

Randomly Monitored Quantum Codes

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Recent studies of monitored quantum dynamics have revealed that projective measurements, traditionally viewed as decohering operations, can instead generate and sustain long-range entanglement. Motivated by these, we ask how many physical qubits must be measured in random basis to irreversibly destroy quantum information encoded in a quantum error-correcting code. We study this problem for a broad class of stabilizer and subsystem codes, derive necessary and sufficient conditions for measurement-induced information destruction, and show that many codes, including concatenated and topological codes, achieve the maximal measurement threshold $p_m^{\text{th}} = 1$, meaning that the encoded information survives as long as arbitrarily small but finite fraction of physical qubits remain unmeasured. Beyond this surprising robustness, we find a structural relation underlying maximal thresholds. Namely, we prove that if the Pauli basis of the logical operator measured at full measurement does not concentrate on a single Pauli, then the measurement threshold always satisfies $p_m^{\text{th}} = 1$. This result uncovers a structural relation between logical measurement statistics and stability under partial measurement, revealing a general mechanism by which access to measurement outcomes enhances decodability under monitored dynamics.

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I. INTRODUCTION

One of the long-standing puzzles in modern physics is why we perceive the world as classical, even though the fundamental laws of physics are quantum mechanical [1]. A canonical example illustrating this tension is Schrödinger's cat, which involves a macroscopic superposition of a dead and a live cat. Although such a Schrödinger's cat state is, in principle, allowed by quantum mechanics, the puzzle lies in why we rarely observe such superpositions in the macroscopic world. This paradox can be partially resolved by recognizing that information about whether the cat is dead or alive can be revealed by a simple measurement of its quantum state,

$$\alpha|\text{dead}\rangle + \beta|\text{alive}\rangle \xrightarrow{\text{measurement}} |\text{dead}\rangle \text{ or } |\text{alive}\rangle, \quad (1)$$

implying instability of the superposition under decoherence induced by the environment.

This phenomenon admits a natural information-theoretic interpretation. Imagine preparing a qubit in the state

$\alpha|0\rangle_{\text{dead}} + \beta|1\rangle_{\text{alive}}$ and encoding it into a macroscopic cat state,

$$\alpha|0\rangle_{\text{dead}} + \beta|1\rangle_{\text{alive}} \xrightarrow{\text{encode}} \alpha|\text{dead}\rangle + \beta|\text{alive}\rangle. \quad (2)$$

The fragility of Schrödinger's cat can then be understood as the loss of encoded quantum information under measurement. A simple analogue is the n -qubit cat code, $|0\rangle_{\text{dead}} \mapsto |0\rangle^{\otimes n}$, $|1\rangle_{\text{alive}} \mapsto |1\rangle^{\otimes n}$, for which a single local Pauli Z measurement collapses the encoded state,

$$\alpha|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n} \xrightarrow{Z_1 \text{ measurement}} |0\rangle^{\otimes n} \text{ or } |1\rangle^{\otimes n}, \quad (3)$$

destroying the encoded quantum information. These observations are often invoked to explain the absence of macroscopic superpositions in our world [2].

Recent studies of monitored hybrid quantum circuits, however, offer a complementary perspective [3–13], challenging this conventional wisdom. These works suggest that quantum entanglement and encoded information may tolerate a surprisingly large amount of decoherence from projective measurements. In particular, encoded quantum information prepared by short-depth quantum circuits can persist even when an overwhelming majority of qubits are measured [6]. At the heart of this unexpected persistence of long-range entanglement lies the intrinsic quantum error-correcting structure and its stability against local projective

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measurements. Related phenomena have also been discussed in the context of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [14,15]. These developments motivate a systematic investigation of the fate of encoded information in quantum error-correcting codes (QECCs) under local random measurements.

In this work, we consider a QECC encoding k logical qubits into n physical qubits. Each physical qubit is independently measured in a local Pauli basis (X , Y , or Z) with probabilities (p_X, p_Y, p_Z) . Specifically, both the measurement basis and the measurement outcome ± 1 are recorded rather than discarded. We denote the total measurement rate by

$$p = p_X + p_Y + p_Z \leq 1,$$

so that a fraction $1 - p$ of qubits remains unmeasured. Specifically, fixing the relative measurement frequencies $\vec{\alpha} = (\alpha_X, \alpha_Y, \alpha_Z)$ with $\alpha_X + \alpha_Y + \alpha_Z = 1$ and $(p_X, p_Y, p_Z) = (p\alpha_X, p\alpha_Y, p\alpha_Z)$, we increase p . We then define a *measurement threshold* p_m^{th} below which the encoded information remains recoverable with high probability. More precisely, we will consider a family of quantum error-correcting codes with increasing n and ask whether the encoded information remains recoverable for $p < p_m^{\text{th}}$ with probability approaching unity as n increases.

Our results establish two main conclusions. First, we demonstrate that a broad class of QECCs achieve the maximal measurement threshold

$$p_m^{\text{th}} = 1$$

for suitable relative measurement frequencies. In these codes, encoded information remains recoverable with a probability converging to unity even when qubits are measured with probability $1 - \epsilon$ for arbitrarily small but finite $\epsilon > 0$. Examples include concatenated five-qubit and seven-qubit codes, two-dimensional topological codes, Haar-random codes, and holographic QECCs.

Second, and more fundamentally, we reveal a structural relation between maximal threshold and the statistics of logical Pauli measurements at full measurement $p = 1$. Recall that, when all physical qubits are locally measured in a stabilizer code, the resulting measurement pattern effectively measures a logical Pauli operator of the code. We show that maximal threshold is intrinsically tied to an *uncertainty in the Pauli basis* of this measured logical operator. Namely, when there exists a probabilistic uncertainty in the Pauli basis of the induced logical operator measurement, then the code always achieves the maximal measurement threshold $p_m^{\text{th}} = 1$. More precisely, in the large- n limit, if more than one logical Pauli operator (e.g., $\bar{X}$, $\bar{Y}$, or $\bar{Z}$) occurs with nonvanishing probability at full measurement $p = 1$, then the code always achieves $p_m^{\text{th}} = 1$. This result, proven in Sec. V, shows that maximal robustness against local measurements is not accidental or

model-specific, but is constrained by a structural property of the logical operator algebra.

The notion of measurement threshold has appeared previously in related contexts. Reference [6] analyzed the measurement threshold of scrambling encoding maps, including Haar-random encodings, which we review in Sec. VI. Reference [16] studied the threshold of the 2D toric code, overlapping with our discussion in Sec. IV. Our main contribution, however, is to demonstrate that the maximal threshold $p_m^{\text{th}} = 1$ can be achieved by a broad class of QECCs that do not rely on information scrambling, and to establish a general theorem relating maximal threshold to intrinsic logical measurement uncertainty.

This paper is organized as follows. In Sec. II, we present general results on local measurements in QECCs. Section III analyzes concatenated quantum codes. Section IV studies two-dimensional topological codes and illustrates the relation between threshold and logical measurement uncertainty. Section V proves a general theorem connecting maximal threshold to logical measurement statistics. Section VI discusses Haar-random encodings and the effect of local versus global random measurements, and Sec. VII briefly considers holographic codes. We conclude in Sec. VIII.

II. SOME THEOREMS ABOUT MONITORED QUANTUM CODES

In this section, we present general results regarding projective measurements in stabilizer and subsystem codes. Namely, we prove that encoded information in a stabilizer code is destroyed (i.e., logically measured) if and only if the Pauli subgroup generated by the measured operators contains a nontrivial logical Pauli operator. This necessary-and-sufficient criterion extends naturally to subsystem codes, where information destruction is characterized in terms of dressed Pauli logical operators. These results provide a precise algebraic criterion for when local measurements irreversibly destroy encoded quantum information, and will form the basis for our subsequent analysis.

A. Stabilizer update rule

We begin by discussing how the stabilizer group $\mathcal{S}$ is updated after projective Pauli operator measurements. For our purposes, it is convenient to formulate the result for a stabilizer mixed state $\rho_{\mathcal{S}}$, since this directly applies to stabilizer codes.

Let $\rho_{\mathcal{S}}$ be a stabilizer maximally mixed state stabilized by the stabilizer group $\mathcal{S}$, namely,

$$U\rho_{\mathcal{S}} = \rho_{\mathcal{S}}U = \rho_{\mathcal{S}} \quad \text{for all } U \in \mathcal{S}. \quad (4)$$

Up to normalization, this state can be expressed as

$$\rho_S \propto \sum_{U \in \mathcal{S}} U. \quad (5)$$

This represents the maximally mixed state on a stabilizer codespace.

Now suppose that projective measurements are performed on a set of mutually commuting independent Pauli operators $\{P_1, \dots, P_t\}$. In this section, we do not constrain the P_j to be single-qubit Pauli operators. Our goal is to determine the *updated stabilizer group* $\mathcal{S}'$ after the measurements.

Let $m_j = \pm 1$ denote the measurement outcomes of P_j , and define the measured Pauli operators M_j by

$$M_j \equiv m_j P_j \quad (6)$$

so that each M_j has eigenvalue $+1$ on the postmeasurement state.

Define the *measurement group* $\mathcal{M}$ generated by the measured operators by

$$\mathcal{M} \equiv \langle M_1, \dots, M_t \rangle. \quad (7)$$

Note that $\mathcal{M}$ depends on the measurement outcomes $\{m_j\}$. Define the *centralizer* of $\mathcal{M}$ within the Pauli group by

$$\mathcal{C}(\mathcal{M}) \equiv \{U \in \text{Pauli} : [U, M] = 0, \text{ for all } M \in \mathcal{M}\}. \quad (8)$$

Up to normalization, the postmeasurement state is given by

$$\rho'_{S'} \propto \left(\prod_{i=1}^t (I + M_i) \right) \rho_S \left(\prod_{i=1}^t (I + M_i) \right). \quad (9)$$

We now determine its stabilizer group $\mathcal{S}'$.

Theorem 1. Let ρ_S be a stabilizer state with stabilizer group $\mathcal{S}$, and let $\mathcal{M}$ be the measurement group generated by the measured Pauli operators. Then, the postmeasurement state $\rho'_{S'}$ is stabilized by

$$\mathcal{S}' = \langle \mathcal{S} \cap \mathcal{C}(\mathcal{M}), \mathcal{M} \rangle. \quad (10)$$

In particular, since ρ_S is the maximally mixed state on a stabilizer codespace, the same formula describes the updated stabilizer code $\mathcal{S}'$ after measurement by $\mathcal{M}$.

A version of this statement can be found in standard references (e.g., Ref. [17]), and we include a proof here for completeness. We emphasize that the outcomes $\{m_j\}$ are assumed to be physically consistent with the initial stabilizer condition, so that the postmeasurement state is

nonzero (equivalently, $-I$ is not generated in the updated stabilizer group).

Proof. Using the stabilizer representation $\rho_S \propto \sum_{U \in \mathcal{S}} U$, we compute the postmeasurement state:

$$\begin{aligned} \rho'_{S'} &\propto \left(\prod_{i=1}^t (I + M_i) \right) \rho_S \left(\prod_{j=1}^t (I + M_j) \right) \\ &\propto \sum_{M, M' \in \mathcal{M}} \sum_{U \in \mathcal{S}} MUM'. \end{aligned} \quad (11)$$

Consider the sum over $\mathcal{M}$ for a fixed Pauli U :

$$\sum_{M, M' \in \mathcal{M}} MUM' = \left(\prod_{i=1}^t (I + M_i) \right) U \left(\prod_{i=1}^t (I + M_i) \right). \quad (12)$$

If $U \notin \mathcal{C}(\mathcal{M})$, then there exists some i such that $M_i U = -U M_i$, and hence $(I + M_i)U(I + M_i) = 0$. Therefore,

$$\sum_{M, M' \in \mathcal{M}} MUM' = 0 \quad (U \notin \mathcal{C}(\mathcal{M})). \quad (13)$$

It follows that

$$\begin{aligned} \rho'_{S'} &\propto \sum_{M, M' \in \mathcal{M}} \sum_{U \in \mathcal{S} \cap \mathcal{C}(\mathcal{M})} MUM' \propto \sum_{M \in \mathcal{M}} \sum_{U \in \mathcal{S} \cap \mathcal{C}(\mathcal{M})} MU \\ &\propto \sum_{U \in (\mathcal{S} \cap \mathcal{C}(\mathcal{M}), \mathcal{M})} U. \end{aligned} \quad (14)$$

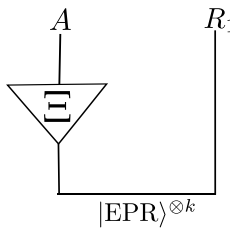
Hence we have $\mathcal{S}' = \langle \mathcal{S} \cap \mathcal{C}(\mathcal{M}), \mathcal{M} \rangle$. ■

B. Information preservation for stabilizer codes

We now derive necessary and sufficient conditions for preserving encoded information under projective measurements in stabilizer codes.

Suppose that k logical qubits are encoded into n physical qubits using a stabilizer code with stabilizer group $\mathcal{S}$, and that projective measurements are performed with respect to a set of commuting Pauli operators $\{M_1, \dots, M_m\}$. Intuitively, encoded information is preserved if and only if the measurements do not reveal any logical information of the code. Although this statement appears natural, we provide a rigorous formulation and proof. The key idea is to use the Choi-Jamiołkowski isomorphism to convert the problem of information preservation into a question about entanglement between the system and reference qubits.

Choi state: Let Ξ be the encoding isometry that maps an input k -qubit state $|\psi\rangle$ to an encoded n -qubit state $|\bar{\psi}\rangle$. The *Choi state* $|\Phi_{\text{Choi}}\rangle$ is constructed by applying Ξ to k maximally entangled (EPR) pairs:

$$|\Phi_{\text{Choi}}\rangle \equiv (\Xi \otimes I)|\text{EPR}\rangle^{\otimes k} =$$


$$(15)$$

Here, the Choi state $|\Phi_{\text{Choi}}\rangle$ is an $(n+k)$ -qubit state supported on the system A and the reference R .

Let $\mathcal{S}_{\text{code}}$ denote the stabilizer group of the original code. The stabilizer group $\mathcal{S}_{\text{Choi}}$ for the Choi state $|\Phi_{\text{Choi}}\rangle$ is

$$\mathcal{S}_{\text{Choi}} = \langle \mathcal{S}_{\text{code}} \otimes I, \{\bar{X}_j \otimes X_j, \bar{Z}_j \otimes Z_j\}_{j=1,\dots,k} \rangle, \quad (16)$$

where $\bar{X}_j, \bar{Z}_j$ are logical X_j, Z_j operators. The fact that k logical qubits are encoded can be seen by noting that the system A and the reference R are maximally entangled in the Choi state $|\Phi_{\text{Choi}}\rangle$. Specifically, the mutual information is maximal; $I(A, R) \equiv S_A + S_R - S_{AR} = 2k$.

