THE MAGNETIC TRION

In the limit of low exciton density and hole doping ($n_x, n_h \ll l_B^{-2}$ with $l_B = \sqrt{\hbar c/eB}$ the magnetic length), the system can be effectively described by a few-body problem

involving a single exciton interacting attractively with a single hole in Landau levels. Understanding this few-body interaction is crucial as we show later, since it provides the foundation for modeling the exciton-quantum Hall system in the many-body limit, where the exciton behaves as a pointlike impurity and the dynamics of the quantum Hall system can be projected onto the lowest Landau level. The few-body physics also plays a central role in understanding exciton-polaron formation in doped TMDs, where the exciton-polaron emerges as a coherent superposition of a bare exciton and collective trion-like excitations of the Fermi sea [54,55].

The dynamics is governed by the *magnetic trion Hamiltonian* $\hat{H}_{\text{MT}} = \hat{H}_e + \hat{H}_h + \hat{H}_Q + \hat{V}_{eh} + \hat{V}_{eQ} + \hat{V}_{hQ}$ where $\hat{H}_{e(h)}$ and $\hat{H}_Q$ are the kinetic terms and $\hat{V}_{eh}$, $\hat{V}_{eQ}$ and $\hat{V}_{hQ}$ denote the corresponding interaction potentials (Appendixes A–H) [56]. Hereafter, physical quantities with subscripts e, h, Q , and X belong to the exciton's electron, hole, the graphene hole and the exciton, respectively. To represent the few-body wave functions, it is convenient to express particle positions in terms of the trion center-of-mass $\mathbf{R}_{\text{MT}}$, $\mathbf{r}_{eh} = \mathbf{r}_e - \mathbf{r}_h$, and $\mathbf{r}_{QX} = \mathbf{r}_Q - \mathbf{R}_X$, where $\mathbf{R}_X$ is the exciton center-of-mass. After a suitable gauge transformation of $\hat{H}_{\text{MT}}$ (Appendixes A–H), $\hat{\mathbf{R}}_{\text{MT}}$ can be removed from the Hamiltonian. Thus, we only consider states with vanishing trion center-of-mass momentum, i.e., $\mathbf{P}_{\text{MT}} = 0$.

To obtain the bound state energy, we compare the ground state energy E_{gs} of $\hat{H}_{\text{MT}}$ obtained via exact diagonalization, in the presence of exciton-hole interaction to the ground state $E_{\text{gs},0}$ when $V_{eQ} = V_{hQ} = 0$ and define the magnetic trion binding energy by $E_{\text{MT}} = E_{\text{gs},0} - E_{\text{gs}}$ (Appendixes A–H). The magnetic trion binding energy is depicted in Fig. 2(a) as a function of B for different values of TMD bilayer spacing d . The binding energy increases with B , and asymptotically vanishes as $B \rightarrow 0$. We emphasize that the formation of the magnetic trion is directly tied to the quenching of the hole's kinetic energy by the magnetic field, as in the absence of a magnetic field, the exciton cannot bind a massless Dirac particle in graphene. Figure 2(a) also reveals that at sufficiently large B , E_{MT} reaches a few percent of the exciton binding energy $E_{B,X}$, which is significant for experimental detection: given that typical interlayer exciton linewidths are $\sim 1\%$ of $E_{B,X}$, the magnetic trion resonance should be observable. This demonstrates that sizable trion binding energies can be achieved at moderate magnetic fields.

To reveal the microscopic structure of the magnetic trion, we analyze the admixture of excitonic Rydberg states (n_X, l_X) —principal and angular momentum quantum numbers—and graphene hole Landau level states (n, m) , denoting Landau level index and magnetic degeneracy, respectively. To this end, we define the weight of a quantum number $\alpha \in \{n_X, l_X, n, m\}$ in a magnetic trion state $|\Psi_{\text{MT}}\rangle$. We denote a state of the exciton-quantum Hall

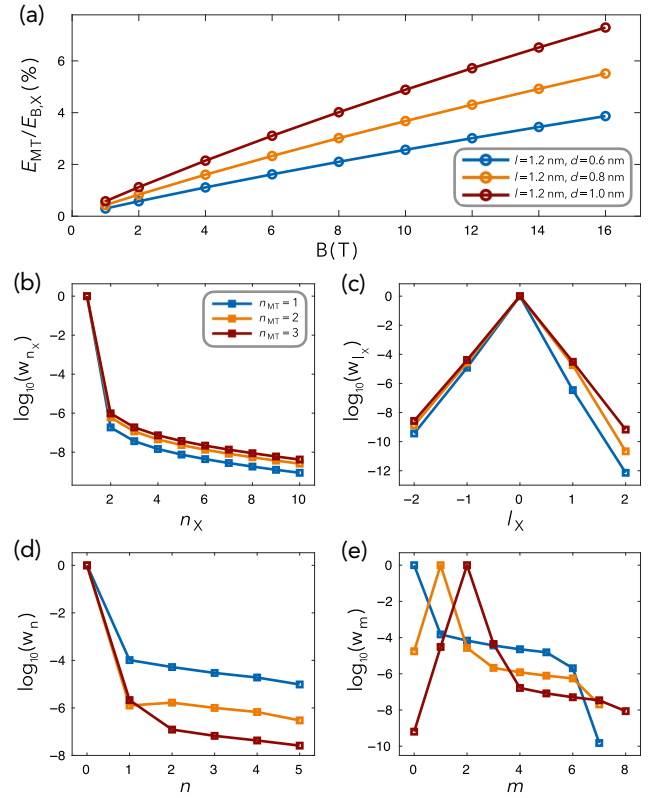


FIG. 2. Few-body states of an interlayer exciton coupled to a quantum Hall hole. (a) Magnetic trion binding energy vs magnetic field and various values of the spacing d between the MoSe₂ and WSe₂ layers, for a fixed distance of $l = 1.2$ nm from the graphene layer. The binding energy vanishes as $B \rightarrow 0$. (b)–(e) Weights w_α for the three lowest eigenstates on a logarithmic scale, resolved by (b) exciton principal quantum number n_X , (c) angular momentum l_X , (d) Landau level index n , and (e) magnetic quantum number m . Weights are defined as $w_\alpha = \sum_{\bar{\alpha}} |\langle \alpha, \bar{\alpha} | \Psi_{\text{MT}} \rangle|^2$, where α is the corresponding quantum number (n_X, l_X, n, m) , and $\bar{\alpha}$ are all the quantum numbers other than α . The data show that the magnetic trion is primarily a $1s$ exciton bound to a hole in the lowest Landau level. Parameters: $B = 16$ T, $l = 1.2$ nm, $d = 0.6$ nm.

system by $|\alpha, \bar{\alpha}\rangle$, where $\bar{\alpha}$ is the set of all the quantum numbers of the joint system other than α . The weight of α in the state $|\Psi_{\text{MT}}\rangle$ is then defined as $w_\alpha = \sum_{\bar{\alpha}} |\langle \alpha, \bar{\alpha} | \Psi_{\text{MT}} \rangle|^2$. For instance, the weight w_{n_X} is of the form $w_{n_X} = \sum_{l_X, n, m} |\langle n_X, l_X; n, m | \Psi_{\text{MT}} \rangle|^2$, where $|n_X, l_X; n, m\rangle$ is a state of the exciton-quantum Hall system where the exciton is in the state (n_X, l_X) and the hole has quantum numbers (n, m) (Appendixes A–H). The resulting weights are presented in Figs. 2(b)–2(e).

Figures 2(b) and 2(c) reveal that the dominant excitonic contribution to the magnetic trion originates from the $1s$ exciton. As shown in Fig. 2(d), the graphene hole remains predominantly in the LLL, with negligible contributions from higher Landau levels. Furthermore, for each of the three lowest trion eigenstates ($n_{\text{MT}} = 1, 2, 3$), the

wave function exhibits significant weight only for a single value m , which characterizes the orbital of the charge carrier relative to the exciton's center-of-mass. This analysis suggests a physically transparent picture: the lowest trion states can be accurately described as a bound state of a free $1s$ exciton and a Landau-quantized carrier occupying a single orbital in the LLL relative to the exciton.

This structure reflects the key interplay between the characteristic length and energy scales, and the Landau quantization. In graphene, the Landau level spacing $\hbar v_F/l_B \sim 28.2\sqrt{B[\text{T}]}$ reaches ~ 112.9 meV at $B = 16$ T—comparable to the interlayer exciton binding energy $E_{B,X} \sim 100$ meV [45,57,58]. Interaction terms coupling to higher excitonic states and Landau levels scale as $(a_X/l_B)^2 \sim 0.01$ at this field, strongly suppressing such excitations. Here, a_X is the exciton Bohr radius. Optical excitation thus primarily creates $1s$ excitons, and the resulting few-body states remain adiabatically connected to them, with the holes in the quantum Hall state confined to the lowest Landau level. Based on this, we model the many-body problem of an exciton interacting with a fractional quantum Hall liquid in the next section by treating the exciton as a pointlike impurity interacting with charges in the lowest Landau level.

III. EXCITONS IN FQH LIQUIDS: ANYON-TRIONS

We now turn to the many-body regime in which the graphene layer hosts an electronic FQH liquid at filling factor $\nu = 1/3$ [see Fig. 1(c)]. In this setting, we demonstrate the emergence of a novel type of bound state: an anyon-trion, formed by the binding of an interlayer exciton to a quasihole of the FQH fluid. To model the coupled exciton-quantum Hall system, we employ the standard spherical geometry, which contains an exciton and N electrons on the surface of a sphere of radius R threaded by a magnetic flux from a monopole of strength Q at its center. The full system is governed by the Hamiltonian

$$\hat{H} = \frac{\hat{\mathbf{L}}_X^2}{2M_X R^2} + \sum_{i=1}^N V_{Xe}(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_X) + \hat{H}_{\text{FQH}}, \quad (1)$$

where M_X , $\hat{\mathbf{r}}_X$, and $\hat{\mathbf{L}}_X$ denote the exciton's effective mass, position, and angular momentum on the sphere, respectively. The term V_{Xe} represents the exciton-electron interaction for the considered heterostructure (Appendixes A–H) [56], $\hat{\mathbf{r}}_i$ is the position of the i 'th electron, and $\hat{H}_{\text{FQH}} = \hat{\mathcal{P}}_{\text{LLL}} \sum_{i < j} V(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j) \hat{\mathcal{P}}_{\text{LLL}}$ describes the interacting electron Hamiltonian projected onto the lowest Landau level via the projector $\hat{\mathcal{P}}_{\text{LLL}}$.

