

Theory of Quantum-Enhanced Interferometry with General Markovian Light Sources

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Quantum optical systems comprising quantum emitters interacting with engineered optical modes generate non-classical states of light that can be used as resource states for quantum-enhanced interferometry. However, outside of well-controlled systems producing either single-mode states (e.g. Fock states or squeezed states) or highly symmetric multi-mode states (e.g. superradiant states), their potential for quantum advantage remains uncharacterized. In this work, we develop a framework to analyze quantum enhanced interferometry with general Markovian quantum light sources. First, by connecting the quantum Fisher information (QFI) of the photons emitted by a source to the multi-time correlation functions and the source dynamics, we elucidate the link between the level structure and spectrum of the source to a potential quantum advantage in interferometry. Second, we analyze optimal measurement protocols that can be used to achieve this quantum advantage with experimentally available optical elements. In particular, we show that tunable optical elements with Kerr non-linearity can always be harnessed to implement the optimal measurement for any given source. Simultaneously, we also outline general conditions under which linear optics and photodetection is enough to implement the optimal measurement.

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I. INTRODUCTION

Leveraging quantum many-body correlations is well known to provide a quadratic advantage in sensing [1–4]. Using classical sensors with N probes, the precision for estimating an unknown parameter is limited to the standard quantum limit (SQL) which scales as $1/\sqrt{N}$. However, employing probes in a highly correlated state, such as a Greenberger-Horne-Zeilinger (GHZ) state [5], allows us to surpass this limit and achieve Heisenberg-limited (HL) scaling of $1/N$ [4,6,7]. A specific and practically relevant application of quantum-enhanced sensing is interferometry, where the probes are photons emitted from a non-classical source, and the parameter to be sensed is an unknown phase. This phase is typically applied in one arm of a Mach-Zehnder interferometer (MZI). The interferometer is then illuminated with the probe photons, and the photons at the MZI output are detected to infer the

unknown phase. Such a scheme, using “classical” photonic states, such as coherent states, senses the unknown phase at the standard quantum limit. In contrast, quantum photonic states, such as NOON states [8] or twin Fock states [9], can achieve the Heisenberg limit.

However, while several theoretical proposals exist for generating quantum states such as NOON states or Fock states [10–18], they largely require strong photon blockade as well as a high degree of controllability in the quantum emitters [19–25], which is hard to achieve at optical frequencies [26,27]. This has limited the number photons that can be prepared in such quantum states [28–31].

Rather than asking which states saturate the Heisenberg limit and then finding ways to prepare them, a practically oriented approach is to ask which quantum-light sources are both simple and able to deliver a quantum advantage in metrology (but not necessarily Heisenberg scaling). Notably, significant experimental efforts have focused on efficiently generating squeezed states [32–34] that go beyond the standard quantum limit [35,36] and are generated by strongly pumping bulk optical non-linearities [37–40]. Another recent line of work has theoretically analyzed the metrological potential of photonic states generated by the collective superradiant decay in the waveguide Dicke limit [41] and have shown that these

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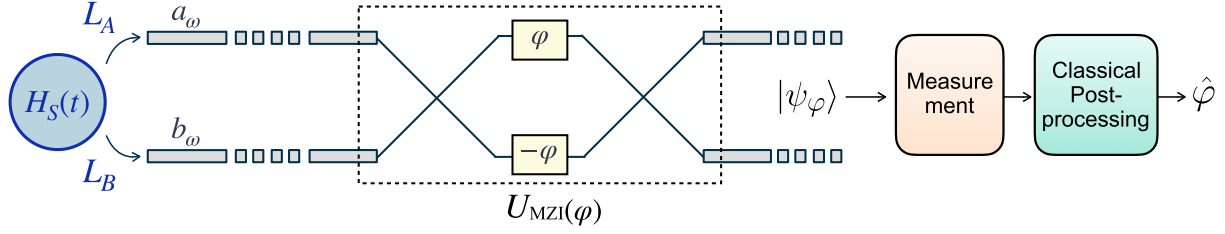


FIG. 1. Schematic depiction of the setup considered in this paper. A source emits photons into input ports of the MZI through the jump operators L_A and L_B . The generated photonic state passes through a MZI that includes an unknown sensing parameter, φ and the output state $|\psi_\varphi\rangle$ then depends on φ . This state is then subsequently measured, and the measurement outcome is classically post-processed to output an estimate $\hat{\varphi}$ of φ .

states achieve a Heisenberg-limited scaling with the number of photons and are only $\sim 18\%$ worse than Fock states [42].

Going beyond individual examples, it would be desirable to assess more generally the metrological potential of different types of quantum-optical light sources. An important quantity to calculate is the quantum Fisher information (QFI) [43], which characterizes the maximum sensitivity the interferometer can achieve [44,45]. A key obstacle here is that the QFI for photonic states is, in general, challenging to compute since it usually requires being able to calculate the full photonic state, which often has a large number of photons in possibly multiple modes. While this question has been studied in the context of spectroscopy with continuous-field measurements [46–48], computing QFI for interferometry has been largely limited to either single-mode states [4] or certain specific highly symmetric multi-mode states [42]. Furthermore, the QFI only provides an upper bound on the sensitivity achievable in the spectroscope. Whether an optically implementable measurement protocol can be devised to achieve this maximal sensitivity also remains an important open question and has been answered only for very specific set of photonic states [42,49–51].

In this paper, we develop a framework to address these problems in a setup where a general source emits photons into each arm of the Mach-Zehnder interferometer to sense an unknown phase φ [Fig. 1]. The only assumption that we make on the source is that its coupling with the output photon field is well described within the Markov approximation. We explicitly connect the QFI of the emitted photons to their two-time correlation functions, which can be computed by only tracking the internal dynamics of the source. Unlike previous works [42], we do not require an explicit computation of the output many-body photonic state, which allows us to numerically analyze a much larger class of photon sources [52–55]. Furthermore, using this result for the QFI, we then analyze how the source spectrum impacts the possibility of quantum metrological advantage and demonstrate that the photons generated by driving a source with a single ground state will always exhibit a SQL scaling. However, sources with multiple

ground states can generate photons with HL scaling. For instance, we show that a simple π -level system with a continuous-wave drive can generate photons with quantum metrological advantage.

Finally, we explore the optimal measurement protocol that is needed to achieve the QFI-limited sensitivity. We show that, irrespective of the source, such a measurement can always be implemented with a dynamically driven non-linear resonator and photodetectors. Furthermore, since it is experimentally much easier to implement linear optics, we also study the optimality of measurements that can be implemented with only dynamically controllable linear optical elements and photodetectors and outline conditions when such measurements can be optimal.

II. QFI FOR GENERAL LIGHT SOURCES

A. Formalism

1. Setup

The interferometric setup that we consider throughout this paper is shown in Fig. 1—a source of photons emits into the two input ports (labeled as port A and port B) of the MZI, which has the unknown phase φ to be sensed. After passing through the MZI, the photons are in a φ -dependent state $|\psi_\varphi\rangle$ which is then measured and the measurement outcome is classically post-processed to produce an estimate of the unknown phase φ . We model the input ports of the MZI as one-dimensional propagating fields (e.g., fields in a waveguide mode or a collimated free space beam) with annihilation operators a_ω and b_ω corresponding to the frequency ω . Within the Markov approximation [56], the source-port Hamiltonian can be expressed as

$$H(t) = H_S(t) + H_P + V_{SP}, \quad (1a)$$

where $H_S(t)$ is a possibly time-dependent Hamiltonian acting on the source, H_P is the Hamiltonian describing photon propagation in the ports, i.e.,

$$H_P = \sum_{x \in \{a,b\}} \int_{-\infty}^{\infty} \omega \tilde{x}_\omega^\dagger \tilde{x}_\omega d\omega, \quad (1b)$$

and

$$V_{SP} = \sum_{x \in \{a,b\}} \int_{-\infty}^{\infty} (\tilde{x}_\omega^\dagger L_X + \text{H.c.}) \frac{d\omega}{\sqrt{2\pi}}, \quad (1c)$$

is the interaction Hamiltonian between the source and port. Here L_A, L_B are the operators, acting only on the Hilbert space of the source, through which the source couples to the two output ports. Equation (1) follows the Markov approximation for the model by (i) assuming that the frequencies in the port extend from $-\infty$ to ∞ , with the addition of fictitious negative frequencies being justified when the natural resonant frequency of the source is much larger than other frequency scales in the problem and (ii) assuming that the source interacts identically with each frequency ω in the output ports.

It is convenient to introduce the time-domain annihilation operator for the port, $x_\tau = \int_{\mathbb{R}} \tilde{x}_\omega e^{-i\omega\tau} d\omega / \sqrt{2\pi}$, which can be interpreted as destroying excitations at the time τ in the port. Working in the interaction picture with respect to the port Hamiltonian H_P , we obtain the source-port Hamiltonian $H_I(t)$:

$$H_I(t) = H_S(t) + \sum_{x \in \{a,b\}} (x_t L_X^\dagger + \text{H.c.}).$$

Assuming that the source is initially in state $|\phi_{S,i}\rangle$ and the ports are initially in a vacuum state, and finally projecting the source onto the state $|\phi_{S,f}\rangle$, the state of the light channel after time T is expressed as

$$|\psi\rangle = \frac{1}{\mathcal{N}} \langle \phi_{S,f} | U_I(T, 0) | \phi_{S,i}, \text{vac} \rangle, \quad (2a)$$

where

$$U_I(T, 0) = \vec{T} \exp \left(\int_0^T H_I(s) ds \right), \quad (2b)$$

with $\vec{T}$ being the time-ordering operator that orders times in decreasing order and $\mathcal{N}$ chosen to ensure that $|\psi\rangle$ is normalized. The projection onto a final state $|\phi_{S,f}\rangle$ maybe performed explicitly via a projective measurement on the source and post-selecting on the measurement outcome. More realistically, in an experiment, we would often wait for a sufficiently long time T for the source to have decayed into its ground state which, if unique, would disentangle from the photons and thus effectively projecting the source onto its ground state. Finally, while we consider the more general setting of the same source emitting into both the ports, in many experimental setups

(such as twin-Fock state interferometry [9]), it is more common to consider two independent sources emitting separately into the two ports. In this case, $H_S(t) = H_A(t) \otimes I_B + I_A \otimes H_B(t)$ and the operator L_A will act only on the Hilbert space of the source coupling to port A and the operator L_B will act only on the Hilbert space of the source coupling to port B. Consequently, the state of photons $|\psi\rangle$ will be a separable state in between the two ports i.e. $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$.

The ports then pass through an MZI which implements a unitary transformation dependent on the phase φ that is to be sensed and can be expressed as

$$U_{\text{MZI}}^\dagger(\varphi) \begin{bmatrix} a_t \\ b_t \end{bmatrix} U_{\text{MZI}}(\varphi) = \begin{bmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{bmatrix} \begin{bmatrix} a_t \\ b_t \end{bmatrix}. \quad (3)$$

Consequently, the final photonic state at the output of the MZI is given by

$$|\psi_\varphi\rangle = U_{\text{MZI}}(\varphi) |\psi\rangle, \quad (4)$$

which depends on the phase φ . In the next subsection, we start from this expression for the state $|\psi_\varphi\rangle$ and compute its QFI with respect to φ . Moreover, in Sec. B we extend the results of this subsection to general linear optical systems, but here we restrict attention to the balanced MZI case.

2. Computing QFI efficiently

The QFI of the φ -dependent state $|\psi_\varphi\rangle$ evaluated at $\varphi = \varphi_0$ is formally given by [57]

$$\text{QFI} = 4 \left(\left\| \frac{\partial}{\partial \varphi} |\psi_\varphi\rangle \right\|^2 - \langle \psi_\varphi | \frac{\partial}{\partial \varphi} |\psi_\varphi\rangle \right)_{\varphi=\varphi_0}. \quad (5)$$

As is typical while studying quantum enhanced interferometry, we will consider the QFI at $\varphi = 0$ since the measurement of a non-zero φ can always be done in two steps—first making an approximate estimate of φ and then biasing the MZI around $\varphi = 0$ using the estimated phase [58–60].

One approach to developing an explicit expression for QFI, which has been adopted in Ref. [42], is to first calculate the wave functions

$$\psi_\varphi(t_1, t_2 \cdots t_n; s_1, s_2 \cdots s_m) = \langle \text{vac} | \prod_{i=1}^n a_{t_i} \prod_{i=1}^m b_{s_i} | \psi_\varphi \rangle,$$

associated with $|\psi_\varphi\rangle$, which can be obtained entirely in terms of the source's effective Hamiltonian $H_{\text{eff}}(t) = H_S(t) - i \sum_{x \in \{a,b\}} L_x^\dagger L_x / 2$, using input-output formalism [25,61], and then evaluate QFI. The complexity of evaluating $\psi_\varphi(t_1, t_2 \cdots t_n; s_1, s_2 \cdots s_m)$ grows exponentially with n, m in the absence of any special symmetries in

the source, making it hard to employ this approach beyond very special cases [42].

We take an alternative route and first express QFI in terms of the correlation functions of $|\psi_\varphi\rangle$. We begin by noting that the MZI unitary in Eq. (4) can be considered as being generated by the time-dependent Hamiltonian [62]:

$$H_{\text{MZI}}(t, \varphi) = 2 \tan\left(\frac{\varphi}{2}\right) h(t) \quad \text{with} \quad h(t) = i(a_t^\dagger b_t - a_t b_t^\dagger). \quad (6)$$

Since $|\psi_\varphi\rangle = \vec{T} \exp(-i \int_0^T H_{\text{MZI}}(s, \varphi) ds) |\psi\rangle$, we can now use Duhamel's formula [63] to evaluate $\partial|\psi_\varphi\rangle/\partial\varphi$ at $\varphi = 0$:

$$\frac{\partial}{\partial\varphi} |\psi_\varphi\rangle|_{\varphi=0} = -i H_d |\psi\rangle \quad \text{where} \quad H_d = \int_0^T h(s) ds. \quad (7)$$

It then follows from Eq. (5) that $\text{QFI} = 4(\langle H_d^2 \rangle - \langle H_d \rangle^2)$ or equivalently:

$$\begin{aligned} \text{QFI} = & 8 \int_0^T \int_0^T \text{Re}(\mathcal{C}_{a,b;a}^{(2)}(t, s) - \mathcal{C}_{b,b;a}^{(2)}(t, s)) dt ds \\ & + 4 \int_0^T (\mathcal{C}_{a;a}^{(1)}(t) + \mathcal{C}_{b;b}^{(1)}(t)) dt \\ & + 16 \left| \int_0^T \text{Im}(\mathcal{C}_{a;b}^{(1)}(t)) dt \right|^2, \end{aligned} \quad (8)$$

where $\mathcal{C}_{x^1 \dots x^n; y^1 \dots y^n}^{(n)}(t_1 \dots t_n)$, for $x^i, y^i \in \{a, b\}$ is an n -point correlation function given by

$$\mathcal{C}_{x^1 \dots x^n; y^1 \dots y^n}^{(n)}(t_1 \dots t_n) = \left\langle \prod_{i=1}^n x_{t_i}^{i\dagger} \prod_{i=1}^n y_{t_i}^i \right\rangle.$$

Equation (8) expresses the QFI of the emitted photonic state in terms of its correlation functions—as we detail in Appendix A1, this also yields a method to extract the QFI from two-photon correlation functions without necessarily having to perform the full sensing measurement to measure the unknown phase.

From the perspective of numerical calculations of QFI it is convenient to express it in terms of the internal dynamics of the source [64], which in turn is captured by the Lindbladian $\mathcal{L}_S(t)$:

$$\mathcal{L}_S(t) = -i[H_S(t), \cdot] + \sum_{X \in \{A, B\}} \mathcal{D}_{L_X}, \quad (9)$$

where $\mathcal{D}_L(X) = LXL^\dagger - \{L^\dagger L, X\}/2$, and the associated channel $\mathcal{E}_S(t, s) = \vec{T} \exp(\int_s^t \mathcal{L}_S(\tau) d\tau)$. In Appendix A2, we show that this can be accomplished by using the input-output formalism [65] and the quantum regression theorem

[66,67]. In particular, we show that the correlator $\mathcal{C}_{x^1 \dots x^n; y^1 \dots y^n}^{(n)}(t_1 \dots t_n)$ can be expressed as

$$\begin{aligned} \mathcal{C}_{x^1 \dots x^n; y^1 \dots y^n}^{(n)}(t_1 \dots t_n) \\ = \frac{1}{\mathcal{N}^2} \langle \phi_{S,f} | \left[\prod_{i=1}^n \mathcal{E}_S(t_{i-1}, t_i) \mathcal{M}_i \right] \mathcal{E}_S(t_n, 0) (\rho_{S,i}) | \phi_{S,f} \rangle, \end{aligned} \quad (10)$$

where we have assumed $T = t_0 \geq t_1 \geq t_2 \geq \dots \geq t_n \geq 0$, $\rho_{S,i} = |\phi_{S,i}\rangle\langle\phi_{S,i}|$ and $\mathcal{M}_i(\cdot) = L_{Y^i}(\cdot)L_{X^i}^\dagger$. Furthermore, the normalization constant $\mathcal{N}$ can also be expressed as $\mathcal{N} = (\langle \phi_{S,f} | \mathcal{E}_S(T, 0) (\rho_{S,i}) | \phi_{S,f} \rangle)^{1/2}$.

We emphasize that, for a source with D levels, the computation of the QFI from Eqs. (8) and (10) requires time that scales as $D^2 T^2$, whereas computing the QFI via first computing the output photon wave-packet would typically require time that scales exponentially with T . An alternative route also to performing the QFI computation efficiently for this problem would be to formulate an effective two-sided master equation by following the formalism in Refs. [46–48,68]. These works have focused on analyzing the sensitivity of continuous measurement for the spectroscopy problem, i.e., where the parameter to be sensed is encoded in the Hamiltonian of the source (e.g., resonant frequency of the emitting atoms). In Sec. C, we make this connection explicit by showing that the parameter φ , although corresponding to a linear-optical element applied outside the source, can be equivalently viewed as a parameter governing the jump operators with no linear-optical element, thereby allowing us to apply the framework of Refs. [46–48].

3. Accounting for losses and decoherence

Thus far, we have only considered sources which are coupling only to the ports of the MZI and do not lose photons into any additional channels. In many experimental settings, this may not exactly be the case—the source might emit into multiple output channels which may correspond to different propagation directions, or even channels corresponding to decohering processes (such as non-radiative photon absorption from the source, or dephasing). When the source couples to additional loss channels, its dynamics is described instead by the Lindbladian

$$\mathcal{L}_S(t) = -i[H_S(t), \cdot] + \sum_{X \in \{A, B\}} \mathcal{D}_{L_X} + \sum_{\alpha} \mathcal{D}_{K_{\alpha}}, \quad (11)$$

where the jump operators $K_1, K_2 \dots$ model photon emission into additional channels. When the photons emitted into these channels are not discarded (for e.g., if the different channels correspond to photon emission in

different propagation directions and these photons are optically collected), then the expression for the QFI in Eqs. (8) and (10) continue to hold. However, when the photons in the additional channels are discarded, the result given in Eqs. (8) and (10) no longer accurately captures the QFI of the MZI output, since it is derived under the assumption that the photons emitted are in a pure-state. In the presence of the loss channels, however, the photons in the output ports would be in a mixed state for which computing the QFI is a significantly harder task. However, Eqs. (8) and (10) still provide the QFI for a purification of the mixed state of the photons in the output ports, and consequently still sets an upper bound on the QFI of the photons in the output ports [43].

This upper bound can be supplemented with a lower bound by computing the sensitivity of different measurements on the MZI output. In particular, for a Hermitian measurement operator O , the sensitivity for the estimation of φ from the photonic state is defined by

$$\xi_\varphi(O) = \frac{\partial_\varphi \langle O \rangle^2}{\langle O^2 \rangle - \langle O \rangle^2}, \quad (12)$$

where the expectation is taken with respect to the photonic state at the output of the MZI. When the measurement is optimal, the sensitivity is exactly equal to the QFI. For suboptimal measurements, the sensitivity furnishes a lower bound on the QFI. Similar to the computation of the QFI in Eqs. (8) and (10), we show in Sec. A 3 that for selected measurement operators O (such as the photon number operator or photon-number parity operator), the sensitivity in Eq. (12) can be computed efficiently by tracking the evolution of the source.

III. EXAMPLES

As examples of applying the formalism introduced in the previous section, we consider some paradigmatic systems used as quantum light sources and study the QFI of the emitted photons. In all of our examples, we will consider the setting of two identical and independent sources emitting into the two input ports of the MZI. More specifically, the system Hamiltonian will be assumed to be of the form $H_S(t) = H(t) \otimes I + I \otimes H(t)$, the jump operators are of the form $L_A = L \otimes I$, $L_B = I \otimes L$ and the initial and final source states are of the form $|\phi_{S,i}\rangle = |\phi_i\rangle \otimes |\phi_i\rangle$, $|\phi_{S,f}\rangle = |\phi_f\rangle \otimes |\phi_f\rangle$.

A. Driven cavity, 2-level and π -level source

Cavity source We first consider a driven single-mode cavity with annihilation operator c , initialized in the vacuum state. The source Hamiltonian is given by $H = \Omega(c + c^\dagger)$, where Ω is the drive amplitude, and the coupling to the output port is described by the jump operator $L = \sqrt{\kappa}c$ with κ being the cavity decay-rate

[Fig. 2(a)]. Because the dynamics is linear and the input field is vacuum, the output field of each source is a coherent state for all times. Consequently, the emitted-photon QFI scales linearly with the emission time T (equivalently, linearly with the mean photon number). In particular, for a coherent output one obtains the simple form $\text{QFI} = 8\bar{N}(T)$ where $\bar{N}(T)$ is the mean number of photons emitted by the source into its output port over time T (an explicit expression is given in Appendix E). Figure 2(b) shows the linear scaling of the QFI with T . Moreover, since the output is coherent, photon-number measurement is optimal: the sensitivity of photon-number measurement matches the QFI, as shown in Fig. 2(c).

Two-level source Next, we consider a continuously driven two-level system [Fig. 2(d)]. The source Hamiltonian is given by $H = \Omega\sigma + \text{H.c.}$, where $\sigma = |g\rangle\langle e|$ and Ω is the drive amplitude, and the coupling to the output port is described by the jump operator $L = \sqrt{\gamma}\sigma$ where γ is the emitter decay-rate. We take the initial and final states of the two-level emitter to be the ground state i.e., $|\phi_i\rangle = |\phi_f\rangle = |g\rangle$. Figure 2(e) shows the QFI numerically for several values of Ω : we observe that in the long-time regime $\gamma T \gg 1$, the QFI scales linearly with time. This can be seen to be a consequence of the emitter emitting photons that become uncorrelated if the emission times are temporally separated by more than $1/\gamma$. The absence of long-time correlations thus results in the QFI being proportional to the number of emitted photons, which itself grows linearly with T . Furthermore, in the short-time regime, we see oscillations in the QFI when $\Omega \gg \gamma$ reflecting the underlying Rabi oscillations of the driven system.

Since the photons emitted by this system have only short-time correlations, we can expect that a photon-number measurement to be a good measurement. In Fig. 2(f), we plot the sensitivity of $n_{\text{ph}} = \int_0^T a_t^\dagger a_t dt$ computed using Eq. (12): we see that its long-time scaling is linear in T , thereby matching the scaling of the QFI with T .

π -level source We next consider a driven π -level system with two excited states $|e_1\rangle, |e_2\rangle$ and two ground states $|g_1\rangle, |g_2\rangle$, with both the transitions ($|e_1\rangle \rightarrow |g_1\rangle$ and $|e_2\rangle \rightarrow |g_2\rangle$) emitting collectively through the jump operator $L = \sqrt{\gamma}(|g_1\rangle\langle e_1| + |g_2\rangle\langle e_2|)$ [Fig. 2(g)]. We drive the two transitions coherently but with different phases—this is modelled by a Hamiltonian

$$H = \Omega(\sigma_1 + \text{H.c.}) + \Omega_0(\sigma_2 e^{i\theta} + \text{H.c.}), \quad (13)$$

where $\sigma_i = |g_i\rangle\langle e_i|$ and θ is the phase-difference between the drives on the two transitions. We take the initial and final states for the source to be $|\phi_i\rangle = |\phi_f\rangle = (|g_1\rangle + |g_2\rangle)/\sqrt{2}$. Figure 2(h) shows numerically calculated QFI as a function of T —we see that for $\theta \neq 0$ the QFI

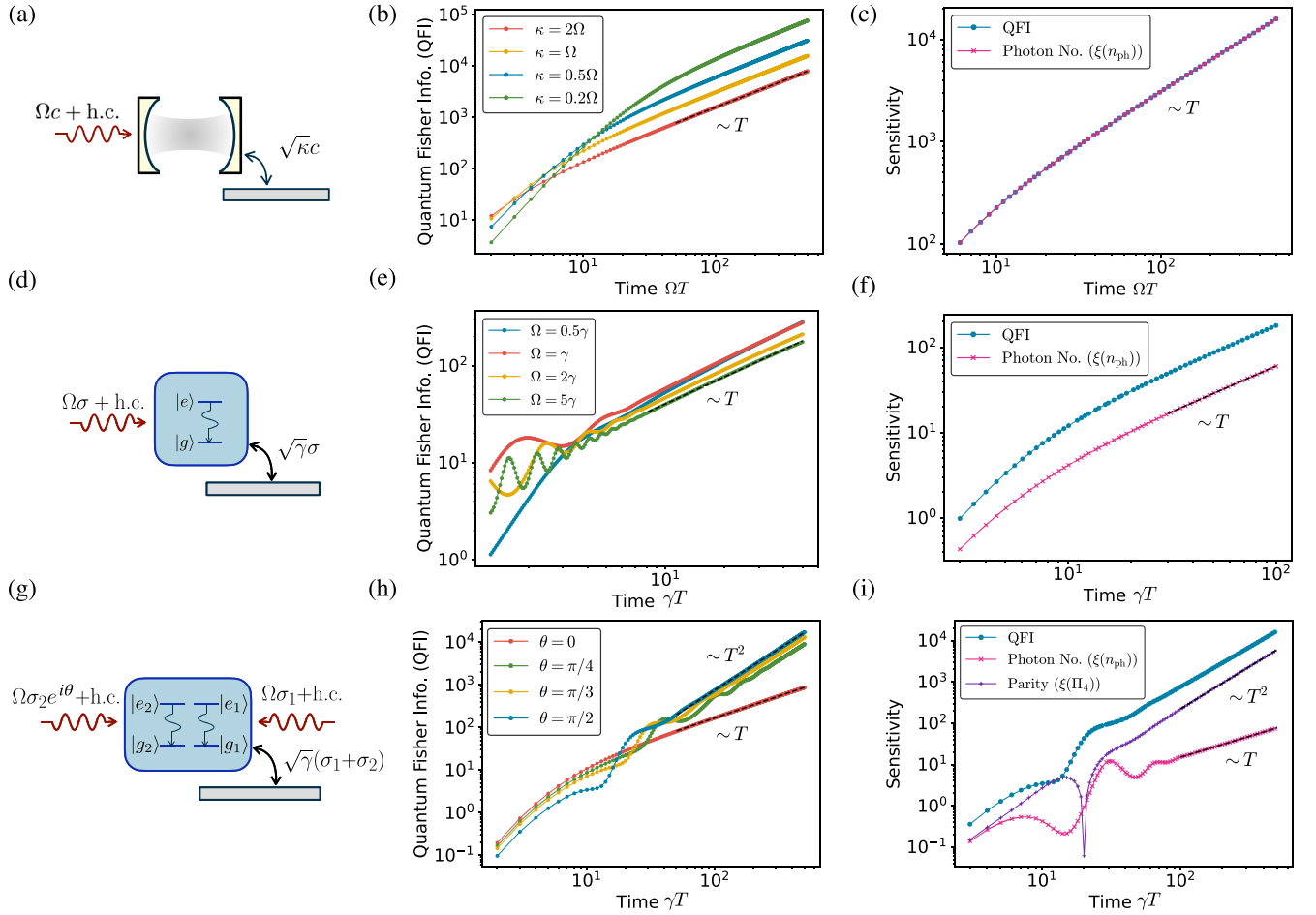


FIG. 2. (a) Schematic of the driven cavity set up, where the system is driven by a laser with amplitude Ω and emits photons into the port at a decay rate κ . (b) Scaling of the QFI with time for different values of κ/Ω . (c) QFI, and the sensitivity of the photon number measurement for $\kappa = 2\Omega$. The QFI, and measurement sensitivity both scale linearly with T in the large time limit. (d) Schematic of the 2-level system setup, where the system is driven by a laser and emits photons into the port at a decay rate γ . (e) Scaling of the QFI with time for the 2-level system. The QFI scales linearly with time for different values of Ω . (f) Simulation of the QFI, and photon number sensitivity with time for the 2-level system with $\Omega = 0.2\gamma$. (g) Schematic of the π -level system setup, where the system is driven by two lasers with amplitude Ω and phase difference θ , emitting photons into the port at a decay rate γ . (h) Scaling of the QFI with time for the π -level system with $\Omega = 0.2\gamma$. The QFI scales quadratically with time for non-zero θ , while for $\theta = 0$ it scales linearly. (i) Simulation of the QFI, photon number sensitivity and the sensitivity of the parity-like measurement with time for the π -level system with $\theta = \pi/2$.

exhibits Heisenberg-like scaling $\sim T^2$, while for $\theta = 0$ it scales only linearly with T . The QFI, when γT is large, can also be analytically calculated (see Appendix E), to obtain

$$\text{QFI} = \frac{128\gamma^4\Omega_0^4}{(\gamma^2 + 8\Omega_0^2)^4} T^2 \sin^2\theta + O(T), \quad (14)$$

which agrees with Fig. 2(h). This can physically be understood as follows: for the π -level source, if initialized in and projected on the state $(|g_1\rangle + |g_2\rangle)/\sqrt{2}$, the emitted photonic state $|\psi\rangle$ is a macroscopic superposition (i.e., “cat” like state) of two photonic states: $|\phi\rangle$ and $|\phi_\theta\rangle$, which are the photonic states that would have been emitted by a two-level system with drive Ω_0 and $\Omega_0 e^{i\theta}$ respectively. For $\theta \neq 0$,

$|\phi\rangle, |\phi_\theta\rangle$ become asymptotically orthogonal to each other as $T \rightarrow \infty$, and their macroscopic superposition inherits long-time correlations that make them useful for quantum-enhanced interferometry. When $\theta = 0$, the two states $|\phi\rangle, |\phi_\theta\rangle$ become identical and thus the emitted photonic state is the same as the state emitted by a driven two-level system which has a QFI scaling only linearly with T .

Next, we analyze concrete measurements in Fig. 2(i). We focus on the case of $\theta = \pi/2$ which as per Eq. (14) yields the highest QFI for large γT . We note that if $|\phi\rangle = \sum_{n=0}^{\infty} |\phi_n\rangle$, where $|\phi_n\rangle$ is the n -photon component of $|\phi\rangle$, then $|\phi_\theta\rangle = \sum_{n=0}^{\infty} e^{in\theta} |\phi_n\rangle$; thus, when $\theta = \pi/2$, the state $|\phi\rangle + |\phi_\theta\rangle$ has no component on the $n = 2, 6, 10, 14 \dots$ photon subspaces. This motivates us to

consider the generalized parity-operator Π_4 given by

$$\Pi_4 = f(n_{\text{ph}}), \quad \text{where } f(n) = \begin{cases} -1 & \text{if } \exists k: n = 4k + 2, \\ 1 & \text{otherwise.} \end{cases} \quad (15)$$

This operator can also be written as linear combination of exponentials of n_{ph} :

$$\Pi_4 = \frac{1}{2} \left(I + e^{i\frac{\pi}{2}n_{\text{ph}}} - e^{i\pi n_{\text{ph}}} + e^{i\frac{3\pi}{2}n_{\text{ph}}} \right). \quad (16)$$

By construction, when the MZI phase $\varphi = 0$, measuring this operator would yield +1 with probability 1, whereas a non-zero phase φ in the MZI would result in an outcome of -1 with some probability dependent on φ . Figure 2(i) shows that the sensitivity associated with Π_4 : we observe that, like the QFI, this sensitivity also grows quadratically with T thus showing that Π_4 is a good-measurement. Figure 2(i) also shows the sensitivity of photon-number observable n_{ph} which only scales linearly with T despite the QFI showing a quadratic scaling.

While the π -level source is a good toy example to show how a quadratically growing QFI can arise, it also

motivates a concrete experimental implementation. In particular, a two-level emitter with an additional spin degree of freedom (e.g., two-level transitions in a quantum dot [69] or in color centers [70,71]) naturally realizes the π -level structure [Fig. 3(a)]. The Hilbert space of this system is spanned by two excited states $|e_1\rangle = |e, \uparrow\rangle, |e_2\rangle = |e, \downarrow\rangle$ and two ground states $|g_1\rangle = |g, \uparrow\rangle, |g_2\rangle = |g, \downarrow\rangle$. Since photon-emission cannot change the spin-degree of freedom, the jump operator modeling photon-emission is given by $L = \sqrt{\gamma}\sigma \otimes I = \sqrt{\gamma}(\sigma_1 + \sigma_2)$ where $\sigma = |g\rangle\langle e|, \sigma_i = |g_i\rangle\langle e_i|$.

To achieve the quadratic scaling of the QFI with T , the two transitions ($|e, \uparrow\rangle \rightarrow |g, \uparrow\rangle$ and $|e, \downarrow\rangle \rightarrow |g, \downarrow\rangle$) need to be driven asymmetrically. Imparting a different phase on the drives applied on the two transitions is not directly possible since a laser field cannot couple to the spin-degree of freedom. An alternative way to create this asymmetry is to apply a magnetic field that produces different Zeeman splittings on the spin manifold in the excited and ground states [Fig. 3(a)], thus making the transition frequency of $|e, \uparrow\rangle \rightarrow |g, \uparrow\rangle$ different from $|e, \downarrow\rangle \rightarrow |g, \downarrow\rangle$. This is modeled by the Hamiltonian

$$H = \omega_0 |e\rangle\langle e| \otimes I + \Omega(\sigma \otimes I + \text{H.c.}) + \sum_{a \in \{e, g\}} \frac{\Delta_a}{2} |a\rangle\langle a| \otimes (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|), \quad (17)$$

where Δ_e, Δ_g are the Zeeman-splittings corresponding to the excited and ground state. In Fig. 3(b), we simulate the QFI as a function of time for different values of Δ_e/Δ_g . As expected, when $\Delta_e = \Delta_g$ the two transitions are identical and the QFI scales linearly with T . However, for $\Delta_e \neq \Delta_g$ the QFI exhibits quadratic scaling in the long-time limit.

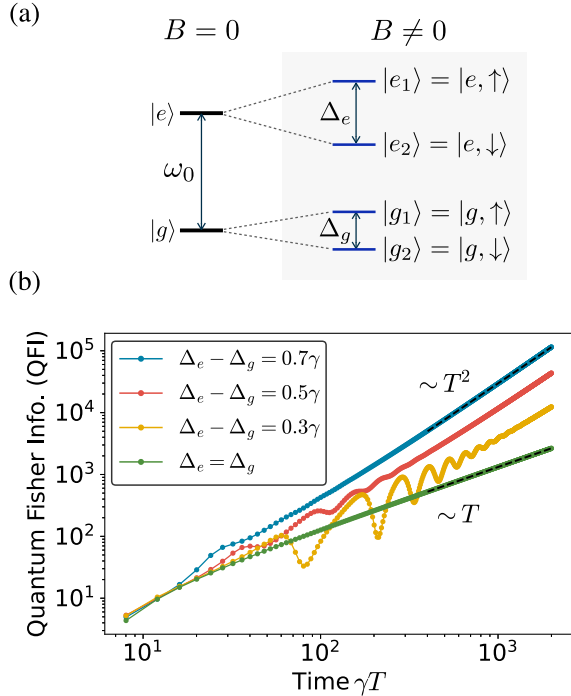


FIG. 3. (a) Spin-based implementation of the π -level system: a two-level optical transition with spin sublevels yields two spin-conserving transitions $|e_1\rangle \rightarrow |g_1\rangle$ and $|e_2\rangle \rightarrow |g_2\rangle$. A magnetic field induces Zeeman splittings, producing distinct detunings Δ_e and Δ_g for the two levels. (b) QFI of the emitted photons versus time for several detuning differences $\Delta_e - \Delta_g$, with $\Delta_e = \gamma$ and $\Omega = \gamma/2$.