The role of the reference system R is to keep track of the original k logical qubits. Accordingly, we define the post-measurement Choi state $|\Phi'_{\text{Choi}}\rangle$ by performing the same projective measurements M_j on subsystem A of the Choi state:

$$|\Phi'_{\text{Choi}}\rangle \propto \prod_{j=1}^t (I + M_j) |\Phi_{\text{Choi}}\rangle, \quad (17)$$

where each M_j acts on A and trivially on R . Note that after measurement, the effective number of logical qubits may decrease. Nevertheless, the fixed reference R provides a record for determining whether the original encoded information is preserved [18].

The condition for retaining the encoded information after projective measurement can then be expressed concisely in terms of the updated Choi state.

Lemma 1. The encoded information in the original k -logical-qubit stabilizer code is retained after projective measurements if and only if

$$I(A, R) = 2k \quad (18)$$

in the postmeasurement Choi state $|\Phi'_{\text{Choi}}\rangle$.

This follows because a quantum channel preserves the encoded information if and only if it preserves maximal entanglement with a reference system.

It is useful to rewrite the condition $I(A, R) = 2k$ in terms of stabilizer generators. Since the postmeasurement state

$|\Phi'_{\text{Choi}}\rangle$ is pure, the condition is equivalent to $S_R = k$. Using the entropy formula for stabilizer states [19], we have

$$S_R = |R| - \lambda_R, \quad (19)$$

where λ_R is the number of independent stabilizer generators supported entirely on R . Here, $|R| = k$, so the condition holds if and only if $\lambda_R = 0$, i.e., there is no nontrivial stabilizer element supported entirely on R .

Lemma 2. The encoded information in a stabilizer code is retained after projective measurements if and only if, in the Choi state, there is no nontrivial stabilizer operator supported only on R in the updated stabilizer group $\mathcal{S}'_{\text{Choi}}$.

Cleaning lemma for measurement: Define a group of Pauli logical operators by

$$\mathcal{L} \equiv \langle U \in \text{Pauli} : [U, \mathcal{S}] = 0 \rangle. \quad (20)$$

Note that stabilizer operators can be viewed as trivial elements of $\mathcal{L}$.

Theorem 2. The following three statements, concerning Pauli measurements on a stabilizer code, are equivalent:

- (i) The measured Pauli operator group $\mathcal{M}$ does not contain any nontrivial logical operator, namely,

$$\mathcal{M} \cap \mathcal{L} \subseteq \mathcal{S}. \quad (21)$$

- (ii) For any nontrivial logical operator ℓ , there always exists an equivalent logical operator $\ell' \sim \ell$ that commutes with $\mathcal{M}$, namely, $\ell' \in \mathcal{C}(\mathcal{M})$.
- (iii) The stabilizer code retains the encoded information after the measurements with respect to $\{M_1, \dots, M_m\}$. Namely, in the Choi state,

$$I(A, R) = 2k. \quad (22)$$

Intuitively, the statement (i) asserts that projective measurement defined by $\mathcal{M}$ should not reveal any encoded information of the code, and statement (ii) asserts that encoded logical qubits must be immune to projective measurements in the sense that their logical operators commute with $\mathcal{M}$. These two statements are equivalent, and together constitute the necessary and sufficient conditions for information preservation. The proof is provided in Appendix A.

This theorem can be interpreted as a generalization of the *cleaning lemma* [20]. Recall that, for stabilizer codes, the cleaning lemma states that the following two statements are equivalent.

- (a) A subset of qubits, denoted by A , is correctable, meaning that no logical operator can be supported

on A . In other words, the code can tolerate an erasure error that removes the qubits on A .

- (b) For any nontrivial logical operator ℓ , one can always find an equivalent expression ℓ' which is fully supported on the complement A^c . In other words, one can find a stabilizer operator U so that $\ell' = \ell U$ is supported exclusively on A^c .

Returning to the measurement setting, let us specialize to the case where $\mathcal{M}$ consists of projective measurements on each qubit in a subset A . By combining the statement (i) and (ii) from the theorem, we arrive at the following result.

Lemma 3. The following two statements are equivalent:

- (a) The stabilizer code retains the encoded information after projective measurements on qubits in A , which generate the group $\mathcal{M}$.
- (b) For any nontrivial logical operator ℓ , there exists an equivalent logical operator $\ell' \sim \ell$ which can be written as

$$\ell' = \ell'_{A^c} \otimes \ell'_A, \quad \text{where } \ell'_{A^c} \neq I_{A^c}, \quad \ell'_A \in \mathcal{M}. \quad (23)$$

Furthermore, $\ell'_{A^c} \otimes I_A$ is a logical operator for the postmeasurement code.

C. Information preservation for subsystem codes

We now extend the information preservation condition from stabilizer codes to subsystem codes [21], where logical and gauge degrees of freedom need to be carefully distinguished.

Subsystem code: Let $\mathcal{S}$ and $\mathcal{G}$ be the stabilizer and gauge groups of the subsystem code, respectively. Let $S_1, \dots, S_{n-k-g}$ be independent stabilizer generators of $\mathcal{S}$. Consider the subspace stabilized by $\mathcal{S}$; $\mathcal{H}_{\text{sub}} = \{|\psi\rangle : S_j|\psi\rangle = +|\psi\rangle\}$, which contains $g+k$ logical qubits. The central idea of subsystem codes is to use the k *bare logical* qubits only while ignoring g *gauge logical* qubits associated with the gauge group $\mathcal{G}$. The total Hilbert space can be written as a direct sum $\mathcal{H} = \bigoplus_{\vec{s}} \mathcal{H}^{(\vec{s})}$, where $\vec{s}$ represents a $(n-k-g)$ -component vector, which records eigenvalues of the S_j , and the code subspace is given by $\mathcal{H}_{\text{sub}} = \mathcal{H}^{(\vec{1})}$ with $s = \vec{1}$, where $\vec{1}$ denotes the all +1 eigenvalue sector. The Hilbert space further factorizes as follows [22]:

$$\mathcal{H} = \bigoplus_s \mathcal{H}_{\text{gauge}}^{(s)} \otimes \mathcal{H}_{\text{bare}}^{(s)}, \quad (24)$$

where the action of a gauge operator $U_{\text{gauge}} \in \mathcal{G}$ takes the following form:

$$U_{\text{gauge}} = \bigoplus_s U_{\text{gauge}}^{(s)} \otimes I_{\text{bare}}^{(s)}. \quad (25)$$

The stabilized code subspace $\mathcal{H}_{\text{sub}}$ can be factorized as $\mathcal{H}_{\text{sub}} = \mathcal{H}^{(\vec{1})} = \mathcal{H}_{\text{gauge}}^{(\vec{1})} \otimes \mathcal{H}_{\text{bare}}^{(\vec{1})}$, where $\mathcal{H}_{\text{gauge}}^{(\vec{1})}$ and $\mathcal{H}_{\text{bare}}^{(\vec{1})}$ support gauge and bare logical qubits, respectively [23].

Following Ref. [24], a group of *bare* logical operators is defined by

$$\mathcal{L}_{\text{bare}} \equiv \langle U \in \text{Pauli} : [U, \mathcal{G}] = 0 \rangle, \quad (26)$$

and its action has the following form:

$$U_{\text{bare}} = \bigoplus_s I_{\text{gauge}}^{(s)} \otimes U_{\text{bare}}^{(s)}. \quad (27)$$

Nontrivial bare Pauli logical operators must satisfy $U_{\text{bare}}^{(\vec{1})} \not\propto I$. Two bare logical operators are said to be equivalent when $\ell_{\text{bare}} \ell'_{\text{bare}} \in \mathcal{S}$. A group of *dressed* logical operators is defined by

$$\mathcal{L}_{\text{dressed}} \equiv \langle U \in \text{Pauli} : [U, \mathcal{S}] = 0 \rangle \quad (28)$$

and its action has the following form:

$$U_{\text{dressed}} = \bigoplus_s V_{\text{gauge}}^{(s)} \otimes U_{\text{bare}}^{(s)}. \quad (29)$$

Nontrivial dressed Pauli logical operators must satisfy $U_{\text{bare}}^{(\vec{1})} \not\propto I$. Two dressed logical operators are said to be equivalent when $\ell_{\text{dressed}} \ell'_{\text{dressed}} \in \mathcal{G}$.

Choi state: The Choi state for a subsystem code is constructed using the encoding isometry Ξ that maps an input $(g+k)$ -qubit state, consisting of k bare logical qubits and g gauge qubits, into an encoded n -qubit state. Introducing reference systems for both bare and gauge qubits allows us to track not only the preservation of logical information but also any possible leakage into gauge degrees of freedom under measurements. Accordingly, the Choi state is defined by applying the isometry Ξ to $(g+k)$ maximally entangled pairs:

$$|\Phi_{\text{Choi}}\rangle = (\Xi \otimes I_{\text{gauge}} \otimes I_{\text{bare}}) |\text{EPR}\rangle^{\otimes g+k} = \text{Diagram} \quad (30)$$

where $|\Phi_{\text{Choi}}\rangle$ is an $(n+g+k)$ -qubit state supported on the system A and the references R_{gauge} and R_{bare} . Let $\mathcal{S}_{\text{code}}$ be the original stabilizer group of the subsystem code.

The stabilizer group $\mathcal{S}_{\text{Choi}}$ for the Choi state $|\Phi_{\text{Choi}}\rangle$ is given by

$$\mathcal{S}_{\text{Choi}} = \langle \mathcal{S}_{\text{code}} \otimes I \otimes I, \bar{P}_{\text{gauge}} \otimes P_{\text{gauge}} \otimes I, \bar{Q}_{\text{bare}} \otimes I \otimes Q_{\text{bare}} \rangle, \quad (31)$$

where $\bar{P}_{\text{gauge}}$ and $\bar{Q}_{\text{bare}}$ are gauge and bare logical operators.

We define the postmeasurement Choi state $|\Phi'_{\text{Choi}}\rangle$ by performing the same projective measurements M_j on subsystem A of the Choi state $|\Phi_{\text{Choi}}\rangle$.

Lemma 4. The encoded information in a subsystem code is retained after projective measurements if and only if

$$I(A, R_{\text{bare}}) = 2k \quad (32)$$

in the postmeasurement Choi state $|\Phi'_{\text{Choi}}\rangle$.

Finally, we represent this entropic condition $I(A, R_{\text{bare}}) = 2k$ in terms of stabilizer generators. For the updated stabilizer group $\mathcal{S}'_{\text{Choi}}$, consider its restrictions to $R_{\text{gauge}}R_{\text{bare}}$, R_{gauge} , and R_{bare} , respectively. Let $\lambda_{R_{\text{gauge}}R_{\text{bare}}}$, $\lambda_{R_{\text{gauge}}}$, and $\lambda_{R_{\text{bare}}}$ denote the number of independent generators of $\mathcal{S}'_{\text{Choi}}|_{R_{\text{gauge}}R_{\text{bare}}}$, $\mathcal{S}'_{\text{Choi}}|_{R_{\text{gauge}}}$, and $\mathcal{S}'_{\text{Choi}}|_{R_{\text{bare}}}$, respectively. We then have following von-Neumann entropies for each subregions:

$$\begin{aligned} S_{R_{\text{gauge}}R_{\text{bare}}} &= g + k - \lambda_{R_{\text{gauge}}R_{\text{bare}}}, & S_{R_{\text{gauge}}} &= g - \lambda_{R_{\text{gauge}}}, \\ S_{R_{\text{bare}}} &= k - \lambda_{R_{\text{bare}}}. \end{aligned} \quad (33)$$

In order for R_{bare} to be maximally entangled with some other subsystem, we must have $S_{R_{\text{bare}}} = k$, and therefore

$$\lambda_{R_{\text{bare}}} = 0. \quad (34)$$

Furthermore, since the postmeasurement Choi state is pure, we have

$$\begin{aligned} I(A, R_{\text{bare}}) &= S_A + S_{R_{\text{bare}}} - S_{AR_{\text{bare}}} \\ &= S_{R_{\text{gauge}}R_{\text{bare}}} + k - S_{R_{\text{gauge}}}, \end{aligned} \quad (35)$$

which implies $S_{R_{\text{gauge}}R_{\text{bare}}} - S_{R_{\text{gauge}}} = k$. Hence, we arrive at the following condition:

$$\lambda_{R_{\text{gauge}}R_{\text{bare}}} = \lambda_{R_{\text{gauge}}}. \quad (36)$$

Lemma 5. The encoded information in a subsystem code is preserved if and only if, in the Choi state, every stabilizer element supported on $R_{\text{gauge}}R_{\text{bare}}$ acts trivially on R_{bare} .

Information preservation condition: We now state our main result concerning the condition for information preservation in a subsystem code.

Theorem 3. The following three statements, concerning Pauli measurements on a subsystem code, are equivalent:

- (i) The measured Pauli operator group $\mathcal{M}$ does not contain any nontrivial dressed logical operator, namely,

$$\mathcal{M} \cap \mathcal{L}_{\text{dressed}} \subseteq \mathcal{G}. \quad (37)$$

- (ii) For any nontrivial bare logical operator ℓ_{bare} , there always exists an equivalent bare logical operator $\ell'_{\text{bare}} \sim \ell_{\text{bare}}$ that commutes with $\mathcal{M}$, namely, $\ell'_{\text{bare}} \in \mathcal{C}(\mathcal{M})$.
- (iii) The subsystem code retains the encoded information after measurement of $\{M_1, \dots, M_m\}$, namely, in the Choi state,

$$I(A, R_{\text{bare}}) = 2k, \quad (38)$$

where A is the system and R_{bare} is the reference for the logical qubits.

The proof is provided in Appendix A. As a corollary, we obtain a version of cleaning lemma for measurements in a subsystem code.

Lemma 6. The following two statements are equivalent:

- (a) The subsystem code retains the encoded information after projective measurements that acts on the qubits in A with respect to $\mathcal{M}$.
- (b) For any nontrivial bare logical operator ℓ_{bare} , there always exists an equivalent logical operator $\ell'_{\text{bare}} \sim \ell_{\text{bare}}$, which can be written as

$$\begin{aligned} \ell'_{\text{bare}} &= \ell'_{\text{bare}, A^c} \otimes \ell'_{\text{bare}, A}, \quad \text{where } \ell'_{\text{bare}, A^c} \neq I_{A^c}, \\ \ell'_{\text{bare}, A} &\in \mathcal{M}. \end{aligned} \quad (39)$$

Furthermore, $\ell'_{\text{bare}, A^c} \otimes I_A$ is a bare logical operator for the postmeasurement code.

D. Measurement threshold versus erasure threshold

In the study of QECCs, one is often interested in erasure errors. Suppose that physical qubits are removed from the code randomly with probability p_e . The central question is whether the remaining $(1 - p_e)n$ qubits retain the information. Specifically, for code families whose asymptotic limit is well-defined, one is typically interested in the erasure threshold p_e^{th} below which the encoded information

is preserved with high probability. Suppose that physical qubits in a subset Q_e are removed from a QECC. The remaining system retains the encoded information if and only if Q_e does not support any (dressed) logical operator [25,26]. Comparing this condition with the condition for information preservation under measurement, we arrive at the following simple yet fundamental relation.

