

Spectral Properties and Coding Transitions of Haar-Random Quantum Codes

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A quantum error-correcting code with a nonzero error threshold undergoes a mixed-state phase transition when the error rate reaches that threshold. We explore this phase transition for Haar-random quantum codes, in which the logical information is encoded in a random subspace of the physical Hilbert space. We focus on the spectrum of the encoded system density matrix as a function of the rate of uncorrelated, single-qudit errors. For low error rates, this spectrum consists of well-separated bands, representing errors of different weights. As the error rate increases, the bands for high-weight errors merge. The evolution of these bands with increasing error rate is well described by a simple analytic ansatz. Using this ansatz, as well as an explicit calculation, we show that the threshold for Haar-random quantum codes saturates the hashing bound, and thus coincides with that for random *stabilizer* codes. For error rates that exceed the hashing bound, typical errors are uncorrectable, but postselected error correction remains possible until a much higher *detection* threshold. Postselection can in principle be implemented by projecting onto subspaces corresponding to low-weight errors, which remain correctable past the hashing bound.

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I. INTRODUCTION

The threshold theorem for quantum error-correcting codes is a landmark result in quantum information theory [1–5]. In recent work, the error correction threshold has drawn interest as an example of a *mixed-state phase transition* [6–8]. Encoding a logical state in a finite-threshold error-correcting code, and subjecting the encoded state to local decoherence of strength p , defines a family of mixed states $\rho^{(p)}$. When p is below an error threshold p_c , the mixed state $\rho^{(p)}$ retains logical information; past the threshold, logical information cannot be reliably recovered even with an optimal decoder [9,10]. The distinction between these regimes is an intrinsic *information-theoretic* property of the mixed states. In particular, conventional local correlation functions are insensitive to this transition: all the mixed states $\rho^{(p)}$ are derived from the error-free

state $\rho^{(p=0)}$ by one round of single-qudit channels, which can only modify local correlation functions analytically as p is varied through p_c , even in the large-system limit [11].

A large body of recent work [11–16] has focused on identifying changes in the entanglement structure of $\rho^{(p)}$ across the error correction threshold, primarily focusing on topological codes such as the toric code [17–19]. For example, it has been argued that $\rho^{(p)}$ can be written as an ensemble of short-range entangled states only if $p > p_c$ [14,20], the conditional correlation length (Markov length) of $\rho^{(p)}$ is thought to diverge at p_c [21,22], and the von Neumann entropy of $\rho^{(p)}$ is nonanalytic at p_c [15,23]. Moreover, it was pointed out in Refs. [11,15,23] that there are further transitions in the entanglement structure that occur *past* the error correction threshold, associated with singularities in the Rényi entropies of $\rho^{(p)}$. The information-theoretic implications (if any) of these further transitions have not yet been explained.

In the present work we explore the nature of this mixed-state phase transition, not for highly structured models like the toric code, but for *random* codes. A random code embeds the logical Hilbert space into a subspace of the (typically much larger) physical Hilbert space via a Haar-random isometry. The physical Hilbert space is taken to

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have a tensor-product structure, consisting of N qudits (each of dimension q), and the noise model we consider is single-qudit depolarization acting on each physical qudit with a strength p (defined more precisely below). The only structure in the model comes from the tensor product structure and the single-qudit nature of the noise.

The unstructured nature of these codes enables a novel semianalytic treatment of the spectrum of the mixed state $\rho^{(p)}$. For very small p , this spectrum consists of “bands” of eigenvalues that are well-separated on a logarithmic scale. To our knowledge, this band structure has gone unnoted in the literature until this work. The w th band consists of states that (to a good approximation) are of weight w , meaning that they can be created from the encoded logical state by applying errors on w sites. Thus, for qudits there are $(q^2 - 1)^w \binom{N}{w}$ states in the w th band and their eigenvalues scale as $\sim p^w$ at small p . Each error maps the logical subspace to approximately orthogonal subspaces; small- w errors are typically correctable since these (relatively dilute) subspaces are not forced to overlap. Since for N qudits there are q^{2N} possible errors, and only $O(q^N)$ mutually orthogonal subspaces [for $O(1)$ encoded logical qudits], not all errors can be corrected: instead, errors with weight $w > w_c \approx 0.189N$ for qubits (the quantum hashing bound [24]) do not map the logical subspace to dilute approximately orthogonal subspaces, and thus cannot be corrected. The band with the smallest eigenvalues is associated with these uncorrectable errors.

The description above is parametrically accurate as $p \rightarrow 0$. As one increases p , the bands become less well separated, and interband mixing of error weights becomes important. We treat this band mixing perturbatively and find good agreement with numerics for a single logical qubit (Fig. 3). Crucially, for any subthreshold $p < p_c$, we find that the bands corresponding to correctable errors remain well-defined and nonoverlapping; our perturbation theory remains well-defined up to at least second order below the critical window $w_c - w \sim \sqrt{N}$.

As the weight of the typical errors [25] increases, the spectral weight in $\rho^{(p)}$ is dominated by bands of weight pN , as long as $p < p_c$. The number of states in such bands grows with p until at $p = p_c$ it reaches $\sim q^N$ (i.e., the physical Hilbert space dimension) and thus cannot continue to grow. Past this point, typical errors are uncorrectable, and the von Neumann entropy density of $\rho^{(p)}$ reaches that of the maximally mixed state.

The structure of well-defined bands gives a physical mechanism for why the error correction transition in random codes should saturate the hashing bound, consistent with our numerics, and also explains the finite-size scaling behavior at this transition. We confirm that the hashing bound is saturated by an explicit calculation using the Weingarten calculus (see Sec. IV of the Supplemental Material [26]). The hashing bound was originally derived

for random stabilizer codes [24,27]; we show here that Haar-random codes also saturate it.

