

Low-Density Parity-Check Codes as Stable Phases of Quantum Matter

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Phases of matter with robust ground-state degeneracy, such as the quantum toric code, are known to be capable of robust quantum information storage. Here, we address the converse question: given a quantum error-correcting code, when does it define a stable gapped quantum phase of matter, whose ground-state degeneracy is robust against perturbations in the thermodynamic limit? We prove that a low-density parity-check (LDPC) code defines such a phase, robust against all few-body perturbations, if its code distance grows at least logarithmically in the number of degrees of freedom, and it exhibits “check soundness.” Many constant-rate quantum LDPC expander codes have such properties, and define stable phases of matter with a constant zero-temperature entropy density, violating the third law of thermodynamics. Our results also show that quantum toric-code phases are robust to spatially nonlocal few-body perturbations. Similarly, phases of matter defined by classical codes are stable against symmetric perturbations. In the classical setting, we present improved locality bounds on the quasiadiabatic evolution operator between two nearby states in the same code phase.

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I. INTRODUCTION

The past two decades have seen a revolution in our understanding of quantum phases of matter [1–7].

First and foremost, we now have a useful definition of a gapped phase of quantum matter: two states $|\psi_1\rangle$ and $|\psi_2\rangle$ are in the same phase if there is some unitary U , obtained by finite-time evolution with a (quasi)local Hamiltonian, such that $|\psi_1\rangle = U|\psi_2\rangle$. Notice that this is very different from the textbook thermodynamic definition of a phase of matter, characterized by passing from one system to another without encountering any thermodynamic singularities in, e.g., a free energy density.

Secondly, we have discovered new kinds of quantum phases that are completely robust to all local perturbations. A simple example is the quantum toric code [8], a simple model exhibiting local topological order [4,9]. Local topological order implies that all ground states are indistinguishable to any local operators, in stark contrast to the Landau paradigm for phases of matter, wherein *local* order

parameters (such as the magnetization, for a ferromagnetic phase) can be used to diagnose a symmetry-breaking pattern.

The quantum toric code both exhibits local topological order, and also highlights why the thermodynamic definition of a phase falls short: it is thermodynamically indistinguishable from a trivial paramagnet [10,11] yet corresponds to a topologically ordered *zero temperature* phase of matter. It also represents the simplest example of both a topological quantum phase, *and* a quantum error-correcting code robust against finitely many errors, in the thermodynamic limit. Indeed, the ability of the toric code to correct against errors is a simple consequence of local topological order, as no local perturbation can transform one ground state to another, i.e., induce a logical error on quantum information protected in the ground state(s). Many experiments [12–17] have aimed to realize scalable error correction of a logical qubit using the toric (or related surface) code.

Since the toric code only stores a handful of logical qubits in the thermodynamic limit, however, many researchers are also looking beyond the toric code for a more general family of low-density parity-check (LDPC) codes. LDPC codes, like the toric code, can be defined in terms of quantum Hamiltonians with few-body interactions. Like the toric code, the ground states of these Hamiltonians represent the entangled logical codewords; unlike the toric code, the interactions are not local in any finite spatial dimension. Instead, LDPC codes are “*q* local”: the

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Hamiltonian is a sum of products of q or fewer Pauli matrices acting on n qubits, but there are no constraints on which q qubits can interact in any given term.

There has been a flurry of recent activity on discovering interesting quantum LDPC codes [18–24], understanding the physics of these new quantum codes [25–27], and adapting them in real experiments [28–31]. At low temperature, some of these codes represent highly entangled phases of matter where all low-energy states lie in a nontrivial phase of matter [32] (verifying the “NLTS conjecture” of Ref. [33]). These low-energy states are capable of quantum self-correction [25]. It has further been conjectured that the good LDPC codes, defined on expander graphs, form well-defined phases of matter [34,35]; a proof for nonexpander codes is in Ref. [34]. These would represent unusual phases, with a completely robust violation of the third law of thermodynamics, with an extensive ground-state degeneracy in the thermodynamic limit, as emphasized in Ref. [34]. However, because LDPC codes are only q local—not spatially local—the mathematical tools, which have, until now, been able to prove the stability of topological order, cannot be applied.

Here, we prove that many quantum LDPC expander codes indeed form stable phases of matter to q -local perturbations. In particular, we prove that (i) the gap between the ground states, which encode logical information in the code, and excited states, does not close for arbitrary small perturbations; (ii) the splitting of the energy levels between these code states is stretched exponentially small in n (for many codes of interest); (iii) a quasilocal unitary can rotate from the unperturbed to the perturbed ground-state sector efficiently. These ideas are summarized in Fig. 1. Our proof was inspired by the techniques we developed in a recent theory of quantum metastable states and slow false vacuum decay [36]. We identify a sufficient condition on a quantum code for stability, coined “check soundness,” which is obeyed by finite-dimensional toric codes, hypergraph product expander codes [18], and

the “good” constant-rate quantum LDPC codes [22–24]. The latter two constant-rate quantum codes represent a truly “infinite-dimensional” phase of matter without a third law of thermodynamics, whose stability to perturbations is established here.

Even in finite-dimensional models, our methods lead to a number of new results. We establish the stability of quantum toric-code phases to q -local perturbations, which is consistent with the nontrivial circuit depth lower bounds [3,7,37] required to prepare such topological states. As in Ref. [38], it is also simple to extend our results to study the stability of phases of matter defined by classical codes, in the presence of symmetric perturbations. An important application of this result concerns the construction of quadiabatic evolution operators that connect two different states in the same phase. By avoiding spectral filtering [39,40], we are able to find very strong bounds on the locality of these operators, one application of which will be detailed elsewhere [41].

The remainder of the main text gives a high-level and nonrigorous introduction to quantum LDPC codes, our main theorem, and some implications of the result. Appendices contain technical definitions along with our main theorem’s statement and proof.

II. ERROR-CORRECTING CODES

We first provide a high-level introduction to (quantum) error-correcting codes, from a physicist’s point of view.

The simplest kind of error-correcting code—the repetition code—can be understood quite readily as the classic Ising model from statistical physics. On a d -dimensional lattice, we have Hamiltonian

$$H_0 = - \sum_{i \sim j} Z_i Z_j, \quad (1)$$

where $i \sim j$ denotes that i, j are neighbors on the lattice, and $Z_i \in \{\pm 1\}$ are binary variables. H_0 has a doubly degenerate ground state, corresponding to either all $Z_i = +1$ or all $Z_i = -1$. These two ground states encode a logical classical bit. In $d > 1$ dimensions, even when coupling H_0 to a thermal bath, the ferromagnetic low-temperature phase of the Ising model ensures that the time it takes to move from one ground state to another is exponentially large in the system size [42]; this is the basis of magnetic memory. We say that the Ising model stores $k = 1$ logical bit in n physical bits. To get from one logical codeword to another, we must flip every single bit, so the code distance $d = n$: if we have $\frac{1}{2}(d - 1)$ errors in the system, then an optimal decoder could correctly return us to our original state. In fact, thinking of H_0 as a quantum Hamiltonian acting on a Hilbert space of n qubits, with Pauli matrices

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (2)$$

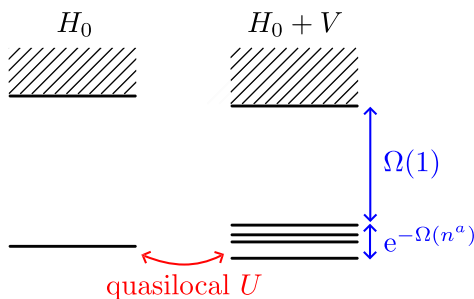


FIG. 1. We prove that many quantum LDPC codes are stable against perturbations, in the sense that (i) the gap between the encoded ground states and excited states remains $\Omega(1)$, (ii) the energy splitting of the ground states is exponentially small, (iii) the perturbed ground states are connected to the unperturbed ones by a sufficiently local unitary.

we can interpret code distance d as the size of the smallest logical operator $X_1 \cdots X_n$ that converts one code state $|00 \cdots 0\rangle$ to the other: $|11 \cdots 1\rangle$.

The simplest quantum error-correcting code whose distance d scales with n is the two-dimensional toric (or surface) code [8]. Consider a two-dimensional square lattice, with qubits placed on the edges e of the graph. The codewords are the ground states of the Hamiltonian

$$H_0 = - \sum_{\text{vertex } v} \prod_{e \sim v} X_e - \sum_{\text{face } f} \prod_{e \sim f} Z_e = - \sum_{\text{vertex } v} A_v - \sum_{\text{face } f} B_f, \quad (3)$$

where $e \sim v$ means that edge e is adjacent to vertex v , and $e \sim f$ means that edge e lies on the boundary of face (plaquette) f . We have also introduced the A_v and B_f notation, as is common in the literature. A simple calculation shows that

$$[A_v, B_f] = 0 \quad (4)$$

for all v and f , because these operators always share an even number of edges in common, and thus an even number of Pauli X and Z on the same edges (whose product therefore commutes). Equation (4) implies that the ground state of H_0 is the simultaneous $+1$ eigenspace of the operators A_v and B_f , which are called the parity checks of the code. They are analogous to $Z_i Z_j$ in the Ising model, but due to the presence of both Pauli X and Z , imply the existence of large entanglement in the ground states of H_0 . The ground-state subspace is also the $+1$ eigenspace of any stabilizer, i.e., a Pauli string that is a product of the checks A_v, B_f . Logical operators for the toric code become products of Pauli X and Z that commute with all stabilizers, and correspond to string operators across the system.

In a pioneering set of works, Refs. [4,38] showed that for any V , which is a sum of small terms (but which may be thermodynamically extensive), the toric-code Hamiltonian H_0 is robust against perturbations:

$$H = H_0 + V \quad (5)$$

lies in the same phase as H_0 , with the same ground-state degeneracy (up to exponentially small corrections in system size), and with a quasilocal unitary that can rotate from the ground states of H_0 to the ground states of H . To find such an absolutely stable phase of matter is quite nontrivial at first glance: after all, the ground-state degeneracy of the Ising H_0 is *not* stable in this way: taking

$$V = \frac{1}{n} \sum_i Z_i \quad (6)$$

already completely breaks the ground-state degeneracy. The absence of stability of the Ising model to these longitudinal fields is quite well known, and related to the physics of the false vacuum [36].

The language of error correction helps to explain the discrepancy. The toric code is a *quantum code*: acting on a codeword of the toric code with $\ll d$ errors, either X type or Z type, can be detected and corrected by identifying the locations of the flipped parity checks A_v and B_f . Viewing the classical Ising model as a quantum code, we see that it has a large distance $d_X = n$ for X errors, but a tiny distance $d_Z = 1$ for Z errors, as

$$Z_1 \frac{|00 \cdots 0\rangle + |11 \cdots 1\rangle}{\sqrt{2}} = \frac{|00 \cdots 0\rangle - |11 \cdots 1\rangle}{\sqrt{2}}. \quad (7)$$

If, therefore, we study the robustness of the Ising model H_0 to perturbations V that are products of X operators, we might expect a similar stability to the toric code, and this has indeed been shown rigorously [38,43]. The intuitive result is much older, and dates back to the classic solution of the transverse-field Ising model, which undergoes a quantum phase transition out of the ferromagnetic phase only for a finite perturbation strength in V [44].

It thus appears that whenever we have a quantum Hamiltonian H_0 , which can be interpreted as a quantum error-correcting code, we might expect a stable phase of matter. Along these lines, we might ask what happens when we study more exotic codes than those that correspond to geometrically local Hamiltonians H_0 . The most famous example of such a code is an LDPC code. The details of how to construct H_0 for an LDPC code will not be essential to understanding our main results, but at a high level, one writes

$$H_0 = - \sum_{\text{parity check } C_X} \prod_{i \in C_X} X_i - \sum_{\text{parity check } C_Z} \prod_{i \in C_Z} Z_i, \quad (8)$$

subject to (i) the requirement that each C_X and C_Z commute and (ii) the requirement that each $C_{X,Z}$ is a product of $O(1)$ Paulis, and that each Pauli X_i or Z_i shows up in an $O(1)$ number of parity checks [45]. Remarkably, it is known how to construct such codes, which are constant rate: in the thermodynamic limit $n \rightarrow \infty$, one finds $k = \Theta(n)$, while d also grows with n , ideally linearly. From an error-correcting perspective, such codes only use a constant number of redundant physical bits to encode information, yet it is an encoding that is extremely robust to large errors. It is known [46] that such constant-rate codes (with growing distance) can only exist in the limit of infinitely-many spatial dimensions. Mathematically, one often refers to such a code as defined on an expander graph, where two qubits share an edge if they share a common parity check, and the resulting graph has the property that

$$\begin{aligned} \text{number of qubits within distance } r \text{ of qubit } i &\gtrsim \exp[\alpha r], \\ \text{if } r &\lesssim \log n. \end{aligned} \quad (9)$$

for some positive $\alpha > 0$.

The purpose of this paper is to make this intuition precise, and to prove that a quantum LDPC Hamiltonian H_0 indeed forms a stable phase of matter. A classical LDPC code (with only Z -type checks) will similarly form a robust phase of matter to X -type perturbations.

III. CHECK SOUNDNESS: A SUFFICIENT CONDITION FOR STABILITY

To prove the stability of an LDPC code, we must first ask whether there is any way to make a genuine code unstable. While the “reasonable” error-correcting toric-code Hamiltonian in Eq. (3) indeed represents a stable phase of matter, a “stupid” toric code can fail to have this property [4]. Consider modifying Eq. (3) to

$$H_0 = - \sum_v A_v - B_{f_0} - \sum_{f \sim f'} B_f B_{f'}, \quad (10)$$

where A_v, B_f are the vertex and plaquette checks introduced before, f_0 is one “special” face anywhere in the lattice, and faces $f \sim f'$ are adjacent if and only if they share a single edge. In the modified Hamiltonian Eq. (10), only the single plaquette term at p_0 remains to be a check, and the other Z checks are Ising interactions among all nearest-neighbor plaquettes $f \sim f'$. Although the modified H_0 has the same error-correcting ground states as the standard toric code, its gap is unstable to the same kind of “longitudinal field” perturbation as Eq. (6): $V = n^{-1} \sum_f B_f$.

As proposed in Ref. [4] and refined in Ref. [9], a sufficient condition for stability in finitely many spatial dimensions is local topological order (LTO), which states that for any *ball* region S of qubits with sufficiently small radius, any operator $\mathcal{O}$ supported on S satisfies

$$P_{\tilde{S}} \mathcal{O} P_{\tilde{S}} = c_{\mathcal{O}} P_{\tilde{S}}, \quad (11)$$

for some number $c_{\mathcal{O}}$, where $\tilde{S} \supset S$ is a slightly larger region that includes all checks touching S , and $P_{\tilde{S}}$ is a local ground-state projector that projects onto the simultaneous $+1$ eigenspace of all checks inside $\tilde{S}$. Equation (11) is the definition of a detectable error $\mathcal{O}$ in an error-correcting code (Knill-Laflamme condition) condition by replacing S in the subscripts by the whole system, so Eq. (11) can be viewed as a *local* version of correctability. For stabilizer codes, Eq. (11) holds if the volume of S is smaller than the code distance, and any stabilizer in S can be expanded as a product of checks in $\tilde{S}$ [4].

Observe that LTO indeed rules out the Ising toric code Eq. (10): around any ball S faraway from the particular plaquette f_0 , the local ground states $P_{\tilde{S}}$ contain two sectors where $B_f = +1$ ($\forall f \in \tilde{S}$) for the first sector and -1 for the second. Equation (11) is thus violated by $\mathcal{O} = B_{f_0}$ that differentiates the local ground states; this

operator is precisely the kind of perturbation that would make the model unstable. Equation (11) naturally generalizes to LDPC codes, where one can still define a ball S as containing all vertices of a bounded graph distance to a given vertex. This is indeed the condition used in the proof for LDPC stability on nonexpander graphs [34].

We now argue that LTO is not enough to get a *good* bound on the stability of a phase of matter defined by an LDPC code. (1) Since S in the local indistinguishability condition is restricted to balls, it does not control general operators that act on $> L_* + 1$ sites, where L_* is the maximal diameter of S that satisfies Eq. (11). Even if the original perturbation V is local, such uncontrolled operators would appear at order approximately L_* in perturbation series. Such operators are only suppressed by factor approximately e^{-L_*} , so that the energy splitting of the perturbed ground states is only bounded by a polynomial $1/\text{poly}(n)$, as L_* is bounded by the diameter of the system approximately $\log(n)$ for exponential-expanding graphs [47]. This bound is much weaker than the stretched-exponential bound $e^{-\sqrt{n}}$ for the perturbed two-dimensional toric code [4]. (2) In order for the indistinguishability diameter L_* to scale as approximately $\log(n)$, the distance of the code would typically need to grow at least as a power law of n . Therefore the result may not hold for codes with $\text{polylog}(n)$ distance, which are potentially interesting because this distance is sufficient to have a finite error-correction threshold [48]. (3) The local indistinguishability for ball regions does not directly control operators that are supported on disjoint regions. Therefore, it is not obvious how this condition can be used to prove stability under perturbations that are not geometrically local with respect to the original code. This includes important scenarios such as power-law decaying interactions in finite dimensions with sufficiently small power-law decay exponent, where the stability of the two-dimensional toric-code phase has not yet been established.

To deal with these caveats, we propose a suitable generalization Eq. (11) to more general, possibly disconnected regions S , as long as its *volume* (instead of diameter) is bounded by, roughly speaking, the code distance d . However, the naive generalization of Eq. (11) would fail: even for the toric code, Eq. (11) does not hold for S being an annulus, where a Z -loop operator $\mathcal{O}$ in S cannot be expanded to checks in $\tilde{S}$, which is still an annulus if the “radius” of S is larger than a constant. In this case, we need a larger region $\bar{S}$ that “fills the hole” of $\tilde{S}$, so that Eq. (11) is recovered by replacing the subscripts $\tilde{S}$ to $\bar{S}$:

$$P_{\bar{S}} \mathcal{O} P_{\bar{S}} = c_{\mathcal{O}} P_{\bar{S}}. \quad (12)$$

This is because stabilizers in S can be expanded into product of checks inside $\bar{S}$, as illustrated in Fig. 2. Note that the volume of $\bar{S}$ can be much larger than $\tilde{S}$; conceptually we need that the growth of $|\bar{S}|$ with respect to $|S|$ is not

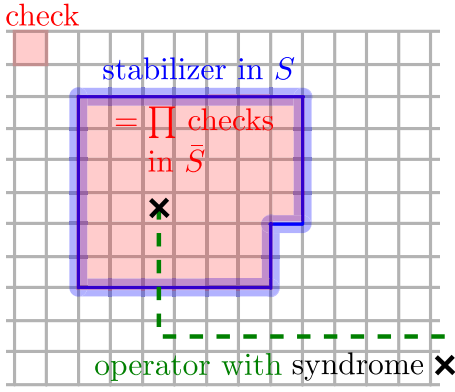


FIG. 2. For the toric code, an annulus region S (shaded blue region) contains a Z -loop stabilizer that is a product of all Z checks in the shaded red region $\tilde{S}$, which is the annulus S with its hole filled. As a result, this Z -loop stabilizer may anticommute with X strings (shown by green) that has syndrome (checks that anticommute with the operator) in $\tilde{S}$ but has no syndrome in S . In our proof of stability for general LDPC codes, we require that $|\tilde{S}|$ does not grow too quickly with $|S|$, so that in a nutshell, the stabilizer in S does not anticommute with too many operators whose syndrome are faraway from S . We call this condition “check soundness.” For example, the toric code has quadratic check soundness because $|\tilde{S}| \lesssim |S|^2$ due to the area-versus-diameter relation.

“too fast,” so that the condition Eq. (12) is still sufficiently local. For the toric code (Fig. 2), it makes sense to identify $\tilde{S}$ as the “filled in” (simply connected) version of S , but in infinite dimensions it is not obvious how to pinpoint the “holes” of a region S that we want $\tilde{S}$ to fill. Moreover, requiring Eq. (12) for any shape of S would naively require one to examine exponentially many sets S .

Because of these challenges, we have identified a distinct condition that is sufficient for stability, which we will call *check soundness*.

Definition 1. We say a stabilizer code has (d_c, f) -check-soundness, if any stabilizer acting on $M < d_c$ sites is equal to a product of $\leq f(M)$ parity checks with an increasing function f .

Here the weight cutoff d_c is often of the same order as code distance d that scales with n . The more crucial parameter is the function f , which are usually chosen as a power law $f(M) \sim M^a$ [49].

For the two-dimensional toric code, we have check soundness with $a = 2$, because the product of parity checks in region R acts along the boundary of the region R , and area $\lesssim \text{perimeter}^2$ in two spatial dimensions.