Performing ED using a spherical geometry is a standard method for studying fractional quantum Hall systems [48,59]. Introducing a mobile exciton greatly enlarges the

Hilbert space, making ED computationally challenging. Instead of diagonalizing the full Hamiltonian in Eq. (1), we reduce the degrees of freedom by exploiting the system's full rotational and translational symmetry.

This is achieved via a unitary transformation analogous to those applied in Bose polaron physics [49–51], known as the LLP transformation. In spherical geometry, it takes the form

$$\hat{U}_{\text{LLP}} = \exp(-i\hat{\varphi} \otimes \hat{L}_z) \exp(-i\hat{\theta} \otimes \hat{L}_y) \exp(-i\hat{\gamma} \otimes \hat{L}_z), \quad (2)$$

where $(\hat{\varphi}, \hat{\theta}, \hat{\gamma})$ are exciton orientation operators and $\hat{\mathbf{L}}$ is the total electronic angular momentum (Appendixes A–H). Intuitively, $\hat{U}_{\text{LLP}}$ applies a rigid-body rotation into the frame comoving with the exciton, which is then localized at the north pole.

After applying $\hat{U}_{\text{LLP}}$, the transformed Hamiltonian $\hat{H}_{\text{LLP}} = \hat{U}_{\text{LLP}}^\dagger \hat{H} \hat{U}_{\text{LLP}}$ is suitable for ED (Appendixes A–H). Figure 3(a) shows the low-energy spectrum for $N = 5$ –8 electrons and a total monopole charge $Q_{\text{qh}} = Q_{\text{LN}} + 1/2$, corresponding to a Laughlin $\nu = 1/m$ FQH state at $Q_{\text{LN}} = m(N - 1)/2$ with a single quasihole (here $m = 3$). At $Q = Q_{\text{qh}}$, the quasihole configurations form an $L = N/2$ multiplet of $N + 1$ degenerate gapped ground states. Energies are measured in units of the magnetic Coulomb energy $E_C = e^2/4\pi\epsilon_0\epsilon_{\text{hBN}}l_B$ ($\epsilon_{\text{hBN}} = 4.5$ the dielectric constant of hBN) relative to the quasihole ground state and plotted versus the total angular momentum J relative to that of the isolated quasihole $J_{\text{qh}} = N/2$. Hereafter we set the parameters $B = 16$ T, $l = 1.2$ nm and $d = 0.6$ nm in our numerical calculations.

The ground state consistently appears at $J = J_{\text{qh}}$, confirming the adiabatic connection to the noninteracting ground state comprised of a zero angular momentum exciton bound to a quasihole. The quadratic center-of-mass dispersion of the anyon-trion with respect to $J - J_{\text{qh}}$ indicated in Fig. 3(a) originates from its finite mass (Appendixes A–H). The spectrum exhibits two robust gaps across all N : $E_{\text{Gap},1}$ separates the ground state from the first excited state, while $E_{\text{Gap},2}$ separates the latter from the continuum. The scaling of $E_{\text{Gap},1}$ and $E_{\text{Gap},2}$ with $1/N$ indicates that the gaps remain finite in the thermodynamic limit [see Fig. 3(b)]. For a detailed justification of the $1/N$ scaling, see Appendixes A–H.

To demonstrate the binding of the exciton to the quasihole and the formation of anyon-trions, a crucial quantity is the exciton-electron spatial correlation function $g^{(2)}(\mathbf{r})$, defined by

$$g^{(2)}(\mathbf{r}) = \frac{1}{\rho N} \sum_{i=1}^N \langle \delta^{(2)}(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_X - \mathbf{r}) \rangle, \quad (3)$$

where ρ is the density of the uniform electron gas, and the expectation value $\langle \cdot \rangle$ is evaluated for an eigenstate $|\Psi\rangle$ of

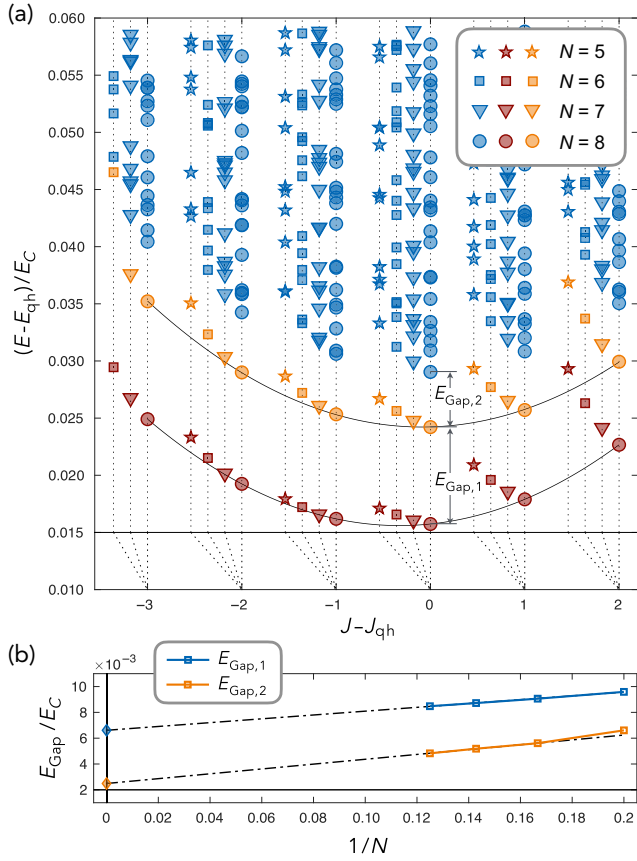


FIG. 3. (a) Many-body energies, relative to the quasihole energy E_{qh} of the exciton interacting with a $\nu = 1/3$ system of $N = 5$ –8 electrons at the magnetic monopole Q_{qh} , as a function of the total angular momentum J relative to the quasihole angular momentum $J_{qh} = N/2$. Two energy gaps $E_{Gap,1}$ and $E_{Gap,2}$ are visible in the spectrum for different angular momenta and particle numbers. Here we take the exciton creation energy $E_x = 0$. (b) The scaling of gaps $E_{Gap,1}$ and $E_{Gap,2}$ with $1/N$.

the full system. The $g^{(2)}(\mathbf{r})$ function measures the probability of finding a particle in the quantum Hall system at position $\mathbf{r}$ relative to the exciton.

Figure 4 shows $g^{(2)}(\theta)$ for $N = 8$ electrons as a function of the polar angle θ . Without exciton-electron interaction, $g^{(2)}(\theta) = 1$, reflecting the absence of correlations. When the repulsive interaction is included, the ground state at $Q = Q_{qh}$ and $J = J_{qh}$ exhibits $g^{(2)}$ closely matching that of a free quasihole at $L_z = -N/2$, indicating the formation of a bound state. The first excited state exhibits slightly enhanced electron density near the exciton (Appendixes A–H).

In contrast, the lowest-energy state at the onset of the continuum exhibits a pronounced enhancement of the electron density at the exciton position. Analysis of its structure reveals that the weight of the $L_z = -N/2$ quasihole state is around 0.0037, whereas quasihole states with $-N/2 < L_z \leq N/2$ account for roughly 99.16% of the total weight

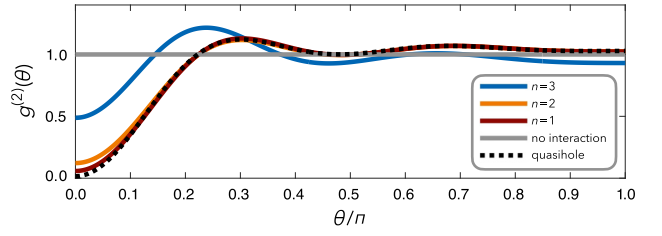


FIG. 4. Exciton-electron two-particle correlation function $g^{(2)}(\theta)$ for $N = 8$ electrons as a function of polar angle. The $g^{(2)}$ of a quasihole at $Q = Q_{qh}$, $L_z = -N/2$ (dashed black) closely matches that of a mobile exciton repulsively interacting with electrons at the same filling and total angular momentum $J = J_{qh}$ (solid red, $n = 1$). The first excited state ($n = 2$) with an internal edge excitation shows similar correlations (solid orange, $n = 2$) (Appendixes A–H), while the continuum onset state differs markedly (solid blue, $n = 3$). Without exciton-electron interaction, $g^{(2)}(\theta) = 1$, indicating no spatial correlations.

of the quantum Hall system's wave functions in this state. An additional 0.47% originates from contributions from the gapped quantum Hall excitations. This significant admixture leads to the observed increase in the local electron density and explains the energy gap separating this state from the first excited state.

We conclude by noting that, although our analysis here focused on an exciton interacting with a single quasihole, the framework naturally extends to situations involving multiple anyons. In a realistic setting with a finite density of quasiholes, an optically injected interlayer exciton would be dynamically dressed by many anyons to form an *anyon-polaron*; since the strength of the anyon-polaron transition scales linearly with the anyon density, it may be possible to extract the actual ground state anyon density for a given filling of the Landau level.

IV. QUASIHOLE DETECTION USING A QTM

In a translationally invariant van der Waals heterostructure depicted in Fig. 1(a), excitons are mobile which reduces the exciton-(quasi)hole binding due to kinetic energy cost of localizing the exciton around a (quasi)hole. As we discuss in the next section, the resulting binding energy is only ~ 0.5 meV—smaller than the typical linewidth of mobile interlayer excitons, ~ 1 meV—which makes direct observation of mobile anyon-trions challenging.

A more promising approach which circumvents the kinetic energy limitation and takes advantage of the dramatically narrower exciton linewidths of order 25 μeV [52] is to use localized interlayer excitons or quantum emitters: given that we expect (quasi)holes to be localized due to unavoidable disorder, it is important to be able to

scan the localized exciton and ensure its spatial overlap with a (quasi)hole. This challenge can be addressed using a quantum optical version of the recently demonstrated quantum twisting microscope (QTM) [53]. The setup we envision features a localized exciton at the microscope tip. In the limit where the exciton-electron interaction energy is small compared to the gap of the incompressible IQH and FQH state, scanning the tip over a graphene sample in high magnetic fields will show spatially uniform exciton resonances. Since quasiholes appear as local charge depletions, moving the localized exciton so that it spatially overlaps with a quasihole, will lead to a redshift of the exciton resonance.