This band structure also sheds light on the physics for $p > p_c$. The error correction threshold is a transition in the structure of the bands that dominate the probability in $\rho^{(p)}$. However, the low- w bands remain well isolated from each other across p_c and their character does not change across this transition. In particular, the density matrix projected onto the bands with weight below some $w^*(p)$ retains logical information even for $p > p_c$, so errors remain correctable after postselecting on these bands, up to a higher detection threshold p_d (although this postselection is exponentially unlikely to succeed). Moreover, for $p_c < p < p_d$ individual eigenvalues in these bands are parametrically larger than the typical eigenvalue of q^{-N} , so these bands dominate the Rényi entropies $S_\alpha \equiv (1 - \alpha)^{-1} \log_q \text{Tr}(\rho^\alpha)$ with $\alpha > \alpha^*(p) > 1$. As p increases, each low-weight band hits its own threshold, until at the detection threshold p_d , postselected error correction fails even for $w = 0$. We estimate p_d through an elementary combinatorial argument, and confirm our estimate using an exact duality called a quantum MacWilliams identity [28–32] (defined originally for classical error-correcting codes [33]). (Sections VII–IX of the Supplemental Material [26] contain an extensive discussion of quantum MacWilliams identities that may be of independent interest.)

The rest of this paper is structured as follows. In Sec. II we specify the codes and information-theoretic diagnostics that will be considered. In Sec. III we introduce a “fixed-weight” noise model that helps us isolate and analyze the physics of an individual spectral “band.” We then leverage these results to analyze the standard depolarizing noise model, which can be written as a convex sum over fixed-weight noise models, and compute its threshold, in Sec. IV. In Sec. V we explore the behavior of the spectrum for noise rates exceeding threshold, and compute thresholds for postselected error correction. Finally, in Sec. VI, we summarize our results and discuss open questions that they raise.

II. SETUP

We will discuss the encoding procedure using a standard technique for studying information-theoretic thresholds for quantum codes. Our objective is to encode k logical qudits in a physical system Q comprising N qudits. To this end, we take a subsystem L containing k qudits, and place each of the k qudits in a generalized Bell pair with a qudit of the “reference” system R , producing the state $|\Phi_{RL}\rangle$. The other $N - k$ qudits in Q (subsystem A) are initialized in an arbitrary pure state $|0_A\rangle$. To this initial state, we apply a random encoding unitary U on Q sampled from the Haar measure. The resulting encoded pure state on RQ is

$$|\Psi_{RQ}\rangle = (\mathbb{1}_R \otimes U_Q) (|\Phi_{RL}\rangle |0_A\rangle), \quad (1)$$

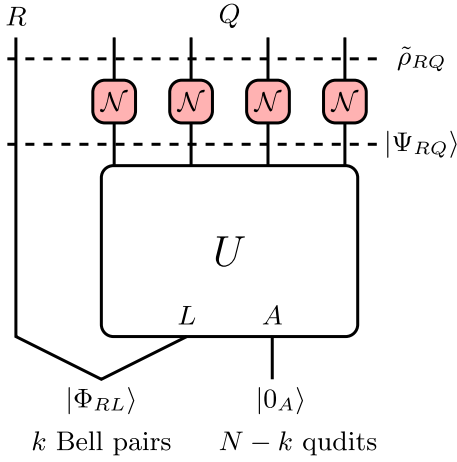


FIG. 1. Schematic circuit diagram of the Haar-random codes studied in this work. k logical qudits are isometrically encoded in an N -qudit system via a Haar-random unitary U . Each system qudit i is then subjected to a depolarizing channel $\mathcal{N}_{ip}$ of strength p [see Eq. (3)].

as shown in Fig. 1. Its reduced density matrix on Q , denoted ρ_Q , is the maximally mixed state on the code space, which is a random q^k -dimensional subspace of the physical Hilbert space.

The density matrix ρ_{RQ} is then sent through an error channel acting on Q to create a corrupted density matrix $\tilde{\rho}_{RQ}$; tracing out R gives $\tilde{\rho}_Q$. The optimal capacity of a (hypothetical) decoder to recover the logical information from Q is quantified by the coherent information [34]:

$$I_c(R)Q = S_{\text{vN}}(\tilde{\rho}_Q) - S_{\text{vN}}(\tilde{\rho}_{RQ}),$$

where $S_{\text{vN}}(\rho) = -\text{Tr}[\rho \log_q(\rho)]$, (2)

which ranges from k to $-k$. Maximal quantum coherent information between the physical system, Q , and the reference, R , indicates the information is safely recoverable [35,36]. As Q is decohered, Q and R become entangled with the (implicit) environment, reducing $I_c(R)Q$. Once R is maximally entangled with the (implicit) environment, $I_c(R)Q = -k$, and none of the originally encoded information is recoverable.

The noise model we are primarily interested in is depolarizing noise that acts independently on each qudit. The single-qudit channel can be represented in Kraus form as

$$\mathcal{N}_{ip}(\rho) = (1-p)\rho + \frac{p}{q^2-1} \sum_{\mu=1}^{q^2-1} E_i^\mu \rho E_i^{\mu\dagger}, \quad (3)$$

where $\{E_i^\mu\}$ are the $(q^2 - 1)$ nonidentity generalized Pauli operators on qudit i , and $E^0 = \mathbb{1}$. The global depolarizing channel can be written as $\mathcal{N}_p = \bigotimes_{i \in Q} \mathcal{N}_{ip}$. If one expands out the product, one arrives at a Kraus decomposition of

$\mathcal{N}_p$ in terms of global errors $E_\mu = \bigotimes_{i \in Q} E_i^{\mu_i}$. Here the vector μ lists each qudit on which the error acts nontrivially, and the specific operator μ_i acting on that qudit.

