

Predicting Magic from Very Few Measurements

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The nonstabilizerness of quantum states is a necessary resource for universal quantum computation, yet its characterization is notoriously demanding. Quantifying nonstabilizerness typically requires an exponential number of measurements and a doubly exponential classical post-processing cost to evaluate its standard monotones. In this work, we show that nonstabilizerness is, to a large extent, in the eyes of the beholder: it can be witnessed and quantified using *any* set of mn -qubit Pauli measurements, provided the set contains anti-commuting pairs. We introduce a general framework that projects the stabilizer polytope onto the subspace defined by these observables and provide an algorithm that estimates magic from Pauli expectation values with runtime exponential in the number of measurements m and polynomial in the number of qubits n . By relating the problem to a stabilizer-restricted variant of the quantum marginal problem, we also prove that deciding membership in the corresponding reduced stabilizer polytope is NP-hard. In particular, unless $P = NP$, no algorithm polynomial in m can solve the problem in full generality, thus establishing fundamental complexity-theoretic limitations. Finally, we employ our framework to compute nonstabilizerness in different Hamiltonian ground states, demonstrating the practical performance of our method in regimes beyond the reach of existing techniques.

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I. INTRODUCTION

A central problem in quantum information science is to identify and quantify the resources that allow quantum systems to outperform their classical counterparts [1,2]. In the context of quantum computation, this distinction is sharply captured by the theory of stabilizer states and operations [3]. While stabilizer states may exhibit large amounts of entanglement, the Gottesman-Knill theorem guarantees their efficient classical simulability [4,5], so they are insufficient for universal quantum computation. The essential ingredient behind quantum advantage is then called nonstabilizerness, or magic, which has therefore emerged as a fundamental resource for fault-tolerant quantum computation [6] as well as in other quantum information processing tasks [7–11].

Despite its importance, the characterization of nonstabilizerness remains well known to be difficult. At the heart of the problem lies the complex geometry of the stabilizer polytope, the convex hull of all stabilizer states [12]. Determining whether a given quantum state lies inside or outside this polytope or quantifying its distance from it poses two major bottlenecks. The first bottleneck is experimental. Most existing measures and witnesses of nonstabilizerness rely on complete state tomography, which requires estimating 4^n expectation values for an n -qubit system and is therefore infeasible beyond small system sizes. The second bottleneck is computational. Even assuming full tomographic data were available, deciding whether a state lies inside the stabilizer polytope is itself an intractable task: the number of its vertices grows super-exponentially with the number of qubits [13]. As a consequence, existing approaches must rely on strong symmetry assumptions, highly structured families of states, or severe dimensional restrictions [14,15]. This exponential overhead contrasts with the scale of current quantum platforms, which already operate at—and in some cases beyond—hundreds of qubits [16–19].

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Together, these obstacles severely limit our ability to detect and quantify nonstabilizerness in large quantum systems, and raise a fundamental question: Can one quantify the degree of nonstabilizerness of a quantum state using only a limited number of measurements, while simultaneously avoiding computationally intractable optimization over the full stabilizer polytope? In this work, we answer this question in the affirmative. Inspired by the philosophy of classical shadows [20], we show that it is possible to capture essential information about nonstabilizerness by probing only a low-dimensional subspace of the full Hilbert space. Rather than reconstructing the entire quantum state, we project the stabilizer polytope onto this subspace and study the resulting reduced marginal polytope. Remarkably, this projection preserves enough geometric structure to allow for detection and quantification of nonstabilizerness, while drastically reducing both measurement and computational complexity.

We develop a near-optimal algorithm to construct such projected stabilizer polytopes from a number of measurements that scales polynomially with the system size, while the classical post-processing cost scales exponentially only in the number m of selected Pauli observables. This approach simultaneously addresses the two aforementioned bottlenecks: it replaces full tomography with a shadow-like measurement scheme, and it reduces the complexity of the convex optimization problem by working with a significantly smaller polytope. It is characterized by the commutativity graph of Pauli measurements, similar to the ideas developed in Refs. [21–24]. We illustrate the general framework with several explicit examples, demonstrating that our method reliably quantifies nonstabilizerness in regimes far beyond the reach of existing techniques.

The paper is organized as follows. In Sec. II, we review the essentials of stabilizer resource theory and the robustness of magic (RoM), the central monotone of nonstabilizerness employed throughout this work. In Sec. III, we introduce the reduced stabilizer polytope and present a near-optimal algorithm for its construction, together with the associated nonstabilizerness monotone. We also provide a detailed analysis of several reduced polytopes of particular interest. In Sec. IV, we address the computational complexity of the problem, proving that even for reduced stabilizer polytopes there exists no polynomial-time algorithm to decide whether a given set of correlations lies inside the polytope. In Sec. V B, we apply our general framework to selected cases of interest, computing the nonstabilizerness of ground states of several Hamiltonians using only the local observables required to determine their energy. Finally, in Sec. VI, we summarize our findings and highlight directions for future research.

II. STABILIZER RESOURCE THEORY AND MONOTONES

In this section, we review the basic ingredients underlying the definition of stabilizer states and summarize the relevant resource monotones. We consider systems of n qubits, described by the Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$. We denote by $\mathcal{P}_n$ the n -qubit Pauli group, defined as the group generated by tensor products of single-qubit Pauli operators, including overall phases. When convenient, we will refer to $\tilde{\mathcal{P}}_n$ as the Pauli group modulo phases, i.e., the Pauli group without signs.

A stabilizer group $\mathcal{S} \subset \mathcal{P}_n$ is an Abelian subgroup of the Pauli group such that $-1 \notin \mathcal{S}$. If $\text{rank}(\mathcal{S}) = n$, the group $\mathcal{S}$ uniquely defines a pure stabilizer state $|\psi^{\mathcal{S}}\rangle$ as the joint $+1$ eigenstate of all its elements, that is,

$$s|\psi^{\mathcal{S}}\rangle = +|\psi^{\mathcal{S}}\rangle \quad \forall s \in \mathcal{S}. \quad (1)$$

Because pure stabilizer states are in one-to-one correspondence with stabilizer groups of maximal rank, we denote both sets by stab_n .

The set of (possibly mixed) stabilizer states is defined as the convex hull of pure stabilizer states,

$$\begin{aligned} \text{STAB}_n &\equiv \text{conv}(\text{stab}_n) \\ &= \left\{ \sum_{\mathcal{S} \in \text{stab}_n} p_{\mathcal{S}} \psi^{\mathcal{S}} \mid \vec{p} \in \Delta_{|\text{stab}_n|} \right\}, \end{aligned} \quad (2)$$

where $\Delta_{|\text{stab}_n|} = \{\vec{p} \in \mathbb{R}^{|\text{stab}_n|} \mid p_{\mathcal{S}} \geq 0, \sum_{\mathcal{S}} p_{\mathcal{S}} = 1\}$ denotes the probability simplex and $\psi^{\mathcal{S}} = |\psi^{\mathcal{S}}\rangle\langle\psi^{\mathcal{S}}|$. The set STAB_n thus forms a polytope in the space of quantum states; the above characterization as the convex hull of finitely many extremal points is known as its V -representation [25].

There exists a natural class of unitaries that preserve the stabilizer polytope and form a group, known as the Clifford group. These unitaries are defined as those that conjugate Pauli operators into Pauli operators,

$$\mathcal{C}\ell_n \equiv \{U \in U(\mathcal{H}) \mid \forall P \in \mathcal{P}_n, \exists P' \in \mathcal{P}_n \text{ s.t. } UPU^\dagger \propto P'\}. \quad (3)$$

A quantum channel $\mathcal{E}$ that preserves STAB_n —that is, $\mathcal{E}(\psi) \in \text{STAB}_n$ for all $\psi \in \text{STAB}_n$ —is called a *stabilizer operation*. Any such channel can be decomposed into a sequence of Clifford unitaries, computational-basis measurements, and classical post-processing. Together, the stabilizer polytope and the set of stabilizer operations define the stabilizer resource theory [12], in which STAB_n constitutes the set of free states and stabilizer operations are the free operations.

A generic quantum state ψ is said to be *nonstabilizer* if $\psi \notin \text{STAB}_n$. Nonstabilizerness constitutes the resource of the stabilizer resource theory, in direct analogy with entanglement, coherence, nonlocality, and other quantum resources [26]. To quantify this resource, one introduces a *stabilizer monotone*, namely a real-valued function f on quantum states satisfying the following properties: (i) all stabilizer states attain the same value f_0 , and (ii) $f(\mathcal{E}(\psi)) \leq f(\psi)$ for all stabilizer operations $\mathcal{E}$. We say that f is *faithful* if $f(\psi) \geq f_0$ for all states, with equality if and only if $\psi \in \text{STAB}_n$.

Various stabilizer monotones have been proposed in the literature. These include measures based on the trace distance [27] and the relative entropy [12], which are generic constructions for convex quantum resource theories. Such monotones can typically be expressed as

$$f(\psi) = \min_{\sigma \in \text{STAB}_n} D(\psi, \sigma), \quad (4)$$

where $D(\cdot, \cdot)$ denotes a convex distance measure (or, in the case of concave entropic measures, an analogous expression in which the optimization is a maximization over stabilizer states). For simpler, non-optimizing entropic quantifiers, one must restrict to the pure-state stabilizer resource theory [28], as is also common in the resource theory of entanglement [1].