To model the QTM setup, we consider a static exciton fixed at the north pole of the sphere, corresponding to the limit $M_X \rightarrow \infty$. In this setting, the z axis projection of the total quantum Hall system's angular momentum, L_z , is conserved, allowing diagonalization of the full Hamiltonian within each L_z sector. The resulting low-energy spectrum is shown in Fig. 5 for $N = 8$ and $Q = Q_{\text{qh}}$. Introducing the exciton lifts the ground state degeneracy, yielding a nondegenerate gapped ground state and a band of in-gap states separated from the excitation continuum. These states, connected to the original quasihole manifold, experience an energy increase and contain an admixture of the higher excited states at the same L_z of the order of $\sim 10^{-4}$ – 10^{-3} . Clearly, the quasihole localized on the north pole, with $L_z = -N/2$ is bound to the exciton. This highlights the QTM as a powerful, noninvasive tool capable of directly detecting quasiholes with nanometer-scale spatial resolution.

V. EXTRACTION OF THE QUASIHOLE FRACTIONAL CHARGE FROM THE EXCITON RESONANCE SHIFTS

Exact diagonalization allows the extraction of exciton resonance blueshifts arising from repulsive interactions with electrons in a quantum Hall system. Thus, we define the blueshift, ΔE , as the difference in the ground state energy of the exciton-quantum Hall Hamiltonian [Eq. (1)] with and without exciton-electron interaction, i.e., with $V_{Xe} \neq 0$ and $V_{Xe} = 0$, respectively. Since ΔE reflects the local electron density probed by the exciton, it depends on the filling factor ν and whether a quasihole is present. In the presence of a quasihole, the local charge depletion leads to a reduced ΔE compared to the uniform Laughlin state. In spherical geometry, introducing a single quasihole corresponds to increasing the magnetic monopole strength by $1/2$ from its Laughlin value: $Q_{\text{qh}} = Q_{\text{LN}} + 1/2$. We therefore denote the exciton blueshift as $\Delta E(\nu, Q)$, where Q characterizes the magnetic flux and indirectly the presence of a quasihole.

In our numerical calculations, we extract ΔE at $\nu = 1$ and $\nu = 1/3$ when the quantum Hall system is either in

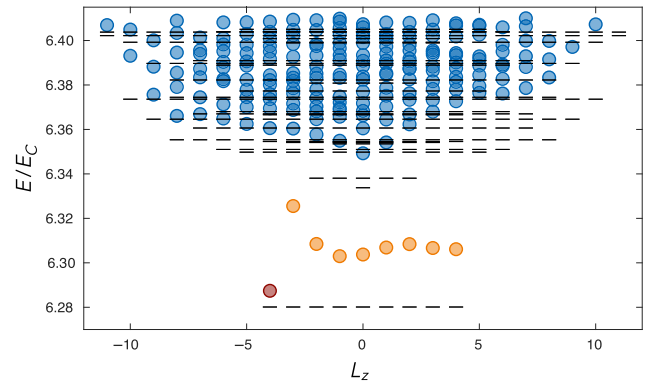


FIG. 5. Low-energy spectrum of a localized exciton on the tip of a quantum twist microscope, in the presence of a fractional quantum Hall state of $N = 8$ electrons at $Q = Q_{\text{qh}}$. Dashed lines indicate the eigenenergies of the pure quantum Hall system, where the degenerate ground state subspace corresponds to the quasihole configurations with $-N/2 \leq L_z \leq N/2$. The $L_z = -N/2$ state corresponds to the quasihole localized on the north pole. Circles depict the spectrum in the presence of the exciton. The bound state of the exciton and the $L_z = -N/2$ quasihole appears as the unique ground state (red circle), while other quasihole configurations form a band of states (orange circles) with a gap to the continuum (blue circles).

the Laughlin ground state or hosts a single quasihole. For $N = 7, 8$ particles at $\nu = 1/3$, the Laughlin state is realized by setting the magnetic monopole to $Q_{\text{LN}} = 3(N - 1)/2$. To obtain the $\nu = 1$ Laughlin state, we keep the same monopole value and increase the particle number until the lowest Landau level is completely filled. Likewise, to realize a single hole at $\nu = 1$, we set the monopole to $Q_{\text{qh}} = Q_{\text{LN}} + 1/2$ —corresponding to a quasihole on top of the $\nu = 1/3$ Laughlin state—and again increase the particle number until the system reaches $\nu = 1$ with one excess flux. Table I summarizes $\Delta E(\nu, Q)$ for both $\nu = 1$ and $1/3$, evaluated at Laughlin and quasihole flux values, using fixed Q from the $\nu = 1/3$ configurations and the corresponding particle numbers at $\nu = 1$.

In this context, a natural definition of the anyon-trion binding energy is $E_{\text{B,AT}}(\nu) = \Delta E(\nu, Q_{\text{LN}}) - \Delta E(\nu, Q_{\text{qh}})$. Using this relation, we evaluate the anyon-trion binding energy $E_{\text{B,AT}}(1/3) = 1.295 - 0.8 = 0.495$ meV for $N = 7$ and $E_{\text{B,AT}}(1/3) = 1.27 - 0.785 = 0.485$ meV for $N = 8$ (see the data in Table I). Similarly, the QTM exciton-quasihole binding energy is obtained as $E_{\text{B,AT}}(1/3) = 1.295 - 0.37 = 0.925$ meV for $N = 7$ and $E_{\text{B,AT}}(1/3) = 1.27 - 0.365 = 0.905$ meV for $N = 8$. Therefore, we estimate the mobile anyon-trion binding energy to be ~ 0.5 meV and that of the QTM exciton-quasihole to be ~ 0.9 meV. Importantly, the binding energy of the static anyon-trion far exceeds the linewidth of localized excitons (~ 25 μeV [52]). This hierarchy makes the experimental detection of static anyon-trions promising.

Although the above definition of $E_{\text{B,AT}}(\nu)$ is plausible, we find that a more accurate and experimentally robust definition is $E_{\text{B,AT}}(\nu) = \nu \Delta E(1, Q_{\text{LN}}) - \Delta E(\nu, Q_{\text{qh}})$. This is because, at $\nu = 1/3$ and Q_{LN} , the exciton induces weak excitations on top of the Laughlin state, which nonetheless retains approximately 92% of the weight in the anyon-trion ground state. In contrast, at $\nu = 1$, such excitations are suppressed due to the absence of low-energy bulk modes, making $\Delta E(1, Q_{\text{LN}})$ a more versatile reference. Linear scaling with ν takes into account the reduced electron density in the bulk at filling $\nu < 1$. Meanwhile, the exciton's perturbation of the quasihole state is minimal: its potential is short-ranged compared to the magnetic length, and its peak coincides with the charge depletion at the quasihole center. Consequently, $\Delta E(\nu, Q_{\text{qh}})$ also scales approximately linearly with ν . Using the data in Table I, the anyon-trion binding energy is given by $E_{\text{B,AT}}(1/3) = (4.7550/3) - 0.8 = 0.785$ meV for $N = 7$ and $E_{\text{B,AT}}(1/3) = (4.7350/3) - 0.785 = 0.7933$ meV for $N = 8$, and similarly for the QTM exciton-quasihole bound state, $E_{\text{B,AT}}(1/3) = (4.7550/3) - 0.37 = 1.215$ meV for $N = 7$ and $E_{\text{B,AT}}(1/3) = (4.7350/3) - 0.365 = 1.2133$ meV for $N = 8$.

Since the anyon-trion binding energy at $\nu = 1$ ($\nu = 1/3$) originates from the interaction of the charge e (e^*) hole (quasihole) with the dipole field created by the exciton, we expect $E_{\text{B,AT}}(\nu) \propto e(e^*)$, respectively. Using our estimates for $E_{\text{B,AT}}(\nu)$ from above we thus obtain (Appendixes A–H)

$$\frac{e^*}{e} = \frac{E_{\text{B,AT}}(\nu)}{E_{\text{B,AT}}(1)}. \quad (4)$$

Inserting values from Table I, we obtain $e^*/e \simeq 0.3366$, within $\sim 1\%$ of the true fractional charge. The required measurement data $\Delta E(\nu, Q_{\text{LN}})$, $\Delta E(\nu, Q_{\text{qh}})$, and $\Delta E(1, Q_{\text{qh}})$ are all accessible using QTM measurements. This demonstrates that QTM enables direct measurement

TABLE I. Exciton resonance blueshift $\Delta E(\nu, Q)$ in meV defined as the blueshift of the exciton resonance in the presence of different quantum states: integer quantum Hall ($\nu = 1$) for $N = 19, 22$ electrons, and fractional quantum Hall ($\nu = 1/3$) for $N = 7, 8$ electrons, all at fillings corresponding to the Laughlin state $Q_{\text{LN}} = (N - 1)/2\nu$ and a single quasihole state $Q_{\text{qh}} = Q_{\text{LN}} + 1/2$. In the presence of a single quasihole, the blueshift is reported for the localized exciton in the QTM setup as well as for the lowest energy state of the mobile exciton bound to the quasihole (anyon-trion).

ν	N	ΔE (Laughlin)	ΔE (single quasihole)	
			QTM exciton	Mobile exciton
1	19	4.7550	1.1450	1.5650
	22	4.7350	1.1350	1.5550
1/3	7	1.2950	0.3700	0.8000
	8	1.2700	0.3650	0.7850

of fractional charge with high accuracy and nanoscale resolution.

For mobile excitons, the observed blueshift is consistently larger due to contributions from exciton kinetic energy. This effect originates from the finite exciton mass, which induces nontrivial correlations between the exciton's motional degree of freedom and the many-body states of the quantum Hall system (Appendixes A–H). As a result, the anyon-trion binding energy is systematically smaller than the exciton-quasihole binding energy extracted in the QTM configuration. We find that this reduction in binding energy is approximately 0.42 meV at both $\nu = 1$ and $\nu = 1/3$ (see Table I). This kinetic energy offset leads to a significant underestimation of the fractional quasihole charge when applying Eq. (4), yielding $e^*/e \simeq 0.2461$ for $N = 7$ and $e^*/e \simeq 0.2495$ for $N = 8$ —a 25%–26% deviation from the expected value. We conclude that, although the anyon-trion constitutes a robust bound state of a mobile exciton and a quasihole, its binding energy is best interpreted as a qualitative indicator rather than a direct quantitative probe of the quasihole charge.

