

Fast Mixing of Weakly Interacting Fermionic Systems at Any Temperature

Yu Tong^{1,2,3,*} and Yongtao Zhan^{4,5,†}

¹Department of Mathematics, [Duke University](#), Durham, North Carolina 27708, USA

²Department of Electrical and Computer Engineering, [Duke University](#), Durham, North Carolina 27708, USA

³Duke Quantum Center, [Duke University](#), Durham, North Carolina 27701, USA

⁴Institute of Quantum Information and Matter, [California Institute of Technology](#), California 91125, USA

⁵Department of Physics, [California Institute of Technology](#), California 91125, USA



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We study the mixing time of a recently proposed efficiently implementable Lindbladian designed to prepare the Gibbs states in the setting of weakly interacting fermionic systems. We show that at any temperature, the Lindbladian spectral gap for even parity observables is lower bounded by a constant Δ , when the interaction strength (e.g., the on-site interaction strength for the Fermi-Hubbard model) is below a constant threshold U_β . Both Δ and U_β are independent of the system size. This leads to a mixing time estimate that is at most linear in the system size, thus showing that the corresponding Gibbs states can be prepared efficiently on quantum computers. Our result also implies exponential decay of correlation in these Gibbs states and efficient learnability of their properties.

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I. INTRODUCTION

Simulating nonintegrable quantum systems is a notoriously difficult task. Numerical methods designed for this purpose generally suffer from the curse of dimensionality, the sign problem, or are not efficiently scalable either in the system size or achievable precision. Nonintegrable quantum systems that are efficiently simulable either on a classical or quantum computer are hard to find. Some examples include ground states of one-dimensional (1D) gapped local Hamiltonians [1], Gibbs states that have exponentially decaying correlation and are approximately Markov states [2], Gibbs states of certain commuting Hamiltonians [3–6], ground states of quantum impurity models [7], and time evolution of 1D many-body localized systems [8]. For the last two only quasipolynomial time classical algorithms are known in some parameter regimes.

The recently proposed efficiently implementable Lindbladian [Eq. (11)] that satisfies the Kubo-Martin-Schwinger (KMS) detailed balance condition (a quantum generalization of the detailed balance condition for

Markov semigroups) in Refs. [9,10] offers a new and promising approach to prepare the Gibbs states of a wider class of nonintegrable quantum systems. Similar to classical Markov chain Monte Carlo, the efficiency of the method crucially depends on the *mixing time* of the Lindbladian dynamics, i.e., the time needed for the system to evolve close to the stationary state. Due to its importance in characterizing natural quantum systems and algorithmic performance, the mixing time has been the central topic of many studies [5,6,11–26].

As an example of the above approach, in Ref. [27], the authors analyzed the mixing time of the above Lindbladian for preparing high-temperature Gibbs states of quantum spin systems, and thereby showing that these Gibbs states can be efficiently prepared on quantum computers. By considering the parent Hamiltonian of the Lindbladian, the authors were able to use the stability result for the spectral gap of frustration-free Hamiltonians [28] to show that the spectral gap of the Lindbladian is lower bounded by a constant when the temperature is above a constant threshold that is independent of the system size. This mixing-time bound was later improved in Ref. [16]. We should note, however, that this is not the only approach to prepare such high-temperature Gibbs states. In a subsequent work, Ref. [29], the authors showed that for temperature beyond a certain threshold, the Gibbs state of a quantum spin system becomes unentangled, and they propose an efficient classical algorithm to sample from the Gibbs state. It is at this moment unclear how the temperature thresholds in these works compare with each other.

*Contact author: yu.tong@duke.edu

†Contact author: yzhan@caltech.edu

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From the above discussion we can see that, to study Gibbs states that have intrinsically quantum properties, such as entanglement, one may need to go to lower temperature. Moreover, many quantum systems of interest are not spin systems, but are fermionic in nature, which requires us to deal with a different notion of locality. We therefore ask the following question:

Can we prove fast mixing of fermionic systems at any temperature, given sufficiently weak interaction?

In this work, we will give an affirmative answer to this question. We note that the Lindbladian we work with, proposed in Ref. [10], is artificially constructed to be efficiently implementable on a quantum computer, and may not reflect how a fermionic open quantum system evolves in nature.

A. Main result

The quantum system we consider is a D -dimensional cubic lattice Λ consisting of n sites. On each site we have one Dirac fermion, which is equivalent to two Majorana modes. This gives rise to $2n$ Majorana operators γ_j , $j = 1, 2, \dots, 2n$, satisfying the anticommutation relation $\{\gamma_j, \gamma_k\} = 2\delta_{jk}$. The Hamiltonian we study takes the following form:

$$H = H_0 + V, \quad (1)$$

where $H_0 = \sum_{jk} h_{jk} \gamma_j \gamma_k$ is the noninteracting (quadratic) part, and V is a general fermionic operator describing the interaction. V here preserves parity, i.e., it only contain terms that are products of an even number of Majorana operators. Moreover, we assume that H_0 and V are $(1, r_0)$ -geometrically local and (U, r_0) -geometrically local, respectively, following the definition below:

Definition 1 (((ξ, r_0) -geometrically local operators). We say an operator W is (ξ, r_0) -geometrically local if

$$W = \sum_{B \in \mathcal{B}(r_0)} W_B,$$

where $\mathcal{B}(r)$ denotes the set of balls of radius r , W_B is supported on B , and $\|W_B\| \leq \xi$.

For H_0 this means that $h_{jk} = 0$ if the distance between j and k is greater than r_0 . U here is the *interaction strength*. One can see that the Hamiltonian is quadratic when $U = 0$. Our main result concerns the mixing time of fermionic systems, which we define in a slightly different way from the usual definition to take into account the fermionic superselection rule:

Definition 2 (Mixing time for fermions). For a Lindbladian operator $\mathcal{L}$ with unique stationary state in $D(\mathcal{H})_{\text{even}}$,

we define its ϵ -mixing time to be

$$t_{\text{mix}}(\epsilon) = \inf\{t \geq 0 : \|e^{s\mathcal{L}}(\rho) - \sigma\|_1 \leq \epsilon, \forall s \geq t, \rho \in D(\mathcal{H})_{\text{even}}\}. \quad (2)$$

Here σ is the stationary state of $\mathcal{L}$.

In the above $D(\mathcal{H})_{\text{even}}$ [defined in Eq. (55)] is the set of all density operators satisfying the fermionic superselection rule. With this definition of the mixing time, our main result is as follows:

Theorem 1 (Main result). Let $H = H_0 + V$ be a Hamiltonian on a D -dimensional cubic lattice Λ consisting of n sites. H_0 is quadratic in Majorana operators and is $(1, r_0)$ -geometrically local, and V is parity-preserving and is (U, r_0) -geometrically local. Then for any inverse temperature $\beta > 0$, there exists $U_\beta > 0$ such that when $U < U_\beta$, the Lindbladian defined in Eq. (11) has a unique stationary state satisfying the fermionic superselection rule, and the ϵ -mixing time (Definition 2) of the Lindbladian is at most

$$C(n + \log(1/\epsilon)).$$

where $C, U_\beta > 0$ are constants that depend only on r_0, D, β ,

The proof can be found in Sec. VIB. Note that here we only focus on quantum states that satisfy the fermionic superselection rule, i.e., there is no superposition between even- and odd-parity states. This affects the way we define the mixing time (see Definition 2).

B. Proof ideas

We adopt the approach in Ref. [27] of mapping to a parent Hamiltonian and then lower bounding the spectral gap of that parent Hamiltonian [30]. The hope is that for a fermionic Hamiltonian with weak interaction, it can be mapped to a parent Hamiltonian that consists of a large noninteracting part plus small and quasilocal interaction terms (though these terms can add up to be as large as linear in n). With such a parent Hamiltonian, we then use the stability result about the gap of free-fermion Hamiltonian [31] to obtain a gap lower bound for the parent Hamiltonian, which implies a gap lower bound for the Lindbladian. A mixing-time bound then follows. However, we will see that we need to overcome several difficulties to use this approach.

The first difficulty is that the parent Hamiltonian construction in Ref. [10] does not work for fermions. In Ref. [27], for bosonic spin systems, this construction was sufficient because it is based on a “vectorization” mapping from superoperators to operators on an enlarged Hilbert space that preserves both *locality* and the *spectrum* using

the tensor product of two algebras. The mapping preserves locality because the tensor product preserves locality in bosonic spin systems. The mapping preserves the spectrum because it is a $*$ isomorphism between two C^* algebras. For fermions, we have the additional requirement that the parent Hamiltonian needs to be generated by a set of Majorana operators satisfying the canonical anticommutation relations. This immediately calls into question the use of the tensor product: one would need to redefine it as a native fermionic operation. In Appendix D, we will use an example to show that the most obvious attempt to redefine the tensor product will fail to make the mapping an algebra isomorphism. While we cannot rule out the possibility that with sufficient effort one can get a usable redefinition, it seems to be a highly nontrivial task, and may end up yielding something resembling the third quantization [32], which is what we are going to use to partially solve the problem.

The third quantization is a method to map a quasi-free Lindbladian to a quadratic operator in an enlarged fermionic system, which we extend to arbitrary Lindbladians. However, it does not offer as clean a solution as in the bosonic scenario, because in order to preserve locality, the third quantization mapping fails to be an algebra isomorphism, and thereby does not preserve the whole spectrum. In fact, one needs to decide to preserve the spectrum of either the even- or the odd-parity sector, which leads to two different mappings. Fortunately, the fermionic supers-election rule, which prohibits the superposition of even and odd particle numbers, allows us to focus on observables that are even polynomials of Majorana operators, which correspond to the even-parity sector. This is sufficient for establishing the mixing time. Several other questions that are obvious for the vectorization mapping in Ref. [10] also require careful study in the new setting, such as the following:

- (1) Do errors in the Lindbladian result in amplified errors in the parent Hamiltonian through the third quantization?
- (2) Is the parent Hamiltonian self-adjoint?
- (3) Can the parent Hamiltonian have a odd-parity eigenstate with non-negative eigenvalue thus making the top eigenstate no longer uniquely correspond to the Gibbs state?

We answer the first question in the negative in Appendix G, the second question in the affirmative in Sec. V (note that it does not follow directly from the quantum detailed balance condition, which only ensures self-adjointness in the even-parity sector), and the third question in the negative in Theorem 3.

With the parent Hamiltonian constructed from third quantization, we then need to show that the noninteracting parent Hamiltonian, denoted by H_0^{parent} , has a constant

gap lower bound when the jump operators are generated from the full set of single Majorana operators. This is suggested by the numerical results in Ref. [33]. We compute the spectral gap of H_0^{parent} as a function of the inverse temperature β and the single-particle eigenenergies of H_0 , thus obtaining the needed lower bound (Proposition 1).

Next, we show that the interacting part introduces a quasilocal perturbation to the parent Hamiltonian. More precisely, we show that the parent Hamiltonian H^{parent} can be written as $H_0^{\text{parent}} + V^{\text{parent}}$, where H_0^{parent} is quasilocal (Proposition 2), and the interacting part V^{parent} consists of quasilocal terms, each of which scales linearly in the interaction strength U (Proposition 3). The proof makes extensive use of the Lieb-Robinson bound [34], which as discussed in Ref. [35] works equally well for bosonic and fermionic systems. This structure of H^{parent} then enables us to use the spectral gap stability result in Ref. [31] to lower bound the spectral gap of H^{parent} . The spectral gap bound then yields a mixing-time bound following standard analysis as done in Corollary 6. We note that carefully studying the structure of V^{parent} is necessary because our result does not follow from an eigenvalue perturbation theory: even though the perturbation is locally small in U , the entire interacting part V^{parent} has spectral norm on the order of $nU \gg 1$.

C. Organization

The rest of the paper is organized as follows: In Sec. II we discuss the application of our main result to preparing the Gibbs state and learning the Gibbs state properties and its implication on correlation decay. In Sec. III we introduce the basic setup, and in particular the Lindbladian for which we prove the mixing-time bound. In Sec. IV we introduce several theoretical tools that will be needed in the proof: third quantization, the stability of free-fermion Hamiltonians, and the Lieb-Robinson bound. In Sec. V we construct the parent Hamiltonian using third quantization, and study its properties (e.g., locality and how its spectrum reflects that of the Lindbladian). In Sec. VI we state the results about the quasilocality and the spectral gap of the parent Hamiltonian, which imply fast mixing.

II. APPLICATIONS AND IMPLICATIONS

Our result covers a wide range of quantum systems. As an example, we can consider the Fermi-Hubbard model on a D -dimensional cubic lattice Λ :

$$H = - \sum_{\langle i,j \rangle, \varsigma} (c_{i,\varsigma}^\dagger c_{j,\varsigma} + c_{j,\varsigma}^\dagger c_{i,\varsigma}) - \mu \sum_i (n_{i,\uparrow} + n_{i,\downarrow}) + U \sum_i \left(n_{i,\uparrow} - \frac{1}{2} \right) \left(n_{i,\downarrow} - \frac{1}{2} \right), \quad (3)$$

where $c_{i,\varsigma}$ is the fermionic annihilation operator on site i with spin $\varsigma \in \{\uparrow, \downarrow\}$, $n_{i,\varsigma} = c_{i,\varsigma}^\dagger c_{i,\varsigma}$ is the corresponding number operator, U is the interaction strength, and μ is the chemical potential. Theorem 1 immediately implies the following:

Corollary 1. For H given in Eq. (3), for any $\beta > 0$, there exists $U_\beta > 0$ such that if $|U| \leq U_\beta$, then the mixing time of the Lindbladian given in Eq. (11) is at most $C(n + \log(1/\epsilon))$, where $C, U_\beta > 0$ are constants that depend only on D, β, μ .

The mixing-time estimate implies an efficient quantum algorithm to prepare the Gibbs state.

Corollary 2 (Preparing Fermi-Hubbard model Gibbs state). Under the same assumption as Theorem 1, we can prepare the Gibbs state $\sigma = (1/Z)e^{-\beta H}$ where $Z = \text{Tr}[e^{-\beta H}]$ to trace distance ϵ on a quantum computer with runtime

$$\mathcal{O}(n^3 \text{polylog}(n/\epsilon))$$

for any inverse temperature $\beta = \mathcal{O}(1)$ if $U < U_\beta = \mathcal{O}(1)$.

We will explain how we arrive at the $\mathcal{O}(n^3 \text{polylog}(n/\epsilon))$ runtime: using the algorithm in Ref. [10, Theorem I.2] we can see that preparing the Gibbs state requires simulating the dynamics e^{-iHt} for total time $\mathcal{O}(nt_{\text{mix}} \text{polylog}(nt_{\text{mix}}/\epsilon))$ (the factor n comes from normalizing the jump operators as is required in Ref. [10]) [36], where t_{mix} is the mixing time, and this is the dominant part of the cost. Using the mixing-time bound in Theorem 1 we can see that the require evolution time is $\mathcal{O}(n^2 \text{polylog}(n/\epsilon))$. Using the Hamiltonian simulation algorithm for lattice Hamiltonians in Ref. [37], we can simulate the dynamics with a $\mathcal{O}(n)$ overhead up to a polylogarithmic factor in terms of the gate complexity. We then arrive at the total runtime of $\mathcal{O}(n^3 \text{polylog}(n/\epsilon))$ measured in gate complexity. The circuit depth is $\mathcal{O}(n^2 \text{polylog}(n/\epsilon))$.

Besides algorithmic applications, our result also reveals important information about properties of the Gibbs state itself using the correspondence between the top eigenstate of the parent Hamiltonian and the Gibbs state as discussed in Proposition 4. As an example, we have the following corollary about correlation decay:

Corollary 3 (Exponential decay of correlation). Under the same assumptions as Theorem 1, there exists $C, \xi > 0$ that depends only on β, r_0, D , such that for an observable $X \in B(\mathcal{H})$ supported on a constant number of adjacent sites, and any observable $Y \in B(\mathcal{H})$ with support S ,

$$|\text{Tr}[\sigma XY] - \text{Tr}[\sigma X]\text{Tr}[\sigma Y]| \leq C\|X\|\|Y\||S|e^{-d/\xi},$$

where d is the distance between the supports of X and Y .

A proof is provided in Sec. VI C. We note that decay of correlation for Gibbs states were previously only proved for a limited number of settings, typically for 1D systems or for high temperatures [38–40], whereas our result is not constrained by the geometry of the lattice or by the temperature, but is instead constrained by the nonquadratic interaction strength.

The correlation decay immediately implies the learnability of Gibbs state observable expectation values as functions of Hamiltonian parameters through [41, Theorem 2], which is related to the ground state learning in Refs. [42–44]. For concreteness we assume

$$V(x) = \sum_{l=1}^L x_l V_l, \quad (4)$$

where $x = (x_1, x_2, \dots, x_L) \in [-\chi, \chi]^L$, and each V_l is supported on a ball with radius r_0 independent of n , with $\|V_l\| \leq 1$. We also assume that each Majorana mode is involved in at most constant number of terms V_l , and $c_1 n \leq L \leq c_2 n$ for some universal constants c_2, c_1 . These conditions ensure that V is geometrically local. We consider the family of Hamiltonians $H(x) = H_0 + V(x)$, with H_0 defined the same way as before, and denote the normalized Gibbs state corresponding to $H(x)$ at temperature β by $\sigma_\beta(x)$. Given samples of $(x, \tilde{\sigma}_\beta(x))$, where $\tilde{\sigma}_\beta(x)$ is the measurement outcomes from one copy of $\sigma_\beta(x)$ with classical postprocessing (as described in Ref. [41] in the paragraph following Theorem 2 in that paper) and x is drawn uniformly randomly from $[-\chi, \chi]^L$, our goal is to learn the function

$$f_O(x) = \text{Tr}[\sigma_\beta(x)O], \quad (5)$$

for any given observable $O = \sum_{i=1}^M O_i$, where each O_i is supported on a ball with radius independent of the system size. We then have the following corollary as a result of the exponential decay of correlation in Corollary 3 and [41, Theorem 2]:

Corollary 4 (Learning Gibbs state properties). In the setting described in the paragraph above, there exists a constant $\chi_0 > 0$ that is independent of n , such that if $0 < \chi \leq \chi_0$, given

$$\mathcal{O}\left(\log\left(\frac{M}{\delta}\right)\log\left(\frac{n}{\delta}\right)e^{\text{polylog}(1/\epsilon)}\right)$$

samples of $(x, \tilde{\sigma}_\beta(x))$, there exists a classical algorithm that learns a function $f_O^\sharp(x)$, which satisfies

$$\sup_{x \in [-\chi, \chi]^L} |f_O^\sharp(x) - f_O(x)| \leq \epsilon \sum_{i=1}^M \|O_i\|,$$

with probability at least $1 - \delta$.

III. PROBLEM SETUP

The quantum system we consider is a D -dimensional cubic lattice Λ consisting of n sites. We use $d(x, y)$ to denote the Euclidean distance between sites x and y . The Hilbert space we denote by $\mathcal{H}$, and the operator algebra by $B(\mathcal{H})$. The operators on $B(\mathcal{H})$, which we call superoperators, also form an algebra that we denote by $B(B(\mathcal{H}))$. The set of quantum states we denote by $D(\mathcal{H})$, i.e.,

$$D(\mathcal{H}) = \{\rho \in B(\mathcal{H}) : \rho \succcurlyeq 0, \text{Tr}[\rho] = 1\}. \quad (6)$$

On each site we have one Dirac fermion corresponding to two Majorana modes. This gives rise to $2n$ Majorana operators $\gamma_j, j = 1, 2, \dots, 2n$, satisfying the anticommutation relation

$$\{\gamma_j, \gamma_k\} = 2\delta_{jk}. \quad (7)$$

These Majorana operators can be combined to define the fermionic creation and annihilation operators through

$$c_j = \frac{1}{2}(\gamma_{2j-1} + i\gamma_{2j}), \quad c_j^\dagger = \frac{1}{2}(\gamma_{2j-1} - i\gamma_{2j}). \quad (8)$$

For a Majorana mode j , we denote by $\mathfrak{s}(j)$ its corresponding lattice site. Therefore, the distance between two Majorana modes j, k is $d(\mathfrak{s}(j), \mathfrak{s}(k))$.

A. The Hamiltonian

The Hamiltonian we focus on takes the form

$$H = H_0 + V, \quad (9)$$

where

$$H_0 = \sum_{jk} h_{jk} \gamma_j \gamma_k, \quad (10)$$

and V is a general fermionic operator that preserves parity. In other words V only contain terms that are products of an even number of Majorana operators. Moreover, we assume that H_0 and V are $(1, r_0)$ -geometrically local and (U, r_0) -geometrically local, respectively, as defined in Definition 1. For H_0 this means that $h_{jk} = 0$ if $d(\mathfrak{s}(j), \mathfrak{s}(k)) > r_0$.

B. The Lindbladian

We use the Lindbladian $\mathcal{L} : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ constructed in Ref. [10] that was also used in Ref. [27].

$$\begin{aligned} \mathcal{L}(\rho) = & -i[B, \rho] + \sum_j \int_{-\infty}^{\infty} d\omega \eta(\omega) \\ & \times \left(A_j(\omega) \rho A_j(\omega)^\dagger - \frac{1}{2} \{A_j(\omega)^\dagger A_j(\omega), \rho\} \right), \end{aligned} \quad (11)$$

where

$$A_j(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \gamma_j(t) e^{-i\omega t} f(t) dt, \quad (12)$$

and $B = \sum_j B_j$,

$$B_j = \int_{-\infty}^{\infty} dt b_1(t) \int_{-\infty}^{\infty} dt' b_2(t') \gamma_j(\beta(t' - t))^\dagger \gamma_j(-\beta(t + t')). \quad (13)$$

In the above $\gamma_j(t) = e^{iHt} \gamma_j e^{-iHt}$, and

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{\beta} \sqrt{\pi/2}} e^{-(t^2/\beta^2)}, \\ b_1(t) &= 2\sqrt{\pi} e^{1/8} \left(\frac{1}{\cosh(2\pi t)} *_t \sin(-t) e^{-2t^2} \right), \\ b_2(t) &= \frac{1}{2\pi^{3/2}} e^{-4t^2 - 2it}, \quad \eta(\omega) = e^{-((\beta\omega+1)^2/2)}. \end{aligned} \quad (14)$$

Here $*_t$ denotes convolution in the variable t .

