Low-Density Parity-Check Stabilizer Codes as Gapped Quantum Phases: Stability under Graph-Local Perturbations

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We generalize the proof of stability of topological order, due to Bravyi, Hastings, and Michalakis, to stabilizer Hamiltonians corresponding to low-density parity-check (LDPC) codes without the restriction of geometric locality in Euclidean space. We consider Hamiltonians H_0 defined by $[\![N, K, d]\!]$ LDPC codes, which obey certain topological quantum order conditions: (i) code distance $d \geq c \log(N)$, implying local indistinguishability of ground states, and (ii) a mild condition on local and global compatibility of ground states—these include good quantum LDPC codes and the toric code on a hyperbolic lattice, among others. We consider stability under weak perturbations that are quasilocal on the interaction graph defined by H_0 and that can be represented as sums of bounded-norm terms. As long as the local perturbation strength is smaller than a finite constant, we show that the perturbed Hamiltonian has well-defined spectral bands originating from the $O(1)$ smallest eigenvalues of H_0. The band originating from the smallest eigenvalue has 2^K states, is separated from the rest of the spectrum by a finite energy gap, and has exponentially narrow bandwidth $\delta = CNe^{-\Theta(d)}$, which is tighter than the best-known bounds even in the Euclidean case. We also obtain that the new ground-state subspace is related to the initial-code subspace by a quasilocal unitary, allowing one to relate their physical properties. Our proof uses an iterative procedure that performs successive rotations to eliminate non-frustration-free terms in the Hamiltonian. Our results extend to quantum Hamiltonians built from classical LDPC codes, which give rise to stable symmetry-breaking phases. These results show that LDPC codes very generally define stable gapped quantum phases, even in the non-Euclidean setting, initiating a systematic study of such phases of matter.

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I. INTRODUCTION

Over the past two decades, ideas from quantum information theory have been transformational in our understanding of quantum phases of matter [1]. This has been particularly notable in the study of *zero-temperature topological order*. Stabilizer codes [2–4], the most well-studied examples of error-correcting codes, also define gapped and exactly solvable Hamiltonians that are sums of commuting projectors, such that the ground-state space of the Hamiltonian is the same as the code space. Encoding a number of qubits $K > 0$ corresponds to a 2^K-fold ground-state degeneracy for the corresponding Hamiltonian [5]; likewise, features that endow the code with robustness against errors—such as local indistinguishability of code states [6]—also define nontrivial topological order in the ground states. Kitaev's toric code [7] stands as a paradigmatic example of $\mathbb{Z}_2$ topological order [8], while various other examples such as fracton models [9,10] describe distinct types of quantum order.

This correspondence between codes and quantum order is powerful because stabilizer Hamiltonians can serve as representative fixed points that capture the properties of an entire *phase* of matter. This has been established in the seminal work of Bravyi, Hastings, and Michalakis (BHM) [11] (see also Refs. [12–14]), which has shown that zero-temperature topological order in gapped commuting-projector Hamiltonians remains *stable* to *all* sufficiently weak perturbations comprised of a sum of sufficiently local

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terms (as long as the local perturbation strength is below a constant threshold, which remains finite in the thermodynamic limit). More specifically, BHM have shown that (i) the gap above the ground-state subspace remains finite in the perturbed model and (ii) the degeneracy of the ground states is split into a narrow spectral band, the width δ of which scales to zero exponentially with increasing system size, so that the ground-state degeneracy is robust. Furthermore, (iii) the low-energy subspaces of the unperturbed and perturbed Hamiltonians are related by quasilocal unitary transformations, which cannot destroy the patterns of long-range entanglement characteristic of topologically ordered ground states [15,16]. In other words, despite being solvable, stabilizer models are *not* fine-tuned points in parameter space with respect to their physical properties. The results of BHM apply to geometrically local Hamiltonians on D-dimensional Euclidean lattices that obey certain topological quantum order conditions (referred to as TQO-I and TQO-II). Nowadays, the most widely used results on ground-state stability are Refs. [14,17,18], which, in contrast to BHM, are not restricted to commuting-projector Hamiltonians. That restriction is replaced by local versions of the gap condition and the TQO conditions. However, as it stands, this result is also restricted to Euclidean lattices.

Recently, there have been a number of breakthroughs in quantum error correction (QEC) on the topic of low-density parity-check (LDPC) codes defined on *non-Euclidean* expander graphs [19–27]. The LDPC condition imposes that each stabilizer check acts on only a constant number of qubits and that every qubit is only part of a constant number of checks; besides these requirements, completely general interaction graphs are allowed. Thus, the LDPC condition imposes a notion of graph locality but otherwise allows for spatially nonlocal interactions and/or non-Euclidean geometries, which are outside the scope of BHM's results. Of particular interest are "good" LDPC codes defined on *higher-dimensional expanders* [20–27], which exhibit optimal scaling for certain metrics of error correction that can provably not be realized in local Euclidean geometries [28,29]. Expanders have the property that the surface area of a region scales proportionally to its volume; notably, the volume of a ball of radius r on an expander graph can grow *exponentially* with r. In good codes, both K (the number of logical qubits encoded) and the code distance d (the size of the smallest operator that distinguishes states in the code space) are proportional to N (the number of physical qubits), which is the optimal scaling for the density and robustness of encoded information. Quantum codes that satisfy this property have been discovered only recently, after a decades-long quest. Moreover, such non-Euclidean geometries are rapidly becoming experimentally accessible in various quantum computing and simulation platforms [30–33].

These developments raise a natural question, central to this work:

> *Can we associate nonlocal and/or non-Euclidean LDPC stabilizer codes with stable phases of quantum matter?*

It is already apparent that non-Euclidean quantum LDPC (qLDPC) codes can display novel and interesting properties from the perspective of physics. A subset of the present authors have recently embarked on the program of understanding these codes from a physics perspective and have shown that these models can be understood as unconventional gauge theories [34–36]. In Refs. [37,38], it has been shown that good qLDPC codes can exhibit a particularly striking and strong form of topological order (called "no low-energy trivial states," or NLTSs), which crucially relies on the bulk and boundary scaling proportionally on expander graphs. The question remains: are the novel properties of the Hamiltonians defined by these codes *stable* to weak graph-local perturbations, in analogy with BHM's results for local Euclidean models? Establishing stability under perturbations is a necessary first step for associating these codes with stable *phases*.

On one hand, it is physically intuitive to expect all LDPC codes to be stable. They are gapped and the LDPC condition ensures that some aspects of local physics is retained, including extensivity of energy and generalized Lieb-Robinson bounds on graphs [39] (such bounds play an important role in BHM's results). Most importantly, if codes have a code distance d that diverges with N, it means that ground states are locally indistinguishable because operators must have support on at least d qubits to distinguish different ground states. Thus, if the perturbation is comprised of graph-local terms of strength $\epsilon < 1$, an operator of weight d is only generated at dth order in degenerate perturbation theory, suggesting that ground states get split by an amount ϵ^d that vanishes as $N \to \infty$. In line with this intuition, the local indistinguishability condition is precisely the TQO-I condition that enters BHM's rigorous stability proof for local Euclidean models. A related point is that it is proven that $d \geq c \log(N)$ for a constant $c > 0$ is sufficient to ensure that LDPC codes have a finite *decoding threshold* [40,41] to all local errors, even in nonlocal and non-Euclidean settings. This is a different notion of stability (and is the property of a nonequilibrium error channel rather than a local Hamiltonian) but this property is also intrinsically tied to local indistinguishability and the topological character of the code space. In Ref. [36], a subset of us have been able to use the existence of a decoding threshold to physically argue for stability of the gap and ground-state degeneracy of LDPC Hamiltonians, but our results have been for a limited class of

models and perturbations [Calderbank-Shor-Steane (CSS) LDPC codes perturbed by purely X or Z fields] and these arguments do not constitute a rigorous proof.

Despite these intuitive reasons for optimism, there are various issues that also make the stability analysis subtle, as recently noted in Ref. [42]. Good codes have $K \propto N$, corresponding to an exponentially large ground-state degeneracy and an extensive ground-state entropy. Thus, within degenerate perturbation theory, even exponentially small matrix elements between these exponentially many states could produce large energy differences, thereby destroying the ground-state degeneracy. More physically, stability of the ground-state degeneracy would imply a stable violation of the third law of thermodynamics, which is certainly unconventional from the perspective of statistical physics [43].

Lavasani *et al.* [42] have extended the approach of Ref. [13] to prove the stability of a large class of k-local (but geometrically nonlocal) LDPC codes. Their results have significantly extended BHM's work, but have fallen short of proving stability on expander graphs in which the volume of a ball of radius r grows exponentially with r. This has left open the stability question for important examples such as the surface code on hyperbolic tilings [44] and "good" qLDPC codes [25,45–47]. The reason why the analysis of Ref. [42] did not extend to expander graphs was because, following Ref. [13], the authors used the formalism of quasiadiabatic continuity in their analysis. This converts local operators to quasilocal operators; however, the bounds on locality (and Lieb-Robinson velocities) were not strong enough to produce operators with exponentially decaying tails to counteract the exponential growth of volumes.

A. Informal statement of conditions and results

In this work, we provide a definitive affirmative answer to the perturbative stability of *all* qLDPC stabilizer codes with a sufficiently large code distance and under reasonable TQO conditions; these conditions are stated precisely in Secs. II B and III E but they are a natural generalization of the TQO conditions in BHM:

TQO-I The ground subspace is the code space of a quantum code with code distance $d \geq c \log(N)$ for constant $c > 0$.

TQO-II Local ground subspaces are consistent with the global one.

The first statement is a statement of local indistinguishability, and BHM discuss via the example of an "unstable toric code" why **TQO-I** is necessary but not sufficient [11].

Informally, we prove that for such Hamiltonians, all *graph-local perturbations perturb graph locally* i.e., the ground states of the perturbed Hamiltonian and of the unperturbed one are related by quasilocal unitary transformations. In particular, for weak and extensive perturbations comprised of sums of bounded-norm interactions with an appropriate local strength smaller than a finite threshold, we show that (i) the spectral gap remains finite in the perturbed model and (ii) the ground-state splitting is exponentially small in the code distance, scaling as $\delta \leq CN \exp(-c'd)$ for finite constants $C, c' > 0$. This splitting decays to zero in the thermodynamic limit, so the ground-state degeneracy and gap remain robust.

When applied to local LDPC codes on Euclidean lattices, our results significantly strengthen parametric bounds on the ground-state splitting relative to those obtained by BHM. As an example, when applied to the two-dimensional (2D) toric code on an $L \times L$ lattice with $d \propto L$, our results predict the optimal scaling $\delta \leq CL^2 \exp\{-c'L\}$, while BHM obtain a weaker bound $\delta = \text{poly}(L) \exp(-c_2 L^{3/8})$ with $c', c_2 > 0$. We also require a weaker distance bound $d \geq c \log(N)$ relative to BHM, who require $d \geq cL$.

With minimal additional effort, we extend our results to the following:

(i) Topologically trivial and local gapped paramagnetic stabilizer models with a unique ground state, on any geometry. Indeed, these models belong to the class of models that we specify in Sec. II, with the specification that $K = 0$, and their stability is established by noting that nowhere in our proof do we invoke conditions on K.

(ii) Classical LDPC codes with symmetry-breaking order, under symmetry-respecting perturbations. Examples range from perturbed Ising models on any graph to perturbed "good" classical LDPC codes on expander graphs. In general, a classical code that encodes K bits has a symmetry group $\mathbb{Z}_2^K$ (for a discussion of this point, see Ref. [34]), which should be respected by the perturbations. We detail this discussion in Sec. IX.

The key technical tool enabling our result is an iterative procedure that performs successive rotations to iteratively eliminate non-frustration-free terms in the Hamiltonian (as it is inspired by the iterative procedure in the proof of the KAM theorem [48–50]; we call this procedure "KAM" for Kolmogorov, Arnold, and Moser). We generalize the approach in De Roeck and Schütz [51], which has considered the stability of noninteracting paramagnetic Hamiltonians comprised of single-site stabilizers, to topologically ordered quantum stabilizer codes. Notably, this approach provides much stronger control over the locality of rotations by tracking the growth of the qubit support of rotated operators at each stage and the number of stabilizers excited by rotated operators at each stage (in contrast to only tracking the spatial extent of the operator). We note that there are several alternative (though related) schemes

for proving the stability of the gap for many-body systems [52–54].

Our results rigorously establish that two cornerstones of zero-temperature many-body physics—the stability of gapped phases and the idea of quasiadiabatic continuation—remain well-defined for large classes of models beyond the setting of geometrically local models in finite-dimensional Euclidean spaces. This opens the door to a plethora of mostly unexplored questions in quantum many-body physics, such as the classification of non-Euclidean and k-local (but geometrically nonlocal) phases, and a systematic study of their robust physical properties. This is particularly topical in light of stunning experimental developments giving us access to tunable interactions and geometries [30–33], opening up a wide new frontier for many-body physics.

II. MODELS AND SUMMARY OF RESULTS

Before presenting a formal statement of our results in Sec. II D, we discuss the types of Hamiltonians and perturbations considered in this work. In Sec. II A, we review a special class of Hamiltonians, stabilizer Hamiltonians, with frustration-free commuting Pauli terms. In Sec. II B, we discuss natural conditions on these Hamiltonians, including the LPDC condition giving a generalized notion of locality. Our main theorems assume these conditions. In Sec. II C, we define an operator norm that penalizes nonlocality and we discuss restrictions on the class of perturbations that we consider in terms of this norm.

Section II D gives the formal statement of our results, including our results on energy gaps and splittings (Theorem 1), relationships between perturbed and unperturbed ground states (Theorem 2), and analyticity properties of the ground states (Theorem 3).

A. Stabilizer codes, checks, and graphs

We first recall a few standard notions for stabilizer codes.

We are given a set Λ of $|\Lambda| \equiv N$ sites, and to each site $x \in \Lambda$ we associate a qubit. The Hilbert space of all qubits in Λ is $\mathcal{H}_\Lambda := \bigotimes_{x\in\Lambda} \mathcal{H}_x \cong (\mathbb{C}^2)^{\otimes N}$. We denote by $\mathcal{P}_\Lambda$ the *Pauli group* supported on Λ. Elements of $\mathcal{P}_\Lambda$ are called *Pauli strings* [55], which are operators on $\mathcal{H}_\Lambda$. A Pauli string $p \in \mathcal{P}_\Lambda$ takes the form

$$p = \bigotimes_{x\in\Lambda} p_x, \quad \text{where } p_x \in \left\{ \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathsf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathsf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \mathsf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{1}$$

The *qubit support of a Pauli string* p is defined as

$$\mathrm{supp}(p) = \{x : p_x \neq \mathbb{1}\}. \tag{2}$$

A stabilizer code on Λ is uniquely defined by an Abelian subgroup $\mathcal{G} \subseteq \mathcal{P}_\Lambda$. The *code space* P is the subspace of $\mathcal{H}_\Lambda$, defined as follows:

$$P := \{|\psi\rangle \in \mathcal{H}_\Lambda : \mathcal{G}\,|\psi\rangle = |\psi\rangle\}. \tag{3}$$

We have that $K = \log_2 \dim P = N - \log_2 |\mathcal{G}|$. The number K is often referred to as *the number of logical qubits*. The *code distance* of $\mathcal{G}$ is defined as

$$d = \min\{|\mathrm{supp}(p)| : PpP \not\propto P\}. \tag{4}$$

We are interested in subsets of $\mathcal{G}$ that generate $\mathcal{G}$: $\langle\{C_\alpha\}_{\alpha\in\mathcal{E}}\rangle = \mathcal{G}$, where $C_\alpha \in \{C_\alpha\}_{\alpha\in\mathcal{E}} \subseteq \mathcal{G}$, and we use the notation $\mathcal{E}$ for an index set of Pauli operators in a generating subset.

We focus on *stabilizer Hamiltonians* of the following commuting-projector form:

$$H_0 = \sum_{\alpha\in\mathcal{E}} E_\alpha, \quad \text{where } E_\alpha = (1 - C_\alpha)/2. \tag{5}$$

Here, each $C_\alpha \in \{C_\alpha\}_{\alpha\in\mathcal{E}} \subseteq \mathcal{G}$ is called a *check* and we henceforth define $N_c := |\mathcal{E}|$. Given $\mathcal{G}$, the choice of $\mathcal{E}$ is equivalent to the choice of Hamiltonian H_0. We note in passing that different choices of $\mathcal{E}$ for a given $\mathcal{G}$ can have different stability properties and that the dependence on $\mathcal{E}$ is reflected in the **TQO-II** condition given in Sec. II B.

In other words, the checks generate the stabilizer group. Therefore,

$$\prod_{\alpha\in\mathcal{E}} G_\alpha = P, \quad \text{where } G_\alpha = (1 + C_\alpha)/2. \tag{6}$$

Here, we use P to refer to both the code space and the projector onto this space. The ground-state sector of H_0 corresponds to all checks being satisfied, which is identical to P.

We define the *check support of a qubit* x as follows:

$$\mathrm{supp}_c(x) := \{\alpha : x \in \mathrm{supp}(C_\alpha)\}. \tag{7}$$

We similarly define the *check support of a Pauli string* p as

$$\mathrm{supp}_c(p) := \bigcup_{x\in\mathrm{supp}(p)} \mathrm{supp}_c(x). \tag{8}$$

The checks define a graph structure on Λ. We introduce an edge (x, y) for $x, y \in \Lambda$ if $x \in \mathrm{supp}(C_\alpha)$ and $y \in \mathrm{supp}(C_\alpha)$ for some check C_α. The distance between any two qubits is the usual graph distance. We similarly define a graph structure on $\mathcal{E}$ with the checks as the vertices and we introduce an edge $(C_\alpha, C_{\alpha'})$ if $x \in \mathrm{supp}(C_\alpha)$ and $x \in \mathrm{supp}(C_{\alpha'})$ for some qubit x.

Let $S \subseteq \mathcal{E}$. We define the qubit support of S to be

$$\mathrm{supp}(S) := \bigcup_{\alpha \in S} \mathrm{supp}(C_\alpha). \tag{9}$$

We define the $\mathcal{G}_S \subseteq \mathcal{G}$ as the subgroup generated by checks in S,

$$\mathcal{G}_S := \langle \{C_\alpha\}_{\alpha \in S} \rangle. \tag{10}$$

Meanwhile, let $\mathcal{G}(S) \subseteq \mathcal{G}$ be the following subgroup:

$$\mathcal{G}(S) := \{p \in \mathcal{G} : \mathrm{supp}(p) \subseteq \mathrm{supp}(S)\}. \tag{11}$$

We hence have $\mathcal{G}_S \subseteq \mathcal{G}(S)$ but the inclusion is in general strict. For example, consider an annulus-shaped region of qubits A in the 2D toric code and take S to be the set of all plaquette checks that overlap with A. The group $\mathcal{G}(S)$ contains a loop operator that is the product of all plaquettes enclosed by A but this operator is not within $\mathcal{G}_S$.

In the remainder of this paper, we always use Greek letters $(\alpha, \beta, \ldots)$ to refer to checks, and Latin letters $(x, y, \ldots)$ to refer to qubits.

B. Conditions on stable unperturbed stabilizer Hamiltonians

Our main result concerns the stability of Hamiltonians H_0 associated with stabilizer codes, according to Eq. (5). We are interested in families of codes with growing N. When discussing statistical mechanics in Euclidean spaces, the limit $N \to \infty$ is called the *thermodynamic limit* and this limit is essential for defining a sharp notion of phases and phase transitions. For codes on general graphs, it may not be possible to smoothly change N (in units of 1) while maintaining local properties such as graph degree. Nevertheless, to associate codes to many-body phases, we have in mind families of models in which N can be made arbitrarily large, even if the increase in N does not happen in regular steps of 1.

With these definitions, we are now ready to state the conditions on unperturbed stabilizer Hamiltonians H_0 that obey the stability theorems in Sec. II D:

LDPC We say that the code family is LDPC if the following conditions hold on the qubit weight w_q and the check weight w_c:

$$\begin{aligned} w_q &\equiv \max_{x \in \Lambda} |\mathrm{supp}_c(x)| = \Theta(1), \\ w_c &\equiv \max_{\alpha \in \mathcal{E}} |\mathrm{supp}(C_\alpha)| = \Theta(1). \end{aligned} \tag{12}$$

Throughout, we will use $\Theta(1)$ to mean asymptotically upper and lower bounded by constants as $N \to \infty$. It follows from the definition of the graphs Λ and $\mathcal{E}$ that their graph degrees are both upper bounded by $w_c w_q$. The **LDPC** condition guarantees that the energy density of H_0 for all states in $\mathcal{H}_\Lambda$ is upper bounded by a constant. If we additionally impose that each qubit is part of at least one check (so that no qubits are "idle"), then the LDPC condition guarantees that $N_c \propto N$ corresponding to an extensive bandwidth $\propto N$ for the spectrum of H_0, analogous to conventional thermodynamics.