By Eq. (2), optimization over stabilizer states requires $|\text{stab}_n| = 2^{\Theta(n^2)}$ data even to specify $\sigma \in \text{STAB}_n$ in the V -representation [5]. Moreover, computing the monotone itself typically requires tomographically complete information about the state ψ —as is the case for both the relative entropy [12] and the trace distance [27], for example—entailing an experimental overhead scaling as $O(4^n)$.

Indeed, recently in Ref. [13], it was noted that deciding if a quantum state is in the stabilizer polytope requires $\exp(n^2)$ time complexity, under standard complexity-theoretic assumptions. They have also shown that this lower bound is actually reached by computing a monotone, the RoM, defined for a quantum state ψ as [29]

$$\text{RoM}(\psi) = \min_{\vec{x}} \left\{ \|\vec{x}\|_1 \mid \psi = \sum_{S \in \text{stab}_n} x_S \psi^S \right\}, \quad (5)$$

where the vector $\vec{x} \in \mathbb{R}^{|\text{stab}_n|}$ specifies a stabilizer pseudomixture of ψ , and $\|\vec{x}\|_1 = \sum_{S \in \text{stab}_n} |x_S|$ is referred to as its negativity. Since $\text{tr}\psi = 1$, the coefficients necessarily satisfy $\sum_{S \in \text{stab}_n} x_S = 1$, implying the lower bound $\text{RoM}(\psi) \geq 1$. We will use this monotone as our figure of merit.

The robustness of magic is a faithful stabilizer monotone [29], that is, the minimum value is attained if and only if ψ is a stabilizer state, and the minimization ensures monotonicity under all stabilizer operations. It was also

shown that the evaluation of the RoM can be formulated as a linear program (LP),

$$\text{RoM}(\psi) = \min_{\vec{x}} \|\vec{x}\|_1, \text{ s.t. } \mathbf{A}\vec{x} = \vec{b}, \quad (6)$$

where the elements of the rectangular matrix $\mathbf{A}$ are given by $A_{P,S} = \text{tr}(P\psi^S)$ and the vector $\vec{b}$ is defined as $b_P = \text{tr}(P\psi)$. Despite this reformulation, the problem remains plagued by the same fundamental limitations discussed above. In its current form, it requires an exponential amount of experimental data and computational resources to be evaluated.

Beyond its role as a resource monotone, the RoM also quantifies the classical simulability of quantum states via quasiprobability methods [30]. In particular, it was shown that expectation values of the form $\text{tr}(P\mathcal{E}(\psi))$ for stabilizer operations $\mathcal{E}$ can be estimated within additive error δ and with failure probability at most ϵ in $\text{poly}(n)$ time, provided that

$$N \geq \frac{2}{\delta^2} \text{RoM}(\psi)^2 \ln\left(\frac{2}{\epsilon}\right), \quad (7)$$

independent samples are drawn from the probability distribution defined by $p_S = |x_S|/\|\vec{x}\|_1$ [29].

In the next section, we turn to the projection of the stabilizer resource theory in settings where tomographic completeness is not available. In particular, we discuss how to construct suitable projections of the stabilizer polytope and define the corresponding RoM measure.

III. REDUCED STABILIZER POLYTOPE AND ROBUSTNESS

In realistic experimental settings, access to tomographically complete data is rarely available. Instead, measurements are typically constrained by the physical architecture of the platform, calibration requirements, and noise considerations, resulting in access to only a limited set of observables. These observables are often fixed in advance, for instance, by native measurement bases, locality constraints, or readout fidelities, and may scale at most polynomially with the system size. As a consequence, any characterization of nonstabilizerness that fundamentally relies on full state tomography is incompatible with the operational regime of contemporary quantum devices. This motivates the following shift in perspective: rather than attempting to reconstruct the full quantum state, we ask what can be inferred about nonstabilizerness from partial information alone. In particular, given expectation values of a restricted set of experimentally accessible observables, can one still certify and quantify nonstabilizerness in a meaningful and resource-efficient way? Addressing this question requires rethinking the geometry of the stabilizer polytope under projection

onto lower-dimensional subspaces, which is the approach we will develop in the following. The key idea behind it is to construct a *projected stabilizer polytope*.

We begin by assuming that the experimenter has access to only m Pauli operators, collected into the set

$$\mathcal{M} = \{P_1, P_2, \dots, P_m\}, \quad (8)$$

where we always take $m \leq \text{poly}(n)$. Given a quantum state ψ , we may represent it by its Pauli expectation values,

$$\vec{\psi} \equiv (\langle P \rangle_\psi)_{P \in \tilde{\mathcal{P}}_n} \in [-1, 1]^{4^n}. \quad (9)$$

In this representation, the stabilizer polytope can be written as

$$\text{conv}(\vec{\psi}^S)_{S \in \text{stab}_n} \subset [-1, 1]^{4^n}.$$

Denoting by $\Pi_{\mathcal{M}}$ the projection that maps a vector $\vec{v} \in [-1, 1]^{4^n}$ onto the components corresponding to the observables in $\mathcal{M}$, we define the *projected stabilizer polytope* as the image of STAB_n under this projection,

$$\text{STAB}_n(\mathcal{M}) \equiv \Pi_{\mathcal{M}}[\text{STAB}_n]. \quad (10)$$

Clearly, if $\mathcal{M} = \tilde{\mathcal{P}}_n$, one recovers the full stabilizer polytope, i.e., $\text{STAB}_n(\tilde{\mathcal{P}}_n) = \text{STAB}_n$.

A noteworthy property of this construction is that the resulting projected polytope depends only weakly on the total number of qubits. In particular, we have the following result.

Lemma 1. Let $\{P_1, P_2, \dots, P_m\}$ be a set of n -qubit Pauli operators, and let $n' \geq n$. Then

$$\text{STAB}_n(\{P_i\}_{i=1}^m) = \text{STAB}_{n'}(\{P_i \otimes \mathbb{1}_{n'-n}\}_{i=1}^m), \quad (11)$$

where the identity acts on the remaining $n' - n$ qubits.

The proof is given in Appendix A 1. This result suggests that the projected stabilizer polytope depends only on the algebraic relations among the Pauli operators in $\mathcal{M}$, rather than on their explicit tensor-product embedding into the full Hilbert space. We will make this intuition precise in the next subsection.

Since specifying the measurement set $\mathcal{M}$ already fixes the number of qubits required to realize the corresponding observables, we will henceforth suppress the explicit dependence on n and write $\text{STAB}(\mathcal{M}) \equiv \text{STAB}_n(\mathcal{M})$. We further denote by $\text{stab}(\mathcal{M}) \subseteq \text{STAB}(\mathcal{M})$ a generating set for the V -representation of the projected polytope, namely

$$\text{STAB}(\mathcal{M}) = \text{conv}[\text{stab}(\mathcal{M})] \equiv \text{conv}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{|\text{stab}(\mathcal{M})|}). \quad (12)$$

Because the projection can map distinct stabilizer states to the same point—or render some points redundant via

convex combinations—one generally has $\text{stab}(\mathcal{M}) \subsetneq \Pi_{\mathcal{M}}\text{stab}_n$. For instance, it may occur that

$$\Pi_{\mathcal{M}}\vec{\psi}^S = \sum_{S' \neq S} p_{S'} \Pi_{\mathcal{M}}\vec{\psi}^{S'}.$$

We postpone the explicit construction of the V -representation $\text{stab}(\mathcal{M})$ to the next subsection and treat it as an input for the remainder of the present discussion.

Observing correlations that lie outside the projected stabilizer polytope provides a bona fide witness of non-stabilizerness. Indeed, for a given state ψ ,

$$(\langle P_1 \rangle_\psi, \dots, \langle P_m \rangle_\psi) \notin \text{STAB}(\mathcal{M}) \Rightarrow \psi \notin \text{STAB}_n. \quad (13)$$

While this criterion suffices for witnessing nonstabilizerness, it does not provide a quantitative measure. To go beyond mere detection, we seek an analog of a stabilizer monotone that can be evaluated solely from the expectation values of the observables in $\mathcal{M}$.

To this end, we introduce a natural generalization of the robustness of magic in Eq. (5), which we call the *reduced robustness of magic*,

$$\text{RoM}_{\mathcal{M}}(\psi) = \min_{\vec{x} \in \mathbb{R}^{|\text{stab}(\mathcal{M})|}} \|\vec{x}\|_1, \quad (14)$$

$$\begin{aligned} \text{s.t. } \Pi_{\mathcal{M}}\vec{\psi} &= \sum_{j=1}^{|\text{stab}(\mathcal{M})|} x_j \vec{v}_j, \\ \sum_{j=1}^{|\text{stab}(\mathcal{M})|} x_j &= 1. \end{aligned} \quad (15)$$

As in the full RoM, this definition leads to a linear program, but now expressed exclusively in terms of the measurement data associated with $\mathcal{M}$. It is clear that $\mathcal{M} = \tilde{\mathcal{P}}_n$ implies $\text{RoM}_{\mathcal{M}} = \text{RoM}$. Also, the normalization condition is automatically satisfied if we include $\mathbb{1} \in \mathcal{M}$, which we assume from now on.