Theorem 4. The measurement threshold, for both stabilizer and subsystem codes, is always greater than or equal to the erasure threshold. That is,

$$p_e^{\text{th}} \leq p_m^{\text{th}}. \quad (40)$$

Note that a measurement threshold can be defined only for code families with a well-defined asymptotic limit. In this work, we take the asymptotic limit either by concatenating the code or by increasing the lattice size on which the code is defined. Also, note that the erasure threshold must satisfy $p_e^{\text{th}} \leq \frac{1}{2}$ due to the no-cloning theorem. Several examples of QECCs considered in this paper achieve $p_e^{\text{th}} = \frac{1}{2}$ and $p_m^{\text{th}} = 1$ for certain relative measurement frequencies $\vec{\alpha} = (\alpha_X, \alpha_Y, \alpha_Z)$.

III. RANDOMLY MONITORED CONCATENATED CODES

In this section, we study the measurement threshold for several examples of concatenated quantum codes and find that the measurement threshold p_m^{th} is significantly higher than the erasure threshold p_e^{th} . Specifically, we find that both the five-qubit code and the seven-qubit code achieve $p_m^{\text{th}} = 1$ under uniformly random Pauli measurements.

To recap the notations, each physical qubit is independently measured in a local Pauli basis (X , Y , or Z) with probabilities (p_X, p_Y, p_Z) , where the total measurement rate is given by $p = p_X + p_Y + p_Z$. Specifically, fixing the relative measurement frequencies $\vec{\alpha} = (\alpha_X, \alpha_Y, \alpha_Z)$ with $\alpha_X + \alpha_Y + \alpha_Z = 1$ and $(p_X, p_Y, p_Z) = (p\alpha_X, p\alpha_Y, p\alpha_Z)$, we increase p and ask if the encoded information is retained or not. We also denote the measurement probability of each Pauli logical operator $\bar{X}, \bar{Y}, \bar{Z}$ by $p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}}$.

A. Five-qubit code

We begin by studying the concatenated five-qubit code. In particular, we find that the measurement threshold is $p_m^{\text{th}} = 1$ under random measurements for arbitrary relative measurement frequencies $\vec{\alpha} = (\alpha_X, \alpha_Y, \alpha_Z)$. This suggests that the logical information is preserved under random measurements as long as $p = p_X + p_Y + p_Z < 1$.

Measurement threshold: The stabilizer group and logical operators are

$$\begin{aligned} \mathcal{S} &= \langle X_1 Z_2 Z_3 X_4 I_5, I_1 X_2 Z_3 Z_4 X_5, X_1 I_2 X_3 Z_4 Z_5, Z_1 X_2 I_3 X_4 Z_5 \rangle, \\ \bar{X} &= X_1 X_2 X_3 X_4 X_5, \quad \bar{Y} = Y_1 Y_2 Y_3 Y_4 Y_5, \quad \bar{Z} = Z_1 Z_2 Z_3 Z_4 Z_5. \end{aligned} \quad (41)$$

Under a uniformly random probability distribution ($\alpha_X = \alpha_Y = \alpha_Z = \frac{1}{3}$) with $p_X = p_Y = p_Z = \frac{p}{3}$, we find the probabilities of measuring X, Y, Z logical operators are also uniform with $p_{\bar{X}} = p_{\bar{Y}} = p_{\bar{Z}}$. Specifically, we find that the logical measurement probability p' is given by

$$p' = \frac{1}{9}(10p^3 - p^5) \quad (42)$$

after one concatenation, where $(p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}}) = (p'/3, p'/3, p'/3)$. This equation has two fixed points satisfying $p' = p$ at $p = 0, 1$ with $p = 1$ being an unstable fixed point. Namely, whenever $p < 1$, the updated probability p' decreases and eventually approaches to 0 in the limit of many concatenations. Hence, the measurement threshold of the five-qubit code is $p_m^{\text{th}} = 1$ under uniformly random measurements.

In fact, we find that, for *arbitrary* measurement frequency $\vec{\alpha} = (\alpha_X, \alpha_Y, \alpha_Z)$, the measurement threshold remains $p_m^{\text{th}} = 1$. In other words, as long as $p_X + p_Y + p_Z < 1$, the probability of measuring a logical operator approaches $(p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}}) \rightarrow (0, 0, 0)$ in the limit of many concatenations. We have numerically confirmed this by computing the flow of the information preservation probabilities (Fig. 1). Specifically, we started from the initial

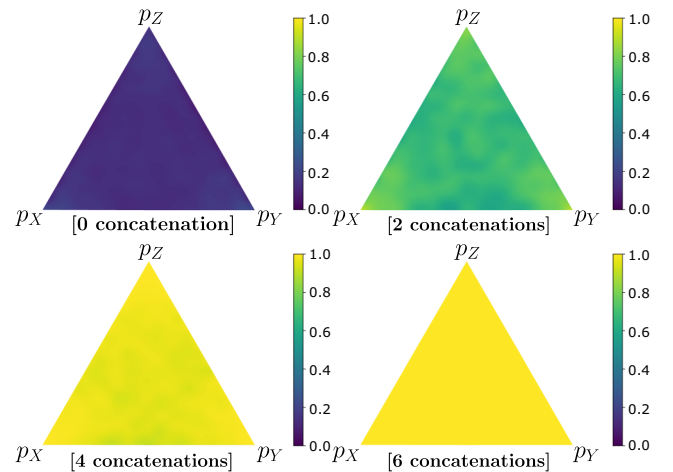


FIG. 1. Information preservation probability of the five-qubit code when $p = 0.95$. To obtain the data, we recursively computed the logical measurement probabilities at each round of concatenation by performing random measurements, using the logical measurement probability obtained in the preceding round. The random sampling size at each round of concatenation was 10^3 .

probability with $p = 0.95$ and computed the information preservation probability after successive concatenations of the code. After multiple rounds of concatenations, we find that the information preservation probability approaches 1 in all the probability distributions (p_X, p_Y, p_Z) . Hence, the concatenated five-qubit code exhibits extraordinary robustness against decoherence caused by measurements, satisfying $p_m^{\text{th}} = 1$ for relative measurement frequencies $\vec{\alpha} = (\alpha_X, \alpha_Y, \alpha_Z)$. This is in contrast with the erasure threshold of the concatenated five-qubit code; which is $p_e^{\text{th}} = \frac{1}{2}$.

It is worth mentioning that a hyperbolic tiling of the five-qubit code is often used as a toy model of the AdS/CFT correspondence [27]. For such a toy model of quantum error-correcting codes, previous works have found $p_m^{\text{th}} = 1$ and $p_e^{\text{th}} = \frac{1}{2}$ through a combination of analytical arguments and numerical analysis [15,27].

Logical measurement probability: Even though the five-qubit code exhibits an extraordinary information preservation ability under random measurements, the encoded quantum information will be eventually measured when all physical qubits are measured. Here, we study the probability distribution $(p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}})$ for the measured logical qubit when each physical qubit is measured with the probability distribution (p_X, p_Y, p_Z) satisfying $p_m = p_X + p_Y + p_Z = 1$. We will find that, for all the distributions of (p_X, p_Y, p_Z) except those with $p_X = 0, p_Y = 0$, and $p_Z = 0$, the probability distribution of the logical qubit measurement approaches a uniform distribution of $(p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ in the limit of many concatenations.

When all the qubits are measured in the same basis, the logical qubit is measured in one particular basis. Specifically, for $(p_X, p_Y, p_Z) = (1, 0, 0), (0, 1, 0), (0, 0, 1)$, we find $(p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}}) = (1, 0, 0), (0, 1, 0), (0, 0, 1)$. Next, when the physical qubits are measured in a pair of randomly chosen Pauli bases—say, in the X and Y basis—we find that the logical qubit will be measured in the two Pauli basis with equal probabilities. Namely, for $(p_X, p_Y, 0)$ with $p_X, p_Y \neq 0$, we find that each round of concatenation generates a

flow of the probability distributions toward the fixed point of $(p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}}) = (\frac{1}{2}, \frac{1}{2}, 0)$. Finally, for any other distribution of (p_X, p_Y, p_Z) with $p_X, p_Y, p_Z \neq 0$, we find that the probability distribution flows to the uniform distribution $(p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The flows of the probability distributions $(p_X, p_Y, p_Z) \rightarrow (p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}})$ under concatenation are illustrated in Fig. 2.

B. Seven-qubit code

We also study the concatenated seven-qubit code, another example that achieves $p_m^{\text{th}} = 1$ under uniformly random measurements.

Measurement threshold: The stabilizer group and logical operators are

$$S = \langle X_1 X_2 X_3 X_4, X_2 X_3 X_5 X_6, X_3 X_4 X_6 X_7, Z_1 Z_2 Z_3 Z_4, \\ Z_2 Z_3 Z_5 Z_6, Z_3 Z_4 Z_6 Z_7 \rangle,$$

$$\bar{X} = X_1 X_2 \cdots X_7, \quad \bar{Y} = Y_1 Y_2 \cdots Y_7, \quad \bar{Z} = Z_1 Z_2 \cdots Z_7. \quad (43)$$

The seven-qubit code can be viewed as the smallest 2D color code with a triangular boundary.

We have numerically computed the probability of measuring logical operators. Starting from the initial probability distribution (p_X, p_Y, p_Z) with $p_m = 0.95$ and obtained the logical probability distribution $(p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}})$ after seven rounds of concatenations. Figure 3 shows the probability of retaining the logical qubit, suggesting that the information preservation condition is $p_X, p_Y, p_Z < \frac{1}{2}$, and $p_m < 1$. Namely, the measurement threshold under uniformly random measurements is $p_m^{\text{th}} = 1$. This is in contrast with the erasure threshold of the seven-qubit code, which is $p_e^{\text{th}} = \frac{1}{2}$. In Sec. IV B, we will find that this condition qualitatively coincides with the one for the 2D color code.

Logical measurement probability: Next, we study the logical measurement probability distribution $(p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}})$ when all the physical qubits are measured with (p_X, p_Y, p_Z)

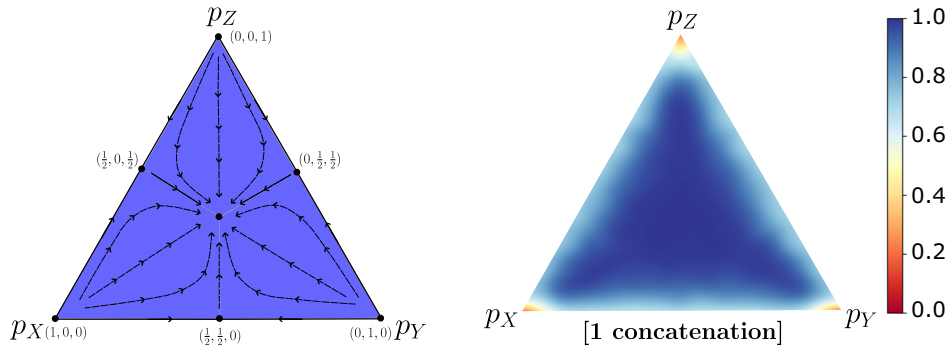


FIG. 2. (Left) Flow of logical measurement probability of the concatenated five-qubit code. (Right) Uncertainty of measured logical operators after one round of concatenation. Uncertainty of measured logical operators is quantified by Renyi-2 entropy, $S^{(2)}(p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}}) = -\log_3(p_{\bar{X}}^2 + p_{\bar{Y}}^2 + p_{\bar{Z}}^2)$. The data were obtained by recursively computing logical measurement probability with random sampling size 10^3 at each round of concatenation.

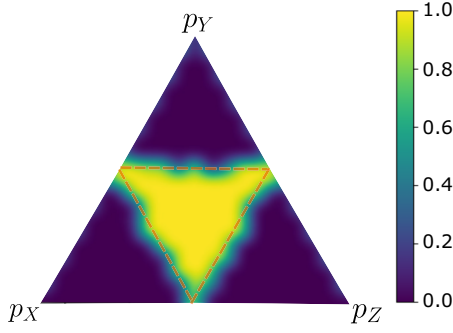


FIG. 3. Information preservation probability of the seven-qubit code after seven levels of concatenation when $p_m = 0.95$. The dashed lines indicate phase boundaries corresponding to $p_X = \frac{1}{2}$, $p_Y = \frac{1}{2}$, and $p_Z = \frac{1}{2}$. The data were obtained by recursively computing logical measurement probability with random sampling size 1.5×10^3 at each round of concatenations.

at $p_m = 1$ in the seven-qubit code. Unlike in the five-qubit code, we find that the uniform probability distribution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is an unstable fixed point in the flow generated by concatenation in the seven-qubit code. In fact, for generic distributions of (p_X, p_Y, p_Z) , the probability distributions converge to $(1, 0, 0)$, $(0, 1, 0)$, or $(0, 0, 1)$. The phase boundaries of the measured logical operator are shown in Fig. 4. Here it is useful to characterize the suppression of randomness with an entropic quantity. Specifically, we consider the Rényi-2 entropy:

$$S^{(2)}(p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}}) = -\log_3(p_{\bar{X}}^2 + p_{\bar{Y}}^2 + p_{\bar{Z}}^2), \quad (44)$$

where uniformly random distributions yield $S^{(2)}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 1$. Figure 4 shows how $S^{(2)}(p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}})$ evolves under concatenation.

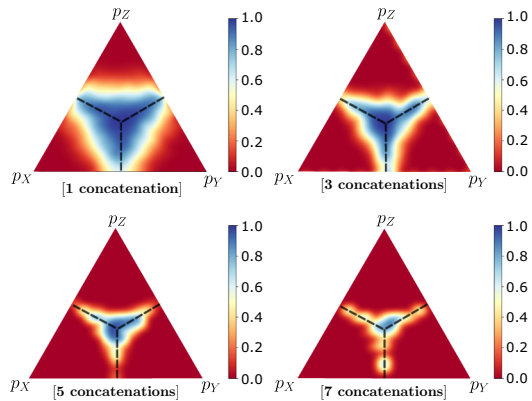
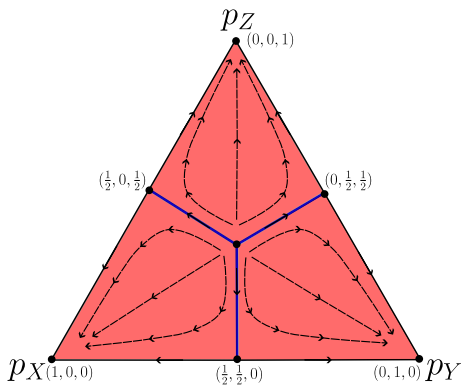


FIG. 4. (Left) The flow of logical measurement probability of the concatenated seven-qubit code. (Right) The uncertainty of the basis of measured logical operators after concatenations, measured in terms of the Rényi-2 entropy. The data were obtained by recursively computing logical measurement probability with random sampling size 10^3 at each round of concatenations.

C. 15-qubit code

Finally, we study a concatenated 15-qubit Reed-Muller code. We find that this code has measurement threshold strictly less than 1 under uniformly random measurements.

Measurement threshold: A convenient way to characterize the 15-qubit code is to view it as the 3D color code [28] supported on a tetrahedron as shown in Fig. 5, where qubits reside on vertices. The X stabilizer generators are defined by weight-eight X operators acting on each body cell, while Z stabilizer generators are defined by weight-four Z operators acting on each face cell.