We find that $a < 2$ is sufficient to prove the stability of a phase defined by a general quantum LDPC code, while in finite spatial dimensions LTO guarantees a finite a that is also sufficient. Intriguingly, we show in Appendix B 3 that many constant-rate quantum LDPC codes, including

“good” codes where $d = \Theta(n)$, have the best-possible linear check soundness $a = 1$, so our stability proof applies.

IV. MAIN RESULT

For an operator V that is an extensive sum of local terms, we define its local strength ϵ as, roughly speaking, the maximum ∞ -norm ($\|\cdot\|$) of operators in V that act nontrivially on a given qubit. More precisely,

$$\epsilon = \max_i \sum_{S \ni i} \sum_s \|V_{S,s}\| e^{\kappa|S|}, \quad (13)$$

where $V = \sum_{S,s} V_{S,s}$ with $V_{S,s}$ acting on qubit set S and having error syndrome $\mathbf{s}$ (all terms in $V_{S,s}$ anticommute with the same parity checks), and $\kappa = O(1)$ is a constant used to penalize terms of large support. Note that having a finite κ -norm ensures that perturbations decay exponentially with *volume*, not diameter, of the set S . See Appendix A for the formal definition. Such a local strength does not scale with the system size, and we desire stability against perturbations V with a small but finite local strength. We are ready to state our main result illustrated in Fig. 1.

Theorem 1 (Informal version). Let H_0 be the Hamiltonian of a quantum LDPC code with gap 1 and code parameters $[[n, k, d]]$. Suppose

$$d = \Omega(\log n), \quad (14)$$

and H_0 has better-than-quadratic check soundness $a < 2$. There exists a positive constant ϵ_0 , such that for any perturbation V whose local strength ϵ is small: $\epsilon < \epsilon_0$, the perturbed Hamiltonian $H := H_0 + V$ still has 2^k low-energy states, separated by a gap $\geq 1/2$ from the remainder of the spectrum. These lowest-energy states are almost degenerate with small energy difference

$$\delta E = \epsilon e^{-\Omega(d)}. \quad (15)$$

Moreover, the perturbed lowest-energy states are connected to the unperturbed ground states by a quasilocal unitary whose generator has small $O(\epsilon)$ local strength.

The above conclusions hold for the known constructions of good quantum LDPC codes [22–24] with $d = \Theta(n)$, quantum expander codes from hypergraph product [18,50] with $d = \Theta(\sqrt{n})$, and finite-dimensional codes with local topological order [51].

See Appendix C for the formal statements and the proof, where we employ the Schrieffer-Wolff transformation (SWT) scheme used to prove stability of finite-dimensional codes [4]. This is in contrast to the scheme based on quadiabatic evolutions [9,34,38]. We do not employ the

latter scheme because it inevitably invokes Lieb-Robinson bounds [52] for Hamiltonian evolution whose locality is exponential decaying with the *diameter*, not the *volume*, which seems to only produce a quasipolynomial bound on energy splitting [34] for expander graphs.

As we do not require the perturbation V to be geometrically local, Theorem 1 proves stability against general q -local perturbations, as long as $q = \Theta(1)$ and the local perturbation strength ϵ that each qubit feels is smaller than some constant. In particular, this holds for finite-dimensional codes with local topological order, and significantly improves upon the previous result [9], which proves stability with a polynomial energy splitting $\delta E = O(n^{-\alpha})$ for power-law decaying perturbations with a sufficiently fast decay exponent. Here we achieve exponential energy splitting, e.g., $\delta E = e^{-\Omega(\sqrt{n})}$ for the toric code, for all power-law decaying exponents. Note that Ref. [9] does work with the weaker assumption that H_0 is frustration-free, rather than requiring it to come from a stabilizer code, as in this paper.

A. Proof sketch for a simple example

Here we sketch the proof of Theorem 1 using a simple example, and point out our technical ingredients along the way. Consider a nongeometrically local perturbation on a magnetic-field Hamiltonian H_0 :

$$H_0 = -\frac{1}{2} \sum_{i=1}^n Z_i, \quad V_1 = \frac{1}{n} \sum_{1 \leq i < j \leq n} \epsilon_{ij} (X_i X_j + Z_i Z_j), \quad (16)$$

where $|\epsilon_{ij}| \leq \epsilon$. Note that we have normalized the perturbation V_1 so that its local strength acting on any one qubit is bounded by ϵ .

H_0 has a single gapped ground state $|0\rangle$, the all-zero state, so can be viewed as a trivial $[[n, 0, \infty]]$ quantum code. Since H_0 satisfies LTO in a trivial way, and can be viewed as in one spatial dimension, it is stable for small ϵ according to Theorem 1, which goes beyond previous stability proofs for topological orders due to the nongeometrically local perturbation. An alternative way of applying Theorem 1 to this example is to observe H_0 has the best-possible linear check soundness, because the checks Z_i just act on different qubits.

V_1 couples the ground state $|0\rangle$ of H_0 to excited states. We wish to find a unitary U that block diagonalizes the Hamiltonian $H = H_0 + V_1$ such that $|0\rangle$ is an eigenstate of the dressed Hamiltonian $U^\dagger H U$. Here the two blocks correspond to $|0\rangle$ and its orthogonal subspace. However, such a unitary U is hard to find exactly, and may be very nonlocal.

Instead, one can first aim something weaker: to find a unitary U_1 that block diagonalizes H up to second order:

$$U_1^\dagger H U_1 = (H_0 + D_2) + V_2, \quad (17)$$

where $D_2 |0\rangle \propto |0\rangle$ and $V_2 \propto \epsilon^2$. Note that V_2 is still an extensive sum of many local terms so $\|V_2\| \sim \epsilon^2 n \gg 1$. For each local term $V_{1;ij} \propto X_i X_j + Z_i Z_j$ of V_1 , it is not hard to find a unitary $U_{1;ij}$ acting on sites i, j that rotates away the off-block-diagonal part $X_i X_j$ of $V_{1;ij}$, such that Eq. (17) holds for $U_1 = U_{1;ij}$ if V_1 only consists of this single term. Moreover, $U_{1;ij} = e^{A_{1;ij}}$ for an anti-Hermitian operator $A_{1;ij}$ supported on i and j that is as small as $V_{1;ij}: A_{1;ij} \sim \epsilon/n$. As a result, even if V_1 contains many local terms $V_{1;ij}$, we have $A_{1;ij}$ for each term so that we can define the SWT unitary $U_1 = e^{A_1}$ with $A_1 = \sum_{1 \leq i < j \leq n} A_{1;ij}$. This U_1 is quasilocal in the sense that A_1 has small local strength ϵ . On the other hand, this U_1 achieves Eq. (17) because at first order every $V_{1;ij}$ is independently rotated by the corresponding $A_{1;ij}$ in A_1 . This SWT procedure can be understood as a nondegenerate perturbation theory applied to local operators.

Since U_1 is quasilocal, D_2, V_2 are also expected to be sufficiently local. For example, V_2 is roughly $[A_1, V_1]$ so a typical local term of it acts on only ≤ 3 sites, as A_1, V_1 are both two local. However, V_2 strictly speaking contains Heisenberg-evolved operators like $e^{-A_1} V_1 e^{A_1}$, so in principle contains nonlocal operators that act on the whole system. We thus need to control the locality of V_2 by examining how a term in it decays with the support (or operator size) of that term. Since the operators like A_1 are nongeometrically local, we cannot invoke Lieb-Robinson bounds. Instead, we use locality bounds obtained from cluster-expansion techniques [36, 53]. This enables us to show that, a term in V_2 decays exponentially with its support volume, so V_2 is still a sufficiently local operator dominated by q -local terms with small $q \approx 3$ (albeit nongeometrically local). We point out that this exponential decay with volume (instead of, e.g., diameter of the support) is crucial to obtain the final bound Eq. (15) that is exponential in the code distance.

We refer to Lemma C3 in the Appendix for the mathematical cluster-expansion bound we use; here we provide a rough intuition on how it works. For a local operator $\mathcal{O}$ at site i_0 , one can Taylor expand $e^{-A_1} \mathcal{O} e^{A_1} = \mathcal{O} - [A_1, \mathcal{O}] + \dots$ and then replace each A_1 by the sum of its local terms. Then $e^{-A_1} \mathcal{O} e^{A_1}$ will be a sum over terms $[\dots [A_{1;i'j'}, [A_{1;ij}, \mathcal{O}]] \dots]$ organized by their “clusters,” i.e., the supports $(\dots, i'j', ij, i_0)$. This expansion clearly has the property we want that each cluster has an operator norm that decays exponentially with the cluster volume, because each $A_{1;ij}$ is small. The technical part is then to show this expansion converges when truncating the sum at some cutoff cluster volume. The idea is that a cluster can be viewed as appending and growing one extra $A_{1;ij}$ on a smaller cluster; the cluster-expansion technique establishes a condition for each local growth, under which the whole expansion would indeed converge. This condition translates to a small local strength of A_1 in our context, which is satisfied from the discussions above.

Observe that we have achieved Eq. (17) that is already an order of improvement over the original problem where $V_1 \propto \epsilon$. Moreover, the dressed Hamiltonian in Eq. (17) looks almost exactly like the original problem, where now we are perturbing $H_0 + D_2$ by V_2 . We then want to follow the procedure of obtaining U_1 to suppress the perturbation V_2 to an even higher order so that the Hamiltonian becomes

$$U_2^\dagger U_1^\dagger H U_1 U_2 = (H_0 + D_3) + V_3, \quad (18)$$

where U_2 is another quasilocal SWT unitary. However, the dressed Hamiltonian in Eq. (17) is *not* exactly the original problem because of the extra D_2 that makes the major part $H_0 + D_2$ unsolvable. This step thus requires extra conditions of H_0 to guarantee that $H_0 + D_2$ is, roughly speaking, still a “code” such that there remains a $\Omega(1)$ gap above $|\mathbf{0}\rangle$.

Here $D_2 = n^{-1} \sum_{ij} \epsilon_{ij} Z_i Z_j$ is the block-diagonal part of V_1 . Although $\|D_2\| \gg 1$, it cannot close the gap of $H_0 + D_2$. To see this, consider a particular state $|\psi\rangle = |\mathbf{0}\rangle_{1,\dots,n/2} \otimes |\mathbf{1}\rangle_{n/2+1,\dots,n}$ (supposing n is even) and a particular choice of $\epsilon_{ij} = \epsilon$ ($i \leq n/2, j \geq n/2$) and $\epsilon_{ij} = 0$ otherwise. This choice makes $|\psi\rangle$ gain a large extensive energy from D_2 comparing to $|\mathbf{0}\rangle$:

$$\langle \mathbf{0} | D_2 | \mathbf{0} \rangle - \langle \psi | D_2 | \psi \rangle = \epsilon n/2 \gg 1, \quad (19)$$

much larger than the gap of H_0 . However, this does not make $|\psi\rangle$ the new ground state: it has an extensive energy penalty $\langle \psi | H_0 | \psi \rangle - \langle \mathbf{0} | H_0 | \mathbf{0} \rangle = n/2$ from the original Hamiltonian H_0 , which is larger than Eq. (19)!

The gap of $H_0 + D_2$ can be attributed to the check soundness property of H_0 , and this turns out to be a general sufficient condition for stability. Intuitively, D_2 contains small stabilizers, and check soundness implies that each stabilizer $\mathcal{S}$ in D_2 can be expanded to a product of a small number of checks $C_1, \dots, C_M$ in H_0 . As a result, in order for a state to gain energy (relative to $|\mathbf{0}\rangle$) from stabilizer $\mathcal{S}$, it has to flip one of the few checks in $C_1, \dots, C_M$. As M is small from check soundness, this means that when considering all stabilizers in D_2 , a sufficiently large fraction approximately $1/M$ of the checks in H_0 need to be flipped, so that the energy penalty from H_0 always wins and the gap is preserved at small but constant ϵ .

Having figured out how to do a single step of SWT using check soundness, the goal is then to repeat the SWT procedures Eqs. (17) and (18) to very high orders. We do not need to compute D_m, V_m, U_m explicitly, but we need to control their locality as they become more nonlocal at higher orders. With such technicalities addressed by the cluster-expansion bounds, we finally arrive at a sufficiently high order $m_* \sim n$ for the example Eq. (16), so that the dressed Hamiltonian becomes

$$U^\dagger H_0 U = H_0 + D_{m_*} + V_{m_*}, \quad (20)$$

where $U = U_1 U_2 \dots U_{m_*-1}$ is still quasilocal, $H_0 + D_{m_*}$ has $|\mathbf{0}\rangle$ as its gapped ground state from check soundness,

and $V_{m_*} \sim \epsilon^n$ is so small that it does not close the gap and only causes an exponentially small energy splitting [if the model has multiple ground states unlike Eq. (16)]. This establishes the stability of the original problem Eq. (16).

B. High-level summary of the proof of Theorem 1

Our goal is to find a quasilocal unitary U that achieves Eq. (20), where H_0 is the Hamiltonian of an LDPC code with check soundness, and D_{m_*}, V_{m_*} are still quasilocal Hamiltonians. We desire D_{m_*} to be block diagonal between the codespace and the rest of the Hilbert space, and that $H_0 + D_{m_*}$ retains the codespace as a gapped 2^k -fold degenerate ground-state subspace. Lastly we require

$$\|V_{m_*}\| = e^{-\Omega(d)}. \quad (21)$$

The existence of such a U immediately implies Theorem 1. Since V_{m_*} is exponentially small, the spectrum of H agrees with that of $H_0 + D_{m_*}$ up to energy differences given by Eq. (15), while $H_0 + D_{m_*}$ has all the desired properties, including a preserved $\Omega(1)$ gap. To establish all of these facts, there are three crucial steps:

- (1) To find U , we formally perform a perturbation theory in ϵ by iterated SWTs, up to order $m_* \sim d$. $U = U_1 U_2 \dots U_{m_*}$, where at each order $U_l = \exp[A_l]$ where A_l is chosen carefully by using the decomposition Eq. (13). Terms $V_{S,l}$, which violate more parity checks, can be removed by terms in A_l with smaller coefficients.
- (2) We then prove the convergence of this series up to the desired order: in particular, that D_{m_*} and V_{m_*} are quasilocal in the sense of Eq. (13) with a finite κ . The cluster expansion proves useful here, along with check soundness, which constrains the number of nonvanishing possible terms in the cluster expansion.
- (3) Lastly, we verify that $H_0 + D_l$ remains gapped at each step of the SWT. Check soundness is again crucial to ensure that this gap cannot close, by forbidding pathological examples such as Eq. (10).

V. APPLICATION TO CLASSICAL CODES AND QUASIADIABATIC EVOLUTION

Our result also generalizes to prove (in Appendix D) stability of classical LDPC codes with check soundness.

Corollary 1 (Informal version). Theorem 1 also holds for H_0 being a classical LDPC code, as long as each term in perturbation V is symmetric under conjugation by any logical operator (made out of Pauli X s), as discussed around Eq. (7).

Here we need to restrict to symmetric perturbations, because otherwise the gap is in general unstable to Z -field

perturbations that locally differentiate the ground states, lifting some of the ground states to false vacuum states [36,54,55] in the spectrum of excited states, as previously discussed. There do exist classical codes with check soundness (see Appendix B 3) to which this result applies.

It would be interesting to compare this ground-state stability result to the provably robust eigenstate localization in classical LDPC codes on expander graphs [56], which can be viewed as a weaker sense of stability, albeit one that holds for all states in the spectrum with a bounded energy density. The stability theorem of Ref. [56] is weaker in the sense that, the eigenstates form groups where different groups corresponding to different codewords do not mix with each other; perturbation can cause strong mixing within groups. The guarantee of Corollary 1 is closer, in spirit, to the conjecture for “1-bits”—namely, that there would exist *local unitary* rotations [57] from a product state into a many-body localized eigenstate. Of course, Corollary 1 only holds for the ground states, whereas the localization guarantee of Ref. [56] also holds for excited states.

A limitation of Corollary 1 is that it does not apply to classical codes without check soundness. An important example of a classical code without check soundness is the Ising model Eq. (1). If the shortest path in the interaction graph from site 1 to M is $1 \rightarrow 2 \rightarrow \dots \rightarrow M$, then stabilizer $Z_1 Z_M$ expands to at least $M - 1$ checks $Z_1 Z_2, Z_2 Z_3, \dots, Z_{M-1} Z_M$ along the path. This lack of check soundness is physical. The Ising model is known to be unstable against long-range $Z_i Z_j$ perturbations [43,58,59]. The idea is that although $Z_i Z_j$ is globally symmetric, it “fractionalizes” into two parts Z_i and Z_j that each is charged under the symmetry and locally destroys the error-correction phase. Intriguingly, a similar instability also exists for symmetry-protected topological states [7].

Nevertheless, we can generalize our result as follows, focusing on the Ising model:

Corollary 2 (Informal version). The conclusions of Theorem 1 hold for H_0 being the Ising model Eq. (1) on any graph, so long as the perturbation V is a sum of sufficiently local terms, which act on connected subsets of qubits.

Here we crucially restrict to *connected* sets, ruling out long-range perturbations that decay slower than exponential with respect to the connected support. Corollary 2 improves previous results [38,43] in that (1) we do not require finite dimensions; (2) we obtain an energy splitting exponentially small in the system *volume* $\delta E = e^{-\Omega(n)}$ instead of diameter; (3) we find a quasilocal unitary $U = \mathcal{T} \exp \left[\int_0^1 dt A(t) \right]$ connecting the old and new ground states whose generator $A(t) = \sum_S A_S(t)$ decomposed to connected sets decays exponentially with volume

$|S|$ instead of diameter:

$$\|A_S(t)\| = \epsilon e^{-\Omega(|S|)}. \quad (22)$$

To understand this improved bound, we recall that when two gapped Hamiltonians H_1, H_2 are in the same phase, i.e., there exists a smooth path of Hamiltonians $\{H(s) : s \in [0, 1], H(0) = H_1, H(1) = H_2\}$ connecting the two without closing the gap, a quasi-local unitary U is known to exist that connects the two gapped subspaces [39,40], which is called quasiadiabatic evolution (continuation). However, even if $H(s)$ is strictly local (i.e., interactions have finite range) in finite dimensions, the generator $A(t)$ of the quasiadiabatic evolution is only known to decay as [40]

$$\|A_S(t)\| = \exp \left[-\Omega \left(\frac{r}{\log^2 r} \right) \right], \quad (23)$$

where $r = \text{diam } S := \max_{i,j \in S} d(i,j)$ is the diameter of S . Our results Theorem 1 and Corollary 2 significantly improves this almost exponential decay with the diameter, to an exponential decay with the volume $|S|$ in Eq. (22), with a restriction that the Hamiltonians H_1, H_2 are close enough to a solvable fixed point H_0 as an LDPC code. This improved tail bound enables us to show much stronger symmetry properties for gapped ground states around fixed points in the phase space, which are discussed elsewhere [41].

Beyond classical codes under symmetric perturbation, Eq. (22) also holds for quantum code Hamiltonians under general perturbation. This fact follows from Theorem 1.

VI. CONCLUSION

We have established that a broad family of quantum LDPC codes define stable gapped phases of matter, whose (approximate) ground-state degeneracy is robust to perturbation. Under sufficiently small perturbations, quasilocal unitaries can relate one ground state to another in the same phase. We have further argued that check soundness, rather than local topological order, is the critical property that renders a quantum code a stable phase of matter. LDPC codes then represent a setting in which certain mathematical definitions about what constitutes a robust phase of matter can be extended into “infinite dimensions,” beyond the paradigms of Landau theory or statistical and quantum field theory.

These results represent a valuable contribution to a rapidly growing body of literature [26,27,29,34,35,56] on the *physics* of quantum LDPC codes. Like classical LDPC codes [60], it increasingly appears that quantum LDPC codes form a kind of quantum glass with ergodicity breaking at low temperature, perhaps related to the complex entanglement (NLTS) of the low-energy states [32]. In particular, our work combined with Ref. [29]

suggests that these quantum LDPC codes are *stable* self-correcting quantum memories, where both coupling to a low-temperature thermal bath and perturbing the Hamiltonian can not corrupt the encoded quantum information. These unusual phases of matter break our intuition about thermodynamics in finitely many spatial dimensions, including the existence of a third law of thermodynamics: namely, that any well-defined quantum *phase* should have a vanishing entropy density at zero temperature. We expect that this new family of phases of matter will continue to push the limits of our understanding of the foundations of statistical mechanics, and may indeed represent an ideal setting to “stress test” the lore of quantum statistical mechanics that was developed with our three-dimensional universe in mind.

Note added. Recently, we became aware of related work [61]. Our preprints appeared in the same arXiv posting.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: PRELIMINARIES

1. Graphs

We define the floor function $\lfloor x \rfloor$ as the largest integer that is not larger than x .