Growth of balls Let $B_r(x)$ be the set of all sites within a distance r of x in the graph Λ. Similarly, let $B_r(\alpha)$ be all checks within a distance r of α in the graph $\mathcal{E}$. We consider Hamiltonians for which there exists $\kappa = \Theta(1)$ such that

$$\begin{aligned} &\forall x \in \Lambda, \quad |B_r(x)| \le e^{\kappa r}; \\ &\forall \alpha \in \mathcal{E}, \quad |B_r(\alpha)| \le e^{\kappa r}. \end{aligned} \tag{13}$$

We note that **growth of balls** follows from **LDPC** with any $\kappa \ge \log(w_c w_q + 1)$.

Note that on D-dimensional local Euclidean graphs, $B_r \propto r^D$; so our condition allows for much stronger growth of volumes, thereby encompassing a large family of graphs including the expanders.

TQO-I For every set of qubits $A \subseteq \Lambda$ with $|A| < d$ and for every Pauli string p such that $\mathrm{supp}(p) \subseteq A$, we have

$$\forall g \in \mathcal{G} : [p, g] = 0 \quad \Rightarrow \quad p \in \mathcal{G}. \tag{14}$$

In other words, there are no logicals supported on A. For stabilizer codes, **TQO-I** follows directly from the definition of the code distance d.

TQO-II For every connected set S in the check graph $\mathcal{E}$, if $|S| < \widetilde{d} = \Theta(d)$, then we have

$$\mathcal{G}(S) \subseteq \mathcal{G}_{B_{\ell|S|}(S)} \tag{15}$$

where $\ell = \Theta(1)$, and $B_{\ell|S|}(S) = \{\alpha : \mathrm{dist}(\alpha, S) \le \ell |S|\}$.

We define $d_* := \min(d/w_c, \widetilde{d}) = \Theta(d)$. The theorems in Sec. II D are meaningful, i.e., they give a stable gap and ground-state degeneracy if $d_* \ge c \log(N)$ for $c > 0$.

For local stabilizer codes on Euclidean geometries, our **TQO-II** condition is closely related to the **TQO-II** condition of BHM in Ref. [11, Lemma 2.1]. Therefore, we expect our conditions to hold for a wide class of codes in Euclidean spaces. In Appendix A, we argue that this

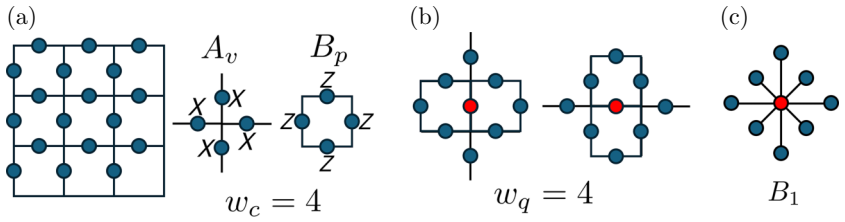


FIG. 1. Examples of a few geometric quantities illustrated via the vanilla toric code. (a) The stabilizer group $\mathcal{G}$ of the toric code is generated by the set $\{A_v\}_v \cup \{B_p\}_p$ of X stars and Z plaquettes. In this figure, we consider $\mathcal{E}$ [see Eq. (5) and surrounding discussion] such that $\{C_\alpha\}_{\alpha\in\mathcal{E}} = \{A_v\}_v \cup \{B_p\}_p$. This is the usual set of checks of the toric code. For this $\mathcal{E}$, the maximum qubit support of any check is $w_c = 4$, since each check contains exactly four qubits. (b) With this choice of $\mathcal{E}$, the maximum check support of the qubits is $w_q = 4$, since a given qubit (red) touches exactly four stabilizers (two stars and two plaquettes). (c) The checks define a graph structure on the qubits, where two qubits are connected by an edge if they are both part of the same check. Pictured is a ball of radius 1 surrounding the red qubit (at site, say, x), which we denote $B_1(x)$.

condition indeed holds for various important examples of non-Euclidean qLDPC codes; in particular, for known constructions of good qLDPC codes and for hypergraph products of good classical LDPC codes. We further note that the condition **TQO-II** here is a very mild one; we are able to use such a mild condition because of how we track our rotated perturbations throughout the proof. We note that the hyperbolic toric code satisfies both **TQO-I** and **TQO-II** with $\ell = \Theta(1)$ and $d_* = \Theta(\log(n))$ and so we are able to prove stability of the hyperbolic toric code.

In Fig. 1, we use the familiar example of the vanilla toric code to illustrate some of our notation, particularly the important quantities w_c and w_q that encode the LDPC nature of the Hamiltonian.

C. Conditions on allowed perturbations

We will consider the stability of Hamiltonians H_0 obeying the conditions in Sec. II B to extensive perturbations $Z^{(0)}$ the energy bandwidth of which $\propto N$. In the simplest case, $Z^{(0)}$ is a sum of $\Theta(N)$ strictly graph-local operators the support of which is bounded within balls of finite size on the graph. However, we allow for more general quasilocal perturbations subject to the norm below; informally, perturbations that are local enough and small enough will suffice.

To state the class of perturbations that we can handle, we will use a norm that penalizes nonlocal operators. Given the (unique) decomposition of an operator $O = \sum_p c_p p$ into the Pauli basis, we define an **intensive Pauli norm** $|||\cdot|||_\mu$:

$$|||O|||_\mu = \sup_x \sum_{p:x\in\mathrm{supp}(p)} |c_p| e^{\mu|\mathcal{M}(p)|}, \tag{16}$$

where $\mathcal{M}(p)$ is a connected set of qubits that covers the support of p, $\mathrm{supp}(p) \subset \mathcal{M}(p)$, with $|\mathcal{M}(p)| = \min_{\mathcal{A}:\mathrm{supp}(p)\subset\mathcal{A},\,\mathcal{A}\text{ connected}} |\mathcal{A}|$.

The supremum over x, and more particularly the restriction that $\mathrm{supp}(p)$ must contain x, means that this norm can and will be *intensive* for the *extensive* perturbations that we consider.

As an example, if O is a field $h\sum_x \mathsf{X}_x$, then $|||O|||_\mu = he^\mu$. If O consists of a single Pauli string with support on all sites, such as $O = \prod_x \mathsf{X}_x$, then $|||O|||_\mu = e^{\mu N}$. Having a small $|||O|||_\mu$ requires that O cannot contain too much weight on Pauli strings with large support, and requiring small $|||O|||_\mu$ for larger values of μ is more stringent.

To have control of the perturbation $Z^{(0)}$ for all system sizes N, we require that

$$|||Z^{(0)}|||_{\widetilde{\mu}_0} = \Theta(1) \tag{17}$$

for some $\widetilde{\mu}_0 > \mu_\#$, where

$$\mu_\# := w_q\mu_* + (w_q\log(2) + \kappa + \log(2w_c)) \tag{18}$$

and, letting c be the maximal number such that $d_* \geq c\log(N)$ for all system sizes N,

$$\mu_* := \max(\kappa\ell + \log(4)(w_c+1), c^{-1}). \tag{19}$$

Recall that κ is the geometric constant appearing in Eq. (13), ℓ is the constant appearing in **TQO-II**, and d_* appears underneath **TQO-II** (cf. Sec. II B). For all $\widetilde{\mu}_0 > \mu_\#$, there is a threshold strength $\widetilde{\epsilon} = \Theta(1)$ that is a function of $\widetilde{\mu}_0, \mu_\#$ and the geometrical constants w_q, w_c, and κ, but not the system size N (cf. Sec. IV D). We further require that

$$\widetilde{\epsilon}_0 := |||Z^{(0)}|||_{\widetilde{\mu}_0} \leq \widetilde{\epsilon}. \tag{20}$$

Roughly speaking, $\widetilde{\mu}_0$ controls the locality of $Z^{(0)}$, i.e., how fast the perturbation terms decay with increasing operator weight, while $\widetilde{\epsilon}_0$ controls the overall local strength of the perturbation. We require $\widetilde{\mu}_0 > \mu_\#$ to ensure that the perturbation terms are decaying sufficiently rapidly. If the perturbation $Z^{(0)}$ satisfies Eq. (17), then multiplying $Z^{(0)}$ by a sufficiently small system-size-independent constant will make it satisfy Eq. (20).

Many perturbations satisfy Eq. (17). Extensive and strictly graph-local $Z^{(0)}$ (i.e., a sum of terms supported within finite-sized balls and with uniformly bounded operator norms) satisfy Eq. (17). More generally, quasilocal perturbations with sufficiently short "tails" will satisfy Eq. (17). Note that the definition of locality relevant here is locality on the graphs Λ and $\mathcal{E}$, defined directly in terms of the unperturbed Hamiltonian (which defines a notion of distance between qubits). Thus, we do not allow for perturbations by, e.g., some other arbitrary LDPC code but only for those the structure of which is compatible with that of H_0. One context in which this arises naturally is if H_0 is defined in terms of some fixed underlying graph (such as many of the known constructions of good qLDPC codes, such as quantum Tanner codes and their relatives [23,25,47]) and the perturbations are required to also be local on the same graph.

D. Statement of main theorems

In what follows, we will use the generic notation c and C for constants that can depend on the $\Theta(1)$-model parameters, i.e., on w_q, w_c, κ, and ℓ. We use c whenever we want to stress that $c > 0$ and C whenever we want to stress that $C < \infty$. The precise values of c and C can change from line to line. Whenever we allow constants to depend on some additional parameters, we indicate this in the notation. We are now ready to state our main results.

Theorem 1. Suppose that H_0 obeys the conditions of Sec. II B and that $Z^{(0)}$ obeys the conditions of Sec. II C. Constants $\widetilde{\epsilon}_0$, $\widetilde{\epsilon}$, $\widetilde{\mu}_0$, and $\mu_\#$ are defined in Sec. II C and obey $\widetilde{\mu}_0 > \mu_\#$ and $\widetilde{\epsilon}_0 \leq \widetilde{\epsilon}$. N_c is defined in Sec. II A and d_* is defined in Sec. II B.

Then, there are constants b, $\mu_\infty > \mu_*$, and $\epsilon_0 \leq \widetilde{\epsilon}_0$, such that the spectrum of the perturbed Hamiltonian,

$$H = H_0 + Z^{(0)} - b, \tag{21}$$

is contained in a union of intervals $\bigcup_{k\geq 0} I_k$, where $k \in \{0, 1, 2, \ldots\}$ runs over the spectrum of H_0 and

$$I_k = [k - (kC'\epsilon_0 + \delta), k + (kC'\epsilon_0 + \delta)] \tag{22}$$

with $\delta = CN_c\epsilon_0 e^{-\mu_\infty d_*}$.

Each interval is centered on k and has width $2(kC'\epsilon_0 + \delta)$. The nearly degenerate ground states correspond to the interval $I_{k=0}$ with splitting at most 2δ, which exponentially decays in d_*. Thus, for small ϵ_0, the spectral gap above I_0 remains $\Theta(1)$ and the splitting δ of the eigenvalues in I_0 rapidly decays to zero as $CN_c\epsilon_0 e^{-\mu_\infty d_*}$.

Let us denote the spectral projector corresponding to I_0 by $\widetilde{P}$ and the spectral projector corresponding to the ground-state sector of H_0 by P.

Theorem 2. Under the same assumptions as in Theorem 1 above, we have

$$\widetilde{P} = UPU^{-1} \tag{23}$$

for a unitary U that is locality preserving and locally close to identity, in the following sense. For any operator O supported in a connected set $Y \in \Lambda$, we can write

$$U^{-1}OU = O + \sum_{r=1}^{\mathrm{diam}(\Lambda)} O_r + O_{\mathrm{bg}}, \tag{24}$$

where $\mathrm{diam}(\Lambda)$ is the diameter of the graph Λ, and

(1) O_r is supported in $\{x : \mathrm{dist}(Y, x) \leq r\}$.
(2) O_r satisfies

$$||O_r|| \leq \epsilon_0 C(\mu_0, Y) e^{-cr} ||O|| \tag{25}$$

for some $c > 0$ and constant $C(\mu_0, Y)$ that can depend on μ_0 and Y.
(3) O_{bg} (with bg standing for "background") is bounded as $||O_{\mathrm{bg}}|| \leq CN_c\epsilon_0 e^{-\mu_\infty d_*}||O||$.

The same statement holds true for UOU^{-1}.

If the number $N_c\epsilon_0 e^{-\mu_\infty d_*}$ is upper bounded by $e^{-c\mathrm{diam}(\Lambda)}$, for some $c > 0$, then the background term O_{bg} can be omitted in the statement of the theorem, at the cost of increasing the constant $C(\mu_0, Y)$. Indeed, in that case, $||O_{\mathrm{bg}}||$ is small enough to be included in O_r for $r = \mathrm{diam}(\Lambda)$. This is the case for all examples that we are aware of.

Finally, we state an analyticity property of the spectral projector $\widetilde{P}$.

Theorem 3. Suppose that H_0 and $Z^{(0)}$ satisfy the conditions of Theorem 1. If the perturbation $Z^{(0)}$ is analytic in some parameter γ on a complex domain $D \subset \mathbb{C}$, such that Eq. (20) is satisfied uniformly in $\gamma \in D$, then the ground-state space projector $\widetilde{P}$ can be continued analytically to a function of $\gamma \in D$. It is of the form $\widetilde{P} = UPU^{-1}$ with the invertible operator U such that Eq. (24) holds and items (1), (2), and (3) of Theorem 2 still hold (upon redefining the constants $C(\mu_0, Y)$ and C), uniformly in $\gamma \in D$.

This theorem is a simple consequence of the fact that our results continue to hold when the perturbation is no longer Hermitian. A corresponding statement about the spectrum of the non-Hermitian perturbed operator H can be made as well (for details, see Sec. IV G).

III. GENERAL SETUP

In Sec. IV, we will describe the iterative KAM procedure for iteratively performing unitary transformations to

eliminate non-frustration-free terms. A key book-keeping tool of this procedure, which allows us to provide much stronger control on the locality of rotated operators, is that the procedure keeps track of the number of stabilizers excited by operators at each stage of the rotation, even if the stabilizer is deexcited at a later stage. Our procedure is an adaptation of Ref. [51], which considers the stability of paramagnetic Hamiltonians comprised of sums of single-site stabilizers, of the form $H_0 = \sum_x \mathsf{X}_x$. In the paramagnetic case, a simple basis change on every site allows all operators to be represented by their "raising" and "lowering" action on the stabilizers, which helps track the number of excitations. When the stabilizers are multisite, as they will be for general quantum codes, we need more formalism to represent operators in a way that allows us to track their action on the stabilizers, which is described in this section.

A. Operators and "words"

In Sec. II, we have defined projectors on whether a stabilizer is satisfied or not, namely,

$$G_\alpha = (1 + C_\alpha)/2, \quad E_\alpha = (1 - C_\alpha)/2, \quad G_\alpha + E_\alpha = 1. \tag{26}$$

The notation G (E) refers to ground state (excited). Recall that our unperturbed Hamiltonian is

$$H_0 = \sum_{\alpha \in \mathcal{E}} E_\alpha. \tag{27}$$

We now set up a formalism to deal with the operators that will appear in the perturbation and those generated in the iterative KAM procedure. We will consider quadruples ("words") $\mathbf{S} = (S_+, S_-, S_e, S_g)$ of disjoint sets of checks and we will write

$$S := S_+ \cup S_- \cup S_e \cup S_g. \tag{28}$$

Here, $\{e, g, +, -\}$ should be read as "{excited, ground, raising, lowering}." We use words to keep track of the action of each operator on the checks and we group operators into different classes depending on their action on the checks.

We define a class of operators $\mathcal{X}_\mathbf{S}$. An operator $X \in \mathcal{X}_\mathbf{S}$ if and only if there exists an operator $\widetilde{X}$ such that

$$X = \left(\prod_{\alpha \in S} \mathrm{left}_\alpha(\mathbf{S}) \right) \widetilde{X} \left(\prod_{\alpha \in S} \mathrm{right}_\alpha(\mathbf{S}) \right), \tag{29}$$

where

$$\mathrm{left}_\alpha(\mathbf{S}) = \begin{cases} G_\alpha, & \text{if } \alpha \in S_- \cup S_g, \\ E_\alpha, & \text{if } \alpha \in S_+ \cup S_e, \end{cases} \tag{30}$$

$$\mathrm{right}_\alpha(\mathbf{S}) = \begin{cases} G_\alpha, & \text{if } \alpha \in S_+ \cup S_g, \\ E_\alpha, & \text{if } \alpha \in S_- \cup S_e, \end{cases} \tag{31}$$

and we require that

$$\bigcup_{x \in \mathrm{supp}(\widetilde{X})} \mathrm{supp}_c(x) \subseteq S \quad \Leftrightarrow \quad \Lambda \setminus \mathrm{supp}(\widetilde{X}) \supseteq \mathrm{supp}(\mathcal{E} \setminus S). \tag{32}$$

We note that $\widetilde{X}$ can be expanded as a sum of Pauli operators and by $\mathrm{supp}(\widetilde{X})$ we mean the union of the qubit support of all nonzero Pauli operators in the expansion of $\widetilde{X}$; $\mathrm{supp}(S)$ is defined in Eq. (9). Here, we require that all checks that overlap with $\widetilde{X}$ are also in S.

In what follows, we will use $X_\mathbf{S}$ to denote an element of $\mathcal{X}_\mathbf{S}$.

The above definition also shows how to treat perturbations built out sums of local terms. Each local perturbation term should be thought of as $\widetilde{X}$. One decomposes them in operators corresponding to classes $\mathcal{X}_\mathbf{S}$. We note that the decomposition of a given operator is not unique. For strictly local perturbations (i.e., those living in finite-sized balls), it is always possible to expand them into *connected* words, i.e., words where S is connected in the graph $\mathcal{E}$, such that the expansions have a finite norm [to be defined below in Eq. (38)].

Let us provide an example. In a one-dimensional Ising model, the stabilizers are $C_{x+\frac{1}{2}} = \mathsf{Z}_x \mathsf{Z}_{x+1}$, where Z_x is the Pauli-Z operator on site x. To consider a local perturbation such as X_x, we decompose it into different words that describe the raising or lowering of X_x on its adjacent stabilizers, which in turn is conditioned on the state of the adjacent stabilizers prior to acting with X_x. So, we write

$$\begin{aligned} \mathsf{X}_x &= (G_{x+\frac{1}{2}} + E_{x+\frac{1}{2}})(G_{x-\frac{1}{2}} + E_{x-\frac{1}{2}}) \mathsf{X}_x (G_{x-\frac{1}{2}} \\ &\quad + E_{x-\frac{1}{2}})(G_{x+\frac{1}{2}} + E_{x+\frac{1}{2}}) \\ &= E_{x+\frac{1}{2}} E_{x-\frac{1}{2}} \mathsf{X}_x G_{x-\frac{1}{2}} G_{x+\frac{1}{2}} + E_{x+\frac{1}{2}} G_{x-\frac{1}{2}} \mathsf{X}_x E_{x-\frac{1}{2}} G_{x+\frac{1}{2}} \\ &\quad + G_{x+\frac{1}{2}} E_{x-\frac{1}{2}} \mathsf{X}_x G_{x-\frac{1}{2}} E_{x+\frac{1}{2}} + G_{x+\frac{1}{2}} G_{x-\frac{1}{2}} \mathsf{X}_x E_{x-\frac{1}{2}} E_{x+\frac{1}{2}}, \end{aligned} \tag{33}$$

where we have used Eq. (26), the fact that X_x flips adjacent stabilizers from ground to excited and vice versa, and the fact that $G_e E_e = 0$. The four terms in the second line of

TABLE I. Multiplication table of words.

α in	$\mathcal{E}\setminus S$	S_g	S_e	S_+	S_-
$\mathcal{E}\setminus S'$	$\mathcal{E}\setminus(S'S)$	$(S'S)_g$	$(S'S)_e$	$(S'S)_+$	$(S'S)_-$
S'_g	$(S'S)_g$	$(S'S)_g$	0	0	$(S'S)_-$
S'_e	$(S'S)_e$	0	$(S'S)_e$	$(S'S)_+$	0
S'_+	$(S'S)_+$	$(S'S)_+$	0	0	$(S'S)_e$
S'_-	$(S'S)_-$	0	$(S'S)_-$	$(S'S)_g$	0

Eq. (33) belong, in order, to the classes

$$\begin{aligned}
&E_{x+\frac{1}{2}}E_{x-\frac{1}{2}}\mathsf{X}_x G_{x-\frac{1}{2}}G_{x+\frac{1}{2}} \in \mathcal{X}_{\mathbf{S}=\{S_+=\{C_{x+\frac{1}{2}},C_{x-\frac{1}{2}}\},\emptyset,\emptyset,\emptyset\}},\\
&E_{x+\frac{1}{2}}G_{x-\frac{1}{2}}\mathsf{X}_x E_{x-\frac{1}{2}}G_{x+\frac{1}{2}} \in \mathcal{X}_{\mathbf{S}=\{S_+=\{C_{x+\frac{1}{2}}\},S_-=\{C_{x-\frac{1}{2}}\},\emptyset,\emptyset,\}},\\
&G_{x+\frac{1}{2}}E_{x-\frac{1}{2}}\mathsf{X}_x G_{x-\frac{1}{2}}E_{x+\frac{1}{2}} \in \mathcal{X}_{\mathbf{S}=\{S_+=\{C_{x-\frac{1}{2}}\},S_-=\{C_{x+\frac{1}{2}}\},\emptyset,\emptyset,\}},\\
&G_{x+\frac{1}{2}}G_{x-\frac{1}{2}}\mathsf{X}_x E_{x-\frac{1}{2}}E_{x+\frac{1}{2}} \in \mathcal{X}_{\mathbf{S}=\{\emptyset,S_-=\{C_{x+\frac{1}{2}},C_{x-\frac{1}{2}}\},\emptyset,\emptyset\}},
\end{aligned} \tag{34}$$

where $\emptyset$ is the empty set. We will always decompose operators into classes defined by words in this way.