The number of qudits $w(\mu)$ on which E_μ acts nontrivially is called the weight of the error. The global depolarizing channel contains errors of all weights, but it will be helpful in what follows to decompose it in terms of channels $\mathcal{N}^{(w)}$ consisting of errors of fixed weight w . There are $\Omega(w) \equiv (q^2 - 1)^w \binom{N}{w}$ distinct errors of weight w , and each of these is equally likely, so we can write

$$\mathcal{N}^{(w)}(\rho) = \frac{1}{\Omega(w)} \sum_{\{\mu | w(\mu)=w\}} E_\mu \rho E_\mu^\dagger. \quad (4)$$

Defining $P_w \equiv p^w (1-p)^{N-w} \binom{N}{w}$ as the probability of a weight- w error, one can decompose the depolarizing channel as a convex sum of fixed-weight error channels as follows:

$$\mathcal{N}_p(\rho) = \sum_{w=0}^N P_w \mathcal{N}^{(w)}(\rho). \quad (5)$$

III. FIXED-WEIGHT ERROR ENSEMBLE

We first discuss the error correction threshold for the family of fixed-weight error channels $\mathcal{N}^{(w)}$, as a function of the error weight w . As we will see, this ensemble allows for a simpler interpretation of the error correction threshold than the standard local error channel. Moreover, since the error-weight distribution for the local error channel $\mathcal{N}_p$ is tightly concentrated about its mean (with width scaling as $\sqrt{N}$), one might expect $\mathcal{N}^{(pN)}$ to be a reasonable proxy for the product of local channels $\mathcal{N}_p$, at least for some physical quantities.

The channel $\mathcal{N}^{(w)}$ maps any initial pure state to a mixture of $\Omega(w)$ random pure states. Random vectors in a high-dimensional space are approximately orthogonal, so long as the number of random vectors is much smaller than the dimension of the space. On the other hand, when the number of random vectors is much larger than the dimension of the space, the resulting mixed state is close to maximally mixed.

As a first approximation, therefore, $\mathcal{N}^{(w)}$ maps ρ_Q to a mixed state consisting of $q^k \Omega(w)$ orthogonal vectors if $q^k \Omega(w) \leq q^N$, and to the maximally mixed state otherwise. Meanwhile, ρ_{RQ} is mapped to a state consisting of $\Omega(w)$ orthogonal vectors if $\Omega(w) \leq q^{N+k}$, and to the maximally mixed state otherwise. In this approximation, there are three regimes of behavior. First, when $q^k \Omega(w) \leq q^N$, $\tilde{\rho}_Q$ is proportional to a projector onto a $q^k \Omega(w)$ -dimensional subspace, and $\tilde{\rho}_{RQ}$ to a projector onto an $\Omega(w)$ -dimensional subspace. From the definition of the coherent information, it follows that $I_c(R)Q = k$. Second, when $\Omega(w) \geq q^{N+k}$, both $\tilde{\rho}_Q$ and $\tilde{\rho}_{RQ}$ are maximally

mixed, and one has $I_c(R|Q) = -k$ qudits. Finally, for $q^{N-k} \leq \Omega(w) \leq q^{N+k}$, $\tilde{\rho}_Q$ is maximally mixed, but $\tilde{\rho}_{RQ}$ has entropy $S_{vN}(\tilde{\rho}_{RQ}) = \log_q \Omega(w)$.

Expressing these results in terms of the Shannon entropy of the uniform error distribution of the fixed-weight error ensemble, we obtain the leading-order ansatz:

$$I_c(R|Q) = \begin{cases} k & H(p) \leq 1 - k/N, \\ -k & H(p) \geq 1 + k/N, \\ N(1 - H(p)) & \text{otherwise,} \end{cases} \quad (6)$$

where $p = w/N$ and we have defined the entropy of the error distribution

$$H(p) = -(1-p)\log_q(1-p) - p\log_q\left[\frac{p}{q^2-1}\right], \quad (7)$$

and used the relation

$$\Omega(pN) \sim q^{NH(p)}. \quad (8)$$

The condition $H(p) \leq 1 - k/N$ for perfect information recoverability is the familiar *hashing bound*, which traditionally arises in the context of nondegenerate stabilizer codes [37]. In the Supplemental Material, we provide an *exact* derivation of Eq. (6) as the leading- N behavior of the coherent information under the full channel $\mathcal{N}_p$, using the Weingarten calculus. For finite-rate codes ($k/N = r = O(1)$), the coherent information decreases over an extensive interval of weights, crossing zero at p^* such that $H(p^*) = 1$, while for zero-rate codes ($k = O(1)$), $p^* = p_c \equiv w_c/N \approx 0.189$ (for qubits) marks a transition directly from the “coding” to the “noncoding” phase.

Above, we oversimplified by treating the channel $\mathcal{N}^{(w)}$ as mapping each random vector onto a set of strictly orthogonal vectors. In fact, for the basis of orthogonal errors $\{E_\mu\}$ introduced above, and a basis of q^k orthogonal codewords $\{|\psi^{(n)}\rangle\}$ spanning the support of ρ_Q , the pairwise overlap $|\langle E_\mu \psi^{(n)} | E_{\mu'} \psi^{(n')} \rangle| \sim q^{-N/2}$ for $\mu \neq \mu'$ or $n \neq n'$. Mixed states consisting of ensembles of ϕ random vectors in a d -dimensional Hilbert space have been considered in the literature [38,39]; the eigenvalue density of these mixed states follows the (appropriately rescaled) Marchenko-Pastur distribution with parameter $c = d/\phi$. We note that the identification of Marchenko-Pastur statistics in the case of Haar-random codes remains unproven, and neglects certain matrix element correlations, which are discussed briefly in the Supplemental Material [26].