Let Λ denote a set of vertices on an undirected graph with edge set $\mathcal{E}$. When discussing subsets $S \subset \Lambda$, the complement of S will be denoted as S^c . $S \setminus S'$ denotes the set $\{i \in S : i \notin S'\}$. A set S is connected if for any two $i \in \Lambda$ and $j \in \Lambda$, there exists a subset of $\mathcal{E}$ of the form $(i, k_1), (k_1, k_2), \dots, (k_l, j)$ such that $\{k_1, \dots, k_l\} \subseteq \Lambda$; otherwise, S is disconnected. We assume that Λ is connected.

Definition A1 (Distances on graphs). For each pair of vertices $i, j \in \Lambda$, the distance $d(i, j) \in \mathbb{Z}_{\geq 0}$ between them is the length of the shortest path in graph $(\Lambda, \mathcal{E})$ between them, with $d(i, j) = 0$ if and only if $i = j$. We similarly define the distance from a vertex to a set $d(i, S) := \min_{j \in S} d(i, j)$, and between two sets $d(S, S') := \min_{i \in S, j \in S'} d(i, j)$.

Definition A2 (Ball around a vertex). A ball $B_{i,r}$ of center i and radius r is the set of vertices that are of distance $\leq r$ to i . Let $\Gamma_i(r) := |B_{i,r}|$ be the upper bound on the volume of a radius- r ball, and $\gamma_i(r) := |B_{i,r}| - |B_{i,r-1}|$ be the

bound on the surface volume. We often drop the subscript $\Gamma_i \rightarrow \Gamma$ when i is clear from context.

Definition A3 (Degree and dimensionality). The **degree** of a vertex i is $\gamma_i(1)$. If a graph is “constant degree” with $\gamma_i(1) \leq \Delta$ for all i , then we always have a bound

$$\gamma(r) \leq \Gamma(r) \leq \Delta^r. \quad (\text{A1})$$

If $\Gamma(r) \geq \exp[\alpha r]$ for some finite $\alpha > 0$, up to $r = O(\log n)$, we say that the graph is an **expander graph**.

If there exist constants $\mathcal{D}, c_{\mathcal{D}}$ such that

$$\Gamma(r) \leq c_{\mathcal{D}} r^{\mathcal{D}}, \quad \gamma(r) \leq c_{\mathcal{D}} r^{\mathcal{D}-1}, \quad \forall r \geq 1, \quad (\text{A2})$$

we say the graph G is of finite spatial dimension $\mathcal{D}$. Note that for any connected graph we require $\mathcal{D} \geq 1$.

2. Quantum stabilizer codes

In this paper, we study quantum many-body systems consisting of n qubits, i.e., those whose Hilbert space

$$\mathcal{H} := (\mathbb{C}^2)^{\otimes n}. \quad (\text{A3})$$

We will usually denote $\Lambda := \{1, \dots, n\}$ to be the set of qubits, and $i \in \Lambda$ to denote one particular qubit. We let X_i , Z_i and $Y_i = iZ_iX_i$ denote the three Pauli matrices acting on qubit i .

We now restrict our study further to H , which arise from quantum (Pauli) stabilizer codes. We are deliberately restricting our discussion of quantum codes to only the bare minimal ingredients necessary to understand the proof of our main result.

Definition A4 (Stabilizer code). Let Λ_c denote a set of **parity checks** (checks for short): **Pauli strings** of the form $\prod_{j \in S} A_j$ where each $A_j \in \{X_j, Y_j, Z_j\}$ and $S \subset \Lambda$, which obey

$$[\mathcal{Q}_1, \mathcal{Q}_2] = 0, \quad \text{if } \mathcal{Q}_{1,2} \in \Lambda_c. \quad (\text{A4})$$

If for some $\mathcal{Q}_1, \dots, \mathcal{Q}_n \in \Lambda_c$, $\prod_{a=1}^n \mathcal{Q}_a = I$ (the identity), the stabilizer code is **redundant**. In what follows, we will usually refer to checks $\mathcal{Q}_C$ by the subset $C \subset \Lambda$ on which they act; $|C|$ refers to the length of the Pauli string $\mathcal{Q}_C$. A product of parity checks is called a **stabilizer**.

A **logical operator** is a Pauli string, which commutes with all stabilizers, yet is not a product of elements of Λ_c . A **quantum code** has $2k$ logical operators whose algebra is isomorphic to that generated by $X_1, Z_1, \dots, X_k, Z_k$. The length of the smallest nonidentity logical operator is called the **code distance** d . If $|\Lambda| = n$, we call this an $[[n, k, d]]$ code.

A **LDPC code** is a stabilizer code for which (1) any $\mathcal{Q}_C \in \Lambda_c$ has the property that $|C| \leq q$, for some $O(1)$

value q , and (2) for any $i \in \Lambda$, $|\{Q_C \in \Lambda_c : i \in C\}| \leq q'$ for $O(1) q'$. In other words, each qubit is involved in a finite number of checks, and each check involves a finite number of qubits.

The **Tanner graph** of a stabilizer code is a bipartite graph with vertices $\Lambda \cup \Lambda_c$, where there is an edge (i, Q_C) with $i \in \Lambda$, $Q_C \in \Lambda_c$ if and only if $i \in C$.

A **CSS code** is a stabilizer code where any $Q_C \in \Lambda_c$ is a product of all Pauli X s or all Pauli Z s. k logical operators will be products of all X s, and k will be products of all Z s.

A **classical code** is a stabilizer code where any Q_C is a product of Pauli Z s. In a classical code, we restrict study to logical operators that are a product of Pauli X s.

For any CSS code, all of its Z checks in Λ_c form the checks of a classical code. The same holds for X checks with a local basis change $X_i \leftrightarrow Z_i$ for each qubit.

Definition A5 (Code Hamiltonian). Given an LDPC code, we define an unperturbed **code Hamiltonian**

$$H_0 = \sum_{Q_C \in \Lambda_c} \lambda_Q \frac{I - Q_C}{2}, \quad (\text{A5})$$

where, for some finite $a < \infty$,

$$1 \leq \lambda_Q \leq a. \quad (\text{A6})$$

We define a **codespace (ground-state subspace) projector**

$$P := \prod_{Q_C \in \Lambda_c} \frac{I + Q_C}{2}. \quad (\text{A7})$$

$|\psi\rangle$ is a ground state of H_0 , obeying $H_0|\psi\rangle = 0$, if and only if $P|\psi\rangle = |\psi\rangle$. There are 2^k linearly independent ground states. Notice that H_0 has a spectral gap ≥ 1 between the ground state and excited states.

It will often be useful to call upon a *local* version of the ground-state projector P , restricted to set $S \subset \Lambda$:

$$P_S := \prod_{Q_C \in \Lambda_c : C \subset S} \frac{I + Q_C}{2}. \quad (\text{A8})$$

Similarly, we define the local projector onto excited states $Q_S := I - P_S$, the global excited-state projector $Q := I - P$, and the local patch of the Hamiltonian in set S :

$$H_S := \sum_{Q_C \in \Lambda_c : C \subset S} \lambda_Q \frac{I - Q_C}{2}. \quad (\text{A9})$$

Definition A6 (Edge set for the code graph). Given a quantum code, we define the edge set $\mathcal{E}$ such that

$$\mathcal{E} := \{(i, j) = (j, i) : \exists Q_C \in \Lambda_c \text{ with } \{i, j\} \subseteq C\}. \quad (\text{A10})$$

Notice that the code Hamiltonian H_0 thus couples $i \neq j$ if and only if $d(i, j) = 1$.

Given this convention for the edge set on the interaction graph of the code Hamiltonian, notice that the parameter Δ from Definition A3 obeys $\Delta \leq qq' - 1$ [62].

3. Operator decomposition into syndromes

Definition A7 (Syndromes). Let $\mathcal{O}$ be a Pauli string. We write $\mathcal{O} = \mathcal{O}_s$ where the **syndrome** $s \in \mathbb{F}_2^{\Lambda_c}$ is defined such that

$$\mathcal{O}_s Q = \begin{cases} -Q\mathcal{O}_s, & Q \in s \\ Q\mathcal{O}_s, & Q \notin s \end{cases}. \quad (\text{A11})$$

Notice that if the code is redundant, not all syndromes s are possible. More generally, we write $\mathcal{O} = \mathcal{O}_s$ if operator $\mathcal{O}$ can be decomposed as a sum of Pauli strings all having the same syndrome s .

Definition A8 (Strongly supported operators). For any operator $\mathcal{O}$, we can decompose it as

$$\mathcal{O} = \sum_{S, s} \mathcal{O}_{S, s}, \quad (\text{A12})$$

where each $\mathcal{O}_{S, s}$ has syndrome s and is **strongly supported** in S . More precisely, we demand that $\mathcal{O}_{S, s}$ acts trivially on S^c , and any check that does not commute with $\mathcal{O}_{S, s}$ is contained in S :

$$[Q_C, \mathcal{O}_{S, s}] = 0, \quad \text{if } C \not\subset S. \quad (\text{A13})$$

By definition, if an operator is strongly supported in S , it is supported in S , i.e., it does not act on S^c . On the other hand, if an operator is supported in S' , it is strongly supported in a larger set S that includes S' and its neighborhood sites that are connected to S' by a check.

$\mathcal{O}_{S, s}$ can anticommute with operators that flip syndromes s whose checks are not contained in S . An example using the toric code was stressed in the main text (Fig. 2), and accounting for this possibility will be an important issue in our proof. We also stress that $\mathcal{O}_{S, s}$ is not in general a simple Pauli string, and in general it can include arbitrary functions of stabilizers contained within S . When the syndrome is not important, we will sometimes also use

$$\mathcal{O} = \sum_S \mathcal{O}_S, \quad (\text{A14})$$

to simplify notation, where $\mathcal{O}_S := \sum_s \mathcal{O}_{S, s}$.

Definition A9 (κ -norm (local strength in the main text)). For any $\kappa > 0$, we define a κ -norm

$$\|\mathcal{O}\|_\kappa := \max_i \sum_{S \ni i} \sum_s \|\mathcal{O}_{S,s}\| e^{\kappa|S|}. \quad (\text{A15})$$

We will frequently use that

$$\|\mathcal{O}\|_{\kappa'} \leq \|\mathcal{O}\|_\kappa, \quad \forall \kappa' \leq \kappa. \quad (\text{A16})$$

Since one can expand any operator into Pauli strings where each Pauli has a well-defined syndrome, the decomposition Eq. (A12) always exists; furthermore, locality is well preserved in this procedure.

Proposition A1 (Adapted from Proposition 18 in Ref. [36]). For any operator $\mathcal{O} = \sum_S \mathcal{O}_S$ where $\mathcal{O}_S$ is supported in S (not necessarily strongly supported) with

$$\max_i \sum_{S \ni i} \|\mathcal{O}_S\| e^{\kappa|S|} \leq 1, \quad (\text{A17})$$

one can always decompose it to the form (A12), which induces local norm $\|\mathcal{O}\|_{\kappa'} \leq c_\Delta$ with $\kappa' = c'_\Delta \kappa - \log(2)\Delta$, where the constants c_Δ, c'_Δ are determined by Δ .

In words, it is always easy to decompose an extensive operator $\mathcal{O}$ into the form $\sum_S \mathcal{O}_S$ such that the corresponding local strength of this decomposition [left-hand side of Eq. (A17)] is bounded by a constant [1 in Eq. (A17) by normalizing the operator]. Proposition A1 ensures that one can furthermore find a decomposition $\mathcal{O}_{S,s}$ that keeps track of the syndrome information, which induces a finite κ -norm defined in Eq. (A15) as long as the original $\mathcal{O}_S$ decays exponentially with $|S|$ in a sufficiently fast way.

APPENDIX B: CHECK SOUNDNESS

A critical property of a quantum stabilizer code will be the existence of check soundness. The main purpose of this Appendix is to collect known properties about check soundness, which are usually expressed in a different form in the existing coding literature.

1. Formal definition

Definition 1 (Check soundness). We say a stabilizer code has (d_c, f) -check-soundness, if for any (Pauli string) stabilizer $\mathcal{O}$, obeying $[H_0, \mathcal{O}] = 0$, acting on $M < d_c$ sites, there exist $M' \leq f(M)$ parity checks $\mathcal{Q}^{(1)}, \dots, \mathcal{Q}^{(M')} \in \Lambda_c$ whose product produces $\mathcal{O}$: $\mathcal{O} = \mathcal{Q}^{(1)} \dots \mathcal{Q}^{(M')}$. Here $f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a monotonically increasing function.

To understand the above definition, although each stabilizer $\mathcal{O}$ may be expanded to checks in many ways (if the checks have redundancies), there is one (not necessarily

unique) minimal expansion where the number of checks is smallest. Check soundness requires this minimal number to be bounded by a function of the weight of the original stabilizer.

We desire $f(M)$ to grow not too fast with M , and to not depend on n : this immediately rules out the unstable Ising version of toric code Eq. (10), where $f(4)$ is already the order of the linear system size. Furthermore, this definition is closely related to local indistinguishability:

- (1) It is straightforward to show that LTO Eq. (12) in finite dimensions guarantees check soundness for a power function f . We prove this claim in Proposition B2.
- (2) Check soundness implies local indistinguishability in the following sense: A first direct implication is that, for any S with $|S|$ sufficiently small, there exists a region $\bar{S}$ satisfying LTO Eq. (12) with a bounded volume

$$|\bar{S}| < 4^{|S|} f(|S|). \quad (\text{B1})$$

Here the $4^{|S|}$ factor is a very loose bound accounting for all possible Pauli strings in region S , while $f(|S|)$ assumes that each separate Pauli string might be “check sound” on a different set of qubits. Clearly, this bound is very loose, but it does imply that with check soundness, local operators ($|S| = O(1)$) cannot differentiate the local ground states of a local part of H_0 in $\bar{S}$ near its support. Nevertheless, this simply derived local indistinguishability condition, even without the $4^{|S|}$ factor, turns out to be insufficient for our proof of stability. Importantly, check soundness gives us much tighter constraints than Eq. (B1), because loosely speaking it also constrains the “shape” of $\bar{S}$; we will formulate this fact more precisely in Lemma B1.

For CSS codes, one can look for check soundness separately for the X part and Z part:

Proposition B1. For a CSS code, if any X stabilizer acting on $M < d_c$ sites can be expanded as a product of at most $f(M)$ number of X checks, and the same holds for Z stabilizers, then the code has $(d_c, 2f)$ -check-soundness, where $(2f)(M) := 2 \cdot f(M)$.

Proof. For any stabilizer $\mathcal{O} = \mathcal{O}_X \cdot \mathcal{O}_Z$ of the code acting on $M < d_c$ sites, its X -part stabilizer $\mathcal{O}_X$ acts on at most M sites, so can be expanded to at most $f(M)$ X checks. The same holds for the Z part, so the whole stabilizer $\mathcal{O}$ can be expanded to at most $f(M) + f(M) = 2f(M)$ checks. ■

2. Finite-dimensional codes with local topological order

We have introduced local topological order (LTO) in the main text; here we provide the formal definition for convenience.

Definition B1 (Local topological order). For a stabilizer code that is local on a graph with vertex set Λ , we say it has LTO of distance d_{TO} , if the following holds. For any ball $S_0 \subset \Lambda$ with radius smaller than d_{TO} , any operator $\mathcal{O}$ supported on S_0 satisfies

$$P_{\tilde{S}} \mathcal{O} P_{\tilde{S}} = c_{\mathcal{O}} P_{\tilde{S}}, \quad (\text{B2})$$

for some number $c_{\mathcal{O}}$, where $\tilde{S} \supset S_0$ is a larger region that includes all checks touching S_0 .

The maximum size d_{TO} of S_0 could scale with the system size. We clearly need $d_{\text{TO}} < d$, as a logical operator $\mathcal{O}$ would violate Eq. (B2). Nevertheless, notice that LTO implies check soundness with a power-law function f :

Proposition B2. For a local stabilizer code in $\mathcal{D}$ spatial dimensions with code distance d , if it has LTO of distance $d_{\text{TO}} < d$, then it has (d_{TO}, f) -check-soundness, where

$$f(M) = \Delta c_{\mathcal{D}} M^{\mathcal{D}}, \quad (\text{B3})$$

for some $\mathcal{O}(1)$ constant $c_{\mathcal{D}}$.

Proof. For any stabilizer $\mathcal{O}$ acting on S with $|S| \leq d_{\text{TO}}$, decompose the support into its connected components: $S = S_1 \cup \dots \cup S_m$, such that each pair (S_μ, S_ν) is disconnected on the graph G . The Pauli string $\mathcal{O} = \mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_m$ can also be decomposed. Since $|S_\mu| \leq |S| < d$, each $\mathcal{O}_\mu$ is not a logical. Moreover, $\mathcal{O}_\mu$ commutes with all the checks, because otherwise the whole $\mathcal{O}$ would fail to be a stabilizer. As a result, each $\mathcal{O}_\mu$ is also a stabilizer.

Since S_μ is connected on the $\mathcal{D}$ -dimensional lattice, there exists a ball B_μ containing S_μ that has radius $|S_\mu| \leq d_{\text{TO}}$. Using the LTO condition, $\mathcal{O}_\mu$ can be expanded as a product of checks inside B_μ , whose number is at most $\Delta c_{\mathcal{D}} |S_\mu|^{\mathcal{D}}$ using Eq. (A2) and the LDPC condition. This holds for all $\mu = 1, \dots, m$, so the stabilizer $\mathcal{O}$ can be expanded to checks whose number is upper bounded by

$$\Delta c_{\mathcal{D}} \sum_{\mu=1}^m |S_\mu|^{\mathcal{D}} \leq \Delta c_{\mathcal{D}} \left(\sum_{\mu=1}^m |S_\mu| \right)^{\mathcal{D}} = \Delta c_{\mathcal{D}} |S|^{\mathcal{D}}. \quad (\text{B4})$$

This establishes check soundness. $\blacksquare$

Since most finite-dimensional codes, like the toric code, have LTO, they also have check soundness with a power-law function f , up to d_c that is of the order of the

linear system size. As we will see, any power law f is sufficient for perturbative stability in finite dimensions. As an explicit example, we illustrate in Fig. 2 that the two-dimensional toric code has quadratic check soundness $f(M) \sim M^2$, and this arises from the product of M^2 checks in a connected region reducing to a product of Pauli matrices along the perimeter of that region, which has length $\gtrsim M$.

3. Constant-rate quantum codes

Quantum codes with a constant rate and a distance growing with n have to live in “infinite dimensions” [46], i.e., on expander graphs. The most interesting codes satisfy the scaling in Eq. (A1). Unfortunately, we could not prove stability for a power-law soundness function for these expanding codes; without further assumptions, we require it to grow slower than quadratic:

$$f(M) \leq c_f M^{2-\beta}, \quad (\text{B5})$$

for positive constants c_f, β .

Intriguingly, Eq. (B5) is satisfied for all three known constructions of good quantum LDPC codes [22–24], where linear check soundness holds with $\beta = 1$, up to $d_c = \Theta(n)$. Linear check soundness is the best scaling of soundness one can hope for in an LDPC code. Although the constructions for good codes are involved and we do not attempt to describe them in this paper, properties are proven for these codes that straightforwardly lead to linear check soundness. Since these codes are all CSS codes, it suffices to investigate the X part and Z part of the code separately due to Proposition B1.

The most obvious reference is Ref. [24], where linear check soundness is just a special case of its Definition 2.13 for small-set (co)boundary expansion, which is satisfied for the code by combining its Theorem 3.8 and Corollary 3.10. It is also pointed out there that linear check soundness leads to local testability [63] when viewing the three-term chain complex associated with the CSS code, as a classical code with redundant checks.

As the constructions for good quantum codes in Refs. [22,23] also generate good local testable classical codes, these quantum codes are expected to also have linear check soundness. Indeed, linear check soundness can be verified more directly. For the good quantum code in Ref. [22], it is shown in its proof of Theorem 2 that $d_{\text{LM}}^{(1)}$ scales linearly with n , where the locally minimal distance $d_{\text{LM}}^{(1)}$ is defined (in its Definition 5) as the size of the smallest stabilizer or logical Z -string $\mathcal{O}$ that is locally minimal, i.e., multiplying any single Z check on $\mathcal{O}$ cannot decrease its support. The same holds for the locally minimal distance $d_{\text{LM}}^{(1),*}$ for X strings. The locally minimal distance lower bounds the code distance to guarantee the goodness of the code. Here we show that it also yields linear check soundness in a straightforward way.

Proposition B3. Let $d_c = \min(d_{\text{LM}}^{(1)}, d_{\text{LM}}^{(1),*})$ be the smaller locally minimal distance of a CSS code. The CSS code has (d_c, f) -check-soundness for $f(M) = M$.