This Lindbladian is designed to satisfy the KMS-detailed balance condition (KMS-DBC), which we will introduce below. The KMS-inner product corresponding to a full-rank quantum state σ is defined as

$$\langle X, Y \rangle_{\text{KMS}} = \text{Tr}[\sigma^{1/2} X^\dagger \sigma^{1/2} Y], \quad (15)$$

for any $X, Y \in B(\mathcal{H})$. A superoperator $\mathcal{X}$ satisfies the KMS-DBC if

$$\langle \mathcal{X}(X), Y \rangle_{\text{KMS}} = \langle X, \mathcal{X}(Y) \rangle_{\text{KMS}}, \quad (16)$$

i.e., it is self-adjoint under the KMS-inner product. For the Lindbladian $\mathcal{L}$ in Eq. (11), $\mathcal{L}^\dagger$ is self-adjoint under the KMS-inner product and therefore satisfies the KMS-DBC.

IV. BACKGROUND

In this section we will introduce some tools that are needed in the proof of our main result.

A. Third quantization

Third quantization is a useful method to map a Lindbladian superoperator to an operator in an enlarged Hilbert space [32]. It was originally used to study quasi-free Lindbladians, i.e., those with jump operators that are linear in the Majorana operators, and coherent parts that are quadratic. Here we extend it to more general Lindbladians.

We define a 2^{2n} -dimensional Hilbert space $\mathcal{H}'$ spanned by vectors v_α , where $\alpha \in \{0, 1\}^{2n}$. Define map

$\varphi : B(\mathcal{H}) \rightarrow \mathcal{H}'$,

$$\varphi(\gamma_1^{\alpha_1} \gamma_2^{\alpha_2} \cdots \gamma_{2n}^{\alpha_{2n}}) = v_\alpha. \quad (17)$$

We let v_α form an orthonormal basis by defining the inner product $\langle \cdot, \cdot \rangle$ through

$$\langle v_\alpha, v_{\alpha'} \rangle = \delta_{\alpha\alpha'}. \quad (18)$$

This ensures that

$$\langle \varphi(X), \varphi(Y) \rangle = \frac{1}{2^n} \text{Tr}[X^\dagger Y], \quad (19)$$

for any $X, Y \in B(\mathcal{H})$. Define a new set of creation and annihilation operators through

$$\begin{aligned} \hat{c}_j v_\alpha &= \delta_{1,\alpha_j} \varphi(\gamma_j \gamma_1^{\alpha_1} \gamma_2^{\alpha_2} \cdots \gamma_{2n}^{\alpha_{2n}}), \\ \hat{c}_j^\dagger v_\alpha &= \delta_{0,\alpha_j} \varphi(\gamma_j \gamma_1^{\alpha_1} \gamma_2^{\alpha_2} \cdots \gamma_{2n}^{\alpha_{2n}}). \end{aligned} \quad (20)$$

One can check that they satisfy the canonical anticommutation relation. Following Ref. [32], we call the fermionic particles defined through these creation and annihilation operators a fermions. From these operators we then define the corresponding Majorana operators:

$$\hat{\gamma}_{2j-1} = \hat{c}_j + \hat{c}_j^\dagger, \quad \hat{\gamma}_{2j} = i(\hat{c}_j^\dagger - \hat{c}_j). \quad (21)$$

Remark 1 (a fermions on the lattice Λ). There is an underlying geometry in this new set of fermionic operators. For each site x in the original lattice Λ associated with Majorana operators γ_j and γ_k , we associate $\hat{c}_j$ and $\hat{c}_k$ with the same site. Therefore, each site contains two a -Dirac fermion modes and four a -Majorana modes. We use $\mathfrak{s}'(j)$ to indicate the site in Λ that corresponds to the a -Majorana mode j .

φ induces a representation of the superoperators on $B(\mathcal{H})$, where each superoperator $\mathcal{X}$ is represented by $\varphi \circ \mathcal{X} \circ \varphi^{-1}$, which is an operator on $\mathcal{H}'$. In particular, we consider the left- and right-multiplication with an operator A defined as

$$L_A(X) = AX, \quad R_A(X) = XA. \quad (22)$$

We have the following rules for the their multiplication:

$$L_{AB} = L_A L_B, \quad R_{AB} = R_B R_A. \quad (23)$$

From Ref. [32], and more directly from Eqs. (20) and (21), we have

$$\varphi \circ L_{\gamma_j} \circ \varphi^{-1} = \hat{\gamma}_{2j-1}, \quad \varphi \circ R_{\gamma_j} \circ \varphi^{-1} = -i\hat{\gamma}_{2j}(-1)^{\hat{n}}, \quad (24)$$

where $\hat{n} = \sum_{j=1}^{2n} \hat{c}_j^\dagger \hat{c}_j$ is the a -fermion number operator, which implies that $(-1)^{\hat{n}}$ is the a -fermion parity operator.

From this we can get the corresponding representations of more general operators:

$$\begin{aligned} \varphi \circ L_{\gamma_{j_1} \cdots \gamma_{j_k}} \circ \varphi^{-1} &= \hat{\gamma}_{2j_1-1} \cdots \hat{\gamma}_{2j_k-1}, \\ \varphi \circ R_{\gamma_{j_1} \cdots \gamma_{j_k}} \circ \varphi^{-1} &= -i^k \hat{\gamma}_{2j_k} \cdots \hat{\gamma}_{2j_1} (-1)^{k\hat{n}}. \end{aligned} \quad (25)$$

The locality of the operators obtained through the above mapping depends crucially on their parities, for which we give a precise definition.

Definition 3. We define $\mathcal{H}_{\text{even}}$ and $\mathcal{H}_{\text{odd}}$ to be the 1-eigenspace and (-1) -eigenspace of the parity operator $i^n \gamma_1 \gamma_2 \cdots \gamma_{2n}$, respectively. We say that the states in $\mathcal{H}_{\text{even}}$ have even parity while those in $\mathcal{H}_{\text{odd}}$ have odd parity.

We can similarly define $\mathcal{H}'_{\text{even}}$ and $\mathcal{H}'_{\text{odd}}$ for the enlarged Hilbert space as well. For operators, we can also define the following:

Definition 4. We define $B(\mathcal{H})_{\text{even}}$ to be the subspace of $B(\mathcal{H})$ spanned by $\gamma_1^{\alpha_1} \gamma_2^{\alpha_2} \cdots \gamma_{2n}^{\alpha_{2n}}$ for even $|\alpha|$. Similarly we define $B(\mathcal{H})_{\text{odd}}$ to be the subspace spanned by $\gamma_1^{\alpha_1} \gamma_2^{\alpha_2} \cdots \gamma_{2n}^{\alpha_{2n}}$ for odd $|\alpha|$.

Remark 2. These two notions of parity are related, as $v \in \mathcal{H}'$ has even and odd parity if and only if $\varphi^{-1}(v) \in B(\mathcal{H})$ has even and odd parity. One can see this from the mapping φ defined in Eq. (17) and by observing that $\mathcal{H}'_{\text{even}}$ and $\mathcal{H}'_{\text{odd}}$ are spanned by v_α for even $|\alpha|$ and odd $|\alpha|$, respectively.

We will focus on superoperators that are parity preserving, as defined below.

Definition 5. We say $\mathcal{X} \in B(B(\mathcal{H}))$ is parity-preserving if $\mathcal{X}(B(\mathcal{H})_{\text{even}}) \subset B(\mathcal{H})_{\text{even}}$ and $\mathcal{X}(B(\mathcal{H})_{\text{odd}}) \subset B(\mathcal{H})_{\text{odd}}$.

We then use third quantization to build a map that preserves locality and also preserves the spectrum for the even sector.

Definition 6. We define a linear map $\Phi : B(B(\mathcal{H})) \rightarrow B(\mathcal{H}')$ such that, for any $W, W' \in B(\mathcal{H})$,

$$\begin{aligned} \Phi(L_W) &= \varphi \circ L_W \circ \varphi^{-1}, \\ \Phi(R_W) &= \begin{cases} \varphi \circ R_W \circ \varphi^{-1} & \text{if } W \text{ is even,} \\ (\varphi \circ R_W \circ \varphi^{-1})(-1)^{\hat{n}} & \text{if } W \text{ is odd,} \end{cases} \\ \Phi(L_W R_{W'}) &= \Phi(L_W) \Phi(R_{W'}). \end{aligned}$$

Remark 3. Because $L_W R_{W'}$ with even and odd W and even and odd W' span disjoint subspaces of $B(B(\mathcal{H}))$, the above map is well defined. Because $L_W R_{W'}$ for W and W' with even and odd parities span the entire $B(B(\mathcal{H}))$, the

above definition uniquely defines the map $\Phi(\mathcal{X})$ for all $\mathcal{X} \in B(B(\mathcal{H}))$.

Remark 4. Φ defined in Definition 6 strictly preserves locality. We can see this by computing

$$\begin{aligned}\Phi(L_{\gamma_{j_1}\gamma_{j_2}\dots\gamma_{j_k}}) &= \hat{\gamma}_{2j_1}\hat{\gamma}_{2j_2}\dots\hat{\gamma}_{2j_k}, \\ \Phi(R_{\gamma_{j_1}\gamma_{j_2}\dots\gamma_{j_k}}) &= -i^k \hat{\gamma}_{2j_k}\hat{\gamma}_{2j_{k-1}}\dots\hat{\gamma}_{2j_1}.\end{aligned}$$

Note that Φ is not a homomorphism between algebras because even though L_W and R_W commute with each other, $\Phi(L_W)$ and $\Phi(R_W)$ do not necessarily commute with each other. However, it still preserves part of the spectrum, as stated in the following lemma:

Lemma 1. Φ satisfies

$$\Phi(\mathcal{X})|_{\mathcal{H}'_{\text{even}}} = \varphi \circ \mathcal{X} \circ \varphi^{-1}|_{\mathcal{H}'_{\text{even}}}.$$

Moreover, when $\mathcal{X}$ is parity preserving, $\Phi(\mathcal{X})|_{\mathcal{H}'_{\text{even}}}$ and $\mathcal{X}|_{B(\mathcal{H})_{\text{even}}}$ share the same spectrum.

Proof. Note that for $v \in \mathcal{H}'$ with even parity, $(-1)^{\hat{n}}v = v$. Therefore $\Phi(L_W R_{W'})v = \Phi(L_W)\Phi(R_{W'})v = (\varphi \circ L_W R_{W'} \circ \varphi^{-1})v$. From this the equality follows by linearity. Because φ preserves parity as discussed in Remark 2, we can then see that $\Phi(\mathcal{X})|_{\mathcal{H}'_{\text{even}}}$ and $\mathcal{X}|_{B(\mathcal{H})_{\text{even}}}$ must share the same spectrum if $\mathcal{X}$ preserves parity. ■

B. Stability of free-fermion Hamiltonians

We will introduce the stability result for the gap of free-fermion Hamiltonians proved in Ref. [31]. It relies crucially on notions of locality, which we will discuss below.

Definition 7 (Quasilocality around x). Let $x \in \Lambda$ be a lattice site. We say an operator W is (C, μ) -quasi-local around x if $\|W\| \leq C$, and for every integer $r \geq 1$, there exists W_r supported on the ball of radius r centered at x such that

$$\|W - W_r\| \leq Ce^{-\mu r},$$

for some constants $C, \mu > 0$.

We adopt the same definition in Ref. [31] of the decay property of an operator, which characterize the quasilocal-ity of an extensive operator in the following way.

Definition 8 (Decay of operator). We say an operator W has (C, μ) -decay if W can be decomposed as

$$W = \sum_{r \geq 1} \sum_{B \in \mathcal{B}(r)} W_{r,B},$$

where $\mathcal{B}(r)$ denotes the set of balls of radius r and $W_{r,B}$ is an operator supported on a ball B and where

$$\max_{B \in \mathcal{B}(r)} \|W_{r,B}\| \leq Ce^{-\mu r},$$

for some constants $C, \mu > 0$.

Naturally, an operator that is the sum of quasilocal terms has nice decay property:

Lemma 2. For any $C, \mu > 0$, if an operator W can be written as $W = \sum_{x \in \Lambda} W_x$ where each W_x is (C, μ) -quasi-local around x , then W has $(2Ce^\mu, \mu)$ decay.

Proof. Since $W = \sum_{x \in \Lambda} W_x$, and W_x is (C, μ) -quasi-local around x , we have

$$W = \sum_{r \geq 0} \sum_{x \in \Lambda} (W_{x,r} - W_{x,r-1}),$$

where $\|W_x - W_{x,r}\| \leq Ce^{-\mu r}$, and we let $W_{x,-1} = 0$. Note that $W_{x,r} - W_{x,r-1}$ is supported on the ball with radius r centered at x , and $\|W_{x,r} - W_{x,r-1}\| \leq 2Ce^{-\mu(r-1)}$ by the triangle inequality. This tells us that W has $(2Ce^\mu, \mu)$ decay. ■

For quadratic operators Ref. [31] defined a somewhat different notion of decay, which we state here.

Definition 9 (Decay of coefficient matrix). For a matrix $B = (B_{jk})$ where each j and k is associated with a lattice site $x \in \Lambda$, and no more than n_0 j or k is associated with the same site, then we say B has $[K, \nu]$ -decay if

$$\|B_{xy}\| \leq Ke^{-\nu d(x,y)},$$

where $x, y \in \Lambda$, and B_{xy} is the submatrix of B whose rows and columns are associated with x and y , respectively. Here n_0 is a constant that is independent of $|\Lambda|$.

We remark that in this work we only need to have $n_0 \leq 4$. With the above notions of locality, we will state the main result in Ref. [31]:

Theorem 2 (Corollary 1 of Ref. [31]). Let $H_{FF} = \sum_{jk} A_{jk} \gamma_j \gamma_k$ be a Hamiltonian on lattice Λ with spectral gap Δ . We assume that $A = (A_{jk})$ has $[K, \nu]$ decay. Let $H = H_{FF} + V$, where we assume V has (C, μ) decay. Then there exists constants $C_0, c_1 > 0$ depending only on

K, ν, μ, Δ, D such that when $C \leq C_0$, the spectral gap of H is at least $\Delta - c_1 C$.

Note that in this work we focus on the spectral gap between the top eigenvalue and the second largest eigenvalue as explained in Ref. [30], while [31] was about the gap between the ground state and the first excited state. These two are however equivalent as one can simply consider $-H$ rather than H .

C. The Lieb-Robinson bound

The Lieb-Robinson bound was used extensively in Refs. [27,31] and is also needed in our proof. We state the version used in Ref. [37, Lemma 5], which follows Refs. [34,45–48], with modification:

Lemma 3. Let H be a fermionic Hamiltonian on Λ with even parity that is (ξ, r_0) -geometrically local (Definition 1). For an operator O_X supported on $X \subset \Lambda$, and $\Omega \subset X$, we let

$$\ell = \lfloor \min\{d(x, y) : x \in X, y \in \Lambda \setminus \Omega\} \rfloor.$$

Then

$$\|e^{iHt} O_X e^{-iHt} - e^{iH't} O_X e^{-iH't}\| \leq |X| \|O_X\| \frac{(2J|t|)^\ell}{\ell!},$$

where H' consists of the terms in H that are supported within Ω , $J \leq C\xi r_0^{2D}$ for some constant $C > 0$ that depends only on D .

We remark that we adapt Ref. [37, Lemma 5] to the fermionic setting, which, as commented in Ref. [37, Sec. 4], works without needing any modification. As pointed out in Ref. [37, Appendix C.6] the above lemma works for all Hamiltonians with bounded-range interactions. J is sometimes known as the Lieb-Robinson velocity. When H is (ξ, r_0) -geometrically local, then by Definition 1 we can write it as $H = \sum_{B \in \mathcal{B}(r_0)} H_B$ with $\|H_B\| \leq \xi$. Then

$$J = \max_{x \in \Lambda} \sum_{B \ni x} |B| \|H_B\| \leq \xi \times C_1 r_0^D \times C_2 r_0^D = C\xi r_0^{2D}. \quad (26)$$

We have used the fact that each term involves at most $|B| \leq C_1 r_0^D$ sites and each site is involved in at most $C_2 r_0^D$ terms, for constants $C_1, C_2 > 0$ that depend only on D . We will also use the Lieb-Robinson velocity for the noninteracting

part H_0 , for which we can get a tighter bound:

$$J_0 = \max_{x \in \Lambda} \sum_{B \ni x} |B| \|(H_0)_B\| \leq 1 \times 2 \times C'_2 r_0^D = C'\xi r_0^D. \quad (27)$$

In the above because H_0 only contains terms that act on two Majorana modes, B for which the corresponding term $(H_0)_B$ is nonzero can have at most $|B| = 2$, and $\|(H_0)_B\| \leq 1$.

V. CONSTRUCTING THE PARENT HAMILTONIAN

Even though third quantization provides a way to map the Lindbladian to an operator, we still do not have a Hamiltonian because the resulting operator may not be Hermitian. In this section we will use the stationary state σ to construct a Hermitian parent Hamiltonian. We define superoperator $\Gamma \in B(B(\mathcal{H}))$ as follows:

$$\Gamma(X) = \sigma^{1/4} X \sigma^{1/4}. \quad (28)$$

We then define the map

$$\tilde{\varphi} = \varphi \circ \Gamma. \quad (29)$$

This map is an isometry, i.e., it satisfies

$$\begin{aligned} \langle \tilde{\varphi}(X), \tilde{\varphi}(Y) \rangle &= \frac{1}{2^n} \text{Tr}[\Gamma(X)^\dagger \Gamma(Y)] \\ &= \frac{1}{2^n} \text{Tr}[\sigma^{1/4} X^\dagger \sigma^{1/2} Y \sigma^{1/4}] \\ &= \frac{1}{2^n} \langle X, Y \rangle_{\text{KMS}}, \end{aligned} \quad (30)$$

where the first equality comes from Eq. (19). We then define, for any $\mathcal{X} \in B(B(\mathcal{H}))$,

$$\tilde{\Phi}(\mathcal{X}) = \Phi(\Gamma \circ \mathcal{X} \circ \Gamma^{-1}). \quad (31)$$

By the definition of Φ in Definition 6, we have

$$\tilde{\Phi}(\mathcal{X})|_{\mathcal{H}'_{\text{even}}} = \tilde{\varphi} \circ \mathcal{X} \circ \tilde{\varphi}^{-1}|_{\mathcal{H}'_{\text{even}}}. \quad (32)$$

This immediately implies the following:

Lemma 4. Let $\mathcal{X} \in B(B(\mathcal{H}))$ be a superoperator that preserves parity according to Definition 5, then $\tilde{\Phi}(\mathcal{X})|_{\mathcal{H}'_{\text{even}}}$ and $\mathcal{X}|_{B(\mathcal{H})_{\text{even}}}$ share the same spectrum.

In particular for left and right multiplication we have

$$\tilde{\Phi}(L_A) = \Phi(L_{\sigma^{1/4} A \sigma^{-1/4}}), \quad \tilde{\Phi}(R_A) = \Phi(R_{\sigma^{-1/4} A \sigma^{1/4}}). \quad (33)$$

We are now ready to define the parent Hamiltonian through

$$H^{\text{parent}} = \tilde{\Phi}(\mathcal{L}^\dagger), \quad (34)$$

where $\mathcal{L}$ is the Lindbladian defined in Eq. (11) that satisfies the KMS-DBC as defined in Eq. (16).

Lemma 5. The parent Hamiltonian H^{parent} has the following properties:

- (i) H^{parent} is self-adjoint in $\mathcal{H}'$.
- (ii) 0 is an eigenvalue of both H^{parent} and $\mathcal{L}^\dagger|_{B(\mathcal{H})_{\text{even}}}$. Let Δ and Δ' be the gap between 0 and the largest nonpositive eigenvalue for H^{parent} and $\mathcal{L}^\dagger|_{B(\mathcal{H})_{\text{even}}}$, respectively. Then $\Delta \leq \Delta'$.