We now characterize the multiplication of operators $X_{\mathbf{S}'} \in \mathcal{X}_{\mathbf{S}'}$ and $X_{\mathbf{S}} \in \mathcal{X}_{\mathbf{S}}$. The multiplication is defined as the multiplication of operators. By Eq. (29), the multiplication might result in zero, due to incompatible projectors. Whenever the product is nonzero, we have that

$$X_{\mathbf{S}'}X_{\mathbf{S}} \in \mathcal{X}_{\mathbf{S}'\mathbf{S}}, \tag{35}$$

where the multiplication (or concatenation) of words $\mathbf{S}'\mathbf{S}$ is defined as follows. As a set, we have $S'S = S' \cup S$. For each $\alpha \in S'S$, we assign it to exactly one of $(S'S)_{g,e,+,-}$. The rule of assignment is summarized in the "multiplication table" presented in Table I. An entry 0 signifies when $X_{\mathbf{S}'}X_{\mathbf{S}} = 0$, in which case we say that $\mathbf{S}'\mathbf{S}$ is undefined.

Finally, we note the following:

(1) In general, it holds that

$$[X_{\mathbf{S}}, X_{\mathbf{S}'}] = 0 \quad \text{whenever } S \cap S' = \emptyset. \tag{36}$$

(2) Given a quadruple $\mathbf{S}_i$, we write $(S_i)_{+,-,g,e}$ for its components. If the subscript is omitted, i.e., S_i, then we mean the union of components as in Eq. (28).

(3) If two words $\mathbf{S}$ and $\mathbf{S}'$ have $S \cap S' \neq \emptyset$, and $\mathbf{S}'' = \mathbf{S}'\mathbf{S}$ is defined, then

$$|S''| + |S \cap S'| \leq |S| + |S'|. \tag{37}$$

This relation will be used throughout.

B. Operator collections and their norms

Since we will consider extensive operators such as Hamiltonians, it is convenient to define norms that are insensitive to the total volume (sometimes also called intensive or local norms) and that encode the spatial-decay properties of the local Hamiltonian terms. We use several different norms in this paper, including (i) the **intensive Pauli norm** $|||\cdot|||_{\tilde{\mu}}$, defined in Eq. (16); (ii) the **intensive word norm** $||\cdot||_{\mu}$, defined below in Eq. (38); and (iii) the usual nonintensive **operator norm** $||\cdot||$, defined to be the largest singular value of the operator.

We will consider *operator collections* $O = (O_{\mathbf{S}})_{\mathbf{S}}$, indexed by words $\mathbf{S}$ and such that $O_{\mathbf{S}} \in \mathcal{X}_{\mathbf{S}}$. We define the **intensive word norm** $||\cdot||_{\mu}$ of an operator collection:

$$||O||_{\mu} = \sup_{\alpha} \sum_{\mathbf{S}:\alpha\in S} ||O_{\mathbf{S}}|| \cdot e^{\mu|S|}, \tag{38}$$

for some parameter $\mu \geq 0$. Note that this is a different norm (distinguished by $||_{\mu}$ rather than $|||_{\mu}$) from the intensive Pauli norm given in Eq. (16).

To such an operator collection, we associate an *extensive operator*

$$O = \sum_{\mathbf{S}} O_{\mathbf{S}}, \tag{39}$$

which is denoted by the same symbol. Note that

$$||O|| \leq N_c \cdot ||O||_0, \tag{40}$$

where N_c is the number of terms in the Hamiltonian (the number of checks); $N_c := |\mathcal{E}|$.

It is important to realize that an operator collection determines the extensive operator via the above sum in Eq. (39) but the converse does not hold. Therefore, whenever we add extensive operators or take their commutator, the associated operator collection should be explicitly defined, as we do now. Let us have a pair of operator collections $(O_{\mathbf{S}})_{\mathbf{S}}$ and $(O'_{\mathbf{S}})_{\mathbf{S}}$:

(1) We define the interaction corresponding to the sum $O + O'$ by point-wise addition, i.e.,

$$(O + O')_{\mathbf{S}} = (O_{\mathbf{S}} + O'_{\mathbf{S}}). \tag{41}$$

(2) We define the interaction corresponding to the commutator as

$$([O, O'])_{\mathbf{S}} = \sum_{\substack{\mathbf{S}_1,\mathbf{S}_2 \\ \mathbf{S}=\mathbf{S}_1\mathbf{S}_2, S_1\cap S_2\neq\emptyset}} (O_{\mathbf{S}_1}O'_{\mathbf{S}_2} - O'_{\mathbf{S}_1}O_{\mathbf{S}_2}), \tag{42}$$

where we define the summand to be zero whenever $\mathbf{S}_1\mathbf{S}_2$ is not well defined.

C. Initial choice of operator collections

We write the operator $H_0 = \sum_\alpha E_\alpha$ as an operator collection through the choice

$$(H_0)_{\mathbf{S}} = \begin{cases} E_\alpha, & (S_+ = \emptyset, S_- = \emptyset, S_e = \{\alpha\}, S_g = \emptyset), \\ 0, & \text{otherwise.} \end{cases} \tag{43}$$

For the perturbation $Z^{(0)}$, we will first write it as a sum of Paulis, $Z^{(0)} = \sum_p c_p p$. We break each given p into an operator collection $p = \sum_{\mathbf{S}} p_{\mathbf{S}}$. To construct this decomposition for a given p, we first find its qubit support $\mathrm{supp}(p)$. We then find a minimal connected set $\mathcal{M}(p)$ of qubits such that $\mathrm{supp}(p) \subset \mathcal{M}(p)$. The $\mathbf{S}$ that are included in the sum have $S = \mathrm{ext}(p) := \cup_{x \in \mathcal{M}(p)} \mathrm{supp}_c(x)$. Each $p_{\mathbf{S}}$ is constructed by directly sandwiching p by the left($\mathbf{S}$) and right($\mathbf{S}$) operators in Sec. III A. Finally, we choose $(Z^{(0)})_{\mathbf{S}} = \sum_{p:\mathrm{ext}(p)=S} c_p p_{\mathbf{S}}$.

This construction ensures that at the initial step, all operator collections are defined so that if $(H_0)_{\mathbf{S}}$ and/or $(Z^{(0)})_{\mathbf{S}}$ are nonzero, then S is a connected set relative to the graph of stabilizers.

D. Relationships between the Pauli norm and the word norm

We have introduced two norms; $|||O|||_\mu$, which is unique, and $||O||_\mu$, which depends on the choice of operator collection used to describe O. While most of our work is in terms of $||O||_\mu$, we sometimes need to change between $|||O|||_\mu$ and $||O||_\mu$ at the cost of a small increase in μ.

Proposition 1. Let O be an operator collection constructed analogously to that of $Z^{(0)}$. Then,

$$||O||_\mu \leq |||O|||_{w_q \mu + (w_q \log(2) + \kappa + \log(2w_c))}. \tag{44}$$

Proposition 2. Let O be an operator the choice of operator collection of which has all S connected. Then,

$$|||O|||_\mu \leq ||O||_{w_c \mu + \log(4w_q)}. \tag{45}$$

We prove these propositions in Appendix B.

E. TQO conditions restated in terms of words

We may restate the **TQO** conditions from Sec. II B in terms of words. These are equivalent to **TQO-I** and **TQO-II**:

TQO-I Let the word $\mathbf{S}$ satisfy $|S| < d/w_c$. If $O_{\mathbf{S}} \in \mathcal{X}_{\mathbf{S}}$ satisfies $[O_{\mathbf{S}}, g] = 0$ for all $g \in \mathcal{G}$, then $O_{\mathbf{S}} \in \mathcal{A}_{\mathcal{G}}$, where $\mathcal{A}_{\mathcal{G}}$ is the algebra generated by $\mathcal{G}$ with complex coefficients. In other words, there are no logicals supported on S.

TQO-II Let the word $\mathbf{S}$ be such that S is connected in the check graph and $|S| < \widetilde{d}$. If $O_{\mathbf{S}} \in \mathcal{A}_{\mathcal{G}}$, then $O_{\mathbf{S}} \in \mathcal{A}_{\mathcal{G}_{B_{\ell(\mathbf{S})}}}$, where we recall that

$$B_{\ell|S|}(S) = \{\alpha : \mathrm{dist}(\alpha, S) \leq \ell |S|\}. \tag{46}$$

Equivalently to Sec. II B, we define $d_* := \min(d/w_c, \widetilde{d})$. As noted there, the theorems in Sec. II D are meaningful, i.e., they give a stable gap and ground-state degeneracy if $d_* \geq c \log(N)$ for $c > 0$.

IV. SCHEME OF THE PROOF

A. Outline of the proof

In this section, we describe the iterative KAM procedure. We first give an informal outline of the proof strategy.

As we detail in Sec. IV B, at each stage of the procedure, we split the Hamiltonian into four parts: H_0, frustration-free terms ($D^{(n)}$ and $M^{(n)}$) that share the ground-state space of H_0, non-frustration-free terms ($V^{(n)}$), and errors ($E^{(n)}$). The error term $E^{(n)}$ only contains words of weight $|S| \geq d_*$, whereas the others only contain words of weight $|S| < d_*$. The distinction between $D^{(n)}$ and $M^{(n)}$ is technical and we do not discuss this explicitly here.

With this splitting, we formally construct a unitary rotation (in particular, its generator $A^{(n)}$; see Sec. V) to eliminate the non-frustration-free term $V^{(n)}$ to its leading order, in a sense made precise in Sec. IV C. Formally, this rotates the Hamiltonian into a new basis, in which the non-frustration-free term $V^{(n+1)}$ is expected to be subleading compared to $V^{(n)}$.

Assuming conditions stated in Sec. II D, we can obtain control over the locality of $A^{(n)}$ for small $V^{(n)}$ (Proposition 6). With this, we further control the norm of the rotated operators, where Proposition 7 is used extensively. The locality of operators is characterized by the word norm $||\dots||_\mu$ and the iterations of the word norms are captured by "flow equations" (see Sec. IV D). Iterating the flow equations, we can show that: (1) $D^{(n)} + M^{(n)}$ retain a finite word norm; (2) $V^{(n)}$ is weakened double exponentially with n; and (3) $E^{(n)}$ remains small (Sec. IV E).

This procedure is carried out iteratively until a scale n_*, where $V^{(n_*)}$ is subleading to $E^{(n_*)}$ and further rotations will no longer be necessary. We use the bounds on each of the terms to obtain control over the spectrum of the perturbed Hamiltonian, as detailed in Secs. IV F and IV G, thereby proving Theorem 1. Finally, in Sec. IV H, we relate the ground-space projectors of the perturbed and unperturbed Hamiltonians with a locality-preserving unitary rotation, thereby proving Theorem 2.

B. Splitting of Hamiltonians

The initial Hamiltonian is written as

$$H = H^{(0)} = H_0 + Z^{(0)}. \tag{47}$$

At each stage, we will have the Hamiltonian written (after rotation) as

$$\begin{aligned} H^{(n+1)} &= \exp\{-i\mathrm{ad}_{A^{(n)}}\}(H^{(n)}) \\ &= H_0 + Z^{(n+1)} + E^{(n+1)} \\ &= H_0 + D^{(n+1)} + V^{(n+1)} + M^{(n+1)} + E^{(n+1)}, \end{aligned} \tag{48}$$

where $\mathrm{ad}_A(B) = [A, B]$ and where $Z^{(n)}$ will be given as an operator collection, i.e.,

$$Z^{(n)} = \sum_{\mathbf{S}} Z^{(n)}_{\mathbf{S}} \tag{49}$$

and similarly for $D^{(n)}, V^{(n)}, M^{(n)}$, with, moreover, a splitting $V^{(n)} = (V^{(n)})^+ + (V^{(n)})^-$. The latter operator collections have particular conditions on the words $\mathbf{S}$ that they contain. We state these conditions here, dropping the superscript n:

(1) Every one of V^+, V^-, M, and D contains only words $\mathbf{S}$ such that $|S| < d_*$.
(2) V^+ contains only words $\mathbf{S}$ such that $S_e = S_- = \emptyset$ but $S_+ \neq \emptyset$.
(3) V^- contains only words $\mathbf{S}$ such that $S_e = S_+ = \emptyset$ but $S_- \neq \emptyset$.
(4) M contains only words $\mathbf{S}$ such that $S_e = S_+ = S_- = \emptyset$ (but hence $S_g \neq \emptyset$).
(5) D contains only words $\mathbf{S}$ for which both $S_e \cup S_- \neq \emptyset$ and $S_e \cup S_+ \neq \emptyset$.

A bit of thought shows that the every word $\mathbf{S}$ with $S < d$ is therefore assigned to one (and only one) of V^+, V^-, M, and D. This is how we will define in practice the operator collections V^+, V^-, M, and D, starting from an operator collection Z.

C. A KAM step

Let us put an index (n) (scale) on all of these terms except H_0 and then the generator of rotations $A^{(n)}$ is defined such that it satisfies

$$\overline{P}(i[H_0 + D^{(n)} + M^{(n)}, A^{(n)}] + V^{(n)})P = 0, \tag{50}$$

where P is the ground-state projector (cf. Sec. II A) and $\overline{P} = (1 - P)$:

$$P(i[H_0 + D^{(n)} + M^{(n)}, A^{(n)}] + V^{(n)})\overline{P} = 0. \tag{51}$$

This leads to definitions of all terms, as follows. Define $\widetilde{Z}^{(0)} = Z^{(0)}$ and

$$\widetilde{Z}^{(n+1)} \equiv Z^{(n)} + \sum_{k=1}^{\infty} \frac{1}{k!}(-i\mathrm{ad}_{A^{(n)}})^k (H_0 + Z^{(n)}). \tag{52}$$

This gives a unique definition of the operator $\widetilde{Z}^{(n+1)}$. By the rules stated in Sec. III B, the expression in Eq. (52) also defines an operator collection $\widetilde{Z}^{(n+1)}$. Next, we construct $Z^{(n+1)}$ so as to satisfy Eq. (48), namely,

$$Z^{(n+1)}_{\mathbf{S}} = \begin{cases} \widetilde{Z}^{(n+1)}_{\mathbf{S}}, & \mathbf{S} : |S| < d_*, \\ 0, & \text{otherwise.} \end{cases} \tag{53}$$

The operator collections $D^{(n+1)}$, $V^{(n+1)}$, and $M^{(n+1)}$ are now defined as indicated in subsection Eq. (IV B), as restrictions of $Z^{(n+1)}$ to the appropriate words $\mathbf{S}$. The crucial thing to note here is that, by virtue of Eqs. (50) and (51), the operator collection $V^{(n+1)}$ receives only contributions from (in other words, "is the restriction of")

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{1}{k!}(-i\mathrm{ad}_{A^{(n)}})^k (V^{(n)}) \\ &\quad + \sum_{k=2}^{\infty} \frac{1}{k!}(-i\mathrm{ad}_{A^{(n)}})^k (H_0 + D^{(n)} + M^{(n)}). \end{aligned} \tag{54}$$

The error operator is then defined as

$$E^{(n+1)} = \exp\{-i\mathrm{ad}_{A^{(n)}}\}E^{(n)} + \sum_{\mathbf{S}:|S|\geq d_*} (\widetilde{Z}^{(n+1)})_{\mathbf{S}} \tag{55}$$

$$= \exp\{-i\mathrm{ad}_{A^{(n)}}\}E^{(n)} + \sum_{\mathbf{S}:|S|\geq d_*} (\widetilde{Z}^{(n+1)} - Z^{(n)})_{\mathbf{S}}. \tag{56}$$

In Sec. IV E, we bound the operator norm of $E^{(n)}$ using that it contains high-weight operators that will be highly suppressed by virtue of the finite intensive word norms.

D. Bounds on running coupling constants

We start from the perturbation strength

$$\epsilon_0 = ||Z^{(0)}||_{\mu_0}, \tag{57}$$

implying also

$$||V^{(0)}||_{\mu_0} \leq \epsilon_0, \quad ||D^{(0)}||_{\mu_0} \leq \epsilon_0, \quad ||M^{(0)}||_{\mu_0} \leq \epsilon_0, \tag{58}$$

and we define the running coupling constants

$$\eta_n := ||D^{(n)}||_{\mu_n} + ||M^{(n)}||_{\mu_n}, \quad \epsilon_n := ||V^{(n)}||_{\mu_n}, \quad n \geq 1 \tag{59}$$

for a decreasing sequence $(\mu_n)_{n=0,1,\ldots}$ that will be fixed below. Let us now derive some flow equations or, rather,

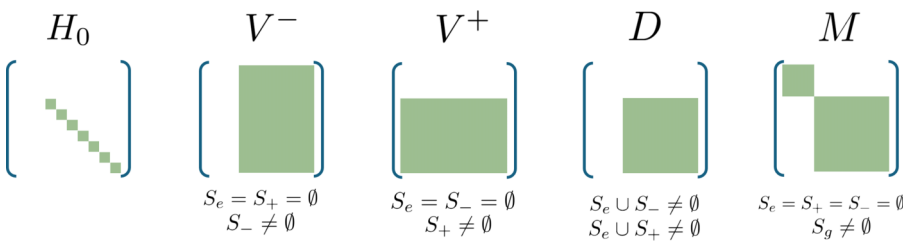


FIG. 2. Depiction of the breakdown of $H^{(n)}$ into H_0, $D^{(n)}$, $(V^{(n)})^+$, $(V^{(n)})^-$, and $M^{(n)}$ in the eigenbasis of H_0, with the upper-left block corresponding to the ground states of H_0. The white regions of the matrices correspond to entries that are necessarily zero by the definitions of these operator collections. When the context is clear, we sometimes omit the superscript (n) for brevity. The aim of the KAM procedure is to rotate away V^+ and V^- into D and M (Sec. IV C). Although D and M are both block diagonal, they contain different words, and we handle M with special care when constructing the generator of rotations (Sec. V). We maintain control over the locality of the rotated operators (Secs. VI and VII) in the sense of the intensive word norm; this control over locality has several benefits. By **TQO-I**, only sufficiently nonlocal operators can be logical operators and hence cause splitting between the ground states. We throw nonlocal operators into E (not pictured, can have any entry nonzero) and we bound the operator norm of E through our control over the locality, allowing us to upper bound the splitting. As an aside, **TQO-I** ensures that M is a linear combination of stabilizers (Sec. VIII) and is then strictly diagonal in the eigenbasis of H_0. Control over locality also plays an important role in arguing (Sec. VIII) that the lower-right blocks of D and M corresponding to the excited states of H_0 cannot affect the low-energy spectrum of $H_0 + D + M$ too much; in particular, in the Hamiltonian $H_0 + D + M$, this block cannot shift the energies of the excited states of H_0 below those of the ground states of H_0.

"flow inequalities," for these running coupling constants. For these to be valid, we need that μ_n is larger than the critical value μ_* [see Eq. (19)], as this is a prerequisite for Proposition 6, which will be used below. We therefore assume that $\mu_n > \mu_*$ for any n and we will fix μ_n more precisely below.

1. Derivation of flow equations

By Proposition 6, we have

$$||A^{(n)}||_{\mu_n} \leq \frac{2\epsilon_n}{1 - e\eta_n}, \tag{60}$$

provided that $e\eta_n < 1$. To obtain a bound on ϵ_{n+1}, we start from Eq. (54) and we use Eq. (60) and Proposition 7. This yields

$$\begin{aligned}\epsilon_{n+1} \leq & \sum_{k=1}^{\infty} \frac{(k+1)!}{k!} \frac{(4e\epsilon_n(1-e\eta_n)^{-1})^k}{|\mu_n - \mu_{n+1}|^{k+1}} \epsilon_n \\ & + \sum_{k=2}^{\infty} \frac{(k+1)!}{k!} \frac{(4e\epsilon_n(1-e\eta_n)^{-1})^k}{|\mu_n - \mu_{n+1}|^{k+1}} (e^{\mu_0} + \eta_n),\end{aligned} \tag{61}$$

where we have also used the equality $||H_0||_{\mu_0} = e^{\mu_0}$. To bound η_n, we directly use Eq. (52) to obtain, again using Propositions 6 and 7,

$$\begin{aligned}\eta_{n+1} \leq & \ \eta_n + \epsilon_n \\ & + \sum_{k=1}^{\infty} \frac{(k+1)!}{k!} \frac{(4e\epsilon_n(1-e\eta_n)^{-1})^k}{|\mu_n - \mu_{n+1}|^{k+1}} (e^{\mu_0} + \eta_n + \epsilon_n).\end{aligned} \tag{62}$$

We will now fix the decay rates μ_n. As already stated, we need $\mu_n > \mu_*$ and so a convenient choice is

$$\mu_n - \mu_{n+1} = \frac{3}{\pi^2}(\mu_0 - \mu_*)\frac{1}{(n+1)^2}, \tag{63}$$

which ensures that

$$\mu_n \geq \mu_\infty := \mu_* + (\mu_0 - \mu_*)/2, \tag{64}$$

for all n.