Proof. For any Z stabilizer $\mathcal{O}$ with $M < d_c$, since $M < d_{\text{LM}}^{(1)}$, it is not locally minimal, so there exists a check that when adding to the stabilizer, decreases its weight by at least 1. One can repeat this argument for the new stabilizer, which is still not locally minimal, so that after at most M steps the stabilizer shrinks to the identity operator. The original $\mathcal{O}$ is thus a product of $\leq M$ checks. The same holds for X stabilizers, so the (d_c, f) -check-soundness follows from Proposition B1. ■

Therefore the quantum code in Ref. [22] has linear check soundness up to $d_c = \Theta(n)$. The same holds for the quantum Tanner code constructed in Ref. [23]; one can see this explicitly by Remark 8.9 in [64], which implies $d_{\text{LM}}^{(1)} = \Theta(n)$ for the quantum Tanner code ($d_{\text{LM}}^{(1),*} = \Theta(n)$ also follows because the code construction is symmetric against exchanging $X \leftrightarrow Z$ parts of the code).

Another notable family of LDPC CSS codes is the quantum expander codes [50] based on the hypergraph product (HGP) [18] of two classical expander codes [65], which is the first quantum LDPC construction that achieves constant rate and a power-law distance $d = \sqrt{n}$. A minimalist definition of an HGP code is defined as follows.

Definition B2 (Hypergraph product code). Consider two classical codes where the bits for the first (second)

code are labeled by b ($\tilde{b}$), and the checks for the first (second) code are labeled by c ($\tilde{c}$). Let $\partial, \tilde{\partial}$ be defined for each code as follows: ∂b is the set of checks $\tilde{b}$ that act on b , while ∂c is the set of bits contained in check c . The hypergraph product of these two classical codes is a quantum CSS code with qubits $\Lambda = \{b\tilde{b}\} \cup \{c\tilde{c}\}$ and checks $\Lambda_c = \Lambda_c^X \cup \Lambda_c^Z$. Here each Z check in Λ_c^Z is labeled by $b\tilde{c}$, while each X check in Λ_c^X is labeled by $c\tilde{b}$, and each are defined as

$$Q_{b\tilde{c}} = \prod_{\tilde{b} \in \tilde{\partial} \tilde{c}} Z_{b\tilde{b}} \cdot \prod_{c \in \partial b} Z_{c\tilde{c}}, \quad Q_{c\tilde{b}} = \prod_{\tilde{c} \in \tilde{\partial} \tilde{b}} X_{c\tilde{c}} \cdot \prod_{b \in \partial c} X_{b\tilde{b}}. \quad (\text{B6})$$

See Fig. 3(a) for a sketch of the Tanner graph (Definition A4) of the quantum HGP code. This relatively simple construction might enable experimental realizations [28,30,31], such as the prototype in Ref. [29]. Intriguingly, the HGP structure naturally leads to check soundness [66]:

Proposition B4 (Lemma 5 in Ref. [66]). Any HGP code generated by taking the product of a classical code with itself, has quadratic check soundness $f(M) = M^2/4$ up to $d_c = \min(d_{\text{cla}}, d_{\text{cla}}^T)$, where $d_{\text{cla}}, d_{\text{cla}}^T$ are the distances of the classical code and its transpose code (by transposing the parity-check matrix).

Although quadratic check soundness is not sufficient for our purpose, here we prove that quantum expander codes actually have linear check soundness up to $d_c = \Theta(\sqrt{n})$. The idea is to exploit a closely related concept.

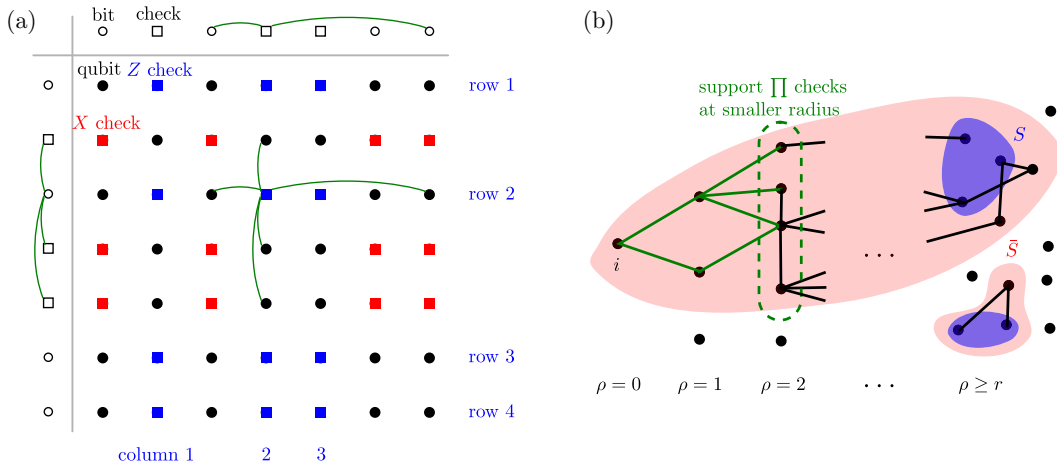


FIG. 3. (a) An illustration of the Tanner graph of a quantum hypergraph product code obtained from two classical codes. Linear check soundness of the quantum code can be obtained from expansion properties of the mother classical codes, as we show in Proposition B5. (b) A sketch of obtaining local indistinguishability Eq. (12) from check soundness. A stabilizer in S (which can be disconnected as shown by the shaded blue regions) is minimally expanded into product of checks in $\tilde{S}$ (shaded red regions), where the checks are shown by edges connecting the qubits (black dots). Here we plot each check as acting on two qubits for simplicity. For $\tilde{S}$ to contain i that is distance r away from S , the checks need to “push” its support outwards to larger radius while constrained the check soundness condition. This results in a r -determined lower bound on $|S|$.

Definition B3 (Check expansion). Let $\eta > 0$. We say a stabilizer code has (d'_c, η) -linear-check-expansion, if for any set of $M' < d'_c$ parity checks $Q^{(1)}, \dots, Q^{(M')} \in \Lambda_c$, their product $Q^{(1)} \dots Q^{(M')}$, as a stabilizer, acts on at least $\eta M'$ sites.

This definition can be easily generalized beyond linear expansion functions like that for soundness; nevertheless this definition for linear case suffices for our purposes. Check expansion for classical codes can be defined in exactly the same way. Notice that check expansion is weaker than check soundness for nonredundant codes: in that check expansion allows in principle for a product of $> d'_c$ checks to act on a small number of qubits. For redundant codes, soundness allows for the product of many checks to be a small stabilizer as long as the product is not the smallest possible representation of the small stabilizer.

Proposition B5. Consider a CSS code on n physical qubits that is the HGP of a classical code and itself, where the classical code has n_{cla} physical bits and $m_{\text{cla}} = \zeta n_{\text{cla}}$ checks. Suppose the classical code, together with its transpose code, have linear distance $d_{\text{cla}}, d_{\text{cla}}^T \geq \xi n_{\text{cla}}$, and has $(\chi n_{\text{cla}}, \eta)$ -linear-check-expansion. Then the HGP quantum code has $(\chi' \sqrt{n}, f)$ -check-soundness with $f(M) = 6M/\eta$, and

$$\chi' = \frac{1}{\sqrt{1 + \zeta^2}} \min(\sqrt{2}\chi, \xi). \quad (\text{B7})$$

This establishes linear check soundness for quantum expander codes with $d_c = \Theta(\sqrt{n})$, so long as its mother classical expander code satisfies all the conditions in Proposition B5 with ζ, ξ, χ, η being constants. More precisely, the linear-check-expansion property comes from the unique neighbor expansion property for a random bipartite graph of degree (Δ_B, Δ_C) , where $\Delta_B, \Delta_C > 2$; see e.g., Theorem A.6 in Ref. [25], or Ref. [67], for details.

Proof of Proposition B5. We first show that the soundness result of Proposition B5 follows if the HGP code has $((\chi')^2 n/4, \eta/6)$ -linear-check-expansion: any stabilizer of weight

$$M < \chi' \sqrt{n} \leq \xi n_{\text{cla}} \leq \min(d_{\text{cla}}, d_{\text{cla}}^T), \quad (\text{B8})$$

is expandable to $M' \leq M^2/4 < (\chi')^2 n/4$ checks due to Proposition B4. Here in Eq. (B8) we have used Eq. (B7), and

$$n = n_{\text{cla}}^2 + m_{\text{cla}}^2 \leq (1 + \zeta^2) n_{\text{cla}}^2, \quad (\text{B9})$$

from Definition B2 of HGP. Because the number of checks M' is bounded by the linear-check-expansion cutoff, we

can apply the check-expansion property to these M' checks and get $M \geq \eta M'/6$, so that $M' \leq 6M/\eta$.

We then use the check expansion of the classical code to prove the linear-check-expansion property of the HGP code. We first focus on Z checks, and aim to show that the Z checks, as checks of a classical code, have (χ_0, η_0) -linear-check-expansion with

$$(\chi_0, \eta_0) = (\chi' 2n/4, \eta/3). \quad (\text{B10})$$

The same property will hold for X checks as well because of the $X \leftrightarrow Z$ symmetry of the construction. Since the Z and X checks are independent, this will lead to the desired $((\chi')^2 n/4, \eta/6)$ -linear-check-expansion for the quantum HGP code, analogously to Proposition B1.

Observe that the (Z) checks are organized in rows and columns, as shown in Fig. 3(a). Consider a given set of $M < (\chi')^2 n/4$ (Z) checks $\mathbf{Q} \subset \Lambda_c$ that multiply to stabilizer $\mathcal{O}$. For each row labeled by $\mathbf{r}$, if it contains

$$M_{\mathbf{r}} < \chi n_{\text{cla}} \quad (\text{B11})$$

checks in $\mathbf{Q}$, then by the check expansion of the classical code, there are at least $\eta M_{\mathbf{r}}$ qubits in this row that $\mathcal{O}$ acts nontrivially on. Here we have used the fact that qubits in this row are not acted on by (Z) checks in other rows. We say row $\mathbf{r}$ is unfull (and full otherwise) if Eq. (B11) is satisfied (violated). The same holds for each column $\mathbf{c}$:

$$M_{\mathbf{c}} < \chi n_{\text{cla}}, \Rightarrow \mathcal{O} \text{ acts on } \geq \eta M_{\mathbf{c}} \text{ qubits in column } \mathbf{c}. \quad (\text{B12})$$

If the set of all unfull rows contains a sufficient portion $\geq 1/3$ of the “weight” M of the checks in $\mathbf{Q}$:

$$\sum_{\mathbf{r} \text{ unfull}} M_{\mathbf{r}} \geq \frac{M}{3}, \quad (\text{B13})$$

then the stabilizer $\mathcal{O}$ acts on at least $\eta M/3$ qubits from the classical check-expansion property for these rows, so that the desired check-expansion Eq. (B10) is satisfied for this particular set of M checks.

Otherwise if Eq. (B13) does not hold, the set of *full* rows must contain a sufficient portion of the check weight:

$$\sum_{\mathbf{r} \text{ full}} M_{\mathbf{r}} \geq \frac{2M}{3}. \quad (\text{B14})$$

For this case, we aim to show that unfull columns contain sufficient check weight:

$$\sum_{\mathbf{c} \text{ unfull}} M_{\mathbf{c}} \geq \frac{M}{3}, \quad (\text{B15})$$

so that the stabilizer still acts on at least $\eta M/3$ qubits from the argument above with rows and columns switched. To

this end, we separate the checks in $\mathbf{Q} = \mathbf{Q}_{\text{full}} \cup \mathbf{Q}_{\text{unfull}}$ into those contained in the full rows, and others in the unfull rows, with

$$|\mathbf{Q}_{\text{unfull}}| < \frac{M}{3}, \quad (\text{B16})$$

from Eq. (B14). The number of full rows is bounded by

$$|\{r : r \text{ full}\}| \leq \frac{M}{\chi n_{\text{cla}}} < \frac{(\chi')^2 n}{4\chi n_{\text{cla}}} \leq \frac{1}{2}\chi n_{\text{cla}}, \quad (\text{B17})$$

using Eqs. (B7) and (B9).

If there are no other checks $|\mathbf{Q}_{\text{unfull}}| = 0$, then because of Eqs. (B17) and (B12), all columns are unfull, which implies that Eq. (B15) holds.

We then think of adding back the checks in $\mathbf{Q}_{\text{unfull}}$ adversarially to make some of the columns full. Equation (B17) implies that each time a column is made full, it consists of $\geq \chi n_{\text{cla}}/2$ checks in $\mathbf{Q}_{\text{unfull}}$ and $< \chi n_{\text{cla}}/2$ checks in $\mathbf{Q}_{\text{full}}$. In other words, by costing x checks in $\mathbf{Q}_{\text{unfull}}$, it only decreases the quantity in Eq. (B15) by at most x . As a result, costing all checks in $\mathbf{Q}_{\text{unfull}}$ only decreases the quantity in Eq. (B15) by an amount bounded by Eq. (B16). Therefore,

$$\sum_{c \text{ unfull}} M_c \geq |\mathbf{Q}_{\text{full}}| - |\mathbf{Q}_{\text{unfull}}| \geq M/3, \quad (\text{B18})$$

from Eq. (B16), which proves Eq. (B15). Recall that this establishes linear-check-expansion Eq. (B10) for the Z and X checks separately, which combines to $((\chi')^2 n/4, \eta/6)$ -linear-check-expansion for the whole quantum code. Proposition B5 then holds by the argument at the beginning of the proof, which boosts the quadratic check soundness of Proposition B4 to linear. ■

4. Local indistinguishability from check soundness

We now address the main concern of this section—showing that check soundness provides strong constraints on locality for stabilizer operators. We will see that this technical issue is crucial in the proof of our main theorem.

Let $S \subset \Lambda$ be a set of distance $r = d(i, S)$ away from a given site $i \in \Lambda$. Suppose there exists a stabilizer $\mathcal{O}$ in S such that its minimal expansion contains a check acting on i . Given check soundness and $|S| \leq d_c$, how large should $|S|$ be for this to happen?

First of all, $|S|$ cannot be a constant if r is large. Any stabilizer in S can be minimally expanded to at most $f^{-1}(|S|)$ number of checks that are all supported in a region $\tilde{S}$. The extra support $\tilde{S} \setminus S$ has to be connected to S , because the expansion is not minimal otherwise. Therefore, in order for

$\tilde{S}$ to contain i as desired, we need it to connect S and i such that

$$r \leq |\tilde{S}| \leq \Delta \cdot f(|S|), \quad (\text{B19})$$

which gives a lower bound on $|S|$ by inverting function f . For linear check soundness, $|S| = \Omega(r)$ for example. Here the second inequality in Eq. (B19) is because the $\leq f(|S|)$ checks in the minimal expansion acts on at most $\Delta f(|S|)$ sites.

The above argument can be “bootstrapped” to yield a much stronger bound than Eq. (B19), for which we sketch the idea in Fig. 3(b). Suppose the minimal expansion of stabilizer $\mathcal{O}$ has the form

$$\mathcal{O} = \prod_{\rho=1}^{r+1} \mathcal{O}_{\rho}, \quad (\text{B20})$$

where $\mathcal{O}_{\rho}$ for $\rho \leq r$ is product of $g(\rho)$ checks that are contained in ball $B_{i,\rho}$ but not $B_{i,\rho-1}$, and $\mathcal{O}_{r+1}$ is product of the remaining checks not contained in $B_{i,r}$.

If $g(1) \geq 1$ checks act on i , then the product of these checks $\mathcal{O}_1$ is supported in $B_{i,1} \setminus B_{i,0}$, on at least $f^{-1}(g(1))$ sites. The reason is that these checks is a minimal expansion of $\mathcal{O}_1$. Otherwise Eq. (B20) is not a minimal expansion for $\mathcal{O}$. The weight of the stabilizer $\mathcal{O}_1$ is then no smaller than $f^{-1}(g(1))$ due to check soundness, if $f^{-1}(g(1)) < d_c$. As a consequence, $\Delta \cdot g(2) \geq f^{-1}(g(1))$ because the $g(2)$ checks in the next layer has to move the support of the product outward (i.e., stabilizer $\mathcal{O}_1 \mathcal{O}_2$ is supported in $B_{i,2} \setminus B_{i,1}$), while each check acts on at most Δ sites.

The above argument holds for any layer, so we have iteration relations

$$\begin{aligned} \Delta \cdot g(\rho + 1) &\geq f^{-1}(g(1) + \dots + g(\rho)), \\ \text{if } f^{-1}(g(1) + \dots + g(\rho)) &< d_c. \end{aligned} \quad (\text{B21})$$

This is solved by $g(1) + \dots + g(\rho) \geq \tilde{g}(\rho)$, where the increasing function $\tilde{g}(\rho)$ is determined by the following iteration relation:

$$\begin{aligned} \tilde{g}(\rho + 1) &= \tilde{g}(\rho) + \min[d_c, f^{-1}(\tilde{g}(\rho))] / \Delta, \\ \text{where } \tilde{g}(1) &= 1. \end{aligned} \quad (\text{B22})$$

Here the minimum function comes from the fact that, if $f^{-1}(\tilde{g}(\rho)) \geq d_c$, the next layer needs at least d_c support to satisfy check soundness.

Since S is minimally expanded with $|S| < d_c$, check soundness guarantees that

$$f(|S|) \geq g(1) + \dots + g(r) \geq \tilde{g}(r), \quad \Rightarrow \quad |S| \geq \tilde{f}(r), \quad (\text{B23})$$

where the increasing function $\tilde{f}(r) = f^{-1}(\tilde{g}(r))$ is a function determined by $\tilde{g}, f$, and thus determined by f using the following iteration relation obtained from Eq. (B22):

$$\tilde{f}(r+1) := f^{-1} \{ f(\tilde{f}(r)) + \min[d_c, \tilde{f}(r)] / \Delta \},$$

where $\tilde{f}(1) = f^{-1}(1)$. (B24)

We summarize the above by the following lemma.

Lemma B1. For a set S , if $|S| \leq \min(d, d_c)$ and $|S| < \tilde{f}(r)$ for some r with function $\tilde{f}$ determined by f in Eq. (B24), then any stabilizer in S can be expanded into product of checks supported in region $\tilde{S} = \{i \in \Lambda : d(i, S) \leq r\}$. Furthermore, the enlarged region $\tilde{S}$ has local indistinguishability in the sense that Eq. (12) holds for any $\mathcal{O}$ in S .

Proof. We have shown the stabilizer property above. There we focused on a given site i , but we have actually shown that the minimal expansion of any stabilizer in S cannot act on any site i that is distance r away, given $|S| < d_c$ and $|S| < \tilde{f}(r)$.

The lemma then follows from this stabilizer property, using an argument closely following Lemma 1 in Ref. [4]. Since Eq. (12) is linear in $\mathcal{O}$, it suffices to investigate the three classes of Pauli strings:

- (1) S does not contain logical operators $\mathcal{O}$ due to $|S| < d$.
- (2) Any error operator $\mathcal{O}$ in S anticommutes with at least one check in S and thus has $P_S \mathcal{O} P_S = 0$.
- (3) Any stabilizer $\mathcal{O}$ in S is expanded into checks contained in $\tilde{S}$, so acts as identity in the subspace $P_{\tilde{S}}$.

A general operator $\mathcal{O}$ in S thus satisfies Eq. (12) by expanding to Paulis. Note that the number $c_{\mathcal{O}}$ is only contributed by stabilizers, which have syndrome $\mathbf{s} = \emptyset$. ■

If f grows slowly, function $\tilde{f}(r)$ will increase with r fast enough. We will need this property in the stability proof to fight against the possible fast expansion of the graph. This is given precisely by the following lemma.

Lemma B2. If either of the following holds:

- (1) f grows slower than quadratic as Eq. (B5) for $\beta > 0$.

- (2) H_0 has finite dimension $\mathcal{D}$ and f grows slower than a polynomial of bounded degree: Eq. (B5) holds for a possibly negative constant β .

Then there exist constants $c'_f, c''_f, \alpha > 0$ determined by Δ, c_f, β (and $\mathcal{D}, c_{\mathcal{D}}$) such that

$$\sum_{r \geq 0: \tilde{f}(r) < d_c} \gamma(r) e^{-\delta \kappa \tilde{f}(r)} \leq c''_f e^{c'_f (\delta \kappa)^{-\alpha}}, \quad \forall \delta \kappa > 0, \quad (\text{B25})$$

where $\tilde{f}$ is the function determined by f in Eq. (B24).