One important observation is that the logical Z operator has a smaller minimum weight compared to the logical X and Y operators. Namely, the minimum weight of logical Z operator is three while that of the logical X and Y operators is seven. Due to the asymmetric weights of different logical operators, we expect that the logical Z operator will be more likely to be measured when uniformly random measurements are performed. Namely, we expect that the measurement threshold p_m^{th} will be lower than 1 for uniform relative measurement frequencies $(\alpha_X, \alpha_Y, \alpha_Z) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. We numerically confirm that this is indeed the case. Namely, we find $p_m^{\text{th}} \approx 0.6$, as shown in Fig. 6. We also numerically studied the measurement threshold p_m^{th} for other relative measurement frequencies $(\alpha_X, \alpha_Y, \alpha_Z)$ after five rounds of concatenation. We find that there is a region of relative measurement frequencies, with less frequent Z measurements, which achieve $p_m^{\text{th}} = 1$.

Logical measurement probability: We numerically studied the logical measurement probability distribution in the 15-qubit code (Fig. 7). We find that there exists a small region of measurement probabilities (p_X, p_Y, p_Z) , where both $\bar{X}$ and $\bar{Y}$ are measured with roughly equal probabilities. This region is clearly visible in the Rényi-2 entropy.

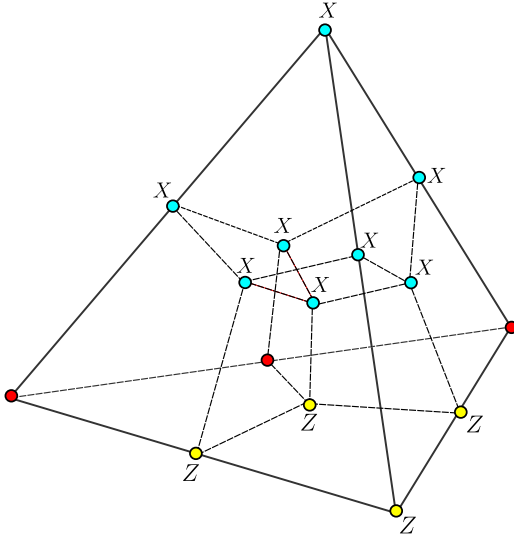


FIG. 5. The 15-qubit Reed-Muller code as the 3D color code. X stabilizer generators act on each body cell (colored blue) and Z stabilizer generators act on each face cell (colored yellow).

IV. RANDOMLY MONITORED TOPOLOGICAL CODE

In this section, we turn our attention to QECCs with geometrically local generators. Specifically, we demonstrate that the 2D toric and color codes achieve $p_m^{\text{th}} = 1$ under uniformly random measurements. We also discover an interesting correspondence between the randomness in measured logical probabilities ($p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}}$) and the measurement threshold p_m^{th} ; we have $p_m^{\text{th}} = 1$ under relative measurement frequencies ($\alpha_X, \alpha_Y, \alpha_Z$) if the measured logical probabilities ($p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}}$) have uncertainty at $p_m = 1$. Indeed, we will provide a proof for this observation in Sec. V. Finally, we study the 2D Bacon-Shor subsystem code and find that $p_m^{\text{th}} = 0$ for any relative measurement frequencies [29].

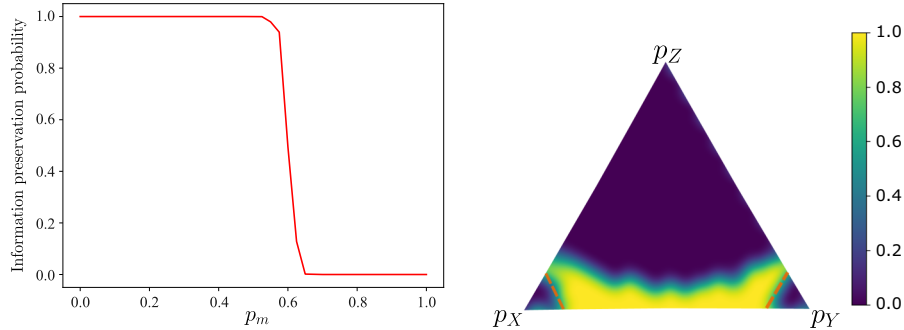


FIG. 6. (Left) Information preservation probability of the 15-qubit code after three rounds of concatenations. (Right) Information preservation probability of the 15-qubit code after five rounds of concatenations when $p_m = 0.95$. The orange dashed lines correspond to $p_X = 0.8$ and $p_Y = 0.8$. The data were obtained by recursively computing logical measurement probability with random sampling size 10^3 at each round of concatenations.

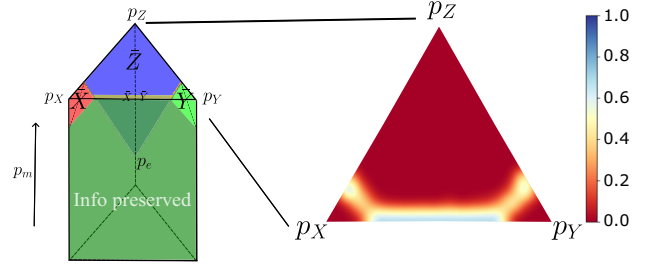


FIG. 7. Measured logical operators in the 15-qubit code after three rounds of concatenations. The numerical data suggest that a particular logical operator is measured in most regions, except for a small region with $p_Z \lesssim 0.95$, $p_X, p_Y \lesssim 0.8$ (colored green brown). The associated Rényi-2 entropy $S^{(2)}(p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}})$ is also shown. The data were obtained by recursively computing logical measurement probability with a random sampling size 10^3 at each round of concatenation.

A. 2D toric code

We show that the 2D toric code, supported on a square lattice, achieves $p_m^{\text{th}} = 1$ under uniformly random projective measurements. Furthermore, we determine the necessary and sufficient condition for retaining logical qubits in terms of (p_X, p_Y, p_Z) . We remark that the same result of this section has been also obtained in Ref. [16].

As shown in Fig. 8, one physical qubit is assigned to each edge of the lattice in the toric code. Stabilizer generators are given by two types of four-body Pauli operators: plaquette operators consisting of Pauli Z are defined on each face, and star operators consisting of Pauli X are defined on each vertex. Logical Pauli Z operators are non-contractible loops consisting of Pauli Z . Similarly, logical Pauli X operators are incontractible loops of Pauli X on the dual lattice.

Claim 1. Suppose random single-qubit Pauli measurements are performed on the 2D toric code on a square lattice with probability (p_X, p_Y, p_Z) . The encoded information

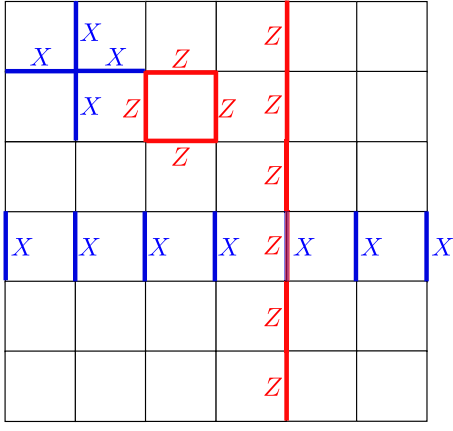


FIG. 8. Stabilizer generators and logical operators of the 2D toric code.

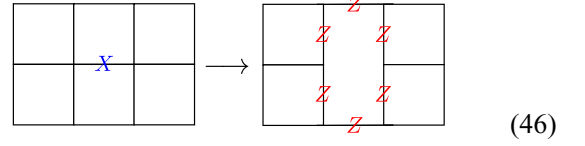
is preserved if and only if

$$p_X < \frac{1}{2}, \quad p_Z < \frac{1}{2}, \quad \text{and} \quad p_m < 1. \quad (45)$$

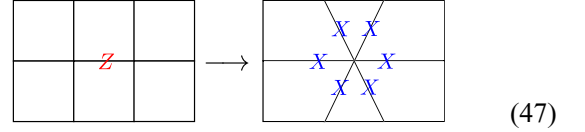
We first show that when Pauli X and Z measurements are randomly performed on the toric code with probabilities $p_X, p_Z < \frac{1}{2}$, the unmeasured qubits still retain the logical qubits of a toric code defined on a deformed 2D lattice. We then argue that Pauli Y measurements with $p_Y < 1$ on the 2D toric code do not destroy the logical qubits.

X and Z measurements: When a single X measurement is performed on an edge e , the two plaquette Z stabilizers that contain this edge do not commute with the measured X_e operator and are therefore removed from the stabilizer group. Instead, their product is added to form the updated stabilizer group. This effect can be visualized as

follows:



where the updated system remains a 2D toric code, but is now defined on a graph with the edge e removed. Similarly, when a single Z measurement is performed on an edge e , a pair of star X operators that include this edge is removed and their product is added to the stabilizer group instead. This leads to



where the edge e is removed and the two vertices connected by e are merged into a single vertex. Examples of such lattice deformations are shown in Fig. 9. It is worth noting that connected clusters of X and Z measurements create the so-called smooth and rough boundaries of the toric code, respectively, where large stabilizers surrounding the clusters can be interpreted as operators that measure the anyon types associated with these boundaries [30].

Recall that the bond percolation threshold of the 2D square lattice is $\frac{1}{2}$. That is, if Pauli X measurements are performed with a probability less than $\frac{1}{2}$, the measured qubits will form a set of disjoint clusters, where the probability of finding a large cluster is exponentially suppressed with respect to its linear size. Namely, most of the clusters have $\mathcal{O}(1)$ sizes although a small number of $\sim \log L$ clusters may still appear. Hence, after the X measurements, one will have a punctured 2D lattice with mostly finite-sized

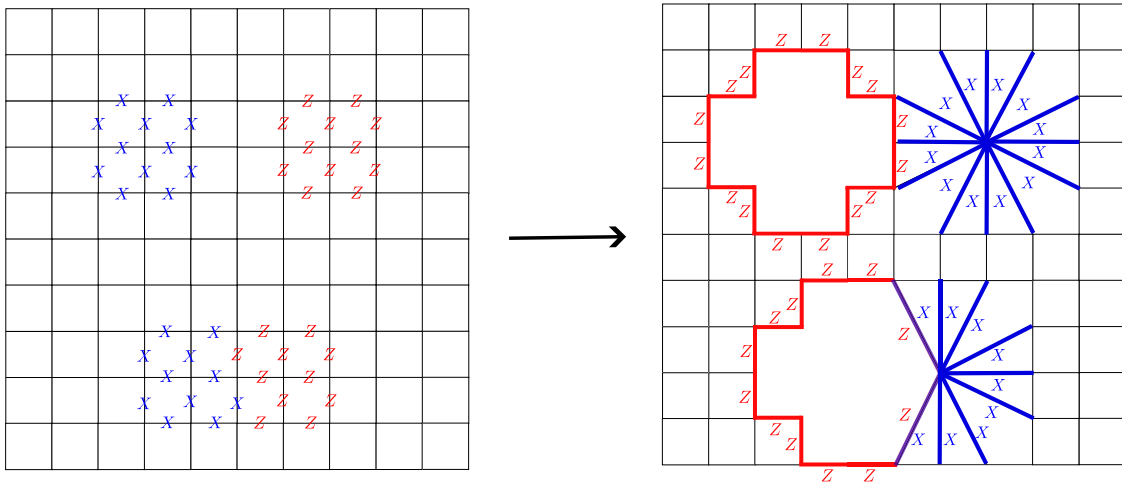


FIG. 9. Lattice deformation induced by Pauli X and Z measurements.

holes. Similarly, Pauli Z measurements with a probability less than $\frac{1}{2}$ will create a punctured lattice with holes whose vertices have been identified at their centers. The upshot is that, when $p_X, p_Z < \frac{1}{2}$ (and $p_Y = 0$), the remaining unmeasured qubits will form the toric code on a deformed lattice that has the topology of a torus.

Y measurements: Next, we analyze the probability of measuring a logical operator by Y measurements while assuming $p_X = p_Z = 0$. The key observation is that Y -type logical operators, those consisting entirely of Pauli Y operators, are exceedingly rare compared to X -type and Z -type logical operators. As such, the probability of measuring a Y -type logical operator remains exponentially suppressed even when p_Y is close to 1.

We begin by finding all the Y -type operators that commute with stabilizer generators. These operators will be referred to as *Y-commutant operators*. Note that a Y -commutant operator can be either a stabilizer or a logical operator. It will be convenient to consider a different lattice in which qubits reside on vertices as shown in Fig. 10. Note that the figure is 45 degree rotated for simplicity of presentation. In this rotated lattice, both X -type and Z -type stabilizer generators are defined on faces and placed in an alternating pattern while physical qubits remain on vertices. By inspecting commutation relations with stabilizer generators, one finds that Y -commutant operators can be generated by straight linelike operators, associated to each row and column of the lattice, as shown in Fig. 10. Namely, one can consider $2L$ of such linelike operators where L is the linear length of the rotated lattice. Note that only $2L - 1$ of them are independent as a product of all the linelike Y -type operators is an identity. It is worth emphasizing that, unlike X - and Z -type logical operators, these linelike Y -commutant operators are not deformable.

Suppose that a Y -commutant operator ℓ is constructed by using a row operators and b column operators ($0 \leq a, b \leq L$). The weight of such an operator is

$$W(a, b) = (L - a)b + (L - b)a. \quad (48)$$

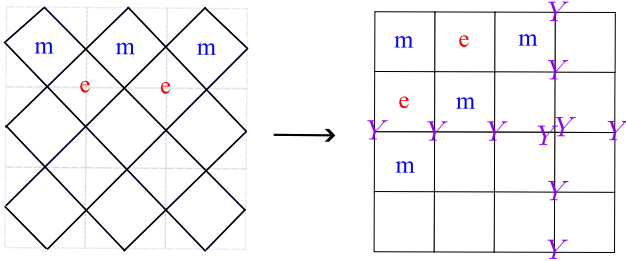


FIG. 10. Stabilizer generators in the 45-degree rotated toric code. Physical qubits are located at each vertex of the lattice. The Y -commutant operators form straight lines, and are not deformable.