VI. CONCLUSION AND OUTLOOK

We introduce an optical approach to probe quantum Hall states using interlayer excitons in TMD heterostructures. In a $\text{MoSe}_2/\text{WSe}_2$ bilayer above a graphene layer, excitons act as sensitive local probes for quasiholes in IQH and FQH states, ideally detectable using a quantum optical microscope.

The emergent bound states we found in this work can serve as a foundation for developing a quantum Hall polaron theory where excitons coupled to quantum Hall fluids form quantum Hall polarons. Understanding the crossover from conventional Fermi to quantum Hall polarons as the magnetic field increases, including possible polaron-to-molecule transitions with fractionalized trions, invites new theoretical developments, such as fractionalized versions of Chevy's ansatz [60] or extension of the Tavis-Cummings theory of exciton-polarons [55]. A full polaron theory would also account for exciton coupling to collective modes of the FQH state, including composite fermion cyclotron and magneto-roton modes, as well as graviton-like geometric excitations [61,62].

Beyond the single-exciton regime, strong optical pumping may drive the system into collective phases—such as exciton condensates or excitonic insulators—in proximity to a quantum Hall fluid. This paves the way for realizing hybrid many-body states comprising electrons, excitons, and trions, potentially giving rise to exotic exciton-composite fermion mixtures. Such strongly interacting quantum Hall Bose-Fermi mixtures offer an exciting platform for exploring novel correlated phases of matter.

Finally, although our analysis focuses on bound states between an interlayer exciton and an anyon in fractional quantum Hall systems, the results can be naturally extended to the binding of anyons in fractional Chern insulator states—such as those observed in twisted MoTe₂ homobilayers [31]—to interlayer (localized) excitons. The stronger binding energies anticipated in these systems should make experimental observation more feasible.

Note added 1. During completion of this work we became aware of a related work by Wagner, Neupert [63] which has been submitted in the same arxiv posting.

Note added 2. A preliminary version of this work was previously published in the thesis [64], including the use of the spherical Lee-Low-Pines transformation for exciton-quantum Hall systems on the sphere.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: ELECTROSTATIC POTENTIAL OF CHARGES IN THE HETEROSTRUCTURE

We derive the interaction potential between charge carriers in the heterostructure considered in this work. We consider a system consisting of N layers stacked along the z axis, with in-plane polarizability $\chi_{2D,i}$ and z coordinate z_i for the i 'th layer. The multilayer is immersed in a medium with dielectric constant ϵ_r , a standard approximation for layered heterostructures [65]. We aim to obtain the electrostatic potential $\Phi(\mathbf{x})$ for charges confined to the layers, where $\mathbf{x} = (\mathbf{r}, z)$ is the three-dimensional position vector and $\mathbf{r}$ is the in-plane coordinate. Exploiting translational symmetry, the potential in layer j due to a charge Q_i in

layer i is computed assuming $\mathbf{r}_i = 0$. The potential satisfies Poisson's equation

$$\nabla^2 \Phi(\mathbf{x}) = -\frac{1}{\epsilon_0 \epsilon_r} \left(Q_i \delta^{(2)}(\mathbf{r}) \delta(z - z_i) + \sum_{j=1}^N \epsilon_0 \chi_{2D,j} \delta(z - z_j) \nabla_{\mathbf{r}}^2 \Phi(\mathbf{r}, z_j) \right), \quad (\text{A1})$$

where $\nabla_{\mathbf{r}}^2$ is the in-plane Laplacian. Fourier transforming Eq. (A1) yields

$$\Phi(\mathbf{q}, q_z) = \frac{Q_i}{\epsilon_0 \epsilon_r} \frac{e^{-iq_z z_i}}{(q^2 + q_z^2)} - \sum_{j=1}^N \frac{\chi_{2D,j}}{\epsilon_r} \frac{q^2 e^{-iq_z z_j}}{(q^2 + q_z^2)} \Phi(\mathbf{q}, z_j), \quad (\text{A2})$$

where $q = |\mathbf{q}|$. Taking the inverse Fourier transform along z and evaluating at z_k gives

$$\Phi(\mathbf{q}, z_k) = \frac{Q_i}{\epsilon_0 \epsilon_r} \frac{e^{-|z_k - z_i|q}}{2q} - \sum_{j=1}^N \frac{\chi_{2D,j}}{2\epsilon_r} q e^{-|z_k - z_j|q} \Phi(\mathbf{q}, z_j). \quad (\text{A3})$$

Defining $\underline{\Phi} = (\Phi(\mathbf{q}, z_1), \dots, \Phi(\mathbf{q}, z_N))^T$ and $\underline{\Phi} = (Q_i/2\epsilon_0 \epsilon_r q) (e^{-q|z_1 - z_i|}, \dots, e^{-q|z_N - z_i|})^T$, Eq. (A3) can be cast in matrix form $\underline{\mathbf{K}} \underline{\Phi} = \underline{\Phi}$, with $[\underline{\mathbf{K}}]_{jk} = \delta_{jk} + (\chi_{2D,j}/2\epsilon_r) q e^{-q|z_k - z_j|}$. Inverting this relation yields the potential in each layer: $\underline{\Phi} = \underline{\mathbf{K}}^{-1} \underline{\Phi}$, and the in-plane inverse Fourier transform gives $\Phi(\mathbf{r}, z_k)$.

This procedure produces the interparticle potentials in the heterostructure [Fig. 1(a)]. For instance, the electron-hole interaction V_{eh} is obtained as the potential in the MoSe₂ layer due to a hole in the WSe₂ layer, while V_{eQ} and V_{hQ} follow from placing a unit charge in the graphene layer and measuring the potential in the MoSe₂ and WSe₂ layers, respectively.

The 2D polarizabilities used are listed in Table II. Due to the strong magnetic field and proximity to incompressible quantum Hall states, the in-plane static polarizability of graphene is negligible, and we set $\chi_{2D,\text{graphene}} = 0$ [66]. This correctly reproduces the Coulomb interaction within graphene and the dipolar potential of excitons sensed by graphene electrons.

APPENDIX B: EXCITONS ON TOP OF AN IQH STATE: THE FEW-BODY PROBLEM

In the following, we detail on the analysis of the exciton-IQH system few-body states. In the presence of a magnetic field, the relevant conjugate momentum of the charged particles is the *kinetic momentum* $\hat{\pi} = \hat{\mathbf{p}} - (q/c)\mathbf{A}(\hat{\mathbf{r}})$, where $\hat{\mathbf{p}} = -i\hbar\nabla_{\mathbf{r}}$, q is the particle's electric

TABLE II. In-plane polarizability and effective mass of electrons and holes in the MoSe₂ and WSe₂ layers, used to obtain the electrostatic potential of the heterostructure considered in this work. The dielectric constant of the surrounding hBN environment is $\epsilon_{\text{hBN}} = 4.5$, and $m_0 = 9.109 \times 10^{-31}$ Kg is the bare electron mass.

	χ_{2D} (nm)	m_e (m_0)	m_h (m_0)
MoSe ₂	9.0	0.84	0.6
WSe ₂	7.8	0.34	0.53

charge ($q_e = -e < 0$ for the electron, $q_h = e$ for the hole, and $q_Q = e$ for the graphene hole, denoted here by Q to distinguish it from the exciton's hole), and $\mathbf{A}(\mathbf{r})$ is the vector potential, which in the symmetric gauge considered in this work reads $\mathbf{A}(\mathbf{r}) = (1/2)\mathbf{B} \times \mathbf{r}$. The dynamics is governed by the magnetic trion Hamiltonian $\hat{H}_{\text{MT}} = \hat{H}_e + \hat{H}_h + \hat{H}_Q + \hat{V}_{eh} + \hat{V}_{eQ} + \hat{V}_{hQ}$ where $\hat{H}_{e(h)} = \hat{\pi}_{e(h)}^2 / 2m_{e(h)}$ and $\hat{H}_Q = \hbar v_F \hat{\pi}_Q \cdot \hat{\sigma}$ are the kinetic terms for the electron (hole), and the graphene hole, respectively, with $v_F = 1.1 \times 10^6$ m/s the Dirac velocity. Here we neglect the graphene valley quantum number and only consider the sublattice pseudospin, represented by the Pauli matrices $\hat{\sigma}$. The electron-hole, electron- Q , and hole- Q interactions are described by $\hat{V}_{eh}$, $\hat{V}_{eQ}$, and $\hat{V}_{hQ}$, respectively. To represent the few-body states, it is convenient to choose the relative and center-of-mass coordinates of the particles, $\mathbf{R}_{\text{MT}} = \mathbf{R}_X = (m_e \mathbf{r}_e + m_h \mathbf{r}_h) / M_X$, $\mathbf{r}_{eh} = \mathbf{r}_e - \mathbf{r}_h$, $\mathbf{r}_{QX} = \mathbf{r}_Q - \mathbf{R}_X$, with M_X the total exciton mass. Note that the center-of-mass of the three-particle system coincides with the exciton's as a result of the vanishing mass of the graphene hole. The conjugate momenta then read $\mathbf{P}_{\text{MT}} = \mathbf{p}_e + \mathbf{p}_h + \mathbf{p}_Q$, $\mathbf{p}_{eh} = (m_h / M_X) \mathbf{p}_e - (m_e / M_X) \mathbf{p}_h$, and $\mathbf{p}_{QX} = \mathbf{p}_Q$. In terms of the new coordinates and momenta, $\hat{H}_{\text{MT}}$ takes a complicated form where all the canonical variables are coupled. To bring the Hamiltonian in a more tractable and physically intuitive form, we follow Ref. [67], and perform a gauge transformation $\hat{U}_g = \exp[-ie/\hbar c \mathbf{R}_T \cdot (\mathbf{A}(\mathbf{r}_{QX}) - \mathbf{A}(\mathbf{r}_{eh}))]$. The gauge-transformed Hamiltonian $\hat{H}_{\text{MT},g} = \hat{U}_g^\dagger \hat{H}_{\text{MT}} \hat{U}_g$ takes the form

$$\begin{aligned} \hat{H}_{\text{MT},g} = & \frac{\hat{\mathbf{P}}_{\text{MT}}^2}{2M_X} + \frac{\hat{\mathbf{P}}_{\text{MT}}}{M_X} \cdot \left[\frac{2e}{c} \mathbf{A}(\hat{\mathbf{r}}_{eh}) - \frac{q_Q}{c} \mathbf{A}(\hat{\mathbf{r}}_{QX}) \right] \\ & + \hat{H}_X + \hat{H}_{QX} - \frac{eq_Q B^2}{2M_X c^2} \hat{\mathbf{r}}_{eh} \cdot \hat{\mathbf{r}}_{QX} + \frac{e^2 B^2}{8M_X c^2} \hat{\mathbf{r}}_{QX}^2 \\ & + V_{eQ} \left(\frac{m_h}{M_X} \hat{\mathbf{r}}_{eh} - \hat{\mathbf{r}}_{QX} \right) + V_{hQ} \left(\frac{m_e}{M_X} \hat{\mathbf{r}}_{eh} + \hat{\mathbf{r}}_{QX} \right). \end{aligned} \quad (\text{B1})$$