Here $\delta \kappa$ is a single variable that will appear in the context of the proof of Theorem 1. In a nutshell, as we do Schrieffer-Wolff transformations the operators in the Hamiltonian become more and more non-local at higher orders. To obtain bounds of the form $\|[A, B]\|_{\kappa - \delta \kappa} \lesssim \|A\|_{\kappa} \|B\|_{\kappa}$ requires reducing the value of κ by a small amount at each order in perturbation theory: see Lemma C3.

Proof. We assume $\beta < 1$ without loss of generality, because the linear check soundness $\beta = 1$ case also satisfies Eq. (B5) with $\beta = 0.9$, for example.

Plugging Eq. (B5) into the iteration for $\tilde{g}(r)$ in Eq. (B22), we have

$$\tilde{g}(r+1) - \tilde{g}(r) \geq \frac{1}{\Delta} \left(\frac{\tilde{g}(r)}{c_f} \right)^{\frac{1}{2-\beta}}. \quad (\text{B26})$$

Here we have focused on regime $r < r_c$ where r_c is the smallest integer r such that $f^{-1}(\tilde{g}(r)) = \tilde{f}(r) \geq d_c$. This is the only regime that contributes to the sum Eq. (B25) that we want to bound, and we have taken $f^{-1}(\tilde{g}(r))$ in Eq. (B22) to be the minimum to get Eq. (B26). Equation (B26) is solved by ansatz $\tilde{g}(r) = c_g r^{(2-\beta)/(1-\beta)}$, because its left-hand side becomes

$$\begin{aligned} \tilde{g}(r+1) - \tilde{g}(r) &\geq c_g \left. \frac{d\tilde{g}}{dr} \right|_r = \frac{2-\beta}{1-\beta} c_g r^{\frac{1}{1-\beta}} \\ &= \frac{2-\beta}{1-\beta} c_g^{1-\frac{1}{2-\beta}} [\tilde{g}(r)]^{\frac{1}{2-\beta}}, \end{aligned} \quad (\text{B27})$$

(the inequality is because the function $d\tilde{g}/dr$ is monotonically increasing), which equals the right-hand side by choosing c_g to be given by $((2-\beta)/(1-\beta)) c_g^{1-1/(2-\beta)} = c_f^{-1/(2-\beta)}/\Delta$. As a result,

$$\tilde{f}(r) = f^{-1}(\tilde{g}(r)) \geq \tilde{c}_f r^{\frac{1}{1-\beta}}, \quad (\text{B28})$$

for some constant $\tilde{c}_f$ determined by Δ, c_f , and β .

We first deal with the general graph case. We use Eqs. (A1) and (B28) to get

$$\begin{aligned}
 & \sum_{r \geq 0: \tilde{f}(r) < d_c} \gamma(r) e^{-\delta \kappa \tilde{f}(r)} \\
 & \leq \sum_{r \geq 0} \Delta^r e^{-\tilde{c}_f \delta \kappa r^{\frac{1}{1-\beta}}} = \sum_{r \geq 0} e^{r \log \Delta - \tilde{c}_f \delta \kappa r^{\frac{1}{1-\beta}}} \\
 & \leq \left(2r_{\max} + \frac{1}{\tilde{c}_f} \right) \max_{r \geq 0} e^{r \log \Delta - \tilde{c}_f \delta \kappa r^{\frac{1}{1-\beta}}} \\
 & = \left(2r_{\max} + \frac{1}{\tilde{c}_f} \right) e^{\beta r_{\max} \log \Delta} \\
 & \leq c_f'' e^{2\beta r_{\max} \log \Delta} \\
 & = c_f'' \exp \left[\beta (\log \Delta) \left(\frac{(1-\beta) \log \Delta}{\tilde{c}_f \delta \kappa} \right)^{\frac{1-\beta}{\beta}} \right], \quad (\text{B29})
 \end{aligned}$$

Here to get the second line, we used the fact that function $e^{r \log \Delta - \tilde{c}_f \delta \kappa r^{\frac{1}{1-\beta}}}$ decays faster than an exponential $\propto (1 - \tilde{c}_f')^r$ for $r > 2r_{\max}$, where $r_{\max} = ((1-\beta) \log \Delta / \tilde{c}_f \delta \kappa)^{(1-\beta)/\beta}$ maximizes the function, and constant $\tilde{c}_f'$ does not depend on $\delta \kappa$. As a result, the sum over r is upper bounded by the maximum times $2r_{\max} + 1/\tilde{c}_f'$, where the second term accounts for the exponentially decaying contribution $\propto 1 + (1 - \tilde{c}_f') + (1 - \tilde{c}_f')^2 + \dots = 1/\tilde{c}_f'$ from $r > 2r_{\max}$. To get the last line, we used $2r_{\max} \leq (c_f'' - 1/\tilde{c}_f') e^{\beta r_{\max} \log \Delta}$ ($\forall r_{\max} \geq 0$) for a constant c_f'' . Equation (B29) is then established by Eq. (B25) for $\alpha = (1-\beta)/\beta$ and $c_f' = 2\beta (\log \Delta) ((1-\beta) \log \Delta / \tilde{c}_f)^{(1-\beta)/\beta}$.

We then treat finite dimensions where β can be negative. Equations (A2) and (B28) lead to

$$\begin{aligned}
 & \sum_{r \geq 0: \tilde{f}(r) < d_c} \gamma(r) e^{-\delta \kappa \tilde{f}(r)} \leq \sum_{r \geq 0} c_{\mathcal{D}} r^{\mathcal{D}-1} e^{-\tilde{c}_f \delta \kappa r^{\frac{1}{1-\beta}}} \\
 & \leq c_{\mathcal{D}} \left(2r_{\max} + \frac{1}{\tilde{c}_f} \right) \max_{r \geq 0} r^{\mathcal{D}-1} e^{-\tilde{c}_f \delta \kappa r^{\frac{1}{1-\beta}}} \\
 & \leq c_{\mathcal{D}} \left(2r_{\max} + \frac{1}{\tilde{c}_f} \right) r_{\max}^{\mathcal{D}-1} \\
 & \leq c_f'' e^{r_{\max}} = c_f'' \exp \left[\left(\frac{(1-\beta)(\mathcal{D}-1)}{\tilde{c}_f \delta \kappa} \right)^{1-\beta} \right]. \quad (\text{B30})
 \end{aligned}$$

Here the function $r^{\mathcal{D}-1} e^{-\tilde{c}_f \delta \kappa r^{\frac{1}{1-\beta}}}$ is analyzed in the same way as Eq. (B29), which decays exponentially (with

exponent $\tilde{c}_f'$ being a constant) for $r > 2r_{\max}$ with $r_{\max} = ((1-\beta)(\mathcal{D}-1)/\tilde{c}_f \delta \kappa)^{1-\beta}$. We then bound the power function of $r_{\max}$ by an exponential with a constant prefactor c_f'' . Equation (B25) is then also satisfied for $\alpha = 1-\beta$ and $c_f' = ((1-\beta)(\mathcal{D}-1)/\tilde{c}_f)^{1-\beta}$. ■

APPENDIX C: STABILITY OF QUANTUM CODES

We are now ready to state our main theorem.

Theorem 1 (Formal version). Let H_0 be a stabilizer code Hamiltonian of the form Eq. (A5) that has (i) LDPC property with bounded degree Δ , (ii) a gap ≥ 1 from Eq. (A6), (iii) code parameters $[[n, k, d]]$, and (iv) (d_c, f) -check-soundness with

$$d_s := \min(d, d_c) \geq c_d \log n, \quad (\text{C1})$$

where $c_d > 0$ is some fixed constant independent of n , and f grows slowly such that Eq. (B25) holds for constants $c_f', c_f'', \alpha > 0$. Let κ_1 be a constant satisfying

$$\kappa_1 > 2/c_d, \quad (\text{C2})$$

There exist positive constants $\epsilon_0, c_1, c_2, c_3$ determined by $\Delta, \kappa_1, c_f', c_f'', \alpha$, such that if $n \geq c_3$, for any perturbation $V = \sum_{S,s} V_{S,s}$ with

$$\|V\|_{\kappa_1} =: \epsilon \leq \min \left(\epsilon_0, \frac{1}{3c_1} \right), \quad (\text{C3})$$

the spectrum of $H := H_0 + V$, up to an overall energy shift, is contained in two disjoint intervals

$$\begin{aligned}
 & [-\epsilon_*, \epsilon_*] \cup [1 - c_1 \epsilon - \epsilon_*, \infty) \\
 & \subset [-\epsilon_*, \epsilon_*] \cup \left[\frac{1}{2} + \epsilon_*, \infty \right), \quad (\text{C4})
 \end{aligned}$$

where there are 2^k eigenstates in the first interval, and

$$\epsilon_* = c_2 n \epsilon e^{-\kappa_1 d_s/2}, \quad (\text{C5})$$

decays to zero as $n \rightarrow \infty$ due to Eqs. (C1) and (C2). In other words, there remains a gap $\geq 1/2$ separating the 2^k ground states from the rest of the spectrum, and the ground states are almost degenerate with small energy difference $\leq$

$2\epsilon_*$. Moreover, the projector P_{new} to the perturbed ground-state subspace is related to the unperturbed P by

$$\|P_{\text{new}} - UPU^\dagger\| \leq 4\sqrt{\epsilon_*}, \quad (\text{C6})$$

where

$$U = \mathcal{T} \exp \left[\int_0^1 dt A(t) \right], \quad (\text{C7})$$

($\mathcal{T}$ means time ordering) is finite-time evolution generated by a time-dependent anti-Hermitian $A(t)$ with local norm

$$\max_t \|A(t)\|_{\kappa_1/2} \leq 2\epsilon. \quad (\text{C8})$$

Our claim that known instances of good quantum LDPC codes, together with the quantum HGP expander codes, form robust phases of matter with finite entropy density at zero temperature, immediately follows from combining Theorem 1 with Lemma B2, along with the results catalogued in Appendix B3.

1. Schrieffer-Wolff transformations

We have reviewed the Schrieffer-Wolff transformation (SWT) scheme for the proof in the main text; here we introduce some notations and explicit calculations on performing one SWT step, which will be useful in the proof.

For any operator $V = \sum_{S,s} V_{S,s}$, define $\mathbb{P}V := \sum_{S,s} (\mathbb{P}V)_{S,s}$ where

$$(\mathbb{P}V)_{S,s} := P_S V_{S,s} P_S + Q_S V_{S,s} Q_S. \quad (\text{C9})$$

$(\mathbb{P}V)_{S,s}$ indeed has syndrome $\mathbf{s}$ because P_S, Q_S commutes with all checks. As a result, since $V_{S,s}$ is strongly supported in S , $(\mathbb{P}V)_{S,s}$ is also strongly supported in S . We will also write $(\mathbb{P}V)_{S,s}$ as $\mathbb{P}V_{S,s}$, although strictly speaking, $\mathbb{P}$ is not a global superoperator that maps operators to operators, because it treats different terms with support S in a different way. $\mathbb{P}V$ is locally block diagonal in the following sense.

Definition C1. We say D is locally block diagonal, if its decomposition $D = \sum_{S,s} D_{S,s}$ satisfies

$$[D_{S,s}, P_S] = 0, \quad \forall S, \mathbf{s}. \quad (\text{C10})$$

On the other hand, $V = \mathbb{P}V + \mathbb{P}^\perp V$ where each local term

$$(\mathbb{P}^\perp V)_{S,s} := P_S V_{S,s} Q_S + Q_S V_{S,s} P_S, \quad (\text{C11})$$

is block-off-diagonal. Both $\mathbb{P}$ and $\mathbb{P}^\perp$ cannot increase operator norm:

$$\begin{aligned} \|\mathbb{P}V_{S,s}\|, \|\mathbb{P}^\perp V_{S,s}\| &\leq \|V_{S,s}\|, \\ \Rightarrow \|\mathbb{P}V\|_\kappa, \|\mathbb{P}^\perp V\|_\kappa &\leq \|V\|_\kappa, \end{aligned} \quad (\text{C12})$$

because, for example,

$$\begin{aligned} \|\mathbb{P}V_{S,s}\| &\leq \max(\|P_S V_{S,s} P_S\|, \|Q_S V_{S,s} Q_S\|) \\ &\leq \max(\|V_{S,s}\|, \|V_{S,s}\|) = \|V_{S,s}\|, \end{aligned} \quad (\text{C13})$$

where the maximization comes from the orthogonality of the subspaces that P_S and Q_S map to.

The following equation

$$[H_0, A] + V = \mathbb{P}V, \quad (\text{C14})$$

is solvable by anti-Hermitian $A = \sum_{S,s} A_{S,s}$ with

$$A_{S,s} = P_S V_{S,s} Q_S H_S^{-1} - H_S^{-1} Q_S V_{S,s} P_S, \quad (\text{C15})$$

as one can verify explicitly. Note that the (pseudo)inverse of H_S above is well defined because it is always multiplied by Q_S that projects out the kernel.

2. Proof of Theorem 1

a. Overview

As already sketched in the main text, our ultimate goal is to construct a unitary Eq. (C7) that approximately block diagonalizes (in terms of the two subspaces P and $I - P$) the Hamiltonian:

$$U^\dagger H U = H_0 + D_* + V_* + c_* I, \quad (\text{C16})$$

where c_* is a number, and we desire the following properties.

- (1) The “remaining perturbation” V_* is (nearly) exponentially small in distance:

$$\|V_*\| \leq \epsilon_*, \quad (\text{C17})$$

with ϵ_* given by Eq. (C5).

- (2) D_* is relatively bounded by H_0 :

$$\|D_* |\psi\rangle\| \leq c_1 \epsilon \|H_0 |\psi\rangle\|, \quad (\text{C18})$$

for some constant c_1 determined by Δ, κ_1 . Equation (C18) is equivalent to the operator inequality $D_*^2 \leq (c_1 \epsilon)^2 H_0^2$, and implies D_* is block diagonal because $D_* |\text{GS}\rangle = 0$ for any ground state $|\text{GS}\rangle$ of H_0 .

- (3) The generator $A(t)$ of U satisfies the locality bound Eq. (C8).

We aim to show these properties, because the theorem follows by the following lemma that we will prove in Appendix C3:

Lemma C1. If there exists a unitary U such that Eq. (C16) holds with the three properties above satisfied, then Theorem 1 holds.

Our proof then involves three parts:

- (1) We first construct U by a perturbation series of iterated SWTs, truncated at a high order $m_* \sim d_s$.
- (2) We show that the perturbation series converges by obtaining strong locality bounds (Lemma C4) on the operators involved in the SWTs.
- (3) Finally, we use the locality bounds to verify the above properties one by one, so that the proof concludes by invoking Lemma C1.

b. Setting up the Schrieffer-Wolff transformation

Recall the strategy of SWTs introduced in Sec. IV. Suppose at the m th step, we have constructed a unitary U_{m-1} such that the Hamiltonian is rotated to

$$U_{m-1}^\dagger H U_{m-1} = H_0 + D_m + V_m + E_m. \quad (\text{C19})$$

Here D_m is the locally block-diagonal effective Hamiltonian at this order that preserves subspace P , and V_m is the remaining perturbation that is roughly suppressed to order ϵ^m . Both of them contain only terms with support smaller than the code distance:

$$D_m = \sum_{S:|S|<d_s} D_{m,S}, \quad V_m = \sum_{S:|S|<d_s} V_{m,S}. \quad (\text{C20})$$

All terms of larger support are contained in the “garbage” E_m .

For example, $m = 1$ corresponds to the original Hamiltonian where $D_1 = 0$, $U_0 = I$, and V_1, E are the terms of V with small (large) supports. At the final step $m = m_*$ with m_* to be determined, Eq. (C19) becomes Eq. (C16) where

$$U = U_{m_*-1}, \quad D_* = D_{m_*} - c_* I, \quad V_* = V_{m_*} + E_{m_*}, \quad (\text{C21})$$

with a suitable number c_* .

To proceed to the next $(m+1)$ th order, we let $V = V_m$ in Eq. (C14) and let the corresponding A be A_m . With

$$U_m = U_{m-1} e^{A_m}, \quad (\text{C22})$$

we have

$$\begin{aligned} e^{-A_m} (H_0 + D_m + V_m + E_m) \\ = H_0 + D_{m+1} + V_{m+1} + E_{m+1}, \quad \text{where } A_m = [A_m, \cdot]. \end{aligned} \quad (\text{C23})$$

Here

$$D_{m+1} := D_m + \mathbb{P} V_m, \quad (\text{C24})$$

and V_{m+1}, E_{m+1} are defined as follows: first define

$$\begin{aligned} V'_{m+1} &= e^{-A_m} (H_0 + D_m + V_m) - H_0 - (D_m + \mathbb{P} V_m) \\ &= (e^{-A_m} - 1) H_0 + \mathbb{P}^\perp V_m + (e^{-A_m} - 1) (D_m + V_m) \\ &= - \int_0^1 ds (e^{-sA_m} - 1) \mathbb{P}^\perp V_m \\ &\quad + (e^{-A_m} - 1) (D_m + V_m), \\ &= [D_m, A_m] + (e^{-A_m} - 1) V_m \\ &\quad + \int_0^1 ds (e^{-sA_m} - 1) ([D_m, A_m] - \mathbb{P}^\perp V_m), \end{aligned} \quad (\text{C25})$$

with local decomposition $V'_{m+1,S}$. Then

$$V_{m+1} = \sum_{S:|S|<d_s} V'_{m+1,S}, \quad (\text{C26a})$$

$$E_{m+1} = e^{-A_m} E_m + \sum_{S:|S|\geq d_s} V'_{m+1,S}. \quad (\text{C26b})$$

Let us first estimate the orders. Assuming $D_m \sim \epsilon$ and $V_m \sim A_m \sim v_m \ll \epsilon$, the last two terms are subdominant because they are at least second order in v_m . It is thus crucial to bound the first term. Naively,

$$[D_m, A_m] \sim \epsilon v_m \times (\text{support volume of } A_m) \sim \epsilon v_m m, \quad (\text{C27})$$

which becomes bad already at a n -independent order $m \sim 1/\epsilon$; note that this is also the key problem for obtaining tight bounds on the lifetime of false vacuum decay [36]. One has to use the fact that code H_0 has large distance to control this term.

Actually, the naive estimate Eq. (C27) for the κ -norm of the whole term $[D_m, A_m]$ cannot be improved. Here is the simplest example: consider the 2D toric code where D_m can possibly be $\epsilon \sum_j X_j (I - Q_{C_j})$ where C_j is a check near j . Then consider a Z string $A_m = Z_1 Z_2 \cdots Z_L$ [or more precisely, the corresponding A for $V = Z_1 Z_2 \cdots Z_L$ in Eq. (C14)], where L is much smaller than the system size (code distance). Then $\|[D_m, A_m]\| \propto L$ because every X_j with $j \in \{1, \dots, L\}$ anticommutes with A_m . This is different to the false vacuum decay setting [36] where $\|[D_m, A_m]\|_\kappa$ can be bounded in a tighter way by detailed analysis.

The resolution for the above toric code example is that, the extensive $\propto L$ part of $[D_m, A_m]$ annihilates the ground states by the $I - Q_{C_j}$ factor, so it is already block diagonal. Only the contributions from the two ends of the string may not be block diagonal (where C_j anticommutes with A_m ; this is not possible in the middle of the string), which should be further rotated away in the next order of SWT.

c. Locality bounds

As suggested by the above estimates, we should keep track of two norms of the perturbation:

$$v_m := \|V_m\|_{\kappa_m}, \quad \tilde{v}_m := \|\mathbb{P}^\perp V_m\|_{\kappa_m} \leq v_m, \quad (\text{C28})$$

where v_m may be parametrically larger than $\tilde{v}_m$. κ_m will be defined shortly in Eq. (C34), which decays with m and controls the growth of operator support when going to higher order. From Eq. (C15) and the gap of H_0 , we obtain

$$\|A_m\|_{\kappa_m} \leq \tilde{v}_m, \quad (\text{C29})$$

bounded by the (eventually at large enough m) smaller (as we will prove) of the two norms. To bound the first term of Eq. (C25), we invoke the following lemma, proved in Appendix C 5:

Lemma C2. For any $\kappa > \kappa'$ with $\delta\kappa = \kappa - \kappa'$,

$$\|[D, A]\|_{\kappa'} \leq \frac{2}{\delta\kappa} \|D\|_{\kappa} \|A\|_{\kappa}. \quad (\text{C30})$$

Moreover, if $D = \sum_{S:|S|<d_s} D_S$ where each D_S is block diagonal, A is given by the solution Eq. (C15) of Eq. (C14) for a given V , and the soundness function f grows slowly such that Eq. (B25) holds, then

$$\begin{aligned} \|\mathbb{P}^\perp [D, A]\|_{\kappa'} &\leq \left(\frac{1}{\delta\kappa} + 2\Delta c_f'' e^{c_f'(\delta\kappa)^{-\alpha}} \right) \|D\|_{\kappa} \|\mathbb{P}^\perp V\|_{\kappa'} \\ &\leq \tilde{c}_f'' e^{c_f'(\delta\kappa)^{-\alpha}} \|D\|_{\kappa} \|\mathbb{P}^\perp V\|_{\kappa'}, \end{aligned} \quad (\text{C31})$$

where constant $\tilde{c}_f'' \geq 2$ is determined by $\Delta, c_f', c_f'', \alpha$.