After rescaling to the mean nonzero eigenvalue $1/\min(d, \phi)$, the distributions for parameters (c, c^{-1}) with $c < 1$ coincide except for a large number of zero eigenvalues in the latter case (which will not concern us). In particular, in our setting, the ratio of the width to the mean is $\sim \min(\sqrt{c}, 1/\sqrt{c})$, where $c \approx q^N/(q^k \Omega(w))$. As

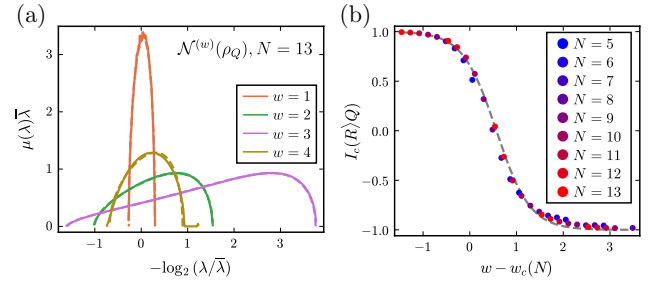


FIG. 2. Fixed-weight error ensemble, $k = 1$ qubit Haar-code. (a) Density of states of the density matrix $\mathcal{N}^{(w)}(\rho_Q)$, for $N = 13$, $w = 1, 2$ (below hashing), and $w = 3, 4$ (above hashing), rescaled to the mean nonzero eigenvalue, $\bar{\lambda} = 1/\min(2^N, 2\Omega(w))$ in each band. Solid curves are data aggregated over 467 independent states, while dashed curves are the Marchenko-Pastur distribution with $c = 2^{N-1}/\Omega(w)$ [26]. (b) Transition in the coherent information of the fixed-weight error ensemble, with $w_c(N) = p_c N$. Each data point is an average over at least 200 independent states, and error bars are smaller than the markers. The dashed curve, shown to guide the eye, is a fit to a sigmoid function using data points with $N \geq 8$.

$N \rightarrow \infty$, therefore, c remains nonzero and finite only if w is within $O(1)$ of the hashing bound. Outside this transition region, the eigenvalue distribution collapses to a delta function, and the effects of nonorthogonality can be neglected for $(w_c - w) \gg 1$.

Numerical evidence for these predictions from Haar-random qubit codes, with $k = 1$ and $N \leq 13$, is shown in Fig. 2: (a) the eigenvalue density follows the Marchenko-Pastur distribution with c specified above, and (b) the width of the transition region remains $O(1)$ in w even as $N \rightarrow \infty$.

IV. DEPOLARIZING ERROR CHANNEL

We now return to the depolarizing error channel at error rate p , $\mathcal{N}_p$, which is a convex sum of the fixed-weight channels considered in the previous section [Eq. (5)]. For ease of discussion, we will focus on zero-rate codes, which undergo a single transition at $p = p_c$, but the arguments can be straightforwardly generalized to finite-rate codes.

The eigenvalue structure of $\mathcal{N}_p(\rho_Q)$ is more complex than that of $\mathcal{N}^{(w)}$, but at sufficiently small p they are simply related. Each $\mathcal{N}^{(w)}$ contributing to $\mathcal{N}_p$ maps ρ_Q onto a mixed state consisting of $q^k \Omega(w)$ random pure states; each of these random states has weight $P_w/(q^k \Omega(w))$ in the ensemble. If we treat all of these states as pairwise orthogonal, we arrive at the following simple description of the eigenvalue spectrum of $\mathcal{N}_p(\rho_Q)$: it consists of many “bands,” each associated with errors of some weight w , and consisting of $q^k \Omega(w)$ roughly degenerate eigenvalues of magnitude $P_w/(q^k \Omega(w))$.

As $p \rightarrow 0$ this picture becomes asymptotically accurate for the bands with $w < w_c$: $P_w \sim p^w$, so bands corresponding to different w have widely separated eigenvalues and can be treated independently. Even in this regime, errors with larger w do contribute to the spectrum: they fill in the “leftover” dimensions of the state space that have not been used by the small- w eigenvalues corresponding to correctable errors, and at low p are well approximated as one more partially filled band with $w = w_c$. We refer to this last band as the “reservoir”; beyond the error threshold it strongly mixes with neighboring bands, degrading the coherent information.

To improve upon this approximation and to capture the behavior at larger p , we need to account for the fact that the states produced by different Pauli errors are not strictly orthogonal. We treat the effects of nonorthogonality perturbatively [26], and find good agreement with the numerically observed eigenvalue density, shown for $N = 13, k = 1$, and $q = 2$ in Fig. 3(a). The effects of this nonorthogonality can be classified into three types: (i) overlaps between states produced by different errors of the same weight, which we accounted for in the previous section using the Marchenko-Pastur distribution; (ii) interband overlaps that rigidly shift each band relative to the others, and which can be computed in first-order perturbation theory; and (iii) interband overlaps that reshape individual bands. We argue in Sec. II of the Supplemental Material that effects of type (iii) are asymptotically subleading in the $N \rightarrow \infty$ limit except for bands very close to the critical hashing weight. We have verified that this simple perturbative ansatz consisting of rigidly shifted bands with Marchenko-Pastur-distributed eigenvalues is in good agreement with our numerical results, and show that the Marchenko-Pastur distribution and rigid shifts capture the leading-order physics (i.e., up to parametrically small corrections) for $w_c - w \gtrsim \sqrt{N}$ up to second order in the Supplemental Material [26].