The probability of measuring a logical qubit can be bounded by summing the probabilities of measuring individual Y -commutant operators. Specifically, we have

$$\begin{aligned} P_{\text{destroy}} &\leq \sum_{0 \leq a, b \leq L} \binom{L}{a} \binom{L}{b} p^{W(a, b)} \\ &= 2 \sum_{0 \leq a, b \leq L/2} \binom{L}{a} \binom{L}{b} p^{W(a, b)} \\ &\quad + 2 \sum_{0 \leq a \leq L/2} \sum_{L/2 < b \leq L} \binom{L}{a} \binom{L}{b} p^{W(a, b)}. \end{aligned} \quad (49)$$

Using the inequality $\binom{n}{k} < (ne/k)^k$, we can bound the first term as

$$\begin{aligned} &\sum_{0 \leq a, b \leq L/2} \binom{L}{a} \binom{L}{b} p^{W(a, b)} \\ &< \sum_{0 \leq a, b \leq L/2} \left(\frac{Le}{a}\right)^a \left(\frac{Le}{b}\right)^b p^{(L-a)b + (L-b)a} \\ &< \sum_{0 \leq a, b \leq L/2} \left(\frac{Le}{a}\right)^a \left(\frac{Le}{b}\right)^b p^{L(a+b)/2} \\ &= \sum_{0 \leq a, b \leq L/2} \left(\frac{Le}{a} p^{L/2}\right)^a \left(\frac{Le}{b} p^{L/2}\right)^b \rightarrow 0 \quad (L \rightarrow \infty), \end{aligned} \quad (50)$$

where we used the fact that $(Le/a)p^{L/2} \approx p^{L/2}$ approaches zero exponentially. The second term can be bounded in a similar fashion. Hence, when $p_Y < 1$, the probability of measuring a Y -type logical operator is exponentially suppressed with respect to L .

X, Y, Z measurements: Finally, we consider the case of $p_X, p_Y, p_Z \neq 0$. Here, we provide a heuristic argument to bound P_{destroy} while providing numerical evidence later. After performing X, Z measurements, we will have a 2D toric code supported on a deformed square lattice with finite-sized punctures (see the discussion above). To find Y logical operators, we begin by constructing a new lattice in which qubits reside on vertices as shown in Fig. 11. This new lattice has a structure locally similar to a square lattice where plaquette and star (m and e) operators are placed in an alternating pattern. Most importantly, vertices are always 4-valent. Due to this special structure, generators of Y -commutant operators can be constructed by linelike operators that always move straight at each vertex as shown in Fig. 11.

In principle, one can construct all the Y -commutant generators by tracing a straight line path until it returns to the starting vertex from the opposite direction. Unlike the original square lattice, however, a straight line moving on a

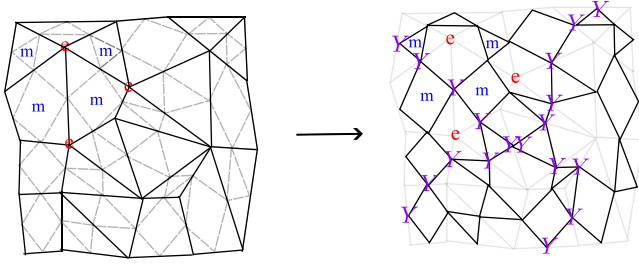


FIG. 11. The toric code on the deformed lattice. The Y -commutant operators form locally straight lines.

deformed lattice may form a closed loop of finite size that does not go around the torus. The probability of measuring such small Y operators remains finite, and thus, our strategy of bounding P_{destroy} by adding all the measurement probabilities would not work. Fortunately, such small Y operators must be stabilizer operators since a logical operator in the deformed toric code must go around the torus. This suggests that, to bound P_{destroy} , it suffices to consider probabilities of measuring Y -commutant operators whose sizes are at least $O(L)$.

In constructing Y -commutant generators, each edge can be used at most once. Hence, there are at most $O(L)$ such generators. Here, we need to find the probability distributions of the weights of Y -commutant operators. We expect that, due to the lattice deformation, the number of large Y generators is actually small, possibly $\sim L^\alpha$ with $\alpha < 1$, since closed loops will be harder to form. We also expect that nontrivial Y generators tend to have larger weights, possibly $\sim L^\beta$ with $\beta > 1$. As such, measuring Y logical operators will be even harder on a deformed lattice. Based on these observations, we claim that the 2D toric code on a randomly deformed lattice retains logical qubits after Y measurements. We numerically confirmed this as shown in Fig. 12.

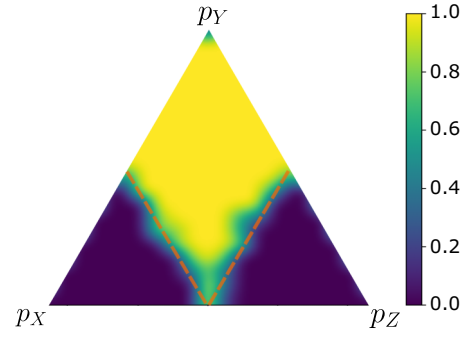


FIG. 12. Information preservation probability in the 2D toric code with $p = 0.95$. The dashed lines indicate the percolation threshold at $p_X = \frac{1}{2}$ and $p_Z = \frac{1}{2}$. Numerical simulations were performed on a 35×35 toric code with a random sampling size 10^2 .

Logical measurement probability: We also studied the logical measurement probability at $p = 1$. The result is summarized in Fig. 13. We find that, for $p_X > \frac{1}{2}$ and $p_Z > \frac{1}{2}$, particular logical operators, namely, $\bar{X}_1, \bar{X}_2$ and $\bar{Z}_1, \bar{Z}_2$, are measured, respectively. Also, for $p_Y \approx 1$, we find that $\bar{Y}_1, \bar{Y}_2$ are always measured at the large lattice size limit for odd L , and $\bar{X}_1 \bar{X}_2, \bar{Z}_1 \bar{Z}_2$ for even L . For all other distributions of (p_X, p_Y, p_Z) , i.e., $p_X < \frac{1}{2}$, $p_Z < \frac{1}{2}$, and $p_Y < 1$, we observed that measured logical operators are uncertain. Specifically, we find that there are ten possible sets of logical operators to be measured; $\{\bar{P}_1, \bar{Q}_2\}$ and $\{\bar{X}_1 \bar{X}_2, \bar{Z}_1 \bar{Z}_2\}$, where $P, Q = X, Y, Z$. The uncertainty is quantified by the Rényi-2 entropy $S^{(2)} = -\log_4(p_{\bar{X}_1}^2 + p_{\bar{Y}_1}^2 + p_{\bar{Z}_1}^2 + p_{\bar{X}_1 \bar{X}_2}^2)$, as shown in the heatmap in Fig. 13. Here $p_{\bar{P}_1}$ represents the probability of measuring $\bar{P}_1$, combining three cases $\{\bar{P}_1, \bar{X}_2\}$, $\{\bar{P}_1, \bar{Y}_2\}$, and $\{\bar{P}_1, \bar{Z}_2\}$.

We have also studied the transition between the uncertain region and the no-uncertainty region by changing p_X while fixing $p_Y = p_Z$. The Rényi-2 entropy $S^{(2)}$ is plotted in Fig. 13 where we observe a transition at $p_X \approx \frac{1}{2}$ as

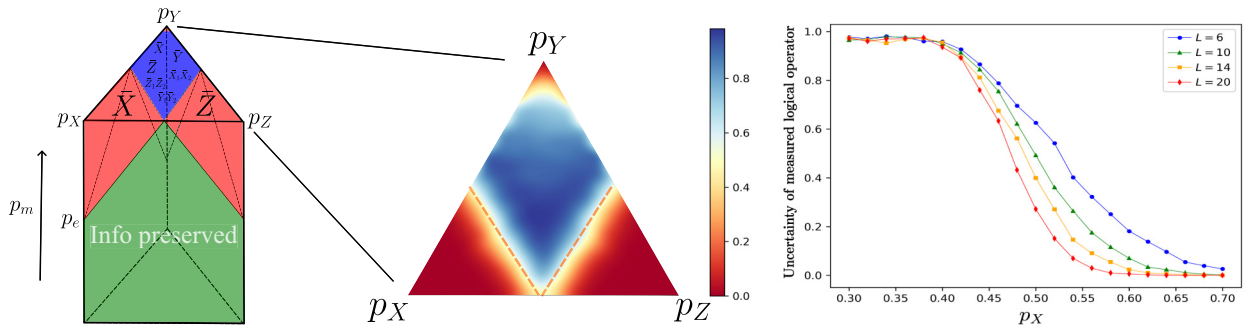


FIG. 13. (Left) A schematic picture showing the information preservation diagram of the 2D toric code under (p_X, p_Y, p_Z) random measurements. Red: encoded information is measured by some particular logical operators. Blue: encoded information is measured with uncertainty in logical measurement probabilities. Green: encoded information is preserved. (Middle) The uncertainty of measured logical operators at $p = 1$ for a 10×10 toric code. The data were obtained using random sampling with sampling size 10^3 . (Right) The uncertainty transition at the $p_Y = p_Z$ line.

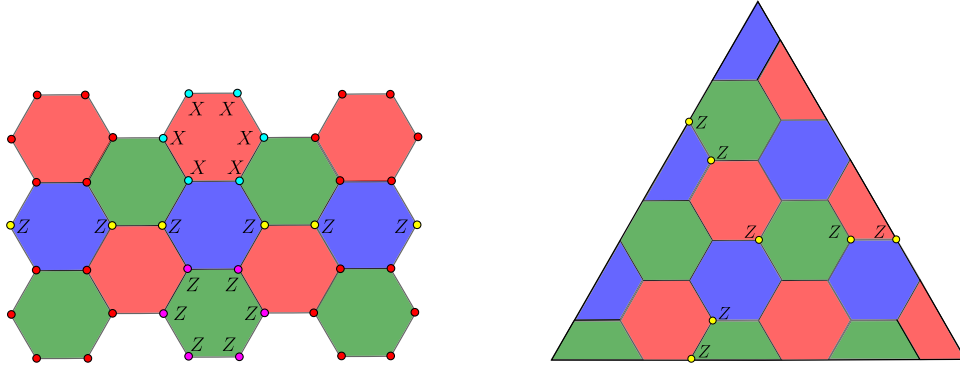


FIG. 14. (Left) A topological color code defined on the 2D hexagonal lattice. (Right) A topological color code with triangular boundaries.

expected. We were not able to determine from the data if this transition is a smooth one or not, and we leave this question for future studies.

Information preservation diagram: We have addressed the information preservation probability and the logical measurement probability. Our result suggests an intriguing relationship between them, as illustrated in Fig. 13. In this diagram, we aim to characterize the fate of encoded information for arbitrary distributions (p_X, p_Y, p_Z) including those with $p < 1$ and $p = 1$ using a 3D figure. The red shaded regions correspond to random measurement probability distributions where encoded information is measured by specific logical operators. This includes two tetrahedral regions with $p_X > \frac{1}{2}$ and $p_Z > \frac{1}{2}$, and a single point corresponding to $p_Y = 1$. The green shaded region corresponds to (p_X, p_Y, p_Z) where encoded information is still preserved with $p_X < \frac{1}{2}$, $p_Z < \frac{1}{2}$, and $p < 1$. Finally, the blue-shaded region emerges at the top boundary of the green region, corresponding to (p_X, p_Y, p_Z) that exhibit uncertainty in logical measurement probabilities. From this figure, one may speculate that there is an intrinsic relationship between the measurement threshold p_m^{th} and the uncertainty $S^{(2)}$ of measured logical operators. Specifically, for relative measurement frequencies $(\alpha_X, \alpha_Y, \alpha_Z)$ that achieve $p_m^{\text{th}} = 1$, we observe $S^{(2)} > 0$ when $p = 1$. On the other hand, when $p_m^{\text{th}} < 1$, we find $S^{(2)} = 0$ when $p = 1$. A similar relation will be seen in the 2D color code.

B. 2D color code

Next, we study an example of 2D topological code that has symmetry under exchanges of X, Y, Z operators. The 2D color code is defined on a graph that satisfies both valence and colorability conditions. Specifically, each vertex of a graph must be connected to three edges, and each face should be three-colorable in a way that two adjacent faces do not share the same color (Fig. 14). A hexagonal lattice is one example that satisfies these conditions. Given

such a graph, physical qubits reside at each vertex and stabilizer generators are defined on each face, consisting of only Pauli Z or Pauli X . Logical operators are given by string operators that connect faces with the same colors.

We numerically studied the information preservation condition for the 2D color code. Our results, summarized in Fig. 15, suggest that encoded information is preserved if and only if the random local measurements are performed with probability $p_X, p_Y, p_Z \lesssim \frac{1}{2}$, and $p < 1$.

We also numerically studied the uncertainty of measured logical operators at $p = 1$. The result is summarized in Fig. 16. We find that, for $p_X, p_Y, p_Z \gtrsim \frac{1}{2}$, the encoded information is measured by specific logical operators. On the other hand, for $p_X, p_Y, p_Z \lesssim \frac{1}{2}$, the logical measurement probabilities exhibit uncertainty, and the encoded information is measured by $\bar{X}, \bar{Y}$, or $\bar{Z}$ with roughly equal probabilities. The associated Rényi-2 entropy is plotted in Fig. 16.

From Figs. 15 and 16, we again observe that relative measurement frequencies $(\alpha_X, \alpha_Y, \alpha_Z)$ that achieve $p_m^{\text{th}} = 1$

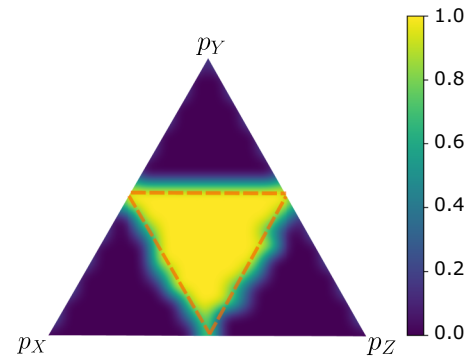


FIG. 15. Information preservation diagram of 2D color code at $p = 0.95$. Simulation was performed using a color code with triangular boundaries of length 35. The orange dashed lines correspond to $p_X = \frac{1}{2}, p_Y = \frac{1}{2}, p_Z = \frac{1}{2}$ lines. The data were obtained by random sampling with sampling size 10^2 .

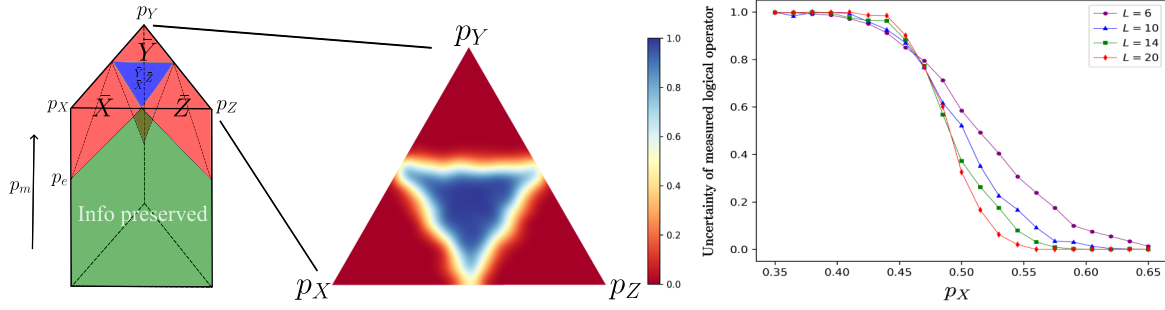


FIG. 16. (Left) Information preservation diagram of the 2D color code. (Middle) Uncertainty of the measured logical operator at $p = 1$. The uncertainty of measured logical information is quantified by $S^{(2)} = -\log(p_X^2 + p_Y^2 + p_Z^2)$. (Right) The uncertainty transition at $p_Y = p_Z$ line.

coincide with those that exhibit uncertainty in $(p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}})$ at $p = 1$.