In Eq. (B1), the first term denotes the magnetic trion kinetic energy, the second term denotes the coupling of the magnetic trion total momentum to the internal coordinates $\mathbf{r}_{eh}$ and $\mathbf{r}_{QX}$, $\hat{H}_X = 1/2m_e(\hat{\mathbf{p}}_{eh} +$

$e/c\mathbf{A}(\hat{\mathbf{r}}_{eh}))^2 + 1/2m_h(\hat{\mathbf{p}}_{eh} - e/c\mathbf{A}(\hat{\mathbf{r}}_{eh}))^2 + V_{eh}(\mathbf{r}_{eh})$ is the exciton's internal Hamiltonian, and $\hat{H}_{QX} = \hbar v_F \hat{\sigma} \cdot \hat{\pi}_{QX}$ with $\hat{\pi}_{QX} = \hat{\mathbf{p}}_{QX} - q_Q/c\mathbf{A}(\hat{\mathbf{r}}_{QX})$. Interestingly, $\hat{\mathbf{R}}_{\text{MT}}$ does not appear in $\hat{H}_{\text{MT},g}$, enabling to restrict $\hat{H}_{\text{MT},g}$ in subspaces with well-defined $\mathbf{P}_{\text{MT}}$. In this work we only consider $\mathbf{P}_{\text{MT}} = 0$.

To obtain the few-body states and their energies, we perform exact diagonalization of $\hat{H}_{\text{MT},g}$ over the basis states $|n_X, l_X; n, m\rangle = |n_X, l_X\rangle \otimes |n, m, +\rangle$ formed by the eigenstates $|n_X, l_X\rangle$ of $\hat{H}_X$, and the positive-energy eigenstates $|n, m, +\rangle$ of $\hat{H}_{QX}$. Here, n_X and l_X are the exciton principle quantum number and the internal angular momentum, respectively, and $|n, m\rangle$ denotes the usual Landau orbitals of a charge- e particle in the symmetric gauge, with $n \geq 0$ the Landau level index and $m \geq 0$ the magnetic quantum number. To obtain the bound state energy, we compare the ground state energy E_{gs} of $\hat{H}_{\text{MT},g}$ in the presence of exciton-carrier interaction to the ground state $E_{\text{gs},0}$ when $V_{eQ} = V_{hQ} = 0$. The magnetic trion binding energy is then defined as $E_{\text{MT}} = E_{\text{gs},0} - E_{\text{gs}}$.

APPENDIX C: CHARGED PARTICLES IN A UNIFORM MAGNETIC FIELD IN THE SPHERICAL GEOMETRY

In numerical studies of quantum Hall systems, it is a standard practice to consider spherical geometry for particles' motion [68]. In this setting, charged particles with mass m and charge q are constrained to move on a sphere with radius R , threaded by a magnetic field $\mathbf{B} = (\Phi/4\pi R^2)\hat{\Omega}$, where Φ is the total magnetic flux threading the sphere and $\hat{\Omega}$ is the unit normal vector. To have single-valued wave functions, an integer number of elementary flux quanta must thread the system, thus $\Phi = (2Q)\phi_0$, where $\phi_0 = hc/|q|$, and Q is called the *monopole strength* and increases in half integer steps. Similar to the planar geometry, the cyclotron motion of a charged particle in a uniform magnetic field is described by the Hamiltonian

$$\hat{H}_C = \frac{\hat{\Lambda}^2}{2mR^2}, \quad (\text{C1})$$

where $\hat{\Lambda}$ is called the *kinetic angular momentum* and is related to the *kinetic momentum* $\hat{\Pi} = \hat{\mathbf{p}} - (q/c)\mathbf{A}(\hat{\mathbf{r}})$ by $\hat{\Lambda} = \hat{\mathbf{r}} \times \hat{\Pi}$. The kinetic angular momentum is related to the spatial angular momentum $\hat{\mathbf{L}}$ by $\hat{\mathbf{L}} = \hat{\Lambda} - \text{sgn}(q)Q\hat{\Omega}$. In particular, $\hat{\mathbf{L}}^2 = \hat{\Lambda}^2 + Q^2$, since $\hat{\Omega} \cdot \hat{\Lambda} = 0$. Thus, the eigenstates of $\hat{\Lambda}^2$ are the same as $\hat{\mathbf{L}}^2$ and are given by the *monopole spherical harmonics* $Y_{Qlm}(\theta, \varphi)$. The monopole spherical harmonics Y_{Qlm} form an angular momentum l irreducible representation of $SO(3)$ for fixed Q , and satisfy

$$\begin{cases} \hat{\mathbf{L}}^2 Y_{Qlm} = \hbar^2 l(l+1) Y_{Qlm}, \\ \hat{L}_z Y_{Qlm} = \hbar m Y_{Qlm}, \end{cases} \quad (\text{C2})$$

where $l = |Q|, |Q| + 1, \dots$ and $m = -l, l + 1, \dots, l$. Moreover, the n 'th Landau level is identical to the angular momentum shell $l = |Q| + n$, with energy $E_n = (n + 1/2)\hbar\omega_C$. The lowest Landau level thus has a degeneracy $2Q + 1$.

To study quantum Hall systems of N particles interacting with a potential V with fractional lowest Landau level filling, the kinetic energy is fully quenched, and the lowest-Landau-level-projected Hamiltonian reads

$$\hat{H}_{\text{FQH}} = \frac{1}{2} \sum_{l=0}^{\infty} V_l \sum_{m=-l}^l (-1)^m : \hat{\rho}_{l(-m)} \hat{\rho}_{lm} :, \quad (\text{C3})$$

where $\hat{\rho}(\mathbf{\Omega}) = \hat{\mathcal{P}}_{\text{LLL}} \hat{\rho}(\mathbf{\Omega}) \hat{\mathcal{P}}_{\text{LLL}}$ is the LLL-projected density operator, $\hat{\rho}_{lm} = \int d\Omega \hat{\rho}(\mathbf{\Omega})$, and the Haldane pseudopotentials are given by

$$V_l = 2\pi \int_0^\pi d\theta \sin(\theta) P_l(\cos \theta) V(2R \sin(\theta/2)). \quad (\text{C4})$$

For the numerical simulations, we set $R = \sqrt{Q_{\text{LN}}} l_B$ and keep it fixed when a quasihole is introduced into the quantum Hall system.

APPENDIX D: LEE-LOW-PINES TRANSFORMATION IN THE SPHERICAL GEOMETRY

Here we elaborate on the application of the Lee-Low-Pines transformation to the exciton-quantum Hall system which was crucial to obtain the results in this work. To this end, we consider the problem of an exciton as a neutral mobile impurity X interacting with a quantum Hall system of N electrons with mass m and charge $q = -e < 0$ with fractionally filled lowest Landau level. Both systems are put on a sphere with radius R enclosing a magnetic monopole with strength Q . The full Hamiltonian reads

$$\hat{H} = \frac{\hat{\mathbf{L}}_X^2}{2M_X R^2} + \sum_{i=1}^N V_{Xe}(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_X) + \hat{H}_{\text{FQH}}. \quad (\text{D1})$$

In Eq. (D1), M_X , $\hat{\mathbf{r}}_X$, and $\hat{\mathbf{L}}_X$ are, respectively, the exciton's mass, position, and angular momentum, V_{Xe} is the exciton-electron interaction potential, $\hat{\mathbf{r}}_i$ is the position of the i 'th electron, and $\hat{H}_{\text{FQH}}$ is given in Eq. (C3). The full rotational symmetry of the exciton-quantum Hall system implies that the spectrum is invariant under the rotation which bring the exciton at a position $\mathbf{r}_X$ to the north pole $\mathbf{r}_0 = R\mathbf{e}_z$, where $\mathbf{e}_z$ is the unit vector along the z axis. This rotation is performed by the following unitary:

$$\hat{U}_{\text{LLP}} = e^{-i\hat{\phi} \otimes \hat{L}_z} e^{-i\hat{\theta} \otimes \hat{L}_y} e^{-i\hat{\gamma} \otimes \hat{L}_z}. \quad (\text{D2})$$

In Eq. (D2), $\hat{U}_{\text{LLP}}$ acts on the whole quantum Hall system with total angular momentum $\hat{\mathbf{L}} = \sum_{i=1}^N \hat{\mathbf{L}}_i$ to

rotate it by a rotation which brings a rigid body in the state $|\varphi, \theta, \gamma\rangle$ characterized with Euler angles $(\varphi, \theta, \gamma)$ to $|\varphi = 0, \theta = 0, \gamma = 0\rangle$. Note that, although for a pointlike impurity the Euler angle γ is not defined, it should be kept throughout the calculations to maintain the entire group structure of the rotations of the quantum Hall system.

To make use of the LLP transformation, it is necessary to understand its effect on the full Hamiltonian $\hat{H}$ as well as an arbitrary state $|J, M, n, \alpha\rangle_{\text{X,QH}}$ of the whole system where J , M , and n denote angular momentum quantum numbers (we discuss the meaning of the quantum number n later), and α denotes the rest of quantum numbers. First, we inspect different terms in the Hamiltonian. The quantum Hall Hamiltonian $\hat{H}_{\text{FQH}}$ is obviously rotationally invariant. Application of $\hat{U}_{\text{LLP}}$ on $V_{Xe}(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_X)$ yields

$$\hat{U}_{\text{LLP}}^\dagger V_{Xe}(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_X) \hat{U}_{\text{LLP}} = V_{Xe}(\hat{\mathbf{r}}'_i - \mathbf{r}_0), \quad (\text{D3})$$

where $\mathbf{r}'_i = \mathcal{R}(\varphi, \theta, \gamma)[\mathbf{r}_i]$ is the rotated position of the i 'th particle. In Eq. (D3), it is noticeable that the action of $\hat{U}_{\text{LLP}}$ has restricted the exciton degree of freedom to the north pole, thus instead of interacting with a mobile exciton, all the electrons in the quantum Hall system interact with a static potential localized around $\mathbf{r}_0$.