Our most important result is then Proposition 3 below. It is derived from the above estimates by inspection.

Proposition 3. Let $\mu_0 > \mu_*$ and let μ_n be fixed as explained above. There is an $\epsilon(\mu_0, \mu_*) > 0$, such that, if $\epsilon_0 \leq \epsilon(\mu_0, \mu_*)$, then

$$\eta_n \leq 2\epsilon_0, \quad \epsilon_n \leq \epsilon_0^{(3/2)^n}. \tag{65}$$

We now connect this to the assumptions made in our results.

Proposition 4. The conditions of Proposition 3 follow from the conditions of Theorem 1.

Proof. For ease, define $f(z) = (1/w_q)(z - (w_q \log(2) + \kappa + \log(2w_c)))$. Note that $\widetilde{\mu}_0 > \mu_\# \leftrightarrow f(\widetilde{\mu}_0) > \mu_*$, and $|||Z^{(0)}|||_{\widetilde{\mu}_0} < \widetilde{\epsilon}(\widetilde{\mu}_0) \to ||Z^{(0)}||_{f(\widetilde{\mu}_0)} < \widetilde{\epsilon}(\widetilde{\mu}_0)$ by Proposition 1. In Theorem 1, we choose $\widetilde{\epsilon}(\widetilde{\mu}_0, \mu_\#) := \epsilon(f(\widetilde{\mu}_0), f(\mu_\#)) = \epsilon(f(\widetilde{\mu}_0), \mu_*)$. Then, the conditions of Theorem 1 being satisfied imply that the conditions of Proposition 3 are satisfied with the choice $\mu_0 = f(\widetilde{\mu}_0)$ and $\epsilon_0 = ||Z^{(0)}||_{\mu_0}$. ■

E. Bounds on error terms $E^{(n)}$

Since the error term is the only term that can contain logical operators, it is ultimately responsible for the splitting between ground states of the rotated Hamiltonian. We can strongly bound the error term E, starting from Eq. (56):

$$||E^{(n+1)}|| \leq ||E^{(n)}|| + \sum_{\mathbf{s}:|S|\geq d_*} ||(\widetilde{Z}^{(n+1)} - Z^{(n)})_{\mathbf{s}}|| \tag{66}$$

$$\leq ||E^{(n)}|| + N_c e^{-\mu_{n+1} d_*} ||\widetilde{Z}^{(n+1)} - Z^{(n)}||_{\mu_{n+1}}. \tag{67}$$

Our bound for $||\widetilde{Z}^{(n+1)} - Z^{(n)}||_{\mu_{n+1}}$ is the same as the right-hand side of Eq. (62) but without the term $(\eta_n + \epsilon_n)$ (because now we are subtracting $Z^{(n)}$). By using Proposition 3, we can bound the resulting series as

$$||\widetilde{Z}^{(n+1)} - Z^{(n)}||_{\mu_{n+1}} \leq C n^4 \epsilon_n. \tag{68}$$

Therefore,

$$||E^{(n+1)}|| \leq ||E^{(n)}|| + C N_c e^{-\mu_n d_*} n^4 \epsilon_n. \tag{69}$$

Since $E^{(0)} = 0$, and using Proposition 3 ($\epsilon_n \leq \epsilon_0^{(3/2)^n}$), we obtain the uniform-in-n bound

$$||E^{(n)}|| \leq C N_c e^{-\mu_\infty d_*} \epsilon_0. \tag{70}$$

In principle, we could run the renormalization scheme an infinite number of times, to obtain the Hamiltonian

$$H^{(\infty)} = H_0 + D^{(\infty)} + M^{(\infty)} + E^{(\infty)}, \tag{71}$$

i.e., where the V term has been completely eliminated. However, there is no need to do this, since after n_* KAM steps, with n_* the smallest natural number such that

$$\epsilon_{n_*} \leq C' \epsilon_0 e^{-\mu_\infty d_*}, \tag{72}$$

we already obtain the operator $V^{(n_*)}$, such that $||V^{(n_*)}|| \leq ||E^{(n_*)}|| \leq C N_c \epsilon_0 e^{-\mu_\infty d_*}$, which leads to the same result.

F. Relative boundedness

Up to now, we have passed from $H^{(0)}$ to $H^{(n_*)}$. By construction, the operators

$$H_0 + D^{(n)} \tag{73}$$

satisfy

$$(H_0 + D^{(n)})P = P(H_0 + D^{(n)}) = 0. \tag{74}$$

Once one adds the term $M^{(n)}$, this is no longer true, as the energy of the ground-state sector shifts. Either way, one needs a nontrival argument to show that the spectral gap of $H_0 + D^{(n)} + M^{(n)}$ remains open. For any operator O, we define the ground-space expectation

$$\langle O \rangle = \frac{\mathrm{Tr} POP}{\mathrm{Tr} P}. \tag{75}$$

Then,

$$K^{(n)} = H_0 + D^{(n)} + M^{(n)} - \langle M^{(n)} \rangle \tag{76}$$

also satisfies

$$K^{(n)} P = P K^{(n)} = 0. \tag{77}$$

This is because every term $M_{\mathbf{s}}$ satisfies

$$P M_{\mathbf{s}} = M_{\mathbf{s}} P = \langle M_{\mathbf{s}} \rangle P \tag{78}$$

as a direct consequence of **TQO-I**. Furthermore, it is manifest [cf. Eq. (40)] that

$$|\langle M^{(n)} \rangle| \leq ||M^{(n)}||_0 N_c \leq C \epsilon_0 N_c, \tag{79}$$

uniformly in n, so the shift in energy density is small. To prove that the gap remains open, we establish the following.

Proposition 5. The operator $K^{(n)} - H_0$ is relatively bounded with respect to H_0, with relative bound $C_0 \epsilon_0$. More precisely, it holds that

$$||(K^{(n)} - H_0)\psi|| \leq C_0 \epsilon_0 ||H_0 \psi|| \tag{80}$$

for any $\psi \in \mathcal{H}_\Lambda$ and a constant C_0 uniform in n.

This proposition is proven in Sec. VIII. As is well known, relative boundedness leads to some stability of the spectrum. In particular, one deduces that the resolvent $(z - K^{(n)})^{-1}$ enjoys the bound

$$||(z - K^{(n)})^{-1}|| \leq \frac{1}{r_z} \frac{1}{1 - C_0 \epsilon (1 + (|z|/r_z))}, \qquad r_z = \mathrm{dist}(z, \mathrm{spec}(H_0)). \tag{81}$$

Given that

$$\mathrm{spec}(H_0) \subset \{0, 1, 2, \ldots\},$$

we deduce that, provided that $C_0 \epsilon_0 < 1$ is small enough, the spectrum of $K^{(n)}$ is contained in disks centered on $m = 0, 1, 2, \ldots$ with radius $C m \epsilon_0$. In particular, if $K^{(n)}$ is Hermitian, then

$$\mathrm{spec}(K^{(n)}) \subset \bigcup_{m=0,1,2,\ldots} [(1 - C\epsilon_0)m, (1 + C\epsilon_0)m]. \tag{82}$$

Note that we have already explicitly checked that 0 is an eigenvalue in Eq. (77) and the above inclusion says, moreover, that 0 is an isolated part of the spectrum.

G. Effect of $E^{(n)}$ on spectrum

Finally, we discuss the spectral problem for the operator $H^{(n_*)}$, which we cast as

$$H^{(n_*)} = K^{(n_*)} + (V^{(n_*)} + E^{(n_*)}) + \langle M^{(n_*)} \rangle. \tag{83}$$

The last term is a multiple of identity and the operator $(V^{(n_*)} + E^{(n_*)})$ is bounded as

$$||V^{(n_*)} + E^{(n_*)}|| \leq CN_c\epsilon_0 e^{-\mu_\infty d_*} =: \delta, \tag{84}$$

by the results of Sec. IV E. Hence, the resolvent $(z - H^{(n_*)} - \langle M^{(n_*)} \rangle)^{-1}$ exists for z such that

$$\delta < ||(z - K^{(n)})^{-1}||^{-1}.$$

By inspection of Eq. (81), we deduce that the spectrum of $H^{(n_*)} - \langle M^{(n_*)} \rangle$ is contained in disks centered on $m = 0, 1, 2, \ldots$ with radius $Cm\epsilon_0 + C\delta$. If all operators are Hermitian, then these results can be slightly strengthened (since in that case, $||(z - K^{(n)})^{-1}|| = (1/\text{dist})(z, \text{spec}(K^{(n)}))$ and we obtain that

$$\text{spec}(H^{(n_*)} - \langle M^{(n_*)} \rangle) \subset \bigcup_{m=0,1,2,\ldots} [(1 - C_0\epsilon_0)m - \delta, (1 + C_0\epsilon_0)m + \delta], \tag{85}$$

and since $\langle M^{(n_*)} \rangle$ is a number, this implies Theorem 1.

Moreover, it follows that, for m such that $(1 + C_0\epsilon_0)m + \delta < (1 - C_0\epsilon_0)(m + 1) - \delta$, the spectral patch contained in $[(1 - C_0\epsilon_0)m - \delta, (1 + C_0\epsilon_0)m + \delta]$ has the same cardinality as the unperturbed eigenvalue m of H_0.

H. The perturbed spectral projector

We recall that H_0 and $K^{(n_*)}$ have an isolated eigenvalue at zero, with corresponding spectral projector P. For the operator $H^{(n_*)}$, the eigenvalue and spectral projector are perturbed and we call the latter P'. By standard rigorous perturbation theory (see, e.g., Ref. [56]), we can construct an operator $A^{(\infty)}$ with

$$||A^{(\infty)}|| \leq CN_c\epsilon_0 e^{-\mu_\infty d_*} \tag{86}$$

such that

$$P' = \exp\{-i\text{ad}_{A^{(\infty)}}\}(P). \tag{87}$$

Ultimately, we are interested in the spectral projector $\widetilde{P}$ of the original Hamiltonian $H = H_0 + Z^{(0)}$. To relate it to the spectral projector P' of $H^{(n_*)}$, we simply need to undo the successive rotations, i.e.,

$$\begin{aligned}\widetilde{P} &= \exp\{-i\text{ad}_{A^{(0)}}\} \ldots \exp\{-i\text{ad}_{A^{(n_*)}}\}(P') \\ &= \exp\{-i\text{ad}_{A^{(0)}}\} \ldots \exp\{-i\text{ad}_{A^{(n_*)}}\} \exp\{-i\text{ad}_{A^{(\infty)}}\}(P) \\ &=: UPU^{-1}.\end{aligned} \tag{88}$$

The operator U in the last line can simply be defined as the product $e^{-iA^{(0)}} \ldots e^{-iA^{(n_*)}} e^{-iA^{(\infty)}}$. It is unitary if the perturbation is Hermitian, since then $A^{(n)}$ and $A^{(\infty)}$ can be chosen Hermitian. We now come to the proof of Theorems 2 and 3, i.e., we investigate the locality properties and analyticity of the map $O \mapsto UOU^{-1}$ and its inverse $O \mapsto U^{-1}OU$. For the sake of explicitness, let us focus on the latter, as the former is treated analogously. It is important that we allow the perturbation to be non-Hermitian, as we also did in the previous subsections. We use the shorthand

$$\mathcal{U}^{(j)} = \exp\{i\text{ad}_{A^{(j)}}\}, \quad \mathcal{U}^{(\infty)} = \exp\{i\text{ad}_{A^{(\infty)}}\}, \tag{89}$$

so that the object of interest reads

$$U^{-1}OU = \mathcal{U}^{(\infty)}\mathcal{U}^{(n_*)} \ldots \mathcal{U}^{(0)}(O). \tag{90}$$

We write this as

$$U^{-1}OU = O + (O' - O) + (\mathcal{U}^{(\infty)}(O') - O'), \tag{91}$$

where $O' = \mathcal{U}^{(n_*)} \ldots \mathcal{U}^{(0)}(O)$. The three terms on the right-hand side of Eq. (91) correspond to the three terms on the right-hand side of Eq. (24). The third term, which is hence identified as the background term O_{bg}, is bounded in norm by $C||A^{(\infty)}||||O'|| \leq CN_c\epsilon_0 e^{-\mu_\infty d_*}||O'||$ by using the smallness of $||A^{(\infty)}||$, i.e., Eq. (86). This will yield the claimed bound once we establish that $||O'|| \leq C||O||$, which will follow from the treatment below. Note that if the perturbation is Hermitian, we have $||O'|| = ||O||$ by unitarity.

To estimate $||O' - O||$, we will interpret O' and O as operator collections and we first establish locality preservation for $\mathcal{U}^{(n)} \ldots \mathcal{U}^{(0)}$ when acting on operator collections.

Lemma 1. For any operator collection B and any n,

$$||\mathcal{U}^{(n)} \ldots \mathcal{U}^{(0)}(B) - B||_{\mu_{n+1}} \leq C\epsilon_0||B||_{\mu_0}. \tag{92}$$

Proof. We write

$$\mathcal{U}^{(n)} \ldots \mathcal{U}^{(0)} - 1 = \sum_{j=0}^{n} (\mathcal{U}^{(n)} \ldots \mathcal{U}^{(j+1)})(\mathcal{U}^{(j)} - 1) \tag{93}$$

and we apply this map to an operator collection B. We use Eq. (148) to bound $(\mathcal{U}^{(j)} - 1)(B)$ in the norm $||\cdot||_{\mu_{j+1}}$

and then successively Eq. (149) to bound $\mathcal{U}^{(i)}(B')$ in the norm $||\cdot||_{\mu_{i+1}}$ with $i > j$ for the resulting operator collection B'. In applying these bounds, we exploit the estimate $||A^{(n)}||_{\mu_n} \leq C\epsilon_n$ [see Eq. (60)]. The lemma then follows by the fast decay of ϵ_n. ■

Finally, we pass from the action of $\mathcal{U}^{(0)} \ldots \mathcal{U}^{(n)}$ on operator collections to its action on local operators O supported in a given set Y. In particular, we want to argue that the second term on the right-hand side of Eq. (91) is exponentially quasilocal (which, as is argued, also settles the third term).

To that end, we encode O as an operator collection O according to the procedure for $Z^{(0)}$ in Sec. III C. We write $O = \sum_p c_p p$ and note that $|c_p| < ||O||$. We break each p into an operator collection $p = \sum_{\mathbf{S}} p_{\mathbf{S}}$. To construct this decomposition for a given p, we first find its qubit support $\mathrm{supp}(p)$ and a minimal connected set $\mathcal{M}(p)$ of qubits such that $\mathrm{supp}(p) \subset \mathcal{M}(p)$. We will choose $\mathcal{M}(p)$ entirely within Y, which we can do by the connectedness of Y; $|\mathcal{M}(p)| \leq |Y|$. The $\mathbf{S}$ that are included in the sum have $S = \mathrm{ext}(p)$ and, accordingly, $|S| \leq w_q|\mathcal{M}(p)| \leq w_q|Y|$. For a given p, there are at most $2^{\mathrm{ext}(p)}$ $\mathbf{S}$ such that $p_{\mathbf{S}}$ is nonvanishing and all of them satisfy $||p_{\mathbf{S}}|| \leq 1$. Together, these ensure that $||O||_{\mu_0} \leq ||O||2^{w_q|Y|}e^{\mu_0 w_q|Y|} = ||O||e^{(w_q\mu_0+\log(2))|Y|}$. By the above lemma, we know that

$$||\mathcal{U}^{(n_*)} \ldots \mathcal{U}^{(0)}(O) - O||_{\mu_n} \leq C\epsilon_0||O||e^{(w_q\mu_0+\log(2))|Y|}. \tag{94}$$

Moreover, by the explicit definition of commutators of operator collections in Eq. (42), we know that the support of every word $\mathbf{S}$ that contributes to $\mathcal{U}^{(n_*)} \ldots \mathcal{U}^{(0)}(O) - O$, intersects $\widetilde{Y}$. This yields that $\mathcal{U}^{(n_*)} \ldots \mathcal{U}^{(0)}(O) - O$ is exponentially localized around $\widetilde{Y}$ in the sense of words $\mathbf{S}$. Upon translating this to a decay on the qubit graph, we conclude the proof of Theorems 2 and 3.

V. CONSTRUCTION OF THE GENERATOR OF ROTATIONS A

We need to find $A = A^+ + A_-$ such that it satisfies the equations

$$\overline{P}(i[H_0 + D + M, A^+] + V^+)P = 0, \tag{95}$$

$$P(i[H_0 + D + M, A_-] + V_-)\overline{P} = 0, \tag{96}$$

where P projects on the ground-state sector and $\overline{P} = 1 - P$. This decomposition of A corresponds to choosing A^+ so that the off-block-diagonal part of V^+ is rotated away at lowest order and choosing A_- so that the off-block-diagonal part of V_- is rotated away at lowest order. Together, these ensure that the off-block-diagonal part of V is rotated away at lowest order. Furthermore, the decomposition is also useful for constraining the kinds of words $\mathbf{S}$ that appear in solutions to these equations.

In the following, we will only explicitly construct A^+ as the case of A_- is completely analogous. Indeed, if D and M are self-adjoint, then $A_- = (A^+)^\dagger$. In the general case, we set $A_- = (A'_+)^\dagger$, where A'_+ is the solution to

$$\overline{P}(i[H_0 + D^\dagger + M^\dagger, A'_+] + V^+)P = 0. \tag{97}$$

Note that there are many possible solutions to Eqs. (95) and (96). Our goal is to find a solution that has good locality properties, so that the KAM steps will not change the locality of the rotated Hamiltonian too much. In the following subsections, we will construct a formal solution A^+ satisfying Eq. (95) and then we check its convergence and its locality properties in Sec. VI.

A. Ghost terms

Recall that M is an operator collection that consists purely of ghosts satisfying the **TQO** condition in Sec. II B. Recall also from Sec. IV B that M only contains words $\mathbf{S}$ with $|S| < d_*$.

Lemma 2. Let $\mathbf{S}'$ be a word with only $+$ and g. Then, for any $X_{\mathbf{S}'} \in \mathcal{X}_{\mathbf{S}'}$, there is a complex number $q(\mathbf{S}', \mathbf{S})$ such that

$$[M_{\mathbf{S}}, X_{\mathbf{S}'}]P = q(\mathbf{S}', \mathbf{S})X_{\mathbf{S}'}P. \tag{98}$$

Moreover, provided that μ satisfies $4^{w_c+1}e^{\kappa\cdot\ell-\mu} < 1$, we have

$$\frac{1}{|S'_+|}\sum_{\mathbf{S}} |q(\mathbf{S}', \mathbf{S})| \leq ||M||_\mu. \tag{99}$$

Proof. The operator $M_{\mathbf{S}}$ can be decomposed in Pauli strings p such that $\mathrm{supp}_c(p) \subseteq S$ [cf. Eqs. (8) and (32)]. Therefore,

$$M_{\mathbf{S}} = \sum_{p:\mathrm{supp}_c(p)\subseteq S} c_p p. \tag{100}$$

Note that $||M_{\mathbf{S}}|| = \text{largest singular value of } M_{\mathbf{S}} \geq \sqrt{\text{average of the squared singular values}} = \sqrt{\sum_p |c_p|^2} \geq |c_p|, \forall p$. Therefore, each of the coefficients c_p is bounded as

$$|c_p| \leq ||M_{\mathbf{S}}||. \tag{101}$$

By construction, the operator $M_{\mathbf{S}}$ commutes with all checks $\{C_\alpha\}_{\alpha\in\mathcal{E}}$. Using that each Pauli string either commutes or anticommutes with C_α, we derive that any p for which $c_p \neq 0$ also commutes with all stabilizers and, therefore, because of **TQO-I**, such p belongs to $\mathcal{G}$. Next, we use **TQO-II** to recast every such p as a product

$\widetilde{p} = \prod_{\alpha \in \hat{S}(p)} C_\alpha$, where $\hat{S}(p) \subseteq B_{\ell|S|}(S)$. For each such product $\widetilde{p}$, we have

$$[p, X_{\mathbf{S}'}]P = [\widetilde{p}, X_{\mathbf{S}'}]P = \eta(1 - (-1)^{|\hat{S}(p) \cap S'_+|}) X_{\mathbf{S}'} P, \quad (102)$$

where $\eta \in \pm 1$. Therefore, Eq. (98) holds with

$$|q(\mathbf{S}', \mathbf{S})| \leq \sum_{p: \mathrm{supp}_c(p) \subseteq S} 2|c(p)| \leq 2 \times 4^{w_c|S|} ||M_{\mathbf{S}}||, \quad (103)$$

where we use that there are at most $w_c|S|$ qubits touching the stabilizers in S (and hence at most $4^{w_c|S|}$ Pauli strings p in the Pauli decomposition of $M_{\mathbf{S}}$). We also note that $q(\cdot, \cdot)$ vanishes whenever $\hat{S}(p) \cap S'_+ = \emptyset$. Therefore, we obtain Eq. (99) by

$$\begin{aligned}
\sum_{\mathbf{S}} |q(\mathbf{S}', \mathbf{S})| &\leq \sum_{\mathbf{S}: B_{\ell|S|}(S) \cap S'_+ \neq \emptyset} 2 \times 4^{w_c|S|} ||M_{\mathbf{S}}|| \\
&\leq \sum_{\alpha \in S'_+} \sum_{r=1}^{\infty} \sum_{\beta: \mathrm{dist}(\alpha,\beta) \leq \ell \cdot r} \sum_{\mathbf{S}: |S|=r, \beta \in S} 2 \times 4^{w_c r} ||M_{\mathbf{S}}|| \\
&\leq 2|S'_+| ||M||_\mu \sum_{r=1}^{\infty} e^{\kappa \cdot \ell \cdot r} (4^{w_c} e^{-\mu})^r \\
&\leq |S'_+| 2 \frac{4^{w_c} e^{-(\mu - \kappa l)}}{1 - 4^{w_c} e^{-(\mu - \kappa l)}} ||M||_\mu \\
&\leq |S'_+| \, ||M||_\mu, \quad (104)
\end{aligned}$$

which ends the proof. ■

We note that it is here that we use that H_0 is a stabilizer model.