The reduced robustness of magic satisfies the following properties.

Lemma 2. Given two sets of n -qubit Pauli operators $\mathcal{M}, \mathcal{N}$ and a quantum state ψ , the reduced robustness of magic obeys:

- (1) Faithfulness: $\text{RoM}_{\mathcal{M}}(\psi) \geq 1$, with equality if and only if there exists a stabilizer state $\phi \in \text{STAB}_n$ such that $\langle P \rangle_\psi = \langle P \rangle_\phi$ for all $P \in \mathcal{M}$.
- (2) Stabilizer monotonicity: $\text{RoM}_{\mathcal{M}}(\mathcal{E}(\psi)) \leq \text{RoM}_{\mathcal{M}}(\psi)$ for all trace-preserving stabilizer operations $\mathcal{E}$.

(3) Submultiplicativity and convexity:

$$\begin{aligned} \text{RoM}_{\mathcal{M} \otimes \mathcal{N}}(\psi \otimes \rho) &\leq \text{RoM}_{\mathcal{M}}(\psi) \text{RoM}_{\mathcal{N}}(\rho), \\ \text{RoM}_{\mathcal{M}}\left(\sum_j p_j \psi_j\right) &\leq \sum_j |p_j| \text{RoM}_{\mathcal{M}}(\psi_j). \end{aligned}$$

(4) Measurement monotonicity:

$$\mathcal{M}' \subseteq \mathcal{M} \Rightarrow \text{RoM}_{\mathcal{M}}(\psi) \geq \text{RoM}_{\mathcal{M}'}(\psi).$$

In particular, given any measurement set $\mathcal{M}$, we have the lower bound $\text{RoM}(\psi) \geq \text{RoM}_{\mathcal{M}}(\psi)$.

The proof is provided in Appendix A 2. Properties 2 and 3 show that the reduced robustness of magic is indeed a valid stabilizer monotone, inheriting many of the structural features of the full robustness of magic [29]. The most significant insights, however, arise from properties 1 and 4.

Property 1 highlights that $\text{RoM}_{\mathcal{M}}$ is generally *not* faithful: the condition $\text{RoM}_{\mathcal{M}}(\psi) = 1$ guarantees only that there exists a stabilizer state reproducing the statistics of ψ on the restricted measurement set $\mathcal{M}$. Consequently, depending on the choice of observables, nonstabilizer states may remain indistinguishable from stabilizer states at the level of accessible Pauli expectations. A precise characterization of this phenomenon will be developed in Sec. IV.

Property 4 ensures that the measurement-limited robustness nonetheless provides a meaningful lower bound on the nonstabilizerness of the full state. In particular, it implies a corresponding lower bound on the classical simulation cost: by measuring only $m \leq \text{poly}(n)$ Pauli operators, one can already infer a sample-complexity bound of the form $N \geq f(\epsilon, \delta) \text{RoM}_{\mathcal{M}}(\psi)^2$ for quasiprobability-based simulation, directly from Eq. (7).

With the reduced robustness of magic, we thus resolve the first of the two central obstacles in nonstabilizerness estimation: the requirement of tomographically complete data. The monotone $\text{RoM}_{\mathcal{M}}$ can be evaluated using any experimentally accessible set of measurements, albeit with a strength that depends on the choice of $\mathcal{M}$. However, the second obstacle, the computational complexity, still remains at this point. Indeed, evaluating $\text{RoM}_{\mathcal{M}}$ still requires access to a V -representation of the projected stabilizer polytope.

A straightforward upper bound on the size of the linear program in Eq. (14) is obtained via a direct “top-down” construction,

$$\text{stab}(\mathcal{M}) \subseteq \Pi_{\mathcal{M}}(\text{stab}_n), \quad (16)$$

namely by projecting all $|\text{stab}_n| = 2^{\Theta(n^2)}$ vertices of the full stabilizer polytope. This approach makes clear that, in the worst case, evaluating the reduced RoM is no easier than computing the original RoM: although the ℓ_1 -norm

minimization can be solved in time polynomial in the number of constraints [31], the number of vertices remains exponentially large. In the following, we show that by exploiting the algebraic structure of the measurement set $\mathcal{M}$, one can construct compressed V -representations of the projected stabilizer polytope, thereby addressing the remaining computational bottleneck.

A. Bottom-up reconstruction

We will now show that there is a purely combinatorial construction of the reduced polytopes. First, define the frustration graph [21–24] of $\mathcal{M}$ as the graph $G_{\mathcal{M}}$ with $V(G_{\mathcal{M}}) = \mathcal{M}$, and edges given by

$$E(G_{\mathcal{M}}) \equiv \{\{P_i, P_j\} \in \mathcal{M}^{\times 2} | P_i P_j = -P_j P_i\}. \quad (17)$$

Then, we can show:

Theorem 3. Let $\mathcal{M}$ be a set of mn -qubit Pauli strings. Then, the reduced polytope has a V -representation given by

$$\text{STAB}(\mathcal{M}) = \text{conv}\{\vec{v}_{S,f}\}_{S \in I_{\max}(G_{\mathcal{M}}), f \in B_S}, \quad (18)$$

where $I_{\max}(G_{\mathcal{M}})$ is the set of (maximally) independent sets of $G_{\mathcal{M}}$, and

$$B_S = \left\{ f \in \{\pm\}^S \mid \prod_{P \in S'} f_P P \neq -1, \forall S' \subseteq S \right\}, \quad (19)$$

denotes all bitstring assignments over the independent set S that do not have a negative Pauli product on the subsets, and the vector $\vec{v}_{S,f}$ is defined as

$$(\vec{v}_{S,f})_P = \begin{cases} f_P & \text{if } f_P P \in S, \\ 0 & \text{otherwise.} \end{cases} \quad (20)$$

Furthermore, the set $\text{stab}(\mathcal{M}) = \{\vec{v}_{S,f}\}$ has size:

$$|\text{stab}(\mathcal{M})| \leq 2^{\min\{n,m\} + \min\{cm, (1/2 + o(1))n^2\}}, \quad (21)$$

where $c = \log 3/3$. Without further restrictions on $\mathcal{M}$, this bound is tight and reached by the set of marginals

$$\mathcal{M} = \bigcup_{i=1}^n \{X_i, Y_i, Z_i\}. \quad (22)$$

We present the full proof in Appendix B, and here we outline the main ideas underlying the construction. Starting from the original “top-down” V -representation of the reduced polytope in Eq. (16), one may view the extremal points of $\text{STAB}(\mathcal{M})$ as arising from the correlations of Paulis drawn from $\mathcal{M}$ of pure stabilizer states. This perspective naturally leads to the question of which stabilizer structures are already encoded in the

measurement set itself. Namely, which subsets of $\mathcal{M}$ can simultaneously fix expectation values $\langle P_i \rangle = \pm 1$.

This intuition is formalized through the notion of *closed commuting subsets* [32], defined as

$$\mathcal{C}(\mathcal{M}) = \{Q \subseteq \pm\mathcal{M} \mid Q = \langle Q \rangle \cap \pm\mathcal{M}, -1 \notin \langle Q \rangle\}, \quad (23)$$

where $\pm\mathcal{M} = \mathcal{M} \sqcup (-\mathcal{M})$ and $-\mathcal{M} = \{-P_1, -P_2, \dots, -P_m\}$. The terminology reflects the two defining properties of any $Q \in \mathcal{C}(\mathcal{M})$. First, the set is *closed*, in the sense that the subgroup it generates, $\langle Q \rangle$, intersects $\pm\mathcal{M}$ only on the elements of Q itself. Second, it is commuting, since the condition $-1 \notin \langle Q \rangle$ implies that $\langle Q \rangle$ is an Abelian subgroup of the Pauli group and therefore defines a stabilizer group [33]. Furthermore, we will consider *maximal commuting subsets* (MCSs), where we impose the additional property that, given $Q \subseteq Q'$, then $Q = Q'$. We denote this set by $\mathcal{C}_{\max}(\mathcal{M})$.

Operationally, the elements of $\mathcal{C}_{\max}(\mathcal{M})$ may be interpreted as measurement contexts [34]: maximal subsets of Pauli observables within $\mathcal{M}$ that can simultaneously attain definite values. The central and somewhat surprising result, proved in Appendix B, is that these MCSs suffice to generate the reduced stabilizer polytope. More precisely, one can construct a V-representation such that

$$\text{stab}(\mathcal{M}) \cong \mathcal{C}_{\max}(\mathcal{M}). \quad (24)$$

This characterization makes explicit that the complexity of the reduced polytope is governed by the algebraic and combinatorial structure of the measurement set $\mathcal{M}$, rather than by the exponentially large set of pure stabilizer states appearing in the top-down construction.

The remainder of the proof establishes a connection between MCSs and a purely combinatorial problem: finding consistent independent sets in an associated frustration graph. This reformulation makes clear why we refer to the present approach as bottom-up. Rather than projecting the full stabilizer polytope without regard to measurement structure, the stabilizer correlations accessible to a restricted measurement set are reconstructed directly from the relations among the measurements themselves. Moreover, the combinatorial expression of the convex hull in Eq. (18) naturally suggests an efficient algorithmic strategy: construct the frustration graph and enumerate its consistent independent sets.