Proof. For (i), note that by Eq. (32) and the fact that $\mathcal{L}$ satisfies KMS-DBC [as defined in Eq. (16)], we can see that $H^{\text{parent}}|_{\mathcal{H}'_{\text{even}}}$ is self-adjoint. This implies

$$H^{\text{parent}}|_{\mathcal{H}'_{\text{even}}} = (H^{\text{parent}})^\dagger|_{\mathcal{H}'_{\text{even}}}. \quad (35)$$

We consider enlarging the Hilbert space $\mathcal{H}$ by one Dirac mode, but not adding anything to the Hamiltonian H . Then going through the same mapping from H to $\mathcal{L}$ [through Eq. (11)] and then $\tilde{\Phi}(\mathcal{L}^\dagger)$, we obtain a new parent Hamiltonian $\bar{H}^{\text{parent}}$ in an enlarged Hilbert space $\bar{\mathcal{H}}'$ involving $2n+2$ Dirac modes. By Eq. (25) and the fact that σ does not involve the modes $2n+1$ and $2n+2$, we can see that $\bar{\mathcal{H}}'$ only acts on the Dirac modes 1 to $2n$, and is of exactly the same form as H^{parent} when written as a multivariate polynomial of Majorana operators, even though they act on different Hilbert spaces. Moreover, by the same argument as for H^{parent} we have

$$\bar{H}^{\text{parent}}|_{\bar{\mathcal{H}}'_{\text{even}}} = (\bar{H}^{\text{parent}})^\dagger|_{\bar{\mathcal{H}}'_{\text{even}}}. \quad (36)$$

We revert back to the creation and annihilation operators in Eq. (21), where $\hat{c}_j = (\hat{\gamma}_{2j-1} + i\hat{\gamma}_{2j})/2$. This also gives us a vacuum state $|\text{vac}\rangle$ for $\mathcal{H}'$ that is in the kernel of every $\hat{c}_j$ for $j = 1, 2, \dots, 2n$. For $\bar{\mathcal{H}}'$, we can likewise get a vacuum state $|\bar{\text{vac}}\rangle$ that is also in the kernel of $\hat{c}_{2n+1}$ and $\hat{c}_{2n+2}$. For any state

$$v = \sum_{\alpha \in \{0,1\}^{2n}} v_\alpha (\hat{c}_1^\dagger)^{\alpha_1} \dots (\hat{c}_{2n}^\dagger)^{\alpha_{2n}} |\text{vac}\rangle \in \mathcal{H}',$$

we define

$$\bar{v} = \sum_{\alpha \in \{0,1\}^{2n}} v_\alpha (\hat{c}_1^\dagger)^{\alpha_1} \dots (\hat{c}_{2n}^\dagger)^{\alpha_{2n}} |\bar{\text{vac}}\rangle \in \bar{\mathcal{H}}'.$$

Because H^{parent} and $\bar{H}^{\text{parent}}$ are represented by the same polynomial of Majorana operators, we have

$$\langle v, H^{\text{parent}} w \rangle = \langle \bar{v}, \bar{H}^{\text{parent}} \bar{w} \rangle, \quad (37)$$

for any $v, w \in \mathcal{H}'$.

Then we consider $v, w \in \mathcal{H}'_{\text{odd}}$. Note that $\hat{c}_{2n+1}^\dagger \bar{v}, \hat{c}_{2n+1}^\dagger \bar{w} \in \bar{\mathcal{H}}'_{\text{even}}$. We then have

$$\begin{aligned} \langle v, H^{\text{parent}} w \rangle &= \langle \bar{v}, \bar{H}^{\text{parent}} \bar{w} \rangle = \langle \hat{c}_{2n+1}^\dagger \bar{v}, \bar{H}^{\text{parent}} \hat{c}_{2n+1}^\dagger \bar{w} \rangle \\ &= \langle \hat{c}_{2n+1}^\dagger \bar{v}, (\bar{H}^{\text{parent}})^\dagger \hat{c}_{2n+1}^\dagger \bar{w} \rangle = \langle \bar{v}, (\bar{H}^{\text{parent}})^\dagger \bar{w} \rangle \\ &= \langle v, (H^{\text{parent}})^\dagger w \rangle, \end{aligned}$$

where in the third equality we have used Eq. (37), and in the second and fourth equalities we have used the fact that $\bar{H}^{\text{parent}}$ and $\bar{v}, \bar{w}$ do not involve any operator on Dirac mode $2n+1$. This then implies

$$H^{\text{parent}}|_{\mathcal{H}'_{\text{odd}}} = (H^{\text{parent}})^\dagger|_{\mathcal{H}'_{\text{odd}}}.$$

Because $\mathcal{H}'_{\text{even}}$ and $\mathcal{H}'_{\text{odd}}$ are both invariant subspaces of H^{parent} , the above equation combined with Eq. (35) yields $H^{\text{parent}} = (H^{\text{parent}})^\dagger$.

For (ii), let Δ'' be the gap between 0 and the largest nonpositive eigenvalue of $H^{\text{parent}}|_{\mathcal{H}'_{\text{even}}}$. Since the spectrum of $H^{\text{parent}}|_{\mathcal{H}'_{\text{even}}}$ is a subset of the spectrum of H^{parent} , and 0 is an eigenvalue of both of them [corresponding to eigenvector $v = \varphi(\sigma^{1/2}) \in \mathcal{H}'_{\text{even}}$], we have $\Delta \leq \Delta''$. By Lemma 4 we have $\Delta'' = \Delta'$. Therefore, $\Delta' \geq \Delta$. ■

The spectral gap of $\mathcal{L}^\dagger|_{B(\mathcal{H})_{\text{even}}}$ is sufficient for establishing the convergence towards stationary state.

Lemma 6. We assume that $\mathcal{L}^\dagger$ satisfies the KMS-DBC (16), the spectral gap between 0 and the second largest eigenvalue of $\mathcal{L}^\dagger|_{B(\mathcal{H})_{\text{even}}}$ is $g > 0$, and $\ker(\mathcal{L}^\dagger|_{B(\mathcal{H})_{\text{even}}}) = \text{span}(I)$. Let $Y \in B(\mathcal{H})$ be self-adjoint with even parity (Definition 4) satisfying $\text{Tr}[\sigma Y] = 0$, then, denoting $Y(t) = e^{t\mathcal{L}^\dagger}(Y)$,

$$\|Y(t)\|_{\text{KMS}} \leq e^{-gt} \|Y\|_{\text{KMS}},$$

where for any $X \in B(\mathcal{H})$, $\|X\|_{\text{KMS}} = \langle X, X \rangle_{\text{KMS}}^{1/2}$.

Proof. Because $\mathcal{L}^\dagger$ satisfies the KMS-DBC (16), and that $\ker(\mathcal{L}^\dagger|_{B(\mathcal{H})_{\text{even}}}) = \text{span}(I)$, for the spectral gap we

have the variational form

$$-g = \sup_{\substack{X \in B(\mathcal{H})_{\text{even}} \\ \langle X, I \rangle_{\text{KMS}} = 0}} \frac{\langle X, \mathcal{L}^\dagger|_{B(\mathcal{H})_{\text{even}}} X \rangle_{\text{KMS}}}{\langle X, X \rangle_{\text{KMS}}}.$$

Therefore, when $\langle X, I \rangle_{\text{KMS}} = 0$, we have

$$\langle X, \mathcal{L}^\dagger|_{B(\mathcal{H})_{\text{even}}} X \rangle_{\text{KMS}} \leq -g \langle X, X \rangle_{\text{KMS}}.$$

The inequality we want to prove follows immediately from

$$\begin{aligned} \frac{d}{dt} \langle Y(t), Y(t) \rangle_{\text{KMS}} &= 2 \langle Y(t), \mathcal{L}^\dagger Y(t) \rangle_{\text{KMS}} \\ &= 2 \langle Y(t), \mathcal{L}^\dagger|_{B(\mathcal{H})_{\text{even}}} Y(t) \rangle_{\text{KMS}} \\ &\leq -2g \langle Y(t), Y(t) \rangle_{\text{KMS}}, \end{aligned}$$

where we have used the fact that $\langle Y(t), I \rangle_{\text{KMS}} = \text{Tr}[\sigma Y(t)] = \text{Tr}[\sigma Y] = 0$ for all t . ■

Corollary 5. We assume that $\mathcal{L}^\dagger$ satisfies the KMS-DBC (16), the spectral gap between 0 and the second largest eigenvalue of $\mathcal{L}^\dagger|_{B(\mathcal{H})_{\text{even}}}$ is $g > 0$, and $\ker(\mathcal{L}^\dagger|_{B(\mathcal{H})_{\text{even}}}) = \text{span}(I)$. Let $X \in B(\mathcal{H})$ be self-adjoint with even parity (Definition 4). For an initial state $\rho \in B(\mathcal{H})_{\text{even}}$ (i.e., satisfying the fermion superselection rule), we have

$$|\text{Tr}[X(t)\rho] - \text{Tr}[X\sigma]| \leq \frac{e^{-gt}}{\sqrt{\sigma_{\min}}} \|X\|,$$

where $\sigma_{\min}$ is the minimum eigenvalue of σ .

Proof. We let $Y = X - I \langle I, X \rangle_{\text{KMS}}$. Then $\text{Tr}[Y\sigma] = 0$. By Lemma 6 we have

$$\|X(t) - I \langle I, X \rangle_{\text{KMS}}\| = \|Y(t)\|_{\text{KMS}} \leq e^{-gt} \|X\|,$$

where we have used $\|Y\|_{\text{KMS}} \leq \|X\|_{\text{KMS}} \leq \|X\|$. Note that

$$\begin{aligned} \text{Tr}[X(t)\rho] - \text{Tr}[X\sigma] &= \langle X(t) - I \langle I, X \rangle_{\text{KMS}}, \sigma^{-1/2} \rho \sigma^{-1/2} \rangle_{\text{KMS}}, \end{aligned}$$

by the Cauchy-Schwarz inequality we have

$$|\text{Tr}[X(t)\rho] - \text{Tr}[X\sigma]| \leq \|\sigma^{-1/2} \rho \sigma^{-1/2}\|_{\text{KMS}} e^{-gt} \|X\|.$$

The result follows by

$$\begin{aligned} \|\sigma^{-1/2} \rho \sigma^{-1/2}\|_{\text{KMS}}^2 &= \text{Tr}[\rho \sigma^{-1/2} \rho \sigma^{-1/2}] \\ &\leq \|\sigma^{-1/2} \rho \sigma^{-1/2}\| \text{Tr}[\rho] \leq \frac{1}{\sigma_{\min}}. \end{aligned}$$

■

Corollary 6. We assume that $\mathcal{L}^\dagger$ satisfies the KMS-DBC (16), the spectral gap between 0 and the second largest eigenvalue of $\mathcal{L}^\dagger|_{B(\mathcal{H})_{\text{even}}}$ is $g > 0$, and $\ker(\mathcal{L}^\dagger|_{B(\mathcal{H})_{\text{even}}}) = \text{span}(I)$. For an initial state $\rho \in B(\mathcal{H})_{\text{even}}$ (i.e., satisfying the fermion superselection rule), we have

$$\|\rho(t) - \sigma\|_1 \leq \frac{e^{-gt}}{\sqrt{\sigma_{\min}}},$$

where $\rho(t) = e^{t\mathcal{L}}(\rho)$, $\sigma_{\min}$ is the minimum eigenvalue of σ , and $\|\cdot\|_1$ denotes the Schatten-1 norm.

Proof. For any observable $X \in B(\mathcal{H})$, we can decompose $X = X_{\text{even}} + X_{\text{odd}}$ where X_{even} and X_{odd} have even and odd parities as defined in Definition 4. We can readily see that $\text{Tr}[\rho(t)X] = \text{Tr}[\rho(t)X_{\text{even}}]$ because $\rho \in B(\mathcal{H})_{\text{even}}$, and the same is true for σ . Also, by the eigenvalue interlacing theorem, $\|X_{\text{even}}\| \leq \|X\|$. Therefore, by Corollary 5 we have

$$|\text{Tr}[(\rho(t) - \sigma)X]| \leq \frac{e^{-gt}}{\sqrt{\sigma_{\min}}} \|X_{\text{even}}\| \leq \frac{e^{-gt}}{\sqrt{\sigma_{\min}}} \|X\|.$$

Since this is true for any $X \in B(\mathcal{H})$, by the duality between the Schatten 1-norm and the Schatten ∞ -norm we have the desired inequality. ■

VI. STRUCTURE OF THE PARENT HAMILTONIAN

We can write $\mathcal{L}^\dagger$ in the superoperator form

$$\begin{aligned} \mathcal{L}^\dagger &= i \sum_j (L_{B_j} - R_{B_j}) + \sum_j \int_{-\infty}^{\infty} d\omega \eta(\omega) \\ &\quad \times \left(L_{A_j(\omega)^\dagger} R_{A_j(\omega)} - \frac{1}{2} L_{A_j(\omega)^\dagger} A_j(\omega) - \frac{1}{2} R_{A_j(\omega)^\dagger} A_j(\omega) \right). \end{aligned} \quad (38)$$

This gives us a corresponding expression for the parent Hamiltonian through Eq. (34):

$$\begin{aligned} H^{\text{parent}} &= \Phi \left(i \sum_j (L_{\tilde{B}_j} - R_{\tilde{B}_j}^\dagger) \right. \\ &\quad + \sum_j \int d\omega \eta(\omega) \left(L_{\tilde{A}_j(-\omega)} R_{\tilde{A}_j(-\omega)^\dagger} \right. \\ &\quad \left. \left. - \frac{1}{2} L_{\tilde{A}_j(-\omega)} \tilde{A}_j(\omega) - \frac{1}{2} R_{\tilde{A}_j(\omega)^\dagger} \tilde{A}_j(-\omega)^\dagger \right) \right), \end{aligned} \quad (39)$$

where

$$\tilde{A}_j(\omega) = \sigma^{1/4} A_j(\omega) \sigma^{-1/4}, \quad \tilde{B}_j = \sigma^{1/4} B_j \sigma^{-1/4}, \quad (40)$$

and we have used the fact that $A_j(\omega)^\dagger = A_j(-\omega)$, which is a consequence of the hermiticity of γ_j .

We then separate the quadratic part from the rest. To do this we first define

$$\gamma_j^{\text{free}}(t) = e^{iH_0 t} \gamma_j e^{-iH_0 t}. \quad (41)$$

Then corresponding to $\tilde{A}_j(\omega)$ we define

$$\tilde{A}_j^{\text{free}}(\omega) = e^{-\beta H_0/4} \left(\frac{1}{\sqrt{2\pi}} \int \gamma_j^{\text{free}}(t) f(t) e^{-i\omega t} dt \right) e^{\beta H_0/4}. \quad (42)$$

In other words $\tilde{A}_j^{\text{free}}(\omega)$ is the jump operator we would have if we use H_0 as the Hamiltonian instead of H . Similarly, we define

$$\begin{aligned} \tilde{B}_j^{\text{free}} &= e^{-\beta H_0/4} \int_{-\infty}^{\infty} dt b_1(t) \int_{-\infty}^{\infty} dt' b_2(t') \\ &\times \gamma_j^{\text{free}}(\beta(t' - t)) \gamma_j^{\text{free}}(-\beta(t + t')) e^{\beta H_0/4}. \end{aligned} \quad (43)$$

We now have the quadratic parts of both $\tilde{A}_j(\omega)$ and $\tilde{B}_j$, and this allows us to define the interacting parts:

$$\tilde{A}_j^{\text{int}}(\omega) = \tilde{A}_j(\omega) - \tilde{A}_j^{\text{free}}(\omega), \quad \tilde{B}_j^{\text{int}} = \tilde{B}_j - \tilde{B}_j^{\text{free}}. \quad (44)$$

We will decompose the parent Hamiltonian into 2 parts, the coherent part and the dissipative part:

$$H^{\text{parent}} = H_C^{\text{parent}} + H_D^{\text{parent}}, \quad (45)$$

where

$$H_C^{\text{parent}} = \Phi \left(i \sum_j \left(L_{\tilde{B}_j} - R_{\tilde{B}_j^\dagger} \right) \right), \quad (46)$$

and

$$\begin{aligned} H_D^{\text{parent}} &= \Phi \left(\sum_j \int d\omega \eta(\omega) \left(L_{\tilde{A}_j(-\omega)} R_{\tilde{A}_j(-\omega)^\dagger} \right. \right. \\ &\quad \left. \left. - \frac{1}{2} L_{\tilde{A}_j(-\omega)} \tilde{A}_j(\omega) - \frac{1}{2} R_{\tilde{A}_j(\omega)^\dagger} \tilde{A}_j(-\omega)^\dagger \right) \right). \end{aligned} \quad (47)$$

Each part is in turn decomposed into two parts, the free part and the interacting part:

$$\begin{aligned} H_C^{\text{parent}} &= H_{C,\text{free}}^{\text{parent}} + H_{C,\text{int}}^{\text{parent}}, \\ H_D^{\text{parent}} &= H_{D,\text{free}}^{\text{parent}} + H_{D,\text{int}}^{\text{parent}}, \end{aligned} \quad (48)$$

where

$$H_{C,\text{free}}^{\text{parent}} = \Phi \left(i \sum_j (L_{\tilde{B}_j^{\text{free}}} - R_{(\tilde{B}_j^{\text{free}})^\dagger}) \right), \quad (49)$$

$$H_{C,\text{int}}^{\text{parent}} = H_C^{\text{parent}} - H_{C,\text{free}}^{\text{parent}}, \quad (50)$$

$$\begin{aligned} H_{D,\text{free}}^{\text{parent}} &= \Phi \left(\sum_j \int d\omega \eta(\omega) \left(L_{\tilde{A}_j^{\text{free}}(-\omega)} R_{\tilde{A}_j^{\text{free}}(-\omega)^\dagger} \right. \right. \\ &\quad \left. \left. - \frac{1}{2} L_{\tilde{A}_j^{\text{free}}(-\omega)} \tilde{A}_j^{\text{free}}(\omega) - \frac{1}{2} R_{\tilde{A}_j^{\text{free}}(\omega)^\dagger} \tilde{A}_j^{\text{free}}(-\omega)^\dagger \right) \right), \end{aligned} \quad (51)$$

and

$$H_{D,\text{int}}^{\text{parent}} = H_D^{\text{parent}} - H_{D,\text{free}}^{\text{parent}}. \quad (52)$$

We then reassemble the parts to get

$$H_0^{\text{parent}} = H_{C,\text{free}}^{\text{parent}} + H_{D,\text{free}}^{\text{parent}}, \quad V^{\text{parent}} = H_{C,\text{int}}^{\text{parent}} + H_{D,\text{int}}^{\text{parent}}. \quad (53)$$

We can that H_0^{parent} is quadratic. Both H_0^{parent} and H^{parent} are Hermitian by Lemma 5, which implies that V^{parent} is also Hermitian. For H_0^{parent} we also have the following explicit expression:

$$\begin{aligned} H_{\text{free}}^{\text{parent}} &= \Phi \left(i \sum_j (L_{\tilde{B}_j^{\text{free}}} - R_{(\tilde{B}_j^{\text{free}})^\dagger}) \right. \\ &\quad \left. + \sum_j \int d\omega \eta(\omega) \left(L_{\tilde{A}_j^{\text{free}}(-\omega)} R_{\tilde{A}_j^{\text{free}}(-\omega)^\dagger} \right. \right. \\ &\quad \left. \left. - \frac{1}{2} L_{\tilde{A}_j^{\text{free}}(-\omega)} \tilde{A}_j^{\text{free}}(\omega) - \frac{1}{2} R_{\tilde{A}_j^{\text{free}}(\omega)^\dagger} \tilde{A}_j^{\text{free}}(-\omega)^\dagger \right) \right). \end{aligned} \quad (54)$$

A. Quasilocality and the spectral gap

We will next prove the following: First, the noninteracting part is gapped.

Proposition 1. For H_0^{parent} defined in Eq. (53) corresponding to a $(1, r_0)$ -geometrically local H_0 , 0 is its nondegenerate top eigenvalue, and is separated from the rest of its spectrum by a constant gap $\Delta > 0$ that depends only on β, r_0, D .

A proof is given at the end of Appendix A. The noninteracting part also has nice locality.

Proposition 2. For $H_0^{\text{parent}} = \sum_{j,k=1}^{4n} m_{jk} \hat{\gamma}_j \hat{\gamma}_k$ defined in Eq. (53) corresponding to a $(1, r_0)$ -geometrically local H_0 , its coefficient matrix (m_{jk}) has $[K, \nu]$ decay as defined in Definition 9. Here $K, \mu > 0$ are constants that depend only on β, r_0, D .

A proof is given at the end of Sec. B 2. Next, we focus on the interacting part V^{parent} , and show that it can be treated as quasilocal perturbation.

Proposition 3. For $\mathcal{V}^{\text{parent}}$ defined in Eq. (53) corresponding to a $(1, r_0)$ -geometrically local H_0 and (U, r_0) -geometrically local V for $U < U_0$, it has (CU, μ) -decay as defined in Definition 8. Here $C, \mu > 0$ are constants that depend only on β, r_0, D, U_0 .

A proof is provided at the end of Appendix C. Combining these three propositions, we then have the following result about the spectrum of the parent Hamiltonian

Theorem 3. For any $\beta, r_0 > 0$, and integer $D > 0$, there exists $U_\beta > 0$ that depends only on β, r_0, D such that when H_0 and V in Eq. (1) are $(1, r_0)$ -geometrically local and (U, r_0) -geometrically local, respectively, for $U < U_\beta$, then the parent Hamiltonian H^{parent} given in Eq. (34) has 0 as its nondegenerate top eigenvalue, and the spectral gap separating 0 from the rest of its spectrum is lower bounded by a constant $\Delta > 0$ that depends only on $\beta, r_0, D > 0$.

Proof. Instead of directly considering $H = H_0 + V$ as in Eq. (1), we consider $H(u) = H_0 + uV$, $u \in [0, 1]$. We then write H^{parent} and $\mathcal{V}^{\text{parent}}$ in Eq. (53) as functions of u .

First, Lemma 5 ensures that H_0^{parent} in Eq. (53) and $\mathcal{V}^{\text{parent}}(u)$ are both Hermitian. By Proposition 1, we know that the spectral gap of H_0^{parent} is lower bounded by a constant Δ_0 . By Proposition 2, we know that the coefficient matrix of H_0^{parent} as a quadratic fermionic Hamiltonian has $[K, \nu]$ decay. By Proposition 3, setting $U_0 = 1$, we know that $\mathcal{V}^{\text{parent}}(u)$ has (CUu, μ) decay for $U < 1$. Combining all these results, by Theorem 2, we can then see that there exists $U'_\beta > 0$, such that when $U \leq U'_\beta$, the spectral gap between the top eigenvalue and the second largest eigenvalue of $H^{\text{parent}}(u) = H_0^{\text{parent}} + \mathcal{V}^{\text{parent}}(u)$ is lower bounded by $\Delta_0 - c_1 CUu$, for any $u \in [0, 1]$.