This picture of well-defined bands, each associated with errors of some weight w , implies that the threshold transition at p_c should have width $\sim 1/\sqrt{N}$ for zero-rate codes. This follows because the probability mass of the density matrix $\mathcal{N}_p(\rho_Q)$ is dominated by errors of weight $w \in (pN - O(\sqrt{N}), pN + O(\sqrt{N}))$. When $|p - p_c| < O(1/\sqrt{N})$, the dominant errors have an appreciable probability of being either correctable or uncorrectable. Therefore, error correction fails with an $O(1)$ probability throughout this regime, and the local error model has a parametrically wider transition region than the fixed-weight error model (though both regions shrink to zero as a function of p when $N \rightarrow \infty$). This prediction for the width of the transition region is consistent with numerics, as seen from the scaling collapse of the coherent information as a function of $(p - p_c)N^{1/\nu}$ with $\nu = 2$ in Fig. 3(b). This scaling also satisfies Chayes *et al.*’s general bound on finite-size scaling exponents in the presence of disorder

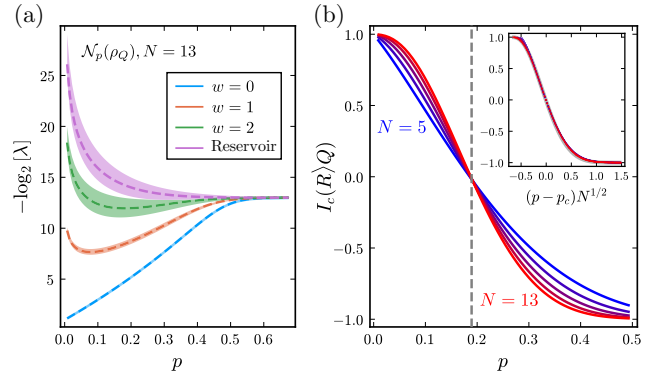


FIG. 3. Depolarizing error channel, $k = 1$ qubit Haar-code. (a) Eigenvalues of the density matrix $\mathcal{N}_p(\rho_Q)$, as a function of depolarization rate p . Filled curves are data aggregated from ≈ 100 samples at $N = 13, k = 1$. Dashed curves are the ansatz. (b) Coherent information for $k = 1$, odd N ranging from 5 to 13. Inset: scaling collapse with $p_c = p_{\text{hash}} \approx 0.1893$ for all N in the interval $[7, 13]$. Gray curve shows the ansatz for $N = 13$.

[40], which in our setting reads $\nu \geq 2$. Since the scaling exponent in the fixed-weight error channel is convincingly less than 2 (indeed, the assumption of Marchenko-Pastur statistics suggests $\nu = 1$ in that case), we expect the depolarizing channel to saturate this inequality: the transition is broadened solely by the variance of error weights.

V. POSTSELECTION PAST THRESHOLD

Above the hashing bound, error correction fails because there are too many typical errors to distinguish: the spectrum of ρ_Q is dominated by states of high w that cannot be associated with specific errors. As shown in Fig. 3(a), the bands with w near w_c also merge as one increases the error rate past p_c . Nevertheless, the bands associated with low-weight errors remain well-defined, and their states do not strongly mix with states produced by uncorrectable errors. This motivates us to consider two forms of postselection, by which quantum information can be recovered past the hashing bound.

A. Hard postselection

We first consider *hard postselection*: If we define a projector Π_w onto the space of states spanned by all codewords acted upon by errors of weight w , then the density matrix $\sigma_{p,w} \propto \Pi_w \tilde{\rho}_Q \Pi_w$ projected onto this space may contain only correctable errors [41], even for $p > p_c$. In this regime, postselecting on errors of weight w is exponentially unlikely to succeed, but if it does, then the quantum information remains protected. This projector Π_w is not possible to construct in a scalable fashion, but consideration of this postselection is conceptually informative.

Note also that when the bands are spectrally separated, this projector is approximately the projection onto one band.

An instructive limiting case is postselecting on the space spanned by errors of weight zero, i.e., on the logical subspace [42], a common tactic in both near-term experiments and proposed fault-tolerant protocols [43]. An initial logical state can map into this subspace by one of two mechanisms: (i) the state was not subject to any errors or (ii) an arbitrary error occurred, but mapped the initial state to one that was not strictly orthogonal to the logical subspace. In our model, case (i) occurs with probability $(1 - p)^N$, and case (ii) with probability $(1 - (1 - p)^N)q^{k-N}$ (where the factor q^{k-N} is the trace of a random state on the logical subspace, i.e., the probability that a random state will project onto a subspace consisting of q^k states). When $-\log_q(1 - p) \leq 1 - k/N$, conditional on postselection succeeding, it is overwhelmingly likely in the limit of large N that no error acted, and the quantum information remains protected. In this sense, the success or failure

of postselection provides sharp information on whether an error occurred, and errors are therefore “detectable.”

When this inequality fails, errors are not detectable: even if one successfully postselects on the “no-error” syndrome, it yields no information about which error acted. For $O(1)$ logical qubits encoded in N physical qubits, this detection threshold occurs at $p_d = 1/2$. The argument above is consistent with a more explicit calculation of the detection threshold using a high-low temperature duality, laid out in the Supplemental Material [26].

This argument for the detection threshold readily generalizes to error thresholds upon postselecting for errors of weight $w < w_c$. The probability that an error of precisely weight w acted is P_w , while the fraction of states that are in the band of weight w scales as $f_w \equiv q^{k-N} \Omega(w)$. These curves cross when

$$-\frac{w}{N} \log_q \frac{p}{q^2 - 1} - \left(1 - \frac{w}{N}\right) \log_q(1 - p) = 1 - k/N, \quad (9)$$

which can be solved for p to determine a postselected error threshold. This postselected threshold is plotted for zero-rate qubit codes in Fig. 4(a) (blue curve); it interpolates smoothly between the hashing weight w_c (for $p \leq p_c$) and the detection threshold at $p_d = 1/2$. Note that this postselected threshold is accompanied by a nonanalytic change in the probability that postselection succeeds: below (above) the postselected threshold this probability scales as $P_w(f_w)$.