We defer a detailed discussion of this algorithm to Sec. III D. Instead, in the next section, we illustrate the geometry of reduced stabilizer polytopes for small measurement sets, which provides valuable intuition for how their structure emerges from the underlying algebraic relations among the observables.

B. On small polytopes and Pauli commutativity

We now explore reduced polytopes for $m \leq 3$, since these are the cases that can be visualized and hence easily understood. All three cases are plotted in Fig. 1.

For $m = 1$, we have $\mathcal{M} = \{P_1\}$. We have the frustration graph of one vertex, which also composes the single maximal independent subset. Since we can either assign $f_{P_1} = 1$ or $f_{P_1} = -1$, we have:

$$\text{STAB}(\{P_1\}) = \text{conv}(-1, 1) = [-1, 1], \quad (25)$$

which is indeed the full range of $\langle P_1 \rangle$, since $\|P_1\| = 1$. A single Pauli expectation value cannot witness non-stabilizerness.

The first nontrivial case is for $m = 2$. If $[P_1, P_2] = 0$, the graph is fully disconnected and there is a single maximally independent set, $I_{\max}(G_{\mathcal{M}}) = \{\mathcal{M}\}$. We have two further cases: If $P_1 \neq -P_2$, they are fully independent and:

$$\text{STAB}(\{P_1, P_2\}) = \text{conv}(\{(\pm 1, \pm 1)\}) = [-1, 1]^2. \quad (26)$$

Otherwise, if $P_1 = -P_2$, we obtain $\text{conv}((1, -1), (-1, 1))$, forming the red line in Fig. 1(a), since those two are the consistent sign assignments for the two Paulis. In all those cases, again, $\langle P_1 \rangle$ and $\langle P_2 \rangle$ fully obtain all the admissible expectation values, and no nonstabilizerness can be detected. The other case is when $\{P_1, P_2\} = 0$, and then the independent sets are formed by single Paulis. In this case, we have:

$$\text{STAB}(\{P_1, P_2\}) = \text{conv}(\{(\pm 1, 0), (0, \pm 1)\}), \quad (27)$$

as illustrated by the second example in Fig. 1(a). Now, one can see that there is a range of correlations for which nonstabilizerness can be witnessed, since the corresponding polytope does not fully span the operator spectrum of both Paulis.

For $m = 3$, different commutation structures can emerge. We might have $|E(G_{\mathcal{M}})| \in \{0, 1, 2, 3\}$, which labels the 4 possible graph permutation classes based on the number of edges. The cases with $|E(G_{\mathcal{M}})| \in \{0, 2, 3\}$ are illustrated in Fig. 1(b). Note that assuming negative Pauli product constraints, such as $P_1 P_2 = -1$, it would correspond also to a projection, since it does correspond to imposing the extra constraint $\langle P_1 \rangle + \langle P_2 \rangle = 0$. For $m \geq 3$ however, more interesting constraints start to appear. For example, we can have $P_1 P_2 P_3 = \pm 1$, and it cannot be written linearly in the expectation values. In the figure, the fully commuting case is shown with and without this extra constraint. Note that, for the fully anticommuting case, we have the octahedron that also defines STAB_1 , since indeed it is the reduced polytope for the single-qubit Pauli triple: $\text{STAB}_1 = \text{STAB}(\{X, Y, Z\})$.

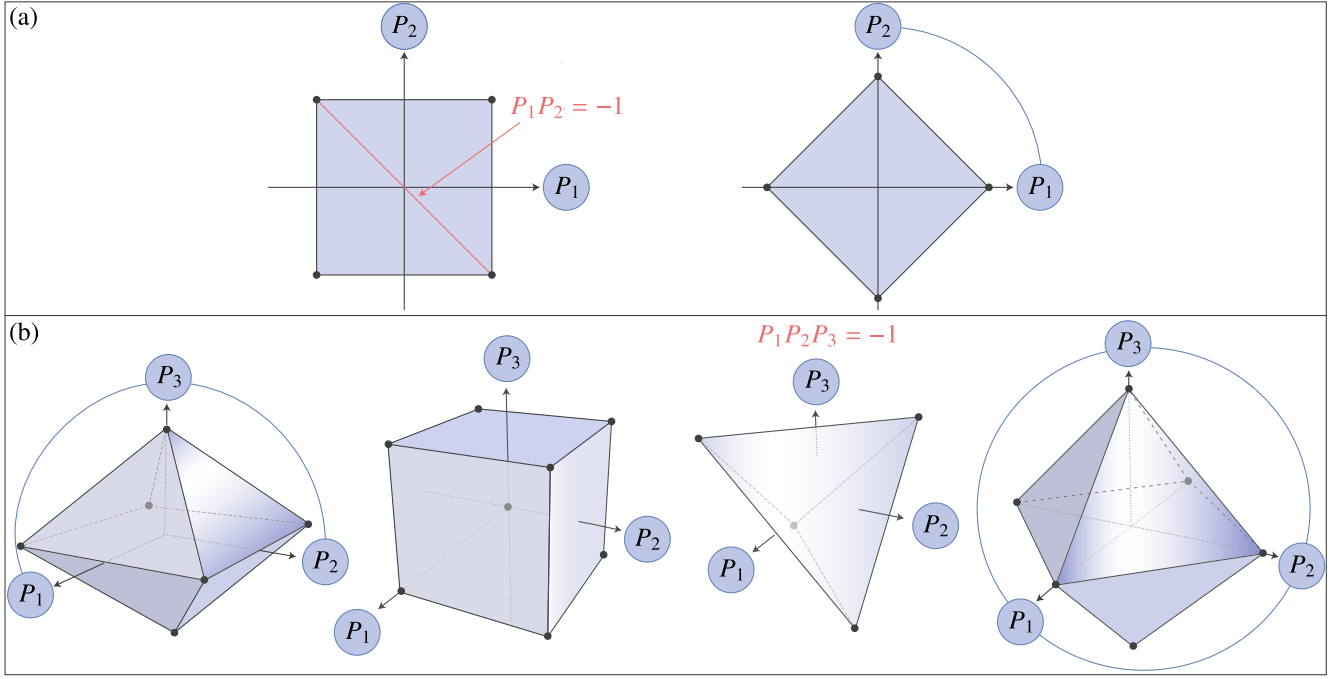


FIG. 1. Illustration of some of the reduced polytopes for $m \in \{2, 3\}$. We draw the polytopes in $[-1, 1]^m$, with the axis labels as vertices of the frustration graph drawn in overlap. In (a), we show the two reduced polytopes coming from the two possible frustration graphs, where in red we show the corresponding reduced polytope if the extra condition $P_1 P_2 = -1$ is added in the commuting case. In (b), we highlight four possibilities: The case of a graph of two edges, the fully commuting case, where the correlations span the full $[-1, 1]^3$, the case where the commuting tuple also satisfies the constraint $P_1 P_2 P_3 = -1$ and the fully anti-commuting case that reduces to the definition of STAB_1 .

A very clear message that those examples show is that (independent) commuting sets of Paulis are trivial as witnesses. Indeed, we can show that this is a generic result:

Proposition 4. Let $\mathcal{M} = \{P_1, P_2, \dots, P_m\}$ be a set of n -qubit Pauli strings that are independent, meaning that there is no $S \subseteq [m]$ such that $\prod_{j \in S} P_j = \pm 1$. If $\mathcal{M}$ is commuting,

$$\text{STAB}(\mathcal{M}) = [-1, 1]^m. \quad (28)$$

Proof. The proof follows from recognizing there is a single maximally independent subset, $I_{\max}(G_{\mathcal{M}}) = \{\mathcal{M}\}$. By independence, every $f \in \{\pm 1\}^{|\mathcal{M}|=m}$ is in $\mathcal{B}_{\mathcal{M}}$, since all of the sign assignments do satisfy the product condition. Hence, by Theorem 3, the reduced polytope is the convex hull of the Hamming cube:

$$\text{STAB}(\mathcal{M}) = \text{conv}\{\underbrace{(\pm 1, \pm 1, \dots, \pm 1)}_{\text{length } m}\}, \quad (29)$$

which is exactly $[-1, 1]^m$. ■

Then, a natural question to ask is whether the amount of anticommutativity generates more constrained polytopes. We can show that the reduced polytopes of fully anti-commuting sets are the most basic building blocks of reduced polytopes:

Lemma 5. Let $\mathcal{M} = \{P_1, P_2, \dots, P_m\}, \mathcal{M}' = \{P'_1, P'_2, \dots, P'_m\}$ be two sets of n -qubit Pauli operators with $P_j \neq -P_i$ for all $i, j \in [m]$. Then, if $\mathcal{M}$ is fully anti-commuting,

$$\text{STAB}(\mathcal{M}) \subseteq \text{STAB}(\mathcal{M}'). \quad (30)$$

Moreover, $\mathcal{M}$ can only be realized with $n \geq \lfloor m/2 \rfloor$ qubits, where the inequality is tight.