B. Dissipative Tavis-Cummings and Dicke model

Next, we consider a multi-emitter cavity-QED system as a photon source: we consider N two-level emitters coupling collectively to a cavity mode which in turn emits photons into the output port [Fig. 4(a)]. This model describes a number of recently developed solid-state quantum optical systems [72–75] that can potentially be used as quantum light sources. Assuming the emitters to be identical, the Hamiltonian of the source is given by the Tavis-Cummings model [72,76]

$$H = g \sum_{i=1}^N (c^\dagger \sigma_i + \sigma_i^\dagger c), \quad (18)$$

where g is the cavity-emitter coupling strength and the jump operator coupling to the output port is given by $\sqrt{\kappa}c$ where κ is the cavity decay rate. We will analyze the setting where all the emitters are excited at $t = 0$ to the state $|e\rangle^{\otimes N}$

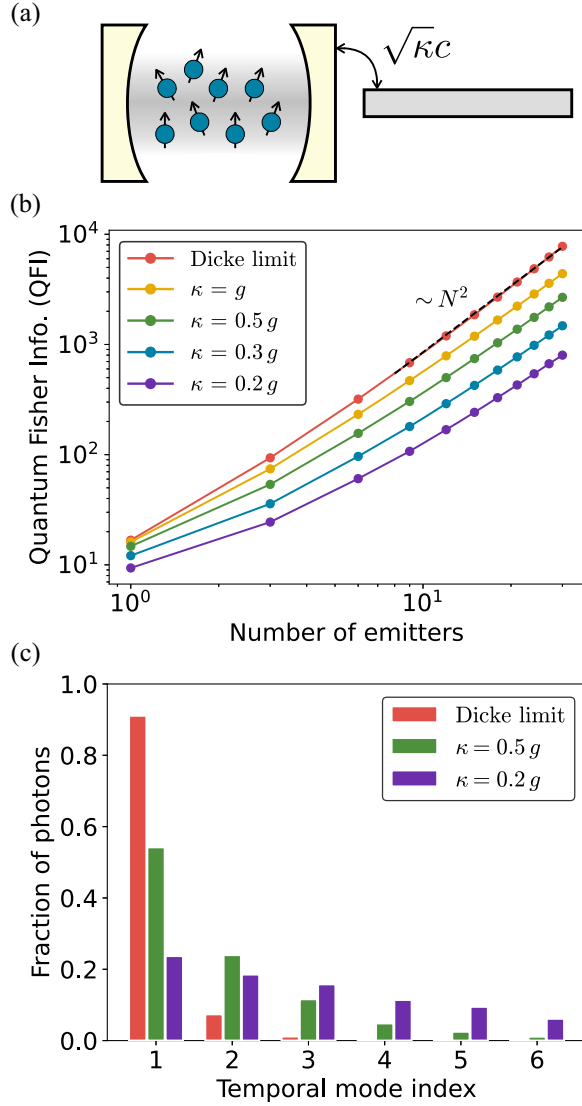


FIG. 4. (a) Tavis-Cummings model setup. All atoms are initially excited and interact with a cavity mode with an interaction strength g . The annihilation operator of the cavity mode is denoted as c . Photons are emitted from the cavity into the port at a decay rate κ . (b) Scaling of the QFI with the number of emitters for the Dicke and Tavis-Cummings setup. The Dicke limit is defined by taking $\kappa \rightarrow \infty$ with $g^2/\kappa = \gamma$. (c) Occupations of the first six orthogonal temporal modes of the output photonic state for 10 number of emitters. The Dicke emission is strongly dominated by a single mode, indicating a state close to a single-mode Fock state. In the Tavis-Cummings case, the output becomes more multi-mode as κ/g is reduced.

and subsequently decay and emit N photons into the output port and decay to the ground state $|g\rangle^{\otimes N}$.

In Fig. 4(b), we plot the QFI as a function of the number of emitters, N (which is also equal to the number of emitted photons). We find that the QFI scales quadratically with N . In the bad cavity regime (i.e., $\kappa \rightarrow \infty$), which is described by the Dicke model, this quadratic scaling agrees with the QFI calculation performed in Ref. [42].

While the QFI seems to have a quadratic scaling for all values of g/κ , we see that, surprisingly, the QFI becomes worse as the coupling between the emitters and the cavity is increased, i.e., weak coupling regime is more favorable. To explain this behavior, we follow Ref. [77] and analyze the temporal-mode distribution of the emitted photons: we obtain the temporal eigenmodes of the emitted photons by first computing the first-order correlator $G^{(1)}(t, t') = \kappa \langle c^\dagger(t) c(t') \rangle$ and then computing its eigendecomposition: the eigenvalues of $G^{(1)}(t, t')$ measure the relative occupation of different eigenmodes. Figure 4(c) shows these eigenvalues for different g/κ . In the bad-cavity/Dicke-limit, we see that one temporal mode is dominantly occupied over other modes thus making the photonic state closer to a single-mode Fock state which is known to be optimal for quantum-enhanced interferometry [9, 77]. As g/κ is increased, the emitted photons are more spread out into several temporal modes. This multi-mode nature of the emitted photons can be attributed to the polaritonic splittings that develop when the emitters and the cavity mode are strongly coupled and results in a lower QFI.

C. Lossy driven 2-level and π -level source

Finally, we revisit the examples studied in Sec. III A when unwanted loss channels are taken into account. As described in Sec. II A 3, the output photonic state is mixed and the computation of QFI is harder. However, it still remains possible to compute upper and lower bounds on the QFI. In particular, the QFI expressions developed in Sec. II A 2 provide us with an upper bound on the actual QFI while a lower bound can be derived by computing the sensitivity of specific measurement operators. We numerically evaluate both bounds in both the examples of driven 2-level and π -level systems.

Two-level system. We consider the same driven two-level setup studied previously, now with an additional loss channel $K = \sqrt{\gamma_0} \sigma_-$. For the lower bound, we use the photon-number observable n_{ph} at the output of the MZI and evaluate its sensitivity using Eq. (12). As shown in Fig. 5(a), in the presence of loss both the upper and lower bounds scale linearly with the total evolution time T . We therefore conclude that the QFI of the noisy case scales linear in T .

π -level system. Next, we return to the π -level system with $\theta = \pi/2$ and consider two loss models:

- (I) $K \in \{\sqrt{\gamma_n} |g_1\rangle\langle g_2|, \sqrt{\gamma_n} |g_2\rangle\langle g_1|\}$,
- (II) $K \in \{\sqrt{\gamma_n} \sigma_1, \sqrt{\gamma_n} \sigma_2\}$.

Physically, model (I) corresponds to unwanted incoherent coupling between the two ground states at rate γ_0 and model (II) corresponds to emitted photons being lost at rate γ_0 into unwanted loss channels. For each model, we compute two measurement-based lower bounds: we

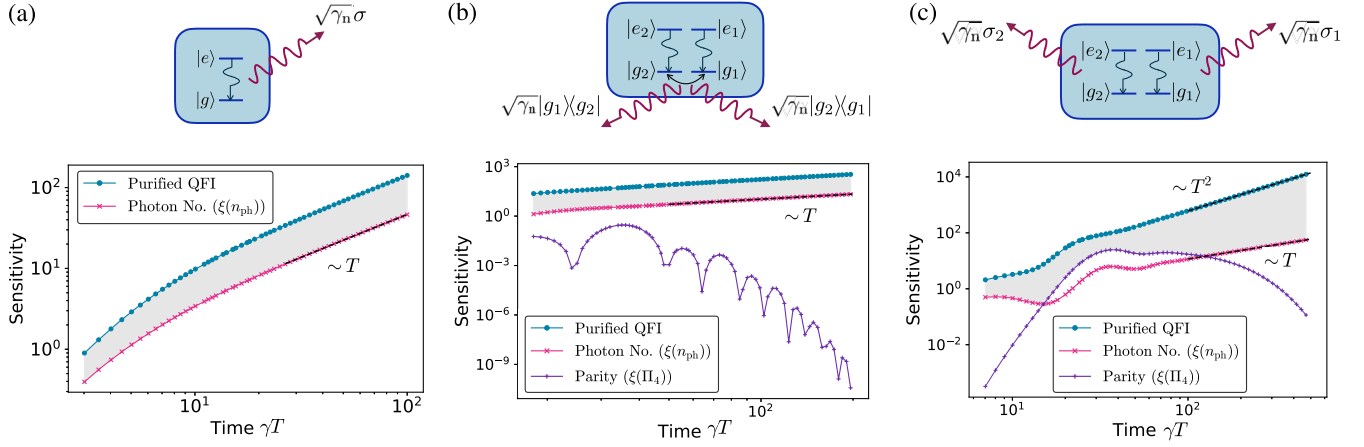


FIG. 5. (a) The range for the noisy 2-level system QFI with $K = \sqrt{\gamma_n} \sigma$ with time. The QFI of the mixed photonic state lies within the shaded gray region. The upper bound is obtained by evaluating the QFI assuming access to the loss channel, while the lower bound is given by the sensitivity of photon-number measurement. (b),(c) The range for the QFI of the lossy π -level system with time assuming $\theta = \pi/2$, and noise models I) $K_\alpha \in \{\sqrt{\gamma_n} |g_1\rangle\langle g_2|, \sqrt{\gamma_n} |g_2\rangle\langle g_1|\}$, and II) $K_\alpha \in \{\sqrt{\gamma_n} \sigma_1, \sqrt{\gamma_n} \sigma_2\}$. The lower bound is given by the maximum sensitivity of photon-number measurement (n_{ph}), and parity-like measurement (Π_4). For all three plots we have $\Omega = 0.2\gamma$, and $\gamma_n = 0.1\gamma$.

evaluate the sensitivities for the photon-number observable n_{ph} as well as the parity observable Π_4 and compare them with the upper bound for the QFI. The resulting bounds for models (I) and (II) are shown in Figs. 5(b) and 5(c). In each panel, the gray band indicates the admissible region for the true QFI of the mixed output state, bounded below by the best measurement sensitivity and above by the QFI for the purification of the mixed state. Interestingly, for loss model (I) the upper bound itself scales only linearly with T , indicating that the quadratic advantage is lost at the level of the best possible strategy. In contrast, for loss model (II) the upper bound retains a quadratic scaling with T , leaving open the possibility of a quantum-enhanced scaling. Moreover, for noise model (II), the two measurement sensitivities are complementary: depending on time, either n_{ph} or Π_4 can provide the tighter lower bound.

We emphasize that the loss channels considered here act only on the internal Hilbert space of the light source. We assume that the loss corresponding to the emitted photons passing through the MZI is negligible. If photon loss is present in the ports, with jump operators proportional to a_i and b_i , the asymptotic scaling of the QFI with time T can be restricted by general results in noisy quantum metrology [78–80]. In particular, the Hamiltonian in Eq. (6) resides in the Lindblad span generated by a_i and b_i . Thus, in the presence of photon loss in the ports, the QFI is restricted to grow at most linearly with time in the asymptotic limit [80,81]. Therefore, the quadratic scalings with T discussed in this work should not be interpreted as being robust to arbitrary noise models. This behavior is observed in noise model (I) for the π -level system: in the absence of noise, the QFI scales quadratically with time, whereas the addition of noise reduces the asymptotic

scaling to linear in T , showing that the quadratic enhancement is not robust to this particular noise.

IV. SOURCE SPECTRUM AND QUANTUM ADVANTAGE

In this section, we show that the result of the previous section can also be used to understand the constraints put by the level structure and the spectrum of the source on the possible quantum advantage. Throughout this section, we will again restrict ourselves to the case where both the arms of the MZI have independent and identical sources i.e., $H_S(t) = H(t) \otimes I + I \otimes H(t)$, $L_A = L \otimes I$, $L_B = I \otimes L$ and $|\phi_{S,i}\rangle = |\phi_i\rangle \otimes |\phi_i\rangle$, $|\phi_{S,f}\rangle = |\phi_f\rangle \otimes |\phi_f\rangle$. We will denote by $\mathcal{E}(t, s) = \mathcal{T} \exp(\int_s^t \mathcal{L}(\tau) d\tau)$, where $\mathcal{L}(\tau) = -i[H(\tau), \cdot] + \mathcal{D}_L$, to be the channel describing the dynamics of the individual sources, and by $\rho_i(t) = \mathcal{E}(t, 0)(|\phi_i\rangle\langle\phi_i|)$, $\mathcal{P}_f(t) = \mathcal{E}^\dagger(T, t)(|\phi_f\rangle\langle\phi_f|)$. The expression for the QFI in Eqs. (8) and (10) can then be reduced to

$$\text{QFI} = 8(Q^{(2)} + N_{\text{ph}}), \quad (19a)$$

where N_{ph} is the number of photons emitted by the source in time T i.e.,

$$N_{\text{ph}} = \frac{1}{\mathcal{N}^2} \int_0^T n(t) dt, \quad (19b)$$

with $n(t) = \text{Tr}(\mathcal{P}_f(t) L \rho_i(t) L^\dagger)$ and $\mathcal{N}^2 = \langle \phi_f | \rho_i(T) | \phi_f \rangle$. Furthermore, $Q^{(2)}$ is the contribution to the QFI from the two-point correlation functions:

$$Q^{(2)} = 2 \int_0^T \int_0^t (|C^{(g)}(t, s)|^2 - |C^{(x)}(t, s)|^2) dt ds, \quad (19c)$$

with

$$\begin{aligned} C^{(g)}(t, s) &= \langle \psi | a_i^\dagger a_s | \psi \rangle = \frac{1}{\mathcal{N}^2} \text{Tr}(L^\dagger \mathcal{P}_f(t) \mathcal{E}(t, s) (L \rho_i(s))), \\ C^{(x)}(t, s) &= \langle \psi | a_i a_s | \psi \rangle = \frac{1}{\mathcal{N}^2} \text{Tr}(\mathcal{P}_f(t) L \mathcal{E}(t, s) (L \rho_i(s))), \end{aligned} \quad (19d)$$

where $|\psi\rangle$ is the state of the photons emitted into any one of the ports. $C^{(g)}(t, s)$ and $C^{(x)}(t, s)$ are the two-point correlation functions contributing to the QFI.

We will assume each source to have D levels and driven for time T during which it emits photons into the output port. While analyzing the scaling of QFI as a function of T , we will also make the assumption that the normalization constant $\mathcal{N}$, which in general will depend on the total time T , is asymptotically lower bounded by a T -independent constant $p_0 > 0$, i.e., $\mathcal{N}^2 \geq p_0 > 0$ as $T \rightarrow \infty$. Physically, this corresponds to the assumption that the projection of the source onto the final state $|\phi_f\rangle$ at time T succeeds with a probability at least p_0 which does not vanish as $T \rightarrow \infty$. We remark that even when the source has only a few levels (i.e., D is small), it can still be used to emit a large number of photons in a possibly entangled state. In particular, from Eq. (19b), it follows that

$$N_{\text{ph}} \leq \frac{T \|L\|^2}{p_0}, \quad (20)$$

i.e., the number of emitted photons can grow at most linearly with the emission time T . For time-independent sources, N_{ph} in-fact grows linearly with T at long times i.e., $N_{\text{ph}} = r_{\text{ph}} T + O(1)$ for some constant $r_{\text{ph}} \geq 0$ (details in Appendix D) which can be interpreted as the asymptotic photon emission rate. It is possible that $r_{\text{ph}} = 0$ but in this case the source would only emit at most a constant number of photons and if $r_{\text{ph}} \neq 0$ the number of photons will be linear in T . Therefore, a time-independent source cannot exhibit sublinear photon scaling of the form T^ν with $0 < \nu < 1$. While, driving schemes in which the emitted photon number grows sublinearly with T are in principle possible for time-dependent sources, throughout this work we assume that the source continuously emits photons with a non-zero emission rate and thus the number of emitted photons grows linearly with T . It can then be asked what kind of a source is required to emit photons in a state which has a QFI that scales as $\sim (\text{Number of photons})^2 \sim T^2$, and under what conditions would the QFI scale only as $\sim T$, thus forbidding a scaling quantum advantage with T .

With this assumption, we note that a possible quadratic scaling in QFI can only be due to a quadratic scaling in

$Q^{(2)}$ since from Eqs. (19a) and (20)

$$\text{QFI} \leq 8 \left(Q^{(2)} + \frac{T \|L\|^2}{p_0} \right). \quad (21)$$

Therefore, to assess quantum advantage, we aim to understand how $Q^{(2)}$ scales with T in the large T limit.

A. Time-independent sources

For a time-independent source (i.e., $\mathcal{L}(t) = \mathcal{L}$ independent of t), whether $Q^{(2)}$ scales as T^2 or only as T can be related to the spectrum of the source Lindbladian $\mathcal{L}$. Consider first the case where $\mathcal{L}$ has a unique fixed point τ (i.e., $\mathcal{L}(\tau) = 0$) and its decay rates have a gap γ_0 (i.e., all other eigen-values of $\mathcal{L}$ have a negative real part $\leq -\gamma_0$ [Fig. 6(a)]). In this case, using the Jordan decomposition of $\mathcal{L}$, we can decompose $\mathcal{E}(t, s)$ as

$$\mathcal{E}(t, s) = \text{Tr}(\cdot) \tau + \Delta(t - s), \quad (22a)$$

with

$$\|\Delta(\tau)\|_\diamond \leq D_0(\tau) e^{-\gamma_0 \tau}, \quad (22b)$$

where $\|\cdot\|_\diamond$ is the superoperator diamond norm [82]. Moreover, $\gamma_0 > 0$ is a constant that depends on the spectrum of $\mathcal{L}$ and $D_0(\tau) \leq O(\tau^{m_0})$ for some m_0 as $\tau \rightarrow \infty$. Returning to Eq. (19), Eq. (22) implies that the correlation functions $C^{(g)}(t, s), C^{(x)}(t, s)$ approximately factorize into products of a function of t and another function of s when $|t - s|$ is large:

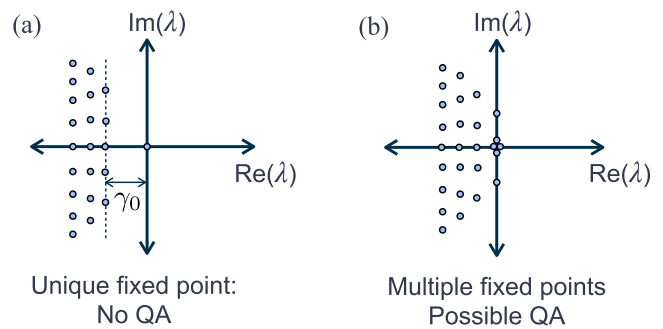


FIG. 6. Schematic depiction of eigenvalues λ of a time-independent Lindbladian $\mathcal{L}$ for the two cases considered in our analysis. (a) If $\mathcal{L}$ has a unique fixed point τ and all other eigenvalues have a strictly negative real part, QFI scales at most linearly with time T , preventing any quantum advantage. (b) If the Lindbladian has multiple fixed points, then a quadratic quantum advantage may be possible with respect to the emission time T .

$$\begin{aligned}
|C^{(g)}(t, s) - \alpha^*(t)\beta(s)| &\leq \frac{\|L\|^2}{p_0} D_0(|t-s|) e^{-\gamma_0|t-s|}, \\
|C^{(\chi)}(t, s) - \alpha(t)\beta(s)| &\leq \frac{\|L\|^2}{p_0} D_0(|t-s|) e^{-\gamma_0|t-s|},
\end{aligned} \quad (23)$$

where $\alpha(t) = \text{Tr}(\mathcal{P}_f(t)L\tau)$ and $\beta(s) = \text{Tr}(L\rho_i(s))$. This factorization of the two-point correlation function suggests that the photon emission at a time t becomes uncorrelated with a previous photon emission at a time $s \ll t$. Using Eq. (23) together with $|C^{(g)}(t, s)|, |C^{(\chi)}(t, s)| \leq \|L\|^2$ and $|\alpha(t)|, |\beta(t)| \leq \|L\|$, we obtain that

$$Q^{(2)} \leq \frac{4\|L\|^4 T}{p_0^2} \int_0^T (2D_0(\tau) + D_0^2(\tau)) e^{-\gamma_0\tau} d\tau. \quad (24)$$

Since $D_0(\tau)$ scales polynomially with τ , for $\gamma_0 T \gg 1$, it follows that $Q^{(2)}$ and thus the QFI scales only linearly with T : forbidding a quadratic quantum advantage. This can be seen as a consequence of such sources only emitting photons with short-time temporal correlations [as shown in Eq. (23)] with correlation times $1/\gamma_0$, while obtaining a quadratic quantum advantage in an interferometry task requires some long-time correlations. On the other hand, when the source Lindbladian does not have a unique fixed point [Fig. 6(b)], then there is the possibility of quantum advantage since such sources are not forbidden from having long-time correlations. Finally, while we have confined our discussion in this section to the case of independent and identical sources, the conclusions continue to be valid even when the source emits photons entangled between the two ports. In particular in Appendix F1, we show that, for a joint-source emitting photons into the two output ports, if the source Lindbladian has a unique steady-state then the QFI of the emitted photonic state can at most scale linearly with T .

We remark that the discussion above is consistent with the numerical examples presented in the previous section. For instance, the Lindbladian describing the driven two-level system has a unique fixed point and thus shows linear scaling of QFI with T (Fig. 2). On the other hand, the Lindbladian describing the driven π -level system has multiple fixed points, which is consistent with the quadratic scaling of its QFI with T . It might, however, superficially appear that this section contradicts the Heisenberg-limited scalings obtained for the dissipative Tavis-Cumming model in Sec. III B: The dissipative Tavis-Cumming model has a unique steady-state (i.e., all emitters in the ground state and the cavity in vacuum) but Sec. III B predicted Heisenberg scaling for the QFI of the emitted photons. However, there is in-fact no contradiction because the analysis of this section applies to sources where the number of levels are not scaled, and the number of emitted photons is increased by increasing T .

On the other hand, in the dissipative Tavis-Cumming model, the number of emitted photons increases with the number of emitters, which corresponds to increasing the number of levels in the source.

B. Time-dependent sources

We next consider sources where the source Hamiltonian itself is dependent on time t . In many experimental setups, while it is often possible to have a completely controllable source Hamiltonian (i.e., the source Hamiltonian can be designed as a function of time to apply any desired unitary on the source Hilbert space), it is typically hard to modulate this interaction between the source and the output port (i.e., the jump operator L). Consequently, we will assume that the jump operator L is time-independent. While it might be physically expected that the ability to apply arbitrary unitaries on the source provides a huge flexibility in designing the output wave-packet, we provide evidence below that the form of the jump operator L places severe restrictions on the achievable scalings of QFI with T .

Similar to the time-independent sources with a unique fixed point and a decay rate gap, time-dependent sources with asymptotically strictly contractive dynamics will have QFI $\sim T$. More specifically, a time-dependent source will be asymptotically strictly contractive if for all states ρ_1, ρ_2 with $X = \rho_1 - \rho_2$,

$$\|\mathcal{E}(t, s)(X)\|_1 \leq C_0 e^{-\gamma_0|t-s|} \|X\|_1 \quad \forall |t-s| \geq \tau_0 \quad (25)$$

for some $C_0, \gamma_0, \tau_0 \geq 0$. Physically, these conditions imply that the dynamics of the source at long-times becomes independent of the initial state of the source. Given that the source dynamics satisfies Eq. (25), we show in Appendix F1 that, similar to the case of time-independent sources with a unique fixed point (Eq. 23), $C^{(g)}(t, s), C^{(\chi)}(t, s)$ also factorize when $t \gg s$ and consequently the QFI $\sim T$. While it is generally a difficult task to show strict contractivity of a Lindbladian with a time-dependent Hamiltonian and a set of time-independent jump operators, it is generally expected to hold for sources with a single ground state and no possible dark states. In Appendix F1, we rigorously show that this condition holds for a two-level source decaying into the output port with $L = \sqrt{\gamma}|g\rangle\langle e|$, as well as a multi-level source where the jump operators have a full Kraus rank, i.e., they form a complete basis for the space of source operators.

We next consider sources where the dynamics is not strictly contractive, and thus a quantum advantage is not forbidden. The simplest and experimentally available sources with nonstrictly contractive dynamics are those that emit via a transition from an excited state $|e\rangle$ to a ground state $|g\rangle$ and additionally have some number of dark states $\{|m_1\rangle, |m_2\rangle, \dots, |m_k\rangle\}$. The jump operator corresponding to the source is $L = |g\rangle\langle e|$, which leaves the

dark states unchanged, but the Hamiltonian $H(t)$ can in general couple the dark states with the excited and ground state. Examples of such sources include Λ and V-level systems with both the transitions coupling to the same output port, as well as multi-emitter systems described by the Dicke model [41]. Several protocols have been devised that leverage the dark states in such sources to create long-time correlated photonic states (such as the photonic GHZ state) [83–86]. As a simple example, consider a source with an excited state $|e\rangle$, ground state $|g\rangle$ and a dark state $|m\rangle$: By initializing the source in $(|g\rangle + |m\rangle)/\sqrt{2}$ and then driving the $|e\rangle - |g\rangle$ transition with $H(t) = \Omega(|e\rangle\langle g| + \text{H.c.})$ and finally projecting the source on $(|g\rangle + |m\rangle)/\sqrt{2}$, we can generate a photonic state

$$|\psi\rangle \propto |\psi_{\text{TLS}}\rangle + |\text{vac}\rangle, \quad (26)$$

where $|\psi_{\text{TLS}}\rangle$ is the state of photons emitted by a driven two-level state. Since $|\psi\rangle$ is a coherent superposition between $|\text{vac}\rangle$ and a state $|\psi_{\text{TLS}}\rangle$ with $O(T)$ photons emitted in the time interval $[0, T]$, it is a state with long-time correlations.

Despite the possibility of generating long-time correlations, we find that the QFI of the photons emitted from such sources $\sim T$. In Fig. 7, we numerically investigate the photonic state emitted by a two-level system, a two-level system with upto two dark states and a π -level system [Fig. 7(a)]. We use a gradient-based optimization algorithm to numerically maximize $Q^{(2)}$ of the emitted photons with respect to the time-dependent source Hamiltonian $H(t)$. To compute the gradient of $Q^{(2)}$ with respect to $H(t)$ for large T , we develop an adjoint variable method [87–89] that significantly speeds up the gradient computation

(Appendix F2). As shown in Fig. 7(b), we find that the optimized $Q^{(2)} \sim T$ for the two-level system, which is expected since the source dynamics of the two-level systems are provably strictly contractive. Furthermore, we also find that for a two-level system with additional dark states, while having a $Q^{(2)}$ higher than that of a two-level system without dark states, still exhibits $Q^{(2)} \sim T$. On the other hand, for a π -level system, the optimized $Q^{(2)}$ scales as T^2 . We remark that the optimized $Q^{(2)}$ shown in Fig. 7(b) are the largest $Q^{(2)}$ obtained from individual optimization trials with randomly chosen guess for the initial $H(t)$ —Fig. 7(c) shows the distribution of the optimized $Q^{(2)}$ obtained from these different trials.

The scaling of $Q^{(2)} \sim T$ for the two-level system with dark states, despite their potential for generating long-time correlated states, can be physically attributed to the correlation functions $C^{(g)}(t, s)$ and $C^{(x)}(t, s)$ that determine the QFI [Eq. (19)] remaining short-range correlated. Specifically, for states of the form as in Eq. (26), $C^{(g)}(t, s) \propto \langle \psi_{\text{TLS}} | a_t^\dagger a_s | \psi_{\text{TLS}} \rangle$, $C^{(x)}(t, s) \propto \langle \psi_{\text{TLS}} | a_t a_s | \psi_{\text{TLS}} \rangle$, i.e., the correlation functions $C^{(g)}(t, s)$ and $C^{(x)}(t, s)$ inherit the correlations of the state $|\psi_{\text{TLS}}\rangle$ which, being a photonic state emitted by a simple two-level system, only has short time correlations thus yielding a QFI $\propto T$. Even with more complicated protocols, we expect that the emitted photonic state that can possibly be generated is of the form of Eq. (26), since such a source can only create correlations by either emitting a photon (when it is in $|e\rangle$) or not emitting a photon (when it is in a dark state). In order to generate a long-time correlated state with QFI $\sim T^2$, we need a source that can generate photons via two distinct and

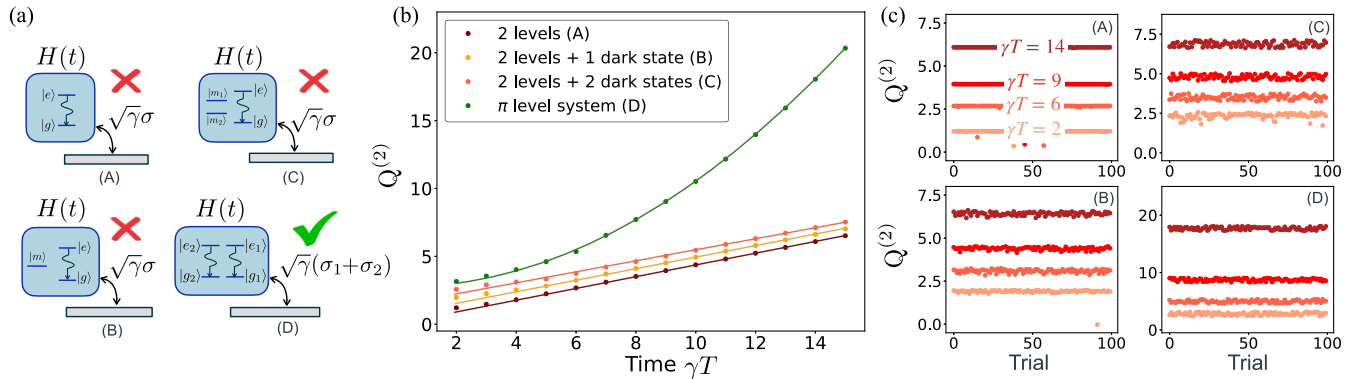


FIG. 7. (a) Schematic illustration of different level structures analyzed for possible quantum advantage. (A) a two-level system, (B) a two-level system with a single dark state, (C) a two-level system with two dark states, and (D) a π -level system. (b) Numerically optimized $Q^{(2)}$, with the optimization variable being the time-dependent source Hamiltonian $H(t)$, as a function of emission time T for different sources. When the source has only one emission pathway ($|e\rangle \rightarrow |g\rangle$), with or without dark states, $Q^{(2)}$ scales linearly with T thus providing no quantum advantage. Since a π -level source has two emission pathways $|e_1\rangle \rightarrow |g_1\rangle$, $|e_2\rangle \rightarrow |g_2\rangle$, $Q^{(2)}$ scales quadratically with T thus providing a quadratic quantum advantage at large T . (c) Distribution of the optimized $Q^{(2)}$ over 100 trials with random initial conditions for $H(t)$. The data plotted in (b) are the maximum $Q^{(2)}$ obtained over the 100 optimization trials to ensure that it is a good estimate of the “global optimum” of the optimization problem.

distinguishable emission paths. Here, distinguishable is meant in the conditional sense that the emitted field is entangled with the source, so that an appropriate measurement on the emitter can identify the emission path based on the measurement outcome, even though the photons are indistinguishable in the output mode. For instance, as discussed previously, using a π -level system where photons can be generated via either the transition $|e_1\rangle \rightarrow |g_1\rangle$ or $|e_2\rangle \rightarrow |g_2\rangle$, we can obtain a state $|\psi\rangle$ that is the coherent superposition of two macroscopic photonic states for which even $C^{(g)}(t, s)$ and $C^{(\chi)}(t, s)$ are long-time correlated thus yielding a QFI $\sim T^2$. Indeed, this is what we observe in Fig. 7(b), where a time-dependent π -level source with the source Hamiltonian $H(t)$ optimized to maximize the QFI, exhibits a quadratic improvement over the QFI obtained from source with one emitting transition and a dark state subspace.

V. OPTIMAL MEASUREMENTS

In the previous section, we characterized the QFI that can be achieved with a Markovian light source. However, even when the QFI has Heisenberg-limited scaling, to achieve the corresponding phase-sensitivity in an actual spectroscopic setup, it is necessary to carefully design the measurement that extracts the unknown phase φ from the state at the output of the MZI. More formally, the measurement is captured by a positive-operator valued measurement E_a corresponding to the outcome a . When the measurement is applied on the state $|\psi_\varphi\rangle$ at the output of the MZI, it yields the outcome x with probability $p_\varphi(a) = \langle\psi_\varphi|E_a|\psi_\varphi\rangle$. The measurement is optimal if the classical Fisher information (CFI) of $p_\varphi(a)$, given by

$$\text{CFI} = \int \left(\frac{d}{d\varphi} \ln p_\varphi(a) \right)^2 p_\varphi(a) d\varphi,$$

is equal to the quantum Fisher information (QFI) of $|\psi_\varphi\rangle$. In general, the optimal measurement is not unique and there can be multiple measurements for which the corresponding CFI is equal to the QFI of $|\psi_\varphi\rangle$.

In the following, we analyze different strategies that can be used to optimally measure the phase φ from the photonic state $|\psi_\varphi\rangle$ obtained at the output of the MZI. Our goal here is to understand what experimental resources are required and how they depend on the photonic state. We first describe a general strategy to optimally obtain φ but which requires a high quality factor Kerr resonator. Then, we go onto understand when linear optics and photodetection are sufficient to perform an optimal measurement, deriving a set of sufficient conditions under which a simple photodetection measurement is optimal. Furthermore, in the case of independent sources emitting into the two input ports of the MZI, we show that if φ can be

measured optimally with linear optics and photodetectors, then only photodetection is also an optimal measurement.

A. Optimal measurement with non-linear optics

We first consider the most general setting in which a time-dependent D -level source emits photons into the two ports of the MZI interferometer. When the source has no further symmetries, the optimal measurement essentially has to undo the source-port dynamics. To illustrate this in a simple setting, we consider the simpler setup of n qubits initially in $|0\rangle^{\otimes n}$ and a unitary U_z , dependent on the unknown parameter z , is applied on them to obtain that state $|\psi_z\rangle = U_z|0\rangle^{\otimes n}$. An optimal measurement to extract the parameter z , assuming it to be in the neighborhood of z_0 , would be to implement the projective measurement given by $\{P_0, P_1, P_2 \dots\}$

$$P_0 = |\psi_{z_0}\rangle\langle\psi_{z_0}| \quad \text{and} \quad \sum_{i \geq 1} P_i = I - |\psi_{z_0}\rangle\langle\psi_{z_0}|. \quad (27)$$

It is easy to check that this measurement is optimal, i.e., the CFI of the probability distribution over the measurement outcomes $\{0, 1, 2 \dots\}$, at z_0 , is equal to the QFI of the state $|\psi_{z_0}\rangle$ at $z = z_0$ (Ref. [43,90], and reviewed in Appendix G1). The simplest strategy to implement this measurement on $|\psi_{z_0}\rangle$ is to first undo the unitary transformation U_{z_0} (i.e., apply the unitary $U_{z_0}^\dagger$) followed by a computational basis measurement on the n qubits. Measuring zero on all qubits is then equivalent to measuring outcome 0 in the projective measurement in Eq. (27).

To apply this measurement strategy to the interferometric setup, where we want to measure the MZI phase φ in the neighborhood of $\varphi = 0$, we need to undo the photon emission from the source, i.e., coherently re-absorb the photons emitted by the source [91,92]. A similar strategy has been used to implement an optimal measurement for spectroscopy using measurements on the emitted photon fields [46]—our focus in this section is to show that this measurement can always be implemented with non-linear optical elements. In particular, we show that a perfectly re-absorbing system can in principle be implemented using a multi-mode optical cavity with a $\chi^{(3)}$ non-linearity interacting with the output ports of the MZI (Fig. 8). Specifically, we use 3 modes of this optical cavity, with corresponding annihilation operators c_A, c_B and c_0 . We assume that the mode c_A is coupled to port A, c_B is coupled to port B and all three modes couple to each other via the non-linear self- and cross-phase modulation induced by the $\chi^{(3)}$ non-linearity. In addition, the three modes will be driven by a tunable time-dependent coherent laser. The Hamiltonian of this reabsorbing system is then given by

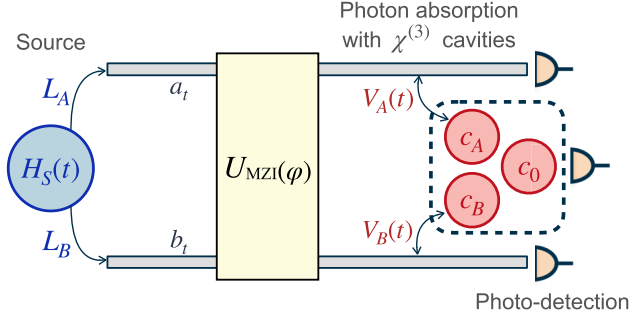


FIG. 8. Schematic depiction of the optimal measurement around $\varphi = 0$ using non-linear optical elements. A $\chi^{(3)}$ cavity with three optical modes (c_A, c_B, c_0), with tunable couplings $V_A(t), V_B(t)$ to the output ports of the MZI, is used to re-absorb the photons emitted by the source, and then a photodetection is performed on both the output ports as well as on the $\chi^{(3)}$ cavity.

$$H_R(t) = \sum_{k \in \{0,A,B\}} (\Omega_k(t) c_k^\dagger + \text{H.c.}) + \frac{\chi}{2} \sum_{k,k' \in \{0,A,B\}} c_k^\dagger c_{k'}^\dagger c_k c_{k'}, \quad (28)$$

with $\Omega_k(t)$ being the coherent field applied on the k^{th} mode. By building upon Ref. [93], we show in Appendix G1 that any unitary on the joint Hilbert space of these cavity modes can be applied by designing the laser field $\Omega_k(t)$. As we show, the speed at which unitaries can be applied is *not* limited by the non-linear strength χ , but only by the magnitude and the rate of change of the coherent drives $\Omega_k(t)$. Thus, gates can be applied very fast on the Hilbert space of the cavity modes. We also assume that the modes c_A and c_B couple linearly with the output ports of the MZI via

$$H_{R,P} = V_A(t) a_t^\dagger c_A + V_B(t) b_t^\dagger c_B + \text{H.c.}, \quad (29)$$

which gives us the ability to sequentially transfer photons from the output ports to the cavity modes during the reabsorption process. We show in Appendix G1 that, by appropriately tuning the parameters $V_A(t), V_B(t), \Omega_A(t), \Omega_B(t), \Omega_0(t)$ as a function of time t , we can coherently absorb the photons emitted from the source into the non-linear cavity. This effectively inverts the emission of photons from the source, and a subsequent photodetection on both the optical cavity as well as the output ports allows us to implement the target optimal measurement.