B. A local solution of Eq. (95)

We consider a sequence

$$\mathbf{S}_i, \quad i = 0, \ldots, k \quad (105)$$

and, whenever it is well defined,

$$\mathbf{S}'_i = \mathbf{S}_i \mathbf{S}'_{i-1}, \quad i = 1, \ldots, k, \quad \mathbf{S}'_0 = \mathbf{S}_0. \quad (106)$$

We say that $\mathbf{S}_i$ is a ghost whenever $(S_i)_{\neg g} = \emptyset$, where $(S_i)_{\neg g} = S_i \setminus (S_i)_g$.

Definition 1. We say that a sequence $\mathbf{S}_{i=0,\ldots,k}$ is admissible if and oniy if:

(1) $\mathbf{S}'_i$, for $i = 0, \ldots, k$, are well defined.
(2) $\mathbf{S}_i$ is not a ghost.
(3) $(S_i)_e \cup (S_i)_- \neq \emptyset$. (i.e., $\mathbf{S}_i$ gives zero when right multiplied by P).
(4) $(S'_i)_e = (S'_i)_- = \emptyset$. (i.e., $\mathbf{S}'_i$ does not give zero when right multiplied by P).

We also need the following definition:

$$\Delta(\mathbf{S}') = \sum_{\text{ghost } \mathbf{S}: S \cap S' \neq \emptyset} \frac{1}{|S'_+|} q(\mathbf{S}', \mathbf{S}), \quad (107)$$

with $q(\cdot, \cdot)$ as defined previously, starting from M. We can then state the following lemma, which is for the moment formal, as it depends on absolute convergence of the series in k in a norm $|| \cdot ||_\mu$. Such convergence will be established in Sec. VI.

Lemma 3. If $\mu > \mu_*$ and $||M||_\mu \leq 1$, then the series

$$\begin{aligned}
A^+ = \sum_{k=0}^{\infty} (-1)^k \sum_{\substack{\mathbf{S}_0, \ldots, \mathbf{S}_k \\ \text{admissible}}} & (1 + \Delta(\mathbf{S}'_k))^{-1} \frac{1}{|(S'_k)_+|} \\
& \times D_{\mathbf{S}_k} (1 + \Delta(\mathbf{S}'_{k-1}))^{-1} \ldots \\
& \ldots \frac{1}{|(S'_2)_+|} D_{\mathbf{S}_2} (1 + \Delta(\mathbf{S}'_1))^{-1} \frac{1}{|(S'_1)_+|} \\
& \times D_{\mathbf{S}_1} (1 + \Delta(\mathbf{S}'_0))^{-1} \frac{1}{|(S'_0)_+|} V^+_{\mathbf{S}_0} \quad (108)
\end{aligned}$$

formally satisfies Eq. (95).

C. Another representation for A^+

To check Lemma 3, it is convenient to reexpand the summands in Lemma 3 and for that we need another definition.

Definition 2. We say that a sequence $\mathbf{S}_0, \ldots, \mathbf{S}_k$ is ghost admissible whenever the following holds:

(1) $\mathbf{S}_0$ is not a ghost.
(2) Let $\widetilde{\mathbf{S}}_{i=0,\ldots,l}$ be the restricted sequence obtained by dropping all ghosts from $\mathbf{S}_{0,\ldots,k}$ and relabeling so that the index set is contiguous. Then, $\widetilde{\mathbf{S}}_{i=0,\ldots,l}$ is admissible.

To a ghost-admissible sequence $\mathbf{S}_0, \ldots, \mathbf{S}_k$, we associate the sequence $\mathbf{S}'_i$ as follows:

$$\mathbf{S}'_i = \begin{cases} \mathbf{S}_0, & i = 0, \\ \mathbf{S}_i \mathbf{S}'_{i-1}, & i > 0 \quad \text{and} \quad \mathbf{S}_i \text{ is not a ghost}, \\ \mathbf{S}'_{i-1}, & i > 0 \quad \text{and} \quad \mathbf{S}_i \text{ is a ghost}. \end{cases} \quad (109)$$

We can now give the alternative form of the series for A^+, namely,

$$A^+ = \sum_{k=0}^{\infty} A_k^+, \qquad A_k^+ = \sum_{\substack{\mathbf{s}_0,\ldots,\mathbf{s}_k \\ \text{ghost admissible}}} (A_k^+)_{\mathbf{s}'_k}, \tag{110}$$

with

$$(A_k^+)_{\mathbf{s}'_k} = \frac{i}{|(S'_k)_+|}\widetilde{D}_{\mathbf{s}_k}\cdots\frac{1}{|(S'_2)_+|}\widetilde{D}_{\mathbf{s}_2}\frac{1}{|(S'_1)_+|} \times \widetilde{D}_{\mathbf{s}_1}\frac{1}{|(S'_0)_+|}V^+_{\mathbf{s}_0}, \tag{111}$$

where

$$\widetilde{D}_{\mathbf{s}_i} = \begin{cases} -D_{\mathbf{s}_i}, & \mathbf{S}_i \text{ is not a ghost}, \\ -q(\mathbf{S}'_{i-1},\mathbf{S}_i), & \mathbf{S}_i \text{ is a ghost}. \end{cases}. \tag{112}$$

By the conditions on μ, we derive from Lemma 2 that $\Delta(\mathbf{S}') < 1$ for any $\mathbf{S}'$. We can now re-sum the sum of ghost terms in Eq. (110) between any pair of consecutive nonghost terms, as $\sum_{j=0}^{\infty}(-\Delta(\mathbf{S}'))^j = (1+\Delta(\mathbf{S}'))^{-1}$. This shows that the above series is the same as the one for A^+ given in Lemma 3, provided that both converge absolutely (cf. the remark before Lemma 3).

It remains to check that Eq. (111) satisfies Eq. (95). This happens order by order, i.e., we claim

$$\overline{P}([H_0,A_k^+]+[D,A_{k-1}^+]+[M,A_{k-1}^+])P = 0, \quad k>0 \tag{113}$$

and

$$\overline{P}(i[H_0,A_0^+]+V^+)P = 0. \tag{114}$$

The latter equation is directly verified from Eq. (111) and we focus on the former. From here on, the index k is traded for m. A first observation [from inspecting Eq. (110) and using the definition of Δ] is that

$$(A_k^+)_{\mathbf{s}'_k} = \frac{1}{|(S'_k)_+|}\left(\sum_{\mathbf{s}'_{k-1},\mathbf{s}_k:\mathbf{S}'_k=\mathbf{s}_k\mathbf{s}'_{k-1}} D_{\mathbf{s}_k}(A_{k-1}^+)_{\mathbf{s}'_{k-1}}\right) - (A_{k-1}^+)_{\mathbf{s}'_k}\Delta(\mathbf{S}'_k) \tag{115}$$

where the sequence is admissible, i.e., $\mathbf{S}_k$, $\mathbf{S}'_{k-1}$, and $\mathbf{S}'_k$ satisfy the conditions of Definition 1.

To check that Eq. (113) is satisfied, we observe

(1)

$$\begin{aligned}\overline{P}[M,(A_{k-1}^+)_{\mathbf{S}'_{k-1}}]P &= \overline{P}(A_{k-1}^+)_{\mathbf{S}'_{k-1}}P \times |(S'_{k-1})_+|\Delta(\mathbf{S}'_{k-1}) \\ &= \overline{P}(A_{k-1}^+)_{\mathbf{S}'_k}P \times |(S'_k)_+|\Delta(\mathbf{S}'_k),\end{aligned} \tag{116}$$

where, in the last equation, we have used the convention $\mathbf{S}'_k = \mathbf{S}'_{k-1}$ whenever $\mathbf{S}_k$ is a ghost.

(2)

$$\overline{P}[D,(A_{k-1}^+)_{\mathbf{S}'_{k-1}}]P = \sum_{\mathbf{S}_k}\overline{P}D_{\mathbf{S}_k}(A_{k-1}^+)_{\mathbf{S}'_{k-1}}P, \tag{117}$$

because all terms in D give zero when right multiplied with P. The important thing here is to realize that on the right-hand side, one can restrict to $\mathbf{S}_k, \mathbf{S}'_{k-1}$ such that the sequence is admissible.

(3)

$$\overline{P}([H_0,(A_k^+)_{\mathbf{S}'_k}])P = |(S'_k)_+|\,\overline{P}(A_k^+)_{\mathbf{S}'_k}P. \tag{118}$$

With these observations, we finally confirm the validity of Eq. (113).

VI. LOCALITY OF A

We will establish locality of the operator collection A, which was constructed in Sec. V. As explained before, it suffices to consider A^+. We will prove the following.

Proposition 6. Provided that μ satisfies $\mu > \mu_* > 1$, and $e||D||_\mu - ||M||_\mu < 1$, we have

$$||A^+||_\mu \le ||V^+||_\mu \frac{1-||M||_\mu}{1-e||D||_\mu - ||M||_\mu}. \tag{119}$$

A. Preliminary considerations

From Lemmas 2 and 3, we obtain

$$\begin{aligned}&||A^+||_\mu \\ &\le \sum_{k=0}^{\infty}(1-||M||_\mu)^{-k}\Bigg[\sup_\alpha \sum_{\substack{\mathbf{s}_0,\ldots,\mathbf{s}_k \\ \text{admissible} \\ \alpha\in S'_k}} \frac{1}{|(S'_k)_+|}||D_{\mathbf{s}_k}|| \\ &\cdots\frac{1}{|(S'_2)_+|}||D_{\mathbf{s}_2}||\frac{1}{|(S'_1)_+|}||D_{\mathbf{s}_1}||\frac{1}{|(S'_0)_+|}||V^+_{\mathbf{s}_0}||\,e^{\mu|S'_k|}\Bigg].\end{aligned} \tag{120}$$

We now decompose the size $|S'_k|$ by using the enhanced subadditivity in Eq. (37), which yields

$$|S'_k| \leq \sum_{i=0}^{k} |S_i| - \sum_{i=1}^{k} |S_i \cap S'_{i-1}|.$$

Using this, we can bound the summand in Eq. (120) by

$$z(\mathbf{S}_0, \ldots, \mathbf{S}_k) := \frac{e^{-\mu|S_k \cap S'_{k-1}|}}{|(S'_k)_+|} d_{\mathbf{S}_k} \ldots \frac{e^{-\mu|S_1 \cap S'_0|}}{|(S'_1)_+|} d_{\mathbf{S}_1} \frac{1}{|(S'_0)_+|} v_{\mathbf{S}_0}, \quad (121)$$

where we have tried to reduce clutter by abbreviating

$$d_{\mathbf{S}_i} = ||D_{\mathbf{S}_i}|| e^{\mu|S_i|}, \quad v_{\mathbf{S}_0} = ||V^+_{\mathbf{S}_0}|| e^{\mu|S_0|}. \quad (122)$$

So we have to bound

$$z(k) := \sup_{\alpha} \sum_{\substack{\mathbf{S}_0, \ldots, \mathbf{S}_k \\ \text{admissible} \\ \alpha \in S'_k}} z(\mathbf{S}_0, \ldots, \mathbf{S}_k). \quad (123)$$

We resolve the condition $x \in S'_k$ by requiring that $x \in S_j$ and summing over j. The sum with the condition $x \in S_j$ is denoted by

$$z(k) \leq \sum_{j=0}^{k} \sup_{\alpha} \sum_{\substack{\mathbf{S}_0, \ldots, \mathbf{S}_k \\ \text{admissible} \\ \alpha \in S_j}} z(\mathbf{S}_0, \ldots, \mathbf{S}_k) =: \sum_{j=0}^{k} z_{k,j}. \quad (124)$$

Here is what we will actually prove.

Lemma 4. For any $0 < j \leq k$,

$$z_{k,j} \leq z(j-1) ||D||_\mu^{k-j+1}. \quad (125)$$

For $j = 0$, we have

$$z_{k,0} \leq ||D||_\mu^k ||V^+||_\mu. \quad (126)$$

Lemma 4 is proven below. For now, we use it to give the following.

Proof of Proposition 6. Using $z(k) \leq \sum_{j=0}^{k} z_{k,j}$ and Lemma 4, we can inductively bound $z(k)$ as

$$\frac{z(k)}{||D||_\mu^k ||V^+||_\mu} \leq \sum_{p=1}^{k} \frac{k^p}{p!} \leq e^k. \quad (127)$$

Then, provided that $e||D||_\mu + ||M||_\mu < 1$, we estimate

$$||A^+||_\mu \leq \sum_{k=0}^{\infty} e^k \left(\frac{||D||_\mu}{1 - ||M||_\mu} \right)^k ||V^+||_\mu \leq ||V^+||_\mu \frac{1 - ||M||_\mu}{1 - e||D||_\mu - ||M||_\mu}. \quad (128)$$

■

B. Proof of Lemma 4

1. Some preparatory work

We now state a few remarks that will be used. They are specific to the sets $\mathbf{S}_0, \ldots, \mathbf{S}_k$ appearing in our construction. In particular, we assume that the sequence is admissible. We start with two anchoring equations:

$$S_i \cap (S'_{i-1})_+ \neq \emptyset \quad (129)$$

and

$$(S_i)_{\neg g} \cap S'_{i-1} \neq \emptyset. \quad (130)$$

Both follow in a rather direct way from the definition of an admissible sequence. The first equation will be used to anchor S_i in $(S'_{i-1})_+$ and it will be used often. The second equation is used to anchor S'_{i-1} in $(S_i)_{\neg g}$ and it will only be used at the end of Sec. VI B 3.

Lemma 5. If $\mu > 1$, then

$$\frac{e^{-\mu|S_k \cap S'_{k-1}|}}{|(S'_k)_+|} \leq \frac{1}{|(S_k)_{\neg g}|}. \quad (131)$$

Proof. We note that

$$(S_k)_{\neg g} \subseteq (S'_k)_+ \cup (S_k \cap S'_{k-1}). \quad (132)$$

Indeed, $S_k \setminus S'_{k-1}$ is a subset of S'_k and hence it consists of $+$ or g. Therefore, for $\mu > 1$, we have

$$\frac{e^{-\mu|S_k \cap S'_{k-1}|}}{|(S'_k)_+|} |(S_k)_{\neg g}| \leq \frac{|(S'_k)_+| + |S_k \cap S'_{k-1}|}{|(S'_k)_+|} e^{-\mu|S_k \cap S'_{k-1}|} \leq 1 \quad (133)$$

and hence the claim follows. ■

Lemma 6. If $\mu > 1$, then

$$\frac{e^{-\mu|S'_{i-1}\cap S_i|}}{|(S'_i)_+|} \le \frac{1}{|(S'_{i-1})_+|}. \tag{134}$$

Proof. From $\mathbf{S}'_i = \mathbf{S}_i\mathbf{S}'_{i-1}$, we obtain

$$(S'_{i-1})_+ \subseteq (S'_i)_+ \cup (S'_{i-1} \cap S_i). \tag{135}$$

This directly implies the statement of the lemma. ■

2. The argument for $j = 0$

We recall that we are estimating

$$z_{k,0} = \sup_\alpha \sum_{\substack{\mathbf{S}_0,\ldots,\mathbf{S}_k \\ \text{admissible} \\ \alpha\in S_0}} z(\mathbf{S}_0,\ldots,\mathbf{S}_k). \tag{136}$$

This case is special in that we can afford to bound

$$\begin{aligned} z(\mathbf{S}_0,\ldots,\mathbf{S}_k) &\le \widetilde{z}(\mathbf{S}_0,\ldots,\mathbf{S}_k) \\ &:= d_{\mathbf{S}_k}\frac{1}{|(S'_{k-1})_+|}d_{\mathbf{S}_{k-1}}\cdots\frac{1}{|(S'_1)_+|}d_{\mathbf{S}_1}\frac{1}{|(S'_0)_+|}v_{\mathbf{S}_0}. \end{aligned} \tag{137}$$

That is, we have dropped the exponential factors in $z(\ldots)$ and we have also dropped the leftmost denominator $1/|(S'_k)_+|$. To perform the sum over admissible sequences $\mathbf{S}_0,\ldots,\mathbf{S}_k$, we note that

$$S_{i+1} \cap (S'_i)_+ \neq \emptyset. \tag{138}$$

We use this to "anchor" the sets S_{i+1} in $(S'_i)_+$. Concretely, starting with $i = k-1$, we rewrite

$$\begin{aligned} \sum_{\mathbf{S}_k : S_k\cap(S'_{k-1})_+\neq\emptyset} d_{\mathbf{S}_k} &\le \sum_{\beta\in(S'_{k-1})_+}\sum_{\mathbf{S}_k:S_k\ni\beta} d_{\mathbf{S}_k} \\ &\le \sum_{\beta\in(S'_{k-1})_+} ||D||_\mu \le |(S'_{k-1})_+|\,||D||_\mu. \end{aligned} \tag{139}$$

Hence we have obtained

$$\sum_{\substack{\mathbf{S}_0,\ldots,\mathbf{S}_k \\ \text{admissible} \\ \alpha\in S_0}} \widetilde{z}(\mathbf{S}_0,\ldots,\mathbf{S}_k) \le ||D||_\mu \sum_{\substack{\mathbf{S}_0,\ldots,\mathbf{S}_{k-1} \\ \text{admissible} \\ \alpha\in S_0}} \widetilde{z}(\mathbf{S}_0,\ldots,\mathbf{S}_{k-1}). \tag{140}$$

We can iterate this argument and this yields the case $j = 0$ in Lemma 4.

3. The argument for $j > 0$

We start from

$$\begin{aligned} z_{k,j} = \sup_\alpha \sum_{\mathbf{S}_j:S_j\ni\alpha} d_{\mathbf{S}_j} \sum_{\mathbf{S}_0,\ldots,\mathbf{S}_{j-1}} \frac{e^{-\mu|S_j\cap S'_{j-1}|}}{|(S'_j)_+|} z(\mathbf{S}_0,\ldots,\mathbf{S}_{j-1}) \\ \times \sum_{\mathbf{S}_{j+1},\ldots,\mathbf{S}_k}\prod_{t=j+1}^{k}\frac{e^{-\mu|S_t\cap S'_{t-1}|}}{|(S'_t)_+|}d_{\mathbf{S}_t}, \end{aligned} \tag{141}$$

where, of course, the sums are still constrained by the requirement that $\mathbf{S}_0,\ldots,\mathbf{S}_k$ is admissible. We now use Lemma 5 with $i = j$ and we use Lemma 6 with $i = j+1,\ldots,k$. We then obtain

$$\begin{aligned} z_{k,j} \le \sup_\alpha \sum_{\mathbf{S}_j:S_j\ni\alpha} d_{\mathbf{S}_j} \sum_{\mathbf{S}_0,\ldots,\mathbf{S}_{j-1}} \frac{1}{|(S_j)_{\neg g}|} z(\mathbf{S}_0,\ldots,\mathbf{S}_{j-1}) \\ \times \sum_{\mathbf{S}_{j+1},\ldots,\mathbf{S}_k}\prod_{t=j+1}^{k}\frac{1}{|(S'_{t-1})_+|}d_{\mathbf{S}_t}. \end{aligned} \tag{142}$$

We first deal with the last sum, i.e.,

$$\sum_{\mathbf{S}_{j+1},\ldots,\mathbf{S}_k}\prod_{t=j+1}^{k}\frac{1}{|(S'_{t-1})_+|}d_{\mathbf{S}_t}. \tag{143}$$

Here, we use the strategy that has already been explained and used in Sec. VI B 2. We sum first over $\mathbf{S}_k$, which is anchored in $(S'_{k-1})_+$. The factor $1/|(S'_{k-1})_+|$ controls the choice of anchoring points, and we get $||D||_\mu$ times the rest. We iterate this and we obtain in this way

$$\sum_{\mathbf{S}_{j+1},\ldots,\mathbf{S}_k}\prod_{t=j+1}^{k}\frac{1}{|(S'_{t-1})_+|}d_{\mathbf{S}_t} \le ||D||_\mu^{k-j}. \tag{144}$$

Plugging this into Eq. (143), we obtain

$$\begin{aligned} z_{k,j} &\le ||D||_\mu^{k-j}\sup_\alpha \sum_{\mathbf{S}_j:\alpha\in S_j} d_{\mathbf{S}_j}\sum_{\mathbf{S}_0,\ldots,\mathbf{S}_{j-1}}\frac{1}{|(S_j)_{\neg g}|} \\ &\quad\times z(\mathbf{S}_0,\ldots,\mathbf{S}_{j-1}) \\ &\le ||D||_\mu^{k-j}\sup_\alpha \sum_{\mathbf{S}_j:\alpha\in\mathbf{S}_j} d_{\mathbf{S}_j}\sum_{\beta\in(S_j)_{\neg g}}\frac{1}{|(S_j)_{\neg g}|}z(j-1) \\ &\le ||D||_\mu^{k-j} z(j-1)\sup_\alpha\sum_{\mathbf{S}_j:\alpha\in\mathbf{S}_j} d_{\mathbf{S}_j} \\ &\le ||D||_\mu^{k-j+1} z(j-1). \end{aligned} \tag{145}$$

To obtain the second inequality, we have used the anchoring given in Eq. (130), i.e., $(S_j)_{\neg g} \cap S'_{j-1} \neq \emptyset$. This concludes the proof.