We will next assume $U < \min\{U'_\beta, \Delta_0/(c_1 C)\}$, and show that the top eigenvalue of $H^{\text{parent}}(u)$ is 0 and nondegenerate for $u \in [0, 1]$. First we observe that 0 is always an eigenvalue of $H^{\text{parent}}(u)$ for $u \in [0, 1]$ [corresponding to $\varphi(\sigma(u)^{1/2})$]. If it becomes degenerate, then because the eigenvalues can only change continuously as a function of u by Weyl's inequality, there must exist $u_\epsilon \in [0, 1]$ such that the spectral gap is smaller than ϵ for any $\epsilon > 0$, which contradicts the gap lower bound $\Delta_0 - c_1 CUu \geq \Delta_0 - c_1 CU > 0$. This shows that 0 must be a nondegenerate eigenvalue. If there is another eigenvalue of $H^{\text{parent}}(u)$ that is positive for any $u \in [0, 1]$, then because 0 is the nondegenerate top eigenvalue of $H^{\text{parent}}(0) = H_0^{\text{parent}}$ by Proposition 1, there must exist $u^* \in [0, 1]$ such that 0 is a degenerate eigenvalue of $H^{\text{parent}}(u^*)$, which contradicts the nondegeneracy of 0 that we just proved. Therefore, 0 is the nondegenerate top eigenvalue of $H^{\text{parent}}(u)$ for all $u \in [0, 1]$.

Letting $u = 1$, $U_\beta = 0.99 \min\{U'_\beta, \Delta_0/(c_1 C)\}$, we then conclude that when $U \leq U_\beta$, the spectral gap of H^{parent} defined in Eq. (34) is lower bounded by

$\Delta = \Delta_0 - c_1 CU > 0$, and that 0 is the nondegenerate top eigenvalue of H^{parent} . In the above $C, K, U'_\beta, \nu, \mu, c_1, \Delta_0 > 0$ are all constants that depend only on β, D, r_0 . This implies that $U_\beta, \Delta > 0$ are also constants that only depend on β, D, r_0 . ■

B. Fast mixing

The mixing time characterizes the maximal time for a quantum state to converge to the stationary state in time evolution. In the fermionic setting, however, the superselection rule places a constraint on what quantum states should be considered. Since no superposition between even- and odd-parity sectors are allowed, we only need to consider quantum states that can be expressed as linear combinations of Majorana operators of even parity. They make up the set

$$D(\mathcal{H})_{\text{even}} = D(\mathcal{H}) \cap B(\mathcal{H})_{\text{even}}, \quad (55)$$

$D(\mathcal{H})$ is the set of all positive semidefinite and trace-1 operators defined in Eq. (6). This requires us to modify the definition of the mixing time as in Definition 2, which we restate here:

Definition (Mixing time for fermions). For a Lindbladian operator $\mathcal{L}$ with unique stationary state in $D(\mathcal{H})_{\text{even}}$, we define its ϵ -mixing time to be

$$t_{\text{mix}}(\epsilon) = \inf\{t \geq 0 : \|e^{s\mathcal{L}}(\rho) - \sigma\|_1 \leq \epsilon, \forall s \geq t, \rho \in D(\mathcal{H})_{\text{even}}\}. \quad (56)$$

Here σ is the stationary state of $\mathcal{L}$.

We are then ready to prove Theorem 1, which we restate here:

Theorem (Main result). Let $H = H_0 + V$ be a Hamiltonian on a D -dimensional cubic lattice Λ consisting of n sites. H_0 is quadratic in Majorana operators and is $(1, r_0)$ -geometrically local, and V is parity-preserving and is (U, r_0) -geometrically local. Then for any inverse temperature $\beta > 0$, there exists $U_\beta > 0$ such that when $U < U_\beta$, the Lindbladian defined in Eq. (11) has a unique stationary state satisfying the fermionic superselection rule, and the ϵ -mixing time (Definition 2) of the Lindbladian is at most

$$C(n + \log(1/\epsilon)),$$

where $C, U_\beta > 0$ are constants that depend only on r_0, D, β .

Proof. By Theorem 3, we know that there exists $U_\beta > 0$ such that when $U < U_\beta$, the spectral gap of H^{parent} is lower bounded by $\Delta > 0$, and that 0 is a nondegenerate

eigenvalue. This implies that 0 is also a nondegenerate eigenvalue of $H^{\text{parent}}|_{\mathcal{H}'_{\text{even}}}$ (since $\varphi(\sigma^{1/2})$, which spans the kernel, is in $\mathcal{H}'_{\text{even}}$), and that the spectral gap of $H^{\text{parent}}|_{\mathcal{H}'_{\text{even}}}$ is also at least Δ . By Eq. (32) and the definition of the parent Hamiltonian in Eq. (34), we can then see that $\mathcal{L}^\dagger|_{B(\mathcal{H})_{\text{even}}}$ has spectral gap at least Δ , and 0 is its nondegenerate eigenvalue, implying that $\ker(\mathcal{L}^\dagger|_{B(\mathcal{H})_{\text{even}}}) = \text{span}(I)$. This shows that σ is the unique fixed point of $\mathcal{L}$ that satisfies the fermionic superselection rule. Therefore, all conditions of Corollary 6 are satisfied, and it tells us that

$$\|e^{t\mathcal{L}}(\rho) - \sigma\|_1 \leq \frac{e^{-\Delta t}}{\sqrt{\sigma_{\min}}},$$

for all $\rho \in D(\mathcal{H})_{\text{even}}$. The ϵ -mixing time is then upper bounded by

$$t_{\text{mix}}(\epsilon) \leq \frac{1}{\Delta} \left(\frac{1}{2} \log(1/\sigma_{\min}) + \log(1/\epsilon) \right).$$

Note that

$$\sigma_{\min} = \frac{1}{Z} e^{-\beta\|H\|} \geq \frac{1}{2^n e^{\beta\|H\|}} e^{-\beta\|H\|} = \frac{1}{2^n} e^{-2\beta\|H\|},$$

and $\|H\| \leq C'n$ for some $C' > 0$ that only depends on D, r_0 , we then have $\log(1/\sigma_{\min}) \leq n \log(2) + 2\beta C'n$, which yields the mixing time bound in the theorem. In the above $\Delta, U_\beta > 0$ are both constants that depend only on β, r_0, D . ■

C. Recovering the Gibbs state from the top eigenstate

In this section we will discuss recovering information about the Gibbs state from the top eigenstate of the parent Hamiltonian. Suppose we want to obtain the expectation value of an observable

$$X = \sum_{\alpha \in \{0,1\}^{2n}} X_\alpha \gamma_1^{\alpha_1} \gamma_2^{\alpha_2} \cdots \gamma_{2n}^{\alpha_{2n}} \in B(\mathcal{H}), \quad (57)$$

then we will show that we can get it from an observable

$$X^\sharp = \Phi(L_X) = \sum_{\alpha \in \{0,1\}^{2n}} X_\alpha \gamma_1^{\alpha_1} \gamma_3^{\alpha_3} \cdots \gamma_{4n-1}^{\alpha_{4n-1}} \in B(\mathcal{H}'). \quad (58)$$

Proposition 4. Let σ be the Gibbs state of $\mathcal{L}$ in Eq. (11) and let v be the (normalized) top eigenstate of the parent Hamiltonian H^{parent} defined in Eq. (34). Then for any observable $X \in B(\mathcal{H})$ as expressed in Eq. (57), we have a corresponding $X^\sharp \in B(\mathcal{H}')$ as defined in Eq. (58) such that $\text{Tr}[\sigma X] = \langle v, X^\sharp v \rangle$.

Proof. By Theorem 3 we know that $v \propto \varphi(\sigma^{1/2})$. We can compute that, by Eq. (19),

$$\langle \varphi(\sigma^{1/2}), \varphi(\sigma^{1/2}) \rangle = \frac{1}{2^n} \text{Tr}[\sigma] = \frac{1}{2^n}.$$

Therefore, we have $v = 2^{n/2} \varphi(\sigma^{1/2})$ (fixing an unimportant global phase). Then we have

$$\begin{aligned} \langle v, X^\sharp v \rangle &= \langle v, \varphi \circ L_X \circ \varphi^{-1}(v) \rangle \\ &= 2^n \langle \varphi(\sigma^{1/2}), \varphi(X \sigma^{1/2}) \rangle \\ &= \text{Tr}[\sigma^{1/2} X \sigma^{1/2}] = \text{Tr}[\sigma X]. \quad \blacksquare \end{aligned}$$

Remark 5. This mapping $X \in B(\mathcal{H}) \mapsto X^\sharp \in B(\mathcal{H}')$ is an algebra homomorphism. One can easily check that for $X, Y \in B(\mathcal{H})$ we have $(XY)^\sharp = X^\sharp Y^\sharp$. Moreover, we have $\|X^\sharp\| = \|X\|$ by Lemma 25.

As a result of this correspondence we can readily prove correlation decay in the Gibbs state, as stated in Corollary 3, which we restate here:

Corollary. Under the same assumptions as Theorem 1, there exists $C, \xi > 0$ that depends only on β, r_0, D , such that for an observable $X \in B(\mathcal{H})$ supported on a constant number of adjacent sites, and any observable $Y \in B(\mathcal{H})$ with support S ,

$$|\text{Tr}[\sigma XY] - \text{Tr}[\sigma X] \text{Tr}[\sigma Y]| \leq C \|X\| \|Y\| |S| e^{-d/\xi},$$

where d is the distance between the supports of X and Y .

Proof of Corollary 3. Let v be the normalized top eigenstate of H^{parent} . By Proposition 4 and Remark 5, we have

$$\begin{aligned} \text{Tr}[\sigma XY] &= \langle v, X^\sharp Y^\sharp v \rangle, \quad \text{Tr}[\sigma X] = \langle v, X^\sharp v \rangle, \\ \text{Tr}[\sigma Y] &= \langle v, Y^\sharp v \rangle. \end{aligned}$$

The exponential clustering theorem for gapped ground states (as stated in Ref. [46, Theorem 2] and also proved in Ref. [47]), tells us that

$$|\langle v, X^\sharp Y^\sharp v \rangle - \langle v, X^\sharp v \rangle \langle v, Y^\sharp v \rangle| \leq C \|X\| \|Y\| |S| e^{-d/\xi}.$$

We therefore have the desired inequality stated in the corollary. ■

VII. DISCUSSIONS

In this work we proved that, at any temperature, when the interacting strength in a geometrically local fermionic Hamiltonian is below a constant threshold that depends

only on the lattice geometry (r_0, D) and the inverse temperature (β) , the mixing time of an efficiently implementable Lindbladian that prepares the Gibbs state can be bounded linearly in the system size. This immediately implies efficient preparation of the Gibbs state on a quantum computer. In the proof, we obtained a constant lower bound for the Lindbladian spectral gap restricted to the even parity subspace by mapping the Lindbladian to a parent Hamiltonian, for which the result from Ref. [31] provides a spectral gap lower bound. The mapping from the Lindbladian to the parent Hamiltonian utilizes the third quantization, and we paid special attention to ensure that the mapping preserves locality and the relevant part of the Lindbladian spectrum.

From the above we can see that, at any constant temperature, our mixing-time result allows us to efficiently prepare the Gibbs states of a wide range of nonintegrable fermionic Hamiltonians on a quantum computer. On the classical side, we are not aware of any such rigorous result. This then raises the question: is this a place where we can demonstrate significant quantum advantage? There are many reasons to be cautious in this regard. First of all, the weak interaction regime we are studying is generally considered easy for classical heuristic algorithms. Commonly used algorithms such as Hartree-Fock achieve good empirical performance even though they generally cannot achieve arbitrarily high accuracy. The recent result on the lack of entanglement for high-temperature Gibbs states of quantum spin systems in Ref. [29] also raises concern that in the weak interaction regime the Gibbs states may have structures that enable efficient classical simulation. However, the techniques in Ref. [29] and its extension to the fermionic setting [49] rely heavily on the high-temperature assumption, which ensures that asymptotic expansion around the infinite-temperature limit has good convergence and is easy to evaluate. It is unclear how these techniques can be extended to the weakly interacting setting without compromising algorithmic efficiency. All of these lead to an intriguing question: can we design a provably efficient classical algorithm to simulate Gibbs state properties in this regime, or can we prove that no classical algorithm can do so? Either outcome would significantly advance our understanding of the simulability of such systems and quantum advantage in general.

We remark that the threshold for interaction strength U_β decays exponentially in the inverse temperature β . Therefore, at low temperature only very small interaction can be analyzed within the current framework, as can be seen from Corollary 7, which states that the spectral gap of the noninteracting parent Hamiltonian is exponentially small in β . This limitation may be due to the construction of the Lindbladian in Ref. [10] and not a fundamental issue. It may be possible to substantially improve the β dependence using the detailed-balanced Lindbladian in Ref. [50], and preliminary calculation shows that the spectral gap of the

corresponding noninteracting parent Hamiltonian does not decay with β .

It is also natural to ask whether one can obtain spectral gap results and hence mixing-time bounds by directly studying the Lindbladian rather than going through the parent Hamiltonian. If possible, this approach would enable us to deal with Lindbladians that may not satisfy the detailed balance condition. Such an approach has been studied for Lindbladian satisfying a special set of conditions in Ref. [24], from which the idea of constructing a Lyapunov functional may be more generally useful.

The quantum algorithm for Gibbs state preparation discussed in Corollary 2 also motivates further development of Lindbladian simulation algorithms. Note that to prepare the Gibbs state we need to simulate Lindbladian dynamics for a time that is linear in n on a system of size that is also linear in n . One would naturally expect the gate complexity to be proportional to n^2 , but we in the end have a n^3 scaling because of an additional n factor from normalizing the Lindbladian. This is in contrast with Hamiltonian simulation algorithms where a gate complexity that is almost linear in the space-time volume can be attained [37,51]. This problem also arises in other Lindbladian simulation algorithms [52–56]. Whether one can attain the space-time volume scaling in Lindbladian simulation remains an interesting open question.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIXES: ORGANIZATION OF THE APPENDIXES

The Appendixes are organized as follows. Appendixes A, B, and C are devoted to proving Propositions 1, 2, and 3. In Appendix A we compute the spectral gap for the noninteracting part of the parent Hamiltonian. In Appendix B we prove the quasilocality of the part of the parent Hamiltonian corresponding to the dissipative part of the Lindbladian, and in Appendix C we deal with the coherent part. The Appendixes contain a table of symbols used in this work (Appendix I), and proofs of several technical lemmas.

Notations. We write $A \succcurlyeq B$ to indicate that $A - B$ is positive semidefinite. We use the following convention for Fourier transforms for a function $f(t)$:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt. \quad (59)$$

This leads to the corresponding definition of the inverse Fourier transform for a function $h(\omega)$:

$$\check{h}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} h(\omega) d\omega. \quad (60)$$

For an operator W , we use $W(t)$ to denote its time evolution in the Heisenberg picture: $W(t) = e^{iHt} W e^{-iHt}$.

APPENDIX A: SPECTRAL GAP OF THE NONINTERACTING PART

We will first show that the noninteracting, coherent part of the parent Hamiltonian is zero.

Lemma 7. $H_{C,\text{free}}^{\text{parent}}$ as defined in Eq. (49) is 0.

Proof. Since

$$H_{C,\text{free}}^{\text{parent}} = \Phi \left(i \sum_j (L_{\tilde{B}_j^{\text{free}}} - R_{(\tilde{B}_j^{\text{free}})^\dagger}) \right),$$

it suffices to show that $\sum_j \tilde{B}_j^{\text{free}} = \sum_j \sigma^{1/4} B_j^{\text{free}} \sigma^{-1/2} = 0$, which is equivalent to showing $B^{\text{free}} = \sum_j B_j^{\text{free}} = 0$. For B^{free} we have

$$\begin{aligned} B^{\text{free}} &= \sum_k \int_{-\infty}^{\infty} b_1(t) e^{-i\beta H_0 t} \left(\int_{-\infty}^{\infty} b_2(t') e^{i\beta H_0 t'} \gamma_k e^{-2i\beta H_0 t'} \gamma_k e^{i\beta H_0 t'} dt' \right) e^{i\beta H_0 t} dt \\ &= \sum_k \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' b_1(t) b_2(t') e^{i\beta H_0 (t'-t)} \gamma_k e^{-i\beta H_0 (t'-t)} e^{-i\beta H_0 (t'+t)} \gamma_k e^{i\beta H_0 (t'+t)}. \end{aligned}$$

Using Lemma 23 we then have

$$\begin{aligned} B^{\text{free}} &= \sum_k \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' b_1(t) b_2(t') \sum_p \gamma_p \left(e^{2i\beta h(t'-t)} \right)_{p,k} \sum_q \gamma_q \left(e^{-2i\beta h(t'+t)} \right)_{q,k} \\ &= \sum_k \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' b_1(t) b_2(t') \sum_p \gamma_p \left(e^{2i\beta h(t'-t)} \right)_{p,k} \sum_q \gamma_q \left(e^{2i\beta h(t'+t)} \right)_{k,q} \\ &= \int_{-\infty}^{\infty} b_1(t) dt \int_{-\infty}^{\infty} dt' b_2(t') \sum_p \sum_q \left(e^{4i\beta h t'} \right)_{p,q} \gamma_p \gamma_q = C \sum_{p,q} \check{b}_2(4i\beta h)_{p,q} \gamma_p \gamma_q, \end{aligned}$$

where $C = \int_{-\infty}^{\infty} b_1(t) dt$ and $\check{b}_2(x) = \int_{-\infty}^{\infty} b_2(t') e^{ixt'} dt'$. Then we compute the constant C .

$$\begin{aligned} C &= \int_{-\infty}^{\infty} 2\sqrt{\pi} e^{1/8} \left(\frac{1}{\cosh(2\pi t)} \right) * \sin(-t) \exp(-2t^2) dt \\ &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds 2\sqrt{\pi} e^{1/8} \left(\frac{1}{\cosh(2\pi s)} \right) \sin(-(t-s)) \exp(-2(t-s)^2) = 0, \end{aligned} \quad (\text{A1})$$

where we have used the fact that $\int_{-\infty}^{\infty} \sin(-(t-s)) \exp(-2(t-s)^2) dt = 0$. This then implies $B^{\text{free}} = 0$. ■

This allow us to focus on the dissipative part in $H_{D,\text{free}}^{\text{parent}}$ (51). We will first analyze it for the simplest case, supposing that the Hamiltonian acts only on one fermionic mode.

Lemma 8. We consider the free parent Hamiltonian $H_{\text{free}}^{\text{parent}}$ defined in Eq. (54) generated by

$$H_0 = i\epsilon(\zeta_1 \zeta_2 - \zeta_2 \zeta_1),$$

where ζ_1 and ζ_2 are two Majorana operators satisfying $\{\zeta_j, \zeta_k\} = 2\delta_{jk}$. Then defining the third quantized

a -Majorana modes $\hat{\zeta}_j, j = 1, 2, 3, 4$, following Eq. (21), we have

$$H_{\text{free}}^{\text{parent}} = Ce^{-4\beta^2\epsilon^2} \left(-i\hat{d}_1^\dagger \hat{d}_2 + i\hat{d}_2^\dagger \hat{d}_1 - \sinh(2\beta\epsilon)\hat{d}_1^\dagger \hat{d}_1 + \sinh(2\beta\epsilon)\hat{d}_2^\dagger \hat{d}_2 - \cosh(2\beta\epsilon) \right), \quad (\text{A2})$$

where $\hat{d}_1 = (\hat{\zeta}_1 + i\hat{\zeta}_3)/2$, $\hat{d}_2 = (\hat{\zeta}_2 + i\hat{\zeta}_4)/2$, and C is a universal constant.

Proof. Since H_0 is noninteracting and only involves one fermionic mode, we can directly compute $\zeta_1^{\text{free}}(t)$ and $\zeta_2^{\text{free}}(t)$ corresponding to Eq. (41). This then allows us to explicitly compute $\tilde{A}_j^{\text{free}}(\omega)$ in Eq. (42). The coherent part can be neglected since we know it is 0 from Lemma 7. The explicit expression for $H_{\text{free}}^{\text{parent}} = H_{D,\text{free}}^{\text{parent}}$ then follows from straightforward calculation. ■

Corollary 7. Under the same setup as in Lemma 8, the top eigenvalue of $H_{\text{free}}^{\text{parent}}$ is 0, and the spectral gap separating it from the second largest eigenvalue is $Ce^{-4\beta^2\epsilon^2} \cosh(2\beta\epsilon)$, where C is the same universal constant as in Lemma 8.