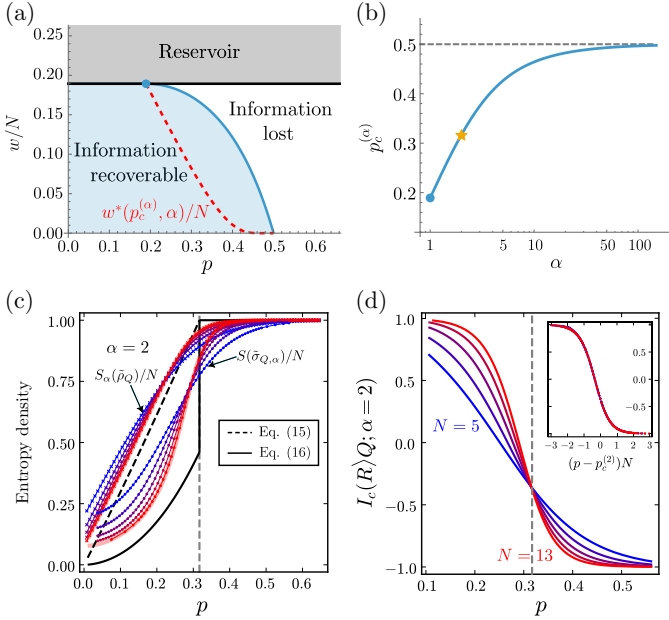


FIG. 4. (a) Theoretical phase diagram for postselection in a zero-rate qubit Haar-code. Information is recoverable at error rate p by postselecting on band w within the blue-shaded region. Dashed curve shows the threshold under the soft postselection defined in Eq. (11), corresponding to the Rényi threshold $p_c^{(\alpha)}$ plotted as a function of α in (b). The star in (b) marks $\alpha = 2, p_c^{(2)} = (3 - \sqrt{3})/4$. This threshold manifests as both (c) a non-analyticity in $S_2(\tilde{\rho}_Q)/N$ (x’s) and $S_{VN}(\tilde{\sigma}_{Q,2})/N$ (circles) and (d) a transition in the coherent information of $\tilde{\sigma}_{Q,2}$. The main panels show odd system sizes ranging from $N = 5$ to $N = 13$. In (c), black lines indicate the leading- N behavior for the two quantities [Eqs. (15) and (16)], while the barely visible red curves are the theoretical calculations for $N = 13$. The inset of (d) shows a scaling collapse of the postselected coherent information, with system sizes ranging from $N = 7$ to $N = 13$.

B. Soft postselection: Rényi reweights

The perspective of postselected error correction also sheds light on the evolution of Rényi entropies with p , as we now discuss. Recall that the α -Rényi entropy is defined as $S_\alpha(\tilde{\rho}_Q) \equiv (1 - \alpha)^{-1} \log_q [\text{Tr}(\tilde{\rho}_Q^\alpha)]$. Singularities in this quantity have been widely explored in the literature [11,15,23], owing to the tractability of integer $\alpha > 1$ in statistical mechanics models of decohered stabilizer codes, but a clear operational meaning for these singularities (apart from the von Neumann limit $\alpha \rightarrow 1$) has been elusive. In this subsection, we show how a “soft” postselection protocol tuned by α offers two information-theoretic interpretations of these singularities.

In place of the “hard” projectors onto individual w considered in the previous subsection, we introduce a family of positive operator-valued measures (POVMs) $\{F_1, F_2\}$ that reweight each eigenvalue of ρ_Q as follows:

$$\{F_1^{(\alpha)} = M_\alpha^\dagger M_\alpha, F_2^{(\alpha)} = \mathbb{1}_Q - F_1^{(\alpha)}\}, \quad (10)$$

where

$$M_\alpha = \sum_{n=1}^{q^k} \left(\frac{\lambda_n}{\lambda_0} \right)^{(\alpha-1)/2} |n\rangle\langle n|, \quad (11)$$

and $\{|n\rangle\}$ is an eigenbasis for $\tilde{\rho}_Q$, with λ_0 the leading eigenvalue. This is a valid (albeit impractical to implement) POVM, provided $\alpha \geq 1$, and reduces to a projector onto the $w = 0$ band when $\alpha = \infty$. Postselecting on obtaining outcome 1 yields

$$\tilde{\sigma}_{Q,\alpha} \equiv \frac{M_\alpha \tilde{\rho}_Q M_\alpha^\dagger}{\text{Tr}[M_\alpha \tilde{\rho}_Q M_\alpha^\dagger]} = \frac{\tilde{\rho}_Q^\alpha}{\text{Tr}[\tilde{\rho}_Q^\alpha]}. \quad (12)$$

For $\alpha < \infty$ and $p < p_d$, the probability that postselection succeeds, $\text{Tr}[\tilde{\rho}_Q^\alpha]/\lambda_0^{\alpha-1} = q^{(\alpha-1)(S_\infty(\tilde{\rho}_Q) - S_\alpha(\tilde{\rho}_Q))}$, is a smooth function of the α -Rényi entropy, which has two regimes of behavior. At large p , $\tilde{\rho}_Q$ is close to the identity matrix, so postselecting with Eq. (11) uniformly rescales all the eigenvalues, and $S_\alpha(\tilde{\rho}_Q) \approx N$. However, for smaller p , the rescaled density matrix is dominated by large eigenvalues corresponding to distinguishable errors (which are individually likely but collectively unlikely past p_c). At the Rényi threshold $p_c^{(\alpha)}$, $\tilde{\rho}_Q^\alpha$ changes from being dominated by correctable errors to uncorrectable errors.

While our quantitative predictions below will focus on Haar-random codes, this mechanism for the nonanalyticity in $S_\alpha(\tilde{\rho}_Q)$ applies qualitatively to any sensible code and error model: (some class of) individual correctable errors are likelier to occur, and hence are associated with larger eigenvalues than individual uncorrectable errors. Biasing toward larger eigenvalues by increasing α thus results in a monotonically increasing threshold $p_c^{(\alpha)}$. Monotonicity may be violated in certain degenerate codes [26], though it has been conjectured to hold in a large family of stabilizer codes [23].