The action of $\hat{U}_{\text{LLP}}$ on the exciton kinetic energy is more involved. To proceed with the analytics, it is more convenient to work with the spherical tensor components of the angular momentum operators. For any vector operator $\hat{\mathbf{O}} = \hat{O}_x \mathbf{e}_x + \hat{O}_y \mathbf{e}_y + \hat{O}_z \mathbf{e}_z$, the spherical tensor components are defined by and $\hat{O}_\mu = (-\mu)/\sqrt{2}(\hat{O}_x + i\mu\hat{O}_y)$, $\mu = \pm 1$,

$$\begin{cases} \hat{O}_0 = \hat{O}_z, & \mu = 0, \\ \hat{O}_\mu = -\frac{\mu}{\sqrt{2}}(\hat{O}_x + i\mu\hat{O}_y), & \mu = \pm 1. \end{cases} \quad (\text{D4})$$

Accordingly, the spherical tensor components of the exciton angular momentum operator in coordinate space read

$$\begin{cases} \hat{L}_{X,0}(\varphi, \theta, \gamma) = -i\partial_\varphi, \\ \hat{L}_{X,\pm 1}(\varphi, \theta, \gamma) = \frac{i}{\sqrt{2}} e^{\pm i\varphi} \left[\pm \cot(\theta) \partial_\gamma + i\partial_\theta \mp \frac{1}{\sin(\theta)} \partial_\varphi \right]. \end{cases} \quad (\text{D5})$$

Applying $\hat{U}_{\text{LLP}}$ to $\hat{L}_{X,\mu}$ directly gives

$$\hat{U}_{\text{LLP}}^\dagger \hat{L}_{X,\mu} \hat{U}_{\text{LLP}} = \hat{L}_{X,\mu} - \sum_v D_{\mu v}^{(1)*}(\hat{\varphi}, \hat{\theta}, \hat{\gamma}) \hat{L}_v, \quad (\text{D6})$$

where $D_{\mu v}^{(1)*}(\varphi, \theta, \gamma)$ is the complex conjugate of the Wigner D-matrix $D^{(1)}$, and L_v is the v -component of the total angular momentum of the quantum Hall system. From Eq. (D6), it is evident that $\hat{U}_{\text{LLP}}$ imprints the effect of the

many-body medium on the exciton through the total angular momentum components $\hat{L}_v$. The LLP transformation of $\hat{L}^2$ is then achieved by using Eq. (D6) as the following:

$$\begin{aligned}\hat{U}_{\text{LLP}}^\dagger \hat{L}_X^2 \hat{U}_{\text{LLP}} &= \hat{L}_X'^2 - \sum_{\mu} (-1)^{\mu} \hat{L}_X'^{(-\mu)} \hat{L}_{\mu} + \hat{L}^2 \\ &= (\hat{L}_X'^{(c)} - \hat{L})^2.\end{aligned}\quad (\text{D7})$$

In Eq. (D7), $\hat{L}_X'^{(c)} = \sum_{\mu} \hat{L}_X'^{\mu} \mathbf{e}_{\mu}$ denotes the *contravariant angular momentum* of the exciton [69], where $\hat{L}_X'^{\mu} = (-1)^{\mu} \hat{L}_{X,-\mu}'$. Here, $\hat{L}_{X,\mu}'$ represents the exciton angular momentum in the *body-fixed frame*, defined as the frame obtained by rotating the space-fixed frame via Euler angles

$(\varphi, \theta, \gamma)$. Although the term “body-fixed” traditionally refers to a frame attached to a rigid body in three dimensions, we adopt it here to refer to a frame whose z axis passes through the exciton position. This usage serves as a natural generalization in the context of exciton motion on the sphere. The components of $\hat{L}_X'$ and $\hat{L}_X$ are related to one another by

$$\begin{cases} \hat{L}_X'^{\mu} = \sum_{\nu} D_{\nu\mu}^{(1)}(\varphi, \theta, \gamma) \hat{L}_{X,\nu}, \\ \hat{L}_{v,X} = \sum_{\mu} D_{v\mu}^{(1)*}(\varphi, \theta, \gamma) \hat{L}_X'^{\mu}. \end{cases}\quad (\text{D8})$$

In order to come from Eqs. (D6) to (D7), the following identities that are the defining differential equations for $D_{\mu\lambda}^{(j)}(\hat{\varphi}, \hat{\theta}, \hat{\gamma})$ must be used:

$$\begin{cases} [\hat{L}_v, D_{\mu\lambda}^{(j)}(\hat{\varphi}, \hat{\theta}, \hat{\gamma})] = (-1)^{1+\nu} \sqrt{j(j+1)} C_{j,\mu;1,-\nu}^{j,\mu-\nu} D_{(\mu-\nu)\lambda}^{(j)}(\hat{\varphi}, \hat{\theta}, \hat{\gamma}), \\ [\hat{L}^v, D_{\mu\lambda}^{(j)}(\hat{\varphi}, \hat{\theta}, \hat{\gamma})] = -\sqrt{j(j+1)} C_{j,\lambda;1,v}^{j,\lambda+v} D_{\mu(\lambda+v)}^{(j)}(\hat{\varphi}, \hat{\theta}, \hat{\gamma}), \end{cases}\quad (\text{D9})$$

where $C_{j_1,\mu_1;j_2,\mu_2}^{j,\mu} = \langle j_1, \mu_1; j_2, \mu_2 | j, \mu \rangle$ are Clebsch-Gordan coefficients.

The next step is to find the form of an arbitrary state $|J, M, n; \alpha\rangle_{\text{X,QH}}$ of the full system with angular momentum quantum numbers J, M , and n , and the rest of the quantum numbers summarized in β . To this end, we note that $|J, M, n; \alpha\rangle_{\text{X,QH}}$ takes the most general form

$$|J, M, n; \alpha\rangle_{\text{X,QH}} = \sum_{j,L,\beta} c_{j,L;\alpha,\beta} \sum_{m,n} C_{j,m;L,N}^{J,M} |j, m, n\rangle_{\text{X}} \otimes |L, N; \beta\rangle_{\text{QH}} = \sum_{j,L,\beta} c_{j,L;\alpha,\beta} |j, L; \beta\rangle_{\text{X,QH}}. \quad (\text{D10})$$

In Eq. (D10), $|j, m, n\rangle_{\text{X}}$ is an angular-momentum- j eigenstate of $\hat{L}_X^2$, and the quantum numbers m and n denote the angular momentum projections along the z axis in the space-fixed and body-fixed frames, respectively. More explicitly,

$$\begin{cases} \hat{L}_X^2 |j, m, n\rangle_{\text{X}} = \hbar^2 j(j+1) |j, m, n\rangle_{\text{X}}, \\ \hat{L}_{X,z} |j, m, n\rangle_{\text{X}} = \hbar m |j, m, n\rangle_{\text{X}}, \\ \hat{L}_{X,z}' |j, m, n\rangle_{\text{X}} = \hbar n |j, m, n\rangle_{\text{X}}. \end{cases}\quad (\text{D11})$$

The state $|L, N; \beta\rangle_{\text{QH}}$ is an eigenstate of $\hat{H}_{\text{FQH}}$ with angular momentum quantum numbers L and N and labeled by β . It is clear from the form of $|J, M, n; \alpha\rangle_{\text{X,QH}}$ in Eq. (D10) that it suffices to evaluate $\hat{U}_{\text{LLP}}^\dagger$ on $|j, m, n\rangle_{\text{X}} \otimes |L, N; \beta\rangle_{\text{QH}}$,

$$\hat{U}_{\text{LLP}}^\dagger |j, m, n\rangle_{\text{X}} \otimes |L, N; \beta\rangle_{\text{QH}} = \int_0^{2\pi} d\gamma \int_0^\pi d\theta \sin(\theta) \int_0^{2\pi} d\varphi \sqrt{\frac{2j+1}{8\pi^2}} D_{mn}^{(j)*}(\varphi, \theta, \gamma) |\varphi, \theta, \gamma\rangle_{\text{X}} \otimes \hat{U}^\dagger(\varphi, \theta, \gamma) |L, N; \beta\rangle_{\text{QH}}. \quad (\text{D12})$$

To proceed, we use the transformation of angular momentum representations under arbitrary rotations

$$\hat{U}^\dagger(\varphi, \theta, \gamma) |L, N; \beta\rangle_{\text{QH}} = \sum_{N'} D_{NN'}^{(L)*}(\varphi, \theta, \phi) |L, N'; \beta\rangle_{\text{QH}}, \quad (\text{D13})$$

together with several identities involving angular momentum summations and symmetry properties of the Wigner D matrices and Clebsch-Gordan coefficients to finally arrive at

$$\hat{U}_{\text{LLP}}^\dagger |j, L; \beta\rangle_{\text{X,QH}} = \sum_{N'} (-1)^{L+N'} C_{J, -(n+N'); L, N'}^{j, (-n)*} |J, M, n+N'\rangle_{\text{X}} \otimes |L, N'; \beta\rangle_{\text{QH}}. \quad (\text{D14})$$

Inserting Eq. (D14) into Eq. (D10),

$$\begin{aligned} \hat{U}_{\text{LLP}}^\dagger |J, M, n; \alpha\rangle_{\text{X,QH}} \\ = \sum_{L, N, \beta} f_{L, N, n; \alpha, \beta} |J, M, n + N\rangle_{\text{X}} \otimes |L, N; \beta\rangle_{\text{QH}}, \end{aligned} \quad (\text{D15})$$

where $f_{L, N, n; \alpha, \beta} = \sum_j (-1)^{L+N} C_{J, -(n+N); L, N}^{j, (-n)*} C_{j, L; \alpha, \beta}$.

Similar results have been obtained in Ref. [70] in the context of angulons. However, the underlying physics differs significantly: angulon systems involve quantum rotors with internal rotational degrees of freedom in 3D space, while in our case, angular momentum arises from the translational motion of excitons and electrons confined to 2D space. These distinctions lead to fundamentally different thermodynamic limits. Notably, in the angulon model, the quantum number n is fixed at zero, whereas in the impurity-quantum Hall system on a sphere, n reflects genuine angular momentum excitations of the quantum Hall fluid around the impurity.