To implement the coherent photon reabsorption required to apply this measurement strategy, one approach is to time-reverse the source. However, this may not always be possible. *First*, we may not have access to a replica of the physical system used as the source and thus would like to be able to design the re-absorbing system with components that are independent of the source. *Second*, in several sources exhibiting quantum advantage, the source might

not disentangle from the emitted photons on its own and we might have to project the source onto a final target state (e.g., see the example of the driven π -level system discussed in Sec. IV).

We instead design a perfectly reabsorbing system using a completely controllable multi-mode non-linear optical cavity. The key idea that we use to implement this photon absorption is to note that, when expressed in the time-bin basis, the photonic state of the port is approximately a matrix product state (MPS) with a bond-dimension equal to the dimensionality of the Hilbert space of the source [94]. Consequently, by using the canonical form of this MPS, we can compute a sequence of unitaries applied on the qudits in this MPS and an ancilla to map it to the vacuum state [95–97]. To physically implement the reabsorption, we treat the cavity mode c_0 as the ancilla and perform two operations when each time-bin in the output port is incident on the reabsorbing $\chi^{(3)}$ cavity—we first swap the photons in this time-bin from the ports A and B into the cavities c_A and c_B respectively. This operation can be performed by only tuning $V_A(t), V_B(t)$. Next, since control over the parameters $\Omega_A(t), \Omega_B(t), \Omega_0(t)$ allows us to implement any unitary on the joint Hilbert space of the three cavity modes, which now contains both the qudit of the MPS describing the photonic state in the cavities c_A, c_B and the ancilla in c_0 , we design them to implement the gates in the previously computed unitary circuit. The details of this protocol are explicitly laid out in Appendix G1.

We remark that this protocol can be considered to be a generalization of well-known state-transfer protocols [98], which considered the problem of coherently transferring the quantum state of a source, that couples to an output photon field and emits a single photon entangled with the source, to another quantum system also coupling to the photon field. Finally, we remark that this protocol guarantees that there is always a choice of the parameters of the $\chi^{(3)}$ cavity so as to reabsorb the emitted photons, but might not be the most practical implementation of the reabsorption process. In practice, depending on the constraints of the specific experimental system at hand, heuristic gradient-based control design algorithms such as GRAPE [99] can also be used to find the parameters of the $\chi^{(3)}$ cavity that accomplish this reabsorption.

B. Optimal measurement with photodetection and linear optics

Although the previous subsection described a general strategy to implement an optimal measurement irrespective of the source, having a completely controllable non-linear optical cavity is still experimentally challenging. In this subsection, we study under what conditions photodetectors and linear-optical elements alone are enough to implement an optimal measurement.

The simplest measurement that can be performed is a measurement of the total number of photons in the output

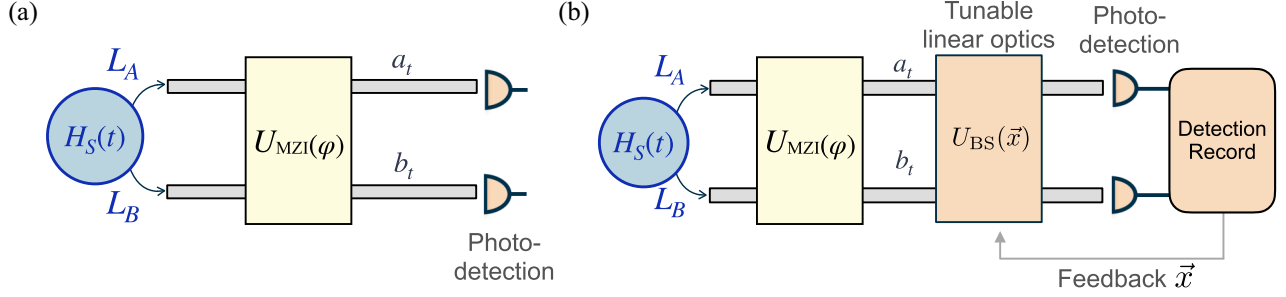


FIG. 9. Schematic depiction of two measurements implementable with linear optical elements and photodetectors studied in Sec. V B. (a) A setup where photodetection is performed at the MZI output, and (b) a setup where a linear-optical element, tuned based on the photodetection record, is applied before the photodetectors.

ports. We first provide a general sufficient condition under which photon-number measurement at the two ports (Fig. 9) is optimal. Suppose Π_{n_A, n_B} is the projector on the subspace where there n_A photons in the port A and n_B photons in the port B. Let $\mathcal{I} = \{(n_A, n_B) : n_A, n_B \in \{0, 1, 2, \dots\}\}$ be a set of photon numbers in the two ports such that for all $(n_A, n_B) \in \mathcal{I}$, $(n_A \pm 1, n_B \pm 1) \notin \mathcal{I}$. Then, if the photonic state incident on the MZI $|\psi\rangle$ satisfies

$$\Pi_{\mathcal{I}}|\psi\rangle = |\psi\rangle, \quad (30)$$

where $\Pi_{\mathcal{I}} = \sum_{(n_A, n_B) \in \mathcal{I}} \Pi_{n_A, n_B}$ is a projector, then photon-number measurement at the output of the MZI is the optimal measurement in the neighborhood of $\varphi = 0$. Physically, this condition implies that addition or subtraction of a single photon in both the output ports to $|\psi\rangle$ makes the resulting state orthogonal to the subspace with projector $\Pi_{\mathcal{I}}$ which contains the state $|\psi\rangle$. Furthermore, since this subspace can be identified by just the number of photons in the two output ports, photon number measurement becomes an optimal measurement for sensing the phase φ .

Examples of states that satisfy this condition are

- (a) $|\psi\rangle = |\psi_A^{(N_A)}\rangle \otimes |\psi_B^{(N_B)}\rangle$ where $|\psi_A^{(N_A)}\rangle$ has N_A photons and $|\psi_B^{(N_B)}\rangle$ has N_B photons. Here $\mathcal{I} = \{(N_A, N_B)\}$, and it is clear that adding or removing a photon from any of the output ports results in state orthogonal to $\Pi_{\mathcal{I}}$. For such states, it is already known that just photon-number measurement is optimal even if the state is not a Fock state [42].
- (b) $|\psi\rangle = |\psi_A^{(N_A)}\rangle \otimes |\psi_B\rangle$ where $|\psi_A^{(N_A)}\rangle$ has N_A photons while $|\psi_B\rangle$ can be arbitrary. In this case, $\mathcal{I} = \{(N_A, n_B) : n_B \in \{0, 1, 2, \dots\}\}$ and therefore adding or removing a photon from port A also results in a state orthogonal to $\Pi_{\mathcal{I}}$. Consequently, photon number measurement at the output ports remains optimal if only one of the ports has a state with a definite number of photons.
- (c) $|\psi\rangle = |\psi_A^{(k_A, r_A)}\rangle \otimes |\psi_B\rangle$ where $|\psi_A^{(k_A, r_A)}\rangle$ has either $r_A, k_A + r_A, 2k_A + r_A, 3k_A + r_A, \dots$ photons and

$|\psi_B\rangle$ can be arbitrary. When $k_A = 2$, the state $|\psi_A\rangle$ is a state with a definite photon number parity (i.e., odd parity if $r_A = 1$ and even parity if $r_A = 0$). Here, again photon number measurement at the output ports is optimal when $k_A \geq 2$ since $\mathcal{I} = \{(r_A + m_A k_A, n_B) : m_A, n_B \in \{0, 1, 2, \dots\}\}$ and adding or removing a photon from port A results in an orthogonal state.

To show the optimality of the photon-number measurement, we consider the projective measurement with outcomes 0, 1 given by $P_0 = \Pi_{\mathcal{I}}, P_1 = I - \Pi_{\mathcal{I}}$. This measurement can clearly be performed with just photodetectors by first measuring the photon numbers (n_A, n_B) at the output port and then recording 0 if $(n_A, n_B) \in \mathcal{I}$ else recording 1. The measurement outcome follows the probability distribution $p_\varphi(0) = \langle \psi_\varphi | \Pi_{\mathcal{I}} | \psi_\varphi \rangle, p_\varphi(1) = 1 - p_\varphi(0)$. We first note that $p_{\varphi=0}(0) = 1, p_{\varphi=0}(1) = 0$ and from Eq. (7),

$$\frac{d}{d\varphi} p_{\varphi=0}(0) = -\frac{d}{d\varphi} p_{\varphi=0}(1) = 2 \text{Im}(\langle \psi | \Pi_{\mathcal{I}} H_d | \psi \rangle) = 0,$$

where we remind the reader that $H_d = i \int_0^T (a_t^\dagger b_t - a_t b_t^\dagger) dt$. Furthermore, we have used $\Pi_{\mathcal{I}} H_d | \psi \rangle = 0$ since $H_d | \psi \rangle$ is a state in which one photon has either been added or subtracted from both the modes and hence, by assumption, is in the subspace orthogonal to $\Pi_{\mathcal{I}}$. Consequently, the CFI of the probability distribution p_φ at $\varphi = 0$ is given by

$$\begin{aligned} \text{CFI} &= \lim_{\varphi \rightarrow 0} \frac{1}{p_\varphi(1)} \left(\frac{d}{d\varphi} p_\varphi(1) \right)^2 = 2 \frac{d^2}{d\varphi^2} p_{\varphi=0}(1) \\ &= 4 \langle \psi | H_d (I - \Pi_{\mathcal{I}}) H_d | \psi \rangle = 4 \langle \psi | H_d^2 | \psi \rangle, \end{aligned} \quad (31)$$

where we have again used $\Pi_{\mathcal{I}} H_d | \psi \rangle = 0$. Finally, noting that $\langle \psi | H_d | \psi \rangle = \langle \psi | \Pi_{\mathcal{I}} H_d | \psi \rangle = 0$, it follows from Eq. (8) that CFI = QFI, thus establishing that photon-number measurement is the optimal measurement.

Next, we analyze if adding linear-optical elements before the photo detectors can allow us to construct optimal measurements for a larger class of quantum photonic states [Fig. 9(b)]. To keep the measurement protocol as general as possible, we allow the linear optical elements before the photodetector to be modulated as a function of time depending on the result of the photodetection. More specifically, the output $\vec{x}$ of the photodetectors is a set of times $\tau_1, \tau_2, \tau_3 \dots$ at which a photon has been detected as well indices $\sigma_1, \sigma_2, \sigma_3 \dots \in \{a, b\}$ where σ_i indicates the port in which the photon at time τ_i is detected. As depicted in Fig. 9(b), we allow the beam splitter $U_{BS}(\vec{x})$, that is applied before the photodetection, to change depending on the photodetector output $\vec{x}$ recorded so far.

We first analyze this measurement protocol for the case of independent sources emitting into the two input ports of the MZI, i.e., the input photonic state $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, where $|\psi_A\rangle$ and $|\psi_B\rangle$ are the photonic states emitted into the ports A and B, respectively. We make an additional assumption that the wave-functions corresponding to the states $|\psi_A\rangle, |\psi_B\rangle$ are non-zero, i.e.,

$$\langle \text{vac} | \prod_{i=1}^n x_{t_i} |\psi_x\rangle \neq 0,$$

for all $x \in \{a, b\}, n \in \{0, 1, 2 \dots\}, t_1, t_2 \dots t_n \geq 0$. For sources satisfying this assumption, we show in Appendix G2 that optimality of a measurement protocol with a tunable beam-splitter and photodetector also implies optimality of a measurement protocol with only photodetectors without using any linear-optical elements. Stated differently, when the sources emitting photons in the two input ports of the MZI are independent, if photodetection is a sub-optimal measurement then adding linear-optics to it alone cannot make it optimal. On the other hand, when the photons emitted in the two input ports of the MZI are entangled, then it is possible that photodetection alone is sub-optimal as a measurement, but becomes optimal when supplemented with linear optical elements. We provide an explicit example of a photonic state illustrating this fact in Appendix G2.

VI. CONCLUSION

In this paper, we provide a general framework for evaluating the quantum advantage of light sources for quantum interferometry. Within the Markov approximation, we first show how to compute the quantum Fisher information (QFI) of the emitted photons in a Mach-Zehnder Interferometer (MZI). In particular, we show that to compute QFI, it is enough to be able to simulate the internal dynamics of the source and we do not need to calculate the entire photon wave-packet. We then use this result to analytically and numerically elucidate the level structure and spectral properties of the photon source needed to obtain a possible quantum advantage in

interferometry. Finally, we turn to the question of how to implement optimal measurements for photons emitted from general light sources and with experimentally available optical elements. We show that a controllable system of coupled $\chi^{(3)}$ cavities together with a photodetector can always be used to implement the optimal measurement irrespective of the light source. We additionally elucidate conditions when photodetectors and linear optics alone are enough to implement the optimal measurement.

An immediate next step is to apply the framework developed in our work to numerically and analytically study the potential of experimentally realistic quantum photonic system to generate metrologically useful photonic states. It would be particularly interesting to understand the impact of experimental non-idealities, such as position and spectral inhomogeneities in the quantum emitters as well as losses [100,101], on the possible quantum advantage in the emitted photons. On the more theoretical side, making rigorous the limitations to quantum advantage imposed by the level structure of the quantum emitters driven by a possible time-dependent Hamiltonian is also an open question—while we have partly addressed this question through analytical calculations and numerical simulations, a fully rigorous treatment of this question remains open. Finally, extending our framework to cases where the light source is not Markovian (e.g., has a time-delay and feedback [83,102–105]) could also allow us to study if and how non-Markovianity can be used as a resource to generate quantum advantage in interferometry.

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VII. DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: QUANTUM FISHER INFORMATION OF THE LIGHT SOURCE

In this section, we provide details on the measurement protocol for QFI of a light source using its correlation functions and Eq. (8), the derivation of Eq. (10), which expresses the QFI in terms of the internal dynamics of the system, as well as an application of Eq. (8) to study the QFI of some paradigmatic quantum optical systems.

1. Measurement of the QFI from correlation functions

To measure the QFI of a light source, it is enough to measure the correlation functions in Eq. (8). In this section, we show that this can be done without a full interferometric measurement of the unknown phase, for both the cases where there is a single source simultaneously emitting into the two input ports of the MZI, as well as when independent and identical sources emit into these input ports and we only have access to one of these sources.

a. Single source emitting into both ports

First, we apply a phase shifter $U_P(\theta_1)$ to one of the output arms of the source (here a_t) and then pass it through an MZI with the known phase θ_2 (Fig. 10). This transforms the annihilation operator a_t to

$$\begin{aligned} c_t &= U_{\text{MZI}}^\dagger(\theta_2) U_P^\dagger(\theta_1) a_t U_P(\theta_1) U_{\text{MZI}}(\theta_2) \\ &= a_t e^{i\theta_1} \cos(\theta_2) + b_t \sin(\theta_2). \end{aligned} \quad (\text{A1})$$

If we measure the photon flux at the output we get,

$$\begin{aligned} n_c(t) &= \langle c_t^\dagger c_t \rangle \\ &= \cos^2(\theta_2) \langle a_t^\dagger a_t \rangle + \sin^2(\theta_2) \langle b_t^\dagger b_t \rangle \\ &\quad + \sin(2\theta_2) \text{Re}(\langle a_t^\dagger b_t \rangle e^{-i\theta_1}). \end{aligned} \quad (\text{A2})$$

Thus, if we choose $\theta_1 = \pi/2$ and measure $n_c(t)$ as a function of θ_2 , we can extract the 1-point correlations $\mathcal{C}_{a;a}^{(1)}(t)$, $\mathcal{C}_{b;b}^{(1)}(t)$, and $\mathcal{C}_{a;b}^{(1)}(t)$. Next, we perform a two-photon correlation measurement at the output port to obtain $G_c^{(2)}(t, s) = \langle c_t^\dagger c_s^\dagger c_t c_s \rangle$. This yields

$$\begin{aligned} G_c^{(2)}(t, s) &= \cos^4(\theta_2) g_{40} + \sin^4(\theta_2) g_{04} \\ &\quad + \cos^2(\theta_2) \sin^2(\theta_2) g_{22} + 2\cos^3(\theta_2) \sin(\theta_2) g_{31} \\ &\quad + 2\cos(\theta_2) \sin^3(\theta_2) g_{13}, \end{aligned} \quad (\text{A3a})$$

where

$$g_{40} = \langle a_t^\dagger a_s^\dagger a_t a_s \rangle, \quad (\text{A3b})$$

$$g_{31} = \text{Re}(\langle a_t^\dagger a_t^\dagger a_s b_s^\dagger \rangle e^{i\theta_1(s)}) + \text{Re}(\langle a_t b_t^\dagger a_s^\dagger a_s \rangle e^{i\theta_1(t)}), \quad (\text{A3c})$$

$$\begin{aligned} g_{22} &= \langle a_t^\dagger b_s^\dagger a_t b_s \rangle + \langle b_t^\dagger a_s^\dagger b_t a_s \rangle \\ &\quad + 2\text{Re}(\langle a_t^\dagger a_t^\dagger b_t b_s \rangle e^{-i(\theta_1(t)+\theta_1(s))}) \\ &\quad + 2\text{Re}(\langle a_t^\dagger b_s^\dagger b_t a_s \rangle e^{-i(\theta_1(t)-\theta_1(s))}), \end{aligned} \quad (\text{A3d})$$

$$\begin{aligned} g_{13} &= \text{Re}(\langle a_t^\dagger b_t b_s^\dagger b_s \rangle e^{-i\theta_1(t)}) \\ &\quad + \text{Re}(\langle b_t^\dagger a_s^\dagger b_t b_s \rangle e^{-i\theta_1(s)}), \quad \text{and} \end{aligned} \quad (\text{A3e})$$

$$g_{04} = \langle b_t^\dagger b_s^\dagger b_t b_s \rangle, \quad (\text{A3f})$$

where we assume that the angle of the phase shifter θ_1 can be time dependent. Choosing a linear function for $\theta_1(t)$, i.e. $\theta_1(t) = \theta_1^* t$ and measuring the value of $G_c^{(2)}(t, s)$ as a function of t, s, θ_1^* , and θ_2 , we can find the 2-point correlations $\mathcal{C}_{a;b;b,a}^{(2)}(t, s) = \text{Re}(\langle a_t^\dagger b_s^\dagger b_t a_s \rangle)$, $\mathcal{C}_{b;b;a,a}^{(2)}(t, s) = \text{Re}(\langle a_t^\dagger a_s^\dagger b_t b_s \rangle)$.

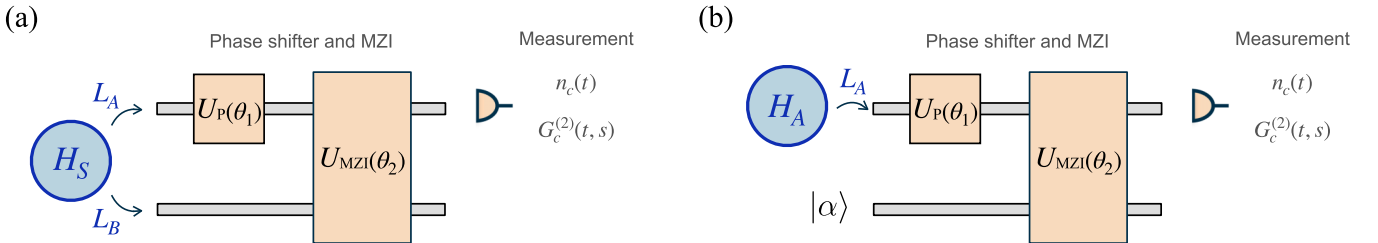


FIG. 10. (a) Schematic representation of the measurement setup for extracting the 2-point and 4-point correlation functions. We measure the photon count, $n_c(t)$, and the second-order correlation function, $G_c^{(2)}(t, s)$, after applying a phase shifter and an MZI unitary for different known values of θ_1 , and θ_2 . This measurement enables the extraction of the correlation functions required for computing QFI. (b) If the source emits into a single port, we can use the same measurement, but with a coherent state as the input to the second port.

b. Single source emitting into one port

If the source emits into one port, we can use the same measurement setup, except that we introduce a coherent state $|\alpha\rangle$ into the second port. Then Eq. (A3d) becomes

$$\begin{aligned} g_{22} &= |\alpha|^2 \langle a_t^\dagger a_t \rangle + |\alpha|^2 \langle a_s^\dagger a_s \rangle \\ &+ 2\text{Re}(\langle a_t^\dagger a_s^\dagger \rangle \alpha^2 e^{-i(\theta_1(t)+\theta_1(s))}) \\ &+ 2\text{Re}(\langle a_t^\dagger a_s \rangle |\alpha|^2 e^{-i(\theta_1(t)-\theta_1(s))}). \end{aligned} \quad (\text{A4})$$

We again choose $\theta_1(t) = \theta_1^* t$ and measure $G_c^{(2)}(t, s)$ for different values of θ_2 and θ_1^* . Thus, we can determine $\langle a_t a_s \rangle$ and $\langle a_t^\dagger a_s \rangle$. Alternatively, we can fix θ_2 and vary $|\alpha|^2$ instead.

2. Quantum Fisher information in terms of the light source dynamics

In this subsection, we show how to compute the functions $C_{x^1 \dots x^n; y^1 \dots y^n}^{(n)}(t_1 \dots t_n)$ by only tracking the internal dynamics of the source. A result that we will use for this analysis is the quantum regression theorem [67], which we review below.

Proposition 1. (Quantum regression theorem). Suppose $H(t)$ is the Hamiltonian of a system with M bosonic ports

$$H(t) = H_S(t) + \sum_{\alpha=1}^M (L_\alpha a_{\alpha,t}^\dagger + \text{H.c.}),$$

where $H_S(t)$, L_α are operators on the system Hilbert space, and $a_{\alpha,t}$ are annihilation operators on the bosonic ports with $[a_{\alpha,t}, a_{\beta,s}^\dagger] = \delta_{\alpha,\beta} \delta(t-s)$. Then, for any system operators $O_1, O_2 \dots O_k, Q_1, Q_2 \dots Q_k, P$, system state $|\phi\rangle$ and $0 < t_1 < t_2 < \dots < t_{k-1} < t_k$,

$$\begin{aligned} &\langle \phi, \text{vac} | \left(\prod_{j=1}^{k-1} O_j(t_j) \right) P(t_k) \left(\prod_{j=k-1}^1 Q_j(t_j) \right) | \phi, \text{vac} \rangle \\ &= \text{Tr} \left(P \left(\prod_{j=k-1}^1 \mathcal{E}(t_{j+1}, t_j) \mathcal{M}_j \right) \mathcal{E}(t_1, t_0) (|\phi\rangle\langle\phi|) \right), \end{aligned} \quad (\text{A5})$$

where for system operator X , $X(t) = U(0, t) X U(t, 0)$ with $U(t, 0) = \bar{T} \exp(-i \int_0^t H(s) ds)$, $\mathcal{E}(t, s) = \bar{T} \exp \times (\int_s^t \mathcal{L}(\tau) d\tau)$ with $\mathcal{L}(\tau) = -i[H_S(\tau), \cdot] + \sum_{\alpha=1}^M \mathcal{D}_{L_\alpha}$ and $\mathcal{M}_j(X) = Q_j X O_j$.

Proof. To prove this statement, we note that in time basis, the Hilbert space of any the ports $\mathcal{H}$ can be expressed as $\mathcal{H} = \mathcal{H}_{(-\infty, 0] \cup (t_k, \infty)} \otimes \mathcal{H}_{(0, t_1]} \otimes \mathcal{H}_{(t_1, t_2]} \otimes \dots \otimes \mathcal{H}_{(t_{k-1}, t_k]}$, where for an interval I , $\mathcal{H}_I$ is the Hilbert space of states with photons created by $a_{\alpha,t}$ with $t \in I$, i.e.

$$|\psi\rangle \in \mathcal{H}_I: |\psi\rangle = |\text{vac}_I\rangle + \sum_{n=1}^{\infty} \int_{t_1, t_2, \dots, t_n \in I} \psi_{\alpha_1, \alpha_2, \dots, \alpha_n}(t_1, t_2, \dots, t_n) a_{t_1}^\dagger a_{t_2}^\dagger \dots a_{t_n}^\dagger |\text{vac}_I\rangle dt_1 dt_2 \dots dt_n,$$

where $|\text{vac}_I\rangle$ is the vacuum corresponding to the Hilbert space $\mathcal{H}_I$. We note that

$$|\text{vac}\rangle = |\text{vac}_{(-\infty, 0] \cup (t_k, \infty)}\rangle \otimes |\text{vac}_{(0, t_1]}\rangle \otimes |\text{vac}_{(t_1, t_2]}\rangle \otimes \dots \otimes |\text{vac}_{(t_{k-1}, t_k]}\rangle \quad (\text{A6})$$

Furthermore, for any system operator Q and for $s < t$, we note that

$$q(t, s) = \langle \text{vac}_{(s, t]} | U(s, t) Q U(t, s) | \text{vac}_{(s, t]} \rangle = \mathcal{E}^\dagger(t, s)(Q), \quad (\text{A7})$$

where $\mathcal{E}^\dagger(t, s)$ is the adjoint of $\mathcal{E}(t, s)$, i.e., it is a super-operator satisfying $\text{Tr}(\mathcal{E}^\dagger(t, s)(X)Y) = \text{Tr}(X\mathcal{E}(t, s)(Y))$ for all X and Y . It is also easy to show that

$$\frac{d}{dt} \mathcal{E}^\dagger(t, s) = \mathcal{E}^\dagger(t, s) \mathcal{L}^\dagger(t) \quad \text{where } \mathcal{L}^\dagger(t) = -i[\cdot, H(t)] + \sum_{\alpha} \mathcal{D}_{L_\alpha}^\dagger,$$

with $\mathcal{D}_L^\dagger(X) = L^\dagger X L - \{L^\dagger L, X\}/2$.

To establish Eq. (A7), we begin by partitioning the interval $(s, t] = (\tau_0, \tau_1] \cup (\tau_1, \tau_2] \cup \dots \cup (\tau_{m-1}, \tau_m]$, where $\tau_j = s + j\varepsilon$ with $\varepsilon = (t-s)/m$. Then, we note that for any system operator Q' ,

$$\begin{aligned}
\langle \text{vac}_{(\tau_{k-1}, \tau_k]} | U(\tau_{k-1}, \tau_k) Q' U(\tau_k, \tau_{k-1}) | \text{vac}_{(\tau_{k-1}, \tau_k]} \rangle &= Q' - i \int_{\tau_{k-1}}^{\tau_k} \langle \text{vac}_{(\tau_{k-1}, \tau_k]} | [Q', H(s)] | \text{vac}_{(\tau_{k-1}, \tau_k]} \rangle ds \\
&\quad - \int_{\tau_{k-1}}^{\tau_k} \int_{\tau_{k-1}}^s \langle \text{vac}_{(\tau_{k-1}, \tau_k]} | [[Q', H(s')], H(s)] | \text{vac}_{(\tau_{k-1}, \tau_k]} \rangle ds dt + O(\varepsilon^2) \\
&= Q' - i \int_{\tau_{k-1}}^{\tau_k} [Q', H_S(s)] ds + \varepsilon \sum_{\alpha} \mathcal{D}_{L_{\alpha}}^{\dagger}(Q') + O(\varepsilon^2) \\
&= \mathcal{E}^{\dagger}(\tau_k, \tau_{k-1})(Q') + O(\varepsilon^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
q(t, s) &= \left(\prod_{k=0}^{m-1} \langle \text{vac}_{(\tau_k, \tau_{k+1}]} | \right) \left(\prod_{k=0}^{m-1} U(\tau_k, \tau_{k+1}) \right) Q \left(\prod_{k=m-1}^0 U(\tau_{k+1}, \tau_k) \right) \left(\prod_{k=m-1}^0 | \text{vac}_{(\tau_k, \tau_{k+1}]} \rangle \right) \\
&= \left(\prod_{k=0}^{m-1} \langle \text{vac}_{(\tau_k, \tau_{k+1}]} | U(\tau_k, \tau_{k+1}) \right) Q \left(\prod_{k=m-1}^0 U(\tau_{k+1}, \tau_k) | \text{vac}_{(\tau_k, \tau_{k+1}]} \rangle \right) = \left(\prod_{k=0}^{m-1} \mathcal{E}^{\dagger}(\tau_{k+1}, \tau_k) \right) Q + O(m\varepsilon^2) \\
&= \mathcal{E}^{\dagger}(t, s)(Q) + O(|t-s|\varepsilon).
\end{aligned} \tag{A8}$$

Taking $\varepsilon \rightarrow 0$ establishes Eq. (A7). Next, using Eqs. (A6) and (A7), we obtain that

$$\begin{aligned}
\langle \phi, \text{vac} | \left(\prod_{j=1}^{k-1} O_j(t_j) \right) P(t_k) \left(\prod_{j=k-1}^1 Q_j(t_j) \right) | \phi, \text{vac} \rangle \\
&= \langle \phi, \text{vac} | U(t_0, t_1) \left(\prod_{j=1}^{k-1} O_j U(t_j, t_{j+1}) \right) P \left(\prod_{j=k-1}^1 U(t_{j+1}, t_j) Q_j \right) U(t_1, t_0) | \phi, \text{vac} \rangle \\
&= \langle \phi | [\langle \text{vac}_{(t_0, t_1]} | U(t_0, t_1) \left(\prod_{j=1}^{k-1} O_j \langle \text{vac}_{(t_j, t_{j+1}]} | U(t_j, t_{j+1}) \right) P \\
&\quad \times \left(\prod_{j=k-1}^1 U(t_{j+1}, t_j) | \text{vac}_{(t_j, t_{j+1}]} \rangle Q_j \right) U(t_1, t_0) | \text{vac}_{(t_0, t_1]} \rangle] | \phi \rangle \\
&= \langle \phi | \mathcal{E}^{\dagger}(t_1, t_0) (O_1 \mathcal{E}^{\dagger}(t_2, t_1) (O_2 \mathcal{E}^{\dagger}(t_3, t_2) (O_3 \cdots \mathcal{E}^{\dagger}(t_{k-1}, t_{k-2}) (O_{k-1} \mathcal{E}^{\dagger}(t_k, t_{k-1})) (P) Q_{k-1} \cdots Q_3) Q_2) Q_1) | \phi \rangle \\
&= \text{Tr} \left(P \left(\prod_{j=k-1}^1 \mathcal{E}(t_{j+1}, t_j) \mathcal{M}_j \right) \mathcal{E}(t_1, t_0) (|\phi\rangle\langle\phi|) \right),
\end{aligned} \tag{A9}$$

which proves the proposition statement. ■

Next, we consider the correlator $\mathcal{C}_{x^1 \dots x^n, y^1 \dots y^n}^{(n)}(t_1 \cdots t_n)$ defined in the main text:

$$\mathcal{C}_{x^1 \dots x^n, y^1 \dots y^n}^{(n)}(t_1 \cdots t_n) = \left\langle \prod_{i=1}^n x_{t_i}^{i\dagger} \prod_{i=1}^n y_{t_i}^i \right\rangle,$$

where $x^i, y^i \in \{a, b\}$. Using Eq. (2), this correlator can be expressed in the Heisenberg picture as

$$\begin{aligned}
\mathcal{C}_{x^1 \dots x^n, y^1 \dots y^n}^{(n)}(t_1 \cdots t_n) &= \frac{1}{\mathcal{N}^2} \langle \text{vac}, \phi_{S,i} | \Phi_{S,f}(T) \prod_{i=1}^n x_{t_i}^{i\dagger}(T) \prod_{i=1}^n y_{t_i}^i(T) | \text{vac}, \phi_{S,i} \rangle, \\
&= \frac{1}{\mathcal{N}^2} \langle \text{vac}, \phi_{S,i} | \tilde{T} \left[\prod_{i=1}^n x_{t_i}^{i\dagger}(T) \right] \Phi_{S,f}(T) \tilde{T} \left[\prod_{i=1}^n y_{t_i}^i(T) \right] | \text{vac}, \phi_{S,i} \rangle,
\end{aligned} \tag{A10}$$

where $O(T) = U^\dagger(T, 0)OU(T, 0)$, $\Phi_{S,f} = |\phi_{S,f}\rangle\langle\phi_{S,f}|$, and $\vec{T}$ orders Heisenberg-picture operators in the decreasing order of time indices while $\tilde{T}$ performs this time-ordering in increasing order of time-indices. Note that in Eq. (A10), the time ordering operators can be inserted fictitiously since all the Heisenberg-picture operators are evaluated at time T . Next, from the input-output formalism [107], it follows that

$$x_t(T) = x_t(0) - iL_X(t)\Theta(0 \leq t \leq T), \quad (\text{A11})$$

where the indicator function $\Theta(0 \leq t \leq T) = 1$ if $t \in (0, T)$, $1/2$ if $t \in \{0, T\}$ and 0 otherwise. We now substitute Eq. (A11) into Eq. (A10): Due to the time-ordering operator, we can then move $y_t^i(0)$ to the right and $x_t^{i\dagger}(0)$ to the left and apply them on the vacuum state in the port. Since $y_t^i(0)|\text{vac}\rangle = 0$ and $\langle\text{vac}|x_t^{i\dagger}(0) = 0$, we obtain that

$$\begin{aligned} \mathcal{C}_{x^1 \dots x^n, y^1 \dots y^n}^{(n)}(t_1 \dots t_n) &= \frac{1}{\mathcal{N}^2} \langle \text{vac}, \phi_{S,i} | \tilde{T} \left[\prod_{i=1}^n L_{X^i}^\dagger(t_i) \right] \\ &\times \Phi_{S,f}(T) \vec{T} \left[\prod_{i=1}^n L_{Y^i}(t_i) \right] | \text{vac}, \phi_{S,i} \rangle. \end{aligned} \quad (\text{A12})$$

Now, applying the quantum regression theorem (Proposition 1) to Eq. (A12), we obtain Eq. (10) from the main text.

3. Sensitivity of parity and photon number measurement

Here, we analyze the estimation variance of φ , when the operator O is measured in the photonic output state. In this case, the sensitivity is given by

$$\xi_\varphi(O) = \frac{|\partial_\varphi \langle O \rangle|^2}{\langle O^2 \rangle - \langle O \rangle^2}, \quad (\text{A13})$$

where the expectation is taken with respect to the state $|\psi_\varphi\rangle$. Next, we show that for the photon number and parity operator, we can obtain the sensitivity by tracking the evolution of the system and not the photonic state.

a. Photon number measurement

First we consider the photon number measurement $O = \int a_t^\dagger a_t dt$. We can write

$$\langle \psi_\varphi | O | \psi_\varphi \rangle = \langle \psi | U_{\text{MZI}}^\dagger O U_{\text{MZI}} | \psi \rangle = \int \langle (a_t^\dagger \cos(\varphi) + b_t^\dagger \sin(\varphi))(a_t \cos(\varphi) + b_t \sin(\varphi)) \rangle dt, \quad (\text{A14})$$

where we used Eq. (3). Thus, at $\varphi = 0$, we get

$$\langle O \rangle = \int_0^T C_{a:a}^{(1)}(t) dt, \quad (\text{A15})$$

$$\langle O^2 \rangle = \int_0^T \int_0^T C_{a,a:a,a}^{(2)}(t,s) dt ds + \int_0^T C_{a:a}^{(1)}(t) dt, \quad (\text{A16})$$

and

$$\partial_\varphi \langle O \rangle = 2 \int_0^T \text{Re}(C_{a:b}^{(1)}(t)) dt. \quad (\text{A17})$$

As a reminder $C_{x_1, \dots, x_n; y_1, \dots, y_n}^{(1)}(t_1, \dots, t_n)$ is an n -point correlation function which can be computed by only tracking the internal dynamics of the source (see Eq. (10)).

b. Parity-like measurement

Second, we analyze the sensitivity of a parity-like measurement defined as

$$O = \exp \left(i\eta \int_0^T a_t^\dagger a_t dt \right). \quad (\text{A18})$$

Note, that η is an arbitrary phase and for $\eta = \pi$, O corresponds to the parity number measurement of the photonic state. We can write

$$\begin{aligned} \langle O \rangle &= \langle \psi | U_{\text{MZI}}^\dagger O U_{\text{MZI}} | \psi \rangle, \\ &= \frac{1}{\mathcal{N}^2} \langle \phi_{S,i}, \text{vac} | U^\dagger(T, 0) | \phi_{S,f} \rangle O_\varphi U(T, 0) O_\varphi^\dagger | \phi_{S,i}, \text{vac} \rangle, \\ &= \frac{1}{\mathcal{N}^2} \langle \phi_{S,i}, \text{vac} | U^\dagger(T, 0) | \phi_{S,f} \rangle \tilde{U}(T, 0) | \phi_{S,i}, \text{vac} \rangle \end{aligned} \quad (\text{A19})$$

where we defined $O_\varphi = U_{\text{MZI}}^\dagger O U_{\text{MZI}}$, and in the second equality we used Eq. (2), and $O_\varphi |\text{vac}\rangle = |\text{vac}\rangle$. Also, $\tilde{U}(T, 0)$ is defined to be the unitary with respect to the Hamiltonian $\tilde{H}_I(t) = O_\varphi H_I(t) O_\varphi^\dagger$. Next, we can write

$$O_\varphi = \exp \left(i\eta \int_0^T a_t^\dagger(\varphi) a_t(\varphi) \right), \quad (\text{A20})$$

with

$$a_t(\varphi) = a_t \cos \varphi + b_t \sin \varphi, \quad (\text{A21a})$$

$$b_t(\varphi) = b_t \cos \varphi - a_t \sin \varphi. \quad (\text{A21b})$$

Since,

$$O_\varphi a_t(\varphi) O_\varphi^\dagger = e^{-i\eta} a_t(\varphi), \quad (\text{A22a})$$

$$O_\varphi b_t(\varphi) O_\varphi^\dagger = b_t(\varphi), \quad (\text{A22b})$$

we obtain

$$O_\varphi a_t O_\varphi^\dagger = a_t (e^{-i\eta} \cos^2 \varphi + \sin^2 \varphi) + b_t ((e^{-i\eta} - 1) \cos \varphi \sin \varphi), \quad (\text{A23a})$$

$$O_\varphi b_t O_\varphi^\dagger = b_t (e^{-i\eta} \sin^2 \varphi + \cos^2 \varphi) + a_t ((e^{-i\eta} - 1) \cos \varphi \sin \varphi). \quad (\text{A23b})$$

Inserting these into $\tilde{H}_I(t) = O_\varphi H_I(t) O_\varphi^\dagger$, we obtain

$$\begin{aligned} \tilde{H}_I(t) = & H_S(t) + \sum_{x \in \{a,b\}} (x_t \tilde{L}_X^\dagger + \text{H.c.}) \\ & + \sum_\alpha (c_{\alpha,t} K_\alpha^\dagger + \text{H.c.}), \end{aligned} \quad (\text{A24})$$

where

$$\tilde{L}_A = L_A (e^{i\eta} \cos^2 \varphi + \sin^2 \varphi) + L_B ((e^{i\eta} - 1) \cos \varphi \sin \varphi), \quad (\text{A25a})$$

$$\tilde{L}_B = L_B (e^{i\eta} \sin^2 \varphi + \cos^2 \varphi) + L_A ((e^{i\eta} - 1) \cos \varphi \sin \varphi). \quad (\text{A25b})$$

Here, K_α is the system jump operator associated with loss, as defined in Eq. (11), and $c_{\alpha,t}$ is the corresponding time-domain annihilation operator describing the loss channel.