In Haar-random codes, one can use the zeroth order ansatz for the spectrum of $\tilde{\rho}_Q$, $\lambda(w) = P_w/(q^k \Omega(w))$, to estimate the weight $w^*(p, \alpha)$ which dominates the spectrum of $\tilde{\rho}_Q^\alpha$ [26]:

$$\frac{w^*(p, \alpha)}{N} = \frac{(q^2 - 1)(p/(q^2 - 1))^\alpha}{(1 - p)^\alpha + (q^2 - 1)(p/(q^2 - 1))^\alpha}. \quad (13)$$

As p increases, so does $w^*(p, \alpha)$, until at $p_c^{(\alpha)}$, the contribution to $\text{Tr}(\tilde{\rho}_Q^\alpha)$ from errors of weight $w^*(p, \alpha)$ becomes subdominant to the contribution $\sim q^{N(1-\alpha)}$ from uncorrectable errors at larger w . For $p > p_c^{(\alpha)}$, therefore, $S_\alpha(\tilde{\rho}_Q) = N$ is pinned to its maximum value.

The threshold $p_c^{(\alpha)}$ is determined by a Rényi generalization of the hashing bound, $H_\alpha(p) = 1 - k/N$, where

$$H_\alpha(p) = \frac{1}{1 - \alpha} \log_q \left[(1 - p)^\alpha + (q^2 - 1) \left(\frac{p}{q^2 - 1} \right)^\alpha \right] \quad (14)$$

is the Rényi version of the classical Shannon entropy [Eq. (7)]. This threshold is shown in Fig. 4(b), again interpolating between the correction threshold p_c at $\alpha = 1$ and the

detection threshold p_d as $\alpha \rightarrow \infty$. A saddle-point approximation, in agreement with a calculation using Weingarten calculus, yields a leading- N behavior of

$$\frac{S_\alpha(\tilde{\rho}_Q)}{N} = \begin{cases} H_\alpha(p) + k/N & p < p_c^{(\alpha)} \\ 1 & p \geq p_c^{(\alpha)} \end{cases}. \quad (15)$$

We note that for $\alpha = 2$ and $\alpha = \infty$, the Rényi transition is pinned to $p_c^{(\alpha)}$ by the quantum MacWilliams identity, a high-low temperature duality which applies to general codes [28]. For these values of α , we can also perform an annealed average to obtain the finite-size rounding of $S_\alpha(\tilde{\rho}_Q)$, shown for $\alpha = 2$ in Fig. 4(c) [26].

So far, we have seen that the Rényi entropy S_α changes nonanalytically at $p_c^{(\alpha)}$, and tied this transition operationally to a change in the probability that postselection succeeds. However, the change in the character of $\tilde{\rho}_Q^\alpha$ also manifests itself as an error-correction threshold for the postselected state $\tilde{\sigma}_{Q,\alpha}$. For $p > p_c^{(\alpha)}$, $\tilde{\sigma}_{Q,\alpha}$ is dominated by uncorrectable errors; for $p < p_c^{(\alpha)}$, it is dominated by correctable errors of weight $w^*(p, \alpha) < w_c$. Its (von Neumann) entropy per qubit, in the limit of infinite system size, is therefore

$$\frac{S_{\text{vN}}(\tilde{\sigma}_{Q,\alpha})}{N} = \begin{cases} H(w^*(p, \alpha)/N) + k/N & p < p_c^{(\alpha)} \\ 1 & p > p_c^{(\alpha)} \end{cases}. \quad (16)$$

Unlike the Rényi entropy, $S_{\text{vN}}(\tilde{\sigma}_{Q,\alpha})$ has a jump discontinuity at $p_c^{(\alpha)}$, which gets rounded out at finite N [Fig. 4(c)].

Intuitively, therefore, one would expect $\tilde{\sigma}_{Q,\alpha}$ to remain correctable as long as $p < p_c^{(\alpha)}$. This is in good agreement with our numerical findings [Fig. 4(d)]. We have not been able to derive an analytic ansatz for the coherent information of the postselected state beyond this saddle-point level: since the postselection protocol involves reweighting the eigenvectors of $\tilde{\rho}_{RQ}$ by the eigenvalues of $\tilde{\rho}_Q$, we do not have an explicit expression for the eigenvalues (and therefore the entropies) of $M_\alpha \tilde{\rho}_{RQ} M_\alpha^\dagger$. However, we observe numerically, and argue heuristically [26], that the width of the transition narrows from $\Delta p = O(1/\sqrt{N})$ ($\nu = 2$) in the absence of postselection to $\Delta p = O(1/N)$ ($\nu = 1$) for strong enough postselection, including $\alpha = 2$.

C. Discussion

In this section we considered two distinct types of postselected transition: directly postselecting on errors of weight w , and smoothly reweighting the density matrix by a parameter α . In both cases, postselected error correction is possible until some threshold that exceeds the hashing bound p_c . Moreover, in both cases, the failure of postselected error correction is accompanied by a nonanalyticity in the probability that postselection succeeds. For

the reweighted problem, this nonanalyticity can be interpreted as a singularity in the Rényi entropy of the state $\tilde{\rho}_Q$.

Interestingly, despite these similarities, the two postselection protocols we considered are inequivalent: for a fixed p between the hashing and detection thresholds, one can define $\alpha_{\min}(p)$ and $w_{\max}(p)$ such that postselected error correction succeeds for $\alpha > \alpha_{\min}(p)$ and for $w < w_{\max}(p)$. However, because of the discontinuity discussed in the previous paragraph, $w^*(p, \alpha_{\min}(p)) \neq w_{\max}(p)$: thus, α and w are not simply proxies for one another. Instead, as shown in Fig. 4(a), the region in the space of $(p, w/N)$ correctable under α -reweighted postselection is strictly smaller than the correctable region under hard postselection; the two thresholds agree only at the hashing bound (no postselection) and detection threshold. This inequivalence is reminiscent of the inequivalence between the microcanonical and canonical ensembles in certain long-range statistical mechanics models [44–46]; it might be interesting to explore whether it also occurs in local quantum codes such as the surface code.