The proof is presented in Appendix A 3. In particular, the proof shows that if $\mathcal{M}$ is fully anticommuting, then $\text{STAB}(\mathcal{M}) = \text{conv}\{\pm e_1, \dots, \pm e_m\} = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\|_1 \leq 1\}$. Equivalently, the facets are $\sum_{j=1}^m s_j \langle P_j \rangle \leq 1, s_j \in \{\pm 1\}$. Thus a fully anticommuting measurement set gives an explicit analytic family of reduced-polytope witnesses. However, that does not imply that fully anticommuting sets are optimal for magic detection in any situation. Even if the inclusion in Eq. (30) holds, we do not necessarily have that $\text{RoM}_{\mathcal{M}}(\psi)$ is greater than $\text{RoM}_{\mathcal{M}'}(\psi)$; the set $\mathcal{M}'$ can be better suited to witness the magic of a particular state ψ than $\mathcal{M}$.

C. Triangle criterion as a reduced stabilizer polytope

We now isolate a class of reduced polytopes corresponding exactly with the triangle criterion, introduced in Ref. [35]:

Theorem 6. Let ρ be a quantum state. Suppose there exist three pure stabilizer states $\{\psi_1, \psi_2, \psi_3\}$ satisfying

$$\text{tr}(\psi_i \psi_j) = \frac{1}{2} \quad \text{for all } i \neq j, \quad (31)$$

such that

$$\text{tr}(\rho \psi_1) > \text{tr}(\rho \psi_2) + \text{tr}(\rho \psi_3). \quad (32)$$

Then ρ is a magic state.

In Appendix E we show that such a triple of neighboring stabilizer states corresponds to a common stabilizer group S of rank $n - 1$, with three additional Pauli generators P_i such that the stabilizer of ψ_i is $S_i := \langle S \cup P_i \rangle$. A natural question is how much information is contained in the reduced polytope built from the measured Pauli set $\mathcal{M} = S_1 \cup S_2 \cup S_3$. In addition to the facet (32), the reduced polytope contains exactly all the facets corresponding to changing the signs of the generators in each S_i .

More precisely, after a Fourier transform over the Abelian group S in the Pauli coefficient coordinates, the new coordinates split into syndrome blocks, corresponding to irreducible characters of S . For each character $\chi: S \rightarrow \{\pm 1\}$, i.e., every possible measurement outcome of the common stabilizer subgroup S , we can define the syndrome probability p_χ and the logical Bloch coordinates q_{χ, P_j} . In these coordinates, the reduced stabilizer polytope is

$$\text{STAB}(\mathcal{M}) = \left\{ (p_\chi, q_{\chi, P_1}, q_{\chi, P_2}, q_{\chi, P_3})_{\chi \in S^*} : \sum_{\chi} p_\chi = 1, \right. \\ \left. |q_{\chi, P_1}| + |q_{\chi, P_2}| + |q_{\chi, P_3}| \leq p_\chi \quad \forall \chi \right\}.$$

Thus each syndrome sector carries a copy of a logical one-qubit stabilizer octahedron. It follows that the facets of $\text{STAB}(\mathcal{M})$ are exactly

$$p_\chi + s_1 q_{\chi, P_1} + s_2 q_{\chi, P_2} + s_3 q_{\chi, P_3} \geq 0, \quad (33)$$

for $\chi \in S^*$ and $s_1, s_2, s_3 \in \{\pm 1\}$. Equivalently, in the original Pauli coordinates, these inequalities read

$$\sum_{\chi \in S} \chi(Q) [r_Q + s_1 r_{QP_1} + s_2 r_{QP_2} + s_3 r_{QP_3}] \geq 0. \quad (34)$$

These are precisely the triangle inequalities for each fixed common syndrome and arbitrary sign choice of P_i . The character χ chooses the syndrome sector of the common stabilizer context S , while the signs (s_1, s_2, s_3) choose one of the eight facets of the $\{P_1, P_2, P_3\}$ logical one-qubit stabilizer octahedron. The proof is given in Appendix E.

D. Explicit algorithm

Now that some intuition is gained on how the combinatorial problem of the frustration graph fixes the polytope, we present the general algorithm. The detailed analysis for the runtime is left for Appendix C.

Theorem 7. Let $\mathcal{M} = \{P_1, P_2, \dots, P_m\}$ be a set of n -qubit Pauli strings. There is a classical algorithm, sketched in Algorithm 1, that takes $\mathcal{M}$ as an input and outputs the V -representation of Theorem 3 with runtime

$$T = p(n, m) 2^{O(\min\{m, n\} + \min\{m, n^2\})}, \quad (35)$$

where $p(n, m)$ is bounded by some polynomial in n and m .

Note that the runtime of the algorithm to list all $|\text{stab}(\mathcal{M})| m$ -dimensional vectors has a similar upper bound as the length of the list itself. This is due to the way Algorithm 1 works: We list all the possible independent subsets and build the sign assignments, that give the exponential scaling. The polynomial overhead mostly comes from building the compatible sign assignments and the vectors $\vec{v}_{S, f}$.

There are some interesting regimes: for a constant number of measurements, $m = O(1)$, the algorithm runs in linear time $T = O(n)$. If we allow logarithmic many measurements $m = O(\log n)$, the time is still polynomial $T = O(\log(n) n^{1+\log 3/3})$. However, if we consider a quadratic (or larger) number of measurements $m = \Omega(n^2)$, then the bottleneck of the algorithm becomes the possibly exponentially many maximal commuting subsets, and the time scales with the size of $|\text{stab}_n|$, giving $T = O(p(m, n) 2^{O(n^2)})$. Modulo a polynomial overhead, this already reaches the mentioned minimum time complexity for determining membership in the full stabilizer polytope when tomographic information is available [13]. The same regimes are captured in the size of the reduced constraint polytope. From Eq. (21), $|\text{stab}(\mathcal{M})| \leq 2^{\min\{n, m\} + \min\{cm, (1/2 + o(1))n^2\}}$. Thus for $m = O(1)$ the number of vertices is bounded independently of n , up to the polynomial cost of manipulating the Pauli strings, while for $m = O(\log n)$ the V -representation has polynomial size. For $m = O(n)$, the reduced polytope may already

ALGORITHM 1. V -representation of $\text{STAB}(\mathcal{M})$.

Input: $\mathcal{M} = \{P_1, P_2, \dots, P_m\}$ as the set of Pauli measurements;

Output: $\text{stab}(\mathcal{M}) = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{|\text{stab}(\mathcal{M})|}\}$ as defined in Theorem 3.

- 1 Initialize $V \leftarrow \{\}$;
 - 2 Compute $E(G_{\mathcal{M}})$;
 - 3 Find $I_{\max}(G_{\mathcal{M}})$;
 - 4 **for** $S \in I_{\max}(G_{\mathcal{M}})$ **do**
 - 5 Compute all $f \in B_S$ and append $\vec{v}_{S, f}$ to V ;
 - 6 **return** V
-

have exponentially many vertices, although still parametrically fewer than the full stabilizer polytope. In the tomographically complete regime, $\mathcal{M} = \tilde{\mathcal{P}}_n$, the reduced polytope coincides with STAB_n , and the size and complexity reduce to those of the full stabilizer-polytope membership problem studied in Ref. [13].

As a corollary, we get the complexity of estimating Eq. (14): It is known that from the theory of linear programs that finding a $\vec{x} \in \mathbb{R}^m$ that ϵ -approximates a linear objective with M constraints can be found in $O(m(m+M)^{3/2} \log(1/\epsilon))$ time [31]. Then, to estimate the reduced RoM, we first obtain the V -representation of the reduced polytope from the set of Pauli measurements via the construction of Theorem 3, realized by Algorithm 1, and then run the linear program with $M = O(2^{\min\{m,n\} + \min\{cm, (1/2+o(1))n^2\}})$. Since the linear programming runtime dominates due to the extra power of $3/2$, we have:

Corollary 8. Given a set of expectation values of n -qubit Pauli strings $\mathcal{M}$ on a quantum state ψ , $(\langle P_1 \rangle_\psi, \langle P_2 \rangle_\psi, \dots, \langle P_m \rangle_\psi)$, with $m \leq \text{poly}(n)$, there is an algorithm that ϵ -approximates $\text{RoM}_{\mathcal{M}}(\psi)$ in $O(2^{c' \min\{m,n^2\}} \log(1/\epsilon))$ runtime, where $c' = O(1)$.

Note that, up to a polynomial-time overhead, our algorithm outputs a compressed representation of the polytope in time proportional to its maximal possible size. Since the 2^{cm} -length representation is known to be tight, this scaling indicates that the algorithm is near-optimal.

IV. STABILIZER MARGINAL PROBLEM AND COMPLEXITY LOWER BOUND

We will show in this section that under the assumption that $P \neq NP$, if there is an algorithm that decides if a set of Pauli correlations is inside the reduced polytope, it must run in

$$T = \omega(\text{poly}(n, m)) \quad (36)$$

time, meaning that there is no poly-time algorithm in the number of Pauli measurements to decide if the corresponding set of expectation values lies inside the reduced polytope or not. Doing so, we will make contact with a classical problem in quantum information theory, the quantum marginal problem.