Here Eq. (C30) is just a combinatorial result; the crucial bound is Eq. (C31) bounding the off-block-diagonal part, where κ' for $\mathbb{P}^\perp V$ indicates that the volume factor in Eq. (C27) is gone. Equation (C31) can be viewed as a generalization of Corollary 3 in Ref. [4], which comes from the fact that D is “relatively bounded” by H_0 . In comparison, our proof of Lemma C2 (presented later) is arguably simpler as we do not need the relative boundedness explicitly; we get rid of this by keeping track of the extra syndrome information in decompositions Eq. (A12). Nevertheless, we will need the relative boundedness for showing that the gap does not close for $H_0 + D_{m*}$, after doing SWTs.

To bound the other terms in Eq. (C25), we do not need the block-diagonal property of D and just invoke the following locality bound based on cluster expansions.

Lemma C3 (Lemma 23 in Ref. [36]). Consider two operators $A, \mathcal{O}$. If

$$\|A\|_{\kappa} \leq \frac{\delta\kappa}{3} := \frac{\kappa - \kappa'}{3}, \quad (\text{C32})$$

then

$$\begin{aligned} \|(e^{-A} - 1) \mathcal{O}\|_{\kappa'}, 2 \left\| \int_0^1 ds (e^{-sA} - 1) \mathcal{O} \right\|_{\kappa'} \\ \leq \frac{18}{\kappa' \delta\kappa} \|A\|_{\kappa} \|\mathcal{O}\|_{\kappa}, \end{aligned} \quad (\text{C33a})$$

$$\|e^{-A} \mathcal{O}\|_{\kappa'} \leq \left(1 + \frac{18}{\kappa' \delta\kappa} \|A\|_{\kappa} \right) \|\mathcal{O}\|_{\kappa}. \quad (\text{C33b})$$

See Lemma 4.1 in Ref. [53] for proof of the situation where the κ norm does not keep track of the syndrome information, and Ref. [36] for the generalization to the case with syndromes. Note that although Refs. [36, 53] focus on constant dimensional lattices where operators $\mathcal{O} = \sum_S \mathcal{O}_S$ are decomposed to only connected subsets S on the lattice, the above lemma therein holds generally for any interaction graph where S can be any subset. This can be checked through the proof in Ref. [53], and can be simply inferred from the fact that Lemma C3 does not involve constants like Δ about the underlying graph.

d. Obtaining the flow equations

We choose a slow decay of κ_m following [36, 53]

$$\kappa_m := \frac{\kappa_1}{2} \left(1 + \frac{1}{1 + \log m} \right) \geq \frac{\kappa_1}{2}, \quad (\text{C34})$$

which is consistent at $m = 1$, and leads to

$$\begin{aligned} \delta\kappa_m &:= \kappa_m - \kappa_{m+1} = \frac{\kappa_1 \log((m+1)/m)}{2(1 + \log m)[1 + \log(m+1)]} \\ &\geq \frac{\kappa_1}{6m(1 + \log^2 m)} \geq \frac{\kappa_1}{6m^2}, \end{aligned} \quad (\text{C35a})$$

$$\begin{aligned} \tilde{\delta\kappa}_m &:= \kappa_m - \kappa_{2m} = \frac{\kappa_1 \log 2}{2(1 + \log m)[1 + \log(2m)]} \\ &\geq \frac{\kappa_1}{5(1 + \log m)^2}, \end{aligned} \quad (\text{C35b})$$

for any $m \geq 1$. The 6 factor in Eq. (C35) comes from numerically plotting the function, and the factor 5 comes from $2(1 + \log 2)/\log 2 < 5$. We will frequently use the fact that $\delta\kappa_m \geq \delta\kappa_{m'}$ for $m \leq m'$.

In the following, we bound the locality of V'_{m+1} in Eq. (C25), and furthermore v_{m+1} and $\tilde{v}_{m+1}$, using the above two lemmas.

For the first term in Eq. (C25), we do not directly plug $D = D_m$ in Eq. (C31) due to the $\delta\kappa$ dependence: $\kappa = \kappa_m$

decays with m and will ultimately make the factors in front of $\|\mathbb{P}^\perp V\|_{\kappa'}$ larger than 1 at an order m independent of system size, where SWT would have to stop. Instead, we know from the iteration Eq. (C24) that

$$D_m = \mathbb{P}V_1 + \cdots + \mathbb{P}V_{m-1} = (\mathbb{P}V_1 + \cdots + \mathbb{P}V_{\lfloor m/2 \rfloor}) + (\mathbb{P}V_{\lfloor m/2 \rfloor + 1} + \cdots + \mathbb{P}V_{m-1}), \quad (\text{C36})$$

where each term is locally block diagonal, and controlled by a different κ . We have also separated D_m into two parts, where we want to use Eq. (C31) for the $m' \leq \lfloor m/2 \rfloor$ part, and Eq. (C30) for the other part. More precisely, we bound the block-off-diagonal part of the first term in Eq. (C25) by

$$\begin{aligned} & \|\mathbb{P}^\perp [D_m, A_m]\|_{\kappa_{m+1}} \\ & \leq \sum_{m'=1}^{m-1} \|\mathbb{P}^\perp [\mathbb{P}V_{m'}, A_m]\|_{\kappa_{m+1}} \\ & \leq \sum_{m'=1}^{\lfloor m/2 \rfloor} \tilde{c}_f'' e^{c_f'(\kappa_{m'} - \kappa_{m+1})^{-\alpha}} \|\mathbb{P}V_{m'}\|_{\kappa_{m'}} \|\mathbb{P}^\perp V_m\|_{\kappa_{m+1}} \\ & \quad + \sum_{m'=\lfloor m/2 \rfloor + 1}^{m-1} \frac{2}{\kappa_m - \kappa_{m+1}} \|\mathbb{P}V_{m'}\|_{\kappa_m} \|\mathbb{P}^\perp V_m\|_{\kappa_m} \\ & \leq \tilde{v}_m \left(\sum_{m'=1}^{\lfloor m/2 \rfloor} \tilde{c}_f'' e^{c_f'(\tilde{\delta}\kappa_{m'})^{-\alpha}} v_{m'} + \frac{2}{\delta\kappa_m} \sum_{m'=\lfloor m/2 \rfloor + 1}^{m-1} v_{m'} \right) \\ & = \tilde{v}_m \left(\mathfrak{d}_m + \frac{1}{\delta\kappa_m} \tilde{\mathfrak{d}}_m \right), \end{aligned} \quad (\text{C37})$$

where we introduce two effective local norms of D_m :

$$\mathfrak{d}_m := \sum_{m'=1}^{\lfloor m/2 \rfloor} \tilde{c}_f'' e^{c_f'(\tilde{\delta}\kappa_{m'})^{-\alpha}} v_{m'}, \quad \tilde{\mathfrak{d}}_m := 2 \sum_{m'=\lfloor m/2 \rfloor + 1}^{m-1} v_{m'}. \quad (\text{C38})$$

On the other hand, the whole first term in Eq. (C25) is bounded by Eq. (C30):

$$\begin{aligned} & \|[D_m, A_m]\|_{\kappa_{m+1}} \\ & \leq \frac{2}{\delta\kappa_m} \|D_m\|_{\kappa_m} \|A_m\|_{\kappa_m} \leq \frac{2}{\delta\kappa_m} \tilde{v}_m \sum_{m'=1}^{m-1} \|\mathbb{P}V_{m'}\|_{\kappa_m} \\ & \leq \frac{2}{\delta\kappa_m} \tilde{v}_m \sum_{m'=1}^{m-1} v_{m'} \leq \frac{1}{\delta\kappa_m} \tilde{v}_m (\mathfrak{d}_m + \tilde{\mathfrak{d}}_m), \end{aligned} \quad (\text{C39})$$

where we have used $\mathfrak{d}_m \geq 2 \sum_{m'=1}^{\lfloor m/2 \rfloor} v_{m'}$ from $\tilde{c}_f'' \geq 2$.

For now, assume the condition of Lemma C3,

$$\tilde{v}_m \leq \delta\kappa_m/3, \quad (\text{C40})$$

holds here (we will verify this condition indeed holds by induction later). Then Lemma C3 bounds the rest of the terms in Eq. (C25) that we denote by $V_{m+1}'' := V_{m+1}' - [D_m, A_m]$:

$$\begin{aligned} & \|\mathbb{P}^\perp V_{m+1}''\|_{\kappa_{m+1}} \leq \|V_{m+1}'\|_{\kappa_{m+1}} \\ & \leq \frac{9}{\kappa_{m+1}\delta\kappa_m} \|A_m\|_{\kappa_m} \\ & \quad \times \left(2 \|V_m\|_{\kappa_m} + 2 \|[D_m, A_m]\|_{\frac{\kappa_m + \kappa_{m+1}}{2}} + \|\mathbb{P}^\perp V_m\|_{\kappa_m} \right) \\ & \leq \frac{9}{\kappa_{m+1}\delta\kappa_m} \tilde{v}_m \left(3v_m + \frac{4}{\delta\kappa_m} \tilde{v}_m (\mathfrak{d}_m + \tilde{\mathfrak{d}}_m) \right). \end{aligned} \quad (\text{C41})$$

Here we have used Eqs. (C12), (C28) and that $\|[D_m, A_m]\|_{\frac{\kappa_m + \kappa_{m+1}}{2}}$ is bounded by Eq. (C39) with an extra factor of 2 due to the different κ -norm.

The perturbation at next order satisfies

$$\begin{aligned} v_{m+1} &= \|V_{m+1}\|_{\kappa_{m+1}} \leq \|V_{m+1}'\|_{\kappa_{m+1}}, \\ \tilde{v}_{m+1} &= \|\mathbb{P}^\perp V_{m+1}\|_{\kappa_{m+1}} \leq \|\mathbb{P}^\perp V_{m+1}'\|_{\kappa_{m+1}}, \end{aligned} \quad (\text{C42})$$

because V_{m+1} is just the support $< d_s$ terms of V_{m+1}' . Combining Eqs. (C25), (C37), (C39), (C41), and (C42), we get a set of iteration relations (so-called flow equations [4]) for variables $(v_m, \tilde{v}_m, \mathfrak{d}_m, \tilde{\mathfrak{d}}_m)$ with $m \geq 1$:

$$\begin{aligned} v_{m+1} &\leq \frac{1}{\delta\kappa_m} \mathfrak{d}_m \tilde{v}_m + \frac{1}{\delta\kappa_m} \tilde{\mathfrak{d}}_m \tilde{v}_m \\ &\quad + \frac{9}{\kappa_{m+1}\delta\kappa_m} \tilde{v}_m \left(3v_m + \frac{4}{\delta\kappa_m} \tilde{v}_m (\mathfrak{d}_m + \tilde{\mathfrak{d}}_m) \right), \end{aligned} \quad (\text{C43a})$$

$$\begin{aligned} \tilde{v}_{m+1} &\leq \mathfrak{d}_m \tilde{v}_m + \frac{1}{\delta\kappa_m} \tilde{\mathfrak{d}}_m \tilde{v}_m \\ &\quad + \frac{9}{\kappa_{m+1}\delta\kappa_m} \tilde{v}_m \left(3v_m + \frac{4}{\delta\kappa_m} \tilde{v}_m (\mathfrak{d}_m + \tilde{\mathfrak{d}}_m) \right), \end{aligned} \quad (\text{C43b})$$

$$\mathfrak{d}_{m+1} \leq \mathfrak{d}_m + \mathbb{I}[m \text{ is odd}] \tilde{c}_f'' e^{c_f'(\tilde{\delta}\kappa_{(m+1)/2})^{-\alpha}} v_{(m+1)/2}, \quad (\text{C43c})$$

$$\tilde{\mathfrak{d}}_m = 2 \sum_{m'=\lfloor m/2 \rfloor + 1}^{m-1} v_{m'}. \quad (\text{C43d})$$

Here $\mathbb{I}[m \text{ is odd}]$ is an indicator function that returns 0, 1 for m being even or odd. The initial values are

$$v_1, \tilde{v}_1 \leq \epsilon, \quad \mathfrak{d}_1 = \tilde{\mathfrak{d}}_1 = \tilde{\mathfrak{d}}_2 = 0. \quad (\text{C44})$$

e. The last SWT step

We analyze the iteration problem above by the following lemma that we prove in Appendix C 4.

Lemma C4. There exist constants $c_{\text{iter}}, \epsilon_0$ determined by Δ, κ_1 such that

$$c_{\text{iter}}\epsilon_0 \leq 1/4. \quad (\text{C45})$$

Furthermore, for any $\epsilon \leq \epsilon_0$, the iteration Eq. (C43) with κ_m defined in Eq. (C34) and initial value Eq. (C44) generates variables that satisfy

$$\begin{aligned} v_m &\leq \frac{\epsilon}{\delta\kappa_{m-1}} (c_{\text{iter}}\epsilon)^{m-1}, \quad \tilde{v}_m \leq \epsilon (c_{\text{iter}}\epsilon)^{m-1}, \\ \mathfrak{d}_m &\leq \frac{2}{3} c_{\text{iter}}\epsilon, \quad \tilde{\mathfrak{d}}_m \leq \frac{3\epsilon}{\delta\kappa_{m-2}} (c_{\text{iter}}\epsilon)^{\lfloor m/2 \rfloor} \end{aligned} \quad (\text{C46})$$

for any $m \geq 2$. The condition Eq. (C40) required to derive the flow equations is also satisfied for all $m \geq 1$.

This lemma indicates that the iteration can go to $m \rightarrow \infty$. However, we truncate at some order m_* since there is no need to proceed once the perturbation V_{m_*} is smaller than the garbage term E_{m_*} , which we cannot rotate away via SWT. We do not care about the local decomposition Eq. (A12) of E_m , and just bound its total operator norm. To this end, we first obtain

$$\begin{aligned} \left\| \sum_{S:|S|\geq d_s} V'_{m+1,S} \right\| &\leq \sum_{i \in \Lambda} \sum_{S \ni i: |S|\geq d_s} \sum_s \|V'_{m+1,S,s}\| \\ &\leq \sum_{i \in \Lambda} e^{-\kappa_{m+1}d_s} \sum_{S \ni i: |S|\geq d_s} \sum_s \|V'_{m+1,S,s}\| e^{\kappa_{m+1}|S|} \\ &\leq n e^{-\kappa_{m+1}d_s} \|V'_{m+1}\|_{\kappa_{m+1}} \leq n e^{-\kappa_{m+1}d_s} \cdot \frac{\epsilon}{\delta\kappa_m} (c_{\text{iter}}\epsilon)^m \\ &\leq n \frac{6\epsilon}{\kappa_1} m^2 (c_{\text{iter}}\epsilon)^m e^{-\kappa_1 d_s/2}, \end{aligned} \quad (\text{C47})$$

for $m \geq 1$, where we have used Eqs. (C34), (C35), and the bound Eq. (C46) on v_{m+1} that also applies to $\|V'_{m+1}\|_{\kappa_{m+1}}$. Similarly, $\|E_1\| \leq n e^{-\kappa_1 d_s} \epsilon$. Then Eq. (C26b) leads to

$$\begin{aligned} \|E_{m_*}\| &\leq \|E_1\| + \sum_{m=1}^{m_*-1} \left\| \sum_{S:|S|\geq d_s} V'_{m+1,S} \right\| \\ &\leq \max\left(1, \frac{6}{\kappa_1}\right) n \epsilon e^{-\kappa_1 d_s/2} \sum_{m=0}^{m_*-1} \max(1, m^2) (c_{\text{iter}}\epsilon)^m \\ &\leq \frac{c_2}{2} n \epsilon e^{-\kappa_1 d_s/2}, \quad \text{where } c_2 := 4 \max\left(1, \frac{6}{\kappa_1}\right), \end{aligned} \quad (\text{C48})$$

independent of m_* . Here we have used Eq. (C45) and $\sum_{m=0}^{\infty} \max(1, m^2) 4^{-m} < 2$. Since $\|V_{m_*}\| \leq n v_{m_*}$ with v_m

bounded in Eq. (C46), we stop the SWTs at step $m_* = \Theta(d_s)$ that is the smallest integer fulfilling the following:

$$\begin{aligned} \frac{6m_*^2}{\kappa_1} (c_{\text{iter}}\epsilon)^{m_*-1} &\leq e^{-\kappa_1 d_s/2}, \\ \Rightarrow \|V_{m_*}\| &\leq n \frac{\epsilon}{\delta\kappa_{m_*-1}} (c_{\text{iter}}\epsilon)^{m_*-1} \leq \frac{c_2}{2} n \epsilon e^{-\kappa_1 d_s/2}. \end{aligned} \quad (\text{C49})$$

The total garbage in the end is thus

$$\|V_{m_*} + E_{m_*}\| \leq c_2 n \epsilon e^{-\kappa_1 d_s/2}. \quad (\text{C50})$$

This establishes the first property Eq. (C17) we want with c_2 determined by κ_1 .

For the second property of relative boundedness, we follow Ref. [4] to show the following, with the proof delayed until Appendix C 6.

Proposition C1. Suppose the soundness function f grows slowly such that Eq. (B25) holds. If $D = \sum_{S:|S|\leq d_s} \sum_s D_{S,s}$ where each $D_{S,s}$ is block diagonal, then there exists a number c_D such that $D - c_D I$ is relatively bounded by H_0 : for any state $|\psi\rangle$,

$$\begin{aligned} \|(D - c_D I) |\psi\rangle\| &\leq \left(2 + \frac{\Delta}{\kappa}\right) \Delta c_f'' e^{c_f'(\kappa/2)^{-\alpha}} \|D\|_{\kappa} \|H_0 |\psi\rangle\|. \end{aligned} \quad (\text{C51})$$

Letting $D = D_{m_*}$ in Proposition C1, there exist constant c_* such that for any state $|\psi\rangle$,

$$\begin{aligned} \|D_* |\psi\rangle\| &\leq \left(2 + \frac{\Delta}{\kappa_1/2}\right) \Delta c_f'' e^{c_f'(\kappa_1/4)^{-\alpha}} \|D\|_{\kappa_1/2} \|H_0 |\psi\rangle\| \\ &\leq 2 \left(1 + \frac{\Delta}{\kappa_1}\right) \Delta c_f'' e^{c_f'(\kappa_1/4)^{-\alpha}} \|H_0 |\psi\rangle\| \sum_{m=1}^{m_*-1} \|\mathbb{P} V_m\|_{\kappa_1/2} \\ &\leq 2 \left(1 + \frac{\Delta}{\kappa_1}\right) \Delta c_f'' e^{c_f'(\kappa_1/4)^{-\alpha}} \|H_0 |\psi\rangle\| \sum_{m=1}^{m_*-1} v_m \\ &\leq 2 \left(1 + \frac{\Delta}{\kappa_1}\right) \Delta e^{c_f'(\kappa_1/4)^{-\alpha}} \|H_0 |\psi\rangle\| (\mathfrak{d}_{m_*} + c_f'' \tilde{\mathfrak{d}}_{m_*}/2) \\ &\leq c_1 \epsilon \|H_0 |\psi\rangle\|, \end{aligned} \quad (\text{C52})$$

where D_* is defined in Eq. (C21). Here we have used Eq. (C36) in the second line, Eq. (C12) and $\kappa_m \geq \kappa_1/2$ in the third line, and Eq. (C38) in the fourth line. The last line

comes from Eqs. (C46) and (C45) with constant

$$c_1 = 2 \left(1 + \frac{\Delta}{\kappa_1} \right) \Delta e^{c'_f (\kappa_1/4)^{-\alpha}} \times \left(\frac{2}{3} c_{\text{iter}} + c''_f \max_{m \geq 3} \frac{3}{\delta \kappa_{m-2}} 4^{-\lfloor m/2 \rfloor} \right), \quad (\text{C53})$$

where the maximum over m is bounded because of the exponential decay in m . The second property Eq. (C18) is thus also established.