Using a similar approach as in the proof for Proposition 1, we can simplify the expression for $\langle O \rangle$ in Eq. (A19). Again, we begin by partitioning the interval $[0, T] = [\tau_0, \tau_1] \cup (\tau_1, \tau_2] \cup \dots \cup (\tau_{m-1}, \tau_m]$, where $\tau_j = j\varepsilon$ with $\varepsilon = T/m$. Then, for any system operator Q ,

$$\begin{aligned} \langle \text{vac}_{(\tau_{k-1}, \tau_k]} | U(\tau_{k-1}, \tau_k) Q \tilde{U}(\tau_k, \tau_{k-1}) | \text{vac}_{(\tau_{k-1}, \tau_k]} \rangle &= Q - i \int_{\tau_{k-1}}^{\tau_k} \langle \text{vac}_{(\tau_{k-1}, \tau_k]} | Q \tilde{H}(s) - H(s) Q | \text{vac}_{(\tau_{k-1}, \tau_k]} \rangle ds \\ &\quad - \int_{\tau_{k-1}}^{\tau_k} \int_{\tau_{k-1}}^s \langle \text{vac}_{(\tau_{k-1}, \tau_k]} | ([Q \tilde{H}(s') - H(s') Q] \tilde{H}(s) \\ &\quad - H(s) [Q \tilde{H}(s') - H(s') Q]) | \text{vac}_{(\tau_{k-1}, \tau_k]} \rangle ds dt + O(\varepsilon^2) \\ &= Q - i \int_{\tau_{k-1}}^{\tau_k} [Q, H_S(s)] ds + \varepsilon \sum_\alpha \mathcal{D}_{K_\alpha}^\dagger(Q) + \varepsilon \sum_{x \in \{a,b\}} L_X^\dagger Q \tilde{L}_X \\ &\quad - \frac{1}{2} (Q \tilde{L}_X^\dagger \tilde{L}_X + L_X^\dagger L_X Q) + O(\varepsilon^2). \end{aligned}$$

Therefore,

$$\begin{aligned} \langle O \rangle &= \frac{1}{\mathcal{N}^2} \langle \phi_{S,i} | \left(\prod_{k=0}^{m-1} \langle \text{vac}_{(\tau_k, \tau_{k+1}]} | \right) \left(\prod_{k=0}^{m-1} U(\tau_k, \tau_{k+1}) \right) \rho_f \left(\prod_{k=m-1}^0 \tilde{U}(\tau_{k+1}, \tau_k) \right) \left(\prod_{k=m-1}^0 | \text{vac}_{(\tau_k, \tau_{k+1}]} \rangle \right) | \phi_{S,i} \rangle \\ &= \frac{1}{\mathcal{N}^2} \langle \phi_{S,i} | \left(\prod_{k=0}^{m-1} \langle \text{vac}_{(\tau_k, \tau_{k+1}]} | U(\tau_k, \tau_{k+1}) \right) \rho_f \left(\prod_{k=m-1}^0 U(\tau_{k+1}, \tau_k) | \text{vac}_{(\tau_k, \tau_{k+1}]} \rangle \right) | \phi_{S,i} \rangle \\ &= \frac{1}{\mathcal{N}^2} \langle \phi_{S,i} | \left(\prod_{k=0}^{m-1} \mathcal{E}_{\varphi,\eta}^\dagger(\tau_{k+1}, \tau_k) \right) \rho_f | \phi_{S,i} \rangle + O(m\varepsilon^2) = \frac{1}{\mathcal{N}^2} \langle \phi_{S,i} | \mathcal{E}_{\varphi,\eta}^\dagger(T, 0) (\rho_f) | \phi_{S,i} \rangle + O(T\varepsilon). \end{aligned} \quad (\text{A26})$$

Taking $\varepsilon \rightarrow 0$ establishes

$$\langle O \rangle = \frac{1}{\mathcal{N}^2} \text{Tr}(\rho_f \mathcal{E}_{\varphi,\eta}(T, 0)(\rho_i)), \quad (\text{A27})$$

where $\mathcal{E}_{\varphi,\eta}(T, 0)$ is the channel associated with the following Lindbladian

$$\begin{aligned} \mathcal{S}_{\varphi,\eta}(t) = & -i[H_S(t), \cdot] + \sum_{\alpha} \mathcal{D}_{K_{\alpha}} + \sum_{X \in \{A, B\}} \tilde{L}_X \cdot L_X^{\dagger} \\ & - \frac{1}{2} (\tilde{L}_X^{\dagger} \tilde{L}_X \cdot + \cdot L_X^{\dagger} L_X), \end{aligned} \quad (\text{A28})$$

$$\begin{aligned} = & -i[H_S(t), \cdot] + \sum_{\alpha} \mathcal{D}_{K_{\alpha}} + \sum_{X \in \{A, B\}} \tilde{L}_X \cdot L_X^{\dagger} \\ & - \frac{1}{2} (L_X^{\dagger} L_X \cdot + \cdot L_X^{\dagger} L_X). \end{aligned} \quad (\text{A29})$$

Note that for $\eta = 0$, and $\varphi = 0$, $\mathcal{S}_{0,0}(t)$ corresponds to the Lindbladian for the internal dynamics of the source.

Finally, using Eq. (A27) the derivative of $\langle O \rangle$ with respect to φ is

$$\partial_{\varphi} \langle O \rangle = \frac{1}{\mathcal{N}^2} \int_0^T \text{Tr}(\rho_f \mathcal{E}_{\varphi,\eta}(T, t) \partial_{\varphi} \mathcal{S}_{\varphi,\eta}(t, 0)(\rho_i)) dt, \quad (\text{A30})$$

where

$$\partial_{\varphi} \mathcal{S}|_{\varphi=0} = (e^{i\eta} - 1)(L_B \cdot L_A^{\dagger} + L_A \cdot L_B^{\dagger}). \quad (\text{A31})$$

Thus, we can compute the sensitivity of the parity-like measurement on the output photonic state by only simulating the dynamics of the source.

APPENDIX B: MACH-ZEHNDER INTERFEROMETER WITH ARBITRARY LINEAR TRANSFORMATION

In the main text we specialized to a balanced (50: 50) Mach-Zehnder interferometer (MZI), which yielded the

real rotation in Eq. (3). Here we generalize the MZI to an arbitrary *lossless* two-port linear optical transformation. In the Heisenberg picture, the MZI acts on the traveling-wave field operators as

$$U_{\text{MZI}}^{\dagger}(\varphi) \begin{bmatrix} a_t \\ b_t \end{bmatrix} U_{\text{MZI}}(\varphi) = S_{\text{MZI}}(\varphi) \begin{bmatrix} a_t \\ b_t \end{bmatrix}, \quad (\text{B1})$$

where $S_{\text{MZI}}(\varphi) \in U(2)$ is a φ -dependent scattering matrix. Throughout this appendix we assume the interferometer satisfies $S_{\text{MZI}}(0) = I$.

1. Point-interaction Hamiltonian

It is convenient to represent a purely scattering element with scattering matrix $S_{\text{MZI}}(\varphi)$ via the Hamiltonian [62]

$$H_{\text{MZI}}(t, \varphi) = \begin{bmatrix} a_t^{\dagger} & b_t^{\dagger} \end{bmatrix} V(\varphi) \begin{bmatrix} a_t \\ b_t \end{bmatrix}, \quad V(\varphi) = V(\varphi)^{\dagger}, \quad (\text{B2})$$

where the Hermitian 2×2 matrix $V(\varphi)$ is related to $S_{\text{MZI}}(\varphi)$ through the transform

$$\begin{aligned} S_{\text{MZI}}(\varphi) &= \left(I - \frac{i}{2} V(\varphi) \right) \left(I + \frac{i}{2} V(\varphi) \right)^{-1} \Leftrightarrow V(\varphi) \\ &= 2i (S_{\text{MZI}}(\varphi) - I)(S_{\text{MZI}}(\varphi) + I)^{-1}. \end{aligned} \quad (\text{B3})$$

Let

$$|\psi_{\varphi}\rangle = \vec{T} \exp \left(-i \int_0^T H_{\text{MZI}}(s, \varphi) ds \right) |\psi\rangle. \quad (\text{B4})$$

Using Duhamel's formula as in the main text, and the assumption $V(0) = 0$ (using $S_{\text{MZI}}(0) = I$), we obtain

$$\frac{\partial}{\partial \varphi} |\psi_{\varphi}\rangle|_{\varphi=0} = -i H_d |\psi\rangle, \quad H_d := \int_0^T \begin{bmatrix} a_t^{\dagger} & b_t^{\dagger} \end{bmatrix} W \begin{bmatrix} a_t \\ b_t \end{bmatrix} dt, \quad W := \partial_{\varphi} V(\varphi)|_{\varphi=0}. \quad (\text{B5})$$

It then follows that the QFI at $\varphi = 0$ is

$$\text{QFI} = 4(\langle H_d^2 \rangle - \langle H_d \rangle^2). \quad (\text{B6})$$

2. QFI in terms of correlation functions for general W

Write the entries of W as W_{xy} with $x, y \in \{a, b\}$. Expanding H_d from Eq. (B5) yields

$$\langle H_d \rangle = \int_0^T \sum_{x,y \in \{a,b\}} W_{xy} \mathcal{C}_{x;y}^{(1)}(t) dt, \quad (\text{B7})$$

$$\begin{aligned} \langle H_d^2 \rangle &= \int_0^T \int_0^T \sum_{x,y,u,v \in \{a,b\}} W_{xy} W_{uv} \mathcal{C}_{x,u;y,v}^{(2)}(t, s) dt ds \\ &+ \int_0^T \sum_{x,v \in \{a,b\}} (W^2)_{xv} \mathcal{C}_{x;v}^{(1)}(t) dt. \end{aligned} \quad (\text{B8})$$

Combining Eqs. (B6)–(B8), we obtain the QFI in terms of correlations:

$$\text{QFI} = 4 \int_0^T \int_0^T \sum_{x,y,u,v \in \{a,b\}} W_{xy} W_{uv} [\mathcal{C}_{x,u;y,v}^{(2)}(t, s) - \mathcal{C}_{x;y}^{(1)}(t) \mathcal{C}_{u;v}^{(1)}(s)] dt ds + 4 \int_0^T \sum_{x,v \in \{a,b\}} (W^2)_{xv} \mathcal{C}_{x;v}^{(1)}(t) dt. \quad (\text{B9})$$

Equation (B9) is valid for any lossless two-port interferometer described by $S_{\text{MZI}}(\varphi)$ with $S_{\text{MZI}}(0) = I$ and depends only on one- and two-point correlation functions of the input state $|\psi\rangle$.

3. Example: Symmetric MZI with beam-splitter angle θ

As an example, consider a symmetric MZI with beam splitters of angle θ . In the convention where the balanced case $\theta = \pi/4$ reproduces Eq. (3), the scattering matrix can be written as

$$\begin{aligned} S_{\text{MZI}}(\varphi; \theta) &= \cos \varphi I + i \sin \varphi A(\theta), \\ A(\theta) &:= \cos(2\theta) \sigma_z + \sin(2\theta) \sigma_y, \end{aligned} \quad (\text{B10})$$

with $A(\theta)^2 = I$. Using Eq. (B3) one finds

$$V(\varphi) = -2 \tan\left(\frac{\varphi}{2}\right) A(\theta) \Rightarrow W = \partial_\varphi V(\varphi)|_{\varphi=0} = -A(\theta). \quad (\text{B11})$$

Introduce the local operators

$$\begin{aligned} h(t) &:= i(a_t^\dagger b_t - a_t b_t^\dagger) = i(a_t^\dagger b_t - b_t^\dagger a_t), \\ z(t) &:= a_t^\dagger a_t - b_t^\dagger b_t. \end{aligned} \quad (\text{B12})$$

Using $[a_t^\dagger b_t^\dagger] \sigma_y [a_t b_t] = -h(t)$ and $[a_t^\dagger b_t^\dagger] \sigma_z [a_t b_t] = z(t)$, Eq. (B5) becomes

$$H_d = \int_0^T (\sin(2\theta) h(t) - \cos(2\theta) z(t)) dt. \quad (\text{B13})$$

Substituting $W = -A(\theta)$ into the general correlation formula (B9) yields the QFI for an MZI with arbitrary splitting ratio θ . In particular, at $\theta = \pi/4$ one has $\cos(2\theta) = 0$ and $\sin(2\theta) = 1$, and Eq. (B9) reduces to the balanced $-50:50$ expression used in the main text.

APPENDIX C: CONNECTION TO TWO-SIDED MASTER EQUATION APPROACH

Here we connect our interferometry set-up to the two-sided master-equation approach developed in Refs. [46,47], which provides an alternative route to the QFI of the emitted field.

As a reminder, defining $\mathbf{x}_t = (a_t, b_t)^T$ and system jump operators $\mathbf{L} = (L_A, L_B)^T$, the joint emitter and light channels unitary is

$$U_I(T, 0) = \tilde{T} \exp\left(-i \int_0^T H_I(t) dt\right), \quad (\text{C1})$$

with interaction Hamiltonian

$$H_I(t) = H_S(t) + \mathbf{x}_t^\dagger \mathbf{L} + \mathbf{L}^\dagger \mathbf{x}_t. \quad (\text{C2})$$

After the emission, the two channels are mixed by a passive linear optical element (the MZI) described by

$$U_{\text{MZI}}^\dagger(\varphi) \mathbf{x}_t U_{\text{MZI}}(\varphi) = S_{\text{MZI}}(\varphi) \mathbf{x}_t. \quad (\text{C3})$$

The final joint state is then

$$|\psi_\varphi\rangle = U_{\text{MZI}}(\varphi) U_I(T, 0) |\phi_{S,i}, \text{vac}\rangle. \quad (\text{C4})$$

Since the MZI is passive and leaves the vacuum invariant, $U_{\text{MZI}}(\varphi) |\text{vac}\rangle = |\text{vac}\rangle$, thus

$$\begin{aligned} |\psi_\varphi\rangle &= \tilde{U}_I(T, 0) |\phi_{S,i}, \text{vac}\rangle, \\ \tilde{U}_I(T, 0) &:= U_{\text{MZI}}(\varphi) U_I(T, 0) U_{\text{MZI}}^\dagger(\varphi). \end{aligned} \quad (\text{C5})$$

The transformed propagator $\tilde{U}_I$ is generated by the Hamiltonian $\tilde{H}_I(t, \varphi) = U_{\text{MZI}}(\varphi) H_I(t) U_{\text{MZI}}^\dagger(\varphi)$. Using Eq. (C3), we have

$$U_{\text{MZI}}(\varphi) \mathbf{x}_t U_{\text{MZI}}^\dagger(\varphi) = S_{\text{MZI}}(\varphi)^{-1} \mathbf{x}_t, \quad (\text{C6})$$

and hence $\tilde{H}_I(t, \varphi)$ retains the same structure as Eq. (C2), but with modified (generally φ -dependent) jump operators:

$$\tilde{H}_I(t, \varphi) = H_S(t) + \mathbf{x}_I^\dagger \tilde{\mathbf{L}}(\varphi) + \tilde{\mathbf{L}}(\varphi)^\dagger \mathbf{x}_I, \quad (\text{C7})$$

$$\tilde{\mathbf{L}}(\varphi) := S_{\text{MZI}}(\varphi)^{-1} \mathbf{L}. \quad (\text{C8})$$

Thus, the emitter + MZI can be viewed as an effective emitter with φ -dependent jump operators $\{\tilde{\mathbf{L}}_A(\varphi), \tilde{\mathbf{L}}_B(\varphi)\}$ and the same system Hamiltonian $H_S(t)$. This establishes a direct mapping from our sensing architecture to the framework of Refs. [46,47], where the interferometric phase φ is treated as a parameter appearing in the system, enabling computation of the QFI via system-only evolution under the two-sided master equation.

APPENDIX D: SCALING OF THE PHOTON NUMBER WITH TIME FOR TIME-INDEPENDENT SOURCES

In this appendix we show that the number of photons scales linearly with time T for a time-independent source. Let the source evolve according to a time-independent channel $\mathcal{E}(t, 0) = e^{\mathcal{L}t}$, where $\mathcal{L}$ is the Lindbladian of the source. The source state at time t is $\rho(t) = \mathcal{E}(t, 0)(\rho(0))$, and the total expected number of emitted photons is

$$N_{\text{ph}}(T) = \int_0^T \text{Tr}(L^\dagger L \rho(t)) dt. \quad (\text{D1})$$

Since the source Hilbert space is finite dimensional, $\mathcal{L}$ admits a Jordan decomposition. Thus we can write

$$e^{\mathcal{L}t} = \sum_{\nu \in \sigma(\mathcal{L})} e^{\nu t} \sum_{m=0}^{d_\nu-1} t^m \mathcal{K}_{\nu,m}, \quad (\text{D2})$$

where ν are the eigenvalues of $\mathcal{L}$, d_ν is the size of the largest Jordan block associated with ν , and the maps $\mathcal{K}_{\nu,m}$ are time-independent.

Because $e^{\mathcal{L}t}$ is a CPTP channel, its eigenvalues satisfies, $\text{Re}(\nu) \leq 0$. Moreover, for any eigenvalue with $\text{Re}(\nu) = 0$, the corresponding Jordan blocks have dimension one $d_\nu = 1$ [108]. Thus, we obtain

$$\text{Tr}(L^\dagger L \rho(t)) = c_1 + \sum_{\omega_j \neq 0} a_j e^{-i\omega_j t} + \sum_{\alpha} \sum_{m=0}^{d_\alpha-1} a_{\alpha,m} t^m e^{\nu_\alpha t}, \quad (\text{D3})$$

where $\text{Re}(\nu_\alpha) < 0$. The first term comes from the zero-eigenvalue subspace which determines the time-averaged emission rate, and the second term comes from eigenvalues with zero real part but non-zero imaginary part. Next,

computing the photon number gives:

$$N_{\text{ph}}(T) = c_1 T + \sum_{\omega_j \neq 0} a_j \int_0^T dt e^{-i\omega_j t} + \sum_{\alpha} \sum_{m=0}^{d_\alpha-1} a_{\alpha,m} \int_0^T dt t^m e^{\nu_\alpha t}. \quad (\text{D4})$$

The oscillatory terms have a constant contribution at most, and the decaying terms converge to constants, with exponentially corrections in T . Thus, defining $\lambda = \min_{\alpha} |\text{Re}(\nu_\alpha)|$, we obtain

$$N_{\text{ph}}(T) = c_0 + c_1 T + \sum_{\omega_j \neq 0} b_j e^{-i\omega_j T} + O(e^{-\lambda T}). \quad (\text{D5})$$

Therefore,

$$\lim_{T \rightarrow \infty} \frac{N_{\text{ph}}(T)}{T} = c_1. \quad (\text{D6})$$

Thus, if $c_1 > 0$, the emitted photon number grows linearly in time.

APPENDIX E: ANALYTICAL QFI CALCULATIONS FOR SELECTED EXAMPLES

In this appendix we provide analytic derivations for several QFI expressions stated in Appendix III. Throughout, we consider two identical and independent sources feeding the two input ports of a balanced MZI, and evaluate the QFI at $\varphi = 0$.

1. Harmonic oscillator

Consider first the simplest setting of an optical cavity with annihilation operator a and frequency ω_c . We consider the source Hamiltonian to be $H = \omega_c a^\dagger a$ and jump operator $L = a$. We will initialize the cavity in an initial state $|\phi_i\rangle$, and set the total emission time $T \rightarrow \infty$. The cavity would then have decayed to the vacuum state $|\text{vac}\rangle$ which we will use as the final state (i.e., $|\phi_f\rangle = |\text{vac}\rangle$).

For simplicity we will choose the reference frequency to be $\omega_c = 0$. For this setup, it is useful to note that $\mathcal{D}_a^\dagger(X) = -X/2$ for $X \in \{a, a^\dagger\}$ and $\mathcal{D}_a^\dagger(X) = -X$ for $X \in \{a^2, a^{\dagger 2}, a^\dagger a\}$ which implies that

$$\begin{aligned} \mathcal{E}^\dagger(t, s)(X) &= e^{-(t-s)/2} X \quad \text{if } X \in \{a, a^\dagger\}, \\ \mathcal{E}^\dagger(t, s)(X) &= e^{-(t-s)} X \quad \text{if } X \in \{a^2, a^{\dagger 2}, a^\dagger a\}, \end{aligned} \quad (\text{E1})$$

where, for a channel $\mathcal{E}(X) = \sum_i K_i X K_i^\dagger$, its adjoint is given by $\mathcal{E}^\dagger(X) = \sum_i K_i X K_i^\dagger$. From Eq. (19), we then obtain that

$$\text{QFI} = 8(|\langle \phi_i | a^\dagger a | \phi_i \rangle|^2 - |\langle \phi_i | a^2 | \phi_i \rangle|^2 + \langle \phi_i | a^\dagger a | \phi_i \rangle).$$

- (a) Suppose $|\phi_A\rangle = |\alpha\rangle$, i.e., the initial state is a coherent state with (mean) photon number $N = |\alpha|^2$. We then obtain

$$\text{QFI} = 8|\alpha|^2 \sim N \text{ for large } N,$$

and matches the expected scaling.

- (b) Suppose $|\phi_A\rangle = |N\rangle$, i.e., the initial state is Fock state, then

$$\text{QFI} = 8(N^2 + N) \sim N^2 \text{ for large } N,$$

and matches the expected scaling.

2. Driven cavity source

Here, we consider the driven cavity model used in Appendix III. The system Hamiltonian and jump operator are

$$H = \Omega(c + c^\dagger), \quad L = \sqrt{\kappa}c, \quad (\text{E2})$$

with initial state $|0\rangle$. The cavity amplitude $\alpha(t) := \langle c(t) \rangle$ obeys

$$\dot{\alpha}(t) = -\frac{\kappa}{2}\alpha(t) - i\Omega, \quad \alpha(0) = 0, \quad (\text{E3})$$

hence

$$\alpha(t) = -\frac{2i\Omega}{\kappa}(1 - e^{-\kappa t/2}). \quad (\text{E4})$$

The output photon flux in the port is $n(t) = \langle a_t^\dagger a_t \rangle = \kappa|\alpha(t)|^2$, so the mean number of emitted photons from one source over $[0, T]$ is

$$\begin{aligned} \bar{N}(T) &:= \int_0^T n(t)dt = \kappa \int_0^T |\alpha(t)|^2 dt \\ &= \frac{4\Omega^2}{\kappa} \int_0^T (1 - 2e^{-\kappa t/2} + e^{-\kappa t}) dt \\ &= \frac{4\Omega^2}{\kappa} \left[T - \frac{4}{\kappa}(1 - e^{-\kappa T/2}) + \frac{1}{\kappa}(1 - e^{-\kappa T}) \right]. \end{aligned} \quad (\text{E5})$$

Since the dynamics is linear and the input is vacuum, the emitted field is coherent and all connected two-time correlations vanish. Plugging the coherent-state correlators into Eq. (8) shows that only the single-integral contribution remains, yielding

$$\text{QFI} = 4 \int_0^T (\mathcal{C}_{a;a}^{(1)}(t) + \mathcal{C}_{b;b}^{(1)}(t))dt = 8\bar{N}(T), \quad (\text{E6})$$

where $\bar{N}(T)$ is given in Eq. (E5). In the long-time limit $\kappa T \gg 1$, this simplifies to

$$\text{QFI} = \frac{32\Omega^2}{\kappa} T + O(1), \quad (\text{E7})$$

which shows that the QFI scales linearly with time T .

3. Driven 2-level and π -level source

Next, we consider a continuously driven two level system [Fig. 2(a)]. The source Hamiltonian for this system is given by $H = (\Omega\sigma + \text{H.c.})$ and the jump operator $L = \sqrt{\gamma}\sigma$ where $|g\rangle\langle e|$. The Lindbladian specified by H and L has a unique fixed point τ given by

$$\begin{aligned} \tau &= \frac{1}{\gamma^2 + 8|\Omega|^2} ((\gamma^2 + 4|\Omega|^2)|g\rangle\langle g| + 4|\Omega|^2|e\rangle\langle e| \\ &\quad + 2i\gamma\Omega|g\rangle\langle e| - 2i\gamma\Omega^*|e\rangle\langle g|). \end{aligned} \quad (\text{E8})$$

Since this fixed point is unique, as per the discussion in Sec. IV, we expect the QFI to scale linearly with T asymptotically. This is numerically demonstrated in Fig. 2(b) where we plot the QFI as a function of time for $\Omega \in \{0.5\gamma, \gamma, 2\gamma, 5\gamma\}$.

Next we consider a π -level system with two excited states $|e_1\rangle, |e_2\rangle$ and two ground states $|g_1\rangle, |g_2\rangle$ [Fig. 2(d)]. The source Hamiltonian H has a coherent drive applied on the two transitions ($|e_1\rangle \rightarrow |g_1\rangle$ and $|e_2\rangle \rightarrow |g_2\rangle$) but with different phases, i.e., $H = \Omega_0(\sigma_1 + \sigma_2 e^{i\theta} + \text{H.c.})$, where $|\sigma_i\rangle = |g_i\rangle\langle e_i|$. The source is coupled to the output port with the jump operator $L = \sigma_1 + \sigma_2$. The Lindbladian specified by H and L has two fixed points, τ_1, τ_2 where τ_1 is given by Eq. (E8) with $|e\rangle \rightarrow |e_1\rangle, |g\rangle \rightarrow |g_1\rangle, \Omega \rightarrow \Omega_0$ and τ_2 is given by Eq. (E8) with $|e\rangle \rightarrow |e_2\rangle, |g\rangle \rightarrow |g_2\rangle, \Omega \rightarrow \Omega_0 e^{i\theta}$. To analyze the QFI of the emitted photons, we note that $\mathcal{E}(t, s)$ can be expressed as

$$\mathcal{E}(t, s)\rho = \text{tr}(P_1\rho)\tau_1 + \text{tr}(P_2\rho)\tau_2 + \mathcal{M}(t, s)\rho, \quad (\text{E9})$$

where $P_i = |e_i\rangle\langle e_i| + |g_i\rangle\langle g_i|$ and $\|\mathcal{M}(t, s)\|_\diamond \leq O(e^{-\gamma_0(t-s)})$ as $|t-s| \rightarrow \infty$ for some $\gamma_0 > 0$. We will consider the initial source state to be $|\phi_i\rangle = (|g_1\rangle + |g_2\rangle)/\sqrt{2}$ and the final source state to be $|\phi_f\rangle = (|g_1\rangle + |g_2\rangle)/\sqrt{2}$. We now estimate $\text{Q}^{(2)}$ defined in Eq. (19). In the limit of $\gamma_0 T \gg 1$, we obtain that

$$Q^{(2)} = 2T^2 \frac{(|\sum_j \text{Tr}(L\tau_j)|^2 \text{Tr}(\tau_j \rho_f) \text{Tr}(P_j \rho_i)|^2 - |\sum_j \text{Tr}(L\tau_j)^2 \text{Tr}(\tau_j \rho_f) \text{Tr}(P_j \rho_i)|^2)}{(\sum_j \text{Tr}(\tau_j \rho_f) \text{Tr}(P_j \rho_i))^2} + O(T), \quad (\text{E10})$$

where $\rho_i = |\phi_i\rangle\langle\phi_i|$ and $\rho_f = |\phi_f\rangle\langle\phi_f|$. Using the explicit expressions for τ_1, τ_2, P_1, P_2 and $|\phi_i\rangle, |\phi_f\rangle$, we obtain that

$$Q^{(2)} = 2\gamma^4 T^2 \frac{64\Omega_0^4 - 16\Omega_0^4 |1 + e^{-2i\theta}|^2}{(\gamma^2 + 8\Omega_0^2)^4} + O(T) = \frac{2^7 \gamma^4 \Omega_0^4}{(\gamma^2 + 8\Omega_0^2)^4} T^2 \sin^2 \theta + O(T). \quad (\text{E11})$$

Figure 2(e) shows numerically simulated QFI for the π -level system as a function θ . As expected from Eq. (E11), the QFI shows Heisenberg-limited scaling ($\sim T^2$) when $\theta \neq 0$, while when $\theta = 0$, the QFI scales as T . Physically, this can be attributed to the fact that when $\theta = 0$, the π -level system becomes indistinguishable from the 2-level system, and thus the QFI $\sim T$. On the other hand, when $\theta \neq 0$, the emitted photons are a superposition of two macroscopic photon states and hence the state exhibits QFI $\sim T^2$.

APPENDIX F: QUANTUM FISHER INFORMATION SCALING WITH TIME

1. Strictly contractive source dynamics implies no quantum advantage in interferometry

a. Strictly contractive dynamics forbid quantum advantage

We recall from the main text that a Lindbladian $\mathcal{L}(t)$ is said to generate strictly contractive dynamics if $\exists C_0, \tau_0, \gamma_0 > 0$ such that $\forall$ states ρ_1, ρ_2 ,

$$\|\mathcal{E}(t, s)(\rho_1 - \rho_2)\|_1 \leq C_0 e^{-\gamma_0 |t-s|} \|\rho_1 - \rho_2\|_1 \quad \forall |t-s| \geq \tau_0, \quad (\text{F1})$$

where $\mathcal{E}(t, s) = \mathcal{T} \exp(\int_s^t \mathcal{L}(\tau) d\tau)$. We first establish that this condition implies that the QFI introduced in Eqs. (19) scales as T and thus exhibits SQL scaling. We begin by noting the following simple fact.

Lemma 1. Suppose $\mathcal{E}(t, s)$ is strictly contractive [Eq. (F1)], then for any traceless (possibly non-Hermitian) operator B ,

$$\|\mathcal{E}(t, s)(B)\|_1 \leq 2C_0 e^{-\gamma_0 |t-s|} \|B\|_1 \quad \forall |t-s| \geq \tau_0. \quad (\text{F2})$$

Proof. Consider first the case where $B = B^\dagger$. Suppose B has eigenvalues and eigenvectors $(\lambda_\alpha, |\alpha\rangle)$ —then $B = B^{(+)} - B^{(-)}$

$$B^{(+)} = \sum_{\alpha: \lambda_\alpha \geq 0} \lambda_\alpha |\alpha\rangle\langle\alpha| \quad \text{and} \quad B^{(-)} = - \sum_{\alpha: \lambda_\alpha < 0} \lambda_\alpha |\alpha\rangle\langle\alpha|.$$

Note that $B^{(+)}, B^{(-)} \geq 0$ and $\text{Tr}(B^{(+)}) = \text{Tr}(B^{(-)}) = \|B\|_1/2$. Therefore, using Eq. (F1) with $\rho_1 = 2B^{(+)}/\|B\|_1, \rho_2 = 2B^{(-)}/\|B\|_1$, we obtain that for $|t-s| \geq \tau_0$,

$$\|\mathcal{E}(t, s)(B)\|_1 \leq C_0 e^{-\gamma_0 |t-s|} \|B\|_1. \quad (\text{F3})$$

Finally, any traceless (but possibly non-Hermitian) operator B can be expressed as $B = B_h + iB_{ah}$ where $B_h = (B + B^\dagger)/2, B_{ah} = i(B^\dagger - B)/2$ are both Hermitian operators. Using Eq. (F3), we then obtain that for $|t-s| \geq \tau_0$

$$\begin{aligned} \|\mathcal{E}(t, s)(B)\|_1 &\leq \|\mathcal{E}(t, s)(B_h)\|_1 + \|\mathcal{E}(t, s)(B_{ah})\|_1 \\ &\leq C_0 e^{-\gamma_0 |t-s|} (\|B_h\|_1 + \|B_{ah}\|_1) \\ &\leq 2C_0 e^{-\gamma_0 |t-s|} \|B\|_1, \end{aligned} \quad (\text{F4})$$

which proves the Lemma. $\blacksquare$

Next, we show that strict contractivity of the dynamics implies that expected values of time-domain annihilation operators in the emitted state approximately factorize. We consider here the setting introduced in the main text of the source coupling to a set of output ports described by the time-domain annihilation operator $a_{\alpha,t}$,

$$H(t) = H_S(t) + \sum_{\alpha} (a_{\alpha,t}^\dagger L_{\alpha} + \text{H.c.}). \quad (\text{F5})$$

The source dynamics will be captured by the Lindbladian $\mathcal{L}(t) = -i[H_S(t), \cdot] + \sum_{\alpha} \mathcal{D}_{L_{\alpha}}$. We will now consider the state $|\psi\rangle$ in the output ports obtained on evolving an initial state $|\phi_i, \text{vac}\rangle$ (i.e., the source being in the state $|\phi_i\rangle$ and the output ports being in the vacuum state) for time T and projecting the source onto the state $|\phi_f\rangle$:

$$|\psi\rangle = \frac{1}{\mathcal{N}} \langle \phi_f | U(T, 0) | \phi_i, \text{vac} \rangle \quad \text{where } U(t, s) = \mathcal{T} \exp\left(-\int_s^t H(\tau) d\tau\right) \quad \text{and} \quad \mathcal{N} = |\langle \phi_f | U(T, 0) | \phi_i, \text{vac} \rangle|^2.$$

We now establish the fact that, when the source dynamics is strictly contractive, the connected correlators in the emitted photonic state at two different time instants decay exponentially with their time difference.

Lemma 2. (Decay of connected correlators). Suppose $\mathcal{E}(t, s) = \mathcal{T} \exp(\int_s^t \mathcal{L}(\tau) d\tau)$, where $\mathcal{L}(\tau)$ corresponding to the source-port Hamiltonian in Eq. (F5), is strictly contractive [Eq. (F1)], then,

$$|\langle \psi | a_{\alpha,t} a_{\beta,s} | \psi \rangle - \langle \psi | a_{\alpha,t} | \psi \rangle \langle \psi | a_{\beta,s} | \psi \rangle| \leq \frac{8C_0}{\mathcal{N}^4} \|L_\alpha\| \|L_\beta\| e^{-\gamma_0(t-s)}, \quad (\text{F6})$$

and

$$|\langle \psi | a_{\alpha,t}^\dagger a_{\beta,s} | \psi \rangle - \langle \psi | a_{\alpha,t}^\dagger | \psi \rangle \langle \psi | a_{\beta,s} | \psi \rangle| \leq \frac{8C_0}{\mathcal{N}^4} \|L_\alpha\| \|L_\beta\| e^{-\gamma_0(t-s)}, \quad (\text{F7})$$

where $0 \leq s < t \leq T$ and $t - s \geq \tau_0$.