First, we describe the task formulated as what is called a decision problem [36]:

Definition 1. We define the reduced stabilizer membership problem as the decision problem (rSMP) defined by

- (i) Inputs: A set of $m \leq \text{poly}(n)$ n -qubit Pauli strings, $\mathcal{M} = \{P_1, P_2, \dots, P_m\}$ and a vector $\vec{V} \in [-1, 1]^m$;
- (ii) Decision: Output YES if $\vec{V} \in \text{STAB}(\mathcal{M})$ and NO otherwise.

And, by Eq. (13), all the YES outputs to this problem match with all YES outputs with the stabilizer membership

problem, which decides if an n -qubit quantum state ψ is stabilizer or not. However, this variation only requires inputs of size $\text{poly}(n)$.

Before proceeding, let us give a quick review of what is called the quantum marginal problem. We assume to have n qubits and some, possibly overlapping, subsets $\{R_1, R_2, \dots, R_N\} \subseteq [n]$ of constant size, $|R_i| = k = O(1)$, and a list of density matrices $\{\rho_1, \rho_2, \dots, \rho_N\}$ supported on those subsets. The problem is to find if there is a global density matrix ρ such that it correctly marginalizes to the inputs within a $1/\text{poly}(n)$ precision. It was shown that this problem is QMA-hard [37], meaning that, even with quantum computers, it is expected that there is no efficient algorithm to solve it, under standard complexity-theoretic assumptions.

The quantum marginal problem is the natural generalization of a classical problem in probability theory, referred to as the classical marginal problem (CMP). The difference is that the inputs are probability distributions $\{\vec{p}_1, \vec{p}_2, \dots, \vec{p}_N\}$ each over a constant number of bits, and we ask if there is a global probability over n bits that marginalizes to each of the inputs. It is known that this is NP-complete [38,39].

Note the close similarity with our task, namely, determining whether there exists a global stabilizer state that reproduces the statistics of a given set of Pauli observables $\mathcal{M}$. This motivates us to formulate a variant of the quantum marginal problem restricted to the stabilizer setting [40]:

Definition 2. We define the stabilizer marginal problem (SMP) on n qubits as the problem defined by

- (i) Input: A set of stabilizer density matrices $\{\phi_1, \phi_2, \dots, \phi_N\}$, with $N \leq \text{poly}(n)$, $\phi_i \in \text{STAB}_{|R_i|}$, specified by

$$\phi_i = \sum_{S \in \text{stab}_{|R_i|}} p_S \psi^S; \quad \vec{p} \in \Delta_{\text{stab}_{|R_i|}}, \quad (37)$$

where $R_i \subseteq [n]$, $|R_i| \leq k = O(1)$ for all $i \in [N]$;

- (ii) Decision: Output YES if there exists a n -qubit stabilizer state $\phi \in \text{STAB}_n$ such that:

$$\text{tr}_{\bar{R}_i} \phi = \phi_i, \quad \forall i \in [N], \quad (38)$$

and NO otherwise.

The standard tool to relate problems of this kind is a polynomial-time reduction [41]. Concretely, if a decision problem A reduces to a problem B , then any algorithm that solves B can be used to solve A with only polynomial overhead. We establish a sequence of such reductions connecting the classical, stabilizer, and reduced membership problems:

Theorem 9. Denoting $\leq_{\text{poly}}$ for the polynomial-time reduction, the following are true:

- (1) $\text{CMP} \leq_{\text{poly}} \text{SMP}$;
- (2) $\text{SMP} \leq_{\text{poly}} \text{rSMP}$.

Thus, $\text{CMP} \leq_{\text{poly}} \text{rSMP}$, and therefore both SMP and rSMP are NP-hard.

The proof is described in Appendix D. This shows that, by lower bounding the hardness of the SMP, one gets a hardness result for the reconstruction problem. This shows our claim in Eq. (36): If $P \neq \text{NP}$, there is no polynomial-time algorithm to solve an NP-hard problem, and thus no way to solve the membership of the reduced stabilizer polytope in $\text{poly}(n, m)$ time.

V. EXAMPLES

In this section, we provide examples that illustrate the effectiveness of our method with some examples, illustrating its power.

A. On unstructured/random states

Before presenting examples that fall within our framework, let us first address the problem of detecting magic in random states. We have the following no-go result:

Lemma 10. Let $\mathcal{M} = \{P_1, P_2, \dots, P_m\}$ be a set of n -qubit Pauli operators and let ν be a state 2-design. Then,

$$\Pr_{\psi \sim \nu}[(\langle P_1 \rangle_\psi, \langle P_2 \rangle_\psi, \dots, \langle P_m \rangle_\psi) \in \text{STAB}(\mathcal{M})] \geq 1 - \Theta\left(\frac{m^3}{2^n}\right). \quad (39)$$

If $\nu = \text{Haar}$, then, the failure probability is $\Theta(m 2^{-c' 2^n / m^2})$, where $c' = O(1)$ is a constant.

The proof is presented in Appendix A 4. Recall that a 2-design is a distribution of pure states whose first two moments match the uniform Haar distribution. In an approximate setting, they appear quite naturally in random quantum circuits [42,43] and in thermalization [44].

We point out that this is an information-theoretic result in the following sense. If one has access only to expectation values, meaning linear observables on the state ψ , then the magic of typical states can only be witnessed when tomographic information is available. Otherwise, if $m \leq \text{poly}(n)$, the bound in Eq. (39) guarantees that the resulting correlations are almost surely stabilizer. We note that average-case upper bounds on linear magic detection for random induced states, such as those of Ref. [35], also upper bound the generic detection power of reduced polytopes, since any reduced-polytope test is a linear witness in $\text{span}(\mathcal{M})$, complementing the average-case no-go of Lemma 10.

This observation also rules out detecting magic from local marginals of random states, as the following immediate corollary shows:

Corollary 11. Let $|\Psi\rangle$ be Haar-random on $n + n_E$ qubits and $\rho = \text{tr}_E |\Psi\rangle\langle\Psi|$ the induced state on n qubits. For $A \subseteq [n]$ with $|A| = o(n)$, let $\mathcal{M}$ be the $m = 4^{|A|} - 1 = 2^{o(n)} - 1$

nontrivial Paulis supported on A . Then

$$\Pr[\Pi_{\mathcal{M}}(\rho) \notin \text{STAB}(\mathcal{M})] \leq \Theta(m 2^{-c' 2^{n+n_E}/m^2}). \quad (40)$$

This follows because each $P \in \mathcal{M}$ is supported on $A \subseteq [n]$, so $\langle P \rangle_\rho = \langle P \rangle_\Psi$. Hence Lemma 10 applies directly to the reduced correlations.

B. On structured examples: Gibbs and ground states

In this section, we illustrate the power of our algorithm by numerically computing the reduced RoM of ground states of Hamiltonians whose nonstabilizerness is recently discussed [15,45,46].

The general approach will be as follows: An n qubit Hamiltonian can be written as

$$H = \sum_{P \in \mathcal{M}'} w_P P, \quad (41)$$

where $\mathcal{M}'$ is a set of Pauli operators and $\{w_P\}_{P \in \mathcal{M}'}$ is a set of coefficients. Denote ψ_{gs} as a (or *the*, in the unique case) corresponding ground state. In experiments and numerical simulations, one of the most natural observables is the ground state energy:

$$E_0 = \langle H \rangle_{\psi_{\text{gs}}} = \sum_{P \in \mathcal{M}(H)} w_P \langle P \rangle_{\psi_{\text{gs}}}. \quad (42)$$

Indeed, we will assume that our measurement set is a subset $\mathcal{M} \subseteq \mathcal{M}(H)$, on the same footing as energy estimation, and is a weaker assumption when the inclusion is strict. For concreteness, we will focus with translationally-invariant, one-dimensional $k \leq 2$ -local Hamiltonians [48], that have the form:

$$H = \sum_{\alpha, \beta \in \{0,1,2,3\}} \sum_i J^{\alpha\beta} X_i^\alpha X_{i+1}^\beta, \quad (43)$$

where we referred to the components of $(X_i^0, X_i^1, X_i^2, X_i^3) = (1, X_i, Y_i, Z_i)$ in the sum. In particular, we will focus in the following family of examples:

$$H_{\text{ANNNI}} = -\sum_i (Z_i Z_{i+1} - k Z_i Z_{i+2} + g X_i), \quad (44)$$

$$H_{\text{XXZ}} = \frac{1}{4} \sum_n (X_n X_{n+1} + Y_n Y_{n+1} + \Delta Z_n Z_{n+1}) - \frac{h}{2} \sum_n X_n, \quad (45)$$

where all the couplings are assumed to be positive. The first Hamiltonian is referred to as the axial next-nearest neighbor Ising (ANNNI) model [49] and the second is referred to as XXZ chain (with a transverse field) [50]. As discussed in the references above, both Hamiltonian families exhibit quantum phase transitions [51]: The