C. 2D Bacon-Shor code

Finally, we discuss the measurement threshold of the 2D Bacon-Shor code. The gauge group is given by

$$\mathcal{G} = \langle X_{ij} X_{i+1,j}, Z_{ij} Z_{i,j+1} \rangle \quad (51)$$

as depicted in Fig. 17. Here, we show that $p_m^{\text{th}} = 0$.

We consider performing X measurement with probability $p_X > 0$. Observe that a tensor product of X operators along the vertical direction commutes with all the gauge operators, and thus constitutes a bare logical operator as shown in Fig. 17. By deforming this bare logical operator by X -type gauge operators, we can create a dressed logical operator that has single X supported at arbitrary location on each row. In other words, having an X operator on each row creates a dressed logical operator. Randomly selecting qubits with probability p_X will pick at least one qubit from each row with high probability. Hence, the measurement group $\mathcal{M}$ will contain a dressed logical operator, suggesting the loss of the encoded information for any $p_X > 0$ and $p_Z > 0$.

Similar arguments can be applied to the case with Y measurements. One can create a Y -type dressed logical

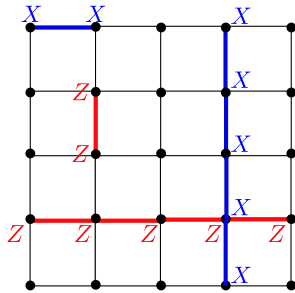


FIG. 17. Gauge operators and bare logical operators of 2D Bacon-Shor code.

operator by multiplying X -type and Z -type dressed logical operators that are supported on the same set of qubits. Specifically, a Y -type dressed logical operator has single Y support on each row and column. When $p_Y > 0$, out of L qubits in each row, $\sim p_Y L$ qubits will be measured in Y . For large L , the probability of having a row with no Y measurements is exponentially suppressed. A similar observation applies to measurements in each column. Therefore, the encoded information will be lost for any $p_Y > 0$.

V. MEASUREMENT UNCERTAINTY AND MEASUREMENT THRESHOLD

In previous sections, we observed that many stabilizer codes achieve the maximal measurement threshold $p_m^{\text{th}} = 1$ for certain choices of measurement frequencies $\vec{\alpha} = (\alpha_X, \alpha_Y, \alpha_Z)$. This includes concatenated five-qubit and seven-qubit codes, as well as the 2D toric and 2D color codes. Beyond this common feature, we also observed an additional interesting pattern. Whenever the Pauli basis of the logical operator measured at $p = 1$ is not uniquely determined (meaning that multiple logical Pauli operators are measured with finite probabilities at the large n limit), the code always appears to have the maximal measurement threshold. Namely, when the logical measurement probability distribution $(p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}})$ exhibits an intrinsic uncertainty, the code has been observed to satisfy $p_m^{\text{th}} = 1$.

Motivated by these observations, we establish a general structural relation between logical measurement uncertainty at $p = 1$ and the measurement threshold p_m^{th} . Specifically, we will prove that, if a family of stabilizer codes has a measurement threshold strictly less than one, $p_m^{\text{th}} < 1$, then the logical measurement outcome at $p = 1$ becomes asymptotically deterministic with one of $p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}}$ being unity. Taking the contrapositive, we obtain

$$\text{uncertainty in } (p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}}) \Rightarrow p_m^{\text{th}} = 1. \quad (52)$$

Throughout this section, we will focus on stabilizer codes encoding a single logical qubit ($k = 1$). This is essentially due to the fact that, for $k > 1$, one may consider

two decoupled codes, one with $p_m^{\text{th}} = 1$ and another with $p_m^{\text{th}} = 0$, whose joint system would not satisfy the relation above. Thus, the statement we prove concerns the measurement statistics of a single logical qubit. Nevertheless, the above relation can be straightforwardly generalized to the $k > 1$ code as long as the uncertainty in the logical measurement probability distributions extend nontrivially to all the k logical qubits.

We will also assume that the family of stabilizer codes under consideration has a well-defined measurement threshold in the thermodynamic limit. More precisely, consider a sequence of $k = 1$ stabilizer codes with increasing n . Fix a Pauli measurement frequency $\vec{\alpha}$, and let $P_{\text{destroy}}(p)$ denote the probability that a nontrivial logical operator is measured when np qubits are independently measured in random Pauli bases according to $\vec{\alpha}$. We assume that there exists a constant threshold p_m^{th} such that

$$P_{\text{destroy}}(p) = O\left(\frac{1}{\text{poly}(n)}\right), \quad p < p_m^{\text{th}}, \quad (53)$$

$$P_{\text{destroy}}(p) = 1 - O\left(\frac{1}{\text{poly}(n)}\right), \quad p > p_m^{\text{th}}. \quad (54)$$

In other words, we assume a sharp transition in the logical destruction probability as n increases. We do not attempt to prove the existence of such a sharp threshold; we assume it holds for the code families under consideration. The polynomial suppression is a mild assumption; in most stabilizer codes studied in previous sections, the suppression is in fact expected to be exponential in the code distance d .

Under these assumptions, we prove the following theorem.

Theorem 5. Consider a family of $k = 1$ stabilizer codes with a constant measurement threshold $p_m^{\text{th}} < 1$ for a fixed measurement frequency $\vec{\alpha} = (\alpha_X, \alpha_Y, \alpha_Z)$ as n increases. Then there exists a fixed Pauli logical operator $\bar{\ell} \in \{\bar{X}, \bar{Y}, \bar{Z}\}$ such that

$$p_{\bar{\ell}} = 1 - O\left(\frac{\log(n)}{\text{poly}(n)}\right). \quad (55)$$

Equivalently, if the logical Pauli measurement probabilities remain uncertain at $p = 1$, then the measurement threshold must satisfy $p_m^{\text{th}} = 1$.

A. Proof

Assume that all the n qubits are measured in the X, Y, Z basis according to the measurement frequency $\vec{\alpha} = (\alpha_X, \alpha_Y, \alpha_Z)$. We will denote the list of chosen Pauli measurement patterns by $M = \{M_1, M_2, \dots, M_n\}$, where $M_j =$

$\pm X_j, \pm Y_j, \pm Z_j$ with $\pm$ indicating the sign of the measurement outcome. Each of M occurs with a probability

$$\text{Prob}(M) = (\alpha_X)^{n_X} (\alpha_Y)^{n_Y} (\alpha_Z)^{n_Z} \quad (56)$$

that depends on the total numbers of X, Y, Z Pauli operators appearing in M . Given the Pauli list M , let us denote the Pauli basis of the measured logical operator by $\bar{\ell}(M) = \bar{X}, \bar{Y}, \bar{Z}$. Given a subsystem R , we denote the restriction of the list M to the subsystem R by $M|_R$. By considering the group $\langle M|_R \rangle$ generated by single-body Pauli operators in the restricted list $M|_R$, one can discuss whether a nontrivial Pauli logical operator has been measured by the projective measurements on R or not.

We begin with a simple observation: if changing the measurement pattern only on a subsystem A^c can change which logical Pauli operator is measured, then the measurement pattern on the complement A does not support any nontrivial logical operator.

Lemma 7. Let M, M' be two Pauli measurement patterns such that $M|_A = M'|_A$, but $\bar{\ell}(M) \neq \bar{\ell}(M')$. Then $M|_A$ supports no nontrivial Pauli logical operator.

Note that this statement applies universally to any stabilizer code with $k = 1$.

Proof. Suppose for contradiction that $M|_A$ supports a nontrivial Pauli logical operator. Since M and M' coincide on A , the same logical operator is supported by $M'|_A$ as well. In that case, the logical Pauli determined by the full patterns M and M' must coincide, i.e., $\bar{\ell}(M) = \bar{\ell}(M')$, contradicting the assumption. ■

We now prove Theorem 5. Focusing on the $k = 1$ stabilizer codes with $p_m^{\text{th}} < 1$ for a fixed measurement frequency $\vec{\alpha}$, we establish the following result.

Lemma 8. Let M, M' be two Pauli measurement patterns sampled randomly with the measurement frequency $\vec{\alpha} = (\alpha_X, \alpha_Y, \alpha_Z)$. Then, the Pauli basis of measured logical operators match with high probability, namely,

$$\text{Prob}\left(\bar{\ell}(M) = \bar{\ell}(M')\right) \geq 1 - O\left(\frac{\log(n)}{\text{poly}(n)}\right). \quad (57)$$

Lemma 8 implies Theorem 5: since M and M' are independently drawn from the $p = 1$ measurement ensemble, the logical Pauli label must concentrate on a single value $\bar{\ell} \in \{\bar{X}, \bar{Y}, \bar{Z}\}$ up to $O(\log(n)/\text{poly}(n))$ corrections.

Proof. Assume that the measurement threshold satisfies $p_m^{\text{th}} = 1 - \epsilon$ for some constant $\epsilon > 0$. Let us choose an

integer m such that

$$\frac{1}{m} < \epsilon \quad \left(\text{equivalently, } 1 - \frac{1}{m} > p_m^{\text{th}} \right).$$

We consider a sequence of l subsystems $A_1, \dots, A_l$, where each subsystem A_j is a randomly drawn $1/m$ fraction of n physical qubits. In other words, the complement A_j^c contains a fraction $1 - 1/m > p_m^{\text{th}}$ of the qubits.

We will compare two independent full-measurement patterns $M = M^{(0)}$ and $M' = M^{(l)}$ by interpolating between them with replacing the Pauli patterns on randomly drawn subsystems A_j . Let $M^{(j)}|_{A_j}$ be independently sampled Pauli measurement patterns on A_j according to the fixed relative frequencies $\tilde{\alpha}$. Define

$$M^{(0)} = M^{(0)}|_{A_1} \cup M^{(0)}|_{A_1^c}, \quad (58)$$

$$M^{(1)} = M^{(1)}|_{A_1} \cup M^{(0)}|_{A_1^c} = M^{(1)}|_{A_2} \cup M^{(1)}|_{A_2^c}, \quad (59)$$

$$M^{(2)} = M^{(2)}|_{A_2} \cup M^{(1)}|_{A_2^c} = M^{(2)}|_{A_3} \cup M^{(2)}|_{A_3^c}, \quad (60)$$

$\vdots$

$$M^{(l)} = M^{(l)}|_{A_l} \cup M^{(l-1)}|_{A_l^c}. \quad (61)$$

Heuristically, we can think of generating $M^{(j)}$ from $M^{(j-1)}$ by replacing the Pauli measurement pattern only on randomly drawn subsystem A_j . We now argue that $M^{(0)}$ and $M^{(l)}$ become independent instances of the random full-measurement pattern at $p = 1$ when $l = O(\log(n))$. Indeed, since A_j is randomly drawn for each step, after l number of rounds, the probability that a fixed single qubit has never been included in any A_j is given by $(1 - 1/m)^l$. Thus, on average, $n(1 - 1/m)^l$ number of qubits are not included in any of A_j . Using Markov's inequality, we obtain

$$\text{Prob} \left(\bigcap_{j=1}^l A_j^c \neq \emptyset \right) \leq n \left(1 - \frac{1}{m} \right)^l. \quad (62)$$

Thus, by taking $l = O(\log n)$, we can make $M^{(0)}$ and $M^{(l)}$ almost surely independent instances of the random full-measurement pattern with probability $1 - O(1/\text{poly}(n))$. It suffices now to bound the probability that the logical Pauli label changes somewhere along the interpolation:

$$\begin{aligned} & \text{Prob} \left(\overline{\ell(M^{(0)})} \neq \overline{\ell(M^{(l)})} \right) \\ & \leq \sum_{j=1}^l \text{Prob} \left(\overline{\ell(M^{(j-1)})} \neq \overline{\ell(M^{(j)})} \right), \end{aligned}$$

where we used the union bound. For fixed j , the patterns $M^{(j-1)}$ and $M^{(j)}$ differ only on A_j , and coincide on A_j^c .

Therefore, by Lemma 7,

$$\begin{aligned} & \left\{ \overline{\ell(M^{(j-1)})} \neq \overline{\ell(M^{(j)})} \right\} \\ & \subseteq \left\{ M^{(j-1)}|_{A_j^c} \text{ supports no nontrivial logical operator} \right\}. \end{aligned}$$

But $M^{(j-1)}|_{A_j^c}$ is a random Pauli measurement pattern on A_j^c with measurement rate $p = 1 - 1/m > p_m^{\text{th}}$. By the sharp-threshold assumption, the probability that A_j^c fails to support a nontrivial logical operator is $O(1/\text{poly}(n))$. Hence for each j ,

$$\text{Prob} \left(\overline{\ell(M^{(j-1)})} \neq \overline{\ell(M^{(j)})} \right) = O \left(\frac{1}{\text{poly}(n)} \right).$$

Summing over $j = 1, \dots, l$ yields

$$\begin{aligned} \text{Prob} \left(\overline{\ell(M^{(0)})} \neq \overline{\ell(M^{(l)})} \right) & \leq l \cdot O \left(\frac{1}{\text{poly}(n)} \right) \\ & = O \left(\frac{\log(n)}{\text{poly}(n)} \right). \end{aligned}$$

This proves the lemma. ■

Our theorem explains why the 2D toric code (defined on a square lattice) and the 2D color code achieve maximum measurement threshold at the uniform measurement frequency $\tilde{\alpha} = (1/3, 1/3, 1/3)$. In case of 2D toric code, if we measure each qubit in X or Z basis with equal probability, the logical $\tilde{X}$ and $\tilde{Z}$ operators should be measured with the same probability since X - and Z -type stabilizer generators are symmetric under the lattice translation. Thus, the measured logical operator cannot be fixed at $p = 1$ and therefore, the 2D toric code achieves the maximum measurement threshold. In case of 2D color code, since it is a self-dual CSS code, all logical $\tilde{X}$, $\tilde{Y}$, and $\tilde{Z}$ operators must be measured with an equal probability at $\tilde{\alpha} = (1/3, 1/3, 1/3)$. Thus, the 2D color code achieves the maximum measurement threshold.

VI. HAAR RANDOM CODE AND MEASUREMENT

In this section, we study the information preservation in QECCs when Haar randomness is involved in either the encoding or the measurement process. Specifically, we study two scenarios. The first scenario considers a Haar-random encoding of quantum information and studies the effect of projective measurements. In this case, we find that the encoded information is retained even when $n - (k + \epsilon)$ qubits are projectively measured. The second scenario considers an arbitrary QECC (not necessarily Haar random code) where a part of the system is measured in globally Haar random basis, as opposed to a locally Haar random basis. For this problem, we will derive a simple condition

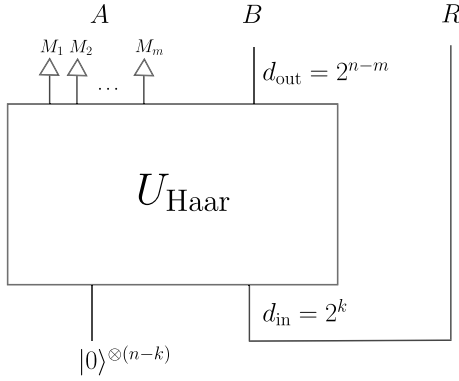


FIG. 18. The Choi state of Haar random code.

under which the original information remains recoverable after projective measurements.