Proof. We note that in Eq. (A2), the part involving fermionic creation and annihilation operators

$$-i\hat{d}_1^\dagger \hat{d}_2 + i\hat{d}_2^\dagger \hat{d}_1 - \sinh(2\beta\epsilon)\hat{d}_1^\dagger \hat{d}_1 + \sinh(2\beta\epsilon)\hat{d}_2^\dagger \hat{d}_2$$

has single-particle eigenenergies $\pm \cosh(2\beta\epsilon)$. Therefore, the top eigenvalue of $H_{\text{free}}^{\text{parent}}$ is

$$Ce^{-4\beta^2\epsilon^2} (\cosh(2\beta\epsilon) - \cosh(2\beta\epsilon)) = 0,$$

and the second largest eigenvalue of it is

$$Ce^{-4\beta^2\epsilon^2} (0 - \cosh(2\beta\epsilon)) = -Ce^{-4\beta^2\epsilon^2} \cosh(2\beta\epsilon).$$

This then gives us the expression for the spectral gap. ■

Next, we will deal with the more general case where the system consists of more than one fermionic modes. However, we will see that the general case can be reduced to this simple case. We first rotate the Majorana operators γ_j to turn the noninteracting part $H_0 = \sum_{jk} h_{jk} \gamma_j \gamma_k$ into a standard form. We choose a $2n \times 2n$ real orthogonal matrix

$Q = (Q_{jk})$ and perform the basis transform

$$\gamma_j = \sum_k Q_{jk} \zeta_k. \quad (\text{A3})$$

It is easy to see that $\zeta_k = \sum_j Q_{jk} \gamma_j$, and that $\langle \zeta_k, \zeta_{k'} \rangle = 2\delta_{kk'}$ and therefore $\{\zeta_k\}$ for a valid set of Majorana operators. The orthogonal matrix Q is chosen so that

$$H_0 = i \sum_{j=1}^n \epsilon_j \zeta_{2j-1} \zeta_{2j}, \quad (\text{A4})$$

for $\epsilon_j \geq 0$. Such a Q always exist because the coefficient matrix (h_{jk}) is Hermitian and pure imaginary. We can see that the free Hamiltonian gets decoupled into terms with nonoverlapping supports. Here $\pm\epsilon_j$ are eigenvalues of the coefficient matrix (h_{jk}) . Because H_0 is $(1, r_0)$ -geometrically local as defined in Definition 1, there are at most Cr_0^D nonzero elements on each row of (h_{jk}) , and therefore

$$|\epsilon_j| \leq \|H\| \leq Cr_0^D, \quad (\text{A5})$$

where the constant $C > 0$ depends only on r_0 .

We will show that this basis transform decouples the free parent Hamiltonian as well. Essentially, this is true because the free parent Hamiltonian is a quadratic function of γ_j .

Recall that the free parent Hamiltonian $H_{\text{free}}^{\text{parent}}$ is of the form, Eq. (54), and because $H_{C,\text{free}}^{\text{parent}} = 0$, we have

$$H_{\text{free}}^{\text{parent}} = \Phi \left(\sum_j \int d\omega \eta(\omega) \left(L_{\tilde{A}_j^{\text{free}}(-\omega)} R_{\tilde{A}_j^{\text{free}}(-\omega)^\dagger} - \frac{1}{2} L_{\tilde{A}_j^{\text{free}}(-\omega)} \tilde{A}_j^{\text{free}}(\omega) - \frac{1}{2} R_{\tilde{A}_j^{\text{free}}(\omega)^\dagger} \tilde{A}_j^{\text{free}}(-\omega)^\dagger \right) \right), \quad (\text{A6})$$

We define

$$\zeta_j^{\text{free}}(t) = e^{iH_0 t} \zeta_j e^{-iH_0 t}. \quad (\text{A7})$$

We then have

$$\gamma_j^{\text{free}}(t) = \sum_k Q_{jk} \zeta_k^{\text{free}}(t). \quad (\text{A8})$$

With this, corresponding to $\tilde{A}_j^{\text{free}}(\omega)$ in Eq. (42) we define

$$\xi_j(\omega) = e^{-\beta H_0/4} \left(\frac{1}{\sqrt{2\pi}} \int \zeta_j^{\text{free}}(t) f(t) e^{-i\omega t} dt \right) e^{\beta H_0/4}, \quad (\text{A9})$$

which is related to $\tilde{A}_j^{\text{free}}(\omega)$ through

$$\tilde{A}_j^{\text{free}}(\omega) = \sum_k Q_{jk} \xi_k(\omega). \quad (\text{A10})$$

Now we can consider how these transformations affect terms in Eq. (54). We have

$$\begin{aligned} \sum_j L_{\tilde{A}_j^{\text{free}}(-\omega)} R_{\tilde{A}_j^{\text{free}}(-\omega)}^\dagger &= \sum_j \sum_{k,k'} Q_{jk} Q_{jk'} L_{\xi_k(-\omega)} R_{\xi_{k'}(-\omega)}^\dagger \\ &= \sum_k L_{\xi_k(-\omega)} R_{\xi_k(-\omega)}^\dagger, \end{aligned}$$

where we have used $\sum_j Q_{jk} Q_{jk'} = \delta_{kk'}$. Similarly,

$$\begin{aligned} \sum_j L_{\tilde{A}_j^{\text{free}}(-\omega)} \tilde{A}_j^{\text{free}}(\omega) &= \sum_k L_{\xi_k(-\omega)} \xi_k(\omega), \\ \sum_j R_{\tilde{A}_j^{\text{free}}(\omega)}^\dagger \tilde{A}_j^{\text{free}}(-\omega)^\dagger &= \sum_k R_{\xi_k(\omega)}^\dagger \xi_k(-\omega)^\dagger. \end{aligned}$$

From the above we can see that

$$H_{\text{free}}^{\text{parent}} = \sum_{k=1}^n H_k^{\text{free}}, \quad (\text{A11})$$

where

$$\begin{aligned} H_k^{\text{free}} &= \Phi \left(L_{\Xi_{kk}} + R_{\Xi_{kk}}^\dagger + \int d\omega \eta(\omega) (L_{\xi_k(-\omega)} R_{\xi_k(-\omega)}^\dagger \right. \\ &\quad \left. - \frac{1}{2} L_{\xi_k(-\omega)} \xi_k(\omega) - \frac{1}{2} R_{\xi_k(\omega)}^\dagger \xi_k(-\omega)^\dagger) \right), \end{aligned} \quad (\text{A12})$$

only involves Majorana modes ζ_{2k-1} and ζ_{2k} .

In the third quantization, we define

$$\hat{\zeta}_{2j-1} = \sum_k Q_{jk} \hat{\gamma}_{2k-1}, \quad \hat{\zeta}_{2j} = \sum_k Q_{jk} \hat{\gamma}_{2k}. \quad (\text{A13})$$

Then by Eq. (25) and linearity, we have

$$\begin{aligned} \varphi \circ L_{\zeta_{j_1} \dots \zeta_{j_k}} \circ \varphi^{-1} &= \hat{\zeta}_{2j_1-1} \dots \hat{\zeta}_{2j_k-1}, \\ \varphi \circ R_{\zeta_{j_1} \dots \zeta_{j_k}} \circ \varphi^{-1} &= -i^k \hat{\zeta}_{2j_k} \dots \hat{\zeta}_{2j_1} (-1)^{k\hat{n}}. \end{aligned}$$

As a result Remark 4 corresponds to

$$\begin{aligned} \Phi(L_{\zeta_{j_1} \dots \zeta_{j_k}}) &= \hat{\zeta}_{2j_1-1} \dots \hat{\zeta}_{2j_k-1}, \\ \Phi(R_{\zeta_{j_1} \dots \zeta_{j_k}}) &= i^k \hat{\zeta}_{2j_k} \dots \hat{\zeta}_{2j_1}. \end{aligned}$$

H_k^{free} is therefore supported only on a -Majorana modes $\hat{\gamma}_{4k-3}$, $\hat{\gamma}_{4k-2}$, $\hat{\gamma}_{4k-1}$, $\hat{\gamma}_{4k}$. Moreover, H_k^{free} is the parent

Hamiltonian one gets through Eq. (54) by replacing H_0 with $i\epsilon_j(\zeta_1\zeta_2 - \zeta_2\zeta_1)$. Therefore, by Lemma 8 we have

$$\begin{aligned} H_k^{\text{free}} &= C e^{-4\beta^2 \epsilon_k^2} \left(-i \hat{d}_{2k-1}^\dagger \hat{d}_{2k} + i \hat{d}_{2k}^\dagger \hat{d}_{2k-1} \right. \\ &\quad \left. - \sinh(2\beta\epsilon_k) \hat{d}_{2k-1}^\dagger \hat{d}_{2k-1} + \sinh(2\beta\epsilon_k) \hat{d}_{2k}^\dagger \hat{d}_{2k} \right. \\ &\quad \left. - \cosh(2\beta\epsilon_k) \right), \end{aligned} \quad (\text{A14})$$

where $\hat{d}_{2k-1} = (\hat{\zeta}_{4k-3} + i\hat{\zeta}_{4k-1})/2$, $\hat{d}_{2k} = (\hat{\zeta}_{4k-2} + i\hat{\zeta}_{4k})/2$, $\hat{\zeta}_j, j = 1, 2, \dots, 4n$, are third quantized a -Majorana operators defined through Eq. (21), and C is the same universal constant as in Lemma 8.

We therefore have the following lemma:

Lemma 9. Let Majorana operators $\zeta_k, k = 1, 2, \dots, 2n$, be as defined in Eq. (A3) so that Eq. (A4) is satisfied. Then the parent Hamiltonian of the free part can be written as

$$H_{\text{free}}^{\text{parent}} = \sum_k H_k^{\text{free}},$$

where each H_k^{free} is given in Eq. (A14).

We are then ready to prove Proposition 1.

Proof of Proposition 1. Because $H_{\text{free}}^{\text{parent}} = \sum_k H_k^{\text{free}}$ and each H_k^{free} are supported on a nonoverlapping set of Majorana operators in the $\{\hat{\zeta}_j\}$ basis, the top eigenvalue of $H_{\text{free}}^{\text{parent}}$ is then the sum of the top eigenvalues of each H_k^{free} , which is 0. The spectral gap g of $H_{\text{free}}^{\text{parent}}$ is the minimum among the spectral gaps of each H_k^{free} , and by Corollary 7,

$$g = \min_k C e^{-4\beta^2 \epsilon_k^2} \cosh(2\beta\epsilon_k),$$

which because of Eq. (A5) is lower bounded by a constant $\Delta > 0$ that depends only on β, D, r_0 . ■

APPENDIX B: STRUCTURE OF THE DISSIPATIVE PART

In this section we will show that both $\tilde{A}_j^{\text{free}}(\omega)$ and $\tilde{A}_j^{\text{int}}(\omega)$ defined in Eqs. (42) and (44) are quasilocal, and that the norm of $\tilde{A}_j^{\text{int}}(\omega)$ is small (on the order U). To do this, we will first express the jump operator $\tilde{A}_j^{\text{free}}(\omega)$ solely in terms of $e^{iHt} \gamma_j e^{-iHt}$ and avoid having imaginary time evolution $e^{-\beta H}$ showing up in the expression.

For an operator $A \in B(\mathcal{H})$, we introduce the following notation: let $\{E_i\}$ be the eigenvalues of H , and $\{\Pi_i\}$ be the corresponding eigenspace projection operators, then we denote

$$A_v = \sum_{ij: E_i - E_j = v} \Pi_i A \Pi_j. \quad (\text{B1})$$

There is only a finite set of v for which $A_v \neq 0$, they are called *Bohr frequencies*.

Using the decomposition $\gamma_j = \sum_{v \in B_H} (\gamma_j)_v$, we have

$$\tilde{A}_j(\omega) = \frac{1}{\sqrt{2\pi}} \sum_v \int e^{-\beta v/4} e^{ivt} (\gamma_j)_v e^{-i\omega t} f(t) dt. \quad (\text{B2})$$

Now we define

$$F_1(v, \omega) = \frac{1}{\sqrt{2\pi}} \int e^{-\beta v/4} e^{i(v-\omega)t} f(t) dt, \quad (\text{B3})$$

then we have

$$\begin{aligned} \tilde{A}_j(\omega) &= \sum_v F_1(v, \omega) (\gamma_j)_v \\ &= \frac{1}{\sqrt{2\pi}} \sum_v \int \check{F}_1(t, \omega) e^{ivt} (\gamma_j)_v dt \\ &= \frac{1}{\sqrt{2\pi}} \int \check{F}_1(t, \omega) e^{iHt} \gamma_j e^{-iHt} dt, \end{aligned} \quad (\text{B4})$$

where

$$\begin{aligned} \check{F}_1(t, \omega) &= \frac{1}{\sqrt{2\pi}} \int F_1(v, \omega) e^{-ivt} dv \\ &= \frac{1}{2\pi} \int dv \int dt' e^{-\beta v/4} e^{i(v-\omega)t'} f(t') e^{-ivt} \\ &= \frac{e^{1/16}}{\sqrt{\beta} \sqrt{\pi/2}} e^{-t^2/\beta^2} e^{-it(\omega-1/(2\beta))} e^{-\beta\omega/4}. \end{aligned} \quad (\text{B5})$$

Similarly, for $\tilde{A}_j^{\text{free}}(\omega)$ we have

$$\tilde{A}_j^{\text{free}}(\omega) = \frac{1}{\sqrt{2\pi}} \int \check{F}_1(t, \omega) e^{iH_0 t} \gamma_j e^{-iH_0 t} dt. \quad (\text{B6})$$

1. Quasilocality of the jump operator

We will first prove that $\tilde{A}_j(\omega)$ [defined in Eq. (40)] can be well approximated by local operators.

Lemma 10. There exists universal constants $C, \mu_1, \mu_2 > 0$ such that for any $r \geq 1$, there exists an operator A^{loc} supported on the ball with radius r centered at j such that

$$\begin{aligned} \|\tilde{A}_j(\omega) - A^{\text{loc}}\| &\leq C \left(\sqrt{\frac{\beta}{r}} e^{-\mu_1 r} + \frac{1}{\sqrt{\beta}} e^{-\mu_2 r^2/(J\beta)^2} \right) e^{-\beta\omega/4}, \end{aligned} \quad (\text{B7})$$

where J is the Lieb-Robinson velocity of H on Λ for which an upper bound is given in Eq. (26). This implies that there exists constants $C', \mu > 0$ that depend only on β and J such that

$$\|\tilde{A}_j(\omega) - A^{\text{loc}}\| \leq C' e^{-\mu r} e^{-\beta\omega/4}. \quad (\text{B8})$$

Proof. We first decompose $\tilde{A}_j(\omega) = \text{I} + \text{II}$, where

$$\begin{aligned} \text{I} &= \frac{1}{\sqrt{2\pi}} \int_{-T}^T \check{F}_1(t, \omega) e^{iHt} \gamma_j e^{-iHt} dt, \\ \text{II} &= \frac{1}{\sqrt{2\pi}} \int_{|t|>T} \check{F}_1(t, \omega) e^{iHt} \gamma_j e^{-iHt} dt. \end{aligned} \quad (\text{B9})$$

In the above $T > 0$ can be arbitrarily chosen. From Eq. (B5), we can see that

$$|\check{F}_1(t, \omega)| \leq C_1 e^{-t^2/\beta^2} e^{-\beta\omega/4}. \quad (\text{B10})$$

Therefore, we have

$$\|\text{II}\| \leq \frac{C_2}{\sqrt{\beta}} e^{-T^2/\beta^2} e^{-\beta\omega/4}. \quad (\text{B11})$$

We then approximate $e^{iHt} \gamma_j e^{-iHt}$ with a local operator. Let H' be the Hamiltonian consisting of terms of H that are supported on the ball around j with radius r , then using the Lieb-Robinson bound [27, Eq. (A1)], we have

$$\|e^{iHt} \gamma_j e^{-iHt} - e^{iH't} \gamma_j e^{-iH't}\| \leq \frac{(2J|t|)^r}{r!}. \quad (\text{B12})$$

Here $e^{iH't} \gamma_j e^{-iH't}$ is supported only on this ball with radius r . We then let

$$A^{\text{loc}} = \frac{1}{\sqrt{2\pi}} \int_{-T}^T \check{F}_1(t, \omega) e^{iH't} \gamma_j e^{-iH't} dt. \quad (\text{B13})$$

This is a good approximate of I:

$$\begin{aligned} \|\text{I} - A^{\text{loc}}\| &\leq \frac{1}{\sqrt{2\pi}} \int_{-T}^T |\check{F}_1(t, \omega)| \frac{(2J|t|)^r}{r!} dt \\ &\leq C_3 \sqrt{\beta} \frac{(2JT)^r}{r!} e^{-\beta\omega/4}. \end{aligned} \quad (\text{B14})$$

Therefore,

$$\begin{aligned} \|\tilde{A}_j(\omega) - A^{\text{loc}}\| &\leq \|\text{I} - A^{\text{loc}}\| + \|\text{II}\| \\ &\leq C_3 \sqrt{\beta} \frac{(2JT)^r}{r!} e^{-\beta\omega/4} \\ &\quad + \frac{C_2}{\sqrt{\beta}} e^{-T^2/\beta^2} e^{-\beta\omega/4}. \end{aligned} \quad (\text{B15})$$

Using Stirling's formula we have $r! \geq \sqrt{2\pi r} (r/e)^r$ for $r \geq 1$, and therefore we further have

$$\begin{aligned} \|\tilde{A}_j(\omega) - A^{\text{loc}}\| &\leq C \left(\sqrt{\frac{\beta}{r}} \left(\frac{2eJT}{r} \right)^r + \frac{1}{\sqrt{\beta}} e^{-T^2/\beta^2} \right) e^{-\beta\omega/4}. \end{aligned} \quad (\text{B16})$$

Since we can arbitrarily choose T , we let $T = r/(2e^2 J)$, and then we have Eq. (B7) with $\mu_1 = 1$ and $\mu_2 = 1/4e^4$. ■

2. Quasilocality of the free part

By the exact same way as for Lemma 10 we can prove the following:

Lemma 11. There exists universal constants $C, \mu_1, \mu_2 > 0$ such that for any $r \geq 1$, there exists an operator $A^{\text{loc}}(\omega)$ supported on the ball with radius r centered at j such that

$$\begin{aligned} \|\tilde{A}_j^{\text{free}}(\omega) - A^{\text{loc}}(\omega)\| \\ \leq C \left(\sqrt{\frac{\beta}{r}} e^{-\mu_1 r} + \frac{1}{\sqrt{\beta}} e^{-\mu_2 r^2 / (J_0 \beta)^2} \right) e^{-\beta \omega / 4}, \end{aligned} \quad (\text{B17})$$

where J_0 is the Lieb-Robinson velocity of H_0 on Λ for which an upper bound is given in Eq. (27). This implies that there exists constants $C', \mu > 0$ that depend only on β and J_0 such that

$$\|\tilde{A}_j^{\text{free}}(\omega) - A^{\text{loc}}(\omega)\| \leq C' e^{-\mu r} e^{-\beta \omega / 4}. \quad (\text{B18})$$

Moreover, both $\tilde{A}_j^{\text{free}}(\omega)$ and $A^{\text{loc}}(\omega)$ are both linear combinations of Majorana operators.

Proof. We only need to prove the statement in the last sentence as the rest is the same as for Lemma 10. We note that because H_0 is quadratic and γ_j is a Majorana operator, $e^{iH_0 t} \gamma_j e^{-iH_0 t}$ is a linear combination of Majorana operators. This ensures that $\tilde{A}_j^{\text{free}}(\omega)$ is also a linear combination of Majorana operators. In the construction of $A^{\text{loc}}(\omega)$, we restrict the Hamiltonian H_0 to a local region, but it is still quadratic. Therefore, the same also holds for $A^{\text{loc}}(\omega)$. ■

We therefore have the following corollary for the coefficients when expanding $\tilde{A}_j^{\text{free}}(\omega)$ as a linear combination of Majorana operators:

Corollary 8. The free part of the jump operator as defined in Eq. (42) satisfies

$$\tilde{A}_j^{\text{free}}(\omega) = \sum_k c_k \gamma_k,$$

where $c_k \in \mathbb{C}$, and for any $r > 0$,

$$\sum_{k: d(j, k) > r} |c_k|^2 \leq C'^2 e^{-2\mu r} e^{-\beta \omega / 2}.$$

Here C' and μ are the same as in Lemma 11 [57].

Proof. By Lemma 11, we can write $\tilde{A}_j^{\text{free}}(\omega) = \sum_k c_k \gamma_k$, and for any $r > 0$, we have

$$A^{\text{loc}}(\omega) = \sum_{k: d(j, k) \leq r} c'_k \gamma_k,$$

such that $\|\tilde{A}_j^{\text{free}}(\omega) - A^{\text{loc}}(\omega)\| \leq C' e^{-\mu r} e^{-\beta \omega / 4}$. By Lemma 24, we have

$$\begin{aligned} \sum_{k: d(j, k) > r} |c_k|^2 &\leq \sum_{k: d(j, k) > r} |c_k|^2 + \sum_{k: d(j, k) \leq r} |c_k - c'_k|^2 \\ &\leq \|\tilde{A}_j^{\text{free}}(\omega) - A^{\text{loc}}(\omega)\|^2 \leq C'^2 e^{-2\mu r} e^{-\beta \omega / 2}. \end{aligned}$$

■

This structure for $\tilde{A}_j^{\text{free}}(\omega)$ gives us a structural result for $H_{D, \text{free}}^{\text{parent}}$, i.e., the part of the parent Hamiltonian corresponding to the free part of the dissipation terms:

Lemma 12. For $H_{D, \text{free}}^{\text{parent}}$ defined in Eq. (51), there exists a constant C that depends only on J_0 in Lemma 11 and β such that

$$H_{D, \text{free}}^{\text{parent}} = \sum_{l, l'=1}^{4n} m_{l, l'} \hat{\gamma}_l \hat{\gamma}_{l'},$$

where

$$|m_{l, l'}| \leq C e^{-\mu d(\mathfrak{s}'(l), \mathfrak{s}'(l'))}.$$

Here $\mathfrak{s}'(l)$ indicate the lattice site in Λ on which the a -Majorana mode l resides. In the above the constant $\mu > 0$ is the same as in Lemma 11.