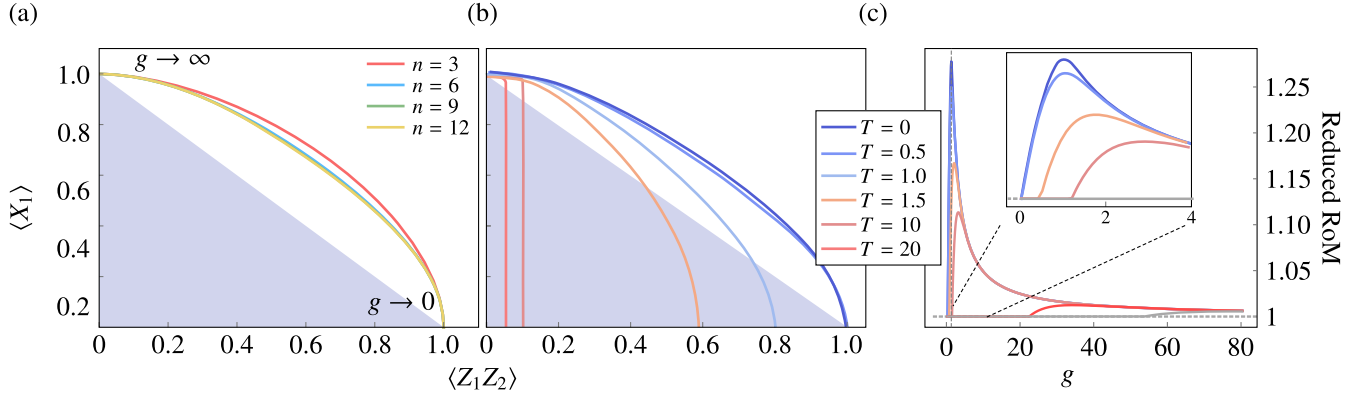


FIG. 2. Reduced nonstabilizerness of the transverse-field Ising model. The shaded region in panels (a) and (b) corresponds to the reduced stabilizer polytope associated with the measurement set $\mathcal{M} = \{Z_1 Z_2, X_1\}$. (a) Ground-state correlations ($\langle Z_1 Z_2 \rangle, \langle X_1 \rangle$) as a function of the transverse field g for different system sizes. The maximal deviation from the stabilizer polytope occurs near the critical point $g = 1$. (b) Correlations for Gibbs states $\rho \propto e^{-H_{\text{TFIM}}/T}$ at different temperatures, showing the suppression of nonstabilizerness with increasing T . (c) Reduced robustness of magic computed from the Hamiltonian observables $\mathcal{M}(H_{\text{TFIM}})$. The peak near the quantum critical point is progressively softened as temperature increases, eventually leading to a stabilizer regime (“sudden death of magic” [47]).

ground state exhibits abrupt changes as the parameters are varied. A natural signature is to look at the energy gap:

$$\Delta(H) = \lim_{n \rightarrow \infty} (E_1 - E_0), \quad (46)$$

where E_1 is the first excited state. A ground state is said to be *gapped* if $\Delta(H) > 0$, and *gapless* if $\Delta(H) = 0$. A *phase diagram* is constructed by identifying states that have their states smoothly varying with the parameters that define the Hamiltonian, although this definition can only be made mathematically sharp in the gapped case [48]. Numerically, this can be probed by looking at the behavior of observables as one takes the system size to be large.

Let us consider the case of the ANNNI. For $k = 0$, the corresponding Hamiltonian is known as the transverse-field Ising model (TFIM). Its nonstabilizerness was analyzed in Ref. [45] by computing the stabilizer Rényi entropy (SRE) [28]. In general, it is necessary to have full tomographic information compute the SRE. It turns out that this model hosts a phase transition at $g = 1$ between two gapped phases, that have stabilizer points: Notice that $\psi_{\text{gs}}(g \rightarrow \infty) = (|+\rangle\langle+|)^{\otimes n}$ is the unique ground state at large fields, stabilized by the group $\langle X_1, X_2, \dots, X_n \rangle$ and $\psi_{\text{gs}}(g = 0)$ is any state formed through the coherent superposition of $|0\rangle^{\otimes n}$ and $|1\rangle^{\otimes n}$, with stabilizer group $\langle Z_1 Z_2, Z_2 Z_3, \dots \rangle$. The transition is a gapless point that tunes into the phases that are adiabatically connected between the two points. The SRE follows this behavior: In the stabilizer points, it vanishes, and acquires a maximum value in the transition between the phases that are connected with the stabilizer points.

Using our approach, we observe that this behavior can be found already with exponentially fewer measurements. Consider:

$$\mathcal{M} = \{Z_1 Z_2, X_1\}, \quad (47)$$

which corresponds to energy measurements on the first pair of qubits. The corresponding reduced polytope is the one corresponding to the second graph in Fig. 1(a). We can compute directly both expectation values as a function of g in the ground state, as shown in Fig. 2(a). As n increases, we can see that the maximal distance of the correlations remains around the critical value $g = 1$, indicating the phase transition.

In order to test our algorithm in the mixed state regime, we will also test our method in Gibbs states of the TFIM model, where our target state is $\rho \propto \exp(-H_{\text{TFIM}}/T)$, now parametrized by (T, g) . As is shown in Fig. 2(b), as temperature increases, the corresponding correlations become more stabilizer-like. In fact, since at $T \rightarrow \infty$, $\rho \rightarrow \frac{1}{2^n} \in \text{STAB}_n$, it is expected that there always will be some critical temperature T_c depending on n and g such that for $T \geq T_c(n, g)$, we have a stabilizer Gibbs state, a “sudden death of magic.” As shown in Ref. [52], this is indeed what happens for geometrically local Hamiltonians, with $T_c = O(1)$. Our method captures this sudden death for the subsystems defined by the measurements. We confirm this expectation by computing the reduced RoM for the Paulis in the Hamiltonian, $\mathcal{M}(H_{\text{TFIM}})$, shown in Fig. 2(c): As temperature increases, the peak at the critical value of the (ground state) phase transition is softened, and, given any temperature T , a discontinuity in the robustness is developed for some g_c , increasing as T increases.

Our results are also model-independent: We extend the TFIM example to the full ground state phase diagram of the models in Eqs. (44) and (45). In the optimal case where we measure all the $\Theta(n)$ Paulis in the Hamiltonian,

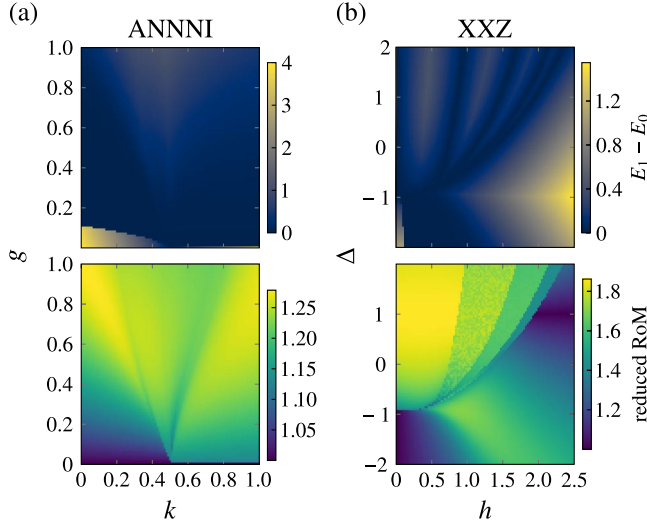


FIG. 3. Comparison between the energy gap and the reduced RoM measure for the ANNNI and XXZ models. For the ANNNI model, 10 qubits were used, while 9 qubits were used for XXZ model. Column (a) presents the results for the ANNNI model as function of the frustration k and the transverse field g . Column (b) details the XXZ model in terms of the magnetic field h and the anisotropy Δ .

meaning that we compute:

$$\mathcal{M}(H_{\text{ANNI}}) \mapsto \text{RoM}_{\mathcal{M}(H_{\text{ANNI}})}[\psi_{\text{gs}}(k, g)], \quad (48)$$

$$\mathcal{M}(H_{\text{XXZ}}) \mapsto \text{RoM}_{\mathcal{M}(H_{\text{XXZ}})}[\psi_{\text{gs}}(h, \Delta)]. \quad (49)$$

The results are shown in Fig. 3. As one can see, the behavior of the energy gap has a similar behavior as the nonstabilizerness as captured by the reduced RoM, where gapless regions are expected to have high nonstabilizerness and gapped phases that host stabilizer points, as is the case for the TFIM, have low nonstabilizerness near stabilizer regions. Although most of the phase diagrams show nonstabilizerness, there are particular regions where it vanishes. For the ANNNI, note that the gapped region at $g = 0$ has vanishing RoM. This is due to the fact that the ground state is in the code stabilized by the group generated by the $Z \otimes Z$ Pauli pairs on first and second neighbors. Similarly, for the XXZ model, we have stabilizer ground states in the ferromagnetic region with $h = 0$ and $\Delta \in [-2, -1]$.

We stress that although the results were derived by exact diagonalization of the Hamiltonians for few qubits, we could use data coming from quantum simulators/computers and more efficient classical simulation algorithms to run the same algorithm for a larger number of qubits, which is a bottleneck for other measures, as previously mentioned. For example, in Ref. [15], the RoM was studied for ≤ 8 -size subsystems of quantum Gibbs states, and although our reduced form does not correspond

to the full information of the RoM, the numerical experiments above indicate that they capture much of the same information with lower computational algorithmic cost.