Similar results for the first scenario have been obtained in the literature [6]. We also would like to highlight conceptually relevant works, often referred to as environment-assisted quantum channels, on these problems, see Ref. [31] for instance. Also, these were recently utilized in the context of monitored circuits in Ref. [32].

A. Haar random code

Suppose that k logical qubits are encoded into n physical qubits using a Haar random unitary, and then m physical qubits are measured in some basis, as shown in Fig. 18. Note that the choice of measurement basis plays no role as the encoding is Haar random. Thus, we measure in computational basis without loss of generality.

The encoded information is recoverable when the subsystems B and R in the postmeasurement state are nearly maximally entangled. Here, we are primarily interested in the average recovery fidelity. For this purpose, it suffices to show that the reduced density matrix $\tilde{\rho}_R$ is close to the maximally mixed state μ_R . Then, the recovery protocol from Ref. [33] achieves a high average recovery fidelity.

The unnormalized wavefunction after projective measurements is given by

$$|\psi\rangle_{BR} = \langle 0|_A^{\otimes m} |\Psi\rangle_{ABR}, \quad (63)$$

where $|\Psi\rangle_{ABR}$ is the Choi state. The normalized density matrix is $\tilde{\rho}_R = \rho_R / \text{Tr}(\rho_R)$. Finally, the 2-norm distance with $\mu_R = I_R / d_R$ is

$$\|\tilde{\rho}_R - \mu_R\|_2^2 = \text{Tr}(\tilde{\rho}_R^2) - \frac{1}{d_R}. \quad (64)$$

We need to evaluate the Haar average of $\text{Tr}(\tilde{\rho}_R^2)$. We have

$$\mathbb{E}(\text{Tr}(\tilde{\rho}_R^2)) = \mathbb{E}\left(\frac{\text{Tr}(\rho_R^2)}{\text{Tr}(\rho_R)^2}\right) \approx \frac{\mathbb{E}(\text{Tr}(\rho_R^2))}{\mathbb{E}(\text{Tr}(\rho_R)^2)}, \quad (65)$$

where the approximation relies on the fact that variances are exponentially suppressed with respect to $n - m$ (i.e., by factors of d_B). Performing the Haar integral (see Ref. [34] for a review), we obtain

$$\begin{aligned} \mathbb{E}(\text{Tr}(\rho_R^2)) &= \frac{1}{d_R^2} \left(\frac{d_R^2 d_B + d_R d_B^2}{d_{AB}^2 - 1} - \frac{d_R^2 d_B^2 + d_R d_B}{d_{AB}(d_{AB}^2 - 1)} \right), \\ \mathbb{E}(\text{Tr}(\rho_R)^2) &= \frac{1}{d_R^2} \left(\frac{d_R^2 d_B^2 + d_R d_B}{d_{AB}^2 - 1} - \frac{d_R d_B^2 + d_R^2 d_B}{d_{AB}(d_{AB}^2 - 1)} \right), \end{aligned} \quad (66)$$

and

$$\frac{\mathbb{E}(\text{Tr}(\rho_R^2))}{\mathbb{E}(\text{Tr}(\rho_R)^2)} \approx \frac{d_R + d_B}{d_R d_B + 1} \approx \frac{1}{d_R} \left(1 + \frac{d_R}{d_B} \right), \quad (67)$$

where the approximation holds for $d_A \gg d_B, d_R$ and $d_B \gg d_R$. This leads to

$$\mathbb{E}(\|\tilde{\rho}_R - \mu_R\|_2^2) \approx \frac{1}{d_B}. \quad (68)$$

Hence, we arrive at

$$\mathbb{E}(\|\tilde{\rho}_R - \mu_R\|_1^2) \leq d_R \mathbb{E}(\|\tilde{\rho}_R - \mu_R\|_2^2) \approx \frac{d_R}{d_B}, \quad (69)$$

which approaches zero when $d_B \gg d_R$.

In summary, we find that the measurement threshold of Haar random code is $p_m^{\text{th}} = 1$. That is, the encoded information is retained even when $n - (k + \epsilon)$ qubits are projectively measured, where the failure probability is suppressed by a factor of $2^{-\epsilon}$.

B. Haar random measurement

Next, we consider an arbitrary QECC and perform a Haar random measurement on subsystem A as shown in Fig. 19.

Let $|\Psi\rangle_{ABR}$ be the Choi state and σ_{ABR} be the corresponding density matrix. The unnormalized wavefunction after projective measurement is

$$|\psi\rangle_{BR} = \langle \text{Haar}|_A |\Psi\rangle_{ABR}, \quad (70)$$

where $|\text{Haar}\rangle_A$ represents a Haar random state on A . This defines the unnormalized post measurement density matrix ρ_R . Expressing the 2-norm as in Eqs. (64) and (65), our task is to evaluate $\mathbb{E}(\text{Tr}(\tilde{\rho}_R^2))$. Performing the Haar integral and

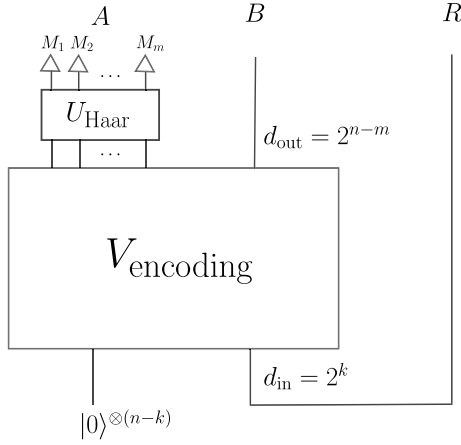


FIG. 19. The Choi state of arbitrary QECC code after global Haar random measurements.

noting that the variances are small, we obtain

$$\begin{aligned}\mathbb{E}(\text{Tr}(\rho_R^2)) &= \frac{1}{d_A(d_A + 1)} (\text{Tr}(\sigma_R^2) + \text{Tr}(\sigma_B^2)), \\ \mathbb{E}(\text{Tr}(\rho_R)^2) &= \frac{1}{d_A(d_A + 1)} (1 + \text{Tr}(\sigma_A^2)),\end{aligned}\quad (71)$$

and

$$\mathbb{E}(\text{Tr}(\tilde{\rho}_R^2)) \approx \frac{\mathbb{E}(\text{Tr}(\rho_R^2))}{\mathbb{E}(\text{Tr}(\rho_R)^2)} \approx \frac{\frac{1}{d_R} + \text{Tr}(\sigma_B^2)}{1 + \text{Tr}(\sigma_A^2)} \leq \frac{1}{d_R} + \text{Tr}(\sigma_B^2). \quad (72)$$

The encoded information will be retained in B when the distance between $\tilde{\rho}_R$ and μ_R is small. This occurs when $\mathbb{E}(\text{Tr}(\tilde{\rho}_R^2)) \approx 1/d_R$, in particular, when $\text{Tr}(\sigma_B^2) \ll 1/d_R$. In other words, if $S_B^{(2)} \gg d_R$, the encoded information will be retained in B .

It is useful to consider two particular mechanisms to achieve it. The first scenario is that B is maximally entangled with R . In this case, the measurement on A has no effects on the entanglement between B and R , and thus, B retains logical qubits after projective measurements. The second scenario is that B is entangled with A , but not with R . In this case, in the original QECC, the subsystem B cannot reconstruct the encoded information. The above result suggests that, as a result of projective measurements on A , the encoded information becomes recoverable from B . One can interpret this as quantum teleportation of logical qubits from A to B by using entanglement between A and B .

VII. MEASUREMENT THRESHOLD IN QUANTUM GRAVITY

In this section, we briefly discuss QECCs under random monitoring in the context of quantum gravity. The AdS/CFT correspondence can be interpreted as a QECC

where the bulk degrees of freedom are holographically encoded into the boundary degrees of freedom. The conceptual pillar behind holographic QECCs is the entanglement wedge reconstruction [35]. Let us focus on the $\text{AdS}_3/\text{CFT}_2$ in order to illustrate the basic idea. Consider an arbitrary subsystem A on the boundary and the minimal surface homologous to A (i.e., the Ryu-Takayanagi (RT) surface), where the bulk region enclosed by A and the RT surface is called the entanglement wedge EA. The entanglement wedge reconstruction is a statement that bulk field operators in EA can be holographically reconstructed on A . Holographic tensor networks provide concrete toy model realizations of this phenomenon.

We begin by studying the fault tolerance of the bulk quantum information b at the center of the AdS space against erasure errors. The erasure threshold for holographic QECCs was found to be $p_e^{\text{th}} = \frac{1}{2}$ [27]. To sketch the argument, consider a subset A of qubits by choosing pn qubits from the boundary with $p < \frac{1}{2}$. We can observe that the RT surface of A is not likely to enclose the bulk field b at the center. Namely, the RT surface of the complementary subset A^c , which consists of $(1-p)n$ qubits with $1-p > \frac{1}{2}$, is more likely to enclose b . This suggests that the bulk quantum information b is protected from erasure of qubits in A , and thus $p_e^{\text{th}} = \frac{1}{2}$.

Next, we study the fault tolerance against local measurements. The measurement threshold for holographic QECCs was argued to be $p_m^{\text{th}} = 1$ in Ref. [15] by two different lines of arguments. First, in the AdS/CFT correspondence, it has been proposed that local measurements on the boundary are dual to insertions of End-of-World (EoW) branes in the bulk at large tension limit where branes will be situated close to the asymptotic boundary. Figure 20 shows an example of geometries with EoW branes that emerge from local projective measurements of subregion A . Suppose that we choose a subregion A that consists of 99% of the boundary qubits. The corresponding EoW brane will still be located near the asymptotic boundary, and the entanglement wedge of A^c contains the entire bulk region, leading to $p_m^{\text{th}} = 1$.

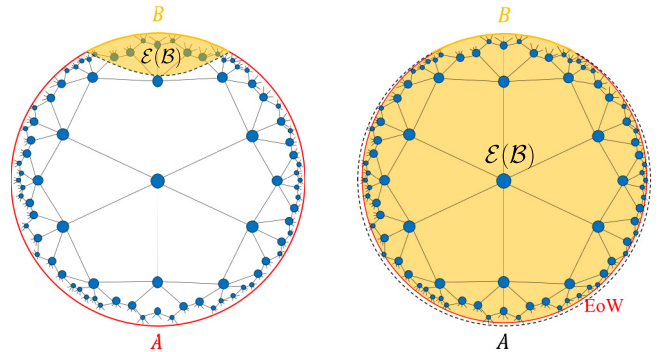


FIG. 20. Holographic description of measurements at the boundary subregion A .

One remarkable property is that we have $p_m^{\text{th}} = 1$ even when the bulk field b is located close to the asymptotic boundary, as long as the EoW does not enclose b [36]. This is not the case for the erasure threshold. Indeed, for bulk information b near the boundary, erasing small subregion A would suffice to lose the quantum information as A 's causal wedge encloses b .

VIII. OUTLOOK

In this paper, we developed a theoretical framework to study the effects of projective measurements on QECCs. We demonstrated that QECCs exhibit exceptional robustness under random local monitoring, achieving the maximal measurement threshold $p_m^{\text{th}} = 1$ in many cases. We also explained the intimate connection between the maximum measurement threshold and the uncertainty of measured logical operator at $p = 1$. Our work suggests that, even a modest degree of quantum encoding enables the system to retain memory of its initial states in the form of QECCs after projective measurements. Below, we provide several relevant discussions and present potential future problems.

Shadow tomography for QECC: The key idea of shadow tomography is to characterize an unknown quantum state ρ by performing measurements in random chosen basis and reconstructing ρ via the inverse channel. When the probability distribution of the measurement basis is chosen appropriately, expectation values of a wide range of observables can be estimated by accessing only a small number of copies of ρ . The original works [37,38] showed that uniformly random global Pauli measurements can achieve efficient sample complexity while subsequent studies [39] have further clarified the connection between the randomness of the measurement basis and the sample efficiency. In the context of QECCs, we may assume that the quantum state ρ of interest is unknown, but partial information is available. Specifically, we know that ρ is a quantum state supported in some subspace $\mathcal{H}_{\text{code}} \subset \mathcal{H}$ that can be interpreted as the codeword subspace. Similar setups were considered in Refs. [40,41]. The shadow tomography for QECCs then requires us to characterize ρ inside $\mathcal{H}_{\text{code}}$ by measuring logical operators in the random basis. Hence, studying the logical measurement probability $(p_{\bar{X}}, p_{\bar{Y}}, p_{\bar{Z}})$ allows us to estimate the sample complexity when the shadow tomography is performed by measuring physical qubits with (p_X, p_Y, p_Z) .

Clifford hierarchy: Calderbank–Shor–Steane (CSS) stabilizer codes that support transversal logical gates from higher levels in the Clifford hierarchy possess smaller erasure thresholds [27]. As we have seen in the seven-qubit code and 15-qubit code, having different sets of transversal gates seems to lead to strikingly different information preservation properties. There may be a

possible connection between transversal logical gates and the information preservation probability.

Decoding by measurement: Our work motivates a problem concerning decoding of QECCs. Namely, it will be interesting to ask if one can efficiently decode the encoded information in QECCs by performing projective measurements in a fault-tolerant manner?

Topological codes: We have demonstrated that the 2D toric code and color code achieve $p_m^{\text{th}} = 1$ under uniformly random local Pauli measurements. This naturally leads us to wonder if $p_m^{\text{th}} = 1$ for uniformly random measurements is a generic property of 2D topological codes. This expectation, however, turns out to be false. In fact, an explicit counterexample can be constructed by considering the toric code on some lattice whose edge percolation threshold p_{perc} is smaller than $\frac{1}{3}$ (for example, the union jack lattice [42,43]), which yields $p_m^{\text{th}} = 3p_{\text{perc}} < 1$. One may formulate a weaker conjecture by observing that the 2D toric code on a square lattice has an extraordinary stability against Y measurements. It will be interesting to ask if there exists some relative measurement frequency $(\alpha_X, \alpha_Y, \alpha_Z)$ which achieves $p_m^{\text{th}} = 1$.

Qudit codes: It is also interesting to consider the information prevention condition for the qudit toric code. For prime dimensional qudit stabilizer codes, local Pauli operators take the form $X^a Z^b$ with $a, b = 0, \dots, p-1$ in terms of generalized Pauli operators. Noting that measurements in the $\mathcal{Q}^c$ basis ($c \neq 0$) are all equivalent, independent measurement basis can be formed by $p+1$ Pauli operators, Z, XZ^b with $b = 0, \dots, p-1$. For uniformly random measurements, Pauli X and Z operators will be measured with probability $1/(p+1)$. Furthermore, for large p , most of measurements will occur in the XZ^b ($b \neq 0$) basis which are analogous to Y operators in the qubit toric code. Accordingly, we expect that the logical information will be preserved under uniformly random measurements for a sufficiently large p qudit toric code on arbitrary lattices.