Proof. We will establish Eq. (F6), the proof of Eq. (F7) will follow similarly. Following the calculation in Sec. A 2, $\langle \psi | a_{\alpha,t} a_{\beta,s} | \psi \rangle$ and $\langle \psi | a_{\nu,\tau} | \psi \rangle$ can be expressed as

$$\langle \psi | a_{\alpha,t} a_{\beta,s} | \psi \rangle = \frac{1}{\mathcal{N}^2} \text{Tr}(P_f(t) L_\alpha \mathcal{E}(t, s) (L_\beta \rho_i(s))) \text{ and } \langle \psi | a_{\nu,\tau} | \psi \rangle = \frac{1}{\mathcal{N}^2} \text{Tr}(P_f(\tau) L_\nu \rho_i(\tau)),$$

where $\rho_i(\tau) = \mathcal{E}(\tau, 0)(\rho_i)$ with $\rho_i = |\phi_i\rangle\langle\phi_i|$ and $\mathcal{P}_f(\tau) = \mathcal{E}^\dagger(T, \tau)(|\phi_f\rangle\langle\phi_f|)$. Using these expressions, we then obtain

$$\begin{aligned} |\langle \psi | a_{\alpha,t} a_{\beta,s} | \psi \rangle - \langle \psi | a_{\alpha,t} | \psi \rangle \langle \psi | a_{\beta,s} | \psi \rangle| &= \frac{1}{\mathcal{N}^4} |\text{Tr}(\mathcal{P}_f(t) L_\alpha \mathcal{E}(t, s) (\mathcal{N}^2 L_\beta \rho_i(s) - \text{Tr}(\mathcal{P}_f(s) L_\beta \rho_i(s)) \rho_i(s))), \\ &\leq \frac{\|L_\alpha\|}{\mathcal{N}^2} \|\mathcal{E}(t, s) (L_\beta \rho_i(s) - \text{Tr}(L_\beta \rho_i(s)) \rho_i(s))\|_1 \\ &\quad + \frac{\|L_\alpha\|}{\mathcal{N}^4} |\text{Tr}(\mathcal{P}_f(s) L_\beta \rho_i(s)) - \mathcal{N}^2 \text{Tr}(L_\beta \rho_i(s))|. \end{aligned} \quad (\text{F8})$$

Similarly, using $\mathcal{N}^2 = \text{Tr}(\mathcal{P}_f \mathcal{E}(T, 0)(\rho_i)) = \text{Tr}(\mathcal{P}_f \mathcal{E}(T, s) \rho_i(s))$, we obtain that

$$\begin{aligned} |\text{Tr}(\mathcal{P}_f(s) L_\beta \rho_i(s)) - \mathcal{N}^2 \text{Tr}(L_\beta \rho_i(s))| &= |\text{Tr}(\mathcal{P}_f \mathcal{E}(T, s) (L_\beta \rho_i(s) - \text{Tr}(L_\beta \rho_i(s)) \rho_i(s)))|, \\ &\leq \|\mathcal{E}(T, s) (L_\beta \rho_i(s) - \text{Tr}(L_\beta \rho_i(s)) \rho_i(s))\|_1. \end{aligned} \quad (\text{F9})$$

Combining Eqs. (F8) and (F9), and using the strict contractivity of $\mathcal{E}(t, s)$, we then obtain that

$$\begin{aligned} |\langle \psi | a_{\alpha,t} a_{\beta,s} | \psi \rangle - \langle \psi | a_{\alpha,t} | \psi \rangle \langle \psi | a_{\beta,s} | \psi \rangle| &\leq 2C_0 \|L_\alpha\| \left(\frac{e^{-\gamma_0(t-s)}}{\mathcal{N}^2} + \frac{e^{-\gamma_0(T-s)}}{\mathcal{N}^4} \right) \|L_\beta \rho_i(s) - \text{Tr}(L_\beta \rho_i(s)) \rho_i(s)\|_1, \\ &\leq \frac{8C_0}{\mathcal{N}^4} \|L_\alpha\| \|L_\beta\| e^{-\gamma_0(t-s)}. \end{aligned}$$

This establishes Eq. (F6). A similar analysis can be performed to establish Eq. (F7). ■

Using lemma 2, we now show that the QFI [given by Eq. (19)] scales as $\sim T$ for large T . Consider the two correlation functions $C^{(g)}(t, s) = \langle a_t^\dagger a_s \rangle$ and $C^{(\chi)}(t, s) = \langle a_t a_s \rangle$. Applying lemma 2 together with the fact that $|\langle a_t \rangle \langle a_s \rangle| = |\langle a_t^\dagger \rangle \langle a_s \rangle|$ and $|\langle a_t a_s \rangle|, |\langle a_t^\dagger a_s \rangle| \leq \|L\|^2, |a_t|, |a_t^\dagger| \leq \|L\|$, we obtain that

$$|C^{(\chi)}(t, s)|^2 - |C^{(g)}(t, s)|^2 \leq \frac{64C_0^2}{\mathcal{N}^8} \|L\|^4 e^{-2\gamma_0|t-s|} + \frac{32C_0}{\mathcal{N}^4} \|L\|^4 e^{-\gamma_0|t-s|} \leq \frac{32\|L\|^4}{p_0^4} (2C_0^2 + C_0) e^{-\gamma_0|t-s|}, \quad (\text{F10})$$

where p_0 is a T -independent constant such that $p_0 \leq \mathcal{N}^2$. Finally, using this together with the expression for $Q^{(2)}$ [Eq. (19b)] we obtain that

$$\begin{aligned} Q^{(2)} &= 2 \int_0^T \int_0^t (|C^{(x)}(t, s)|^2 - |C^{(g)}(t, s)|^2) ds dt = 2 \int_0^T \int_\tau^T (|C^{(x)}(t, t - \tau)|^2 - |C^{(g)}(t, t - \tau)|^2) dt d\tau \\ &\leq \frac{2}{p_0^4} \int_0^{\tau_0} \int_\tau^T \|L\|^4 dt d\tau + \frac{64}{p_0^4} \int_{\tau_0}^T \int_\tau^T \|L\|^4 (2C_0^2 + C_0) e^{-\gamma_0 \tau} dt d\tau \leq \frac{2\|L\|^4}{p_0^4} \left(\tau_0 + \frac{32}{\gamma_0} (2C_0^2 + C_0) \right) T, \end{aligned} \quad (\text{F11})$$

which establishes that $Q^{(2)} \leq O(T)$ as $T \rightarrow \infty$ and consequently from Eq. (19a), $QFI \leq O(T)$.

b. Extension to non-identical emitters with arbitrary linear MZI transformation

Next, we generalize the result for the linear scaling of the QFI with T to the case where the sources are not

necessarily identical and may be entangled. Moreover, we allow the MZI to implement an arbitrary linear transformation. In this setup, the only term that can potentially lead to a quadratic scaling in time is the double-integral term in Eq. (B9), i.e.,

$$Q^{(2)} = \int_0^T \int_0^T \sum_{x,y,u,v \in \{a,b\}} W_{xy} W_{uv} [C_{x,u;y,v}^{(2)}(t, s) - C_{x;y}^{(1)}(t) C_{u;v}^{(1)}(s)] dt ds. \quad (\text{F12})$$

Similar to the approach in the previous section, we show that the connected correlators decay exponentially when the dynamics is strictly contractive. This yields a bound on $Q^{(2)}$, and therefore the QFI scales at most linearly with T .

Lemma 3. (Decay of 2-point correlators). Suppose $\mathcal{E}(t, s) = \tilde{T} \exp(\int_s^t \mathcal{L}(\tau) d\tau)$, where $\mathcal{L}(\tau)$ corresponding to the source-port Hamiltonian in Eq. (F5), is strictly contractive [Eq. (F1)], then,

$$|C_{x,u;y,v}^{(2)}(t, s) - C_{x;y}^{(1)}(t) C_{u;v}^{(1)}(s)| \leq \frac{8C_0}{\mathcal{N}^4} \|L\|^4 e^{-\gamma_0(t-s)}, \quad (\text{F13})$$

where $0 \leq s < t \leq T$, $t - s \geq \tau_0$, $x, y, u, v \in \{a, b\}$, and $\|L\| = \max_x \|L_x\|$.

Proof. From Eq. (10) recall that

$$C_{x,u;y,v}^{(2)}(t, s) = \frac{1}{\mathcal{N}^2} \text{Tr}(\mathcal{P}_f(t) \mathcal{M}_{x,y} \mathcal{E}(t, s) \mathcal{M}_{u,v} \rho_i(s)), \quad (\text{F14})$$

$$C_{x;y}^{(1)}(t) = \frac{1}{\mathcal{N}^2} \text{Tr}(\mathcal{P}_f(t) \mathcal{M}_{x,y} \mathcal{E}(t, s) \rho_i(s)), \quad (\text{F15})$$

$$C_{u;v}^{(1)}(s) = \frac{1}{\mathcal{N}^2} \text{Tr}(\mathcal{P}_f(t) \mathcal{E}(t, s) \mathcal{M}_{u,v} \rho_i(s)), \quad (\text{F16})$$

where $\mathcal{M}_{x,y} = L_x(\cdot) L_y^\dagger$, $\rho_i(\tau) = \mathcal{E}(\tau, 0)(\rho_i)$ with $\rho_i = |\phi_i\rangle\langle\phi_i|$ and $\mathcal{P}_f(\tau) = \mathcal{E}^\dagger(T, \tau)(|\phi_f\rangle\langle\phi_f|)$. Using these expressions, and $\mathcal{N}^2 = \text{Tr}(\mathcal{P}_f(t) \mathcal{E}(t, s) \rho_i(s))$, we then obtain

$$\begin{aligned} |C_{x,u;y,v}^{(2)}(t, s) - C_{x;y}^{(1)}(t) C_{u;v}^{(1)}(s)| &= \frac{1}{\mathcal{N}^2} \text{Tr}(\mathcal{P}_f(t) \mathcal{M}_{x,y} \mathcal{E}(t, s) [\mathcal{M}_{u,v} \rho_i(s) - \text{Tr}(\mathcal{M}_{u,v} \rho_i(s)) \rho_i(s)]) \\ &\quad - \frac{1}{\mathcal{N}^4} \text{Tr}(\mathcal{P}_f(t) \mathcal{M}_{x,y} \mathcal{E}(t, s) \rho_i(s)) \text{Tr}(\mathcal{P}_f(t) \mathcal{E}(t, s) [\mathcal{M}_{u,v} \rho_i(s) - \text{Tr}(\mathcal{M}_{u,v} \rho_i(s)) \rho_i(s)]) \\ &\leq \frac{2\|L_x\| \|L_y\|}{\mathcal{N}^4} \|\mathcal{E}(t, s) [\mathcal{M}_{u,v} \rho_i(s) - \text{Tr}(\mathcal{M}_{u,v} \rho_i(s)) \rho_i(s)]\|_1 \\ &\leq \frac{8C_0}{\mathcal{N}^4} \|L_x\| \|L_y\| \|L_u\| \|L_v\| e^{-\gamma_0(t-s)}. \end{aligned} \quad (\text{F17})$$

This establishes Eq. (F13). ■

Finally, using Lemma 3 together with the expression for $Q^{(2)}$ [Eq. (F12)] we obtain that

$$\begin{aligned}
Q^{(2)} &\leq \frac{32w^2}{p_0^4} \int_0^T \int_\tau^T |\mathcal{C}_{x,u;y,v}^{(2)}(t, t-\tau) - \mathcal{C}_{x;y}^{(1)}(t)\mathcal{C}_{u;v}^{(1)}(t-\tau)| d\tau dt \\
&\leq \frac{32w^2}{p_0^4} \int_0^{\tau_0} \int_\tau^T \|L\|^4 dt d\tau + \frac{2^8 w^2}{p_0^4} \int_{\tau_0}^T \int_\tau^T \|L\|^4 C_0 e^{-\gamma_0 \tau} dt d\tau \leq \frac{32w^2 \|L\|^4}{p_0^4} \left(\tau_0 + \frac{8}{\gamma_0} C_0 \right) T,
\end{aligned} \tag{F18}$$

which establishes that $Q^{(2)} \leq O(T)$ as $T \rightarrow \infty$ and consequently, $QFI \leq O(T)$.

c. Provable strict contractivity for Lindbladians with full Kraus rank

Next, we will consider two settings where we are able to establish that the source dynamics are provably strictly contractive irrespective of the time-dependent Hamiltonian applied on the source. The first setting that we will consider is when the source dynamics are modelled by a Lindbladian with a time-dependent Hamiltonian as well as a set of jump operators which span the whole space of traceless operators. More specifically, for a source with d levels, the source Lindbladian $\mathcal{L}(t)$ is given by

$$\mathcal{L}(t) = -i[H(t), \cdot] + \sum_{\alpha=1}^{d^2-1} \mathcal{D}_{L_\alpha} = -i[H(t), \cdot] + \sum_{\alpha=0}^{d^2} \mathcal{D}_{L_\alpha}, \tag{F19}$$

where $\{L_\alpha\}_{\alpha \in \{1, 2, \dots, d^2-1\}}$ span the space of traceless system operators. Furthermore, we have artificially added the dissipator $\mathcal{D}_{L_0}$ with $L_0 = I$ for convenience without changing the master equation: The set of jump operators $\{L_\alpha\}_{\alpha \in \{0, 1, \dots, d^2-1\}}$ form a complete basis for the space of system operators. Due to this completeness of $\{L_\alpha\}_{\alpha \in \{0, 1, 2, \dots, d^2-1\}}$, $\exists \lambda_0 > 0$ such that

$$\forall \text{ all system operators } X: \sum_{\alpha} |\text{Tr}(L_\alpha^\dagger X)|^2 \geq \lambda_0 \text{Tr}(X^\dagger X). \tag{F20}$$

We first show that the Lindbladian [Eq. (F19)] with the jump operators satisfying Eq. (F20) generates strictly contractive dynamics.

Proposition 2. Suppose $\mathcal{E}(t, s) = \vec{T} \exp(\int_s^t \mathcal{L}(\tau) d\tau)$ is the channel generated by a Lindbladian [Eq. (F19)] whose jump operators satisfy Eq. (F20), then $\forall t > s$ and any two states ρ_1, ρ_2 ,

$$\|\mathcal{E}(t, s)(\rho_1 - \rho_2)\|_1 \leq e^{-\lambda_0 |t-s|} \|\rho_1 - \rho_2\|_1.$$

Proof. We will denote by $\mathcal{D}$ the Lindbladian $\mathcal{D} = \sum_{\alpha=1}^{d^2} \mathcal{D}_{L_\alpha}$. $\mathcal{D}$ contains contribution from all the jump operators in system Lindbladian which are assumed to be time independent. It will also be convenient for us to work

in a discretization of this dynamics: We will discretize the time interval $(t, s] = (t_T, t_{T-1}] \cup (t_{T-1}, t_{T-2}] \cup \dots \cup (t_1, t_0]$ where $t_m = s + m\epsilon$ with $\epsilon = (t-s)/T$ and approximate the channel $\mathcal{E}(t, s)$ as

$$\tilde{\mathcal{E}} = \mathcal{K} \mathcal{U}_T \mathcal{K} \mathcal{U}_{T-1} \dots \mathcal{K} \mathcal{U}_1, \tag{F21}$$

where $\mathcal{U}_\tau(X) = U_\tau X U_\tau^\dagger$, with $U_\tau = \vec{T} \exp(-i \int_{t_{\tau-1}}^{t_\tau} H_S(s) ds)$ and $\mathcal{K}(X) = \sum_{\alpha=0}^{d^2} K_\alpha X K_\alpha^\dagger$ where

$$\begin{aligned}
K_\alpha &= \sqrt{\epsilon} L_\alpha \quad \text{for } i \in \{1, 2, \dots, d^2\} \quad \text{and} \\
K_0 &= \left(I - \epsilon \sum_{\alpha=1}^{d^2} L_\alpha^\dagger L_\alpha \right)^{1/2},
\end{aligned}$$

We note that

$$\|e^{\epsilon \mathcal{D}} - \mathcal{K}\|_\diamond \leq O(\epsilon^2), \tag{F22}$$

and consequently

$$\|\mathcal{E}(t, s) - \tilde{\mathcal{E}}\|_\diamond \leq O(T\epsilon^2) \leq O(|t-s|\epsilon), \tag{F23}$$

and thus the exact dynamics is recovered in the limit of $\epsilon \rightarrow 0$. We denote the fixed point of the Lindbladian $\mathcal{D}$ by τ . Since the jump operators $\{L_\alpha\}_{\alpha \in \{0, 1, 2, \dots, d^2-1\}}$ have full Kraus rank, τ is both the unique fixed point of $\mathcal{D}$ and also positive definite ($\tau > 0$). Furthermore, using Eq. (F22) and the fact that $\mathcal{D}(\tau) = 0$, we obtain that $\|\mathcal{K}(\tau) - \tau\|_1 = \|\mathcal{K}(\tau) - e^{\epsilon \mathcal{D}}(\tau)\|_1 \leq O(\epsilon^2)$. Next, we decompose the channel $\mathcal{K}$ into the convex combination of a superoperator $\tilde{\mathcal{K}}$ and the replacement channel $\mathcal{R}_\tau = \text{Tr}(\cdot)\tau$:

$$\mathcal{K} = (1 - \epsilon\lambda_0)\tilde{\mathcal{K}} + \epsilon\lambda_0\mathcal{R}_\tau. \tag{F24}$$

We first show that $\tilde{\mathcal{K}}$ defined this way is a valid channel. First, we note that $\tilde{\mathcal{K}}$ is trace-preserving since both $\mathcal{K}$ and $\mathcal{R}_\tau$ are trace preserving. Second, to ensure that is also completely positive, by the Choi-Jamiołkowski isomorphism, we need to show that $\Phi_{\tilde{\mathcal{K}}} = (\Phi_{\mathcal{K}} - \epsilon\lambda_0\Phi_{\mathcal{R}_\tau})/(1 - \epsilon\lambda_0) \geq 0$, where $\Phi_{\mathcal{S}} = (\mathcal{S} \otimes \text{id})(|\Phi^+\rangle\langle\Phi^+|)$ is the Choi state of $\mathcal{S}$. We note that for a state $|\psi\rangle = \sum_{i,j} \psi_{i,j} |i\rangle |j\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$, and its corresponding operator $|\Psi\rangle = \sum_{i,j} \psi_{i,j} |i\rangle \langle j|$,

$$\begin{aligned}
\langle \psi | \Phi_{\mathcal{K}} | \psi \rangle &= \frac{1}{d} \sum_{\alpha} \sum_{i,i'} \sum_{p,p'} \psi_{p,i}^* \psi_{p',i'} \langle p | K_{\alpha} | i \rangle \langle i' | K_{\alpha}^{\dagger} | p' \rangle \\
&= \frac{1}{d} \sum_{\alpha} |\text{Tr}(K_{\alpha}^{\dagger} \Psi)|^2 \geq \frac{\varepsilon}{d} \sum_{\alpha \geq 1} |\text{Tr}(L_{\alpha}^{\dagger} \Psi)|^2 \\
&\geq \frac{\varepsilon \lambda_0}{d} \|\Psi\|^2.
\end{aligned} \tag{F25}$$

Similarly,

$$\langle \psi | \Phi_{\mathcal{R}_\tau} | \psi \rangle = \frac{1}{d} \langle \psi | (I \otimes \tau) | \psi \rangle \leq \frac{1}{d} \|\psi\|^2 \|\tau\| \leq \frac{1}{d} \|\psi\|^2. \tag{F26}$$

Therefore, from Eqs. (F25) and (F26), we find that for every $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$,

$$\langle \psi | \Phi_{\tilde{\mathcal{K}}} | \psi \rangle = \frac{1}{1 - \varepsilon \lambda_0} (\langle \psi | \mathcal{K} | \psi \rangle - \varepsilon \lambda_0 \langle \psi | \mathcal{R}_\tau | \psi \rangle) \geq 0,$$

and thus $\tilde{\mathcal{K}}$ is a valid channel. Using Eq. (F24), we can then show that $\mathcal{K}$ is strictly contractive since for any two states ρ_1, ρ_2 ,

$$\begin{aligned}
\|\mathcal{K}(\rho_1 - \rho_2)\|_1 &= (1 - \varepsilon \lambda_0) \|\tilde{\mathcal{K}}(\rho_1 - \rho_2)\|_1 \\
&\leq (1 - \varepsilon \lambda_0) \|\rho_1 - \rho_2\|_1.
\end{aligned}$$

Using this together with Eq. (F21), we obtain that

$$\|\tilde{\mathcal{E}}(\rho_1 - \rho_2)\|_1 \leq (1 - \varepsilon \lambda_0)^T \|\rho_2 - \rho_1\|.$$

Using Eq. (F23), we obtain that

$$\begin{aligned}
\|\mathcal{E}(t, s)(\rho_1 - \rho_2)\|_1 &\leq \|\tilde{\mathcal{E}}(\rho_1 - \rho_2)\|_1 + O(|t - s|\varepsilon) \\
&\leq (1 - \varepsilon \lambda_0)^T \|\rho_1 - \rho_2\| + O(|t - s|\varepsilon).
\end{aligned}$$

Finally, taking the limit $\varepsilon \rightarrow 0$ in this inequality, and setting $(1 - \varepsilon \lambda_0)^T \rightarrow e^{-\lambda_0|t-s|}$, we obtain the proposition. ■

d. Strict contractivity of the two-level system

We next consider the Lindbladian describing a two-level system with states $|e\rangle, |g\rangle$ with a jump operator $L = \sqrt{\gamma}\sigma$ where $\sigma = |g\rangle\langle e|$ and any time-dependent Hamiltonian $H(t)$:

$$\mathcal{L}(t) = -i[H(t), \cdot] + \gamma \mathcal{D}_\sigma. \tag{F27}$$

We show that for this system as well, the dynamics generated by $\mathcal{L}(t)$ is strictly contractive.

Proposition 3. Suppose $\mathcal{E}(t, s) = \vec{T} \exp(\int_s^t \mathcal{L}(\tau) d\tau)$ is the channel generated by the two-level system Lindbladian in Eq. (F27) then $\forall t > s$ and any two states ρ_1, ρ_2 ,

$$\|\mathcal{E}(t, s)(\rho_1 - \rho_2)\|_1 \leq e^{-\gamma|t-s|/2} \|\rho_1 - \rho_2\|_1.$$

Proof. Similar to the proof of proposition 2, we work in a Trotterization of the dynamics: We will discretize the interval $(t, s] = (\tau_T, \tau_{T-1}] \cup (\tau_{T-1}, \tau_{T-2}] \cup \dots \cup (\tau_1, \tau_0]$, where $\tau_m = s + m\varepsilon$ with $\varepsilon = (t - s)/T$. The channel $\mathcal{E}(t, s)$ will be approximated by $\hat{\mathcal{E}}$ where

$$\hat{\mathcal{E}} = \mathcal{K} \mathcal{U}_T \mathcal{K} \mathcal{U}_{T-1} \dots \mathcal{K} \mathcal{U}_1, \tag{F28}$$

where $\mathcal{U}_k(X) = U_k X U_k^\dagger$ with $U_k = \vec{T} \exp(-i \int_{\tau_{k-1}}^{\tau_k} H(s) ds)$ and $\mathcal{K}(X) = K_0 X K_0^\dagger + K_1 X K_1^\dagger$ where

$$\begin{aligned}
K_1 &= \sqrt{\gamma\varepsilon}\sigma = \sqrt{\gamma\varepsilon}|g\rangle\langle e| \quad \text{and} \\
K_0 &= (1 - \gamma\varepsilon\sigma^\dagger\sigma)^{1/2} = |g\rangle\langle g| + \sqrt{1 - \gamma\varepsilon}|e\rangle\langle e|.
\end{aligned}$$

Since $\|\mathcal{K} - \exp(\gamma\varepsilon\mathcal{D}_\sigma)\|_\diamond \leq O(\varepsilon^2)$, it follows that $\|\hat{\mathcal{E}} - \mathcal{E}(t, s)\|_\diamond \leq O(T\varepsilon^2) \leq O(|t - s|\varepsilon)$ and thus the exact continuous-time dynamics can be recovered in the limit of $\varepsilon \rightarrow 0$. Next, we establish that the channel $\mathcal{K}$ is strictly contractive—to show this, we recall a result from Ref. [109] (theorem 2), where it was shown that for any two states ρ_1, ρ_2 such that $\rho_1 \neq \rho_2$,

$$\frac{\|\mathcal{K}(\rho_1 - \rho_2)\|_1}{\|\rho_1 - \rho_2\|_1} \leq \frac{1}{2} \sup_{\substack{|\psi_1\rangle, |\psi_2\rangle \\ \langle \psi_1 | \psi_2 \rangle = 0}} \|\mathcal{K}(|\psi_1\rangle\langle \psi_1| - |\psi_2\rangle\langle \psi_2|)\|. \tag{F29}$$

In the other words, the states ρ_1, ρ_2 that maximize $\|\mathcal{K}(\rho_1 - \rho_2)\|_1$ are pure states $|\psi_1\rangle, |\psi_2\rangle$ that are orthogonal to each other (i.e., $\langle \psi_1 | \psi_2 \rangle = 0$). Setting $|\psi_1\rangle = \cos\theta|e\rangle + e^{i\varphi}\sin\theta|g\rangle$ and $|\psi_2\rangle = \sin\theta|e\rangle - e^{i\varphi}\cos\theta|g\rangle$ we obtain

$$\begin{aligned}
\mathcal{K}(|\psi_1\rangle\langle \psi_1| - |\psi_2\rangle\langle \psi_2|) &= (1 - \gamma\varepsilon) \cos 2\theta (|e\rangle\langle e| - |g\rangle\langle g|) \\
&\quad + (1 - \gamma\varepsilon)^{1/2} \sin 2\theta (e^{i\varphi}|e\rangle\langle g| \\
&\quad + \text{H.c.}),
\end{aligned}$$

and therefore

$$\begin{aligned} \frac{1}{2} \|\mathcal{K}(|\psi_1\rangle\langle\psi_1| - |\psi_2\rangle\langle\psi_2|)\|_1 &= ((1 - \gamma\epsilon)^2 \cos^2 2\theta \\ &\quad + (1 - \gamma\epsilon) \sin^2 2\theta)^{1/2} \\ &\leq \sqrt{1 - \gamma\epsilon}. \end{aligned}$$

Finally, using this together with the fact that unitaries leave the 1-norm invariant, we obtain that for any two states ρ_1, ρ_2

$$\|\hat{\mathcal{E}}(\rho_1 - \rho_2)\| \leq (1 - \gamma\epsilon)^{T/2}.$$

Finally, taking the limit of $\epsilon \rightarrow 0$ and setting $\hat{\mathcal{E}} \rightarrow \mathcal{E}(t, s)$ and $(1 - \gamma\epsilon)^{T/2} \rightarrow e^{-\gamma|t-s|/2}$, we obtain the proposition.

2. Adjoint variable method for optimizing QFI

In this subsection, we will provide details of numerically optimizing the QFI with respect to the time-dependent Hamiltonian applied on the source. We will work with a discretized matrix-product state representation of the emitted photons, which we review first. Then, we will show how the computation of the gradient of the QFI with respect to the Hamiltonian parameters can be sped up using the adjoint variable method.

a. Review: Matrix product state representation

We first show how to approximate the state of the emitted photons with a matrix product state. We recall from the main text that the Hamiltonian of the source interacting with d output ports is given by

$$H(t) = H_S(t) + \sum_{j=1}^d (a_{j,t}^\dagger L_j + \text{H.c.}),$$

where $[a_{\alpha,t}, a_{\alpha',t'}] = \delta_{\alpha,\alpha'} \delta(t - t')$ and $H_S(t)$ and L_α are operators acting on the source Hilbert space. We will denote by $U(t, s)$ and $U_S(t, s)$ the unitaries generated by the Hamiltonians $H(t)$ and $H_S(t)$ respectively:

$$\begin{aligned} U(t, s) &= \tilde{T} \exp\left(-\int_s^t H(\tau) d\tau\right) \quad \text{and} \\ U_S(t, s) &= \tilde{T} \exp\left(-\int_s^t H_S(\tau) d\tau\right). \end{aligned}$$

Assuming that the initial and final states of the source are $|\phi_{S,i}\rangle, |\phi_{S,f}\rangle$, the state of the photons $|\psi\rangle$ in the output ports can be expressed as

$$|\psi\rangle = \frac{1}{\mathcal{N}} \langle\phi_{S,f}|\Psi\rangle \quad \text{where } |\Psi\rangle = U(T, 0)|\text{vac}, \phi_{S,i}\rangle, \quad (\text{F30})$$

where $\mathcal{N} = \|\langle\phi_{S,f}|\Psi\rangle\|$ is a normalization constant. We will approximate $|\Psi\rangle$ by $|\hat{\Psi}\rangle$, which would be a state in the

Hilbert space $(\mathbb{C}^{d+1})^{\otimes M}$ where we will (i) effectively discretize the ports in space as well as (ii) truncate the Hilbert space dimension of each discretized step to having at most a single-excitation, i.e.,

$$|\hat{\Psi}\rangle = U_M^S K_M U_{M-1}^S K_{M-1} \cdots U_1^S K_1 |\phi_{S,i}\rangle, \quad (\text{F31})$$

where M is the number of time steps that $[0, T]$ is discretized into and

$$\begin{aligned} U_k^S &= U_S(k\epsilon, (k-1)\epsilon) \quad \text{and} \\ K_k &= (I - \epsilon Q)^{1/2} |0_k\rangle - i\sqrt{\epsilon} \sum_{j=1}^d L_j |j_k\rangle, \end{aligned} \quad (\text{F32a})$$

with $\epsilon = T/M$ and

$$Q = \sum_{j=1}^d L_j^\dagger L_j. \quad (\text{F32b})$$

Here $|0_k\rangle = |\text{vac}_{((k-1)\epsilon, k\epsilon]}\rangle$ is the vacuum in the output port corresponding to the time interval $((k-1)\epsilon, k\epsilon]$ and $|j_k\rangle$ is a state with a single photon in the j^{th} output port in the time interval $((k-1)\epsilon, k\epsilon]$, i.e.,

$$\begin{aligned} |j_k\rangle &= A_{\alpha,k}^\dagger |\text{vac}_{((k-1)\epsilon, k\epsilon]}\rangle \quad \text{with} \\ A_{\alpha,k} &= \frac{1}{\sqrt{\epsilon}} \int_{(k-1)\epsilon}^{k\epsilon} a_{\alpha,s} ds. \end{aligned} \quad (\text{F33})$$

It can be noted that $[A_{\alpha,k}, A_{\alpha',k'}^\dagger] = \delta_{\alpha,\alpha'} \delta_{k,k'}$. The approximated state of photons $|\hat{\psi}\rangle = \langle\phi_{S,f}|\hat{\Psi}\rangle/\hat{\mathcal{N}}$ where $\hat{\mathcal{N}} = \|\langle\phi_{S,f}|\hat{\Psi}\rangle\|$.

For our analysis, it will be convenient to use the following lemma.

Lemma 4. For any interval $s < \tau < t$,

$$\begin{aligned} &\langle\text{vac}_{(s,t]}|U(t, s)|\text{vac}_{(s,t]}\rangle \\ &= U_{\text{eff}}(t, s) \text{ and } \langle\text{vac}_{(s,t]}|a_{\alpha,\tau} U(t, s)|\text{vac}_{(s,t]}\rangle \\ &= -i U_{\text{eff}}(t, \tau) L_\alpha U_{\text{eff}}(\tau, s), \end{aligned}$$

where

$$\begin{aligned} U_{\text{eff}}(t, s) &= \tilde{T} \exp\left(-i \int_s^t H_{\text{eff}}(\tau) d\tau\right) \quad \text{with} \\ H_{\text{eff}}(t) &= H_S(t) - \frac{i}{2} Q. \end{aligned}$$

Proof. We show this by discretizing the interval $(s, t]$ into $(\tau_0, \tau_1] \cup (\tau_1, \tau_2] \cup \cdots \cup (\tau_{N-1}, \tau_N]$ where $\tau_k = s + k\delta$ with $\delta = (t-s)/N$. We note that an explicit second-order Dyson expansion of $U(\tau_{k+1}, \tau_k)$ yields

$$\langle \text{vac}_{(\tau_k, \tau_{k+1})} | U(\tau_{k+1}, \tau_k) | \text{vac}_{(\tau_k, \tau_{k+1})} \rangle = U_{\text{eff}}(\tau_{k+1}, \tau_k) + O(\delta^2).$$

Consequently,

$$\langle \text{vac}_{(s,t)} | U(t, s) | \text{vac}_{(s,t)} \rangle = \prod_{k=N}^1 \langle \text{vac}_{(\tau_k, \tau_{k+1})} | U(\tau_{k+1}, \tau_k) | \text{vac}_{(\tau_k, \tau_{k+1})} \rangle = U_{\text{eff}}(t, s) + O(|t - s|\delta).$$

Taking the limit of $\delta \rightarrow 0$ yields that $\langle \text{vac}_{(s,t)} | U(t, s) | \text{vac}_{(s,t)} \rangle = U_{\text{eff}}(t, s)$.

Next, we consider deriving an expression for $\langle \text{vac}_{(s,t)} | a_{\alpha, \tau} U(t, s) | \text{vac}_{(s,t)} \rangle$. We begin with the Heisenberg picture operator $a_{\alpha, \tau}(t, s) = U(s, t) a_{\alpha, \tau} U(t, s)$ which satisfies

$$\frac{d}{dt} a_{\alpha, \tau}(t, s) = -i\delta(\tau - t) U(s, t) L_{\alpha} U(t, s).$$

Noting that $a_{\alpha, \tau}(s, s) = a_{\alpha, \tau}$, we can integrate this equation from $s \rightarrow t$ to obtain

$$a_{\alpha, \tau}(t, s) = a_{\alpha, \tau} - iU(s, \tau) L_{\alpha} U(\tau, s),$$

or equivalently

$$a_{\alpha, \tau} U(t, s) = U(t, s) a_{\alpha, \tau} - iU(t, \tau) L_{\alpha} U(\tau, s).$$

Therefore, we obtain that

$$\begin{aligned} \langle \text{vac}_{(s,t)} | a_{\alpha, \tau} U(t, s) | \text{vac}_{(s,t)} \rangle &= -i \langle \text{vac}_{(s,t)} | U(t, \tau) L_{\alpha} U(\tau, s) | \text{vac}_{(s,t)} \rangle \\ &= -i \langle \text{vac}_{(\tau,t)} | U(t, \tau) | \text{vac}_{(\tau,t)} \rangle L_{\alpha} \langle \text{vac}_{(s,\tau)} | U(\tau, s) | \text{vac}_{(s,\tau)} \rangle = -i U_{\text{eff}}(t, \tau) L_{\alpha} U_{\text{eff}}(\tau, s), \end{aligned}$$

which proves the lemma. ■

We next provide an error bound between the exact state and its MPS representation.

Proposition 4. (Error bound on MPS representation). Suppose $J, \ell > 0$ are positive real numbers such that $\forall t \geq 0$: $\|H_S(t)\| \leq J$ and $\sum_{\alpha=1}^d \|L_{\alpha}\|^2 \leq \ell$, and $\eta_0 = 2\ell(J + 2\ell)$ then if $\varepsilon < \mathcal{N}^2/\eta_0 T$

$$\| |\psi\rangle\langle\psi| - |\hat{\psi}\rangle\langle\hat{\psi}| \| \leq \left(1 + \frac{1}{(\mathcal{N}^2 - \eta_0 T \varepsilon)} \right) \frac{\eta_0 T \varepsilon}{\mathcal{N}^2} \leq O\left(\frac{\eta_0 T \varepsilon}{\mathcal{N}^2}\right).$$

Proof. It will be convenient to define J, ℓ via

$$J = \max_{t \in [0, T]} \|H_S(t)\| \quad \text{and} \quad \ell = \sum_{\alpha=1}^d \|L_{\alpha}\|^2.$$

It can be noted that $\|Q\| \leq \ell$, where Q is defined in Eq. (F32b).

We will first show that $\langle \hat{\Psi} | \Psi \rangle$ is close to 1, where $|\Psi\rangle$ and $|\hat{\Psi}\rangle$ are defined in Eqs. (F30) and (F31) respectively. We begin by rewriting $\langle \hat{\Psi} | \Psi \rangle$ as

$$\langle \hat{\Psi} | \Psi \rangle = \langle \phi_{S,i} | V_1^{\dagger} \cdots V_{M-1}^{\dagger} V_M^{\dagger} U_M^S K_M U_{M-1}^S K_{M-1} \cdots U_1^S K_1 | \phi_{S,i} \rangle,$$

where $V_m = U(m\varepsilon, (m-1)\varepsilon) | 0_m \rangle$. It will be convenient to define

$$|\hat{\Psi}_m\rangle = U_m^S K_m U_{m-1}^S K_{m-1} \cdots U_1^S K_1 | \phi_{S,i} \rangle \quad \text{and} \quad |\Psi_m\rangle = V_m V_{m-1} \cdots V_1 | \phi_{S,i} \rangle.$$

It can be noted that $|\Psi_M\rangle = |\Psi\rangle$ and $|\hat{\Psi}_M\rangle = |\hat{\Psi}\rangle$. Furthermore, it is straightforward to see that $\| |\Psi_m\rangle \| = 1$. Additionally, since $\sum_{\alpha} K_{\alpha}^{\dagger} K_{\alpha} = I$ (with the identity being on the system operator), it also follows that $\| |\hat{\Psi}_m\rangle \| = 1$.