VI. DISCUSSION

In this paper we studied how the density matrix $\tilde{\rho}_Q$ of a random quantum code evolves as it is subjected to depolarizing noise acting independently on all qudits. We showed that for error rates below the error correction threshold, the spectrum of $\tilde{\rho}_Q$ consists of well-separated bands. For all but one of these bands, the eigenstates in each band correspond to correctable errors of some particular Pauli weight. Each such band can be interpreted as the image of the logical space under an error model that only includes errors of a fixed Pauli weight w ; the full error model is a convex combination of density matrices decohered by such fixed-weight channels.

In terms of this picture, we were able to derive various spectral and coding properties of these random codes, in particular: (i) the shape and width of the spectrum of each band, (ii) the evolution of each band with error rate, and (iii) the value and finite-size broadening of the error correction threshold. We also showed that the bands corresponding to low-weight errors remain well-defined and correctable between the error correction threshold and the detection threshold: in this regime, we estimated the amount of postselection that is required to make such error correction possible. Finally, we argued that the singularities seen in the Rényi entropies past threshold can be related to the success rates for soft postselection into a correctable subspace. This observation gives a concrete information-theoretic meaning to Rényi-entropy thresholds that have been extensively discussed in the literature [11, 15, 23] and further justifies the conjectured monotonicity of these thresholds [23] (at least for nondegenerate codes).

A. Relation to other works

A threshold at the hashing bound in the fixed-weight error ensemble is equivalent to the statement that Haar-random codes can perform approximate quantum error correction up to the *quantum Hamming bound*. That is, given a flat error probability distribution over a set of trace orthogonal, unitary errors, Haar-random codes can approximately correct errors when the cardinality of the set is far less than q^{N-k} [47]. A rigorous proof of this result appeared nearly simultaneously with our work in Ref. [47], which introduces the notion of an approximately nondegenerate code with respect to an arbitrary unitary error set. A similar result was proven using a different technique—which requires only that the code ensemble be a unitary two-design—in Ref. [48]. We stress that both works are focused on proving a *lower bound* on the fixed-weight/fixed-cardinality threshold, whereas we have argued (using Weingarten calculus, random matrix theory, and numerics) that this bound is tight. While this may seem intuitive on the grounds that *exact* error correction cannot be performed using *exactly* nondegenerate codes past the Hamming bound [49], it may also come as a surprise that, in terms of threshold, Haar-random codes cannot outperform random stabilizer codes.

The connections between our work and Ref. [47] are discussed in detail in the Supplementary Material [26]. There, we also elucidate the connections between band structure, approximately nondegenerate quantum codes, and the Knill-Laflamme conditions. In brief, when a relative bandwidth remains parametrically small, the conditions for forming an approximately nondegenerate quantum code are satisfied.

B. Generalizations

In this work, we have presented an analytically tractable minimal model of a decodability/mixed-state phase transition. While the analysis was facilitated by the simplicity of the code ensemble (Haar-random, with no notion of locality) and the noise model (uncorrelated, depolarizing noise), the insights gleaned from our work have both qualitative and quantitative implications for more structured models.

One avenue for extending this work is by modifying the noise model. One analytically tractable noise model is fully heralded, or erasure, errors. The spectrum of a state with m erased qubits admits a straightforward interpretation: to first approximation, like the fixed-weight error ensemble, it obeys a Marchenko-Pastur distribution [50–52], but now with $d \rightarrow q^{N-m}$, $\phi \rightarrow q^{m+k}$. The threshold and finite-size rounding of the transition, derived using Weingarten calculus in Sec. V of the Supplemental Material [26], are consistent with prior numerical work [53].

A more challenging avenue is the introduction of correlated noise. This would break a central assumption of our perturbation theory, which states that weights are drawn

from Bernoulli trials. Organization into bands around which a perturbation theory could be built is still plausible if certain patterns of errors are dominant, but these bands cannot be organized as simply by many-body weight.

Yet another direction for study is to reintroduce structure into the code ensemble, to determine which aspects of the picture we have presented can generalize to other codes. A simple example are eigenstate codes, where Haar-random states are supplanted by energetically equivalent eigenstates of a chaotic Hamiltonian [54]. Here the Eigenstate Thermalization Hypothesis (ETH) [55,56] suggests that our perturbation theory methods could be applied directly (though the implications for thresholds are not entirely clear).

More nontrivial are local stabilizer codes, such as the toric code, or nonstabilizer codes such as nonabelian topologically ordered states [57–59]. In the toric code specifically, there is also evidence from statistical mechanics mappings for a series of α -dependent singularities in the Rényi entropy $S_\alpha(\rho_Q)$, with the thresholds increasing with α as in random codes. On the other hand, in the toric code and other topological codes, not all errors of a given weight w are equivalent, and the eigenvalue distribution in the large N limit will acquire structure not captured by random matrix theory. Nevertheless, the broad picture of the spectrum splitting into bands may still apply, with the proviso that (for stabilizer codes with degeneracy), one should think of bands as organized by *syndromes*, rather than *errors*. In the Supplemental Material, we develop this perspective by considering the band structure of well-known small stabilizer codes [26]. The utility of this approach is less clear in the large N limit, as both the eigenvalue multiplicity within bands and the band dispersion become nontrivial to calculate. In light of these challenges, understanding whether some notion of universality applies to the eigenvalue spectra of generic nondegenerate decohered codes (and further exploring the broader similarities in decohered codes with degeneracy) is an interesting task for future work.

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DATA AVAILABILITY

The data are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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