Proof. From Eq. (51) we can write

$$H_{D, \text{free}}^{\text{parent}} = \text{I} + \text{II} + \text{III},$$

where

$$\begin{aligned} \text{I} &= \Phi \left(\sum_j \int d\omega \eta(\omega) L_{\tilde{A}_j^{\text{free}}(-\omega)} R_{\tilde{A}_j^{\text{free}}(-\omega)^\dagger} \right), \\ \text{II} &= -\frac{1}{2} \Phi \left(\sum_j \int d\omega \eta(\omega) L_{\tilde{A}_j^{\text{free}}(-\omega)} \tilde{A}_j^{\text{free}}(\omega) \right), \\ \text{III} &= -\frac{1}{2} \Phi \left(\sum_j \int d\omega \eta(\omega) R_{\tilde{A}_j^{\text{free}}(\omega)^\dagger} \tilde{A}_j^{\text{free}}(-\omega)^\dagger \right). \end{aligned}$$

We can simply prove that each of the above three terms has the structure as stated in the lemma. Because the proof for

each of them is essentially the same, we only prove for the term I. By Definition 6 we have

$$I = \sum_j \int d\omega \eta(\omega) \Phi(L_{\tilde{A}_j^{\text{free}}(-\omega)}) \Phi(R_{\tilde{A}_j^{\text{free}}(-\omega)}^\dagger).$$

By Lemma 8, we have

$$\tilde{A}_j^{\text{free}}(\omega) = \sum_k c_{j,k}(\omega) \gamma_k,$$

where $\sum_{k:d(j,k)>r} |c_{j,k}(\omega)|^2 \leq C'^2 e^{-2\mu r} e^{-\beta\omega/2}$. This means $\Phi(L_{\tilde{A}_j^{\text{free}}(\omega)}^\dagger)$ and $\Phi(R_{\tilde{A}_j^{\text{free}}(\omega)})$ admit similar decompositions by Eq. (24):

$$\begin{aligned} \Phi(L_{\tilde{A}_j^{\text{free}}(-\omega)}) &= \sum_k c_{j,k}(-\omega) \hat{\gamma}_{2k-1}, \\ \Phi(R_{\tilde{A}_j^{\text{free}}(-\omega)}^\dagger) &= -i \sum_k c_{j,k}(-\omega)^* \hat{\gamma}_{2k}, \end{aligned} \quad (\text{B19})$$

Therefore,

$$I = -i \sum_{k,k'} \sum_j \int d\omega \eta(\omega) c_{j,k}(-\omega) c_{j,k'}(-\omega)^* \hat{\gamma}_{2k-1} \hat{\gamma}_{2k}. \quad (\text{B20})$$

Let

$$I = \sum_{k,k'} m_{k,k'}^{(\text{I})} \hat{\gamma}_{2k-1} \hat{\gamma}_{2k}, \quad (\text{B21})$$

then

$$\begin{aligned} |m_{k,k'}^{(\text{I})}| &\leq \sum_j \int d\omega \eta(\omega) |c_{j,k}(-\omega) c_{j,k'}(-\omega)^*| \\ &\leq C'^2 \int d\omega \eta(\omega) e^{-\beta\omega/2} \sum_j e^{-\mu(d(j,k)+d(j,k'))} \\ &\leq C'' \sum_j e^{-\mu(d(k,k')+d(j,(k+k')/2))} \\ &\leq C e^{-\mu d(k,k')}. \end{aligned} \quad (\text{B22})$$

Since II and III have the same structure as can be proved in the same way, we have proved the statement in the lemma. ■

We can now prove Proposition 2:

Proof of Proposition 2. By Lemma 12, and the bound for the Lieb-Robinson velocity J_0 in Eq. (27), we can see that $H_{D,\text{free}}^{\text{parent}}$ has $[K, \mu]$ decay for some constants $K, \mu > 0$ that depend only on β, r_0, D . By Lemma 7 we also know that $H_{C,\text{free}}^{\text{parent}} = 0$. Therefore, $H_{\text{free}}^{\text{parent}} = H_{D,\text{free}}^{\text{parent}} + H_{C,\text{free}}^{\text{parent}} = H_{D,\text{free}}^{\text{parent}}$ has $[K, \mu]$ decay. ■

3. The perturbation in the jump operator

In this section we show that the perturbation is also quasilocal and is small in U . We start from the time evolve operators $\gamma_j(t) = e^{iHt} \gamma_j e^{-iHt}$ as compared to $\gamma_j^{\text{free}}(t) = e^{iH_0 t} \gamma_j e^{-iH_0 t}$:

$$\gamma_j(t) = \gamma_j^{\text{free}}(t) + \gamma_j^{\text{int}}(t), \quad (\text{B23})$$

where $\gamma_j^{\text{free}}(t)$ is given in Eq. (41), and

$$\gamma_j^{\text{int}}(t) = i \int_0^t e^{iH(t-s)} [V, e^{iH_0 s} \gamma_j e^{-iH_0 s}] e^{-iH(t-s)} ds. \quad (\text{B24})$$

By Eqs. (B4) and (B6), we then have

$$\tilde{A}_j^{\text{int}}(\omega) := \tilde{A}_j(\omega) - \tilde{A}_j^{\text{free}}(\omega) = \frac{1}{\sqrt{2\pi}} \int dt \tilde{F}_1(t, \omega) \gamma_j^{\text{int}}(t). \quad (\text{B25})$$

Because V is an “extensive” operator in the sense that it acts on the entire system, we need to make sure that it does not blow up the approximation errors by a factor that depends on the system size. Therefore, below we will first focus on $[V, e^{iH_0 s} \gamma_j e^{-iH_0 s}]$.

Lemma 13. Let $\gamma_j^{\text{free}}(s) = \sum_k c_k(s) \gamma_k$, and for some $r > 0$ let

$$X_1(s) = \sum_{k:d(j,k) \leq r} c_k(s) \gamma_k.$$

Then if $r \geq 2e^2 J_0 |s|$ where J_0 is the same as in Lemma 11, we have

$$\|[V, \gamma_j^{\text{free}}(s) - X_1(s)]\| \leq CU(r+1)^D e^{-r},$$

for some constant C that only depends on the lattice dimension D and J_0 .

Proof. By the Lieb-Robinson bound and a derivation similar to Lemma 8 we have

$$\left\| \sum_{k:d(j,k) > r'} c_k(s) \gamma_k \right\| \leq \frac{(2J_0 |s|)^{r'}}{r'!}.$$

By Lemma 24 we then have

$$\sum_{k:d(j,k) > r'} |c_k(s)|^2 \leq \frac{(2J_0 |s|)^{2r'}}{(r'!)^2}.$$

This further implies

$$\begin{aligned} |c_k(s)| &\leq \inf_{r' \in \mathbb{Z}_+ : d(j,k) > r'} \frac{(2J_0 |s|)^{r'}}{r'!} \leq \frac{(2J_0 |s|)^{d(j,k)-1}}{(d(j,k)-1)!} \\ &\leq \frac{1}{\sqrt{2\pi(d(j,k)-1)}} \left(\frac{2\pi J_0 |s|}{d(j,k)-1} \right)^{d(j,k)-1}. \end{aligned} \quad (\text{B26})$$

When $d(j, k) \geq 2e^2 J_0 |s| + 1$, we have

$$|c_k(s)| \leq C_1 e^{-d(j, k)}, \quad (\text{B27})$$

for some constant $C_1 > 0$ that only depends on J_0 . Therefore,

$$\begin{aligned} \sum_{k: d(j, k) \geq r} |c_k(s)| &\leq C_1 e^{-d(j, k)} \leq C_2 \int_r^\infty dr' r'^D e^{-r'} \leq C_3 (r+1)^D e^{-r}, \end{aligned}$$

where $C_2 > 0$ and $C_3 > 0$ are both constants that only depend on J_0 and D .

Because V is (U, r_0) -geometrically local, $\| [V, \gamma_k] \| \leq C_4 U$ for some constant $C_4 > 0$ that depends only on r_0 . Therefore,

$$\| [V, \gamma_j^{\text{free}}(s) - X_1(s)] \| \leq C_3 C_4 U (r+1)^D e^{-r}.$$

We can therefore set $C = C_3 C_4$. $\blacksquare$

This directly allows us to bound the norm of $[V, e^{iH_0 s} \gamma_j e^{-iH_0 s}]$.

Corollary 9. There exists constant $C', \mu' > 0$ that only depends on r_0, J_0, D such that

$$\| [V, \gamma_j^{\text{free}}(s)] \| \leq C' U (|s| + 1)^D (1 + e^{-\mu' |s|}).$$

Proof. In Lemma 13, we can set $r = 2e^2 J_0 |s| + 1$, and then we have

$$\begin{aligned} \left\| \left[V, \sum_{k: d(j, k) \geq r} c_k(s) \gamma_k \right] \right\| &= \| [V, \gamma_j^{\text{free}}(s) - X_1(s)] \| \\ &\leq C' U (|s| + 1)^D e^{-\mu' |s|}. \quad (\text{B28}) \end{aligned}$$

$$\begin{aligned} \left\| \left[V, \sum_{k: d(j, k) < r} c_k \gamma_k \right] \right\| &= \| [V, X_1(s)] \| \\ &\leq C' U (|s| + 1)^D \max_k |c_k(s)| \\ &\leq C' U (|s| + 1)^D. \quad (\text{B29}) \end{aligned}$$

Here we have used Lemma 24 to obtain $\max_k |c_k(s)| \leq \|\gamma_j^{\text{free}}(s)\| = 1$. Adding up the two parts yields the desired bound by the triangle inequality. $\blacksquare$

This bound is useful because it is independent of the system size.

Lemma 14. We define $X_1(s)$ and $X_2(s, t)$ as follows:

$$\begin{aligned} X_1(s) &= \sum_{k: d(j, k) \leq r/2} c_k(s) \gamma_k, \\ X_2(s, t) &= i e^{iH'(t-s)} [V, X_1(s)] e^{-iH'(t-s)}, \end{aligned}$$

where H' consists of the Hamiltonian terms in H that are within r distance from j , and $c_k(s)$ is the same as in Lemma 13. Then $X_2(s, t)$ is supported on the ball with radius r centered at j , and there exists constant C that only depends on r_0, D such that

$$\begin{aligned} \| X_2(s, t) - e^{iH(t-s)} [V, \gamma_j^{\text{free}}(s)] e^{-iH(t-s)} \| &\leq C U (|s| + 1)^D e^{-r/2}, \quad (\text{B30}) \end{aligned}$$

if $r \geq \max\{4e^2 J |t| + 2r_0, 2e^2 J_0 |s|\}$. In the above $J_0, J > 0$ are the Lieb-Robinson velocities defined in Eqs. (26) and (27), respectively.

Proof. By the triangle inequality, we have

$$\begin{aligned} \| X_2(s, t) - i e^{iH(t-s)} [V, \gamma_j^{\text{free}}(s)] e^{-iH(t-s)} \| &\leq \underbrace{\| X_2(s, t) - i e^{iH(t-s)} [V, X_1(s)] e^{-iH(t-s)} \|}_I \\ &\quad + \underbrace{\| e^{iH(t-s)} [V, X_1(s)] e^{-iH(t-s)} - e^{iH(t-s)} [V, \gamma_j^{\text{free}}(s)] e^{-iH(t-s)} \|}_II. \quad (\text{B31}) \end{aligned}$$

For I, by the Lieb-Robinson bound, since $[V, X_1(s)]$ is supported on the ball of radius $r/2 + r_0$ centered at j , when $r \geq r/2 + r_0 + 2e^2 J |t|$, we have

$$I \leq C_1 e^{-r/2} \| [V, X_1(s)] \| \leq C_2 e^{-r/2} U (|s| + 1)^D,$$

where the second inequality comes from Eq. (B29). For II, by Lemma 13 we have

$$II = \| [V, X_1(s)] - [V, \gamma_j^{\text{free}}(s)] \| \leq C_3 e^{-r/2} U (|s| + 1)^D.$$

We thus have Eq. (B30) by combining the two inequalities for I and II above. $\blacksquare$

Corollary 10. Under the same assumptions as in Lemma 14, we have

$$\left\| \gamma_j^{\text{int}}(t) - \int_0^t X_2(s, t) ds \right\| \leq C e^{-r/2} U(|t| + 1)^{D+1}.$$

Here $\int_0^t X_2(s, t) ds$ is supported on the ball with radius r centered at j , satisfying

$$\left\| \int_0^t X_2(s, t) ds \right\| \leq C' U(|t| + 1)^{D+1}.$$

C and C' are constants that depend only on r_0, D . In the above $J_0, J > 0$ are the Lieb-Robinson velocities defined in Eqs. (26) and (27), respectively.

Proof. The first statement is a direct consequence of Lemma 14: one only needs to integrate over s on both sides of the inequality. The second statement is a direct consequence of Eq. (B29) likewise after integrating over s . ■

Lemma 15. Let

$$X_3(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-T}^T \check{F}_1(t, \omega) \int_0^t X_2(s, t) ds dt,$$

where $X_2(s, t)$ is defined in Lemma 14. Then $X_3(\omega)$ is supported on the ball with radius r centered at j , and with $T = (r - 2r_0)/(2e^2(2J + J_0))$, we have

$$\|X_3(\omega) - \tilde{A}_j^{\text{int}}(\omega)\| \leq C U e^{-\mu r} e^{-\beta\omega/4},$$

$$\|X_3(\omega)\|, \|\tilde{A}_j^{\text{int}}(\omega)\| \leq C' U e^{-\beta\omega/4},$$

for all $\omega \in \mathbb{R}$, and $C, C', \mu > 0$ are constants that depend only on r_0, D, β . In the above $J_0, J > 0$ are the Lieb-Robinson velocities defined in Eqs. (26) and (27), respectively.

Proof. By the triangle inequality,

$$\begin{aligned} \|X_3(\omega) - \tilde{A}_j^{\text{int}}(\omega)\| &\leq \underbrace{\left\| \frac{1}{\sqrt{2\pi}} \int_{|t| \geq T} \check{F}_1(t, \omega) \gamma_j^{\text{int}}(t) dt \right\|}_{\text{I}} \\ &\quad + \underbrace{\left\| \frac{1}{\sqrt{2\pi}} \int_{-T}^T \check{F}_1(t, \omega) \left(\int_0^t X_2(s, t) ds - \gamma_j^{\text{int}}(t) \right) dt \right\|}_{\text{II}}. \end{aligned}$$

For I, we have $\text{I} \leq C_1 U e^{-T^2/\beta^2} e^{-\beta\omega/4}$ for some constant C_1 that depends only on r_0, J_0, J, D because Corollary 10 tells

us that

$$\|\gamma_j^{\text{int}}(t)\| \leq C'_1 U(|t| + 1)^{D+1}.$$

for some constant C'_1 . For II, when $T = (r - 2r_0)/(2e^2(2J + J_0))$, we know that $r \geq \max\{4e^2 J|t| + 2r_0, 2e^2 J_0|s|\}$, and therefore by Corollary 10, we have

$$\left\| \gamma_j^{\text{int}}(t) - \int_0^t X_2(s, t) ds \right\| \leq C'_2 U(|t| + 1)^D e^{-r/2} e^{-\beta\omega/4}.$$

This implies

$$\text{II} \leq C_2 U e^{-r/2} e^{-\beta\omega/4},$$

where C_2 is a constant that depends only on r_0, J_0, J, D . Combining the two bounds we have

$$\begin{aligned} \|X_3(\omega) - \tilde{A}_j^{\text{int}}(\omega)\| &\leq C_1 U e^{-T^2/\beta^2} e^{-\beta\omega/4} + C_2 U e^{-r/2} e^{-\beta\omega/4}. \end{aligned}$$

By the fact that $T = (r - 2r_0)/(2e^2(2J + J_0))$, there exists constants $C, \mu > 0$ that only depends on β, r_0, J_0, J, D such that

$$\|X_3(\omega) - \tilde{A}_j^{\text{int}}(\omega)\| \leq C U e^{-\mu r} e^{-\beta\omega/4}.$$

The bound for $\|X_3(\omega)\|$ follows directly from the definition of $X_3(\omega)$ and the norm bound for $\int_0^t X_2(s, t) ds$ in Corollary 10. The bound for $\|\tilde{A}_j^{\text{int}}(\omega)\|$ then follows from the triangle inequality. ■

4. The perturbation in the dissipative part

The above result shows that $\tilde{A}_j^{\text{int}}(\omega)$ can be well approximated by local operators and can be bounded linearly in U . We will then use it to study the perturbation $H_{D,\text{int}}^{\text{parent}}$.

We first decompose $H_{D,\text{int}}^{\text{parent}}$, as defined in Eq. (52), into three parts:

$$H_{D,\text{int}}^{\text{parent}} = H_{LR} + H_{LL} + H_{RR}, \quad (\text{B32})$$

where

$$\begin{aligned}
H_{LR} &= \sum_j \int d\omega \eta(\omega) \Phi \left(L_{\tilde{A}_j(-\omega)} R_{\tilde{A}_j(-\omega)^\dagger} - L_{\tilde{A}_j^{\text{free}}(-\omega)} R_{\tilde{A}_j^{\text{free}}(-\omega)^\dagger} \right), \\
H_{LL} &= -\frac{1}{2} \sum_j \int d\omega \eta(\omega) \Phi \left(L_{\tilde{A}_j(-\omega)} \tilde{A}_j(\omega) - \tilde{A}_j^{\text{free}}(-\omega) \tilde{A}_j^{\text{free}}(\omega) \right), \\
H_{RR} &= -\frac{1}{2} \sum_j \int d\omega \eta(\omega) \Phi \left(R_{\tilde{A}_j(\omega)^\dagger} \tilde{A}_j(-\omega)^\dagger - \tilde{A}_j^{\text{free}}(\omega)^\dagger \tilde{A}_j^{\text{free}}(-\omega)^\dagger \right).
\end{aligned}$$

We will focus on H_{LR} because the other two terms can be dealt with in exactly the same way. For this term we can further decompose it into three parts

$$H_{LR} = H_{LR,1} + H_{LR,2} + H_{LR,3}, \quad (\text{B33})$$

where

$$\begin{aligned}
H_{LR,1} &= \sum_j \int d\omega \eta(\omega) \Phi \left(L_{\tilde{A}_j^{\text{int}}(-\omega)} R_{\tilde{A}_j^{\text{free}}(-\omega)^\dagger} \right), \\
H_{LR,2} &= \sum_j \int d\omega \eta(\omega) \Phi \left(L_{\tilde{A}_j^{\text{free}}(-\omega)} R_{\tilde{A}_j^{\text{int}}(-\omega)^\dagger} \right), \\
H_{LR,3} &= \sum_j \int d\omega \eta(\omega) \Phi \left(L_{\tilde{A}_j^{\text{int}}(-\omega)} R_{\tilde{A}_j^{\text{int}}(-\omega)^\dagger} \right).
\end{aligned}$$

In the above we have used $\tilde{A}_j(\omega) = \tilde{A}_j^{\text{int}}(\omega) + \tilde{A}_j^{\text{free}}(\omega)$ and the linearity of the mappings. We will again only focus on $H_{LR,1}$ because the other terms can be dealt with in exactly the same way.

Lemma 16. Let

$$Y_j(\omega) = \Phi \left(L_{\tilde{A}_j^{\text{int}}(-\omega)} R_{\tilde{A}_j^{\text{free}}(-\omega)^\dagger} \right),$$

and

$$Y_{j,r}(\omega) = \Phi \left(L_{X_3(-\omega)} R_{A^{\text{loc}}(-\omega)^\dagger} \right),$$

where $X_3(\omega)$ and $A^{\text{loc}}(\omega)$ are from Lemma 15 and Lemma 11, respectively, with r chosen to be the same value and ω as given. Then there exists constants $C, C', \mu > 0$ that depend only on β, r_0, D such that

$$\begin{aligned}
\|Y_j(\omega) - Y_{j,r}(\omega)\| &\leq CUe^{-\mu r} e^{\beta\omega/2}, \\
\|Y_j(\omega)\| &\leq C' U e^{\beta\omega/2},
\end{aligned}$$

and $Y_{j,r}(\omega)$ is supported on the ball of radius r centered at j [58].

Proof. We first have

$$\begin{aligned}
&\|Y_j(\omega) - Y_{j,r}(\omega)\| \\
&\leq \|\Phi \left(L_{\tilde{A}_j^{\text{int}}(-\omega) - X_3(-\omega)} R_{\tilde{A}_j^{\text{free}}(-\omega)} \right)\| \\
&\quad + \|\Phi \left(L_{X_3(-\omega)} R_{\tilde{A}_j^{\text{free}}(-\omega)^\dagger - A^{\text{loc}}(-\omega)^\dagger} \right)\| \\
&\leq \|\Phi \left(L_{\tilde{A}_j^{\text{int}}(-\omega)^\dagger - X_3(-\omega)} \right)\| \|\Phi \left(R_{\tilde{A}_j^{\text{free}}(-\omega)} \right)\| \\
&\quad + \|\Phi(L_{X_3(-\omega)})\| \|\Phi \left(R_{\tilde{A}_j^{\text{free}}(-\omega)^\dagger - A^{\text{loc}}(-\omega)^\dagger} \right)\| \\
&\leq \underbrace{\|\tilde{A}_j^{\text{int}}(-\omega) - X_3(-\omega)\|}_{\text{I}} \|\tilde{A}_j^{\text{free}}(-\omega)\| \\
&\quad + \underbrace{\|X_3(-\omega)\| \|\tilde{A}_j^{\text{free}}(-\omega) - A^{\text{loc}}(-\omega)\|}_{\text{II}},
\end{aligned}$$

where in the third inequality we have used $\|\Phi(L_A)\|, \|\Phi(R_A)\| \leq \|A\|$ as stated in Lemma 25.