Our exact diagonalization analysis reveals that the ground state of the anyon-trion corresponds to $n = 0$, while the first excited state is characterized by $n = 1$. This indicates that, in the first excited state, the quantum Hall system carries one additional unit of angular momentum around the exciton. This excitation directly corresponds to the first excited state identified in Ref. [43], in the context of a heavy impurity bound to a quasihole of atomic Laughlin states, which can be understood as an internal edge excitation of the quasihole around the impurity.

APPENDIX E: THERMODYNAMIC LIMIT OF THE ANYON-TRION STATES

Here we demonstrate that the low-energy part of the spectrum of the Hamiltonian in Eq. (1) of the main text

$$\hat{H}_{\text{LLP, Kin}} = \frac{\hbar^2}{2M_X R^2} \left[\left(\hat{L}_{X,0}^2 - \hat{L}_{X,+} \hat{L}_{X,-} - \hat{L}_{X,-} \hat{L}_{X,+} \right) - 2 \left(\hat{L}_{X,0} \hat{L}_0 + \hat{L}_{X,+} \hat{L}_+ + \hat{L}_{X,-} \hat{L}_- \right) + \left(\hat{L}_0^2 - \hat{L}_+ \hat{L}_- - \hat{L}_- \hat{L}_+ \right) \right], \quad (\text{E2})$$

where the spherical tensor components are defined in Eqs. (D4). Acting on the product states $|J, M, n + M_L\rangle_{\text{X}} \otimes |L, M_L; \beta\rangle_{\text{QH}}$ [see Eq. (D15)], we obtain

$$\begin{aligned} \left(\hat{\mathbf{L}}_X^{(c)} - \hat{\mathbf{L}} \right)^2 |J, M, n + M_L\rangle_{\text{X}} \otimes |L, M_L; \beta\rangle_{\text{QH}} = & (J(J+1) + L(L+1) - 2M_L^2 - 2nM_L) |J, M, n + M_L\rangle_{\text{X}} \otimes |L, M_L; \beta\rangle_{\text{QH}} \\ & + (-1) \sqrt{J(J+1) - (n+M_L)(n+M_L+1)} \sqrt{L(L+1) - M_L(M_L+1)} |J, M, n + M_L + 1\rangle_{\text{X}} \otimes |L, M_L + 1; \beta\rangle_{\text{QH}} \\ & + (-1) \sqrt{J(J+1) - (n+M_L)(n+M_L-1)} \sqrt{L(L+1) - M_L(M_L-1)} |J, M, n + M_L - 1\rangle_{\text{X}} \otimes |L, M_L - 1; \beta\rangle_{\text{QH}}. \end{aligned} \quad (\text{E3})$$

The lowest-energy eigenvalues of $\hat{H}_{\text{LLP}}$ for $M_X \rightarrow \infty$ at $Q = Q_{\text{qh}}$ occur for $L_z = -N/2$, corresponding to the quasihole with $L = N/2$ and $L_z = -N/2$ bound to the exciton (see Fig. 5). This quasihole state has a minimal electron density near the north pole (see Fig. 4). Hence, the relevant low-energy states are dominated by configurations with $L \simeq N/2$

exhibits a well-defined thermodynamic limit characterized by a $1/N$ scaling, where N is the number of electrons in the quantum Hall system. Consequently, the energy gaps $E_{\text{Gap},1}$ and $E_{\text{Gap},2}$ inherit the same scaling behavior, as shown in Fig. 3(b).

To this end, we first consider the eigenstates of the Lee-Low-Pines Hamiltonian $\hat{H}_{\text{LLP}}$ given by

$$\begin{aligned} \hat{H}_{\text{LLP}} &= \frac{1}{2M_X R^2} \left(\hat{\mathbf{L}}_X^{(c)} - \hat{\mathbf{L}} \right)^2 + \sum_{i=1}^N V_{Xe}(\mathbf{r}'_i - \mathbf{r}_0) + \hat{H}_{\text{FQH}} \\ &= \hat{H}_{\text{LLP, Kin}} + \hat{V}_X + \hat{H}_{\text{FQH}}, \end{aligned} \quad (\text{E1})$$

where $\hat{H}_{\text{LLP, Kin}}$ and $\hat{V}_X$ denote the first and second terms in the expression for $\hat{H}_{\text{LLP}}$ in Eq. (E1), respectively. In the $M_X \rightarrow \infty$ limit, the first term is negligible, and $\hat{H}_{\text{LLP}}$ describes the quantum Hall system under a static exciton potential localized at the north pole.

The eigenstates of $\hat{H}_{\text{FQH}}$ are denoted by $|L, M_L; \beta\rangle$, as defined in the section “Lee-Low-Pines transformation in the spherical geometry.” Here, L is the total angular momentum, M_L its projection along the z axis, and β labels distinct eigenstates. The exciton potential preserves M_L but mixes states of different L satisfying $|M_L| \leq L$. Since the matrix elements of the exciton potential depend only on local wave function properties near the north pole—over a range comparable to the interlayer spacing and smaller than the magnetic length—the low-energy eigenenergies inherit the $1/N$ scaling of the FQH eigenstates. Thus, the eigenenergies of the static exciton Hamiltonian converge as $1/N$ in the thermodynamic limit.

Next, we verify that including the finite exciton mass, i.e., restoring the kinetic term, does not alter this scaling. The kinetic term can be expressed as

and $L_z \simeq -N/2$. Setting $J = J_{\text{qh}} = N/2$, $L = J_{\text{qh}} + \Delta L$, and $M_L = -J_{\text{qh}} + \Delta M_L$, with $\Delta L, \Delta M_L \ll J_{\text{qh}}$, we find that the matrix elements of $\hat{H}_{\text{LLP,Kin}}$ take the asymptotic forms

$$\begin{aligned} \langle L, M_L | (\hat{\mathbf{L}}_X^{(c)} - \hat{\mathbf{L}})^2 | L, M_L \rangle &\simeq 2J_{\text{qh}} \left[1 + n + \Delta L + 2\Delta M_L + \frac{\Delta L - 2n\Delta M_L}{2J_{\text{qh}}} \right], \\ \langle L, M_L | (\hat{\mathbf{L}}_X^{(c)} - \hat{\mathbf{L}})^2 | L, M_L \pm 1 \rangle &\simeq -J_{\text{qh}} \sqrt{1 + 2(n + \Delta M_L - 1) - \frac{(n + \Delta M_L)(n + \Delta M_L \pm 1)}{J_{\text{qh}}}} \\ &\times \sqrt{1 + 2\Delta L + (2\Delta M_L \pm 1) + \frac{\Delta L(\Delta L + 1) - \Delta M_L(\Delta M_L \pm 1)}{J_{\text{qh}}}}, \end{aligned} \quad (\text{E4})$$

where $|L, M_L\rangle$ in Eq. (E4) is a shorthand for $|J, M, n + M_L\rangle_X \otimes |L, M_L; \beta\rangle_{\text{QH}}$ with the J, L and M_L values determined above, and we neglect the terms of $O(1/J_{\text{qh}})$ in Eq. (E4) and under the square roots. It is important to note that $\hat{H}_{\text{LLP,Kin}}$ is block diagonal in the angular momentum representations J and L and diagonal in the index β .

From Eq. (E4), we note that the matrix elements of $\hat{H}_{\text{LLP,Kin}}$ over the relevant states mentioned above—which we denote by $[\hat{H}_{\text{LLP,Kin}}]_{L, M_L; L', M'_L}$ —are of the form

$$\begin{aligned} [\hat{H}_{\text{LLP,Kin}}]_{L, M_L; L', M'_L} \\ = \frac{\hbar^2 J_{\text{qh}}}{2M_X R^2} \left([\hat{H}_{\text{LLP,Kin}}^{(0)}]_{L, M_L; L', M'_L} + O\left(\frac{1}{N}\right) \right), \end{aligned} \quad (\text{E5})$$

where $[\hat{H}_{\text{LLP,Kin}}^{(0)}]_{L, M_L; L', M'_L}$ is a matrix with entries only dependent on $n, \Delta M_L$, and ΔL . The prefactor $\hbar^2 J_{\text{qh}} / 2M_X R^2$ has a well-defined thermodynamic limit,

$$\begin{aligned} \frac{\hbar^2 J_{\text{qh}}}{2M_X R^2} &= \frac{\hbar^2}{2M_X l_B^2} \frac{N/2}{Q_{\text{LN}}} \\ &= \frac{\hbar^2}{2M_X l_B^2} \frac{N/2}{m(N-1)/2} \rightarrow \frac{\hbar^2}{2M_X l_B^2} \frac{1}{m}. \end{aligned} \quad (\text{E6})$$

From Eqs. (E5) and (E6) we see that the effect of $\hat{H}_{\text{LLP,Kin}}$ over the relevant states $|J, M, n + M_L\rangle_X \otimes |L, M_L; \beta\rangle_{\text{QH}}$ with $J = J_{\text{qh}} = N/2$, $L = J_{\text{qh}} + \Delta L$, $M_L = -J_{\text{qh}} + \Delta M_L$ and $\Delta L, \Delta M_L \ll J_{\text{qh}}$ which dominate the low-energy eigenstates of $\hat{V}_X + \hat{H}_{\text{FQH}}$ has a well-defined $N \rightarrow \infty$ limit plus $1/N$ corrections.

Quantitatively, this prefactor evaluates to $\hbar^2 / (2mM_X l_B^2) \simeq 0.0045 E_C$, significantly smaller than the characteristic gap scale of $\hat{V}_X + \hat{H}_{\text{FQH}}$ ($\sim 0.03 E_C$). This comparison corroborates the numerical observations that for $N = 5-8$, only a few ΔL and ΔM_L values suffice to capture the lowest three eigenstates used to compute $E_{\text{Gap},1}$ and $E_{\text{Gap},2}$ in Figs. 3(a) and 3(b). In fact, as noted in the main text, over 99% of the weight of the third state at $N = 8$ resides in the

quasihole manifold, making $\Delta L = 0$ an excellent approximation. Furthermore, only a few states with $M_L > -J_{\text{qh}}$ contribute to the states up to the third excited state. Note that this explanation is only for argumentation purposes. For the numerical results, we perform exact diagonalization over the entire Hilbert space in the Lee-Low-Pines frame, as discussed in the previous section.