We will also define $T_m = V_m^\dagger U_m^S K_m$: Note that T_m is an operator acting on the Hilbert space of the source. Furthermore, we obtain the recursive inequality

$$|\langle \hat{\Psi} | \Psi \rangle - \langle \hat{\Psi}_{m-1} | \Psi_{m-1} \rangle| \leq |\langle \Psi_{m-1} | (T_m - I) | \hat{\Psi}_{m-1} \rangle| \leq \|T_m - I\|,$$

and therefore

$$|\langle \hat{\Psi}_m | \Psi_m \rangle - 1| \leq \sum_{m=1}^M \|T_m - I\|. \quad (\text{F34})$$

Thus, it is sufficient to bound $\|T_m - I\|$. This can be done with an application of time-dependent perturbation theory. Note that

$$\begin{aligned} T_m &= \langle 0_m | U^\dagger(m\varepsilon, (m-1)\varepsilon) | 0_m \rangle U_m^S (I - \varepsilon Q)^{1/2} - i\sqrt{\varepsilon} \sum_{\alpha=1}^d \langle 0_m | U^\dagger(m\varepsilon, (m-1)\varepsilon) A_{\alpha,m}^\dagger | 0_m \rangle U_m^S L_\alpha \\ &= U_{\text{eff}}^\dagger(m\varepsilon, (m-1)\varepsilon) U_m^S (I - \varepsilon Q)^{1/2} + \sum_{\alpha=1}^d \int_{(m-1)\varepsilon}^{m\varepsilon} U_{\text{eff}}^\dagger(s, (m-1)\varepsilon) L_\alpha^\dagger U_{\text{eff}}^\dagger(m\varepsilon, s) U_m^S L_\alpha. \end{aligned} \quad (\text{F35})$$

Next, we observe that since $\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{4(1-x)^{1/2} + (2-x)^2}$, $(I - \varepsilon Q)^{1/2} \approx I - \varepsilon Q/2$ and for $\varepsilon \leq 1/\ell$

$$\|(I - \varepsilon Q)^{1/2} - \left(I - \frac{\varepsilon Q}{2}\right)\| \leq \varepsilon^2 \|(4(I - \varepsilon Q)^{1/2} + (2 - \varepsilon Q)^2)^{-1} Q^2\| \leq \varepsilon^2 \ell^2. \quad (\text{F36})$$

Furthermore, we note that

$$\begin{aligned} (U_m^S)^\dagger U_{\text{eff}}(m\varepsilon, (m-1)\varepsilon) &= I + \int_{(m-1)\varepsilon}^{m\varepsilon} \frac{d}{dt'} U_S((m-1)\varepsilon, t') U_{\text{eff}}(t', (m-1)\varepsilon) dt' \\ &= I - \frac{1}{2} \int_{(m-1)\varepsilon}^{m\varepsilon} U_S((m-1)\varepsilon, t') Q U_{\text{eff}}(t', (m-1)\varepsilon) dt' \\ &= I - \frac{\varepsilon}{2} Q - \frac{1}{2} \int_{(m-1)\varepsilon}^{m\varepsilon} \int_{(m-1)\varepsilon}^{t'} \frac{d}{dt''} U_S((m-1)\varepsilon, t'') Q U_{\text{eff}}(t'', (m-1)\varepsilon) dt'' dt' \\ &= I - \frac{\varepsilon}{2} Q \\ &\quad - \frac{i}{2} \int_{(m-1)\varepsilon}^{m\varepsilon} \int_{(m-1)\varepsilon}^{t'} U_S((m-1)\varepsilon, t'') (H_S(t'') Q - Q H_{\text{eff}}(t'')) U_{\text{eff}}(t'', (m-1)\varepsilon) dt'' dt'. \end{aligned}$$

Thus, for small ε , $(U_m^S)^\dagger U_{\text{eff}}(m\varepsilon, (m-1)\varepsilon) \approx I - \varepsilon Q/2$ with the equation above implying that

$$\left\| (U_m^S)^\dagger U_{\text{eff}}(m\varepsilon, (m-1)\varepsilon) - \left(I - \frac{\varepsilon Q}{2}\right) \right\| \leq \frac{\varepsilon^2}{4} \ell \left(2J + \frac{1}{2}\ell\right), \quad (\text{F37})$$

where we have used $\|H_S(t)\| \leq J$, $\|H_{\text{eff}}(t)\| \leq J + \|Q\|/2$, $\|U_S(t, s)\| = 1$ and $\|U_{\text{eff}}(t, s)\| \leq 1$.

Finally, we note that

$$\begin{aligned}
U_{\text{eff}}^\dagger(s, (m-1)\varepsilon) L_\alpha^\dagger U_{\text{eff}}^\dagger(m\varepsilon, s) &= L_\alpha^\dagger U_{\text{eff}}^\dagger(m\varepsilon, s) + i \left(\int_{(m-1)\varepsilon}^s U_{\text{eff}}^\dagger(s', (m-1)\varepsilon) H_{\text{eff}}^\dagger(s') ds' \right) L_\alpha^\dagger U_{\text{eff}}^\dagger(m\varepsilon, s) \\
&= L_\alpha^\dagger - i L_\alpha^\dagger \left(\int_{m\varepsilon}^s H_{\text{eff}}^\dagger(s') U_{\text{eff}}^\dagger(m\varepsilon, s') ds' \right) \\
&\quad + i \left(\int_{(m-1)\varepsilon}^s U_{\text{eff}}^\dagger(s', (m-1)\varepsilon) H_{\text{eff}}^\dagger(s') ds' \right) L_\alpha^\dagger U_{\text{eff}}^\dagger(m\varepsilon, s),
\end{aligned}$$

and therefore $U_{\text{eff}}^\dagger(s, (m-1)\varepsilon)L_\alpha^\dagger U_{\text{eff}}^\dagger(m\varepsilon, s) \approx L_\alpha^\dagger$ with the equation above implying that

$$\|U_{\text{eff}}^\dagger(s, (m-1)\varepsilon)L_\alpha^\dagger U_{\text{eff}}^\dagger(m\varepsilon, s) - L_\alpha^\dagger\| \leq \varepsilon \|L_\alpha\| \left(J + \frac{\ell}{2}\right), \quad (\text{F38})$$

where we have used $\|U_{\text{eff}}(t, s)\| \leq 1$ and $\|H_{\text{eff}}(t)\| \leq J + \|Q\|/2$. Using Eqs. (F36), (F37) and (F38) together with Eq. (F35), we obtain that for $\epsilon\|Q\| \leq 1$

$$\|T_m - I\| \leq \frac{5}{4} \varepsilon^2 \ell^2 + \varepsilon^2 \left(J + \frac{\ell}{2}\right) \ell + \frac{\varepsilon^2}{4} \ell \left(2J + \frac{1}{2} \ell\right) \leq \frac{1}{2} \eta_0 \varepsilon^2, \quad (\text{F39})$$

where $\eta_0 = 2\ell(2\ell + J)$. Thus, from Eq. (F34) that

$$|\langle \Psi | \hat{\Psi} \rangle - 1| \leq \frac{1}{2} \eta_0 M \varepsilon^2 \quad \text{or equivalently} \quad \|\Psi - \hat{\Psi}\|^2 = 2\text{Re}(1 - \langle \Psi | \hat{\Psi} \rangle) \leq \eta_0 M \varepsilon^2.$$

Next, we consider the normalized states, $|\psi\rangle = \langle\phi_{S,f}|\Psi\rangle/\mathcal{N}$ and $|\hat{\psi}\rangle = \langle\phi_{S,f}|\hat{\Phi}\rangle/\hat{\mathcal{N}}$. We first note that

$$|\mathcal{N}^2 - \hat{\mathcal{N}}^2| = |\text{Tr}(\langle \phi_{S,f} \rangle \langle \phi_{S,f} | (|\Psi\rangle\langle\Psi| - |\hat{\Psi}\rangle\langle\hat{\Psi}|))| \leq \| |\Psi\rangle\langle\Psi| - |\hat{\Psi}\rangle\langle\hat{\Psi}| \|_1 \leq \eta_0 M \varepsilon^2.$$

Note that this implies that $\hat{N}^2 \geq \mathcal{N}^2 - \eta_0 M \varepsilon^2$. Consequently, we then obtain that if $M \varepsilon^2 < \mathcal{N}^2 / \eta_0$,

$$\left| \frac{1}{\mathcal{N}^2} - \frac{1}{\hat{\mathcal{N}}^2} \right| = \frac{|\mathcal{N}^2 - \hat{\mathcal{N}}^2|}{\mathcal{N}^2 \hat{\mathcal{N}}^2} \leq \frac{\eta_0 M \varepsilon^2}{\mathcal{N}^2 (\mathcal{N}^2 - \eta_0 M \varepsilon^2)}.$$

Finally,

$$||\psi\rangle\langle\psi| - |\hat{\psi}\rangle\langle\hat{\psi}||| \leq \frac{1}{\mathcal{N}^2} |||\Psi\rangle\langle\Psi| - |\hat{\Psi}\rangle\langle\hat{\Psi}|||_1 + \left| \frac{1}{\mathcal{N}^2} - \frac{1}{\hat{\mathcal{N}}^2} \right| \leq \left(1 + \frac{1}{(\mathcal{N}^2 - \eta_0 M \varepsilon^2)} \right) \frac{\eta_0 M \varepsilon^2}{\mathcal{N}^2}, \quad (\text{F40})$$

which proves the lemma. $\square$

b. Gradient computation

We next show how to compute the QFI and the QFI gradient with respect to the system unitaries applied in every time-step. To make the computations easy to visualize, we will use the tensor network diagrammatic notation.

$$|\psi\rangle = \frac{1}{\mathcal{N}} \left(\phi_f^* \right. \left[U_M^S \right. \left[K \right. \dots \left[U_2^S \right. \left[K \right. \left[U_1^S \right. \left[K \right. \left[U_0^S \right. \left. \phi_i \right] \left. \right] \left. \right] \left. \right] \left. \right) , \quad (\text{F41})$$

where

$$\begin{array}{c} \alpha \\ | \\ \text{---} \boxed{K} \text{---} \end{array} = \begin{cases} (I - \varepsilon Q)^{1/2} & \text{if } \alpha = 0, \\ -i\sqrt{\varepsilon}L & \text{if } \alpha = 1. \end{cases} \quad (\text{F42})$$

We will find it convenient to introduce E_m^n, R_m^n, L_m^n for $0 \leq m < n \leq M$ given by

$$\begin{aligned} E_m^n &= \begin{array}{c} \text{---} U_n^{S*} \text{---} \boxed{K^*} \text{---} U_{n-1}^{S*} \text{---} \dots \text{---} \boxed{K^*} \text{---} U_{m+1}^{S*} \text{---} \boxed{K^*} \text{---} U_m^{S*} \text{---} \\ \text{---} U_n^S \text{---} \boxed{K} \text{---} U_{n-1}^S \text{---} \dots \text{---} \boxed{K} \text{---} U_{m+1}^S \text{---} \boxed{K} \text{---} U_m^S \text{---} \end{array}, \\ R_m^n &= \begin{array}{c} \text{---} \boxed{K^*} \text{---} U_n^{S*} \text{---} \dots \text{---} \boxed{K^*} \text{---} U_{m+1}^{S*} \text{---} \boxed{K^*} \text{---} U_m^{S*} \text{---} \\ \text{---} \boxed{K} \text{---} U_n^S \text{---} \dots \text{---} \boxed{K} \text{---} U_{m+1}^S \text{---} \boxed{K} \text{---} U_m^S \text{---} \end{array}, \\ L_m^n &= \begin{array}{c} \text{---} U_n^{S*} \text{---} \boxed{K^*} \text{---} \dots \text{---} U_{m+1}^{S*} \text{---} \boxed{K^*} \text{---} U_m^{S*} \text{---} \boxed{K^*} \text{---} \\ \text{---} U_n^S \text{---} \boxed{K} \text{---} \dots \text{---} U_{m+1}^S \text{---} \boxed{K} \text{---} U_m^S \text{---} \boxed{K} \text{---} \end{array}, \end{aligned} \quad (\text{F43a})$$

where for $n < m$ we interpret $E_m^n, R_m^n, L_m^n = I \otimes I$. We can also note that for any $m \leq p \leq n$,

$$E_m^n = \begin{array}{c} \text{---} \boxed{L_{p+1}^n} \text{---} \boxed{U_p^S} \text{---} \boxed{R_m^{p-1}} \text{---} \\ \text{---} \boxed{L_{p+1}^n} \text{---} \boxed{U_p^{S*}} \text{---} \boxed{R_m^{p-1}} \text{---} \end{array} \quad (\text{F44})$$

The normalization constant $\mathcal{N}$ is given by the diagrammatic expression

$$\mathcal{N}^2 = \begin{array}{c} \phi_f \text{---} \boxed{E_{M,0}} \text{---} \phi_i^* \\ \phi_f^* \text{---} \boxed{E_{M,0}} \text{---} \phi_i \end{array} \quad (\text{F45})$$

We will consider the case of identical and independent sources—in terms of the MPS approximation of the state, the QFI from Eq. (19) can be approximated by

$$\text{QFI} \approx \frac{16}{\mathcal{N}^4} \sum_{n=1}^M \sum_{m=1}^{n-1} (|C_{n,m}^{(g)}|^2 - |C_{n,m}^{(\chi)}|^2) + \frac{8}{\mathcal{N}^2} \sum_{m=1}^M n_m, \quad (\text{F46a})$$

where

$$C_{n,m}^{(g)} = \begin{array}{c} \phi_f \text{---} \boxed{E_n^M} \text{---} \boxed{K_0^*} \text{---} \boxed{E_m^{n-1}} \text{---} \boxed{K_1^*} \text{---} \boxed{E_0^{m-1}} \text{---} \phi_i^* \\ \phi_f^* \text{---} \boxed{E_n^M} \text{---} \boxed{K_1} \text{---} \boxed{E_m^{n-1}} \text{---} \boxed{K_0} \text{---} \boxed{E_0^{m-1}} \text{---} \phi_i \end{array}, \quad (\text{F46b})$$

$$C_{n,m}^{(\chi)} = \begin{array}{c} \phi_f \\ \phi_f^* \end{array} \begin{array}{c} \boxed{E_n^M} \\ \boxed{E_n^M} \end{array} \begin{array}{c} \boxed{K_1^*} \\ \boxed{K_0} \end{array} \begin{array}{c} \boxed{E_{m-1}^{n-1}} \\ \boxed{E_{m-1}^{n-1}} \end{array} \begin{array}{c} \boxed{K_1^*} \\ \boxed{K_0} \end{array} \begin{array}{c} \boxed{E_0^{m-1}} \\ \boxed{E_0^{m-1}} \end{array} \begin{array}{c} \phi_i^* \\ \phi_i \end{array}, \quad (\text{F46c})$$

$$n_m = \begin{array}{c} \phi_f \\ \phi_f^* \end{array} \begin{array}{c} \boxed{E_m^M} \\ \boxed{E_m^M} \end{array} \begin{array}{c} \boxed{K_1^*} \\ \boxed{K_1} \end{array} \begin{array}{c} \boxed{E_0^{m-1}} \\ \boxed{E_0^{m-1}} \end{array} \begin{array}{c} \phi_i^* \\ \phi_i \end{array}. \quad (\text{F46d})$$

We note that Eq. (F46) can be used to numerically compute the QFI in $O(M^2)$ time. To see this, note that $\{E_{n,m}\}_{0 \leq m \leq n \leq M}$ can be pre-computed in $O(M^2)$ time. After this pre-computation, $\mathcal{N}$, $C_{n,m}^{(g)}$, $C_{n,m}^{(\chi)}$ and n_m can be computed in $O(1)$ time. Therefore, the total time required to compute the QFI scales as $O(M^2)$.

We next consider the computation of the gradient of the QFI with respect to the unitaries U_p^S —if the gradient is computed by first computing $\nabla_{U_p^S} C_{n,m}^{(g)}$ and $\nabla_{U_p^S} C_{n,m}^{(\chi)}$, then the total time required to compute the gradient of QFI with respect to all the unitaries U_p^S would be $O(M^3)$. In practice, this can become prohibitively expensive when $M \gtrsim 100$. We next outline an application of the adjoint variable method [87–89] that can reduce the cost of computing $\nabla_{U_p^S} \text{QFI}$ to time $O(M^2)$. We will formulate the adjoint variable method for computing the gradient of both the double summand and the single summand in Eq. (F46), both of which require slightly different treatment.

Cost functions that are single sums. We first consider a cost function $\Gamma^{(1)}(\theta)$ which depends on the parameters $\{\theta_0, \theta_1 \dots \theta_M\}$ through the unitaries $U^S = \{U_0^S(\theta_0), U_1^S(\theta_1) \dots U_M^S(\theta_M)\}$ and can be expressed as a single

summation:

$$\Gamma^{(1)}(\theta) = \sum_{m=1}^N f(\gamma_m), \quad (\text{F47})$$

where $f: \mathbb{C} \rightarrow \mathbb{R}$ is Wirtinger differentiable (i.e., differentiable with respect to the real and imaginary parts of the complex-value argument), and $\gamma_m \in \mathbb{C}$ is given by the following contraction:

$$\gamma_m = \begin{array}{c} \phi_f \\ \phi_f^* \end{array} \begin{array}{c} \boxed{E_m^M} \\ \boxed{E_m^M} \end{array} \begin{array}{c} \boxed{P^*} \\ \boxed{Q} \end{array} \begin{array}{c} \boxed{E_0^{m-1}} \\ \boxed{E_0^{m-1}} \end{array} \begin{array}{c} \phi_i^* \\ \phi_i \end{array}, \quad (\text{F48})$$

for some P and Q . Next, we note that

$$\frac{\partial}{\partial \theta_p} \Gamma(\theta) = 2 \sum_{m=1}^N \text{Re} \left(f'(\gamma_m) \frac{\partial \gamma_m}{\partial \theta_p} \right), \quad (\text{F49a})$$

where $f' \equiv \partial f(z, z^*) / \partial z$. Note that $\partial \gamma_m / \partial \theta_p$ will be given by

$$\begin{array}{c} \frac{\partial \gamma_m}{\partial \theta_p} = \begin{array}{c} \phi_f \\ \phi_f^* \end{array} \begin{array}{c} \boxed{E_m^M} \\ \boxed{E_m^M} \end{array} \begin{array}{c} \boxed{P^*} \\ \boxed{Q} \end{array} \begin{array}{c} \boxed{L_{p+1}^{m-1}} \\ \boxed{L_{p+1}^{m-1}} \end{array} \begin{array}{c} \boxed{\frac{\partial U_p^S}{\partial \theta_p}} \\ \boxed{\frac{\partial U_p^S}{\partial \theta_p}} \end{array} \begin{array}{c} \boxed{R_0^{p-1}} \\ \boxed{R_0^{p-1}} \end{array} \begin{array}{c} \phi_i^* \\ \phi_i \end{array} \quad \text{if } 0 \leq p < m \text{ and,} \\ \frac{\partial \gamma_m}{\partial \theta_p} = \begin{array}{c} \phi_f \\ \phi_f^* \end{array} \begin{array}{c} \boxed{L_{p+1}^M} \\ \boxed{L_{p+1}^M} \end{array} \begin{array}{c} \boxed{\frac{\partial U_p^S}{\partial \theta_p}} \\ \boxed{\frac{\partial U_p^S}{\partial \theta_p}} \end{array} \begin{array}{c} \boxed{R_m^{p-1}} \\ \boxed{R_m^{p-1}} \end{array} \begin{array}{c} \boxed{P^*} \\ \boxed{Q} \end{array} \begin{array}{c} \boxed{E_0^{m-1}} \\ \boxed{E_0^{m-1}} \end{array} \begin{array}{c} \phi_i^* \\ \phi_i \end{array} \quad \text{if } m \leq p \leq M, \end{array} \quad (\text{F49b})$$

where

$$\left[\frac{\partial \mathcal{U}_p^S}{\partial \theta_p} \right] = \frac{\partial}{\partial \theta_p} \left(\begin{array}{c} U_p^{S*} \\ U_p^S \end{array} \right). \quad (\text{F49c})$$

While Eq. (F49) can be used to compute $\partial \Gamma^{(1)}(\theta)/\partial \theta_p$, the total cost to compute this derivative for all p would be $O(M^2)$. To compute these derivatives in time $O(M)$, we first obtain from Eq. (F49a) that

$$\frac{\partial}{\partial \theta_p} \Gamma^{(1)}(\theta) = 2\text{Re} \left(\sum_{m=p+1}^M f'(\gamma_m) \frac{\partial \gamma_m}{\partial \theta_p} \right) + 2\text{Re} \left(\sum_{m=1}^p f'(\gamma_m) \frac{\partial \gamma_m}{\partial \theta_p} \right), \quad (\text{F50a})$$

and using Eq. (F49b), it follows that

$$\sum_{m=p+1}^M f'(\gamma_m) \frac{\partial \gamma_m}{\partial \theta_p} = \left(l_p \right) \left[\frac{\partial \mathcal{U}_p^S}{\partial \theta_p} \right] R_0^{p-1} \begin{pmatrix} \phi_i^* \\ \phi_i \end{pmatrix}, \quad (\text{F50b})$$

where

$$\left(l_p \right) = \sum_{m=p+1}^M f'(\gamma_m) \left(\begin{array}{c} \phi_f \\ \phi_f^* \end{array} \right) \left[E_m^M \right] \begin{pmatrix} P^* \\ Q \end{pmatrix} \left[L_{p+1}^{m-1} \right], \quad (\text{F50c})$$

and

$$\sum_{m=1}^p f'(\gamma_m) \frac{\partial \gamma_m}{\partial \theta_p} = \begin{pmatrix} \phi_f \\ \phi_f^* \end{pmatrix} \left[L_{p+1}^M \right] \left[\frac{\partial \mathcal{U}_p^S}{\partial \theta_p} \right] \left(r_p \right), \quad (\text{F50d})$$

where

$$\left(r_p \right) = \sum_{m=1}^p f'(\gamma_m) \left(\begin{array}{c} \phi_i^* \\ \phi_i \end{array} \right) \left[E_0^{m-1} \right] \begin{pmatrix} P^* \\ Q \end{pmatrix} \left[R_m^{p-1} \right]. \quad (\text{F50e})$$

If the vectors l_p, r_p are pre-computed, then Eqs. (F50a), (F50b) and (F50d) can be used to compute $\partial \Gamma(\theta)/\partial \theta_p$ for all p in $O(M)$ time. Furthermore, l_p, r_p can be computed, for all p , in a total of $O(M)$ time by noting that they satisfy the following recursion:

$$\left(l_{p-1} \right) = \left(l_p \right) \begin{pmatrix} U_p^{S*} & K^* \\ U_p^S & K \end{pmatrix} + f'(\gamma_p) \left(\begin{array}{c} \phi_f \\ \phi_f^* \end{array} \right) \left[E_p^M \right] \begin{pmatrix} P^* \\ Q \end{pmatrix}, \quad (\text{F51a})$$

$$r_p = \begin{array}{c} \text{---} K^* \text{---} U_{p-1}^{S*} \text{---} \\ \text{---} K \text{---} U_p^S \text{---} \end{array} r_p + f'(\gamma_p) \left(\begin{array}{c} \text{---} P^* \text{---} E_0^{p-1} \text{---} \phi_i^* \\ \text{---} Q \text{---} E_0^{p-1} \text{---} \phi_i \end{array} \right), \quad (\text{F51b})$$

together with the boundary conditions $l_M = r_0 = 0$. Therefore, to compute $\{\partial\Gamma(\theta)/\partial\theta_p\}_{p \in \{0,1,2 \dots M\}}$, an algorithm with $O(M)$ run-time is as follows:

- (1) First compute $\{\gamma_m\}_{m \in \{1,2 \dots M\}}$ —this can be done in $O(M)$ time by recursively computing E_0^{m-1} and E_m^M for $m \in \{1, 2 \dots M\}$ using

$$E_{m-1}^M = \begin{array}{c} \text{---} E_m^M \text{---} K^* \text{---} U_{m-1}^{S*} \text{---} \\ \text{---} E_m^M \text{---} K \text{---} U_{m-1}^S \text{---} \end{array} \quad \text{and} \quad E_0^m = \begin{array}{c} \text{---} U_m^{S*} \text{---} K^* \text{---} E_0^{m-1} \text{---} \\ \text{---} U_m^S \text{---} K \text{---} E_0^{m-1} \text{---} \end{array}, \quad (\text{F52})$$

together with $E_{M+1}^M = E_0^{-1} = I$ and then using Eq. (F48).

- (2) Next, we compute $\{l_p\}_{p \in \{0,1,2 \dots M\}}, \{r_p\}_{p \in \{0,1,2 \dots M\}}$ using the recursion in Eq. (F51)—this can be done in $O(M)$ time.
- (3) Finally, we use Eqs. (F50a), (F50b) and (F50d) to compute $\{\partial\Gamma^{(1)}(\theta)/\partial\theta_p\}_{p \in \{0,1,2 \dots M\}}$ —this can again be done in $O(M)$ time.

Cost functions that are double sums. Next, we consider cost functions $\Gamma^{(2)}(\theta)$ that are of the form

$$\Gamma^{(2)}(\theta) = \sum_{n=1}^N \sum_{m=1}^{n-1} f(\gamma_{n,m}), \quad (\text{F53})$$

where $f: \mathbb{C} \rightarrow \mathbb{R}$ is Wirtinger differentiable and $\gamma_{n,m} \in \mathbb{C}$ is given by

$$\gamma_{n,m} = \begin{array}{c} \phi_f \text{---} E_n^M \text{---} P_1^* \text{---} E_{m-1}^{n-1} \text{---} P_2^* \text{---} E_0^{m-1} \text{---} \phi_i^* \\ \phi_f^* \text{---} E_n^M \text{---} Q_1 \text{---} E_{m-1}^{n-1} \text{---} Q_2 \text{---} E_0^{m-1} \text{---} \phi_i \end{array}, \quad (\text{F54})$$

for some fixed tensors P_1, P_2, Q_1, Q_2 . Again, the derivative of $\Gamma^{(2)}(\theta)$, $\partial\Gamma^{(2)}(\theta)/\partial\theta_p$ is given by

$$\frac{\partial}{\partial\theta_p} \Gamma^{(2)}(\theta) = 2 \sum_{n=1}^M \sum_{m=1}^{n-1} \text{Re} \left(f'(\gamma_{n,m}) \frac{\partial\gamma_{n,m}}{\partial\theta_p} \right), \quad (\text{F55a})$$

where $f' \equiv \partial f(z, z^*)/\partial z$. Note that $\partial\gamma_{n,m}/\partial\theta_p$ will be given by

$$\begin{aligned}
\frac{\partial \gamma_{n,m}}{\partial \theta_p} &= \begin{array}{c} \text{Diagram 1: } \phi_f \text{ and } \phi_f^* \text{ enter } E_n^M, \text{ followed by } P_1^*, E_m^{n-1}, P_2^*, L_{p+1}^{m-1}, \frac{\partial \mathcal{U}_p^S}{\partial \theta_p}, R_0^{p-1}, \text{ and } \phi_i^*, \phi_i. \end{array} & \text{if } 0 \leq p < m, \\
\frac{\partial \gamma_{n,m}}{\partial \theta_p} &= \begin{array}{c} \text{Diagram 2: } \phi_f \text{ and } \phi_f^* \text{ enter } E_n^M, \text{ followed by } P_1^*, L_{p+1}^{n-1}, \frac{\partial \mathcal{U}_p^S}{\partial \theta_p}, R_m^{p-1}, P_2^*, E_0^{m-1}, \text{ and } \phi_i^*, \phi_i. \end{array} & \text{if } m \leq p < n \text{ and,} \\
\frac{\partial \gamma_{n,m}}{\partial \theta_p} &= \begin{array}{c} \text{Diagram 3: } \phi_f \text{ and } \phi_f^* \text{ enter } L_{p+1}^M, \text{ followed by } \frac{\partial \mathcal{U}_p^S}{\partial \theta_p}, R_n^{p-1}, P_1^*, E_m^{n-1}, P_2^*, E_0^{m-1}, \text{ and } \phi_i^*, \phi_i. \end{array} & \text{if } n \leq p \leq M.
\end{aligned} \tag{F55b}$$

Note that using Eq. (F55) to compute $\{\partial \Gamma^{(2)}(\theta)/\partial \theta_p\}_{p \in \{0,1,2,\dots,M\}}$ would require $O(M^3)$ time. However, similar to the case of a cost function with a single summation, this computation can be re-organized to require $O(M^2)$ time. To do so, we note that

$$\frac{\partial}{\partial \theta_p} \Gamma^{(2)}(\theta) = 2\text{Re} \left(\sum_{p < m < n \leq M} f'(\gamma_{n,m}) \frac{\partial \gamma_{n,m}}{\partial \theta_p} \right) + 2\text{Re} \left(\sum_{\substack{1 \leq m \leq p \\ p < n \leq M}} f'(\gamma_{n,m}) \frac{\partial \gamma_{n,m}}{\partial \theta_p} \right) + 2\text{Re} \left(\sum_{1 \leq m < n \leq p} f'(\gamma_{n,m}) \frac{\partial \gamma_{n,m}}{\partial \theta_p} \right). \tag{F56a}$$

Using Eq. (F55b), it follows that

$$\sum_{p < m < n \leq M} f'(\gamma_{n,m}) \frac{\partial \gamma_{n,m}}{\partial \theta_p} = \text{Diagram 4: } l_p \text{ (yellow circle) connected to } \frac{\partial \mathcal{U}_p^S}{\partial \theta_p} \text{ (grey box), } R_0^{p-1} \text{ (blue box), and } \phi_i^*, \phi_i. \tag{F56b}$$

where

$$l_p = \sum_{p < m < n \leq M} f'(\gamma_{n,m}) \left(\text{Diagram 5: } \phi_f \text{ and } \phi_f^* \text{ enter } E_n^M, \text{ followed by } P_1^*, E_m^{n-1}, P_2^*, L_{p+1}^{m-1}. \right). \tag{F56c}$$

Furthermore,

$$\sum_{\substack{1 \leq m \leq p \\ p < n \leq M}} f'(\gamma_{n,m}) \frac{\partial \gamma_{n,m}}{\partial \theta_p} = \sum_{1 \leq m \leq p} \left(\text{Diagram 6: } c_{p,m} \text{ (yellow circle) connected to } \frac{\partial \mathcal{U}_p^S}{\partial \theta_p} \text{ (grey box), } R_m^{p-1} \text{ (blue box), } P_2^*, E_0^{m-1}, \text{ and } \phi_i^*, \phi_i. \right), \tag{F56d}$$

where

$$c_{p,m} = \sum_{p < n \leq M} f'(\gamma_{n,m}) \left(\text{Diagram 7: } \phi_f \text{ and } \phi_f^* \text{ enter } E_n^M, \text{ followed by } P_1^*, L_{p+1}^{n-1}. \right). \tag{F56e}$$

Finally,

$$\sum_{1 \leq m < n \leq p} f'(\gamma_{n,m}) \frac{\partial \gamma_{n,m}}{\partial \theta_p} = \begin{array}{c} \phi_f \\ \phi_f^* \end{array} \begin{array}{c} \text{---} \text{L}_{p+1}^M \text{---} \end{array} \begin{array}{c} \frac{\partial \mathcal{U}_p^S}{\partial \theta_p} \end{array} \begin{array}{c} \text{---} r_p \end{array}, \quad (\text{F56f})$$

where

$$r_p = \sum_{1 \leq m < n \leq p} \left(\begin{array}{c} \text{---} R_n^{p-1} \text{---} P_1^* \text{---} E_m^{n-1} \text{---} P_2^* \text{---} E_0^{m-1} \text{---} \phi_i^* \\ \text{---} Q_1 \text{---} Q_2 \text{---} \phi_i \end{array} \right) \quad (\text{F56g})$$

If the vectors $l_p, c_{p,m}, r_p$ are pre-computed, then from Eqs. (F56a), (F56b), (F56d) and (F56f), we can compute $\{\partial \Gamma^{(2)}(\theta)/\partial \theta_p\}_{p \in \{0,1,2 \dots M\}}$ in $O(M^2)$ time. Furthermore, $l_p, c_{p,m}, r_p$ can also be computed in $O(M^2)$ time by noting that they satisfy the following recursions:

$$l_{p-1} = \begin{array}{c} \text{---} U_p^{S*} \text{---} K^* \text{---} \\ \text{---} U_p^S \text{---} K \text{---} \end{array} l_p + \sum_{p < n \leq M} f'(\gamma_{n,p}) \left(\begin{array}{c} \phi_f \text{---} E_n^M \text{---} P_1^* \text{---} E_p^{n-1} \text{---} P_2^* \\ \phi_f^* \text{---} Q_1 \text{---} Q_2 \end{array} \right), \quad (\text{F57a})$$

$$c_{p-1,m} = \begin{array}{c} \text{---} U_p^{S*} \text{---} K^* \text{---} \\ \text{---} U_p^S \text{---} K \text{---} \end{array} c_{p,m} + f'(\gamma_{p,m}) \left(\begin{array}{c} \phi_f \text{---} E_n^M \text{---} P_1^* \\ \phi_f^* \text{---} Q_1 \end{array} \right), \quad (\text{F57b})$$

$$r_p = \begin{array}{c} \text{---} K^* \text{---} U_{p-1}^{S*} \text{---} r_{p-1} \\ \text{---} K \text{---} U_{p-1}^S \end{array} + \sum_{1 \leq m < p} \left(\begin{array}{c} \text{---} P_1^* \text{---} E_m^{p-1} \text{---} P_2^* \text{---} E_0^{m-1} \text{---} \phi_i^* \\ \text{---} Q_1 \text{---} Q_2 \end{array} \right), \quad (\text{F57c})$$

together with the boundary conditions $l_M = c_{M,m} = r_0 = 0$. Therefore, to compute $\{\partial \Gamma(\theta)/\partial \theta_p\}_{p \in \{0,1,2 \dots M\}}$, an algorithm with $O(M)$ run-time is as follows:

- (1) First compute $\{E_{n,m}\}_{0 \leq m \leq n \leq M}$ —this can be done in $O(M^2)$ by using, for every m , the following recurrence in n

$$E_m^n = \begin{array}{c} \text{---} U_n^{S*} \text{---} K^* \text{---} E_m^{n-1} \\ \text{---} U_n^S \text{---} K \end{array}, \quad (\text{F58})$$

together with $E_m^{m-1} = I$ and then using Eq. (F48).

- (2) Next, we compute $l_p, c_{p,m}, r_p$ using the recursion in Eq. (F57)—this can be done in $O(M^2)$ time.
- (3) Finally, we use Eqs. (F56a), (F56b) and (F56d) to compute $\{\partial \Gamma^{(1)}(\theta)/\partial \theta_p\}_{p \in \{0,1,2 \dots M\}}$ —this can again be done in $O(M^2)$ time.

APPENDIX G: OPTIMAL MEASUREMENT

This Appendix provides details on the optimal measurements discussed in the main text. First, we provide details of the optimal measurement protocol which utilizes tunable non-linear optical systems. Next, we complete the details of optimality of photodetection and tunable linear optics from Sec. VB.

1. Optimal measurement with non-linear optics

We first recall a basic fact [110]: A measurement to extract φ from $|\psi_\varphi\rangle$ when $\varphi = \varphi_0$ that saturates the CRB is the projective measurement $P_0 = |\psi_{\varphi_0}\rangle\langle\psi_{\varphi_0}|$, $P_1, P_2, \dots$, where $P_0 + P_1 + P_2 + \dots = I$ and $P_i P_j = \delta_{i,j} P_i$. To see this, note that the probability of the i^{th} measurement outcome is given by $p_i(\varphi) = \langle\psi_\varphi|P_i|\psi_\varphi\rangle$. The classical Fisher information (CFI) for this probability distribution at $\varphi = \varphi_0$ is given by

$$\text{CFI} = \lim_{\varphi \rightarrow \varphi_0} \sum_i \frac{(p'_i(\varphi))^2}{p_i(\varphi)}. \quad (\text{G1})$$

We notice that $p_i(\varphi_0) = \delta_{i,0}$ and $p'_i(\varphi_0) = 0$. Thus, $\text{CFI}(\varphi_0)$ can be expressed as

$$\text{CFI} = \lim_{\varphi \rightarrow \varphi_0} \sum_i \frac{(p'_i(\varphi))^2}{p_i(\varphi)} = 2 \sum_{i \neq 0} p''_i(\varphi_0). \quad (\text{G2})$$

Since for $i \neq 0$, $P_i|\psi_{\varphi_0}\rangle = 0$, it follows that

$$\begin{aligned} p''_i(\varphi_0) &= \langle\psi_{\varphi_0}|P_i|\psi''_{\varphi_0}\rangle + \langle\psi''_{\varphi_0}|P_i|\psi_{\varphi_0}\rangle + 2\langle\psi'_{\varphi_0}|P_i|\psi'_{\varphi_0}\rangle \\ &= 2\langle\psi'_{\varphi_0}|P_i|\psi'_{\varphi_0}\rangle, \end{aligned}$$

and consequently,

$$\begin{aligned} \text{CFI} &= 4 \sum_{i \neq 0} \langle\psi'_{\varphi_0}|P_i|\psi'_{\varphi_0}\rangle = 4(\|\psi'_{\varphi_0}\|^2 - |\langle\psi_{\varphi_0}|\psi'_{\varphi_0}\rangle|^2) \\ &= \text{QFI}, \end{aligned}$$

thus establishing that this measurement is optimal. We emphasize that our construction here establishes local optimality at $\varphi = 0$. However, for a null measurement this equality alone does not guarantee an operationally attainable estimation procedure, since the statistical model may be non-identifiable at the null point. Thus, away from this point a CFI calculation alone is not sufficient to guarantee attainability of the QFI [111]

Next, we continue the analysis in Sec. VA of the main text and provide details on how to implement re-absorption of the photons emitted by the source using a non-linear optical cavity. As was discussed in Appendix F2a, the photons emitted in time T into the two input ports of the MZI can be approximately expressed as an MPS on $M = T/\varepsilon$ qudits with 3 levels, where ε is a small discretization parameter. It is a well known fact that a sequence of unitaries can be designed between an ancilla qudit and the MPS $|\psi\rangle$ to map $|\psi\rangle \rightarrow |0\rangle^{\otimes M}$ [95]. More specifically, given an MPS on M qudits with d -levels and bond dimension D with tensors $A_1, A_2, \dots, A_M$, we can always alternatively expressed in terms of unitaries $U_1, U_2, \dots, U_M$, with $U_i: \mathbb{C}^D \otimes \mathbb{C}^d \rightarrow \mathbb{C}^D \otimes \mathbb{C}^d$ [112]:

$$\phi_f^* \text{---} A_M \text{---} \dots \text{---} A_2 \text{---} A_1 \text{---} \phi_i = |0\rangle \text{---} U_M \text{---} \dots \text{---} U_2 \text{---} U_1 \text{---} (|0\rangle) \quad (\text{G3})$$

In other words, the MPS $|\psi\rangle$ can be generated by initializing a D -level ancilla and the M qudits in $|0\rangle$ and then applying U_M on the ancilla and the M^{th} qudit, then U_{M-1} on the ancilla and the $(M-1)^{\text{th}}$ qudit and so on. Furthermore, since $|\psi\rangle$ is a normalized state, it is always guaranteed that on applying the last unitary in this sequence, U_1 , the ancilla will disentangle from the remaining qudits and can be assumed to be in the state $|0\rangle$. To undo the state preparation, we can thus start from

an ancilla in $|0\rangle$, apply $U_1^\dagger$ to the ancilla and the rightmost qudit, then apply $U_2^\dagger$ to the ancilla and the next qudit and so on—at the end of this sequence of unitaries, the ancilla as well as the remaining qudits will be in $|0\rangle$ state.