VII. CONTROLLING THE TRANSFORMATION GENERATED BY A

We estimate the operator collections defined by repeated applications of commutators.

Proposition 7. Let $B^{(0)}, B^{(1)}, \dots B^{(k)}$ be operator collections and let $\mu' < \mu$. Then,

$$||\mathrm{ad}_{B^{(k)}} \dots \mathrm{ad}_{B^{(1)}}(B^{(0)})||_{\mu'} \leq \frac{(k+1)!(2e)^k}{|\mu-\mu'|^{k+1}} \prod_{t=0}^{k} ||B^{(t)}||_{\mu}. \tag{146}$$

This result will, in particular, be useful to control exponentials: indeed, if

$$\frac{(2e)||B^{(1)}||_{\mu}}{|\mu-\mu'|} \leq 1/2, \tag{147}$$

then it follows that

$$\begin{aligned} &||\exp\{i\mathrm{ad}_{B^{(1)}}\}(B^{(0)}) - B^{(0)}||_{\mu'} \\ &\leq \frac{2||B^{(0)}||_{\mu}}{|\mu-\mu'|} \sum_{k=1}^{\infty} \left(\frac{(2e)||B^{(1)}||_{\mu}}{|\mu-\mu'|}\right)^k \\ &\leq \frac{8e}{|\mu-\mu'|^2} ||B^{(0)}||_{\mu} ||B^{(1)}||_{\mu} \end{aligned} \tag{148}$$

and, similarly,

$$||\exp\{i\mathrm{ad}_{B^{(1)}}\}(B^{(0)})||_{\mu'} \leq ||B^{0}||_{\mu} \left(1 + \frac{8e||B^{(1)}||_{\mu}}{|\mu-\mu'|^2}\right). \tag{149}$$

A. Proof of Proposition 7

We split

$$\mathrm{ad}_{B^{(k)}} \dots \mathrm{ad}_{B^{(1)}}(B^{(0)}) = \sum_{\mathbf{S}_0,\dots,\mathbf{S}_k} [B^{(k)}_{\mathbf{S}_k}, \dots\dots [B^{(1)}_{\mathbf{S}_1}, B^{(0)}_{\mathbf{S}_0}] \dots]. \tag{150}$$

If the condition

$$S_{i+1} \cap (S_0 \cup S_1 \cup \dots S_i) \neq \emptyset, \quad \forall i = 0, \dots, k-1 \tag{151}$$

is not satisfied, then the corresponding term in the sum vanishes. Therefore, we assume that this condition holds and we expand all commutators $[O_1, O_2]$ into the two terms $+O_1O_2$ and $-O_2O_1$. Hence, this gives 2^k terms. For the sake of concreteness, we focus on the expression in which we always choose the term with $'+'$. The other terms are estimated analogously. Hence, we define again, for $i > 0$,

$$\mathbf{S}'_i = \mathbf{S}_i \mathbf{S}'_{i-1}, \quad \mathbf{S}'_0 = \mathbf{S}_0 \tag{152}$$

and we investigate

$$\begin{aligned} &|| \sum_{\mathbf{S}_0,\dots,\mathbf{S}_k} B^{(k)}_{\mathbf{S}_k} \dots B^{(1)}_{\mathbf{S}_1} B^{(0)}_{\mathbf{S}_0} ||_{\mu'} \\ &\leq \sup_{\alpha} \sum_{\substack{\mathbf{S}_0,\dots,\mathbf{S}_k \\ \alpha \in S'_k}} e^{\mu'|S'_k|} ||B^{(k)}_{\mathbf{S}_k}|| \dots ||B^{(1)}_{\mathbf{S}_1}|| ||B^{(0)}_{\mathbf{S}_0}|| \\ &\leq \frac{(k+1)!}{|\mu-\mu'|^{k+1}} \sup_{\alpha} \sum_{\substack{\mathbf{S}_0,\dots,\mathbf{S}_k \\ \alpha \in S'_k}} z(\mathbf{S}_0, \dots, \mathbf{S}_k), \end{aligned} \tag{153}$$

where we have used $e^{-X}X^{k+1} \leq (k+1)!$ for $X > 0$ and we have abbreviated

$$z(\mathbf{S}_0, \dots, \mathbf{S}_k) = \frac{1}{|S'_k|^{k+1}} b^{(k)}_{\mathbf{S}_k} \dots b^{(0)}_{\mathbf{S}_0}, \quad b^{i}_{\mathbf{S}_i} = ||B^{(i)}_{\mathbf{S}_i}|| e^{\mu|S_i|}. \tag{154}$$

We define

$$z(k) = \sup_{\alpha} \sum_{\substack{\mathbf{S}_0,\dots,\mathbf{S}_k \\ \alpha \in S'_k}} z(\mathbf{S}_0, \dots, \mathbf{S}_k) \tag{155}$$

and

$$z(k) \leq \sum_{j=1}^{k} z_{k,j} \equiv \sum_{j=1}^{k} \sup_{\alpha} \sum_{\substack{\mathbf{S}_0,\dots,\mathbf{S}_k \\ \alpha \in S_j}} z(\mathbf{S}_0, \dots, \mathbf{S}_k). \tag{156}$$

We will show the following.

Lemma 7. For $j = 0, \dots, k$,

$$z_{k,j} \leq z(j-1) \prod_{t=j}^{k} ||B^{(t)}||_{\mu}, \tag{157}$$

where $z(-1)$ is defined to be 1.

This will yield the proof of Proposition 7. Indeed, by an inductive argument, very analogous to the reasoning following Lemma 4, we obtain

$$z(k) \leq e^k \prod_{t=0}^{k} ||B^{(t)}||_{\mu}. \tag{158}$$

Then, using Eq. (153) and recalling that there are 2^k terms, we obtain the proposition. It remains to give the proof of the lemma.

B. Proof of Lemma 7

1. The case $j = 0$

We have to bound

$$\sum_{\substack{\mathbf{S}_0,\ldots,\mathbf{S}_k \\ \alpha\in S_0}} z(\mathbf{S}_0,\ldots,\mathbf{S}_k) \leq \sum_{\mathbf{S}_0:S_0\ni\alpha} \frac{1}{|S_k'|} b^{(0)}_{\mathbf{S}_1} \sum_{\mathbf{S}_1:S_1\cap\in S_0'} \frac{1}{|S_k'|} b^{(1)}_{\mathbf{S}_1} \cdots \sum_{\mathbf{S}_k:S_k\cap\in S_{k-1}'} \frac{1}{|S_k'|} b^{(k)}_{\mathbf{S}_k}. \tag{159}$$

To perform the rightmost sum, we bound $1/|S_k'| \leq 1/|S_{k-1}'|$ and we bound the sum by $||B^{(t)}||_\mu$. We continue this process iteratively, obtaining $\prod_{t=0}^{k} ||B^{(t)}||_\mu$. The leftmost factor $1/|S_k'|$ has not been exploited and we have simply bounded it by 1.

2. The case $j > 0$

$$z_{k,j} \leq \sup_\alpha \sum_{\mathbf{S}_j:S_j\ni\alpha} b^{(j)}_{\mathbf{S}_j} \sum_{\mathbf{S}_0,\ldots,\mathbf{S}_{j-1}} \frac{1}{|S_j'|} z(\mathbf{S}_0,\ldots,\mathbf{S}_{j-1}) \times \sum_{\mathbf{S}_{j+1},\ldots,\mathbf{S}_k} \prod_{t=j+1}^{k} \frac{1}{|S_t'|} b^{(t)}_{\mathbf{S}_t}, \tag{160}$$

where we have bounded $1/|S_k'| \leq 1/|S_j'|$ and $1/|S_k'| \leq 1/|S_t'|$. We now use the same reasoning as for case $j = k$ to bound the last sum; we obtain

$$\begin{aligned} z_{k,j} &\leq \sup_\alpha \sum_{\mathbf{S}_j:S_j\ni\alpha} b^{(j)}_{\mathbf{S}_j} \sum_{\beta\in S_j} \sum_{\substack{\mathbf{S}_0,\ldots,\mathbf{S}_{j-1} \\ \beta\in S_{j-1}'}} \frac{1}{|S_j'|} z(\mathbf{S}_0,\ldots,\mathbf{S}_{j-1}) \times \prod_{t=j+1}^{k} ||B^{(t)}||_\mu \\ &\leq \sup_\alpha \sum_{\mathbf{S}_j:S_j\ni\alpha} b^{(j)}_{\mathbf{S}_j} z(j-1) \prod_{t=j+1}^{k} ||B^{(t)}||_\mu \\ &\leq ||B^{(j)}||_\mu z(j-1) \prod_{t=j+1}^{k} ||B^{(t)}||_\mu \leq z(j-1) \times \prod_{t=j}^{k} ||B^{(t)}||_\mu. \end{aligned} \tag{161}$$

This concludes our proof of Lemma 7, which we have used in our proof of Proposition 7. As noted above, Proposition 7 is useful for controlling the locality of an operator after a KAM step and for controlling the difference between an operator and its rotation under a KAM step. This control will allow us to ensure that our operator collections still have good intensive word norms $||\cdot||_{\mu_n}$ for the appropriate μ_n after the KAM steps. It is also important in allowing us to show that $\epsilon_n := ||V^{(n)}||_{\mu_n}$ is rapidly decreasing in n.

VIII. PROOF OF RELATIVE BOUNDEDNESS (PROPOSITION 5)

For any operators A and B acting on the same Hilbert space $\mathcal{H}$, we use the notation $B \leq_{\text{rb}} A$, read "B is relatively bounded by A," to mean that

$$\forall|\psi\rangle \in \mathcal{H}, \quad ||B|\psi\rangle|| \leq ||A|\psi\rangle||. \tag{162}$$

For $c_1, c_2 \geq 0$, if $B_1 \leq_{\text{rb}} c_1 A$ and $B_2 \leq_{\text{rb}} c_2 A$, then $B_1 + B_2 \leq_{\text{rb}} (c_1 + c_2)A$ by the triangle inequality. Additionally, suppose that the $\{A_i\}$ are a set of Hermitian positive-semidefinite operators that mutually commute. Then, for sets of numbers $\{c_i\}$ and $\{c_i'\}$ with $0 \leq c_i \leq c_i'$, $\sum_i c_i A_i \leq_{\text{rb}} \sum_i c_i' A_i$. We will use these facts freely in the following.

It is useful to define some more notation. Let X be a subset of checks and let us define the projector

$$P(X) = \prod_{\alpha\in X} E_\alpha \prod_{\beta\notin X} G_\beta.$$

Summing over all subsets of checks gives the identity, $\sum_X P(X) = 1$. The idea of $P(X)$ is that it projects onto excited stabilizer generators at the check locations contained in X.

We will first relatively bound $D^{(n)}$ following a method inspired by Ref. [57, Sec. 3.2]:

$$\begin{aligned} \langle\psi|(D^{(n)})^\dagger D^{(n)}|\psi\rangle &= \sum_{\mathbf{S}}\sum_{\mathbf{S}'} \langle\psi|(D^{(n)}_{\mathbf{S}})^\dagger D^{(n)}_{\mathbf{S}'}|\psi\rangle \\ &= \sum_{X,Y,Z}\sum_{\mathbf{S}}\sum_{\mathbf{S}'} \langle\psi|P(X)(D^{(n)}_{\mathbf{S}})^\dagger P(Y) D^{(n)}_{\mathbf{S}'} P(Z)|\psi\rangle \\ &\leq \sum_{X,Y,Z}\sum_{\mathbf{S}}\sum_{\mathbf{S}'} ||P(Y)D^{(n)}_{\mathbf{S}}P(X)||\,||P(Y)D^{(n)}_{\mathbf{S}'}P(Z)||\,||P(X)|\psi\rangle||\,||P(Z)|\psi\rangle|| \end{aligned}$$

$$\leq \sum_{X,Y,Z} \sum_{\mathbf{S}} \sum_{\mathbf{S}'} ||P(Y)D^{(n)}_{\mathbf{S}}P(X)||\,||P(Y)D^{(n)}_{\mathbf{S}'}P(Z)||\,\frac{\langle\psi|P(X)|\psi\rangle + \langle\psi|P(Z)|\psi\rangle}{2}$$

$$= \sum_{X,Y,Z} \sum_{\mathbf{S}} \sum_{\mathbf{S}'} ||P(Y)D^{(n)}_{\mathbf{S}}P(X)||\,||P(Y)D^{(n)}_{\mathbf{S}'}P(Z)||\,\langle\psi|P(X)|\psi\rangle. \tag{163}$$

For many choices of X, Y, Z, $\mathbf{S}$, and $\mathbf{S}'$, the summand vanishes. For a given choice of X, $\mathbf{S}$, and $\mathbf{S}'$, there is *at most one* choice for Y and Z such that both $||P(Y)D^{(n)}_{\mathbf{S}}P(X)||$ and $||P(Y)D^{(n)}_{\mathbf{S}'}P(Z)||$ are nonvanishing. For Y, the choice is $Y(X,\mathbf{S}) := S_+ \cup (X/S_-)$. For Z, the choice is $Z(X,\mathbf{S},\mathbf{S}') := S'_- \cup (Y(X,\mathbf{S})/(S_+))$.

Given the structure of $D^{(n)}_{\mathbf{S}}$, with $(S_- \cup S_e)$ and $(S_+ \cup S_e)$ both nonempty, it is also necessary that $S \cap X \neq \emptyset$ and $S' \cap Y \neq \emptyset$ in order that $D^{(n)}_{\mathbf{S}}P(X) \neq 0$ and $P(Y)D^{(n)}_{\mathbf{S}'} \neq 0$, respectively. Note that this precludes $X = \emptyset$.

Putting all these restrictions together,

$$\begin{aligned}
&\langle\psi|(D^{(n)})^\dagger D^{(n)}|\psi\rangle \\
&\quad\leq \sum_{X\neq\emptyset}\langle\psi|P(X)|\psi\rangle \sum_{\mathbf{S}} ||P(Y(X,\mathbf{S}))D^{(n)}_{\mathbf{S}}P(X)|| \\
&\qquad\times \sum_{\mathbf{S}'} ||P(Y(X,\mathbf{S}))D^{(n)}_{\mathbf{S}'}P(Z(X,\mathbf{S},\mathbf{S}'))|| \\
&\quad= \sum_{X\neq\emptyset}\langle\psi|P(X)|\psi\rangle \sum_{\mathbf{S}:S\cap X\neq\emptyset} ||P(Y(X,\mathbf{S}))D^{(n)}_{\mathbf{S}}P(X)|| \\
&\qquad\times \sum_{\mathbf{S}':S'\cap Y(X,\mathbf{S})\neq\emptyset} ||P(Y(X,\mathbf{S}))D^{(n)}_{\mathbf{S}'}P(Z(X,\mathbf{S},\mathbf{S}'))|| \\
&\quad\leq \sum_{X\neq\emptyset}\langle\psi|P(X)|\psi\rangle \sum_{\mathbf{S}:S\cap X\neq\emptyset} ||D^{(n)}_{\mathbf{S}}|| \sum_{\mathbf{S}':S'\cap Y(X,\mathbf{S})\neq\emptyset} ||D^{(n)}_{\mathbf{S}'}|| \\
&\quad\leq \sum_{X\neq\emptyset}\langle\psi|P(X)|\psi\rangle \sum_{\mathbf{S}:S\cap X\neq\emptyset} ||D^{(n)}_{\mathbf{S}}||\,|Y(X,\mathbf{S})|\,||D^{(n)}||_0.
\end{aligned} \tag{164}$$

Note that $|Y(X,\mathbf{S})| \leq |X| + |S| < |X|e^{|S|}$, with the last inequality following by $X \neq \emptyset$. Then,

$$\begin{aligned}
&\langle\psi|(D^{(n)})^\dagger D^{(n)}|\psi\rangle \\
&\quad\leq ||D^{(n)}||_0 \sum_X |X|\langle\psi|P(X)|\psi\rangle \sum_{\mathbf{S}:S\cap X\neq\emptyset} ||D^{(n)}_{\mathbf{S}}||e^{|S|} \\
&\quad\leq ||D^{(n)}||_0||D^{(n)}||_1 \sum_X |X|^2\langle\psi|P(X)|\psi\rangle \\
&\quad\leq ||D^{(n)}||_0||D^{(n)}||_1\langle\psi|(H_0)^2|\psi\rangle.
\end{aligned} \tag{165}$$

Note that $\mu_n > \mu_* > \log(4) > 1$, so $||D^{(n)}||_0 \leq ||D^{(n)}||_1 \leq ||D^{(n)}||_{\mu_n} \leq \eta_n \leq 2\epsilon_0$, implying that

$$D^{(n)} \leq_{\text{rb}} 2\epsilon_0 H_0. \tag{166}$$

Using **TQO-I** and **TQO-II**, $M^{(n)}$ can also be relatively bounded. We will introduce the notation

$$G(\ell,S) = \prod_{\alpha\in B_{\ell|S|}(S)} G_\alpha,$$

which will reduce clutter in the following. Recall the notation $B_{\ell|S|}(S) = \{\alpha : \text{dist}(\alpha,S) \leq \ell|S|\}$.

By construction, $M^{(n)}_{\mathbf{S}}$ is only nonzero for $|S| < d_\star$. That is, **TQO-I** applies to $M^{(n)}_{\mathbf{S}}$, as $M^{(n)}_{\mathbf{S}}$ commutes with all the stabilizers and is not too large, and so $M^{(n)}_{\mathbf{S}}$ is itself a linear combination of terms that are products of stabilizer generators. Then, by **TQO-II**, the locations of these stabilizer generators can be taken to be within $B_{\ell|S|}(S)$ (ℓ is $\Theta(1)$). In particular,

$$M^{(n)}_{\mathbf{S}}G(\ell,S) = G(\ell,S)M^{(n)}_{\mathbf{S}} = \langle M^{(n)}_{\mathbf{S}}\rangle G(\ell,S) \tag{167}$$

This follows from the fact that $M^{(n)}_{\mathbf{S}}$ is a linear combination of terms in $\mathcal{G}_{B_{\ell|S|}(S)}$ and each of those terms reduces to a constant when acting on $G(\ell,S)$.

We will define $\overline{M^{(n)}_{\mathbf{S}}} = M^{(n)}_{\mathbf{S}} - \langle M^{(n)}_{\mathbf{S}}\rangle$. If we put any ground state $|g\rangle$ on the right-hand side of $M^{(n)}_{\mathbf{S}}G(\ell,S) = \langle M^{(n)}_{\mathbf{S}}\rangle G(\ell,S)$, we obtain $M^{(n)}_{\mathbf{S}}|g\rangle = \langle M^{(n)}_{\mathbf{S}}\rangle|g\rangle$, so $||M^{(n)}_{\mathbf{S}}|| \geq |\langle M^{(n)}_{\mathbf{S}}\rangle|$. Accordingly, $\left\|\overline{M^{(n)}_{\mathbf{S}}}\right\| \leq ||M^{(n)}_{\mathbf{S}}|| + |\langle M^{(n)}_{\mathbf{S}}\rangle| \leq 2||M^{(n)}_{\mathbf{S}}||$. Note that $\overline{M^{(n)}_{\mathbf{S}}}\prod_{\alpha\in B_{\ell|S|}(S)} G_\alpha = \prod_{\alpha\in B_{\ell|S|}(S)} G_\alpha\,\overline{M^{(n)}_{\mathbf{S}}} = 0$ and that $\overline{M^{(n)}_{\mathbf{S}}}$ commutes with every E_α and G_α.