Critical mixed state: In the case of the 2D toric code, the measurement frequencies that yield an uncertain logical operator at $p = 1$ coincide with the Pauli noise frequencies known to generate a critical mixed state [44]. Specifically, the authors demonstrated that the Markov length diverges when a flagged Pauli noise channel is applied with corresponding frequencies, while the classical coherent information (for instance, the logical Z operator coupled to a reference qubit) converges to $1/3$. This is consistent with our result that a random logical operator is measured in this frequency regime. Based on the Theorem 5, we conjecture that a relationship between the measurement threshold and the critical mixed phase may hold for general topological codes. Namely, we expect that the flagged Pauli noise channel with frequencies leading to $p_m^{\text{th}} = 1$ will similarly lead to a divergent Markov length.

Measurement uncertainty beyond stabilizer code: We have rigorously established that, for stabilizer codes

encoding a single logical qubit, intrinsic uncertainty in the Pauli basis of the logical operator measured at $p = 1$ implies the maximal measurement threshold $p_m^{\text{th}} = 1$. It would be interesting to investigate whether an analogous structural relation holds for more general (non-stabilizer) quantum error-correcting codes. For example, Haar-random codes exhibit a nearly uniform distribution over logical measurement bases at $p = 1$ and are known to achieve the maximal measurement threshold $p_m^{\text{th}} = 1$. Understanding whether logical-measurement uncertainty universally underlies maximal measurement robustness remains an intriguing open question.

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DATA AVAILABILITY

The data that support the findings of this work are openly available [45].

APPENDIX A: PROOF OF THEOREMS 2 AND 3

We present a proof for subsystem codes since stabilizer codes are special cases of subsystem codes with $\mathcal{G} = \mathcal{S}$. We will proceed by proving (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i).

Proof of (i) $\Rightarrow$ (ii). Recall that a nontrivial dressed logical operator ℓ_{dressed} commutes with $\mathcal{S}$, and is outside $\mathcal{G}$. Hence, the statement (i) suggests that elements in $\mathcal{M}$ must be either (a) inside $\mathcal{G}$, or (b) anticommute with some element in $\mathcal{S}$.

Suppose that $\mathcal{M} \cap \mathcal{G}$ consists of 2^{m_G} elements with $m_G \leq m$. Obviously, $\mathcal{M} \cap \mathcal{G}$ is a subgroup of $\mathcal{M}$; therefore, we can write $\mathcal{M}$ as follows:

$$\mathcal{M} = \langle N_1, \dots, N_{m-m_G}, N_{m-m_G+1}, \dots, N_m \rangle, \quad (\text{A1})$$

where $N_{m-m_G+1}, \dots, N_m \in \mathcal{G}$ and $N_1, \dots, N_{m-m_G}$ anticommute with some element of $\mathcal{S}$.

Given a nontrivial bare logical operator ℓ_{bare} , we present a generic recipe of constructing an equivalent expression ℓ'_{bare} that commutes with $\mathcal{M}$. This recipe relies on the fact that, for N_j with $j \in \{1, \dots, m - m_G\}$, there exist stabilizer operators $\tilde{S}_j$ that anticommute only with N_j while commuting with all N_i with $i \neq j$, namely,

$$N_j \tilde{S}_i = -(-1)^{\delta_{ij}} \tilde{S}_i N_j. \quad (\text{A2})$$

Assuming Eq. (A2), let us consider commutation relations between ℓ_{bare} and N_j . We have $[\ell_{\text{bare}}, N_j] = 0$ for $j \in \{m -$

$m_G + 1, \dots, m\}$ since N_j 's are stabilizer generators. As for N_j with $j \in \{1, \dots, m - m_G\}$, if $\{\ell_{\text{bare}}, N_j\} = 0$, we update the logical operator as $\ell'_{\text{bare}} = \ell_{\text{bare}} \tilde{S}_j$ so that $[\ell'_{\text{bare}}, N_j] = 0$. Repeating this procedure for all N_j 's that anticommute with ℓ , one can construct $\ell'_{\text{bare}} \sim \ell_{\text{bare}}$ that commutes with $\mathcal{M}$, arriving at statement (ii).

Finally, we prove the statement regarding Eq. (A2). Let $S_1, \dots, S_{n-k}$ be independent stabilizer generators. Let $\vec{A}(N_j)$ be a binary vector with $n - k$ entries which records the commutation relations between N_j and S_i . Namely, for $i \in \{1, \dots, n - k\}$, we set

$$\vec{A}(N_j)_i = 0, 1 \quad N_j S_i = (-1)^{\vec{A}(N_j)_i} S_i N_j. \quad (\text{A3})$$

$\vec{A}(N_j)$ are zero vectors for $j \in \{m - m_G + 1, \dots, m\}$ because N_j 's are stabilizer generators, while $\vec{A}(N_j)$ for $j \in \{1, \dots, m - m_G\}$ must contain some nonzero entries. One can show that $\vec{A}(N_j)$ for $j \in \{1, \dots, m - m_G\}$ must be all independent binary vectors. Indeed, if one could construct a zero vector by taking linear combinations of $\vec{A}(N_j)$'s, some product of N_j for $j \in \{1, \dots, m - m_G\}$ would commute with all the stabilizer generators, and therefore be a dressed logical operator or a stabilizer. But both cases lead to contradictions, due to the construction of Eq. (A1) and the statement (i), proving that $\vec{A}(N_j)$ for $j \in \{1, \dots, m - m_G\}$ are independent from each other. Then, by using Gaussian elimination (see Ref. [17] for instance), one can construct stabilizer operators $\tilde{S}_j$ satisfying Eq. (A2). ■

Proof of (ii) $\Rightarrow$ (iii). The statement (ii) suggests that, for any Pauli operator $I \otimes I \otimes P_{\text{bare}}$ acting on the bare reference R_{bare} , there exists a bare logical operator $\bar{P}_{\text{bare}}$ of the code that commutes with $\mathcal{M}$. Namely, we have $\bar{P}_{\text{bare}} \otimes I_{\text{gauge}} \otimes P_{\text{bare}} \in \mathcal{S}_{\text{Choi}}$. Since this operator commutes with all $M \otimes I \otimes I \in \mathcal{M}$, we have $\bar{P}_{\text{bare}} \otimes I \otimes P_{\text{bare}} \in \mathcal{S}'_{\text{Choi}}$. Noting that P_{bare} is an arbitrary Pauli operator on R_{bare} , one notices that $\mathcal{S}'_{R_{\text{gauge}} R_{\text{bare}}}$ cannot contain any nontrivial stabilizer that has nontrivial support on the reference R_{bare} . Hence, due to the Lemma 5, we arrive at the statement (iii). ■

Proof of (iii) $\Rightarrow$ (i). We prove the contraposition. When the statement (i) is not satisfied, $\mathcal{M}$ must contain a nontrivial dressed logical operator. Without loss of generality, assume that $\bar{P}_{\text{dressed}} \in \mathcal{M}$ and $\bar{P}_{\text{dressed}} \otimes P_{\text{gauge}} \otimes P_{\text{bare}} \in \mathcal{S}_{\text{Choi}}$ where $P_{\text{bare}} \neq I$. Since $\mathcal{M}$ is an abelian group, $\bar{P}_{\text{dressed}} \in \mathcal{C}(\mathcal{M})$, and thus $\bar{P}_{\text{dressed}} \otimes P_{\text{gauge}} \otimes P_{\text{bare}} \in \mathcal{C}(\mathcal{M})$. This suggests $\bar{P}_{\text{dressed}} \otimes P_{\text{gauge}} \otimes P_{\text{bare}} \in \mathcal{S}'_{\text{Choi}}$. Noting $\bar{P}_{\text{dressed}} \otimes I \otimes I \in \mathcal{S}'_{\text{Choi}}$, we have $I \otimes P_{\text{gauge}} \otimes P_{\text{bare}} \in \mathcal{S}'_{\text{Choi}}$ with $P_{\text{bare}} \neq I$. Hence, due to the Lemma 5, the statement (iii) is not satisfied. ■

APPENDIX B: RANDOM MEASUREMENTS AS QUANTUM CHANNELS

In this appendix, we review random measurement processes within the framework of quantum channels, clarifying the distinctions depending on the classical measurement record. We analyze three different cases: (i) no classical information is recorded, (ii) only the occurrence of a measurement is recorded, and (iii) both the measurement

basis and the corresponding outcomes are recorded. We note that, this work focused on the third scenario, wherein complete classical information is recorded.

We first begin with the case where a single-qubit state is randomly measured in the Pauli basis $\{X, Y, Z\}$ with probabilities (p_X, p_Y, p_Z) satisfying $p_X + p_Y + p_Z = p \leq 1$. When the measurement basis and outcomes are discarded, the random measurement process reproduces the standard Pauli noise channel:

$$\begin{aligned} \mathcal{N}_P(\rho) &= (1-p)\rho + p_X \left(\frac{I+X}{2} \rho \frac{I+X}{2} + \frac{I-X}{2} \rho \frac{I-X}{2} \right) + p_Y \left(\frac{I+Y}{2} \rho \frac{I+Y}{2} + \frac{I-Y}{2} \rho \frac{I-Y}{2} \right) \\ &\quad + p_Z \left(\frac{I+Z}{2} \rho \frac{I+Z}{2} + \frac{I-Z}{2} \rho \frac{I-Z}{2} \right) \\ &= \left(1 - \frac{p}{2} \right) \rho + \frac{1}{2} (p_X X \rho X + p_Y Y \rho Y + p_Z Z \rho Z). \end{aligned} \quad (\text{B1})$$

Note that when $p_X = p_Y = p_Z$, the random measurement realizes the standard depolarizing channel with depolarizing probability $p/2$:

$$\mathcal{N}_D(\rho) = \left(1 - \frac{p}{2} \right) \rho + \frac{p}{2} \left(\frac{1}{3} X \rho X + \frac{1}{3} Y \rho Y + \frac{1}{3} Z \rho Z \right). \quad (\text{B2})$$

Next, when the measurement event is recorded, the random measurement becomes a flagged Pauli noise channel:

$$\begin{aligned} \mathcal{N}_F(\rho) &= (1-p)\rho \otimes |0\rangle_A \langle 0| + p_X \left(\frac{I+X}{2} \rho \frac{I+X}{2} + \frac{I-X}{2} \rho \frac{I-X}{2} \right) \otimes |1\rangle_A \langle 1| \\ &\quad + p_Y \left(\frac{I+Y}{2} \rho \frac{I+Y}{2} + \frac{I-Y}{2} \rho \frac{I-Y}{2} \right) \otimes |1\rangle_A \langle 1| + p_Z \left(\frac{I+Z}{2} \rho \frac{I+Z}{2} + \frac{I-Z}{2} \rho \frac{I-Z}{2} \right) \otimes |1\rangle_A \langle 1| \\ &= (1-p)\rho \otimes |0\rangle_A \langle 0| + \frac{1}{2} p_X (\rho + X \rho X) \otimes |1\rangle_A \langle 1| + \frac{1}{2} p_Y (\rho + Y \rho Y) \otimes |1\rangle_A \langle 1| + \frac{1}{2} p_Z (\rho + Z \rho Z) \otimes |1\rangle_A \langle 1|. \end{aligned} \quad (\text{B3})$$

Here, an ancilla qubit (with subscript A) acts as a flag, with the state $|1\rangle_A$ indicating that a measurement occurred and $|0\rangle_A$ indicating that it did not. When $p_X = p_Y = p_Z$, this channel is called the flagged depolarizing channel:

$$\mathcal{N}_{FD}(\rho) = (1-p)\rho \otimes |0\rangle_A \langle 0| + p \left(\frac{1}{2} \rho + \frac{1}{6} X \rho X + \frac{1}{6} Y \rho Y + \frac{1}{6} Z \rho Z \right) \otimes |1\rangle_A \langle 1|. \quad (\text{B4})$$

It is instructive to compare this channel to the standard erasure channel:

$$\mathcal{N}_E(\rho) = (1-p)\rho \otimes |0\rangle_A \langle 0| + p \frac{I}{2} \otimes |1\rangle_A \langle 1|, \quad (\text{B5})$$

where the single-qubit state is replaced by the maximally mixed state rather than being subjected to a specific distribution of Pauli errors. Thus, the erasure channel represents a strictly stronger noise model compare to the flagged depolarizing channel $\mathcal{N}_{FD}$.

Finally, when both the choice of measurement basis and the resulting outcomes are recorded, the random measurement can be described by the following quantum channel:

$$\begin{aligned}
\mathcal{N}_{RM}(\rho) = & (1-p)\rho \otimes |0\rangle_A\langle 0| + p_X \left(\frac{I+X}{2}\rho\frac{I+X}{2} \otimes |1,+\rangle_A\langle 1,+| + \frac{I-X}{2}\rho\frac{I-X}{2} \otimes |1,-\rangle_A\langle 1,-| \right) \\
& + p_Y \left(\frac{I+Y}{2}\rho\frac{I+Y}{2} \otimes |2,+\rangle_A\langle 2,+| + \frac{I-Y}{2}\rho\frac{I-Y}{2} \otimes |2,-\rangle_A\langle 2,-| \right) \\
& + p_Z \left(\frac{I+Z}{2}\rho\frac{I+Z}{2} \otimes |3,+\rangle_A\langle 3,+| + \frac{I-Z}{2}\rho\frac{I-Z}{2} \otimes |3,-\rangle_A\langle 3,-| \right). \tag{B6}
\end{aligned}$$

Here, each ancilla qudit records both the choice of measurement basis $\{1, 2, 3\}$ and the corresponding measurement outcome $\{+, -\}$.

Throughout the paper, we focused on the case where each physical qubit is randomly measured, while both the measurement basis and outcomes are recorded. Mathematically, this means the tensor product of single-qubit channels $\bigotimes_{i=1}^N \mathcal{N}_{RM}^{(i)}$ is applied to the N physical qubits to evaluate whether the encoded information is still recoverable.

The high measurement thresholds we showed in this work might appear to be an expected consequence of measurement flags. Indeed, if one retains only the measurement flag, the random measurement process simplifies to a tensor product of flagged Pauli noise channels $\bigotimes_{i=1}^N \mathcal{N}_F^{(i)}$, which is well-known to have higher threshold than standard Pauli noise channel $\bigotimes_{i=1}^N \mathcal{N}_P^{(i)}$ [46,47]. However, our random measurement process $\bigotimes_{i=1}^N \mathcal{N}_{RM}^{(i)}$ provides richer classical information to the decoder than flagged Pauli noise channel $\bigotimes_{i=1}^N \mathcal{N}_F^{(i)}$, fundamentally altering the conditions for decodability. This disparity is evident in the limit where $p \rightarrow 1$ and $p_X = p_Y = p_Z$; here, the flagged Pauli channel reduces to a depolarizing channel $\bigotimes_{i=1}^N \mathcal{N}_D^{(i)}$ with depolarizing probability $\frac{1}{2}$, well beyond the decodability threshold of the 2D toric code [48]. By contrast, Sec. IV establishes that maintaining the complete measurement record (both basis and outcome) allows the 2D toric code to remain decodable even as $p \rightarrow 1$.

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