To physically implement these unitaries, our proposal is to use a multi-mode optical cavity with $\chi^{(3)}$ non-linearity and coherent drives. We assume that we can implement the following Hamiltonian $H_R(t)$ between the three cavities

$$H_R(t) = \frac{\chi}{2} \sum_{k,k' \in \{0,A,B\}} c_k^\dagger c_{k'}^\dagger c_k c_{k'} + \sum_{k \in \{0,A,B\}} (\lambda_k^*(t) e^{-i\phi_k(t)} b_k + \text{H.c.}), \quad (\text{G4})$$

where, as shown in Fig. 8, c_A and c_B are the annihilation operators of the modes coupling to the output ports of the MZI and c_0 is the annihilation operator of an additional mode that will play the role of the ancilla. Furthermore, we also assume that we can tunably and linearly couple the modes c_A and c_B to the MZI output ports via the Hamiltonian

$$H_{R,P}(t) = V_A(t) a_t^\dagger c_A + V_B(t) b_t^\dagger c_B + \text{H.c.} \quad (\text{G5})$$

Now, to implement the unitary that coherently reabsorbs the photons emitted by the source, at every timestep $[m\varepsilon, (m+1)\varepsilon)$, we perform two steps: *first*, transfer the photons from the segment $[m\varepsilon, (m+1)\varepsilon)$ in the ports to the cavity modes c_A, c_B and *second*, apply the unitary in between $U_m^\dagger$ in between the three modes. To perform the first step, we set all the tunable parameters $(\{\lambda_k(t)\})$ to 0, and choose $V_A(t), V_B(t)$ in the interval $t \in [m\varepsilon, (m+1)\varepsilon)$ so as to map

$$(1 + \alpha_0 A_m^\dagger + \beta_0 B_m^\dagger) |\text{vac}\rangle \rightarrow (1 + \alpha_0 c_A^\dagger + \beta_0 c_B^\dagger) |0_A, 0_B\rangle,$$

where, following Appendix F2a, $A_m = \int_{m\varepsilon}^{(m+1)\varepsilon} a_t dt / \sqrt{\varepsilon}$, $B_m = \int_{m\varepsilon}^{(m+1)\varepsilon} b_t dt / \sqrt{\varepsilon}$ and $|0_A\rangle, |0_B\rangle$ are respectively the vacuum states of the oscillators c_A and c_B . Recall from Appendix F2a that for sufficiently small ε , the segment of the ports corresponding to $[m\varepsilon, (m+1)\varepsilon)$ has at most 1-photon upto $O(\varepsilon^2)$ error, so this unitary effectively transfers the photons in the segment $[m\varepsilon, (m+1)\varepsilon)$ of the two ports to the oscillators c_A and c_B . It is also easy to check that the explicit choice $V_A(t), V_B(t) = ((m+1)\varepsilon - t)^{-1/2}$ for $t \in [m\varepsilon, (m+1)\varepsilon)$ accomplishes this transfer of excitations from the ports to the two coupled cavities.

Next, we switch off the couplings to the MZI ports (i.e., set $V_A(t), V_B(t) = 0$) and apply the unitary $U_m^\dagger$: Note that the unitary $U_m^\dagger$ acts on the space $\mathbb{C}^3 \otimes \mathbb{C}^D$, where the qutrit is in between the states $|0_A, 0_B\rangle, c_A^\dagger |0_A, 0_B\rangle, c_B^\dagger |0_A, 0_B\rangle$ and the D -level qudit corresponds to the state of the oscillator c_0 . We also point out that this gate needs to be applied on time-scales much faster than ε , i.e., before the oscillators interact with the next time-bin. However, it would appear from Eq. (G4) that the gate-time is limited to $\sim 1/\chi$. Experimentally it is typically only feasible to have very large coherent drives $\lambda_k(t)$, but χ remains a fixed (and often small) constant. However, we show that by adapting the strategy of Ref. [93], we can use a large drive to apply any

target unitary on this subspace within a time limited only by the strength of drive $\lambda_k(t)$.

Suppose $U_R(t, s) = \mathcal{T} \exp(-\int_s^t H_R(\tau) d\tau)$ is the unitary group corresponding to $H_R(t)$: We first consider this unitary group in the frame rotated with respect to the Hamiltonian $\sum_k \phi_k'(t) c_k^\dagger c_k$ and displaced with respect to the Hamiltonian $\sum_k (\dot{\gamma}_k(t) c_k^\dagger - \text{H.c.})$. Furthermore, given $\gamma_k(t)$ (which we will choose later) we will make the choices

$$\begin{aligned} \dot{\phi}_k(t) &= -\chi \sum_{k'} |\gamma_{k'}(t)|^2 - \frac{\chi}{2} |\gamma_k(t)|^2, \lambda_k(t) \\ &= i\dot{\gamma}_k(t) - \dot{\phi}_k(t) \gamma_k(t) - \chi D \gamma_k(t) - \chi \sum_{k'} |\gamma_{k'}(t)|^2. \end{aligned}$$

The effective Hamiltonian $\tilde{H}_R(t)$ in this new frame is then given by

$$\begin{aligned} \tilde{H}_R(t) &= \frac{\chi}{2} \sum_{k,k'} c_k^\dagger c_{k'}^\dagger c_k c_{k'} + \chi \sum_k (\gamma_k(t) c_k^\dagger (N-D) + \text{H.c.}) \\ &\quad + \frac{\chi}{2} \sum_{k \neq k'} (c_k^\dagger c_{k'}^\dagger \gamma_k(t) \gamma_{k'}(t) + c_k^\dagger c_{k'}^\dagger \gamma_k(t) \gamma_{k'}^*(t) + \text{H.c.}), \end{aligned} \quad (\text{G6})$$

where $N = \sum_k c_k^\dagger c_k$ is the total number of photons in the three cavities. To further simplify $\tilde{H}_R(t)$, we also choose $\gamma_k(t) = \Gamma_k(t) f_k(t)$ where $f_k(t)$ are rapidly oscillating functions that further satisfy $\langle f_k(t) \rangle = 1, \langle f_k^2(t) \rangle = 0$, and for $k \neq k' \langle f_k(t) f_{k'}(t) \rangle = \langle f_k(t) f_{k'}^*(t) \rangle = 0$. A set of possible explicit choices for $f_A(t), f_B(t), f_0(t)$ that satisfy this condition are

$$f_k(t) = 1 + \frac{1}{\sqrt{2}} \sum_{n=1}^6 e^{i2\pi m_k n/7} \cos(n\Omega t) \quad (\text{G7})$$

$$\text{where } m_0 = 1, m_A = 2 \quad \text{and} \quad m_B = 3.$$

In the limit of large Ω , the dynamics due to $\tilde{H}_R(t)$ is well approximated by $\hat{H}_R(t)$

$$\hat{H}_R(t) = \frac{\chi}{2} \sum_{k,k'} c_k^\dagger c_{k'}^\dagger c_k c_{k'} + \hat{H}_R^c(t), \quad (\text{G8a})$$

where

$$\hat{H}_R^c(t) = \chi \sum_k (\text{Re}[\Gamma_k(t)] H_k^{(+)} + \text{Im}[\Gamma_k(t)] H_k^{(-)}), \quad (\text{G8b})$$

with

$$\begin{aligned} H_k^{(+)} &= c_k^\dagger(N - D) + \text{H.c.} \quad \text{and} \\ H_k^{(-)} &= i(c_k^\dagger(N - D) - \text{H.c.}). \end{aligned} \quad (\text{G8c})$$

Here $N = \sum_k c_k^\dagger c_k$ is the total number of photons in the three oscillators. Given $\vec{n} = (n_0, n_A, n_B)$ and denoting by $|\vec{n}\rangle \propto (c_0^\dagger)^{n_0} (c_A^\dagger)^{n_A} (c_B^\dagger)^{n_B} |0\rangle$, we first observe that $\hat{H}_R(t)$ and $\hat{H}_R^c(t)$ are block diagonal in the subspace $\mathcal{S}^{(D)} = \text{span}(\{|\vec{n}\rangle : n_0 + n_A + n_B \leq D\})$ (the subspace with no more than D photons in the three cavities) and the subspace orthogonal to it, i.e., for all $|\psi\rangle \in \mathcal{S}^{(D)}$, $\hat{H}_R(t)|\psi\rangle, \hat{H}_R^c(t)|\psi\rangle \in \mathcal{S}^{(D)}$. We now concentrate on the finite-dimensional subspace $\mathcal{S}^{(D)}$ and show that the Hamiltonian $\hat{H}_R^c(t)$ is universal in this subspace, i.e., any desired unitary U can be implemented up to a global phase with an appropriate choice of $\Gamma_k(t)$. From the standard theory of quantum control [113,114], to establish universality of $H_R^c(t)$, we need to establish that the algebra generated by the operators $\{H_k^{(+)}, H_k^{(-)}\}_{k \in \{0,A,B\}}$ under nested commutator is the full $\mathfrak{su}(\dim(\mathcal{S}^{(D)}))$ algebra, i.e., any traceless Hermitian operator on the subspace $\mathcal{S}^{(D)}$ can be expressed as a linear combination of $\{H_k^{(+)}, H_k^{(-)}\}_{k \in \{0,A,B\}}$ and their nested commutators. Reference [93] considered this problem for the case of a single oscillator and established universality of $\hat{H}_R^c(t)$ using a result from Ref. [115] that described a set of sufficient checkable conditions for a set of operators to generate the full algebra of traceless Hermitian operators. However, the result of Ref. [115] does not apply directly to $\hat{H}_R^c(t)$ when more than one oscillator is present and their argument has to be modified. We do so in the following lemma.

Lemma 5. For a positive integers $k, d > 1$, consider a finite-dimensional Hilbert space $\mathcal{H}$ with orthonormal basis elements $\{|\vec{n}\rangle \text{ where } \vec{n} = \{n_1, n_2 \dots n_k\} \text{ with } n_1 + n_2 \dots n_k \leq d\}$. Consider the Hermitian operators $H_0, \{H_p^{(+)}, H_p^{(-)}\}_{p \in \{1,2 \dots k\}}$ where

$$\begin{aligned} H_0 &= \sum_{\vec{n}: \|\vec{n}\|_1 \leq d} \mu(\vec{n}) |\vec{n}\rangle \langle \vec{n}|, \\ H_p^{(+)} &= \sum_{\vec{n}: \|\vec{n}\|_1 \leq d-1} d_p^{(+)}(\vec{n}) (|\vec{n}\rangle \langle \vec{n} + \vec{e}_p| + \text{H.c.}), \quad \text{and} \\ H_p^{(-)} &= i \sum_{\vec{n}: \|\vec{n}\|_1 \leq d-1} d_p^{(-)}(\vec{n}) (|\vec{n}\rangle \langle \vec{n} + \vec{e}_p| - \text{H.c.}), \end{aligned}$$

where $\vec{e}_p = (\underbrace{0, \dots, 0}_{p-1 \text{ times}}, 1, 0 \dots 0)$. If the coefficients

$d_p^{(\pm)}(\vec{n}), \mu(\vec{n})$ satisfy the conditions

C1. Non-zero off diagonal elements in $H_p^{(\pm)}$: For all $p \in \{1, 2 \dots k\}$ and $\vec{n}, d_p^{(\pm)}(\vec{n}) \neq 0$.

C2. Non-degeneracy in level-spacings of H_0 : For all $p \in \{1, 2 \dots k\}$,

$$\begin{aligned} (\mu(\vec{n} + \vec{e}_p) - \mu(\vec{n}))^2 &= (\mu(\vec{m} + \vec{e}_p) - \mu(\vec{m}))^2 \\ \text{if and only if } \vec{n} &= \vec{m}, \end{aligned}$$

then the algebra generated by the set of operators $\{H_0, H_1^{(\pm)}, H_2^{(\pm)} \dots H_k^{(\pm)}\}$ contains the full $\mathfrak{su}(\dim(\mathcal{H}))$ algebra.

Proof. Unless otherwise mentioned, throughout this proof $\vec{n} = (n_1, n_2 \dots n_k)$ will denote a vector of non-negative integers with $n_1 + n_2 + \dots + n_k \leq d$. For notational convenience, we will define the Hermitian operators $E^{(\pm)}(\vec{n}, \vec{m})$ via

$$\begin{aligned} E^{(0)}(\vec{n}, \vec{m}) &= |\vec{n}\rangle \langle \vec{n}| - |\vec{m}\rangle \langle \vec{m}|, \\ E^{(+)}(\vec{n}, \vec{m}) &= |\vec{n}\rangle \langle \vec{m}| + \text{H.c.} \quad \text{and} \\ E^{(-)}(\vec{n}, \vec{m}) &= i(|\vec{n}\rangle \langle \vec{m}| - \text{H.c.}). \end{aligned}$$

Furthermore, for an operator E , we will denote by $\mathcal{C}_E$ its commutator, i.e., $\mathcal{C}_E(X) = [E, X]$. For a set of operators $\mathcal{Q} = \{E_1, E_2 \dots E_n\}$, we will denote by $\text{alg}(\mathcal{Q})$ to be the algebra generated by $\mathcal{Q}$ under nested commutators.

We first observe that the operators $\mathcal{M} = \{E^{(+)}(\vec{n}, \vec{n} + \vec{e}_p), E^{(-)}(\vec{n}, \vec{n} + \vec{e}_p)\}_{\vec{n}: \|\vec{n}\|_1 \leq d-1}$ generate the full $\mathfrak{su}(\dim(\mathcal{H}))$ algebra (i.e., the algebra of traceless Hermitian operators on $\mathcal{H}$). To see this explicitly, note that for any $\vec{n} \neq \vec{m}$, we can always find $\vec{k}_1, \vec{k}_2 \dots \vec{k}_p$ such that $\|\vec{n} - \vec{k}_1\|_1, \|\vec{k}_1 - \vec{k}_2\|_1, \|\vec{k}_2 - \vec{k}_3\|_1 \dots \|\vec{k}_p - \vec{m}\|_1 = 1$ and consequently $E^{(\pm)}(\vec{n}, \vec{k}_1), E^{(\pm)}(\vec{k}_1, \vec{k}_2), E^{(\pm)}(\vec{k}_2, \vec{k}_3) \dots E^{(\pm)}(\vec{k}_p, \vec{m}) \in \mathcal{M}$. Furthermore, since for any $\vec{n} \neq \vec{m} \neq \vec{k}$, $[E^{(+)}(\vec{n}, \vec{m}), E^{(+)}(\vec{m}, \vec{k})] = -iE^{(-)}(\vec{m}, \vec{k})$, $[E^{(+)}(\vec{n}, \vec{m}), E^{(-)}(\vec{m}, \vec{k})] = iE^{(+)}(\vec{n}, \vec{k})$, $[E^{(-)}(\vec{n}, \vec{m}), E^{(-)}(\vec{m}, \vec{k})] = -E^{(+)}(\vec{n}, \vec{k})$, it follows that for both $\sigma \in \{+, -\}$

$$\begin{aligned} E^{(\sigma)}(\vec{n}, \vec{m}) &= \eta \mathcal{C}_{E^{(\sigma_1)}(\vec{n}, \vec{k}_1)}, \\ \mathcal{C}_{E^{(\sigma_2)}(\vec{k}_1, \vec{k}_2)} \dots \mathcal{C}_{E^{(\sigma_p)}(\vec{k}_{p-1}, \vec{k}_p)}, &E^{(\sigma_{p+1})}(\vec{k}_p, \vec{m}), \end{aligned}$$

for some $\sigma_1, \sigma_2 \dots \sigma_{p+1} \in \{+, -\}$ and $\eta \in \{\pm 1, \pm i\}$ and therefore $E^{(\pm)}(\vec{n}, \vec{m}) \in \text{alg}(\mathcal{M})$. Furthermore, since $[E^{(+)}(\vec{n}, \vec{m}), E^{(-)}(\vec{n}, \vec{m})] = -iE^{(0)}(\vec{n}, \vec{m})$, we conclude that for all $\vec{n}, \vec{m}$, $E^{(0)}(\vec{n}, \vec{m}) \in \text{alg}(\mathcal{M})$. Since the operator set $\{E^{(0)}(\vec{n}, \vec{m}), E^{(+)}(\vec{n}, \vec{m}), E^{(-)}(\vec{n}, \vec{m})\}_{\vec{n} \neq \vec{m}}$ spans the whole set of Hermitian traceless operators on $\mathcal{H}$, we obtain that $\text{alg}(\mathcal{M}) = \mathfrak{su}(\dim(\mathcal{H}))$.

Next, we show that $\mathcal{M} = \{E^{(+)}(\vec{n}, \vec{n} + \vec{e}_p), E^{(-)}(\vec{n}, \vec{n} + \vec{e}_p)\}_{\vec{n}: \|\vec{n}\|_1 \leq d-1}$ is contained in $\text{alg}(\{H_0, H_1^{(\pm)}, H_2^{(\pm)} \dots H_k^{(\pm)}\})$. For this, we observe that

$\mathcal{C}_{H_0}^{2k}(H_p^{(+)}) \in \text{alg}(\{H_0, H_1^{(\pm)}, H_2^{(\pm)} \dots H_k^{(\pm)}\})$ is given by

$$\mathcal{C}_{H_0}^{2k}(H_p^{(+)}) = \sum_{\vec{n}: \|\vec{n}\|_1 \leq d-1} (\mu(\vec{n} + \vec{e}_p) - \mu(\vec{n}))^{2k} d_p^{(+)}(\vec{n}) E^{(+)}(\vec{n}, \vec{n} + \vec{e}_p).$$

We note that this equation can be re-written as

$$\begin{bmatrix} H_p^{(+)} \\ \mathcal{C}_{H_0}^2(H_p^{(+)}) \\ \mathcal{C}_{H_0}^4(H_p^{(+)}) \\ \vdots \end{bmatrix} = \mathbf{V} \begin{bmatrix} E^{(+)}(\vec{n}_1, \vec{n}_1 + \vec{e}_p) \\ E^{(+)}(\vec{n}_2, \vec{n}_1 + \vec{e}_p) \\ E^{(+)}(\vec{n}_3, \vec{n}_1 + \vec{e}_p) \\ \vdots \end{bmatrix},$$

where

$$\mathbf{V} = \begin{bmatrix} 1 & 1 & 1 & \dots \\ (\mu(\vec{n}_1 + \vec{e}_p) - \mu(\vec{n}_1))^2 & (\mu(\vec{n}_2 + \vec{e}_p) - \mu(\vec{n}_2))^2 & (\mu(\vec{n}_3 + \vec{e}_p) - \mu(\vec{n}_3))^2 & \dots \\ (\mu(\vec{n}_1 + \vec{e}_p) - \mu(\vec{n}_1))^4 & (\mu(\vec{n}_2 + \vec{e}_p) - \mu(\vec{n}_2))^4 & (\mu(\vec{n}_3 + \vec{e}_p) - \mu(\vec{n}_3))^4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We note that $\mathbf{V}$ is a Vandermonde matrix and since for $\vec{n} \neq \vec{m}$, $(\mu(\vec{n} + \vec{e}_p))^2 - \mu(\vec{n})^2 \neq (\mu(\vec{m} + \vec{e}_p))^2 - \mu(\vec{m})^2$ by assumption, $\mathbf{V}$ is invertible. Therefore, we obtain that $E^{(+)}(\vec{n}, \vec{n} + \vec{e}_p)$ can be expressed as linear combinations of $H_p^{(+)}, \mathcal{C}_{H_0}^2(H_p^{(+)})$, $\mathcal{C}_{H_0}^4(H_p^{(+)}) \dots$ and thus $E^{(+)}(\vec{n}, \vec{n} + \vec{e}_p) \in \text{alg}(\{H_0, H_1^{(\pm)}, H_2^{(\pm)} \dots H_k^{(\pm)}\})$. A similar argument can be repeated to show that $E^{(-)}(\vec{n}, \vec{n} + \vec{e}_p) \in \text{alg}(\{H_0, H_1^{(\pm)}, H_2^{(\pm)} \dots H_k^{(\pm)}\})$ for all $\vec{n}$. Since we have already previously established that the algebra generated by $\{E^{(+)}(\vec{n}, \vec{n} + \vec{e}_p), E^{(-)}(\vec{n}, \vec{n} + \vec{e}_p)\}$ is the full $\mathfrak{su}(\dim(\mathcal{H}))$, we obtain the lemma statement. $\blacksquare$

Next, we use lemma 5 to show that the Hamiltonian $H_R^c(t)$ in Eq. (G8) is completely controllable within the blockaded subspace $\mathcal{S}^{(D)} = \text{span}(\{|n_0, n_A, n_B\rangle : n_0 + n_A + n_B \leq D\})$. We first re-write it within the subspace $\mathcal{S}^{(D)}$ as

$$\hat{H}_R^c(t) = \chi \sum_{k \in \{0,A,B\}} (\text{Re}[\Gamma_k(t)] H_k^{(+)} + \text{Im}[\Gamma_k(t)] H_k^{(-)}),$$

where

$$\begin{aligned} H_k^{(+)} &= c_k^\dagger (N - D) + \text{H.c.} = \sum_{\vec{n}: \|\vec{n}\|_1 \leq D-1} (\|\vec{n}\|_1 - D) \sqrt{n_k + 1} (|\vec{n}\rangle \langle \vec{n} + \vec{e}_k| + \text{H.c.}), \quad H_k^{(-)} = i(c_k^\dagger (N - D) - \text{H.c.}) \\ &= i \sum_{\vec{n}: \|\vec{n}\|_1 \leq D-1} (\|\vec{n}\|_1 - D) \sqrt{n_k + 1} (|\vec{n}\rangle \langle \vec{n} + \vec{e}_k| - \text{H.c.}). \end{aligned}$$

Note that the operators $H_k^{(\pm)}$ already satisfy the conditions of lemma 5. Next, for any real constants $\vec{\xi} = \{\xi_0, \xi_A, \xi_B\}$, the algebra generated by $\{H_k^{(+)}, H_k^{(-)}\}_{k \in \{0,A,B\}}$ contains the operator H_0 given by

$$H_0 = i \sum_k \xi_k [H_k^{(+)}, H_k^{(-)}] = \sum_{\vec{n}: \|\vec{n}\|_1 \leq D} \mu(\vec{n}) |\vec{n}\rangle \langle \vec{n}|, \quad (\text{G9a})$$

where

$$\mu(\vec{n}) = \sum_k (\xi_k n_k (2\|\vec{n}\|_1 - 2D - 1) + \xi_k (\|\vec{n}\|_1 - D)^2). \quad (\text{G9b})$$

We next show that the constants ξ_0, ξ_A, ξ_B can be chosen to satisfy the non-degeneracy condition in lemma 5. We will choose ξ_k to be irrational numbers such that their products $\{\xi_k \xi_{k'}\}_{k,k' \in \{0,A,B\}}$ are irrational and incommensurate—a concrete choice could be $\xi_0 = 2^{1/3}, \xi_A = 3^{1/3}, \xi_B = 5^{1/3}$. Next, we note that for any integers $\vec{p} = (p_0, p_A, p_B)$ and $\vec{m} = (q_0, q_A, q_B)$,

$$\left(\sum_{k \in \{0,A,B\}} p_k \xi_k \right)^2 = \left(\sum_{k \in \{0,A,B\}} q_k \xi_k \right)^2 \Rightarrow \text{either } \vec{p} = \vec{q} \text{ or } \vec{p} = -\vec{q}. \quad (\text{G10})$$

We use this fact to now show that H_0 in Eq. (G9) satisfies the non-degeneracy condition in lemma 5, i.e., for all k , the set of real numbers $\{(\mu(\vec{n} + \vec{e}_k) - \mu(\vec{n}))^2\}_{\vec{n}: \|\vec{n}\|_1 \leq D}$ are all distinct. We begin by noting that

$$\mu(\vec{n} + \vec{e}_k) - \mu(\vec{n}) = (2n_k + 2(2n - 2D + 1))\xi_k + \sum_{k' \neq k} \xi_{k'} (2n_{k'} + 2n - 2D + 1).$$

Next, suppose that there were $\vec{n}, \vec{m}$ such that $(\mu(\vec{n} + \vec{e}_k) - \mu(\vec{n}))^2 = (\mu(\vec{m} + \vec{e}_k) - \mu(\vec{m}))^2$ —applying Eq. (G10), there are two possibilities. *First*,

$$(2n_k + 2(2n - 2D + 1)) = (2m_k + 2(2m - 2D + 1)), \text{ and} \\ (2n_{k'} + 2n - 2D + 1) = (2m_{k'} + 2m - 2D + 1) \text{ for } k' \neq k,$$

which implies that $\vec{n} = \vec{m}$. *Second*,

$$(2n_k + 2(2n - 2D + 1)) = -(2m_k + 2(2m - 2D + 1)), \text{ and} \\ (2n_{k'} + 2n - 2D + 1) = -(2m_{k'} + 2m - 2D + 1) \text{ for } k' \neq k,$$

which implies that $5(\|\vec{n}\|_1 + \|\vec{m}\|_1) = 4(2D - 1)$ —since we can always choose D such that $2D - 1$ is not divisible by 5, this equation will have no solutions. Therefore, we conclude that as long as $\vec{n} \neq \vec{m}$, $(\mu(\vec{n} + \vec{e}_k) - \mu(\vec{n}))^2 \neq (\mu(\vec{m} + \vec{e}_k) - \mu(\vec{m}))^2$, thus confirming that H_0 as constructed in Eq. (G9) satisfies the non-degeneracy condition in lemma 5.

Thus, we have shown that $\hat{H}_R^c(t)$ as in Eq. (G8) can be used to implement any unitary within the blockaded subspace, thus allowing us to implement the re-absorption unitary required in the optimal measurement setup.

Furthermore, since the magnitude of $\Gamma_k(t)$ in Eq. (G8) is determined only by the coherent drive applied on the oscillators, the speed of applying a unitary on the blockaded subspace is not limited by the non-linear strength χ .

2. Optimality of photodetection and linear optics

a. Independent sources

We recall the setup introduced in the main text Sec. VB—a linear optical element is applied at the output of the MZI and is tuned depending on the output of the photodetectors [Fig. 9(b)]. The result of the photodetection would be a sequence of times $0 \leq \tau_1 < \tau_2 < \dots < \tau_n < \dots$ at which the photons are detected as well as $\sigma_1, \sigma_2, \dots, \sigma_n, \dots \in \{a, b\}$ indicating the port in which a photon has been detected. Having obtained this photodetection record until the n^{th} detection event and before the next detection event, we apply a linear optical element described by the unitary $U_{\vec{\tau}^n}^{\vec{\sigma}^n}(t)$, where $\vec{\sigma}^n = \{\sigma_n, \sigma_{n-1}, \dots, \sigma_1\}$, $\vec{\tau}^n = \{\tau_n, \tau_{n-1}, \dots, \tau_1\}$, for $t > \tau_n$. Suppose that the next detection happens at $\tau_{n+1} > \tau_n$. The annihilation operators at the two output ports at this time, $a_{\tau_{n+1}}$ and $b_{\tau_{n+1}}$, after the application of the unitary $U_{\vec{\tau}^n}^{\vec{\sigma}^n}(t)$ will be given by

$$U_{\vec{\tau}^n}^{\vec{\sigma}^n}(\tau_{n+1}) \begin{bmatrix} a_{\tau_{n+1}} \\ b_{\tau_{n+1}} \end{bmatrix} U_{\vec{\tau}^n}^{\vec{\sigma}^n}(\tau_{n+1}) \\ = \underbrace{\begin{bmatrix} (V_{a,a}(\tau_{n+1}))_{\vec{\tau}^n}^{\vec{\sigma}^n} & (V_{a,b}(\tau_{n+1}))_{\vec{\tau}^n}^{\vec{\sigma}^n} \\ (V_{b,a}(\tau_{n+1}))_{\vec{\tau}^n}^{\vec{\sigma}^n} & (V_{b,b}(\tau_{n+1}))_{\vec{\tau}^n}^{\vec{\sigma}^n} \end{bmatrix}}_{V_{\vec{\tau}^n}^{\vec{\sigma}^n}(\tau_{n+1})} \begin{bmatrix} a_s \\ b_s \end{bmatrix}, \quad (\text{G11})$$

where $V_{\vec{\tau}^n}^{\vec{\sigma}^n}(\tau_{n+1})$ is a 2×2 unitary matrix. We can streamline the notation a lot more by introducing $p_{\vec{\tau}^k}^{\vec{\sigma}^k}, q_{\vec{\tau}^k}^{\vec{\sigma}^k}$ via

$$p_{\vec{\tau}^k}^{\{\vec{\sigma}^{k-1}, a\}} = (V_{a,a}(\tau_k))_{\vec{\tau}^{k-1}}^{\vec{\sigma}^{k-1}}, \quad q_{\vec{\tau}^k}^{\{\vec{\sigma}^{k-1}, a\}} = (V_{a,b}(\tau_k))_{\vec{\tau}^{k-1}}^{\vec{\sigma}^{k-1}}, \\ p_{\vec{\tau}^k}^{\{\vec{\sigma}^{k-1}, b\}} = (V_{b,a}(\tau_k))_{\vec{\tau}^{k-1}}^{\vec{\sigma}^{k-1}} \text{ and } q_{\vec{\tau}^k}^{\{\vec{\sigma}^{k-1}, b\}} = (V_{b,b}(\tau_k))_{\vec{\tau}^{k-1}}^{\vec{\sigma}^{k-1}}. \quad (\text{G12})$$

Using this notation, the measurement of a sequence of photodetection times $\vec{\tau}$ and a sequence of the ports $\vec{\sigma}$ at which these detections happen is equivalent to a projection on the state $|E_{\vec{\tau}}^{\vec{\sigma}}\rangle$ where

$$|E_{\vec{\tau}}^{\vec{\sigma}}\rangle = \prod_{k=1}^n ((p_{\vec{\tau}^k}^{\vec{\sigma}^k})^* a_{\tau_k}^\dagger + (q_{\vec{\tau}^k}^{\vec{\sigma}^k})^* b_{\tau_k}^\dagger) |0\rangle, \quad (\text{G13})$$

where $\vec{\tau}^k = \{\tau_k, \tau_{k-1}, \dots, \tau_1\}$, $\vec{\sigma}^k = \{\sigma_k, \sigma_{k-1}, \dots, \sigma_1\}$. Using Eq. (G11), it also follows that $\langle E_{\vec{\tau}'}^{\vec{\sigma}'} | E_{\vec{\tau}}^{\vec{\sigma}} \rangle = \delta_{\vec{\sigma}, \vec{\sigma}'} \delta(\vec{\tau} - \vec{\tau}')$.

Finally, we remark that if $q_{\vec{\tau}^k}^{\{\sigma k-1,a\}}, p_{\vec{\tau}^k}^{\{\sigma k-1,b\}} = 0$ or $q_{\vec{\tau}^k}^{\{\sigma k-1,b\}}, p_{\vec{\tau}^k}^{\{\sigma k-1,a\}} = 0$ for all $\vec{\sigma}^{k-1}, \vec{\tau}^k$, then this measurement protocol reduces to simply performing photodetection on the two output ports without any linear optics.

To understand the optimality of this measurement protocol, we will use the optimality criteria from Ref. [43]: A rank 1 projective measurement described by orthogonal states $|E_x\rangle$ optimally senses the parameter φ from the state $|\psi_\varphi\rangle$ if and only if

$$\text{Im}(\langle E_x | \psi_\varphi \rangle \langle \psi_\varphi^\perp | E_x \rangle) = 0 \quad \forall x, \quad (\text{G14})$$

where

$$|\psi_\varphi^\perp\rangle = (1 - |\psi_\varphi\rangle \langle \psi_\varphi|) \frac{d}{d\varphi} |\psi_\varphi\rangle.$$

Using the derivative of $|\psi_\varphi\rangle$ from Eq. (7), we obtain that $\{|E_{\vec{\tau}}^{\vec{\sigma}}\rangle \forall \vec{\sigma}, \vec{\tau}\}$ is optimal at $\varphi = 0$ if

$$\langle E_{\vec{\tau}}^{\vec{\sigma}} | \{|\psi\rangle \langle \psi|, H_d\} | E_{\vec{\tau}}^{\vec{\sigma}} \rangle = 2|\langle \psi | E_{\vec{\tau}}^{\vec{\sigma}} \rangle|^2 \langle \psi | H_d | \psi \rangle. \quad (\text{G15})$$

The rest of our analysis will be based on this optimality condition.

We first consider the case where two independent sources emit photons into the two input ports, i.e., the state $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$. It will be convenient to define the wave-functions corresponding to the states $|\psi_A\rangle, |\psi_B\rangle$: Given $\vec{\tau} = \{\tau_1, \tau_2 \dots \tau_n\}$, we will define for $X \in \{A, B\}$

$$\Psi_X(\vec{\tau}) = \langle \text{vac} | (\prod_{i=1}^n x_{\tau_i}) | \psi_X \rangle,$$

with $\Psi_X(\emptyset) = \langle \text{vac} | \psi_X \rangle$. We will additionally assume that

$$\Psi_X(\vec{\tau}) \neq 0 \quad \forall \vec{\tau}, \quad (\text{G16})$$

which is equivalent to assume that the wave-functions corresponding to the states $|\psi_A\rangle, |\psi_B\rangle$ are never exactly zero. Without loss of generality, we will assume $\Psi_X(\emptyset) > 0$ for both $X \in \{A, B\}$, and define phase $\Theta_X(\vec{\tau})$ via

$$\Psi_X(\vec{\tau}) = |\Psi_X(\vec{\tau})| e^{i\Theta_X(\vec{\tau})}.$$

Our main result for the case of independent sources is that if photodetection with time and measurement-record-dependent linear optics is optimal, then photodetection itself is an optimal measurement. To show this, we begin by applying Eq. (G15) for $\vec{\tau}, \vec{\sigma} = \emptyset$, which corresponds to the measurement outcome of no photons being detected. Together with $\Psi_X(\emptyset) \neq 0$, this yields

$$\langle \psi | H_d | \psi \rangle = 0.$$

Next, we consider the case of a single photodetection event and apply Eq. (G15) for $\vec{\tau}^1 = \{\tau_1\}, \vec{\sigma}^1 = \{\sigma_1\}$. We obtain that

$$[p_{\tau_1}^{\sigma_1} \quad q_{\tau_1}^{\sigma_1}] \Lambda^{(1)}(\vec{\tau}^1) \begin{bmatrix} p_{\tau_1}^{\sigma_1*} \\ q_{\tau_1}^{\sigma_1*} \end{bmatrix} = 0, \quad (\text{G17a})$$

where $\Lambda^{(1)}(\vec{\tau}^1)$ is a 2×2 Hermitian matrix depending on τ_1 and whose diagonal elements given by

$$\begin{aligned} [\Lambda^{(1)}(\vec{\tau}^1)]_{1,1} &= [\Lambda^{(1)}(\vec{\tau}^1)]_{2,2} \\ &= \text{Im}(\Psi_A(\tau_1) \Psi_A(\emptyset) \Psi_B^*(\emptyset) \Psi_B^*(\tau_1)). \end{aligned} \quad (\text{G17b})$$

We recall that Eq. (G17a) needs to have two orthogonal solutions corresponding to $\sigma_1 = a$ and b . This is only possible if $\text{Tr}[\Lambda^{(1)}(\vec{\tau}^1)] = 0$ [63], but from Eq. (G17b), this would also imply that $[\Lambda^{(1)}(\vec{\tau}^1)]_{1,1} = [\Lambda^{(1)}(\vec{\tau}^1)]_{2,2} = 0$, i.e., $\Lambda^{(1)}(\vec{\tau}^1)$ is a matrix with 0 diagonal elements. Then, it follows that the possible solutions to Eq. (G17a) are (up to a global phase that can be dropped)

$$\begin{bmatrix} p_{\tau_1}^a \\ q_{\tau_1}^a \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} p_{\tau_1}^b \\ q_{\tau_1}^b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (\text{G18})$$

which would imply that photodetection, without any linear optics, remains optimal until the first photodetection event. We point out that $[p_{\tau_1}^a; q_{\tau_1}^a] = [0; 1]$ and $[p_{\tau_1}^b; q_{\tau_1}^b] = [1; 0]$ are also possible solutions to Eq. (G17a), but this simply corresponds to swapping the two photodetectors before the first photodetection [see Eq. (G11)] and is thus equivalent to the solution in Eq. (G18). Finally, it is useful to note that it follows from Eq. (G17b) and $[\Lambda^{(1)}(\vec{\tau}^1)]_{1,1} = [\Lambda^{(1)}(\vec{\tau}^1)]_{2,2} = 0$ that

$$\Theta_A(\tau_1) = \Theta_B(\tau_1), \quad (\text{G19})$$

which we will use subsequently.