By Lemma 15, we have $\|\tilde{A}_j^{\text{int}}(\omega) - X_3(\omega)\| \leq C_1 U e^{-\mu_1 r} e^{-\beta\omega/4}$ and $\|X_3(\omega)\| \leq C'_1 U e^{-\beta\omega/4}$. By Lemma 11, we have $\|\tilde{A}_j^{\text{free}}(\omega) - A^{\text{loc}}(\omega)\| \leq C_2 e^{-\mu_2 r} e^{-\beta\omega/4}$. By Eq. (42) we also have $\|\tilde{A}_j^{\text{free}}(\omega)\| \leq C'_2 e^{-\beta\omega/4}$. Therefore,

$$\text{I} \leq C_1 C'_2 U e^{-\mu_1 r} e^{\beta\omega/2}, \quad \text{II} \leq C'_1 C_2 U e^{-\mu_2 r} e^{\beta\omega/2}.$$

Adding up the two terms we have

$$\|Y_j(\omega) - Y_{j,r}(\omega)\| \leq CUe^{-\mu r} e^{\beta\omega/2}$$

for some constants $C, \mu > 0$. From Eq. (12) we also have $\|\tilde{A}_j(\omega)\| C_3 e^{\beta\omega/4}$ for some constant C_3 . Therefore,

$$\|Y_j(\omega)\| \leq C' e^{\beta\omega/2}$$

for some constant $C' > 0$ by Lemma 25. In the above all constants depend only on β, J_0, J, r_0, D , and $J_0, J > 0$ defined in Eqs. (26) and (27), respectively, depend only on r_0, D . ■

Lemma 17. $H_{LR,1}$ as defined in Eq. (B33) has (CU, μ) -decay as defined in Definition 8, for some constants $C, \mu > 0$ that depend only on β, r_0, D .

Proof. We can write

$$H_{LR,1} = \sum_j W_j,$$

where

$$W_j = \int d\omega \eta(\omega) Y_j(\omega),$$

in which $Y_j(\omega)$ is defined in Lemma 16. We then define

$$W_{j,r} = \int d\omega \eta(\omega) Y_{j,r}(\omega),$$

where $Y_{j,r}(\omega)$ is also defined in Lemma 16. Note that $W_{j,r}$ is supported on the ball with radius r centered at j . By Lemma 16, we have

$$\|W_j - W_{j,r}\| \leq \int d\omega \eta(\omega) \|Y_j(\omega) - Y_{j,r}(\omega)\| \leq C_1 U e^{-\mu r},$$

and $\|W_j\| \leq C_2 U$. Therefore W_j is $(CU/2, \mu)$ -quasi-local around j by Definition 7. By Lemma 2, we have that $H_{LR,1} = \sum_j W_j$ has (CU, μ) decay. In the above, $C, C_1, C_2, \mu > 0$ are all constants that depend only on β, J_0, J, r_0, D . Since $J_0, J > 0$ in Eqs. (26) and (27), respectively, depend only on r_0, D , all constants only depend on β, r_0, D . ■

Repeating this procedure for $H_{LR,2}$ and $H_{LR,3}$ allows us to show that H_{LR} has $(3C, \mu)$ decay. We then repeat it form H_{LL} and H_{RR} in Eq. (B32), which leads to the following result:

Lemma 18. $H_{D,\text{int}}^{\text{parent}}$ as defined in Eq. (52) has (CU, μ) decay as defined in Definition 8, for some constants $C, \mu > 0$ that depend only on β, r_0, D .

APPENDIX C: STRUCTURE OF THE COHERENT TERMS

In this section we show the coherent terms are quasilo-cal. The coherent term is given by

$$B = \sum_{j \in A} B_j = \sum_{j \in A} \int_{-\infty}^{\infty} b_1(t) \int_{-\infty}^{\infty} b_2(t') e^{i\beta H(t'-t)} \gamma_j \times e^{-2i\beta H t'} \gamma_j e^{i\beta H(t'+t)} dt' dt, \quad (C1)$$

where the functions $b_1(t)$ and $b_2(t')$ are given in Eq. (14). We can decompose any operator $A \in B(\mathcal{H})$ in the energy basis and regroup in Bohr frequency:

$$A = \sum_{E_1, E_2 \in \text{spec}(H)} P_{E_2} A P_{E_1} = \sum_{\nu \in \text{Bohr}(H)} \sum_{E_2 - E_1 = \nu} P_{E_2} A P_{E_1} =: \sum_{\nu \in \text{Bohr}(H)} A_\nu, \quad (C2)$$

where P_{E_1} and P_{E_2} are the projectors on the eigenstates with eigenvalues E_1 and E_2 of H . The Heisenberg evolution operator $A(t)$ can be expressed as

$$A(t) = e^{iHt} A e^{-iHt} = \sum_{\nu} e^{i\nu t} A_\nu. \quad (C3)$$

Using the decomposition, the coherent term can be expressed as

$$\begin{aligned} B_j &= \int_{-\infty}^{\infty} b_1(t) dt \int_{-\infty}^{\infty} b_2(t') dt' \gamma_j(\beta(t' - t)) \gamma_j(-\beta(t + t')) \\ &= \int_{-\infty}^{\infty} b_1(t) dt \int_{-\infty}^{\infty} b_2(t') dt' \sum_{\nu} e^{i\beta \nu(t' - t)} (\gamma_j)_\nu \sum_{\nu'} e^{-i\beta \nu'(t + t')} (\gamma_j)_{\nu'} \\ &= \sum_{\nu, \nu'} (\gamma_j)_\nu (\gamma_j)_{\nu'} \int_{-\infty}^{\infty} b_1(t) dt \int_{-\infty}^{\infty} b_2(t') dt' e^{i\beta \nu(t' - t)} e^{-i\beta \nu'(t + t')}. \end{aligned} \quad (C4)$$

As we show in Eq. (40), the parent Hamiltonian has as its component $\tilde{B}_j = \sigma^{1/4} B_j \sigma^{-1/4}$. Following the decomposition in Eq. (C4), we have

$$\tilde{B}_j = \sum_{\nu, \nu'} (\gamma_j)_\nu (\gamma_j)_{\nu'} e^{-\beta \nu/4} e^{\beta \nu'/4} \int_{-\infty}^{\infty} b_1(t) dt \int_{-\infty}^{\infty} b_2(t') dt' e^{i\beta \nu(t' - t)} e^{-i\beta \nu'(t + t')}. \quad (C5)$$

We define $F_2(v, v')$ such that $\tilde{B}_j = \sum_{v, v'} F_2(\beta v, \beta v') (\gamma_j)_v (\gamma_j)_{v'}$:

$$\begin{aligned} F_2(v, v') &= e^{v/4} e^{-v'/4} \int_{-\infty}^{\infty} b_1(t) dt \int_{-\infty}^{\infty} b_2(t') dt' e^{iv(t'-t)} e^{-iv'(t+t')} \\ &= e^{(v-v')/4} \int_{-\infty}^{\infty} b_1(t) e^{i(v'-v)t} dt \int_{-\infty}^{\infty} b_2(t') e^{-i(v+v')t'} dt' \\ &= 2\pi e^{(v-v')/4} \hat{b}_1(v-v') \hat{b}_2(v'+v). \end{aligned} \quad (C6)$$

We define the function $\check{F}_2(t, t')$, which we call the *kernel function*, to be the inverse Fourier transformation of $F_2(v, v')$:

$$\check{F}_2(t, t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' F_2(v, v') e^{ivt} e^{iv't'} \quad (C7)$$

so we have

$$\begin{aligned} \tilde{B}_j &= \frac{1}{2\pi} \sum_{v, v'} (\gamma_j)_v (\gamma_j)_{v'} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \check{F}_2(t, t') e^{-i\beta vt} e^{-i\beta v't'} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \check{F}_2(t, t') \gamma_j(\beta t) \gamma_j(-\beta t'). \end{aligned} \quad (C8)$$

Below we will show that the kernel function $\check{F}_2(t, t')$ decays rapidly as $|t|, |t'|$ increases.

Lemma 19. For the kernel function $\check{F}_2(t, t')$ defined in Eq. (C7), we have

$$|\check{F}_2(t, t')| \leq C e^{-\lambda(t^2+t'^2)^{1/2}},$$

for some universal constants $C, \lambda > 0$.

For the proof see Appendix H in the Appendix. Recall that in Eq. (48) we decomposed the coherent part of the Hamiltonian into $H_C^{\text{parent}} = H_{C, \text{free}}^{\text{parent}} + H_{C, \text{int}}^{\text{parent}}$. In Lemma 7. Therefore, we only need to show that $H_{C, \text{int}}^{\text{parent}}$ has (CU, μ) decay for some $C, \mu > 0$.

For each j , we decompose

$$\tilde{B}_j^{\text{int}} = \tilde{B}_j - \tilde{B}_j^{\text{free}}, \quad (C9)$$

where

$$\tilde{B}_j^{\text{free}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \check{F}_2(t, t') \gamma_j^{\text{free}}(\beta t) \gamma_j^{\text{free}}(-\beta t'). \quad (C10)$$

Therefore, we have

$$\begin{aligned} \tilde{B}_j^{\text{int}} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \check{F}_2(t, t') \\ &\quad \times \left(\gamma_j(\beta t) \gamma_j(-\beta t') - \gamma_j^{\text{free}}(\beta t) \gamma_j^{\text{free}}(-\beta t') \right). \end{aligned} \quad (C11)$$

We will first show that $\tilde{B}_j^{\text{int}}$ can be well approximated by an integral on the ball with radius T .

Lemma 20. For any $T > 0$, let

$$\begin{aligned} Z(T) &= \frac{1}{2\pi} \int_{t^2+t'^2 \leq T^2} dt dt' \check{F}_2(t, t') \\ &\quad \times \left(\gamma_j(\beta t) \gamma_j(-\beta t') - \gamma_j^{\text{free}}(\beta t) \gamma_j^{\text{free}}(-\beta t') \right). \end{aligned}$$

Then,

$$\|\tilde{B}_j^{\text{int}} - Z(T)\| \leq C' U e^{-\lambda' T},$$

for some constants $C', \lambda' > 0$ that depend only on r_0, D, β .

Proof. We first note that

$$\begin{aligned} &\gamma_j(\beta t) \gamma_j(-\beta t') - \gamma_j^{\text{free}}(\beta t) \gamma_j^{\text{free}}(-\beta t') \\ &= \gamma_j^{\text{int}}(\beta t) \gamma_j^{\text{free}}(-\beta t') + \gamma_j^{\text{free}}(\beta t) \gamma_j^{\text{int}}(-\beta t') \\ &\quad + \gamma_j^{\text{int}}(\beta t) \gamma_j^{\text{int}}(-\beta t'). \end{aligned}$$

Therefore, because $\|\gamma_j^{\text{free}}(s)\| \leq 1$, we have

$$\begin{aligned} &\|\gamma_j(\beta t) \gamma_j(-\beta t') - \gamma_j^{\text{free}}(\beta t) \gamma_j^{\text{free}}(-\beta t')\| \\ &\leq \|\gamma_j^{\text{int}}(\beta t)\| + \|\gamma_j^{\text{int}}(-\beta t')\| + \|\gamma_j^{\text{int}}(\beta t)\| \|\gamma_j^{\text{int}}(-\beta t')\|. \end{aligned}$$

By Corollary 10 we have

$$\|\gamma_j^{\text{int}}(\beta t)\|, \|\gamma_j^{\text{int}}(-\beta t')\| \leq C_1 U(|t| + 1)^{D+1}.$$

Therefore,

$$\|\gamma_j(\beta t)\gamma_j(-\beta t') - \gamma_j^{\text{free}}(\beta t)\gamma_j^{\text{free}}(-\beta t')\| \leq C_1 U(|t| + 1)^{D+1} + C_1 U(|t'| + 1)^{D+1} + C_1^2 U^2(|t| + 1)^{D+1}(|t'| + 1)^{D+1}.$$

By Lemma 19 we have $|\check{F}(t, t')| \leq C e^{-\lambda(t^2+t'^2)^{1/2}}$. Therefore,

$$\begin{aligned} \|\check{B}_j^{\text{int}} - Z(T)\| &\leq \int_{t^2+t'^2 > T^2} dt dt' |\check{F}(t, t')| (C_1 U(|t| + 1)^{D+1} + C_1 U(|t'| + 1)^{D+1} + C_1^2 U^2(|t| + 1)^{D+1}(|t'| + 1)^{D+1}) \\ &\leq C \int_{t^2+t'^2 > T^2} dt dt' e^{-\lambda(t^2+t'^2)^{1/2}} (C_1 U(|t| + 1)^{D+1} + C_1 U(|t'| + 1)^{D+1} + C_1^2 U^2(|t| + 1)^{D+1}(|t'| + 1)^{D+1}) \\ &\leq C' U e^{-\lambda' T}, \end{aligned} \quad (\text{C12})$$

for $U > 0$ below any universal constant. $\lambda' > 0$ is a constant that depends only on D , and $C' > 0$ is a constant that depends only on r_0, D, β . ■

We decompose further the $Z(t)$ introduced in Lemma 20 into three parts, similar to Eq. (B33),

$$Z(T) = Z_1(T) + Z_2(T) + Z_3(T), \quad (\text{C13})$$

where,

$$\begin{aligned} Z_1(T) &= \frac{1}{2\pi} \int_{t^2+t'^2 \leq T^2} dt dt' \check{F}(t, t') \gamma_j^{\text{int}}(\beta t) \gamma_j^{\text{int}}(-\beta t') \\ Z_2(T) &= \frac{1}{2\pi} \int_{t^2+t'^2 \leq T^2} dt dt' \check{F}(t, t') \gamma_j^{\text{int}}(\beta t) \gamma_j^{\text{free}}(-\beta t') \\ Z_3(T) &= \frac{1}{2\pi} \int_{t^2+t'^2 \leq T^2} dt dt' \check{F}(t, t') \gamma_j^{\text{free}}(\beta t) \gamma_j^{\text{int}}(-\beta t'). \end{aligned} \quad (\text{C14})$$

We will show that each of the terms can be well approximated by local operators. We will start with $Z_1(T)$.

Lemma 21. Let $X_1(t)$ and $X_2(s, t)$ be as defined in Lemma 13 and Lemma 14 respectively. We define

$$\begin{aligned} Z_1^{\text{loc}}(T) &= \frac{1}{2\pi} \int_{t^2+t'^2 \leq T^2} dt dt' \check{F}(t, t') \\ &\quad \times \int_0^{\beta t} X_2(s, t) ds X_1(-\beta t'), \end{aligned}$$

which is supported on the ball with radius r centered at j . Then

$$\|Z_1^{\text{loc}}(T) - Z_1(T)\| \leq C U (T + 1)^{D+1} (r + 1)^D e^{-r/2}, \quad (\text{C15})$$

and

$$\|Z_1(T)\| \leq C' U, \quad (\text{C16})$$

when $r \geq C_1(T + 1)$. Moreover, when setting $T = r/C_1 - 1$, we have

$$\|Z_1^{\text{loc}}(T) - Z_1(T)\| \leq C' U e^{-\mu r}. \quad (\text{C17})$$

Here $C, C_1, C', \mu > 0$ are all constants that only depend on r_0, D, β .

Proof. We first recall that by Corollary 10,

$$\left\| \gamma_j^{\text{int}}(\beta t) - \int_0^{\beta t} X_2(s, t) ds \right\| \leq C_2 e^{-r/2} U (|t| + 1)^{D+1},$$

and

$$\left\| \int_0^{\beta t} X_2(s, t) ds \right\| \leq C_3 U (|t| + 1)^{D+1}.$$

By the proof of Lemma 13, we also have

$$\|\gamma_j^{\text{free}}(-\beta t') - X_1(-\beta t')\| \leq C_4 (r + 1)^D e^{-r}.$$

All of the above require $r \geq J_1(|t| + 1)$. These results allow us to approximate $\gamma_j^{\text{int}}(\beta t) \gamma_j^{\text{int}}(-\beta t')$ with $\int_0^{\beta t} X_2(s, t) ds X_1(-\beta t')$. We have

$$\begin{aligned} &\left\| \gamma_j^{\text{int}}(\beta t) \gamma_j^{\text{int}}(-\beta t') - \int_0^{\beta t} X_2(s, t) ds X_1(-\beta t') \right\| \\ &\leq \left\| \gamma_j^{\text{int}}(\beta t) - \int_0^{\beta t} X_2(s, t) ds \right\| \|\gamma_j^{\text{free}}(-\beta t')\| \\ &\quad + \left\| \int_0^{\beta t} X_2(s, t) ds \right\| \|\gamma_j^{\text{free}}(-\beta t') - X_1(-\beta t')\| \\ &\leq C_5 U (T + 1)^{D+1} (r + 1)^D e^{-r/2}, \end{aligned}$$

when $t^2 + t'^2 \leq T^2$. Using the fact that $\int dt dt' |\check{F}_2(t, t')|$ is upper bounded by a constant, we can integrate the two sides of the above inequality to get Eq. (C15). Equation (C17) follows by setting $T = r/J_1 - 1$.

An upper bound for the norm $\|Z_1(T)\|$ can be obtained from the norm bound for $\gamma_j^{\text{int}}(t)$ (available from Corollary 10 by the triangle inequality) and $\|\gamma_j^{\text{int}}(-\beta t')\| \leq 1$, which after multiplying to an exponentially decaying kernel function $\check{F}_2(t, t')$ and integrating over t, t' yield (C16). ■

This procedure can be repeated for $Z_2(T)$ and $Z_3(T)$, yielding

$$\begin{aligned}\|Z_2(T) - Z_2^{\text{loc}}(T)\| &\leq C_2 U e^{-\mu_2 r}, \\ \|Z_3(T) - Z_3^{\text{loc}}(T)\| &\leq C_3 U^2 e^{-\mu_3 r},\end{aligned}\quad (\text{C18})$$

and

$$\|Z_2(T)\| \leq C_2' U, \quad \|Z_3(T)\| \leq C_3' U_2. \quad (\text{C19})$$

Here $Z_2^{\text{loc}}(T)$ and $Z_3^{\text{loc}}(T)$ are both supported on the ball with radius r centered at j . By adding up the three approximations to get

$$Z^{\text{loc}}(T) = Z_1^{\text{loc}}(T) + Z_2^{\text{loc}}(T) + Z_3^{\text{loc}}(T), \quad (\text{C20})$$

we then have

$$\|Z(T) - Z^{\text{loc}}(T)\| \leq C U e^{-\mu r}, \quad \|Z(T)\| \leq C' U. \quad (\text{C21})$$

In the above $C, C', C_2, C_2', C_3, C_3'$ are all constants that depend only on r_0, D . We are now ready to approximate $\tilde{B}_j^{\text{int}}$.

Lemma 22. The term $\tilde{B}_j^{\text{int}}$ can be approximated with a local operator $\tilde{B}_j^{\text{loc}}(r)$. More precisely, we have

$$\|\tilde{B}_j^{\text{int}} - \tilde{B}_j^{\text{loc}}(r)\| \leq C e^{-\mu r} U.$$

Here $\tilde{B}_j^{\text{loc}}(r)$ is supported on the ball with radius r centered at j . Moreover,

$$\|\tilde{B}_j\| \leq C' U.$$

C, C', μ are positive constants that depend only on r_0, D, β .

Proof. We define $\tilde{B}_j^{\text{loc}}(r) = Z^{\text{loc}}(r/J_1 - 1)$ where J_1 is the same as in Lemma 21 and $Z^{\text{loc}}(T)$ is defined in Eq. (C20). Combining Eq. (C21) with Lemma 20 we have the bound for $\|\tilde{B}_j^{\text{int}} - \tilde{B}_j^{\text{loc}}(r)\|$ by the triangle inequality. The bound for $\|\tilde{B}_j\|$ can be derived by the first bound and $\|Z(T)\| \leq C_1 U$ in Eq. (C21). ■

Corollary 11. The part of the parent Hamiltonian $H_{C,\text{int}}^{\text{parent}}$ defined in Eq. (48) has (CU, μ) decay as defined in Definition 8, where $C, \mu > 0$ depend only on r_0, D, β .

Proof. Since $\tilde{B} = \sum_j \tilde{B}_j$, and that $\tilde{B}_j^{\text{int}}$ is $(C_1 U, \mu)$ -quasi-local around j (Definition 7) as proved in Lemma 22,

by Lemma 2 we know that $\tilde{B}$ has $(2C_1 U, \mu)$ decay. Because $H_{C,\text{int}}^{\text{parent}} = \Phi(L_{\tilde{B}_j} - R_{\tilde{B}_j}^\dagger)$ and that locality is preserved in the mapping Φ by Remark 4, we have the (CU, μ) decay of $H_{C,\text{int}}^{\text{parent}}$. ■

We are then ready to prove Proposition 3:

Proof of Proposition 3. By Lemma 18 we know that $H_{D,\text{int}}^{\text{parent}}$ has $(C_1 U, \mu_1)$ decay, and by Corollary 11 we know that $H_{C,\text{int}}^{\text{parent}}$ has $(C_2 U, \mu_2)$ decay. As a result, by setting $C = C_1 + C_2$, $\mu = \min\{\mu_1, \mu_2\}$, we can see that $H_{\text{int}}^{\text{parent}} = H_{C,\text{int}}^{\text{parent}} + H_{D,\text{int}}^{\text{parent}}$ has (CU, μ) decay. ■

APPENDIX D: HOW NOT TO VECTORIZE FERMIONIC SUPEROPERATORS

A crucial step used in Ref. [10] is to “vectorize” a superoperator $\mathcal{C} : X \mapsto AXB$ into an operator $\text{vec}(\mathcal{C}) = A \otimes B^\top$. Now we apply it to fermionic superoperators

$$\mathcal{C}_1 : X \mapsto \gamma_1 X, \quad \mathcal{C}_2 : X \mapsto X \gamma_2.$$

They are then vectorized to be

$$\text{vec}(\mathcal{C}_1) = \gamma_1 \otimes I, \quad \text{vec}(\mathcal{C}_2) = I \otimes \gamma_2.$$

First for the operation $\top$, we can interpret it as the linear involution in the Clifford algebra (as opposed to the antilinear involution $\dagger$).