These arguments explain the numerically observed $1/N$ scaling of the energy gaps $E_{\text{Gap},1}$ and $E_{\text{Gap},2}$ and confirm their finiteness in the thermodynamic limit.

APPENDIX F: PARABOLIC DISPERSION OF ANYON-TRIONS

Here we discuss the observed parabolic dispersion of anyon-trion states indicated in Fig. 3(a). The appearance of the parabola in terms of the angular momentum is a direct consequence of the kinetic term of the free mobile exciton in the Lee-Low-Pines frame, i.e., $\hat{H}_{\text{LLP,Kin}} = 1/2M_X R^2 (\hat{\mathbf{L}}_X^{(c)} - \hat{\mathbf{L}})^2$. It can be shown [64] that the spectrum of $\hat{H}_{\text{LLP,Kin}}$ is exactly $\hbar^2 J(J+1)/2M_X R^2$, where J is the total angular momentum of the system. However, due to the binding of the quasihole to the exciton, the minimum of the dispersion is shifted to the angular momentum of the quasihole. An intuitive picture is provided by taking the expectation value of $\hat{H}_{\text{LLP,Kin}}$ over a state $|\Psi_{\text{AT}}\rangle = |J, M_J, -J\rangle \otimes |J_{\text{qh}}, -J_{\text{qh}}\rangle$ that approximates an anyon-trion wave function in the Lee-Low-Pines frame. The expected value then reads $\langle \Psi_{\text{AT}} | \hat{H}_{\text{LLP,Kin}} | \Psi_{\text{AT}} \rangle = (\hbar^2 / 2M_X R^2) [J(J+1) + J_{\text{qh}}(J_{\text{qh}}+1) - 2JJ_{\text{qh}}] \simeq (\hbar^2 / 2M_X R^2) (J - J_{\text{qh}})^2$ for $J, J_{\text{qh}} \ll (J - J_{\text{qh}})^2$. This result aligns with the expectation that the dispersion of the anyon-trion is a parabola around the quasihole angular momentum. In the thermodynamic limit, the kinetic term approaches that of a free particle in a plane, and the angular momentum eigenstates and eigenvalues resemble partial waves with well-defined radial momenta.

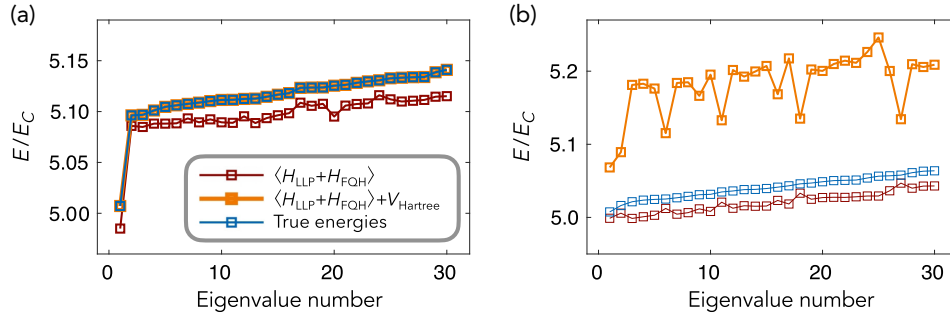


FIG. 6. Comparison between exact eigenenergies of the exciton-quantum Hall system (blue), mean-field estimates E_{MF} (orange), and the expected values of the noninteracting Hamiltonian (red). Results are shown for the 30 lowest eigenstates of an exciton interacting with (a) a $\nu = 1/3$ Laughlin state with $Q = Q_{LN}$, $J = 0$, and (b) a $\nu = 1/3$ state with a single quasihole, $Q = Q_{qh}$, $J = J_{qh}$.

APPENDIX G: IMPACT OF THE EXCITON MASS ON EXCITON-ELECTRON CORRELATIONS IN ANYON-TRION BOUND STATES

Here, we demonstrate that the finite mass of an exciton introduces significant correlations between the exciton's motional degrees of freedom and the many-body states of the surrounding quantum Hall system. Beyond spatial exciton-electron correlations, the finite mass induces non-trivial structure in the joint wave function of the coupled system. Intuitively, a lighter exciton mass enhances correlations with the background medium—in this case, the FQH liquid.

To quantify these correlations, we define an exciton-electron correlation energy analogous to the correlation energy in density functional theory. For a given eigenstate $|\Psi\rangle$ of the full exciton-quantum Hall system, we define the correlation energy as the deviation of its exact energy E_Ψ from a mean-field estimate: $E_{MF} = \langle \hat{H}_{LLP} + \hat{H}_{FQH} \rangle + V_{Hartree}$, where the Hartree shift $V_{Hartree}$ captures the exciton-electron density-density interaction energy, given by

$$V_{Hartree} = \sum_{l,m} V_{Xe,l} Y_{lm}^*(\hat{\mathbf{e}}_z) |\langle \hat{\mathbf{e}}_z | \Psi \rangle|^2 \left\langle \sum_i Y_{lm}(\Omega_i) \right\rangle, \quad (G1)$$

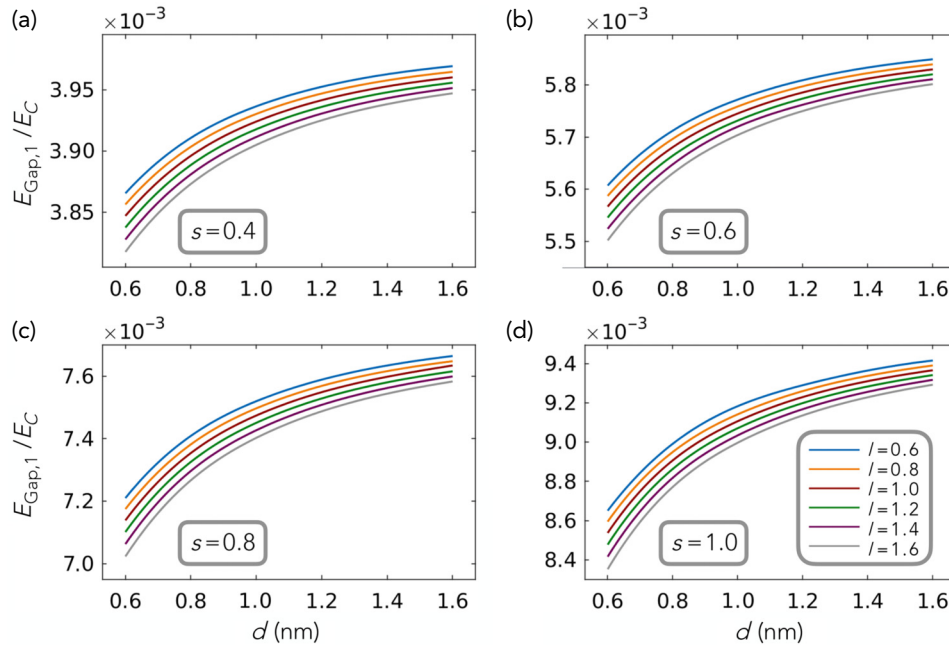


FIG. 7. First excitation gap $E_{Gap,1}$ as a function of exciton mass scaling s , interlayer spacing l , and bilayer separation d . Panels (a)–(d) show that decreasing s (i.e., increasing M_X) reduces $E_{Gap,1}$, indicating weaker exciton-electron correlations. The gap is more sensitive to variations in d than l , reflecting the stronger dependence on the exciton dipole strength. Calculations are for $N = 8$ electrons, $Q = Q_{qh}$, $J = J_{qh}$, and $B = 16$ T.

where $V_{Xe,l}$ are the multipole coefficients of the exciton-electron potential, Y_{lm} are spherical harmonics, and $\mathbf{\Omega}_i$ denotes the position of the i th electron on the sphere.

Figure 6 presents exact diagonalization results for $N = 7$ electrons with interlayer parameters $l = 1.2$ nm and $d = 0.6$ nm and $B = 16$ T. Panel (a) corresponds to a Laughlin state ($Q = Q_{\text{LN}}$) where the system exhibits isotropy and total angular momentum $J = 0$. In this case, the exciton contributes a purely repulsive Hartree shift, well described by mean-field theory. In contrast, panel (b) illustrates the situation in the presence of a quasi-hole, where the many-body eigenstates of the interacting system lie significantly below the mean-field estimate, indicating the presence of strong exciton-electron correlations.

To further investigate the role of the exciton's finite mass, we examine the evolution of the first energy gap as a function of the exciton mass. We introduce a dimensionless scaling parameter $0 < s < 1$, defining a scaled mass $M_X(s) = M_X/s$, and correspondingly scale the kinetic contribution as $\hat{H}_{\text{LLP}}(s) = s\hat{H}_{\text{LLP}}$. This procedure effectively interpolates between a static and a mobile exciton, allowing us to isolate the impact of kinetic motion.

As shown in Fig. 7, the first excitation gap $E_{\text{Gap},1}$ increases with increasing contribution of the exciton kinetic term, underscoring the importance of exciton mobility in generating many-body correlations. The dependence of the gap on the bilayer separation d is particularly pronounced, as it controls the dipole strength of the exciton and, hence, the magnitude of the repulsive interaction.

APPENDIX H: EXTRACTION OF THE FRACTIONAL CHARGE FROM ANYON-TRION BINDING ENERGY

To motivate the expression used in Eq. (4) for extracting the fractional charge from the anyon-trion binding energy, we consider the behavior of a static exciton bound to an unperturbed quasi-hole in a $\nu = 1/3$ FQH state. Given that the exciton potential is short-ranged compared to the magnetic length, it primarily probes the electron density near the quasi-hole core. In this regime, the local electron density scales as $\sim \nu r^2$, where r is the radial distance from the quasi-hole center. As a result, the exciton blueshift $\Delta E(\nu, Q_{\text{qh}})$ appearing as the mean-field energy shift of the exciton is expected to scale linearly with ν , which is also confirmed by numerical calculations. Consequently, the anyon-trion binding energy, defined as $E_{\text{B,AT}}(\nu) = \nu \Delta E(1, Q_{\text{LN}}) - \Delta E(\nu, Q_{\text{qh}})$ also exhibits a linear dependence on ν . This linearity makes $E_{\text{B,AT}}(\nu)$ a sensitive and reliable probe of the quasi-hole's fractional charge.

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