Proceeding similarly, we next consider the case of two photodetection events and apply Eq. (G15) for $\vec{\tau}^2 = \{\tau_1, \tau_2\}$ and $\vec{\sigma}^2 = \{\sigma_1, \sigma_2\}$. We obtain

$$\begin{bmatrix} p_{\tau_1, \tau_2}^{\sigma_1, \sigma_2} & q_{\tau_1, \tau_2}^{\sigma_1, \sigma_2} \end{bmatrix} \Lambda_{\sigma_1}^{(2)}(\vec{\tau}^2) \begin{bmatrix} p_{\tau_1, \tau_2}^{\sigma_1, \sigma_2*} \\ q_{\tau_1, \tau_2}^{\sigma_1, \sigma_2*} \end{bmatrix} = 0, \quad (\text{G20})$$

where $\Lambda_{\sigma_1}^{(2)}(\vec{\tau})$ is again a 2 by 2 Hermitian matrix which depends on τ_1, τ_2 (the times at which the photon is detected) and σ_1 (the port at which the first photon is detected). For $\sigma_1 = 0$,

$$\begin{aligned} [\Lambda_a^{(2)}(\vec{\tau}^2)]_{1,1} &= \text{Im}((\Psi_A(\tau_1)\Psi_B(\tau_2) + \Psi_A(\tau_2)\Psi_B(\tau_1))\Psi_A^*(\tau_1, \tau_2)\Psi_B^*(\emptyset)), \\ [\Lambda_a^{(2)}(\vec{\tau}^2)]_{2,2} &= \text{Im}((\Psi_A(\emptyset)\Psi_B(\tau_1, \tau_2) - \Psi_A(\tau_1, \tau_2)\Psi_B(\emptyset))\Psi_A^*(\tau_1)\Psi_B^*(\tau_2)), \end{aligned} \quad (\text{G21a})$$

and for $\sigma_1 = 1$,

$$\begin{aligned} [\Lambda_b^{(2)}(\vec{\tau}^2)]_{1,1} &= \text{Im}((\Psi_A(\emptyset)\Psi_B(\tau_1, \tau_2) - \Psi_A(\tau_1, \tau_2)\Psi_B(\emptyset))\Psi_A^*(\tau_2)\Psi_B^*(\tau_1)), \\ [\Lambda_b^{(2)}(\vec{\tau}^2)]_{2,2} &= -\text{Im}((\Psi_A(\tau_1)\Psi_B(\tau_2) + \Psi_A(\tau_2)\Psi_B(\tau_1))\Psi_A^*(\emptyset)\Psi_B^*(\tau_1, \tau_2)). \end{aligned} \quad (\text{G21b})$$

Again, since Eq. (G20) needs to have two orthogonal solutions (corresponding to $\sigma_2 = 0$ and 1), we obtain that $\text{Tr}(\Lambda_0^{(2)}(\vec{\tau})) = \text{Tr}(\Lambda_1^{(2)}(\vec{\tau})) = 0$. Using Eq. (G19), this condition can be rewritten as

$$\mathbf{M}^{(2)}(\tau_1, \tau_2) \begin{bmatrix} \text{Im}(\Psi_A^*(\tau_1, \tau_2)\Psi_B^*(\emptyset)e^{i(\chi(\tau_1)+\chi(\tau_2))}) \\ \text{Im}(\Psi_B^*(\tau_1, \tau_2)\Psi_A^*(\emptyset)e^{i(\chi(\tau_1)+\chi(\tau_2))}) \end{bmatrix} = 0, \quad (\text{G22})$$

where $\chi(t) = \Theta_A(t) = \Theta_B(t)$ and

$$\mathbf{M}^{(2)}(\tau_1, \tau_2) = \begin{pmatrix} 2|\Psi_A(\tau_1)||\Psi_B(\tau_2)| + |\Psi_A(\tau_2)||\Psi_B(\tau_1)| & -|\Psi_A(\tau_1)||\Psi_B(\tau_2)| \\ -|\Psi_A(\tau_2)||\Psi_B(\tau_1)| & 2|\Psi_A(\tau_2)||\Psi_B(\tau_1)| + |\Psi_A(\tau_1)||\Psi_B(\tau_2)| \end{pmatrix}. \quad (\text{G23})$$

Since $\Psi_A(\tau), \Psi_B(\tau) \neq 0$ by assumption, by computing the determinant of $\mathbf{M}^{(2)}(\tau_1, \tau_2)$ it can be verified that it is invertible. Therefore, it follows that $\text{Im}(\Psi_A^*(\tau_1, \tau_2)\Psi_B^*(\emptyset)e^{i(\chi(\tau_1)+\chi(\tau_2))}) = 0$ and $\text{Im}(\Psi_B^*(\tau_1, \tau_2)\Psi_A^*(\emptyset)e^{i(\chi(\tau_1)+\chi(\tau_2))}) = 0$. Since we have fixed $\Psi_A(\emptyset), \Psi_B(\emptyset) > 0$, this is equivalent to

$$\Theta_A(\tau_1, \tau_2) = \Theta_B(\tau_1, \tau_2) = \chi(\tau_1) + \chi(\tau_2). \quad (\text{G24})$$

Returning to Eqs. (G20) and (G21), we see that Eq. (G24) implies that $\Lambda^{(2)}(\vec{\tau})$ has 0 diagonal elements, and consequently similar to the case of a single photodetection even, photodetection without any linear optics remains an optimal measurement.

This argument can be continued for any number of photodetection events via induction: We assume that, for $\vec{\tau}^n = \{\tau_1, \tau_2 \dots \tau_n\}$ (which corresponds to n detection events),

$$\Theta_A(\vec{\tau}^n) = \Theta_B(\vec{\tau}^n) = \sum_{j=1}^n \chi(\tau_j), \quad (\text{G25})$$

and apply the optimality condition [Eq. (G15)] for $\vec{\sigma}^{n+1} = \{\sigma_1, \sigma_2 \dots \sigma_{n+1}\}$, and $\vec{\tau}^{n+1} = \{\tau_1, \tau_2 \dots \tau_{n+1}\}$. Doing so, we obtain that

$$[p_{\vec{\tau}^{n+1}}^{\vec{\sigma}^{n+1}} \ q_{\vec{\tau}^{n+1}}^{\vec{\sigma}^{n+1}}] \Lambda_{\vec{\sigma}^n}^{(n+1)}(\vec{\tau}^{n+1}) \begin{bmatrix} p_{\vec{\tau}^{n+1}}^{\vec{\sigma}^{n+1}*} \\ q_{\vec{\tau}^{n+1}}^{\vec{\sigma}^{n+1}*} \end{bmatrix} = 0, \quad (\text{G26})$$

where $\Lambda_{\vec{\sigma}^n}^{(n+1)}(\vec{\tau}^{n+1})$ is a 2 by 2 matrix. Since Eq. (G26) needs to have two orthogonal solutions (corresponding to $\sigma_{n+1} = a$ and $\sigma_{n+1} = b$), it follows that $\text{Tr}(\Lambda_{\vec{\sigma}^n}^{(n+1)}(\vec{\tau}^{n+1})) = 0$. Next, given $l \in \{1, 2 \dots n\}$ we choose $\vec{\sigma}^n = \{\sigma_1, \sigma_2 \dots \sigma_n\}$ where $\sigma_k = a$ for $k \neq l$ and $\sigma_l = b$, i.e., $\vec{\sigma}^n$ corresponds to a photodetection record where all but one of the first n photons are detected at port a , with a single photon detected at port b at time τ_l . For this choice of $\vec{\sigma}^n$, the condition for $\text{Tr}(\Lambda_{\vec{\sigma}^n}^{(n+1)}(\vec{\tau}))$ can be written as

$$\Im(\Psi_A(\vec{\tau}^{n+1})\Psi_B(\emptyset)\Psi_A^*(\vec{\tau}^{n+1} \setminus \tau_l)\Psi_B^*(\{\tau_l\})) = 0. \quad (\text{G27})$$

It thus follows that

$$\Theta_A(\vec{\tau}^{n+1}) = \Theta_A(\vec{\tau}^{n+1} \setminus \tau_l) + \Theta_B(\{\tau_l\}) = \sum_{j=1}^{n+1} \chi(\tau_j), \quad (\text{G28})$$

where, in the last step, we have used the induction hypothesis [Eq. (G25)]. Performing a similar analysis while choosing $\vec{\sigma}^n$ via $\sigma_k = b$ if $k \neq l$ and $\sigma_l = a$, we obtain a similar result for $\Theta_B(\vec{\tau}^{n+1})$. By induction, we thus conclude that Eq. (G25) is satisfied for all n .

Finally, we show that Eq. (G25) implies that photodetection alone, without any linear optical elements, is optimal. We recall that photo-detection corresponds to a projective measurement on the states $|E_{\tau_a, \tau_b}\rangle$ where

$$|E_{\vec{\tau}_a, \vec{\tau}_b}\rangle = \prod_{\tau \in \vec{\tau}_a} a_{\tau}^{\dagger} \prod_{\tau' \in \vec{\tau}_b} b_{\tau'}^{\dagger} |0\rangle, \quad (\text{G29})$$

where $\vec{\tau}_x$ denotes the set of detection times recorded at port $x \in \{a, b\}$. Then, checking the optimality condition in Eq. (G15) is equivalent to checking if

$$\Im \left(\left[\sum_{\tau \in \vec{\tau}_a} \Psi_A(\vec{\tau}_a \setminus \tau) \Psi_B(\vec{\tau}_b \cup \tau) - \sum_{\tau \in \vec{\tau}_b} \Psi_A(\vec{\tau}_a \cup \tau) \Psi_B(\vec{\tau}_b \setminus \tau) \right] \Psi_A^*(\vec{\tau}_a) \Psi_B^*(\vec{\tau}_b) \right) = 0. \quad (\text{G30})$$

It can now easily be verified that this equation holds if Eq. (G25) is true, thus showing that photodetection alone is an optimal measurement.

b. Counterexample: Entangled state

Here, we construct an example where linear optics and photodetection is optimal but photodetection alone is not. We begin by defining

$$\Psi(\vec{\tau}_a; \vec{\tau}_b) = \langle \text{vac} | \prod_{\tau \in \vec{\tau}_a} a_{\tau} \prod_{\tau \in \vec{\tau}_b} b_{\tau} | \psi \rangle. \quad (\text{G31})$$

We will consider a photonic state where a single source emits into both the input ports of the MZI. Consequently, the photons in the two different ports can be entangled and $\Psi(\vec{\tau}_a; \vec{\tau}_b)$ does not necessarily factorize into a product of the form $\Psi_A(\vec{\tau}_a) \Psi_B(\vec{\tau}_b)$.

As an explicit example, consider a two-photon wavepacket given by

$$\Psi(\tau_1, \tau_2; 0) = f(\tau_1, \tau_2), \quad \Psi(\tau_1; \tau_2) = -f(\tau_1, \tau_2), \quad \Psi(0; \tau_1, \tau_2) = (1+i)f(\tau_1, \tau_2), \quad \Psi(\tau_2; \tau_1) = (1+i)f(\tau_1, \tau_2), \quad (\text{G32})$$

where $f(\tau_1, \tau_2)$ is a symmetric function chosen such that the wave function is normalized. First we note that for this specific example, $\langle \psi | H_d | \psi \rangle$ is equal to zero, and the condition in Eq. (G17) is satisfied since it's a two photon wavepacket. The diagonal entries of the matrix $\Lambda_{\sigma_1}^{(2)}(\vec{\tau})$ in Eq. (G20) (corresponding to the optimality of two photodetection events) for $\sigma_1 = a$ and b are

$$\begin{aligned} (\Lambda_a^{(2)}(\vec{\tau}))_{1,1} &= \text{Im}((\Psi(\tau_1; \tau_2) + \Psi(\tau_2; \tau_1)) \Psi^*(\tau_1, \tau_2; 0)) = |f(\tau_1, \tau_2)|^2, \\ (\Lambda_a^{(2)}(\vec{\tau}))_{2,2} &= \text{Im}((\Psi(0; \tau_1, \tau_2) - \Psi(\tau_1, \tau_2; 0)) \Psi^*(\tau_1; \tau_2)) = -|f(\tau_1, \tau_2)|^2, \end{aligned} \quad (\text{G33a})$$

and,

$$\begin{aligned} (\Lambda_b^{(2)}(\vec{\tau}))_{1,1} &= \text{Im}((\Psi(0; \tau_1, \tau_2) - \Psi(\tau_1, \tau_2; 0)) \Psi^*(\tau_2; \tau_1)) = |f(\tau_1, \tau_2)|^2, \\ (\Lambda_b^{(2)}(\vec{\tau}))_{2,2} &= -\text{Im}((\Psi(\tau_1; \tau_2) + \Psi(\tau_2; \tau_1)) \Psi^*(0; \tau_1, \tau_2)) = -|f(\tau_1, \tau_2)|^2. \end{aligned} \quad (\text{G33b})$$

Clearly, we have $\text{Tr}(\Lambda_{\sigma_1}^{(2)}(\vec{\tau})) = 0$, indicating that a measurement using linear optics and photodetection is optimal. However, the diagonal elements of $\text{Tr}(\Lambda_{\sigma_1}^{(2)}(\vec{\tau}))$ are non-zero. This implies that the two orthonormal basis solutions of Eq. (G20) do not correspond to photodetection. Therefore, photodetection is sub-optimal as a measurement but becomes optimal when supplemented with linear optical elements.

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- [1] V. Giovannetti, S. Lloyd, and L. Maccone, *Advances in quantum metrology*, *Nat. Photonics* **5**, 222 (2011).
 - [2] G. Tóth and I. Apellaniz, *Quantum metrology from a quantum information science perspective*, *J. Phys. A* **47**, 424006 (2014).

-
- [3] J. P. Dowling and K. P. Seshadreesan, *Quantum optical technologies for metrology, sensing, and imaging*, *J. Lightwave Technol.* **33**, 2359 (2015).
 - [4] R. Demkowicz-Dobrzański, M. Jarzyna, and J. Kołodyński, *Quantum limits in optical interferometry*, *Prog. Opt.* **60**, 345 (2015).
 - [5] D. M. Greenberger, M. A. Horne, and A. Zeilinger, *Going beyond Bell's theorem*, in *Bell's Theorem, Quantum Theory and Conceptions of the Universe* (Springer, New York, 1989), pp. 69–72.
 - [6] A. M. Fox, *Quantum Optics: An Introduction* (Oxford university press, New York, 2006), Vol. 15.
 - [7] B. C. Sanders and G. J. Milburn, *Optimal quantum measurements for phase estimation*, *Phys. Rev. Lett.* **75**, 2944 (1995).
 - [8] J. J. Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, *Optimal frequency measurements with maximally correlated states*, *Phys. Rev. A* **54**, R4649 (1996).

- [9] M. J. Holland and K. Burnett, *Interferometric detection of optical phase shifts at the Heisenberg limit*, *Phys. Rev. Lett.* **71**, 1355 (1993).
- [10] J. P. Dowling, *Quantum optical metrology—the lowdown on high-NOON states*, *Contemp. Phys.* **49**, 125 (2008).
- [11] H. Lee, P. Kok, and J. P. Dowling, *A quantum Rosetta stone for interferometry*, *J. Mod. Opt.* **49**, 2325 (2002).
- [12] C. C. Gerry and R. A. Campos, *Generation of maximally entangled photonic states with a quantum-optical Fredkin gate*, *Phys. Rev. A* **64**, 063814 (2001).
- [13] K. T. McCusker and P. G. Kwiat, *Efficient optical quantum state engineering*, *Phys. Rev. Lett.* **103**, 163602 (2009).
- [14] K. T. Kapale and J. P. Dowling, *Bootstrapping approach for generating maximally path-entangled photon states*, *Phys. Rev. Lett.* **99**, 053602 (2007).
- [15] H. Cable and J. P. Dowling, *Efficient generation of large number-path entanglement using only linear optics and feed-forward*, *Phys. Rev. Lett.* **99**, 163604 (2007).
- [16] G. J. Pryde and A. G. White, *Creation of maximally entangled photon-number states using optical fiber multi-ports*, *Phys. Rev. A* **68**, 052315 (2003).
- [17] J. I. Cirac, R. Blatt, A. S. Parkins, and P. Zoller, *Preparation of Fock states by observation of quantum jumps in an ion trap*, *Phys. Rev. Lett.* **70**, 762 (1993).
- [18] A. Saxena, E. Abbasgholinejad, A. Majumdar, and R. Trivedi, *Boundary measurement tomography of the Bose Hubbard model on general graphs*, *Phys. Rev. Res.* **6**, 033058 (2024).
- [19] R. Huang, A. Miranowicz, J.-Q. Liao, F. Nori, and H. Jing, *Nonreciprocal photon blockade*, *Phys. Rev. Lett.* **121**, 153601 (2018).
- [20] X. Deng, S. Li, Z.-J. Chen, Z. Ni, Y. Cai, J. Mai, L. Zhang, P. Zheng, H. Yu, C.-L. Zou, S. Liu, F. Yan, Y. Xu, and D. Yu, *Quantum-enhanced metrology with large Fock states*, *Nat. Phys.* **20**, 1874 (2024).
- [21] L. S. Bishop, L. Tornberg, D. Price, E. Ginossar, A. Nunnenkamp, A. A. Houck, J. M. Gambetta, J. Koch, G. Johansson, S. M. Girvin, and R. J. Schoelkopf, *Proposal for generating and detecting multi-qubit GHZ states in circuit QED*, *New J. Phys.* **11**, 073040 (2009).
- [22] Y.-A. Chen, X.-H. Bao, Z.-S. Yuan, S. Chen, B. Zhao, and J.-W. Pan, *Heralded generation of an atomic NOON state*, *Phys. Rev. Lett.* **104**, 043601 (2010).
- [23] K. Müller, A. Rundquist, K. A. Fischer, T. Sarmiento, K. G. Lagoudakis, Y. A. Kelaita, C. Sanchez Munoz, E. Del Valle, F. P. Laussy, and J. Vučković, *Coherent generation of nonclassical light on chip via detuned photon blockade*, *Phys. Rev. Lett.* **114**, 233601 (2015).
- [24] A. Omran, H. Levine, A. Keesling, G. Semeghini, T. T. Wang, S. Ebadi, H. Bernien, A. S. Zibrov, H. Pichler, S. Choi *et al.*, *Generation and manipulation of Schrödinger cat states in Rydberg atom arrays*, *Science* **365**, 570 (2019).
- [25] R. Trivedi, K. Fischer, S. Xu, S. Fan, and J. Vuckovic, *Few-photon scattering and emission from low-dimensional quantum systems*, *Phys. Rev. B* **98**, 144112 (2018).
- [26] J. Tang, W. Geng, and X. Xu, *Quantum interference induced photon blockade in a coupled single quantum dot-cavity system*, *Sci. Rep.* **5**, 9252 (2015).
- [27] H. Snijders, J. Frey, J. Norman, H. Flayac, V. Savona, A. Gossard, J. Bowers, M. Van Exter, D. Bouwmeester, and W. Löffler, *Observation of the unconventional photon blockade*, *Phys. Rev. Lett.* **121**, 043601 (2018).
- [28] M. W. Mitchell, J. S. Lundeen, and A. M. Steinberg, *Super-resolving phase measurements with a multiphoton entangled state*, *Nature (London)* **429**, 161 (2004).
- [29] M. Hofheinz, E. M. Weig, M. Ansmann, R. C. Bialczak, E. Lucero, M. Neeley, A. D. O’Connell, H. Wang, J. M. Martinis, and A. N. Cleland, *Generation of Fock states in a superconducting quantum circuit*, *Nature (London)* **454**, 310 (2008).
- [30] T. Nagata, R. Okamoto, J. L. O’Brien, K. Sasaki, and S. Takeuchi, *Beating the standard quantum limit with four-entangled photons*, *Science* **316**, 726 (2007).
- [31] S. Slussarenko, M. M. Weston, H. M. Chrzanowski, L. K. Shalm, V. B. Verma, S. W. Nam, and G. J. Pryde, *Unconditional violation of the shot-noise limit in photonic quantum metrology*, *Nat. Photonics* **11**, 700 (2017).
- [32] R. Schnabel, *Squeezed states of light and their applications in laser interferometers*, *Phys. Rep.* **684**, 1 (2017).
- [33] H. Vahlbruch, M. Mehmet, K. Danzmann, and R. Schnabel, *Detection of 15 dB squeezed states of light and their application for the absolute calibration of photoelectric quantum efficiency*, *Phys. Rev. Lett.* **117**, 110801 (2016).
- [34] A. H. Safavi-Naeini, S. Gröblacher, J. T. Hill, J. Chan, M. Aspelmeyer, and O. Painter, *Squeezed light from a silicon micromechanical resonator*, *Nature (London)* **500**, 185 (2013).
- [35] C. M. Caves, *Quantum-mechanical noise in an interferometer*, *Phys. Rev. D* **23**, 1693 (1981).
- [36] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge university press, Cambridge, England, 2009).
- [37] U. L. Andersen, T. Gehring, C. Marquardt, and G. Leuchs, *30 years of squeezed light generation*, *Phys. Scr.* **91**, 053001 (2016).
- [38] L. A. Orozco, M. G. Raizen, M. Xiao, R. J. Brecha, and H. J. Kimble, *Squeezed-state generation in optical bistability*, *J. Opt. Soc. Am. B* **4**, 1490 (1987).
- [39] L.-A. Wu, H. J. Kimble, J. L. Hall, and H. Wu, *Generation of squeezed states by parametric down conversion*, *Phys. Rev. Lett.* **57**, 2520 (1986).
- [40] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge university press, Cambridge, England, 1997).
- [41] R. H. Dicke, *Coherence in spontaneous radiation processes*, *Phys. Rev.* **93**, 99 (1954).
- [42] V. Paulisch, M. Pernau-Llobet, A. González-Tudela, and J. I. Cirac, *Quantum metrology with one-dimensional super-radiant photonic states*, *Phys. Rev. A* **99**, 043807 (2019).
- [43] S. L. Braunstein and C. M. Caves, *Statistical distance and the geometry of quantum states*, *Phys. Rev. Lett.* **72**, 3439 (1994).

- [44] C. W. Helstrom, *Quantum detection and estimation theory*, *J. Stat. Phys.* **1**, 231 (1969).
- [45] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (Edizioni della Normale, Pisa, 2011), Vol. 1.
- [46] D. Yang, S. F. Huelga, and M. B. Plenio, *Efficient information retrieval for sensing via continuous measurement*, *Phys. Rev. X* **13**, 031012 (2023).
- [47] S. Gammelmark and K. Mølmer, *Fisher information and the quantum Cramér-Rao sensitivity limit of continuous measurements*, *Phys. Rev. Lett.* **112**, 170401 (2014).
- [48] K. Macieszczak, M. Guță, I. Lesanovsky, and J. P. Garrahan, *Dynamical phase transitions as a resource for quantum enhanced metrology*, *Phys. Rev. A* **93**, 022103 (2016).
- [49] L. Pezzé and A. Smerzi, *Ultrasensitive two-mode interferometry with single-mode number squeezing*, *Phys. Rev. Lett.* **110**, 163604 (2013).
- [50] R. A. Campos, C. C. Gerry, and A. Benmoussa, *Optical interferometry at the Heisenberg limit with twin Fock states and parity measurements*, *Phys. Rev. A* **68**, 023810 (2003).
- [51] H. F. Hofmann, *All path-symmetric pure states achieve their maximal phase sensitivity in conventional two-path interferometry*, *Phys. Rev. A* **79**, 033822 (2009).
- [52] T. Aoki, B. Dayan, E. Wilcut, W. P. Bowen, A. S. Parkins, T. J. Kippenberg, K. J. Vahala, and H. J. Kimble, *Observation of strong coupling between one atom and a monolithic microresonator*, *Nature (London)* **443**, 671 (2006).
- [53] A. V. Akimov, A. Mukherjee, C. L. Yu, D. E. Chang, A. S. Zibrov, P. R. Hemmer, H. Park, and M. D. Lukin, *Generation of single optical plasmons in metallic nanowires coupled to quantum dots*, *Nature (London)* **450**, 402 (2007).
- [54] T. Lund-Hansen, S. Stobbe, B. Julsgaard, H. Thyrestrup, T. Sünner, M. Kamp, A. Forchel, and P. Lodahl, *Experimental realization of highly efficient broadband coupling of single quantum dots to a photonic crystal waveguide*, *Phys. Rev. Lett.* **101**, 113903 (2008).
- [55] A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, R.-S. Huang, J. Majer, S. Kumar, S. M. Girvin, and R. J. Schoelkopf, *Strong coupling of a single photon to a superconducting qubit using circuit quantum electrodynamics*, *Nature (London)* **431**, 162 (2004).
- [56] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
- [57] J. Liu, H. Yuan, X.-M. Lu, and X. Wang, *Quantum Fisher information matrix and multiparameter estimation*, *J. Phys. A* **53**, 023001 (2020).
- [58] O. E. Barndorff-Nielsen and R. D. Gill, *Fisher information in quantum statistics*, *J. Phys. A* **33**, 4481 (2000).
- [59] M. Hayashi, *Comparison between the Cramer-Rao and the mini-max approaches in quantum channel estimation*, *Commun. Math. Phys.* **304**, 689 (2011).
- [60] Y. Yang, G. Chiribella, and M. Hayashi, *Attaining the ultimate precision limit in quantum state estimation*, *Commun. Math. Phys.* **368**, 223 (2019).
- [61] S. Xu and S. Fan, *Input-output formalism for few-photon transport: A systematic treatment beyond two photons*, *Phys. Rev. A* **91**, 043845 (2015).
- [62] R. Trivedi, K. Fischer, S. D. Mishra, and J. Vučković, *Point-coupling Hamiltonian for frequency-independent linear optical devices*, *Phys. Rev. A* **100**, 043827 (2019).
- [63] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge university press, New York, 2012).
- [64] P. Guilmin, P. Rouchon, and A. Tilloy, *Correlation functions for realistic continuous quantum measurement*, *IFAC-PapersOnLine* **56**, 5164 (2023).
- [65] C. Gardiner and P. Zoller, *Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics*, 3rd ed. (Springer, Berlin, Heidelberg, 2004).
- [66] H. Carmichael, *An Open Systems Approach to Quantum Optics: Lectures Presented at the Université Libre de Bruxelles, October 28 to November 4, 1991* (Springer, Berlin, Heidelberg, 2009), Vol. 18.
- [67] H. J. Carmichael, *Statistical Methods in Quantum Optics I: Master Equations and Fokker-Planck Equations* (Springer, Berlin, Heidelberg, 2013).
- [68] F. Bellardo, A. Chu, M. Koppenhöfer, and A. A. Clerk, *Extracting information from a superradiant burst using simple measurements*, *arXiv:2603.13130*.
- [69] A. Majumdar, P. Kaer, M. Bajcsy, E. D. Kim, K. G. Lagoudakis, A. Rundquist, and J. Vučković, *Proposed coupling of an electron spin in a semiconductor quantum dot to a nanosize optical cavity*, *Phys. Rev. Lett.* **111**, 027402 (2013).
- [70] E. I. Rosenthal, C. P. Anderson, H. C. Kleidermacher, A. J. Stein, H. Lee, J. Grzesik, G. Scuri, A. E. Rugar, D. Riedel, S. Aghaieibodi, G. H. Ahn, K. Van Gasse, and J. Vučković, *Microwave spin control of a tin-vacancy qubit in diamond*, *Phys. Rev. X* **13**, 031022 (2023).
- [71] A. E. Rugar, C. Dory, S. Sun, and J. Vučković, *Characterization of optical and spin properties of single tin-vacancy centers in diamond nanopillars*, *Phys. Rev. B* **99**, 205417 (2019).
- [72] R. Trivedi, M. Radulaski, K. A. Fischer, S. Fan, and J. Vučković, *Photon blockade in weakly driven cavity quantum electrodynamics systems with many emitters*, *Phys. Rev. Lett.* **122**, 243602 (2019).
- [73] D. M. Lukin, B. Windt, M. Bello, D. Catanzaro, M. A. Guidry, E. Lustig, S. Biswas, G. Scuri, T. K. Le, J. Yang *et al.*, *Mesoscopic cavity quantum electrodynamics with phase-disordered emitters in a Kerr nonlinear resonator*, *arXiv:2504.09324*.
- [74] D. M. Lukin, M. A. Guidry, J. Yang, M. Ghezellou, S. Deb Mishra, H. Abe, T. Ohshima, J. Ul-Hassan, and J. Vučković, *Two-emitter multimode cavity quantum electrodynamics in thin-film silicon carbide photonics*, *Phys. Rev. X* **13**, 011005 (2023).
- [75] R. E. Evans, M. K. Bhaskar, D. D. Sukachev, C. T. Nguyen, A. Sipahigil, M. J. Burek, B. Machielse, G. H. Zhang, A. S. Zibrov, E. Bielejec, H. Park, M. Lončar, and M. D. Lukin, *Photon-mediated interactions between quantum emitters in a diamond nanocavity*, *Science* **362**, 662 (2018).

- [76] M. Tavis and F. W. Cummings, *Exact solution for an n -molecule—radiation-field Hamiltonian*, *Phys. Rev.* **170**, 379 (1968).
- [77] B. Lemberger and K. Mølmer, *Radiation eigenmodes of Dicke superradiance*, *Phys. Rev. A* **103**, 033713 (2021).
- [78] K. Wan and R. Lasenby, *Bounds on adaptive quantum metrology under Markovian noise*, *Phys. Rev. Res.* **4**, 033092 (2022).
- [79] A. Das, W. Górecki, and R. Demkowicz-Dobrzański, *Universal time scalings of sensitivity in Markovian quantum metrology*, *Phys. Rev. A* **111**, L020403 (2025).
- [80] S. Zhou, M. Zhang, J. Preskill, and L. Jiang, *Achieving the Heisenberg limit in quantum metrology using quantum error correction*, *Nat. Commun.* **9**, 78 (2018).
- [81] R. Demkowicz-Dobrzański, J. Czajkowski, and P. Sekatski, *Adaptive quantum metrology under general Markovian noise*, *Phys. Rev. X* **7**, 041009 (2017).
- [82] The diamond norm is defined as
- $$\|\Delta\|_{\diamond} := \sup_A \sup_{\|X\|_1=1} \|(\Delta \otimes \mathcal{I}_A)(X)\|_1, \quad (32)$$
- where $\|X\|_1 := \text{Tr}\sqrt{X^\dagger X}$ is the trace norm and $\mathcal{I}_A$ is the identity on an auxiliary system A (it suffices to take $\dim A$ equal to the input dimension) [106].
- [83] H. Pichler, S. Choi, P. Zoller, and M. D. Lukin, *Universal photonic quantum computation via time-delayed feedback*, *Proceed. Natl. Acad. Sci.* **114**, 11362 (2017).
- [84] Z.-Y. Wei, D. Malz, A. González-Tudela, and J. I. Cirac, *Generation of photonic matrix product states with Rydberg atomic arrays*, *Phys. Rev. Res.* **3**, 023021 (2021).
- [85] Z.-Y. Wei, J. I. Cirac, and D. Malz, *Generation of photonic tensor network states with circuit QED*, *Phys. Rev. A* **105**, 022611 (2022).
- [86] O. Rubies-Bigorda, S. J. Masson, S. F. Yelin, and A. Asenjo-Garcia, *Deterministic generation of photonic entangled states using decoherence-free subspaces*, *Phys. Rev. Lett.* **134**, 213603 (2025).
- [87] M. B. Giles and N. A. Pierce, *An introduction to the adjoint approach to design*, *Flow Turbul. Combust.* **65**, 393 (2000).
- [88] D. Givoli, *A tutorial on the adjoint method for inverse problems*, *Comput. Methods Appl. Mech. Eng.* **380**, 113810 (2021).
- [89] A. D. White, R. Trivedi, K. Narayanan, and J. Vučković, *Enhancing superradiance in spectrally inhomogeneous cavity QED systems with dynamic modulation*, *ACS Photonics* **9**, 2467 (2022).
- [90] S. Zhou, C.-L. Zou, and L. Jiang, *Saturating the quantum Cramér–Rao bound using LOCC*, *Quantum Sci. Technol.* **5**, 025005 (2020).
- [91] M. Tsang, *Quantum reversal: A general theory of coherent quantum absorbers*, *Quantum* **9**, 1650 (2025).
- [92] F. Girotti, A. Godley, and M. Guta, *Estimating quantum Markov chains using coherent absorber post-processing and pattern counting estimator*, *Quantum* **9**, 1835 (2025).
- [93] M. Yuan, A. Seif, A. Lingenfelter, D. I. Schuster, A. A. Clerk, and L. Jiang, *Universal control in bosonic systems with weak Kerr nonlinearities*, *Phys. Rev. A* **111**, 032606 (2025).
- [94] K. A. Fischer, R. Trivedi, and D. Lukin, *Particle emission from open quantum systems*, *Phys. Rev. A* **98**, 023853 (2018).
- [95] C. Schön, E. Solano, F. Verstraete, J. I. Cirac, and M. M. Wolf, *Sequential generation of entangled multiqubit states*, *Phys. Rev. Lett.* **95**, 110503 (2005).
- [96] C. Schön, K. Hammerer, M. M. Wolf, J. I. Cirac, and E. Solano, *Sequential generation of matrix-product states in cavity QED*, *Phys. Rev. A* **75**, 032311 (2007).
- [97] J. I. Cirac, D. Perez-Garcia, N. Schuch, and F. Verstraete, *Matrix product states and projected entangled pair states: Concepts, symmetries, theorems*, *Rev. Mod. Phys.* **93**, 045003 (2021).
- [98] J. I. Cirac, P. Zoller, H. J. Kimble, and H. Mabuchi, *Quantum state transfer and entanglement distribution among distant nodes in a quantum network*, *Phys. Rev. Lett.* **78**, 3221 (1997).
- [99] N. Khaneja, T. Reiss, C. Kehlet, T. Schulte-Herbrüggen, and S. J. Glaser, *Optimal control of coupled spin dynamics: Design of NMR pulse sequences by gradient ascent algorithms*, *J. Magn. Reson.* **172**, 296 (2005).
- [100] A. Khan, F. Albarelli, and A. Datta, *Tensor network approach to sensing quantum light-matter interactions*, *PRX Quantum* **6**, 040343 (2025).
- [101] D. Yang, M. Ketkar, K. Audenaert, S. F. Huelga, and M. B. Plenio, *Quantum Cramér–Rao precision limit of noisy continuous sensing*, *Phys. Rev. Lett.* **136**, 070802 (2026).
- [102] A. L. Grimsmo, *Time-delayed quantum feedback control*, *Phys. Rev. Lett.* **115**, 060402 (2015).
- [103] G. Calajó, Y.-L. L. Fang, H. U. Baranger, and F. Ciccarello, *Exciting a bound state in the continuum through multiphoton scattering plus delayed quantum feedback*, *Phys. Rev. Lett.* **122**, 073601 (2019).
- [104] R. Trivedi, D. Malz, S. Sun, S. Fan, and J. Vučković, *Optimal two-photon excitation of bound states in non-Markovian waveguide QED*, *Phys. Rev. A* **104**, 013705 (2021).
- [105] D. Tamascelli, A. Smirne, S. F. Huelga, and M. B. Plenio, *Nonperturbative treatment of non-Markovian dynamics of open quantum systems*, *Phys. Rev. Lett.* **120**, 030402 (2018).
- [106] J. Watrous, *The Theory of Quantum Information* (Cambridge university press, Cambridge, England, 2018).
- [107] M. J. Collett and C. W. Gardiner, *Squeezing of intracavity and traveling-wave light fields produced in parametric amplification*, *Phys. Rev. A* **30**, 1386 (1984).
- [108] M. M. Wolf, *Quantum channels and operations-guided tour* (2012).
- [109] M. B. Ruskai, *Beyond strong subadditivity? Improved bounds on the contraction of generalized relative entropy*, *Rev. Math. Phys.* **6**, 1147 (1994).
- [110] F. Albarelli, E. Bisketzi, A. Khan, and A. Datta, *Fundamental limits of pulsed quantum light spectroscopy:*

- Dipole moment estimation*, *Phys. Rev. A* **107**, 062601 (2023).
- [111] F. Girotti, A. Godley, and M. Guță, *Optimal estimation of pure states with displaced-null measurements*, *J. Phys. A* **57**, 245304 (2024).
- [112] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac, *Matrix product state representations*, *Quantum Inf. Comput.* **7**, 401 (2007).
- [113] D. Dong and I. R. Petersen, *Quantum control theory and applications: A survey*, *IET Control Theory Appl.* **4**, 2651 (2010).
- [114] D. d'Alessandro, *Introduction to Quantum Control and Dynamics* (Chapman and Hall/CRC, London, 2021).
- [115] S. G. Schirmer, H. Fu, and A. I. Solomon, *Complete controllability of quantum systems*, *Phys. Rev. A* **63**, 063410 (2001).