Then,

$$\begin{aligned}
\langle\psi|(\overline{M^{(n)}})^\dagger\overline{M^{(n)}}|\psi\rangle &= \sum_{\mathbf{S}}\sum_{\mathbf{S}'}\langle\psi|(\overline{M^{(n)}_{\mathbf{S}}})^\dagger\overline{M^{(n)}_{\mathbf{S}'}}|\psi\rangle \\
&= \sum_{\mathbf{S}}\sum_{\mathbf{S}'}\langle\psi|(1-G(\ell,S'))(1-G(\ell,S))(\overline{M^{(n)}_{\mathbf{S}}})^\dagger\overline{M^{(n)}_{\mathbf{S}'}}(1-G(\ell,S))(1-G(\ell,S'))|\psi\rangle
\end{aligned}$$

$$\leq \sum_{\mathbf{S}}\sum_{\mathbf{S}'} \left\|\overline{M_{\mathbf{S}}^{(n)}}\right\| \left\|\overline{M_{\mathbf{S}'}^{(n)}}\right\| \left\|(1-G(\ell,S))(1-G(\ell,S'))|\psi\rangle\right\|^2$$
$$= \sum_{\mathbf{S}}\sum_{\mathbf{S}'} \left\|\overline{M_{\mathbf{S}}^{(n)}}\right\| \left\|\overline{M_{\mathbf{S}'}^{(n)}}\right\| \langle\psi|(1-G(\ell,S))(1-G(\ell,S'))|\psi\rangle$$
$$= \langle\psi| \left(\sum_{\mathbf{S}} \left\|\overline{M_{\mathbf{S}}^{(n)}}\right\| (1-G(\ell,S))\right)^2 |\psi\rangle, \tag{168}$$

which means that $\overline{M^{(n)}} \leq_{\text{rb}} \sum_{\mathbf{S}} \left\|\overline{M_{\mathbf{S}}^{(n)}}\right\| (1-G(\ell,S)) \leq_{\text{rb}} \sum_{\mathbf{S}} \left\|\overline{M_{\mathbf{S}}^{(n)}}\right\| \sum_{\alpha\in B_{\ell|S|}(S)} E_\alpha$. We can massage this further, freely using that the E_α are positive semidefinite, Hermitian, and commuting:

$$\overline{M^{(n)}} \leq_{\text{rb}} \sum_{\mathbf{S}} \left\|\overline{M_{\mathbf{S}}^{(n)}}\right\| \sum_{\alpha\in B_{\ell|S|}(S)} E_\alpha$$
$$= \sum_\alpha E_\alpha \sum_{\mathbf{S}:\alpha\in B_{\ell|S|}(S)} \left\|\overline{M_{\mathbf{S}}^{(n)}}\right\|$$
$$= \sum_\alpha E_\alpha \sum_{k=1}^{\infty} \sum_{\mathbf{S}:\alpha\in B_{\ell|S|}(S),|S|=k} \left\|\overline{M_{\mathbf{S}}^{(n)}}\right\|$$
$$\leq_{\text{rb}} \sum_\alpha E_\alpha \sum_{k=1}^{\infty} \sum_{\mathbf{S}:\alpha\in B_{\ell|S|}(S),|S|=k} \left(\sum_{\beta:\text{dist}(\alpha,\beta)\leq \ell k} I_{\beta\in S}\right) \left\|\overline{M_{\mathbf{S}}^{(n)}}\right\|. \tag{169}$$

In the last line, we have used that in order for $B_{\ell|S|}(S)$ to contain α, by definition there must be a $\beta \in S$ that is within a distance $\ell|S|$ from α. Rearranging,

$$\sum_{\mathbf{S}} \overline{M_{\mathbf{S}}^{(n)}} \leq_{\text{rb}} \sum_\alpha E_\alpha \sum_{k=1}^{\infty} \sum_{\beta:\text{dist}(\alpha,\beta)\leq \ell k} \sum_{\mathbf{S}:\alpha\in B_{\ell|S|}(S),|S|=k,\beta\in S} \times \left\|\overline{M_{\mathbf{S}}^{(n)}}\right\|$$
$$\leq_{\text{rb}} \sum_\alpha E_\alpha \sum_{k=1}^{\infty} \sum_{\beta:\text{dist}(\alpha,\beta)\leq \ell k} \sum_{\mathbf{S}:\beta\in S,|S|=k} \left\|\overline{M_{\mathbf{S}}^{(n)}}\right\|$$
$$\leq_{\text{rb}} \sum_\alpha E_\alpha \sum_{k=1}^{\infty} e^{-\mu_n k} \sum_{\beta:\text{dist}(\alpha,\beta)\leq \ell k} \sum_{\mathbf{S}:\beta\in S,|S|=k} e^{\mu_n S} \left\|\overline{M_{\mathbf{S}}^{(n)}}\right\|$$
$$\leq_{\text{rb}} \sum_\alpha E_\alpha \sum_{k=1}^{\infty} e^{-\mu_n k} \sum_{\beta:\text{dist}(\alpha,\beta)\leq \ell k} \left\|\overline{M_{\mathbf{S}}^{(n)}}\right\|_{\mu_n}$$
$$\leq_{\text{rb}} \left\|\overline{M_{\mathbf{S}}^{(n)}}\right\|_{\mu_n} \sum_\alpha E_\alpha \sum_{k=1}^{\infty} e^{-\mu_n k} |B_{\ell k}(\alpha)|$$
$$\leq_{\text{rb}} \frac{1}{1-e^{-\mu_n+\ell\kappa}} \left\|\overline{M_{\mathbf{S}}^{(n)}}\right\|_{\mu_n} \sum_\alpha E_\alpha$$
$$\leq_{\text{rb}} \frac{4}{1-e^{-\mu_n+\ell\kappa}} \epsilon_0 H_0$$
$$\leq_{\text{rb}} 8\epsilon_0 H_0. \tag{170}$$

We use $\mu_n \geq \ell\kappa + \log(2)$ to perform the sum on k and to bound the constant out front, and we use $\left\|\overline{M_{\mathbf{S}}^{(n)}}\right\|_{\mu_n} \leq 2||M_{\mathbf{S}}^{(n)}||_{\mu_n} \leq 2\eta_n \leq 4\epsilon_0$ from Proposition 3.

Putting together the bounds on $\overline{M^{(n)}}$ and $D^{(n)}$, we have $(K^{(n)} - H_0) \leq_{\text{rb}} 10\epsilon_0 H_0$.

IX. EXTENSIONS TO CLASSICAL-LDPC-CODE HAMILTONIANS

Here, we explain how our results can be extended to classical code Hamiltonians without topological order, but instead exhibit *symmetry-breaking* orders. These models will not satisfy our **TQO** conditions as stated in Sec. II B. Therefore, we will restate sufficient conditions that allow the application of the same proof strategy, leading to stability results similar to our main Theorems in Sec. II D.

We define the Pauli-Z group $\mathcal{P}_\Lambda^Z \subseteq \mathcal{P}_\Lambda$ as follows:

$$\mathcal{P}_\Lambda^Z := \{p \in \mathcal{P}_\Lambda : p = \bigotimes_x p_x \text{ with } p_x \in \{\mathbb{1}, \mathsf{Z}\}\}. \tag{171}$$

Similarly, we define the Pauli-X group:

$$\mathcal{P}_\Lambda^X := \{p \in \mathcal{P}_\Lambda : p = \bigotimes_x p_x \text{ with } p_x \in \{\mathbb{1}, \mathsf{X}\}\}. \tag{172}$$

A classical code Hamiltonian has the following form:

$$H_0 = \sum_{\alpha\in\mathcal{E}} E_\alpha, \quad \text{where } E_\alpha = (1-C_\alpha)/2, \quad C_\alpha \in P_\Lambda^Z. \tag{173}$$

The checks generate a stabilizer group, $\mathcal{G} = \langle\{C_\alpha\}_{\alpha\in\mathcal{E}}\rangle$, thereby defining a stabilizer code. Compared to general stabilizer Hamiltonians in Sec. II A, the only additional requirement is that each check is a Pauli-Z-string.

Therefore, all results in Sec. II A apply here; in particular, that $K = N - \log_2 |\mathcal{G}|$ and that the ground-state degeneracy of H_0 is 2^K.

The *symmetry group* of H_0 is defined as follows:

$$\mathbf{X} := \mathcal{Z}(\mathcal{G}) \cap \mathcal{P}_\Lambda^X. \tag{174}$$

Here, $\mathcal{Z}(\mathcal{G})$ denotes the *centralizer* of $\mathcal{G}$ within $\mathcal{P}_\Lambda$. Elements of $\mathbf{X}$ are conventionally known as the Pauli-X logical operators of the code $\mathcal{G}$. We have $|\mathbf{X}| = 2^K$.

We define the *symmetric code distance* as follows:

$$d_{\text{sym}} = \min\{|\text{supp}(p)| : p \in \mathbf{X}, p \neq \mathbb{1}\}. \tag{175}$$

This definition coincides with the classical code distance when $\mathcal{G}$ is treated as a classical binary code.

We say that an operator is $\mathbf{X}$ symmetric if it commutes with all elements in $\mathbf{X}$. By definition, H_0 is $\mathbf{X}$ symmetric.

With these, we state conditions on H_0 in parallel to Sec. II B. We require **LDPC** and **growth of balls**, which are identical with those in Sec. II B. We additionally require the following:

TQO-Ic For every set of qubits $A \subseteq \Lambda$ with $|A| < d_{\text{sym}}$ and for every $\mathbf{X}$-symmetric $p \in \mathcal{P}_\Lambda$ such that $\text{supp}(p) \subseteq A$, we have

$$\forall g \in \mathcal{G} : [p, g] = 0 \quad \Rightarrow \quad p \in \mathcal{G}. \tag{176}$$

TQO-Ic follows directly from the definition of d_{sym}.

TQO-IIc For every connected set S in the check graph $\mathcal{E}$ where $|S| < \widetilde{d}$, we have

$$\mathcal{G}(S) \subseteq \mathcal{G}_{B_{\ell|S|}(S)}, \tag{177}$$

where $\ell = \Theta(1)$ and $B_{\ell|S|}(S) = \{\alpha : \text{dist}(\alpha, S) \leq \ell|S|\}$.

We define $d_{*,\text{sym}} = \min(d_{\text{sym}}/w_c, \widetilde{d})$. Theorems 4 and 5 are meaningful i.e., they give a stable gap and ground-state degeneracy if $d_{*,\text{sym}} \geq c \log(N)$ for $c > 0$.

We are interested in the stability of the $\mathbf{X}$-symmetric H_0 under an $\mathbf{X}$-symmetric perturbation $Z^{(0)}$. Viewing $Z^{(0)}$ as an operator collection, we write

$$Z^{(0)} = \sum_{\mathbf{s}} (Z^{(0)})_{\mathbf{s}}. \tag{178}$$

By symmetry, we may choose an operator collection such that each $(Z^{(0)})_{\mathbf{s}}$ is $\mathbf{X}$ symmetric. Similarly, we can choose an operator collection such that each $(H_0)_{\mathbf{s}}$ is $\mathbf{X}$ symmetric. These choices can be made following Sec. III C. We additionally require

$$||Z^{(0)}||_{\mu_0} < \epsilon_0, \tag{179}$$

where we recall that the word norm $||\dots||_{\mu_0}$ is defined in Eq. (38).

We can apply the iterative KAM procedure from Sec. IV to $H_0 + Z^{(0)}$. By induction, each of the operator collections $A^{(n)}$, $D^{(n)}$, $M^{(n)}$, $V^{(n)}$, $E^{(n)}$, and $Z^{(n)}$ is $\mathbf{X}$ symmetric at all scales n. We note that **TQO-I** and **TQO-II** are invoked in Lemma 2 and in Sec. VIII for characterizing $M_{\mathbf{S}}^{(n)}$, and they can be traded for **TQO-Ic** and **TQO-IIc** for $\mathbf{X}$-symmetric $M_{\mathbf{S}}^{(n)}$. The same propositions and lemmas can therefore be used to control their word norms under the flow equations.

With these, we can state our results for classical-LDPC-code Hamiltonians. They are in close parallel to Theorems 1 and 2.

Theorem 4. Assume that $\mu_0 > \mu_*$. Then, there is an $\epsilon(\mu_0)$ that can depend on μ_0 and our $\Theta(1)$-model parameters, such that, for any $\epsilon_0 \leq \epsilon(\mu_0)$, there is a $\mu_\infty > \mu_*$ such that the following holds. There is a number b such that the spectrum of the operator

$$H = H_0 + Z^{(0)} - b \tag{180}$$

is contained in a union of intervals centered on $k = 0, 1, 2, \dots$, where the kth interval is

$$I_k = [k - (kC'\epsilon_0 + \delta), k + (kC'\epsilon_0 + \delta)],$$
$$\delta = CN_c\epsilon_0 e^{-\mu_\infty d_{*,\text{sym}}}. \tag{181}$$

Theorem 5. Under the same assumptions as in Theorem 4 above, we have

$$\widetilde{P} = UPU^{-1} \tag{182}$$

for a unitary U that is locality preserving and locally close to identity, in the following sense. For any operator O supported in a connected set Y, we can write

$$U^{-1}OU = O + \sum_{r=1}^{\text{diam}(\Lambda)} O_r + O_{\text{bg}}, \tag{183}$$

where $\text{diam}(\Lambda)$ is the diameter of the graph Λ and

(1) O_r is supported in $\{x : \text{dist}(Y, x) \leq r\}$.
(2) O_r satisfies

$$||O_r|| \leq \epsilon_0 C(\mu_0, Y) e^{-cr} ||O|| \tag{184}$$

for some $c > 0$ and constant $C(\mu_0, Y)$ that can depend on μ_0 and Y.
(3) O_{bg} (with bg standing for "background") is bounded as $||O_{\text{bg}}|| \leq CN_c\epsilon_0 e^{-\mu_\infty d_{*,\text{sym}}} ||O||$.

The same statement holds true for UOU^{-1}.

X. DISCUSSION OF SOME PHYSICAL CONSEQUENCES

We have proven that classical and quantum LDPC stabilizer codes define robust gapped phases of matter, in the sense that their ground-state degeneracy and gap are both robust to sufficiently small local perturbations. In this section, we comment on some further physical consequences of our proof, discussing what physical properties of the phase remain robust in the vicinity of the unperturbed LDPC Hamiltonian. We will keep the presentation more qualitative in this section.

A. Order parameters and emergent symmetries

Let us start by considering classical LDPC codes. These, when embedded into quantum Hamiltonians, are generalizations of spontaneous symmetry-breaking phases; the logical operators of the code constitute a set of symmetries, which are broken in the ground states. These symmetry-breaking patterns can be characterized by a set of local order parameters. In particular, for a classical code with K logical bits, we can find a (nonunique) set of sites $\{x_i\}_{i=1}^K$ such that the eigenvalues $\mathsf{Z}_{x_i} = \pm 1$ of the corresponding local Pauli matrices can be used to label a full basis set for the 2^K degenerate ground states. If we think of classical codes as a special case of quantum codes, then we can think of these Z_{x_i} as a choice for the K logical Z operators of the code, conjugate to the K logical X operators that correspond to the classical logicals or symmetries. This means that there is a basis choice for the logicals $\{\mathbf{X}_i\}_{i=1}^K$ such that Z_{x_i} anticommutes with $\mathbf{X}_i$ and commutes with $\mathbf{X}_j$ for $j \neq i$.

Now consider the perturbed Hamiltonian $H_0 + V$, where V are perturbations commuting with all the symmetries (classical logicals). By Theorem 2, we can relate the ground-state subspaces as $P = UP_0U^\dagger$. We can use this to define a new "smeared-out" order parameter $\tilde{\mathsf{Z}}_{x_i} = U\mathsf{Z}_{x_i}U^\dagger$ for every $i = 1, \ldots, K$. By properties of U, these are quasilocal operators, localized to an arbitrary precision ϵ within the ball $B_r(x_i)$ around x_i with radius $r = O(\log(1/\epsilon))$, independent of n. By construction, these have the same expectation values in the new ground states as Z_{x_i} in the originals:

$$\langle\tilde{\psi}|\tilde{\mathsf{Z}}_{x_i}|\tilde{\psi}\rangle = \langle\tilde{\psi}|U\mathsf{Z}_{x_i}U^\dagger|\tilde{\psi}\rangle = \langle\psi|\mathsf{Z}_{x_i}|\psi\rangle, \tag{185}$$

where $\left|\tilde{\psi}\right\rangle$ is a ground state of $H_0 + V$ and $|\psi\rangle$ is a ground state of H_0. Thus, the 2^K symmetry-breaking product-state ground states of H_0 labeled by the eigenvalues of Z_{x_i} map onto a set of 2^K ground states labeled by eigenvalues of $\tilde{\mathsf{Z}}_{x_i}$. Moreover, as U is $\mathbf{X}$ symmetric by construction, the smeared-out operators have the same symmetry properties as the old ones. Thus $\tilde{\mathsf{Z}}_{x_i}$ serves as a local order parameter diagnosing the spontaneous breaking of the symmetry $\mathbf{X}_i$.

In fact, we expect that one should also be able to measure the same symmetry breaking using the original unsmeared order parameters, which is more practically relevant. By the analyticity and hence continuity of the ground-state projector $\tilde{P}$, the expectation values of local observables are continuous functions of the local perturbation strength. This implies that for any constant $c < 1$, we can make the perturbation strength sufficiently small (but finite) such that $|\langle\tilde{\psi}|\mathsf{Z}_{x_i}|\tilde{\psi}\rangle| > c$ in the above ground states, which is a more practical way of diagnosing the spontaneous breaking of $\mathbf{X}_i$, without the need to measure the perturbation-dependent smeared order parameters $\tilde{\mathsf{Z}}_{x_i}$.

An advantage of the former approach, using smeared operators, is that it generalizes better to quantum codes. In that case, there are no local order parameters but we can still consider the set of all logical operators, $\mathbf{X}_i$ and $\mathbf{Z}_i$, $i = 1, \ldots, K$, that come in conjugate anticommuting pairs. Now there is no symmetry requirement on the perturbations, so both species of logicals get smeared out by U, to $\tilde{\mathbf{X}}_i = U\mathbf{X}_iU^\dagger$ and $\tilde{\mathbf{Z}}_i = U\mathbf{Z}_iU^\dagger$, but their commutation relations are preserved. These constitute a set of *emergent symmetries* characterizing the phase of matter. These symmetries are also deformable: we can multiply them with the operators $UC_\alpha U^\dagger$ (with which they commute) to find other operators that have the same action on the ground-state subspace. By the cleaning lemma [58], for any region Y with $|Y| < d$, all logicals can be deformed in such a way as to avoid Y; e.g., we can do this for any ball $B_r(x)$ with $r < \log d/\kappa$. Since the smearing out is only by an $O(1)$ factor, one can then also clean out the new logicals, $\tilde{\mathbf{X}}_i$ and $\tilde{\mathbf{Z}}_i$, from any ball with radius $r = o(\log d)$. In this sense, the emergent symmetries of perturbed qLDPC codes have the characteristics of higher-form symmetries [59,60].

Similarly, if one of the original logicals is localized to some subregion Y of size $o(n)$ of the lattice Λ, then its smeared-out version can be approximated, with exponential precision, by an operator supported on a region the size of which is proportional to $|Y|$. One example where this is relevant concerns hypergraph product codes [61]: these come equipped with an abstract "square-grid" structure in which qubits are labeled by a pair of labels (x, y). The code distance is $d = \Theta(\sqrt{n})$ and one can choose a canonical basis set of logical operators that are supported on "rows" and "columns," i.e., sets of qubits where either the first or the second label is held constant (for details, see Ref. [35]), analogous to the familiar logical operators of the toric code. Our results imply that these canonical logicals maintain their "one-dimensional" nature and only get smeared out to an $O(1)$ number of neighboring rows or columns.

B. Excitations

So far, we have focused on ground-state properties. Another way to characterize the phase is in terms of excitations. We will now argue that the local excitations

defined by the unperturbed Hamiltonian remain well defined for small enough perturbations, building on the ideas of BHM [11].

For simplicity, let us consider a stabilizer code the checks of which are all independent. In that case, for any check C_α, we can find an appropriate excitation operator $\mathcal{D}_\alpha$ that anticommutes with C_α and commutes with all other checks. For classical codes, one can think of $\mathcal{D}_\alpha$ as a set of spin flips creating the analog of a domain wall, while for quantum codes, it is the analog of an operator creating an anyon. They create localized excitations that form the lowest band, with energy 1 in H_0. One can also argue that in many cases, the corresponding operator has to be nonlocal, with support that is comparable to the code distance [34,62].

For the perturbed Hamiltonian, we can again use U from Theorem 2 to define smeared-out excitation operators $\tilde{\mathcal{D}}_\alpha = U\mathcal{D}_\alpha U^\dagger$. Then, applied to a ground state $\left|\tilde{\psi}\right\rangle$, these create excitations localized near the support of check C_α. To see this, consider measuring some local operator O, supported on a ball $B_r(x)$, which one can think of as measuring the local energy density near x. We take the ball to be far from the location of C_α, i.e., $r \ll R$, where $R \equiv \mathrm{dist}(x, \mathrm{supp}(C_\alpha))$. We want to argue that if the distance R is large, then O cannot distinguish $\tilde{\mathcal{D}}_\alpha \left|\tilde{\psi}\right\rangle$ from $\left|\tilde{\psi}\right\rangle$. In particular,

$$\langle\tilde{\psi}|\tilde{\mathcal{D}}_\alpha O\tilde{\mathcal{D}}_\alpha|\tilde{\psi}\rangle = \langle\tilde{\psi}|O|\tilde{\psi}\rangle + h(R), \tag{186}$$

where $\lim_{R\to\infty} h(R) = 0$.