We then consider how one should interpret the tensor product $\otimes$. One may be tempted to equate $\gamma_1 \otimes I$ with γ_1 , and $I \otimes \gamma_2$ with γ_{n+2} , which is the “image” of γ_2 in an ancillary system. Adopting this approach, we have

$$\text{vec}(\mathcal{C}_1) = \gamma_1, \quad \text{vec}(\mathcal{C}_2) = \gamma_{n+2}.$$

Now we have a problem: while $\mathcal{C}_1$ and $\mathcal{C}_2$ commute with each other, $\text{vec}(\mathcal{C}_1)$ and $\text{vec}(\mathcal{C}_2)$ do not; in fact they anti-commute with each other. Therefore, this naive vectorization approach fails to preserve the algebraic relations between fermionic superoperators.

With a small modification one can make $\text{vec}(\mathcal{C}_1)$ and $\text{vec}(\mathcal{C}_2)$ commute with each other. Instead of the what was done above, we define

$$\text{vec}(\mathcal{C}_1) = \gamma_1, \quad \text{vec}(\mathcal{C}_2) = \gamma_{n+2}(-1)^N,$$

where N is the particle-number operator and $(-1)^N$ is the parity operator. This is in fact one of the insights of the third quantization approach [32]. However, we can see that this mapping maps an operator that is very local ($\mathcal{C}_2$) to something extremely nonlocal [$\gamma_{n+2}(-1)^N$ acts nontrivially on every mode except for mode $n+2$]. This will prevent us from applying the result in Ref. [31].

The approach used in Ref. [32] and in this work is based on the observation that $(-1)^N$ yields a factor of 1 when restricted to the even-parity sector. However, this does not mean we can treat $(-1)^N$ as if it does not exist: by applying single Majorana operators one will switch between the even- and odd-parity sectors. This is why a great deal of effort of Sec. V was made to prove things that seem obvious (but are not) from Ref. [10].

APPENDIX E: TRANSFORMING THE MAJORANA OPERATORS

Lemma 23. Let $H_0 = \sum_{jk} h_{jk} \gamma_j \gamma_k$, where the matrix (h_{jk}) is Hermitian and pure imaginary, then

$$e^{iH_0 t} \gamma_l e^{-iH_0 t} = \sum_{j=1}^n (e^{4iht})_{jl} \gamma_j. \quad (\text{E1})$$

Proof. We use the notation $\text{ad}_A(B) = [A, B]$,

$$\text{ad}_{H_0}(\gamma_l) = [H_0, \gamma_l] = \sum_{jk} h_{jk} [\gamma_j \gamma_k, \gamma_l]. \quad (\text{E2})$$

Using the commutation relation, we can calculate the commutator

$$\begin{aligned} [\gamma_j \gamma_k, \gamma_l] &= \gamma_j \gamma_k \gamma_l - \gamma_l \gamma_j \gamma_k \\ &= \gamma_j \gamma_k \gamma_l + \gamma_j \gamma_l \gamma_k - \gamma_j \gamma_l \gamma_k - \gamma_l \gamma_j \gamma_k \\ &= \gamma_j \{\gamma_k, \gamma_l\} - \{\gamma_j, \gamma_l\} \gamma_k \\ &= 2(\delta_{kl} \gamma_j - \delta_{jl} \gamma_k). \end{aligned} \quad (\text{E3})$$

Therefore,

$$\begin{aligned} \text{ad}_{H_0}(\gamma_l) &= 2 \left(\sum_j h_{jl} \gamma_j - \sum_k h_{lk} \gamma_k \right) \\ &= 2 \left(\sum_j h_{jl} \gamma_j + \sum_k h_{kl} \gamma_k \right) = 4 \sum_j h_{jl} \gamma_j. \end{aligned} \quad (\text{E4})$$

As a result we have

$$\text{ad}_{H_0}^r(\gamma_l) = \sum_j (4h)_{jl}^r \gamma_j \quad (\text{E5})$$

Using the Baker-Campbell-Hausdorff (BCH) formula we get

$$e^{iH_0 t} \gamma_l e^{-iH_0 t} = \sum_r \frac{1}{r!} \text{ad}_{H_0}^r(\gamma_l) = \sum_{j=1}^n (e^{4iht})_{jl} \gamma_j. \quad (\text{E6})$$

■

APPENDIX F: LINEAR COMBINATION OF MAJORANA OPERATORS

Lemma 24. For $x = (x_j), y = (y_j) \in \mathbb{R}^{2n}$, let

$$A = \sum_j x_j \gamma_j + i \sum_j y_j \gamma_j. \quad (\text{F1})$$

Then

$$\|A\|^2 = \|x\|_2^2 + \|y\|_2^2 + 2\sqrt{\|x\|_2^2 \|y\|_2^2 - (x^\top y)^2}. \quad (\text{F2})$$

Proof. First introduce vectors $a, b \in \mathbb{R}^{2n}$ such that $\|a\|_2 = \|b\|_2 = 1$, $a^\top b = 0$, and

$$(x \ y) = (a \ b) \underbrace{\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}}_M.$$

Then we define ω_1 and ω_2 to be

$$\omega_1 = \sum_j a_j \gamma_j, \quad \omega_2 = \sum_j b_j \gamma_j.$$

We can directly verify that ω_1 and ω_2 are Majorana operators satisfying the canonical anticommutation relation. We can rewrite A as

$$A = M_{11} \omega_1 + M_{21} \omega_2 + i(M_{12} \omega_1 + M_{22} \omega_2).$$

This allows us to compute

$$A^\dagger A = \|M\|_F^2 + i \begin{pmatrix} \omega_1 & \omega_2 \end{pmatrix} \begin{pmatrix} 0 & \det(M) \\ -\det(M) & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Therefore, we can see that $A^\dagger A$ has two eigenvalues $\|M\|_F^2 \pm 2 \det(M)$. We can further compute that

$$\|M\|_F^2 = \|x\|_2^2 + \|y\|_2^2, \quad \det(M)^2 = \|x\|_2^2 \|y\|_2^2 - (x^\top y)^2.$$

Therefore, we have Eq. (F2). ■

APPENDIX G: NORM PRESERVATION IN THIRD QUANTIZATION

In Sec. IV A we have introduced the superoperators L_A and R_A for an operator A , and how these superoperators can be mapped back to operators in an enlarged Hilbert space using Φ defined in Definition 6. In this section we will show that the operator spectral norm is preserved in this process. More precisely, we have the following.

Lemma 25. For $A \in B(\mathcal{H})$ with fixed parity (even or odd as defined in Definition 4), we have

$$\|\Phi(L_A)\| = \|A\|, \quad (\text{G1})$$

and

$$\|\Phi(R_A)\| \leq \|A\|. \quad (\text{G2})$$

Before we present the proof we will first prove some useful lemmas:

Lemma 26. For any $A \in B(\mathcal{H})$, $\varphi \circ R_{A^\dagger} \circ \varphi^{-1} = (\varphi \circ R_A \circ \varphi^{-1})^\dagger$.

Proof. We can write $A = \sum_{\alpha \in \{0,1\}^{2n}} A_\alpha \gamma_1^{\alpha_1} \gamma_2^{\alpha_2} \cdots \gamma_{2n}^{\alpha_{2n}}$. By Eqs. (24), (23), and $\Phi(R_I) = I$, we have

$$\begin{aligned} \varphi \circ R_A \circ \varphi^{-1} &= \sum_{\alpha} A_\alpha \varphi \circ R_{\gamma_{2n}}^{\alpha_{2n}} \cdots R_{\gamma_2}^{\alpha_2} R_{\gamma_1}^{\alpha_1} \circ \varphi^{-1} \\ &= \sum_{\alpha} A_\alpha (-i)^{|\alpha|} \hat{\gamma}_{4n}^{\alpha_{2n}} (-1)^{\alpha_{2n}\hat{n}} \cdots \hat{\gamma}_4^{\alpha_2} (-1)^{\alpha_2\hat{n}} \\ &\quad \times \hat{\gamma}_2^{\alpha_1} (-1)^{\alpha_1\hat{n}}. \end{aligned}$$

Therefore,

$$\begin{aligned} (\varphi \circ R_A \circ \varphi^{-1})^\dagger &= \sum_{\alpha} A_\alpha^* i^{|\alpha|} (-1)^{\alpha_1\hat{n}} \hat{\gamma}_2^{\alpha_1} (-1)^{\alpha_2\hat{n}} \hat{\gamma}_4^{\alpha_2} \cdots (-1)^{\alpha_{2n}\hat{n}} \hat{\gamma}_{4n}^{\alpha_{2n}} \\ &= \sum_{\alpha} A_\alpha^* (-i)^{|\alpha|} \hat{\gamma}_2^{\alpha_1} (-1)^{\alpha_1\hat{n}} \hat{\gamma}_4^{\alpha_2} (-1)^{\alpha_2\hat{n}} \cdots \\ &\quad \times \hat{\gamma}_{4n}^{\alpha_{2n}} (-1)^{\alpha_{2n}\hat{n}}, \end{aligned}$$

where for the second equality we have used $(-1)^{\hat{n}} \hat{\gamma}_{2j} = -\hat{\gamma}_{2j} (-1)^{\hat{n}}$.

On the other hand,

$$\begin{aligned} \varphi \circ R_{A^\dagger} \circ \varphi^{-1} &= \sum_{\alpha} A_\alpha^* \varphi \circ (-i)^{|\alpha|} R_{\gamma_1}^{\alpha_1} R_{\gamma_2}^{\alpha_2} \cdots R_{\gamma_{2n}}^{\alpha_{2n}} \circ \varphi^{-1} \\ &= \sum_{\alpha} A_\alpha^* (-i)^{|\alpha|} \hat{\gamma}_2^{\alpha_1} (-1)^{\alpha_1\hat{n}} \hat{\gamma}_4^{\alpha_2} (-1)^{\alpha_2\hat{n}} \cdots \\ &\quad \times \hat{\gamma}_{4n}^{\alpha_{2n}} (-1)^{\alpha_{2n}\hat{n}}. \end{aligned}$$

We therefore have $\varphi \circ R_{A^\dagger} \circ \varphi^{-1} = (\varphi \circ R_A \circ \varphi^{-1})^\dagger$. ■

Corollary 12. For $A \in B(\mathcal{H})$, if A has even parity, then $\Phi(R_{A^\dagger}) = \Phi(R_A)^\dagger$. If A has odd parity, then $\Phi(R_{A^\dagger}) = -\Phi(R_A)^\dagger$.

Proof. The even case follows directly from the definition of Φ in Definition 6 and Lemma 26. For the odd case, we have

$$\begin{aligned} \Phi(R_A)^\dagger &= \left(\varphi \circ R_A \circ \varphi^{-1} (-1)^{\hat{n}} \right)^\dagger = (-1)^{\hat{n}} \varphi \circ R_{A^\dagger} \circ \varphi^{-1} \\ &= -\varphi \circ R_{A^\dagger} \circ \varphi^{-1} (-1)^{\hat{n}} = -\Phi(R_{A^\dagger}). \end{aligned} \quad \blacksquare$$

Lemma 27. If $A \in B(\mathcal{H})$ is positive semidefinite with even parity, then $\Phi(R_A)$ is also positive semidefinite.

Proof. Since A is positive semidefinite, there exists $Q \in B(\mathcal{H})$ with even parity such that $A = QQ^\dagger$. Therefore, by Lemma 26

$$\begin{aligned} \Phi(R_A) &= \varphi \circ R_{Q^\dagger} R_Q \circ \varphi^{-1} \\ &= (\varphi \circ R_{Q^\dagger} \circ \varphi^{-1}) (\varphi \circ R_Q \circ \varphi^{-1}) \\ &= (\varphi \circ R_Q \circ \varphi^{-1})^\dagger (\varphi \circ R_Q \circ \varphi^{-1}) \end{aligned}$$

and is therefore positive semidefinite. ■

Proof of Lemma 25. We can write

$$A = \sum_{\alpha \in \{0,1\}^{2n}} A_\alpha \gamma_1^{\alpha_1} \gamma_2^{\alpha_2} \cdots \gamma_{2n}^{\alpha_{2n}}.$$

By Eq. (25), we can explicitly write down

$$\Phi(L_A) = \sum_{\alpha \in \{0,1\}^{2n}} A_\alpha \hat{\gamma}_1^{\alpha_1} \hat{\gamma}_3^{\alpha_2} \cdots \hat{\gamma}_{4n-1}^{\alpha_{2n}}.$$

We can see that $\Phi(L_A)$ simply comes from replacing the Majorana operators in A by a different set of Majorana operators, and therefore we have Eq. (G1).

For $\Phi(R_A)$, first we note that $\|A\|^2 I - A^\dagger A$ is positive semidefinite. Therefore, by Lemma 27, we know that

$$\begin{aligned} \Phi(R_{\|A\|^2 I - A^\dagger A}) &= \|A\|^2 I - \Phi(R_{A^\dagger A}) \\ &= \|A\|^2 I - \Phi(R_A) \Phi(R_A)^\dagger \end{aligned}$$

is also positive semidefinite. We therefore have Eq. (G2). In the above, for the case where A has odd parity, we have used

$$\begin{aligned} \Phi(R_A) \Phi(R_A)^\dagger &= (\varphi \circ R_A \circ \varphi^{-1}) (-1)^{\hat{n}} (-1)^{\hat{n}} (\varphi \circ R_A \circ \varphi^{-1})^\dagger \\ &= \varphi \circ R_A R_{A^\dagger} \circ \varphi^{-1} = \Phi(R_{A^\dagger A}). \end{aligned} \quad \blacksquare$$

APPENDIX H: DECAY OF THE KERNEL FUNCTION IN THE COHERENT PART

We will first prove some elementary properties of $b_1(t)$ defined in Eq. (14). We recall that

$$b_1(t) = 2\sqrt{\pi} e^{1/8} \left(\frac{1}{\cosh(2\pi t)} *_t \sin(-t) e^{-2t^2} \right).$$

We will need to use the lemma below:

Lemma 28. If two real functions $f(t)$ and $g(t)$ satisfy $|f(t)| \leq C_1 e^{-\lambda_1 |t|}$ and $|g(t)| \leq C_2 e^{-\lambda_2 |t|}$ with positive constants $C_1, C_2, \lambda_1, \lambda_2$, then

$$|(f * g)(t)| \leq 2C_1 C_2 \left(\frac{e^{-\lambda_2 |t|/2}}{\lambda_1} + \frac{e^{-\lambda_1 |t|/2}}{\lambda_2} \right). \quad (\text{H1})$$

Proof. The convolution can be bounded by

$$\begin{aligned} |(f * g)(t)| &= \left| \int_{-\infty}^{+\infty} ds f(s) g(t-s) \right| < \int_{-\infty}^{+\infty} |f(s) g(t-s)| ds \\ &= C_1 C_2 \int_{-\infty}^{+\infty} e^{-\lambda_1 |s|} e^{-\lambda_2 |t-s|} ds \\ &= C_1 C_2 \left(\int_{|s-t| \leq |t|/2} + \int_{|s-t| > |t|/2} \right) e^{-\lambda_1 |s|} e^{-\lambda_2 |t-s|} ds. \end{aligned} \quad (\text{H2})$$

For the integral for $|s-t| \leq |t|/2$, we have

$$\begin{aligned} \int_{|s-t| \leq |t|/2} e^{-\lambda_1 |s|} e^{-\lambda_2 |t-s|} ds \\ \leq e^{-\lambda_1 |t|/2} \int e^{-\lambda_2 |t-s|} ds \leq \frac{2e^{-\lambda_1 |t|/2}}{\lambda_2}. \end{aligned}$$

For the integral for $|s-t| > |t|/2$, we have

$$\begin{aligned} \int_{|s-t| > |t|/2} e^{-\lambda_1 |s|} e^{-\lambda_2 |t-s|} ds \\ \leq e^{-\lambda_2 |t|/2} \int e^{-\lambda_1 |s|} ds \leq \frac{2e^{-\lambda_2 |t|/2}}{\lambda_1}. \end{aligned}$$

Combining the two parts yields the desired inequality. ■

Lemma 29. Let $b_1(t)$ be as defined in Eq. (14), then there exist universal constants $C, \lambda > 0$ such that

$$b_1(t - i\delta) \leq C \cosh(\delta) e^{2\delta^2} e^{-\lambda t}.$$

Proof. Note that $b_1(t)$ is the convolution between two parts:

$$h_1(t) = 2\sqrt{\pi} e^{1/8} \frac{1}{\cosh(2\pi t)}, \quad h_2(t) \sin(-t) e^{-2t^2}.$$

Therefore,

$$b_1(t - i\delta) = \int h_1(s) h_2(t - i\delta - s) ds.$$

Let $h_{2,\delta}(t) = h_2(t - i\delta)$, then $b_1(t) = (h_1 * h_{2,\delta})(t)$. Therefore, we only need to prove the decay properties of $h_1(t)$ and $h_{2,\delta}(t)$ and then apply Lemma 28.

For $h_1(t)$, we can readily get universal constants $C_1, \lambda_1 > 0$ such that $|h_1(t)| \leq C_1 e^{-\lambda_1 |t|}$. Therefore, we only need to focus on $h_{2,\delta}(t)$. Note that

$$|h_{2,\delta}(t)| = |\sin(-t + i\delta) e^{-2(t-i\delta)^2}| \leq \cosh(\delta) e^{2\delta^2} e^{-2t^2}.$$

Because e^{-2t^2} decays faster than any exponential function, we also have universal constants $C_2, \lambda_2 > 0$ such that $|h_{2,\delta}(t)| \leq C_2 \cosh(\delta) e^{2\delta^2} e^{-\lambda_2 |t|}$. The result then immediately follows from Lemma 28. ■

We are now ready to prove the decay property of $\check{F}_2(t, t')$ defined in Eq. (C7), which we rewrite here

$$\check{F}_2(t, t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' F_2(v, v') e^{ivt} e^{iv't'}.$$

Proof of Lemma 19. We perform a change of variable as follows:

$$\begin{aligned} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v \\ v' \end{pmatrix}, \\ \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} t \\ t' \end{pmatrix}. \end{aligned}$$

This allows us to write

$$\begin{aligned} \check{F}_2 \left(\frac{t_1 + t_2}{\sqrt{2}}, \frac{t_2 - t_1}{\sqrt{2}} \right) &= \frac{1}{2\pi} \int dv_1 \int dv_2 F_2 \left(\frac{v_1 + v_2}{\sqrt{2}}, \frac{v_2 - v_1}{\sqrt{2}} \right) e^{i(v_1 t_1 + v_2 t_2)} \\ &= \int dv_1 e^{iv_1(t_1 - i\sqrt{2}/4)} \hat{b}_1(\sqrt{2}v_1) \int dv_2 e^{iv_2 t_2} \hat{b}_2(\sqrt{2}v_2) \\ &= \frac{1}{2} b_1(t_1/\sqrt{2} - i/4) b_2(t_2/\sqrt{2}). \end{aligned}$$

Since the exponential decay of $b_1(t_1/\sqrt{2} - i/4)$ from Lemma 29 and the exponential decay of $b_2(t_2/\sqrt{2})$ from the definition of $b_2(t)$ in Eq. (14), we have the exponential decay of $\check{F}_2((t_1 + t_2)/\sqrt{2}, (t_2 - t_1)/\sqrt{2})$ in $|t_1| + |t_2|$. Due to norm equivalence we then have the exponential decay of $\check{F}_2(t, t')$ in $(t^2 + t'^2)^{1/2}$. We therefore have

$$|\check{F}_2(t, t')| \leq C e^{-\lambda(t^2 + t'^2)^{1/2}},$$

for some universal constants $C, \lambda > 0$. ■

APPENDIX I: TABLE OF SYMBOLS

Symbol	Meaning
Λ	The physical lattice whose sites each contain a Dirac fermion
D	The physical lattice dimension
$d(x, y)$	The Euclidean distance between lattice sites x and y
γ_j	The j th Majorana operator
$\mathfrak{s}(j)$	The lattice site that contains the j th Majorana mode
$\hat{\gamma}_j$	The j th third quantized a -Majorana operator
$\mathfrak{s}'(j)$	The lattice site that contains the j th a -Majorana mode
$\mathcal{H}$	The Hilbert space of n Dirac fermions
$B(\mathcal{H})$	The operator algebra on $\mathcal{H}$
$\mathcal{H}'$	The enlarged Hilbert space of $2n$ Dirac fermions
$B(B(\mathcal{H}))$	The operator algebra on $B(\mathcal{H})$ (or the algebra of superoperators)
$\mathcal{H}_{\text{even/odd}}$	The even- and odd-parity sector of $\mathcal{H}$ (Definition 3)
$B(\mathcal{H})_{\text{even/odd}}$	The subspace space spanned by Majorana monomials of even and odd degree (Definition 4)
H	The fermionic Hamiltonian (1)
β	The inverse temperature
σ	The Gibbs state $\sigma \propto e^{-\beta H}$
H_0	The noninteracting (quadratic or free) part of H
V	The interacting part of H
r_0	The maximal range at which fermions can interact
U	The interaction strength
J	The Lieb-Robinson velocity for H (26)
J_0	The Lieb-Robinson velocity for H_0 (27)
$\mathcal{L}$	The Lindbladian we study (11)
φ	The isometry from $B(\mathcal{H})$ to $\mathcal{H}'$ used for third quantization (17)
Φ	The third quantization map that preserves locality (Definition 6)
$\tilde{\Phi}$	The map $B(B(\mathcal{H})) \rightarrow B(\mathcal{H}')$ that incorporates σ (31)
L_A/R_A	Left and right multiplication with A (22)
H^{parent}	The parent Hamiltonian (34) [explicit form given in Eq. (39)]
H_0^{parent}	The noninteracting part of the parent Hamiltonian (53)
V^{parent}	The interacting part of the parent Hamiltonian (53)

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- [58] More precisely, because $Y_{j,r}(\omega) \in B(\mathcal{H}')$ is an operator of a fermions, it acts nontrivially only on the a -fermion modes within distance r from Majorana mode j .