Finally, it remains to prove Eq. (C8). Since $U = U_{m_*-1} = e^{A_1} \dots e^{A_{m_*-1}}$ from Eq. (C22), the time-dependent generator in Eq. (C7) can be chosen as

$$A(t) = \begin{cases} 0, & t < 2^{1-m_*} \\ 2^m A_m, & 2^{-m} \leq t < 2^{1-m}, \quad (m = 1, \dots, m_* - 1) \end{cases}. \quad (\text{C54})$$

As a result, for any subset S , Eq. (C8) is satisfied:

$$\begin{aligned} \max_t \|A(t)\|_{\kappa_1/2} &\leq \max_{m=1, \dots, m_*-1} 2^m \|A_m\|_{\kappa_1/2} \leq \max_{m=1, \dots, m_*-1} 2^m \|A_m\|_{\kappa_m} \\ &\leq \max_{m=1, \dots, m_*-1} 2^m \tilde{v}_m \leq \max_{m=1, \dots, m_*-1} 2^m \epsilon 4^{1-m} = 2\epsilon. \end{aligned} \quad (\text{C55})$$

Here in the second line, we have used Eqs. (C29) and (C46) with Eq. (C45).

We have established the three conditions for Lemma C1, so this finishes the proof of Theorem 1.

3. Putting everything together: proof of Lemma C1

Proof of Lemma C1. Due to the gap of H_0 and relative boundedness Eq. (C18), Lemma 2 in Ref. [4] implies that the spectrum of $H_0 + D_*$ is contained in

$$\{0\} \cup [1 - c_1\epsilon, \infty) \subset \{0\} \cup [2/3, \infty), \quad (\text{C56})$$

where the exact energy-0 subspace consists of the unperturbed 2^k ground states in P , and we have used $1 - c_1\epsilon \geq 2/3$ from Eq. (C3).

We then consider V_* as a perturbation to $H_0 + D_*$. According to Weyl's inequality,

$$|\lambda_m(A+B) - \lambda_m(A)| \leq \|B\|, \quad (\text{C57})$$

where $\lambda_m(A)$ is the m th smallest eigenvalue of A . Letting $A = H_0 + D_*$ and $B = V_*$, the spectrum of $U^\dagger H U$ is thus contained in the two disjoint intervals Eq. (C4) due to Eq. (C56), with exactly 2^k ground states. Here to bound the final gap by $1/2$, we have used

$$3\epsilon_* \leq 1/6, \quad (\text{C58})$$

which is achievable if n is larger than a constant c_3 because ϵ_* decays with n in Eq. (C5). The spectrum of H is thus

also contained in Eq. (C4), since conjugation by a unitary does not change eigenvalues.

To prove stability of eigenvectors Eq. (C6), we first show that for any normalized new ground state

$$|\psi\rangle = P'_{\text{new}} |\psi\rangle, \quad \text{where } P'_{\text{new}} := (U^\dagger P_{\text{new}} U) \quad (\text{C59})$$

(here we work in the rotated frame with the new ground states contained in $U^\dagger P_{\text{new}} U$), its support on the unperturbed excited-state subspace $I - P$ of $H_0 + D_*$ is small:

$$\|(I - P) |\psi\rangle\| \leq 2\sqrt{\epsilon_*}. \quad (\text{C60})$$

This comes from a Markov inequality for evaluating the total energy

$$\begin{aligned} \epsilon_* &\geq \langle \psi | (H_0 + D_* + V_*) | \psi \rangle \\ &\geq \langle \psi | (H_0 + D_*) | \psi \rangle - \|V_*\| \\ &\geq (1 - c_1\epsilon) \langle \psi | (I - P) | \psi \rangle - \epsilon_* \\ &\geq \frac{1}{2} \|(I - P) |\psi\rangle\|^2 - \epsilon_*, \end{aligned} \quad (\text{C61})$$

where the first and the third inequality comes from the spectrums Eqs. (C4) and (C56), respectively. We have also used $1 - c_1\epsilon \geq 1/2$ from Eq. (C3). Equation (C60) actually implies

$$\|(I - P) P'_{\text{new}}\| = \|(P'_{\text{new}} - P) P'_{\text{new}}\| \leq 2\sqrt{\epsilon_*}, \quad (\text{C62})$$

due to Eq. (C59).

On the other hand, we can view $H_0 + D_*$ as perturbing $H_0 + D_* + V_*$ by $-V_*$. For any original ground state $|\psi\rangle = P |\psi\rangle$, its support in the new excited states is also small:

$$\|(I - P'_{\text{new}}) |\psi\rangle\| \leq 2\sqrt{\epsilon_*}, \quad (\text{C63})$$

because

$$\begin{aligned} 0 &= \langle \psi | (H_0 + D_*) | \psi \rangle \\ &\geq \langle \psi | (H_0 + D_* + V_*) | \psi \rangle - \|V_*\| \\ &\geq (1 - c_1\epsilon - \epsilon_*) \langle \psi | (I - P'_{\text{new}}) | \psi \rangle \\ &\quad - \epsilon_* \langle \psi | P'_{\text{new}} | \psi \rangle - \epsilon_* \\ &\geq \frac{1}{2} \|(I - P'_{\text{new}}) |\psi\rangle\|^2 - 2\epsilon_*. \end{aligned} \quad (\text{C64})$$

Equation (C63) leads to $\|(P'_{\text{new}} - P) P\| \leq 2\sqrt{\epsilon_*}$, so that combining with Eq. (C62), we have

$$\begin{aligned} \|P'_{\text{new}} - P\| &= \|(P'_{\text{new}} - P)(P + I - P)\| \\ &\leq \|(P'_{\text{new}} - P) P\| + \|(P'_{\text{new}} - P)(I - P)\| \\ &= \|(P'_{\text{new}} - P) P\| + \|P'_{\text{new}}(I - P)\| \leq 4\sqrt{\epsilon_*}. \end{aligned} \quad (\text{C65})$$

Here to the second line comes from $P(I - P) = 0$ and plugging in the bounds above.

Therefore, Theorem 1 holds since Eq. (C8) is also satisfied by assumption. ■

4. Analyzing the flow equations: proof of Lemma C4

Proof of Lemma C4. First observe that if the first bound in Eq. (C46) on v_m holds, then the last bound on $\tilde{\mathbf{d}}_m$ follows because

$$\begin{aligned}\tilde{\mathbf{d}}_m &= 2 \sum_{m'=\lfloor m/2 \rfloor + 1}^{m-1} v_{m'} \leq 2 \sum_{m'=\lfloor m/2 \rfloor + 1}^{m-1} \frac{\epsilon}{\delta \kappa_{m'-1}} (c_{\text{iter}} \epsilon)^{m'-1} \\ &\leq \frac{2\epsilon}{\delta \kappa_{m-2}} \sum_{m'=\lfloor m/2 \rfloor + 1}^{m-1} (c_{\text{iter}} \epsilon)^{m'-1} \\ &\leq \frac{2\epsilon}{\delta \kappa_{m-2}} (c_{\text{iter}} \epsilon)^{\lfloor m/2 \rfloor} \left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots \right) \\ &\leq \frac{3\epsilon}{\delta \kappa_{m-2}} (c_{\text{iter}} \epsilon)^{\lfloor m/2 \rfloor}.\end{aligned}\quad (\text{C66})$$

Therefore, we can focus on the first three bounds in Eq. (C46).

Let

$$\begin{aligned}c_{\text{iter}} &:= \max \left(\frac{3\tilde{c}_f''}{2} \left(e^{c_f'(\tilde{\kappa}_1)^{-\alpha}} + \sum_{m=2}^{\infty} e^{c_f'(\tilde{\kappa}_m)^{-\alpha}} \frac{4^{1-m}}{\delta \kappa_{m-1}} \right), \right. \\ &\quad \left. \times \frac{27}{\kappa_2 \delta \kappa_1} \max(1, \delta \kappa_1) \right),\end{aligned}\quad (\text{C67})$$

be a constant determined by $c_f', \tilde{c}_f'', \alpha, \kappa_1$. In particular, the infinite sum converges because the exponential decay 4^{1-m} dominates, e.g., growth of $e^{c_f'(\tilde{\kappa}_m)^{-\alpha}} = e^{O[(\log m)^{2\alpha}]}$ from

Eq. (C35b). There exists a constant

$$0 < \epsilon_0 < \min(1/(4c_{\text{iter}}), \delta \kappa_1/3), \quad (\text{C68})$$

determined by Δ, κ_1 such that $\forall m \geq 2$,

$$\begin{aligned}&\frac{3}{\delta \kappa_m \delta \kappa_{m-1}} (c_{\text{iter}} \epsilon_0)^{\lfloor m/2 \rfloor} + \frac{9}{\kappa_{m+1} \delta \kappa_m} (c_{\text{iter}} \epsilon_0)^{m-1} \\ &\times \left(\frac{3}{\delta \kappa_{m-1}} + \frac{4\epsilon_0}{\delta \kappa_m} \left(\frac{2}{3} c_{\text{iter}} + \frac{3}{\delta \kappa_{m-1}} \right) \right) \\ &\leq \frac{c_{\text{iter}}}{3} \min \left(1, \frac{1}{\delta \kappa_2} \right).\end{aligned}\quad (\text{C69})$$

The reason is that Eq. (C69) is always suppressed by a positive power of ϵ_0 , and this exponential-decaying factor $(c_{\text{iter}} \epsilon_0)^{m-1}$ dominates over the polynomial dependence of $\kappa_{m+1}, \delta \kappa_{m'}$ ($m' = m, m-1$) at large m , so there always exists a sufficiently small ϵ_0 that does the job.

For $m = 2$, Eq. (C43b) yields

$$\tilde{v}_2 \leq 0 + \frac{9}{\kappa_2 \delta \kappa_1} \epsilon (3\epsilon + 0) \leq c_{\text{iter}} \epsilon^2, \quad (\text{C70})$$

according to Eq. (C67). Equation (C43a) yields $v_2 \leq c_{\text{iter}} \epsilon^2 / (\delta \kappa_1)$ similarly, and Eq. (C43c) leads to

$$\mathbf{d}_2 \leq \tilde{c}_f'' e^{c_f'(\tilde{\kappa}_1)^{-\alpha}} \epsilon \leq \frac{2}{3} c_{\text{iter}} \epsilon, \quad (\text{C71})$$

using Eq. (C67). Therefore, Eq. (C46) is satisfied at $m = 2$.

We then prove Eq. (C46) inductively for any $\epsilon \leq \epsilon_0$ and $c_{\text{iter}}, \epsilon_0$ defined above: if Eq. (C46) holds for all $m = 1, \dots, m'$ with $m' \geq 2$, it suffices to prove that it also holds for $m = m' + 1$. For Eq. (C43b), we have

$$\begin{aligned}\tilde{v}_{m'+1} &\leq \tilde{v}_{m'} \left[\frac{2}{3} c_{\text{iter}} \epsilon + \frac{3\epsilon}{\delta \kappa_{m'} \delta \kappa_{m'-1}} (c_{\text{iter}} \epsilon_0)^{\lfloor m'/2 \rfloor} + \frac{9\epsilon}{\kappa_{m'+1} \delta \kappa_{m'}} (c_{\text{iter}} \epsilon)^{m'-1} \left(\frac{3}{\delta \kappa_{m'-1}} + \frac{4}{\delta \kappa_{m'}} \left(\frac{2}{3} c_{\text{iter}} \epsilon_0 + \frac{3\epsilon_0}{\delta \kappa_{m'-1}} \right) \right) \right] \\ &\leq \tilde{v}_{m'} \left(\frac{2}{3} c_{\text{iter}} \epsilon + \frac{1}{3} c_{\text{iter}} \epsilon \right) \leq c_{\text{iter}} \epsilon \tilde{v}_{m'} \leq \epsilon (c_{\text{iter}} \epsilon)^{m'},\end{aligned}\quad (\text{C72})$$

where we have used Eq. (C46) for $m = m'$, $\delta \kappa_{m'-2} \leq \delta \kappa_{m'-1}$, and that Eq. (C69) holds for ϵ_0 replaced by any $\epsilon \leq \epsilon_0$. One can similarly obtain $v_{m'+1} \leq c_{\text{iter}} \epsilon \tilde{v}_{m'} / (\delta \kappa_{m'})$ using Eq. (C43a) and $\delta \kappa_{m'} \leq \delta \kappa_2$, so that $v_{m'+1}$ also satisfies Eq. (C46). Finally, Eq. (C43c) [or the equivalent Eq. (C38)] leads to

$$\mathbf{d}_{m'+1} \leq \mathbf{d}_2 + \sum_{m=2}^{\lfloor (m'+1)/2 \rfloor} \tilde{c}_f'' e^{c_f'(\tilde{\kappa}_m)^{-\alpha}} \frac{1}{\delta \kappa_{m-1}} v_m \leq \tilde{c}_f'' e^{c_f'(\tilde{\kappa}_1)^{-\alpha}} \epsilon + \tilde{c}_f'' \epsilon \sum_{m=2}^{\infty} e^{c_f'(\tilde{\kappa}_m)^{-\alpha}} \frac{1}{\delta \kappa_{m-1}} 4^{1-m} \leq \frac{2}{3} c_{\text{iter}} \epsilon, \quad (\text{C73})$$

where we have used the bound on v_m in Eq. (C46), $c_{\text{iter}} \epsilon \leq 1/4$ from Eq. (C68), and finally Eqs. (C71) and (C67). Therefore, Eq. (C46) holds inductively.

Finally, since the condition for iteration Eq. (C40) is satisfied at $m = 1$ due to Eq. (C68), and $\delta\kappa_m \geq \delta\kappa_1 4^{1-m}$, Eq. (C40) continues to hold for all $m \geq 1$. ■

5. Application of local indistinguishability: proof of Lemma C2

Proof of Lemma C2. We assign the local terms as

$$([D, A])_{S'', s''} = \sum_{\substack{S, S', s, s': \\ S \cup S' = S'', s + s' = s''}} [D_{S', s'}, A_{S, s}], \quad (\text{C74})$$

where each term is strongly supported in $S' \cup S$, and has syndrome $\mathbf{s} + \mathbf{s}'$.

We first prove the easier Eq. (C30) that does not depend on e.g. the block-diagonal structure of D . For a given $D_{S', s'}$,

$$\begin{aligned} & \sum_{S, s} \|[D_{S', s'}, A_{S, s}]\| e^{\kappa' |S' \cup S|} \\ & \leq \sum_{i \in S'} \sum_{S \ni i} \sum_s 2 \|D_{S', s'}\| \|A_{S, s}\| e^{\kappa' (|S| + |S'|)} \\ & \leq 2 |S'| \|D_{S', s'}\| e^{\kappa' |S'|} \|A\|_{\kappa'} \\ & \leq \frac{1}{\delta\kappa} \|D_{S', s'}\| e^{\kappa' |S'|} \|A\|_{\kappa'}. \end{aligned} \quad (\text{C75})$$

Here in the first line, we have organized the sum by the fact that for any S that overlaps with S' such that the commutator does not vanish, there exists $i \in S'$ such that $S \ni i$. The overlapping condition also leads to $|S' \cup S| \leq |S| + |S'| - 1$ (-1 is not important and ignored). In the second line, we have invoked the definition of κ -norm for A . The last line comes from $2x \leq e^x$ for $x = \delta\kappa |S'|$. Exchanging the roles of D and A in the previous manipulation Eq. (C75), we also have for a given $A_{S, s}$:

$$\sum_{S', s'} \|[D_{S', s'}, A_{S, s}]\| e^{\kappa' |S' \cup S|} \leq \frac{1}{\delta\kappa} \|D\|_{\kappa'} \|A_{S, s}\| e^{\kappa' |S|}. \quad (\text{C76})$$

Observe that if a local term $[D_{S', s'}, A_{S, s}]$ acts on i , either $i \in S$ or $i \in S'$. As a result, Eq. (C30) follows by

$$\begin{aligned} \|[D, A]\|_{\kappa'} & \leq \max_i \left[\sum_{S \ni i} \sum_s \sum_{S', s'} \|[D_{S', s'}, A_{S, s}]\| e^{\kappa' |S' \cup S|} + \sum_{S' \ni i} \sum_{s'} \sum_{S, s} \|[D_{S', s'}, A_{S, s}]\| e^{\kappa' |S' \cup S|} \right] \\ & \leq \max_i \left[\sum_{S \ni i} \sum_s \frac{1}{\delta\kappa} \|D\|_{\kappa'} \|A_{S, s}\| e^{\kappa' |S|} + \sum_{S' \ni i} \sum_{s'} \frac{1}{\delta\kappa} \|D_{S', s'}\| e^{\kappa' |S'|} \|A\|_{\kappa'} \right] \\ & \leq \frac{1}{\delta\kappa} (\|D\|_{\kappa'} \|A\|_{\kappa} + \|D\|_{\kappa} \|A\|_{\kappa'}) \leq \frac{2}{\delta\kappa} \|D\|_{\kappa} \|A\|_{\kappa}. \end{aligned} \quad (\text{C77})$$

Here we have plugged in Eqs. (C76) and (C75) in the second line, and used, e.g., $\|D\|_{\kappa'} \leq \|D\|_{\kappa}$ in the end.

We then turn to Eq. (C31), where we need two crucial observations.

The first is that many terms in Eq. (C74) vanish when projected by $\mathbb{P}^\perp$ (see the toric code example discussed earlier for intuition):

$$\mathbb{P}^\perp [D_{S', s'}, A_{S, s}] = 0, \quad (\text{C78})$$

as long as S' is far from the syndrome $\mathbf{s}$, compared to its own size $|S'|$. More precisely, Eq. (C78) holds if

$$|S'| < \tilde{f}(\mathbf{d}(S', C)), \quad \forall \mathcal{Q}_C \in \mathbf{s}, \quad (\text{C79})$$

where $\tilde{f}$ is the function determined by f in Eq. (B24).

To see this, Lemma B1 implies that (here is why we restrict D to only contain $|S| < d_s$ terms)

$$P_{\bar{S}'} D_{S', s'} P_{\bar{S}'} = c P_{\bar{S}'}, \quad (\text{C80})$$

for constant c (note that $c = 0$ for $\mathbf{s}' \neq \emptyset$), where $\bar{S}'$ does not overlap with the syndrome $\mathbf{s}$ according to Eq. (C79). As a result, for $\bar{S}'' = \bar{S}' \cup S$,

$$\begin{aligned} Q_{\bar{S}''} [D_{S', s'}, A_{S, s}] P_{\bar{S}''} & = Q_{\bar{S}''} [D_{S', s'} - cI, A_{S, s}] P_{\bar{S}''} \\ & = Q_{\bar{S}''} (D_{S', s'} - cI) A_{S, s} P_{\bar{S}''} \\ & = Q_{\bar{S}''} (D_{S', s'} - cI) A_{S, s} P_{\bar{S}'} P_{\bar{S}''} \\ & = Q_{\bar{S}''} (D_{S', s'} - cI) P_{\bar{S}'} A_{S, s} P_{\bar{S}''} = 0. \end{aligned} \quad (\text{C81})$$

Here in the first line, we have inserted an identity cI in the commutator where c is the constant in Eq. (C80) such that [recall Eq. (C10)]

$$P_{\tilde{S}'}(D_{S',s'} - cI) = 0, \quad \Rightarrow \quad P_{\tilde{S}''}(D_{S',s'} - cI) = 0. \quad (\text{C82})$$

In the second line of Eq. (C81), we have commuted $P_{\tilde{S}'}$ through $A_{S,s}$ because the syndrome $\mathbf{s}$ does not touch S' , and used Eq. (C82) again. Equation (C81) implies that the commutator $[D_{S',s'}, A_{S,s}]$ commutes with $P_{\tilde{S}''}$. On the other hand, since $[D_{S',s'}, A_{S,s}]$ is strongly supported in $S'' = S' \cup S$, it commutes with checks in $P_{\tilde{S}''}$ that are not contained in S'' . As a result, in order for the commutator to commute with $P_{\tilde{S}''}$, it must commute with $P_{S''}$:

$$\begin{aligned} Q_{S''}[D_{S',s'}, A_{S,s}]P_{S''} &= 0, \\ \Rightarrow \quad \mathbb{P}^\perp[D_{S',s'}, A_{S,s}] &= Q_{S''}[D_{S',s'}, A_{S,s}]P_{S''} + \text{H.c.} = 0. \end{aligned} \quad (\text{C83})$$

The second observation is that $A_{S,s}$ is itself suppressed by a factor

$$\begin{aligned} \|A_{S,s}\| &\leq \max(\|P_S V_{S,s} Q_S H_S^{-1}\|, \|H_S^{-1} Q_S V_{S,s} P_S\|) \\ &\leq \|P_S V_{S,s} Q_S\| / |\mathbf{s}| = \|\mathbb{P}^\perp V_{S,s}\| / |\mathbf{s}|. \end{aligned} \quad (\text{C84})$$

Here we have used $H_S^{-1} \leq 1/|\mathbf{s}|$ because any state in $V_{S,s} P_S$ flips $|\mathbf{s}|$ checks, and any flipped check cost at least energy 1, from Eq. (A6).