To show this, we note that $\langle\tilde{\psi}|O|\tilde{\psi}\rangle = \langle\psi|\tilde{O}|\psi\rangle$ and $\langle\tilde{\psi}|\tilde{\mathcal{D}}_\alpha O\tilde{\mathcal{D}}_\alpha|\tilde{\psi}\rangle = \langle\psi|\mathcal{D}_\alpha\tilde{O}\mathcal{D}_\alpha|\psi\rangle$, where $|\psi\rangle$ is a ground state of H_0 and $\tilde{O} = U^\dagger OU$. Up to precision $e^{-O(c\tilde{r})}$ we can approximate $\tilde{O}$ by an operator supported on the ball $B_{\tilde{r}}(x)$. It is thus enough to argue that the presence of the excitation at the check C_α cannot be picked up by any such operator, as long as $\tilde{r}$ is not too large. This follows from the condition **TQO-II** [63]. The reduced density matrix of $|\psi\rangle$ on $B_{\tilde{r}}(x)$ involves elements of the local stabilizer group $\mathcal{G}(B_{\tilde{r}}(x))$. The same is true of $\mathcal{D}_\alpha|\psi\rangle$ and the two only differ in the sign of terms involving C_α. However, by **TQO-II** and Eq. (13), $\mathcal{G}(B_{\tilde{r}}(x)) \subseteq \mathcal{G}_{\tilde{r}+\ell e^{\kappa\tilde{r}}}$. As long as this larger ball does not contain C_α, the two reduced density matrices are the same. We can thus choose $\tilde{r} \propto \log(R/\ell)/\kappa$, which makes $h(R)$ in Eq. (186) decay as $h(R) \propto R^{-c/\kappa}$, where c is the constant controlling the locality of U in Theorem 2. We note that the slow decay is due to the very weak **TQO** condition being used; if the radius appearing in **TQO-II** could be strengthened from $\ell|S|$ to $\ell\log|S|$, which we expect in many cases of interest, then we could choose $\tilde{r} = O(R)$, resulting in $h(R) \propto e^{-(c/\kappa\ell)R}$.

Since $\tilde{\mathcal{D}}_\alpha \left|\tilde{\psi}\right\rangle$ only contains excitations within some finite neighborhood of $\mathrm{supp}(C_\alpha)$, it should be spanned, up to good approximation, by low-lying eigenstates of $H_0 + V$. In BHM, a stronger statement is made, using the fact that in their case, there exists a quasilocal unitary U that relates not only the ground-state subspaces but also low-lying excited states, between perturbed and unperturbed Hamiltonians. In particular, let Q be the projector onto eigenstates with energy 1 in H_0 and let $\tilde{Q}$ be the projector onto the corresponding to eigenstates with energies in the interval I_1 defined in Theorem 1, which remains well separated from all the other intervals for sufficiently weak perturbations. Now assume that there exists U such that both $\tilde{P} = UPU^\dagger$ and $\tilde{Q} = UQU^\dagger$, with the same locality properties as in Theorem 2. We then have, following BHM, that

$$\tilde{Q}\tilde{\mathcal{D}}_\alpha \left|\tilde{\psi}\right\rangle = UQ\mathcal{D}_\alpha|\psi\rangle = U\mathcal{D}_\alpha|\psi\rangle = \tilde{\mathcal{D}}_\alpha \left|\tilde{\psi}\right\rangle. \tag{187}$$

This would mean that the excited state created by $\tilde{\mathcal{D}}_\alpha$ lies entirely within the lowest band of $H_0 + V$. We can thus think of this band as being spanned by the localized excitations created by $\tilde{\mathcal{D}}_\alpha$ that are adiabatically connected to the ones of H_0. Whether an appropriate choice of U satisfying this condition exists is an interesting question for future work.

Finally, we note that while here we have considered the case in which we initially excite a single check, the above considerations should also generalize to other low-lying excitations. For example, instead of $\mathcal{D}_\alpha$, we can consider an operator that excites a $\Theta(1)$ number of checks, all spatially far separated from each other. We can then smear out this operator, to get a new operator that creates a set of far-separated localized excitations in the perturbed Hamiltonian. This is relevant for, e.g., the case in which the code exhibits some redundancies, preventing one from creating a single excitation. One would then want to find U that also rotates the projector of the appropriate set of excitations between unperturbed and perturbed Hamiltonians.

XI. SUMMARY AND OUTLOOK

In this paper, we have generalized the stability proof of Ref. [11] to perturbations of a large set of qLDPC stabilizer codes, without the restriction of Euclidean locality, including constructions of good qLDPC codes and other examples living on expander graph geometries. We have proved the robustness of ground-state degeneracy (up to a splitting exponentially suppressed in the code distance), the stability of the spectral gap, and the existence of a unitary relating perturbed and unperturbed ground states that is quasilocal with respect to the interaction distance of the unperturbed Hamiltonian. These results provide a rigorous starting point for defining and studying phases of matter on generic graphs, away from the limit of Euclidean lattices, which is relevant in light of the ongoing experimental

efforts that are currently pushing quantum many-body physics into this new territory [30–33].

Our results raise a number of questions, some of which we have already flagged above. Some of these involve the physical consequences stemming from the quasiadiabatic continuation between ground states. In general, some physical consequences might require strengthening the form of the **TQO-II** condition, which we expect should be possible in many cases of interest. As we have also noted, BHM have shown a stronger quasiadiabatic continuation result [11], where the unitary U relates not only ground states but also low-lying excited states, which was needed to prove statements about the form of low-lying excitations in the perturbed model. Is there a similar adiabatic continuation of excitations in our case?

Another set of questions involves the set of allowed perturbations. Can the locality conditions on V be relaxed while maintaining stability, e.g., to generic k-local perturbations without reference to any graph structure? Our **TQO-II** condition is easy to satisfy and asks that connected stabilizers can be constructed out of stabilizer generators that are not too far away. If we instead ask for a stricter condition (i.e., that **TQO-II** must also apply also to disconnected stabilizers), our proof shows stability to k-local perturbations. The main change to the proof is in the construction of the initial operator collections from the perturbations. In particular, we can construct the initial operator collections via sandwiching the qubit support (rather than a minimal connected set covering the qubit support) with projectors. These words can be disconnected (as can the resulting words generated over the course of the KAM steps); we had previously avoided this, as we were using connectivity of all words to apply **TQO-II** in the proof of relative form boundedness of $M^{(n)}$ in Sec. VIII and in bounding commutators of ghosts with other operators in Lemma 2. However, disconnected words are fine under the stricter **TQO-II** and the corresponding word norms of long-range few-body perturbations are much smaller than if we asked for connected words. As an example, under this looser construction of the initial operator collections, a two-body nonlocal operator separated by a distance r has a word norm weighted by e^2 rather than e^r. This broadens the class of perturbations that have bounded the word norm in the thermodynamic limit to include cases such as $1/L \sum_{ij} Z_i Z_j$. Finally, we had stated some of our results in terms of a Pauli norm for simplicity in calculation; the Pauli norm can now be chosen to be exponentially weighted by qubit support, rather than exponentially weighted in the minimal connected covering. The main change to the results is that there is no longer graph locality of the unitary U that maps the perturbed spectral projector back to the unperturbed one; it is only locality preserving in the sense of word norms. In all, the stricter **TQO-II** condition gives stability to k-local perturbations rather than just graph-local ones—which models satisfy this stricter condition?

Finally, it would be very intriguing to generalize our results from stabilizer codes to more general families of Hamiltonians, either to all commuting-projector models, as was done by BHM, or even to the more general setting of frustration-free Hamiltonians as in Ref. [14]. Other avenues of generalization would involve considering the stability of finite-temperature phases [64,65] in the non-Euclidean setting.

Note added. We have recently become aware that other authors have independently investigated the stability of LDPC codes (see, e.g., Ref. [66]).

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: ARGUMENTS FOR TQO-II CONDITIONS FOR SPECIFIC EXAMPLES

In Sec. II B, we have claimed that the condition **TQO-II** is satisfied by a number of relevant examples of qLDPC codes, such as known construction of good qLDPC codes and hypergraph products of classical expander codes. We will argue for this now.

The codes that we want to consider are all CSS codes. We can thus separately consider the parts of the stabilizer group generated by X and Z checks, which we denote $\mathcal{G}^X$

and $\mathcal{G}^Z$, and similarly for $\mathcal{G}(S)$ and $\mathcal{G}_S$. It is then enough to show **TQO-II** for the X and Z parts of the stabilizer group separately in order for it to be satisfied for the entire stabilizer group.

We will now argue that the claim holds for $\mathcal{G}^X$ (the argument for $\mathcal{G}^Z$ is analogous). The X checks define some classical code with a parity-check matrix $\mathbb{H}_X$. It is more intuitive to consider the question in terms of the transpose code, $\mathbb{H}_X^{\mathrm{T}}$ (which, in the terminology of Refs. [34,67] is *gauge dual* to the quantum code in question), which exchanges the roles of bits and checks. For this classical code, **TQO-II** becomes a question of *energy barriers*. supp(S) becomes, in the transpose code, a set of checks. **TQO-II** then states that any excitation contained in this set can be created by flipping a number of bits with a ball of radius $\ell|S|$ where, in defining the distance, we are now only using the checks of $\mathbb{H}_X^{\mathrm{T}}$.

1. Good qLDPC codes

The key observation, made in Refs. [34,35], is that in all known constructions of good qLDPC codes, the transpose code $\mathbb{H}_X^{\mathrm{T}}$ (and, similarly, the code $\mathbb{H}_Z^{\mathrm{T}}$) is *locally testable*, meaning that there exists a constant $\gamma = O(1)$ such that

$$|\mathbb{H}_X^{\mathrm{T}}\mathbf{x}| \geq \gamma\,|\mathbf{x}| \tag{A1}$$

for any bit string $\mathbf{x}$. In our case, $\mathbb{H}_X^{\mathrm{T}}\mathbf{x}$ is contained within supp(S), so we have $|\mathbf{x}| \leq (w_c/\gamma)|S|$. At the same time, we can assume that $\mathbf{x}$ forms a connected set; otherwise, we would deal with each connected component separately. Thus, by definition, it fits within a ball of radius $r = |\mathbf{x}|/2$. This gives **TQO-II** with $\ell = w_v/(2\gamma)$.

2. Hypergraph products of expander codes

In the case of hypergraph product codes, the classical code $\mathbb{H}_X^{\mathrm{T}}$ in question is the tensor product of two classical input codes. Such tensor product are not locally testable. Nevertheless, if the two input codes are chosen to be classical expander codes (e.g., good classical LDPC codes) than the resulting product will have sufficiently large energy barriers that it still satisfies **TQO-II**. The key point is that for a classical expander code, while Eq. (A1) is not true in general, it is true when restricted to bit strings $\mathbf{x}$ such that $|\mathbf{x}| < \nu N$ (N being the number of bits in the classical code) for some constant $\nu = O(1)$. This property is then satisfied by the product of two such expander codes: i.e., let $\mathbb{H}_A$ and $\mathbb{H}_B$ be the parity-check matrices defining the two input codes that make up the product, with expansion parameters ν_A, ν_B and γ_A, γ_B. In the product, we can restrict ourselves to $|\mathbf{x}| \leq \min(\nu_A N_A, \nu_B N_B)$; for these, we satisfy Eq. (A1) with $\gamma = \min(\gamma_A, \gamma_B)$ (to see this, note that $\mathbf{x}$ can be decomposed into either its rows or its columns; we can count the number of excited checks in each, which amounts to counting checks in $\mathbb{H}_A$ and $\mathbb{H}_B$, respectively). Now, consider bit strings with $d_A d_B/2 > |\mathbf{x}| > \nu_A N_A$. We want to argue that in this case, there exists some constant C such that $|\mathbb{H}_X^{\mathrm{T}}| \geq Cd = \Theta(N^{1/2})$; in this case, we can choose $\widetilde{d}$ in the **TQO-II** condition to be Cd, such that all allowed sets S still satisfy Eq. (A1).

To argue this last point, consider all the nonempty rows R of $\mathbf{x}$ and divide it into two sets, $R = R_< \cup R_>$, where $R_<$ contains rows with $\leq \nu_A N_A$ bits flipped and $R_>$ contains those with more. We now try to construct an $\mathbf{x}$ with energy $|\mathbb{H}_X^{\mathrm{T}}\mathbf{x}| = o(N^{1/2})$ and find that this is impossible when $|\mathbf{x}| > \nu_A N_A = \Theta(N^{1/2})$. First, due to expansion, it must be that $|\mathbf{x}|_{R_<}| = o(N^{1/2})$; otherwise, the energy would already be too high. Thus, almost all nonempty rows have many bits flipped. The same is true when we consider a decomposition into columns, rather than rows. This means that the total number of nonempty rows and columns has to be $|R|, |C| = O(N^{1/2})$. But, by the assumption that $|\mathbf{x}| \leq d_A d_B/2$, most of these rows and columns must contain at least one excitation, bringing the energy up to $O(N^{1/2})$ again.

APPENDIX B: PROOF OF RELATION BETWEEN PAULI NORM AND WORD NORM

1. Proof of Proposition 1

Consider an operator that is a sum of Paulis, $O = \sum_p c_p p$, and consider reexpressing it in terms of an interaction O via the procedure in Sec. III. We show that the resulting norm $||O||_\mu$ can be bounded above by $|||O|||_{\mu'}$ for a $\mu' = a\mu + b$, where a and b are $\Theta(1)$ constants $a = w_q$ and $b = (w_q \log(2) + \kappa + \log(2w_c))$. We will define $\kappa' = \kappa + \log(2w_c)$ for ease.

Before directly bounding $||O||_\mu$, it is useful to note a few preliminary inequalities that we will use freely in the following. For a given p, there are at most $2^{|\mathrm{ext}(p)|}$ $\mathbf{S}$ such that $p_{\mathbf{S}}$ is nonvanishing and all of them satisfy $||p_{\mathbf{S}}|| \leq 1$. Since each qubit touches at most w_q stabilizers, $|\mathrm{ext}(p)| \leq w_q|\mathcal{M}(p)|$. Because $\mathcal{M}(p)$ is connected, every $x, y \in \mathcal{M}(p)$ satisfies $\mathrm{dist}(x,y) \leq |\mathcal{M}(p)|$. Given $\alpha \in \mathrm{ext}(p)$, then there must be an $x \in \mathrm{supp}(p)$ such that $\mathrm{dist}(x, \mathrm{supp}(C_\alpha)) \leq |\mathcal{M}(p)|$. For any α, $\sum_{x:\mathrm{dist}(x,\mathrm{supp}(C_\alpha))<r} 1 \leq w_c e^{\kappa r}$:

$$||O||_\mu = \sup_\alpha \sum_{\mathbf{S}:\alpha\in S} || \sum_{p:\mathrm{ext}(p)=S} c_p p_{\mathbf{S}}||e^{\mu|S|} \leq \sup_\alpha \sum_{\mathbf{S}:\alpha\in S} \sum_{p:\mathrm{ext}(p)=S} |c_p|e^{\mu|S|}$$

$$= \sup_\alpha \sum_{p:\alpha\in\mathrm{ext}(p)} \sum_{\mathbf{S}:S=\mathrm{ext}(p)} |c_p|e^{\mu|\mathrm{ext}(p)|} \leq \sup_\alpha \sum_{p:\alpha\in\mathrm{ext}(p)} |c_p|2^{|\mathrm{ext}(p)|}e^{\mu|\mathrm{ext}(p)|}$$

$$
\begin{aligned}
&\leq \sup_{\alpha} \sum_{p} \left(\sum_{x:\mathrm{dist}(x,\mathrm{supp}(C_\alpha))\leq |\mathcal{M}(p)|} I_{x\in\mathrm{supp}(p)} \right) |c_p| e^{w_q(\mu+\log(2))|\mathcal{M}(p)|} \\
&= \sup_{\alpha} \sum_{r=1}^{\infty} e^{-\kappa' r} \sum_{p:|\mathrm{ext}(p)|=r} \left(\sum_{x:\mathrm{dist}(x,\mathrm{supp}(C_\alpha))\leq r} I_{x\in\mathrm{supp}(p)} \right) |c_p| e^{(\kappa'+w_q(\mu+\log(2)))|\mathcal{M}(p)|} \\
&= \sup_{\alpha} \sum_{r=1}^{\infty} e^{-\kappa' r} \sum_{x:\mathrm{dist}(x,\mathrm{supp}(C_\alpha))\leq r} \sum_{p:|\mathrm{ext}(p)|=r, x\in\mathrm{supp}(p)} |c_p| e^{(\kappa'+w_q(\mu+\log(2)))|\mathcal{M}(p)|} \\
&\leq \sup_{\alpha} \sum_{r=1}^{\infty} e^{-\kappa' r} \sum_{x:\mathrm{dist}(x,\mathrm{supp}(C_\alpha))\leq r} |||O|||_{\kappa'+w_q(\mu+\log(2))} \leq \sum_{r=1}^{\infty} e^{-\kappa' r} w_c e^{\kappa r} |||O|||_{\kappa'+w_q(\mu+\log(2))} \\
&\leq \frac{w_c e^{-(\kappa'-\kappa)}}{1-e^{-(\kappa'-\kappa)}} |||O|||_{\kappa'+w_q(\mu+\log(2))} \leq |||O|||_{w_q\mu+(w_q\log(2)+\kappa+\log(2w_c))}. \qquad \text{(B1)}
\end{aligned}
$$

2. Proof of Proposition 2

Similarly, we can bound $|||O|||_\mu$ in terms of $||O||_{\mu'}$ for a $\mu' = \widetilde{a}\mu + \widetilde{b}$, where $\widetilde{a}$ and $\widetilde{b}$ are $\theta(1)$ constants $\widetilde{a} = w_c$ and $\widetilde{b} = \log(4w_q)$.

We always have connected S and we will use that here. For notational convenience, define $\mathrm{supp}(S) := \cup_{\alpha\in S}\mathrm{supp}(C_\alpha)$. We will collect a few useful inequalities to start, which we will use freely in the following. If $p \in \mathrm{supp}(S)$, then $|\mathcal{M}(p)| \leq w_c|S|$. There are at most $4^{|\mathcal{M}(p)|} \leq 4^{w_c|S|}$ Pauli strings p such that $p \in \mathrm{supp}(S)$. If $x \in \mathrm{supp}(p)$ and $p \in \mathrm{supp}(S)$, then there is an $\alpha \in S$ such that $x \in \mathrm{supp}(C_\alpha)$. If $O = \sum_p c_p p$, then $\forall p$, $|c_p| \leq ||O||$.

We can first expand each term O_S in the Pauli basis as $O_{\mathbf{S}} = \sum_{p:p\in\mathrm{supp}(S)} c_{\mathbf{S},p} p$, so that $O = \sum_p \sum_{\mathbf{S}:p\in\mathrm{supp}(S)} c_{\mathbf{S},p} p$. Then,

$$
\begin{aligned}
|||O|||_\mu &= \sup_x \sum_{p:x\in\mathrm{supp}(p)} \left| \sum_{\mathbf{S}:p\in\mathrm{supp}(S)} c_{\mathbf{S},p} \right| e^{\mu|\mathcal{M}(p)|} \;\& \\
&\leq \sup_x \sum_{p:x\in\mathrm{supp}(p)} \sum_{\mathbf{S}:p\in\mathrm{supp}(S)} ||O_{\mathbf{S}}|| e^{\mu|\mathcal{M}(p)|} \\
&= \sup_x \sum_{\mathbf{S}} \sum_{p:x\in\mathrm{supp}(p), p\in\mathrm{supp}(S)} ||O_{\mathbf{S}}|| e^{\mu|\mathcal{M}(p)|} \\
&\leq \sup_x \sum_{\mathbf{S}} \sum_{p:x\in\mathrm{supp}(p), p\in\mathrm{supp}(S)} ||O_{\mathbf{S}}|| e^{\mu w_c|S|} \\
&\leq \sup_x \sum_{\mathbf{S}} \sum_{p:x\in\mathrm{supp}(p), p\in\mathrm{supp}(S)} \Big(\sum_{\alpha\in\mathrm{supp}_c(x)} I_{\alpha\in S} \Big) ||O_{\mathbf{S}}|| e^{\mu w_c|S|} \\
&= \sup_x \sum_{\alpha\in\mathrm{supp}_c(x)} \sum_{\mathbf{S}:\alpha\in\mathbf{S}} \sum_{p:x\in\mathrm{supp}(p), p\in\mathrm{supp}(S)} ||O_{\mathbf{S}}|| e^{\mu w_c|S|} \\
&\leq \sup_x \sum_{\alpha\in\mathrm{supp}_c(x)} \sum_{\mathbf{S}:\alpha\in\mathbf{S}} 4^{w_c|S|} ||O_{\mathbf{S}}|| e^{\mu w_c|S|} \\
&\leq \sup_x \sum_{\alpha\in\mathrm{supp}_c(x)} ||O||_{w_c\mu+\log(4)} \\
&\leq w_q ||O||_{w_c\mu+\log(4)} \leq ||O||_{w_c\mu+\log(4w_q)}.
\end{aligned}
$$

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