For a given $A_{S,s}$, these two facts lead to

$$\begin{aligned} &\sum_{S',s'} \|\mathbb{P}^\perp[D_{S',s'}, A_{S,s}]\| e^{\kappa'|S' \cup S|} \\ &\leq \sum_{Q_C \in \mathbf{s}} \sum_{i \in C} \sum_{S': |S'| \geq \tilde{f}(\mathbf{d}(i, S'))} \sum_{s'} \|[D_{S',s'}, A_{S,s}]\| e^{\kappa'(|S| + |S'|)} \\ &\leq \sum_{Q_C \in \mathbf{s}} \sum_{i \in C} \sum_{r \geq 0: \tilde{f}(r) < d_s} \sum_{j: \mathbf{d}(i, j) = r} \sum_{S': |S'| \geq \tilde{f}(r)} \\ &\quad \times \sum_{s'} 2 \|D_{S',s'}\| \|A_{S,s}\| e^{\kappa'(|S| + |S'|)} \\ &\leq 2 \|A_{S,s}\| e^{\kappa'|S|} \sum_{Q_C \in \mathbf{s}} \sum_{i \in C} \sum_{r \geq 0: \tilde{f}(r) < d_s} \gamma(r) \|D\|_\kappa e^{-\delta \kappa \tilde{f}(r)} \\ &\leq 2 \|A_{S,s}\| e^{\kappa'|S|} |\mathbf{s}| \Delta \|D\|_\kappa c_f'' e^{c_f'(\delta \kappa)^{-\alpha}} \\ &\leq 2 \Delta c_f'' e^{c_f'(\delta \kappa)^{-\alpha}} \|D\|_\kappa \|\mathbb{P}^\perp V_{S,s}\| e^{\kappa'|S|}. \end{aligned} \quad (\text{C85})$$

Here the first line comes from Eq. (C12) and the first observation above, where for the projected commutator to not vanish, there exists i contained in syndrome $\mathbf{s}$ such that $|S'| \geq \tilde{f}(\mathbf{d}(i, S'))$. This condition is further relaxed to the second line that S' acts on $j \in \Lambda$ such that $|S'| \geq \tilde{f}(\mathbf{d}(i, j))$. The sum over r is restricted by $\tilde{f}(r) < d_s$, because D only contains terms with support $|S'| < d_s$. To get the third line of Eq. (C85), notice that $e^{\kappa'|S'|} \leq e^{\kappa'|S'|} e^{\delta \kappa \tilde{f}(r)}$ for any S' in the sum, so we can use the κ -norm of D with an exponential factor leftover. We also replaced the sum over j by $\gamma(r)$, which was defined in Eq. (A1). We then used condition Eq. (B25) to get the fourth line, and Eq. (C84) for the last line.

For a given $D_{S',s'}$, we use the previous Eq. (C75) to obtain

$$\sum_{S,s} \|\mathbb{P}^\perp[D_{S',s'}, A_{S,s}]\| e^{\kappa'|S' \cup S|} \leq \sum_{S,s} \|[D_{S',s'}, A_{S,s}]\| e^{\kappa'|S' \cup S|} \leq \frac{1}{\delta \kappa} \|D_{S',s'}\| e^{\kappa'|S'|} \|\mathbb{P}^\perp V\|_{\kappa'}, \quad (\text{C86})$$

where we have used $\|A\|_{\kappa'} \leq \|\mathbb{P}^\perp V\|_{\kappa'}$ [see also Eq. (C29)] because each $\|A_{S,s}\| \leq \|\mathbb{P}^\perp V_{S,s}\|$ from Eq. (C84) and $|\mathbf{s}| \geq 1$, since $A_{S,s} = 0$ as long as $\mathbf{s} = \emptyset$. We then follow Eq. (C77) to derive Eq. (C31):

$$\begin{aligned} \|\mathbb{P}^\perp[D, A]\|_{\kappa'} &\leq \max_i \left[\sum_{S \ni i} \sum_{\mathbf{s}} \sum_{S',s'} \|\mathbb{P}^\perp[D_{S',s'}, A_{S,s}]\| e^{\kappa'|S' \cup S|} + \sum_{S' \ni i} \sum_{s'} \sum_{S,s} \|\mathbb{P}^\perp[D_{S',s'}, A_{S,s}]\| e^{\kappa'|S' \cup S|} \right] \\ &\leq \max_i \left[\sum_{S \ni i} \sum_{\mathbf{s}} 2 \Delta c_f'' e^{c_f'(\delta \kappa)^{-\alpha}} \|D\|_\kappa \|\mathbb{P}^\perp V_{S,s}\| e^{\kappa'|S|} + \sum_{S' \ni i} \sum_{s'} \frac{1}{\delta \kappa} \|D_{S',s'}\| e^{\kappa'|S'|} \|\mathbb{P}^\perp V\|_{\kappa'} \right] \\ &\leq \left(\frac{1}{\delta \kappa} + 2 \Delta c_f'' e^{c_f'(\delta \kappa)^{-\alpha}} \right) \|D\|_\kappa \|\mathbb{P}^\perp V\|_{\kappa'}, \end{aligned} \quad (\text{C87})$$

where the second line combines Eqs. (C85) and (C86). The second expression in Eq. (C31) then follows by relaxing the term $1/\delta\kappa$ to an exponential using $\alpha > 0$. We have also assumed $\tilde{c}_f'' \geq 2$ without loss of generality, because otherwise we can set $\max(2, \tilde{c}_f'')$ to be the new constant $\tilde{c}_f''$. ■

6. Relative boundedness: proof of Proposition C1

Proof of Proposition C1. Like operators in Eq. (A12), any state

$$|\psi\rangle = \sum_{\mathbf{s}} \tilde{P}_{\mathbf{s}} |\psi\rangle =: \sum_{\mathbf{s}} |\psi_{\mathbf{s}}\rangle, \quad (\text{C88})$$

can be also decomposed into different syndrome subspaces with orthogonal projectors $\{\tilde{P}_{\mathbf{s}}\}$ for each syndrome $\mathbf{s}$ defined by

$$\tilde{P}_{\mathbf{s}} := \prod_{Q \in \mathbf{s}} \frac{I - Q}{2} \prod_{Q \notin \mathbf{s}} \frac{I + Q}{2}. \quad (\text{C89})$$

In other words, $\tilde{P}_{\mathbf{s}}$ projects onto the simultaneous eigensubspace of the checks with eigenvalues given in $\mathbf{s}$. Note that some $\tilde{P}_{\mathbf{s}} = 0$ if the checks have redundancies.

$D_{S,\mathbf{s}}$ maps a state of syndrome $\mathbf{s}'$ to syndrome $\mathbf{s} + \mathbf{s}'$. Furthermore, similar to the argument around Eq. (C78), we have

$$\begin{aligned} \exists c \in \mathbb{R} : \tilde{P}_{\mathbf{s}+\mathbf{s}'}(D_{S,\mathbf{s}} - c_{S,\mathbf{s}}I)\tilde{P}_{\mathbf{s}'} &= 0, \\ \text{unless } |S| \geq \tilde{f}(r), \quad \text{where } r &= \min_{Q \in \mathbf{s}'} d(C, S). \end{aligned} \quad (\text{C90})$$

This holds because if S is far from syndrome $\mathbf{s}'$ comparing to its size $|S| < \tilde{f}(r)$, $\tilde{P}_{\mathbf{s}'} \propto P_{\tilde{S}}$ where $\tilde{S}$ is the locally indistinguishable region in Lemma B1, so that

$$D_{S,\mathbf{s}}P_{\tilde{S}} = D_{S,\mathbf{s}}P_{\tilde{S}}^2 = P_{\tilde{S}}D_{S,\mathbf{s}}P_{\tilde{S}} = c_{S,\mathbf{s}}P_{\tilde{S}}, \quad (\text{C91})$$

for a number $c_{S,\mathbf{s}}$, which yields Eq. (C90). Here the second equality in Eq. (C91) holds because $D_{S,\mathbf{s}}$ is block diagonal so commutes with $P_{\tilde{S}}$ and furthermore $P_{\tilde{S}}$. Note that $c_{S,\mathbf{s}} \neq 0$ only for $\mathbf{s} = \emptyset$.

Define $c_D = \sum_{S,\mathbf{s}} c_{S,\mathbf{s}}$ to be the total offset of D , and let $\tilde{D} = D - c_D I$.

Similar to the summation arrangement in Eq. (C85), the condition for S in Eq. (C90) implies the existence of site $j \in S$ that is of distance r to the syndrome $\mathbf{s}'$. Equation (C90) then leads to

$$\begin{aligned} \|\tilde{P}_{\mathbf{s}+\mathbf{s}'}\tilde{D}\tilde{P}_{\mathbf{s}}\| &\leq \sum_{Q \in \mathbf{s}} \sum_{i \in C} \sum_{r \geq 0: \tilde{f}(r) < d_S} \sum_{j: d(i,j)=r} \sum_{S \ni j: |S| \geq \tilde{f}(r)} \|D_{S,\mathbf{s}'} - c_{S,\mathbf{s}'}I\| \\ &\leq 2 \sum_{Q \in \mathbf{s}} \sum_{i \in C} \sum_{r \geq 0: \tilde{f}(r) < d_S} e^{-\kappa \tilde{f}(r)} \sum_{j: d(i,j)=r} \sum_{S \ni j: |S| \geq \tilde{f}(r)} \|D_{S,\mathbf{s}'}\| e^{\kappa |S|}, \end{aligned} \quad (\text{C92})$$

where we have used $\|\tilde{P}_{\mathbf{s}+\mathbf{s}'}(D_{S,\mathbf{s}'} - c_{S,\mathbf{s}'}I)\tilde{P}_{\mathbf{s}}\| \leq \|D_{S,\mathbf{s}'} - c_{S,\mathbf{s}'}I\|$ in the first line, and $|c_{S,\mathbf{s}'}| = \|P_{\tilde{S}}D_{S,\mathbf{s}'}P_{\tilde{S}}\| \leq \|D_{S,\mathbf{s}'}\|$ in the second. As a result,

$$\begin{aligned} \sum_{\mathbf{s}'} \|\tilde{P}_{\mathbf{s}+\mathbf{s}'}\tilde{D}\tilde{P}_{\mathbf{s}}\| &\leq 2 \sum_{Q \in \mathbf{s}} \sum_{i \in C} \sum_{r \geq 0: \tilde{f}(r) < d_S} e^{-\kappa \tilde{f}(r)} \sum_{j: d(i,j)=r} \sum_{S \ni j: |S| \geq \tilde{f}(r)} \sum_{\mathbf{s}'} \|D_{S,\mathbf{s}'}\| e^{\kappa |S|} \\ &\leq 2|\mathbf{s}|\Delta \|D\|_{\kappa} \sum_{r \geq 0: \tilde{f}(r) < d_S} \gamma(r) e^{-\kappa \tilde{f}(r)} \leq 2\Delta |\mathbf{s}| \|D\|_{\kappa} c_f'' e^{c_f' \kappa^{-\alpha}} \\ &\leq 2\Delta \|D\|_{\kappa} c_f'' e^{c_f' (\kappa/2)^{-\alpha}} |\mathbf{s}| =: \mathfrak{d}' |\mathbf{s}|, \end{aligned} \quad (\text{C93})$$

where we have used Eq. (B25) and defined $\mathfrak{d}'$ by the final expression. Similarly,

$$\begin{aligned} \sum_{\mathbf{s}'} |\mathbf{s}'| \|\tilde{P}_{\mathbf{s}+\mathbf{s}'}\tilde{D}\tilde{P}_{\mathbf{s}}\| &\leq 2 \sum_{Q \in \mathbf{s}} \sum_{i \in C} \sum_{r \geq 0: \tilde{f}(r) < d_S} e^{-\frac{\kappa}{2} \tilde{f}(r)} \sum_{j: d(i,j)=r} \sum_{S \ni j: |S| \geq \tilde{f}(r)} \sum_{\mathbf{s}'} \|D_{S,\mathbf{s}'}\| e^{\frac{\kappa}{2} |S|} |\mathbf{s}'| \\ &\leq 2 \sum_{Q \in \mathbf{s}} \sum_{i \in C} \sum_{r \geq 0: \tilde{f}(r) < d_S} e^{-\frac{\kappa}{2} \tilde{f}(r)} \sum_{j: d(i,j)=r} \sum_{S \ni j: |S| \geq \tilde{f}(r)} \sum_{\mathbf{s}'} \|D_{S,\mathbf{s}'}\| e^{\frac{\kappa}{2} |S|} \left(\frac{\Delta}{\kappa} e^{\frac{\kappa}{2} |S|} \right) \leq \frac{\Delta}{\kappa} \mathfrak{d}' |\mathbf{s}|. \end{aligned} \quad (\text{C94})$$

Here the last line comes from comparing to Eq. (C93). In the second line of Eq. (C94), we have used $2x \leq e^x$ for $x = \frac{\kappa}{2\Delta} |s'| \leq \kappa |S|/2$, because an operator of support S at most overlaps with $\Delta |S|$ checks that could belong to the syndrome s' , according to the LDPC condition. Note that this is the only place where we need each site to participate in bounded number of checks; all other places using the LDPC condition [bounding $\sum_{i \in C} \leq \Delta$ here and in Eq. (C85)] invoke the other condition that each check acts on bounded number of sites.

For any state ψ expanded by Eq. (C88), we have (here $\|\phi\| := \sqrt{\langle \phi | \phi \rangle}$)

$$\begin{aligned}
\|\tilde{D}|\psi\rangle\|^2 &= \sum_{s_1, s_2, s_3} \langle \psi_{s_3} | \tilde{D} \tilde{P}_{s_2} \tilde{D} | \psi_{s_1} \rangle \leq \sum_{s_1, s_2, s_3} \|\psi_{s_3}\| \|\tilde{P}_{s_3} \tilde{D} \tilde{P}_{s_2}\| \|\tilde{P}_{s_2} \tilde{D} \tilde{P}_{s_1}\| \|\psi_{s_1}\| \\
&\leq \sum_{s_1, s_2, s_3} \frac{1}{2} \left(\|\psi_{s_1}\|^2 + \|\psi_{s_3}\|^2 \right) \|\tilde{P}_{s_3} \tilde{D} \tilde{P}_{s_2}\| \|\tilde{P}_{s_2} \tilde{D} \tilde{P}_{s_1}\| = \sum_{s_1, s_2} \|\psi_{s_1}\|^2 \|\tilde{P}_{s_2} \tilde{D} \tilde{P}_{s_1}\| \sum_{s_3} \|\tilde{P}_{s_3} \tilde{D} \tilde{P}_{s_2}\| \\
&\leq \sum_{s_1, s_2} \|\psi_{s_1}\|^2 \|\tilde{P}_{s_2} \tilde{D} \tilde{P}_{s_1}\| (\mathsf{d}' |s_2|) \leq \mathsf{d}' \sum_{s_1} \|\psi_{s_1}\|^2 \sum_{s'_2} (|s_1| + |s'_2|) \|\tilde{P}_{s_1+s'_2} \tilde{D} \tilde{P}_{s_1}\| \\
&\leq (\mathsf{d}')^2 \sum_{s_1} \|\psi_{s_1}\|^2 |s_1| \left(|s_1| + \frac{\Delta}{\kappa} \right) \leq (\mathsf{d}')^2 \sum_{s_1} \|\psi_{s_1}\|^2 |s_1|^2 \left(1 + \frac{\Delta}{\kappa} \right) \\
&= \left(1 + \frac{\Delta}{\kappa} \right) (\mathsf{d}')^2 \sum_{s_1} |s_1|^2 \langle \psi | \tilde{P}_{s_1} | \psi \rangle \leq \left(1 + \frac{\Delta}{\kappa} \right) (\mathsf{d}')^2 \|H_0 | \psi \rangle\|^2. \tag{C95}
\end{aligned}$$

Here the second line comes from Cauchy-Schwarz inequality and relabeling the variables for the term $\propto \|\psi_{s_3}\|^2$. To get the third line, we observe that the sum over s_3 is equivalent to a sum over syndrome s'_3 : $\sum_{s'_3} \|\tilde{P}_{s_2+s'_3} \tilde{D} \tilde{P}_{s_2}\|$, so we can plug in Eq. (C93). We have defined s'_2 similarly and used $|s_2| = |s_1 + s'_2| \leq |s_1| + |s'_2|$. We have again used Eqs. (C93) and (C94) in the fourth line of Eq. (C95). The last line comes from $H_0^2 \geq \sum_s |s|^2 \tilde{P}_s$ because flipping $|s|$ checks costs at least energy $|s|$ due to Eq. (A6). Equation (C95) finishes the proof of Proposition C1 using d' defined in Eq. (C93), and $\sqrt{1+x} \leq 1+x/2$ for any $x \geq 0$. ■

APPENDIX D: STABILITY OF CLASSICAL CODES UNDER SYMMETRIC PERTURBATIONS

Here we present the proofs of our generalizations of Theorem 1 to classical codes with symmetric perturbations.

Corollary 1 (Formal version). Theorem 1 also holds for H_0 being a classical LDPC code, as long as each term $V_{S,s}$ in the perturbation V is symmetric under conjugation by any logical operator (which are Pauli X s by Definition A4).

Proof. The proof closely follows the argument in the discussion section of Ref. [38]. Here the only difference comparing to the proof of Theorem 1 is that Eq. (12) (where $\tilde{S}$ is defined in the same way as quantum codes from check soundness) does not hold for all operators $\mathcal{O}$, because a single Z_i could differentiate the ground states.

An alternative viewpoint is that Z_i is a logical Z operator when viewing the classical code as a quantum CSS code with only Z checks.

Nevertheless, Eq. (12) holds for any $\mathcal{O}$ that is symmetric under conjugation by X logicals, because this precisely rules out Z logicals that must anticommute with some X logical. If the local perturbation $V_{S,s}$ is symmetric as we assumed, the operators in the proof of Theorem 1 are all symmetric because they are “generated” from the perturbation, along with H_0 , which is also symmetric. More precisely, the initial perturbation V_1 first generates (local terms of) A_1 by Eq. (C15), which is locally symmetric because the P, Q, H factors in Eq. (C15) all commute with X logicals (since each check in H_0 commutes with X logicals). The same holds for D_2 in Eq. (C24). V_2 generated by Eqs. (C25) and (C26a) is also locally symmetric because all components in Eq. (C25) are symmetric. This iterates to arbitrary order m , so that we only encounter symmetric $\mathcal{O}$ when applying Eq. (12). The proof of Theorem 1 thus generalizes here. ■

Corollary 2 (Formal version). The conclusions of Theorem 1 hold for H_0 being the Ising model Eq. (1) on graph G , where the parameter d_s may be taken to be $d_s = n$, if the symmetric perturbation $V = \sum_{S,s} V_{S,s}$ where each S is connected in G .

Proof. We first prove that Eq. (12) holds for any connected S and symmetric $\mathcal{O}$ with $\tilde{S} = \tilde{S}$ being S and its neighborhood. In other words, Eq. (11) holds. Since S is connected, $P_{\tilde{S}} = |\mathbf{0}\rangle_{\tilde{S}} \langle \mathbf{0}| + |\mathbf{1}\rangle_{\tilde{S}} \langle \mathbf{1}|$ projects onto the

all-zero $|0\rangle$ and all-one $|1\rangle$ states in $\tilde{S}$. If $\mathcal{O}$ is symmetric under the X logical $\prod_{i \in \Lambda} X_i$, it is also symmetric under its local version $\mathbf{X} := \prod_{i \in \tilde{S}} X_i$, so that

$$\langle 0 | \mathcal{O} | 0 \rangle_{\tilde{S}} = \langle 0 | \mathbf{X} \mathcal{O} \mathbf{X} | 0 \rangle_{\tilde{S}} = \langle 1 | \mathcal{O} | 1 \rangle_{\tilde{S}}, \quad (\text{D1})$$

and $\langle 1 | \mathcal{O} | 0 \rangle_{\tilde{S}} = 0$ because $\mathcal{O}$ does not act on $\tilde{S} \setminus S$. As a result, $\mathcal{O}$ acts as an identity in the local ground states and Eq. (11) holds.

Observe that by restricting the initial perturbation to contain only connected S , all generated operators A_m, D_m, V_m in the proof of Theorem 1 also only contain connected supports S for their local terms, which can be checked in a similar way as the proof of Corollary 1. In particular, the connectedness pertains when invoking Lemma C3 to generate operators like $(e^{A_m} - 1)V_m$; see Refs. [36,53] for details, which indeed work with connected sets throughout.

Corollary 2 then follows from the above two observations. As we only need to consider connected sets of qubits, we can use Eq. (11) directly that avoids invoking check soundness. More precisely, check soundness was only used to obtain Eq. (B25), but Eq. (B25) now trivially holds because $\tilde{S}$ is only S with its neighborhood so that we can set $\tilde{f}(r) = \infty$ for $r > 1$. ■

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