VI. CONCLUSION AND FINAL REMARKS

In this work, we have shown how nonstabilizerness can be witnessed and quantified even when the number of accessible measurements is limited. By restricting stabilizer resource theory to a subset of observables, we constructed an algorithm that takes as input the expectation values of Pauli operators and decides whether they are compatible with stabilizer statistics. We demonstrated the practical performance of this approach through explicit many-body examples and established its near-optimality from a complexity-theoretic perspective: the superpolynomial dependence on the number of measurements is not an artifact of the method, but is in fact unavoidable unless widely believed complexity-theoretic assumptions fail.

Importantly, the reduced polytope naturally defines a nonstabilizerness monotone, the reduced robustness of magic $\text{RoM}_{\mathcal{M}}$, which lower bounds the magic content of the full state. Operationally, this immediately translates into a lower bound on the classical simulation cost: measuring only $m \leq \text{poly}(n)$ Pauli observables already suffices to infer a quasi-probability sample-complexity bound. Hence, even limited measurement data can certify nontrivial classical hardness.

Several directions for future research naturally emerge. One concerns connections to quantum learning theory, in particular quantum PAC learning, where the goal is to predict expectation values drawn from a distribution using only a linear number of measurements [53]. It was shown in Ref. [54] that stabilizer states are efficiently PAC-learnable, in contrast to generic quantum states. An interesting open question is how this learnability behavior extends to families of states with bounded nonstabilizerness, and whether the tools developed here can help characterize the sample complexity as a function of the magic content. Another promising direction is to relate our framework to recent advances on quantum correlation sets of Pauli observables [24,55], which are likewise governed by algebraic properties of the underlying frustration graph. Understanding these connections may yield a more unified geometric and combinatorial picture of restricted quantum correlations.

Finally, it is natural to explore potential implications for fault-tolerant quantum computation, particularly in the context of magic-state injection [6]. Since the RoM provides quantitative bounds on resources in unitary synthesis [29], it would be interesting to investigate whether our reduced-polytope approach can lead to new resource-estimation or certification protocols tailored to experimentally accessible observables.

Overall, our results offer a new perspective on the geometry of nonstabilizerness and provide a practical route toward its characterization in large-scale quantum systems. More broadly, they suggest that the complexity inherent in

quantum resource theories need not be confronted in its full exponential form, but can instead be meaningfully controlled by projecting the problem onto experimentally and computationally accessible subspaces.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: SHORT PROOFS AND AIDING PROPOSITIONS

A result that we use a lot is the following proposition:

Proposition 12. Let $\mathcal{S} \subseteq \mathcal{P}_n$ be a stabilizer group of generic rank, and let $\psi^{\mathcal{S}} = 2^{-n} \sum_{s \in \mathcal{S}} s$ be the stabilizer projector. Then, for any $P \in \mathcal{P}_n$:

$$\langle P \rangle_{\psi^{\mathcal{S}}} = \text{tr}(P\psi^{\mathcal{S}}) = (P, \mathcal{S}) \equiv 1[P \in \mathcal{S}] - 1[-P \in \mathcal{S}], \quad (\text{A1})$$

where $1[\cdot]$ is the indicator function. Note that $(P, \mathcal{S}) \in \{-1, 0, 1\}$.

Proof. It follows by the definition of $\psi^{\mathcal{S}}$ and the Hilbert-Schmidt orthogonality of Pauli matrices, which imposes $\text{tr}(PQ) = \delta_{PQ}2^n$ for all $P, Q \in \mathcal{P}_n$:

$$\text{tr}(P\psi^{\mathcal{S}}) = \frac{1}{2^n} \sum_{s \in \mathcal{S}} \text{tr}(Ps) = \sum_{s \in \mathcal{S}} (\delta_{P,s} - \delta_{-P,s}), \quad (\text{A2})$$

from which the result follows by recognizing $1[\pm P \in \mathcal{S}] = \sum_{s \in \mathcal{S}} (\delta_{P,s} - \delta_{-P,s})$. ■

This result implies the concentration of Pauli strings of stabilizer pseudomixtures/mixtures: Let $\psi = \sum_{\mathcal{S}} x_{\mathcal{S}} \psi^{\mathcal{S}}$, and let P be a Pauli string. Then, by the proposition above:

$$\langle P \rangle_{\psi} = \sum_{\substack{\mathcal{S} \in \text{stab}_n \\ P \in \pm \mathcal{S}}} x_{\mathcal{S}} (P, \mathcal{S}) = \sum_{\substack{\mathcal{S} \in \text{stab}_n \\ P \in \pm \mathcal{S}}} x_{\mathcal{S}} (P, \mathcal{S}), \quad (\text{A3})$$

where now the sum is restricted to stabilizer groups that include either P or $-P$.

Yet another result is the 1-norm bound on changing V -representations of a polytope:

Proposition 13. Let $P_1, P_2 \subseteq [-1, 1]^N$ be two polytopes with the following convex hulls:

$$P_1 = \text{conv}(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k_1}); \quad P_2 = \text{conv}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k_2}), \quad (\text{A4})$$

such that $P_1 \subseteq P_2$. Then, given $\mathbf{v} \in \mathbb{R}^N$, we have

$$\min \left\{ \|\alpha\|_1 : \mathbf{v} = \sum_{i=1}^{k_1} \alpha_i \mathbf{u}_i \right\} \geq \min \left\{ \|\beta\|_1 : \mathbf{v} = \sum_{j=1}^{k_2} \beta_j \mathbf{v}_j \right\}. \quad (\text{A5})$$

Proof. Given that the $\mathbf{u}$'s are in the convex hull of the $\mathbf{v}$'s:

$$\mathbf{u}_i = \sum_{j=1}^{k_2} q_j^i \mathbf{v}_j; \quad \mathbf{q}^i \in \Delta_{k_2}, \quad \forall i \in [k_1]. \quad (\text{A6})$$

Suppose that we have $\mathbf{v} = \sum_i \alpha_i \mathbf{u}_i$ as a feasible decomposition in the optimization in the lhs of Eq. (A5). Then, defining $\beta_j^{\alpha} \equiv \sum_i q_j^i \alpha_i$, note that $\mathbf{v} = \sum_j \beta_j^{\alpha} \mathbf{v}_j$ is a feasible decomposition in the rhs. We have:

$$\|\beta^{\alpha}\|_1 = \sum_{j=1}^{k_2} |\beta_j^{\alpha}| \leq \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} q_j^i |\alpha_i| = \|\alpha\|_1, \quad (\text{A7})$$

where we have used $\sum_j q_j^i = 1$. Since the inequality holds for every α , Eq. (A5) is satisfied upon optimization. ■

Another geometric fact about the reduced stabilizer polytope is the following:

Proposition 14. Let $\mathcal{M} = \{P_1, P_2, \dots, P_m\}$ be a set of n -qubit Pauli operators, such that $P_j \neq -P_i$ for every pair $i, j \in [m]$. Then, the corresponding reduced polytope includes the 1-norm closed ball of radius 1:

$$B_1(1) \equiv \{\vec{x} \in \mathbb{R}^m : \|\vec{x}\|_1 \leq 1\} \subseteq \text{STAB}(\mathcal{M}). \quad (\text{A8})$$

Proof. For a given $i \in [m]$, there is the following pair of n -qubit stabilizer states:

$$\rho_i^{\pm} = \frac{1 \pm P_i}{2^n} \in \text{STAB}_n. \quad (\text{A9})$$

Since $\text{tr}(P_j \rho_i^\pm) = \pm \delta_{ij}$ for all $j \in [m]$, the corresponding vector of correlations becomes:

$$(\langle P_1 \rangle_{\rho_i^\pm}, \langle P_2 \rangle_{\rho_i^\pm}, \dots, \langle P_i \rangle_{\rho_i^\pm}, \dots, \langle P_m \rangle_{\rho_i^\pm}) = (0, 0, \dots, \pm 1, \dots, 0) \equiv \hat{e}_i^\pm \in \text{STAB}(\mathcal{M}). \quad (\text{A10})$$

1-norm (closed) balls admit a presentation as the following convex hull (which is also referred to as the m -dimensional crosspolytope [25]):

$$B_1(1) = \text{conv}\{\hat{e}_i^\sigma\}_{i \in [m], \sigma \in \pm}, \quad (\text{A11})$$

and since $\text{conv}\{\hat{e}_i^\sigma\}_{i \in [m], \sigma \in \pm} \subseteq \text{STAB}(\mathcal{M})$ by convexity, the claim is shown. ■

1. Proof of Lemma 1

Let us denote $\mathcal{M} = \{P_i\}_{i=1}^m$ and $\mathcal{M} \otimes \mathbb{1} = \{P_i \otimes \mathbb{1}\}_{i=1}^m$. Let $\Pi_{\mathcal{M}} \vec{\psi} \in \text{STAB}_n(\mathcal{M})$. Then:

$$\Pi_{\mathcal{M}} \vec{\psi} = \sum_{S \in \text{stab}_n} p_S \Pi_{\mathcal{M}} \vec{\psi}^S, \quad (\text{A12})$$

where $\Pi_{\mathcal{M}} \vec{\psi}^S = (\langle P_1 \rangle_{\psi^S}, \langle P_2 \rangle_{\psi^S}, \dots, \langle P_m \rangle_{\psi^S})$. By tensoring with the identity, we construct the stabilizer group $S \otimes \mathbb{1}$ by tensoring all Paulis in S with $\mathbb{1}_{n'-n}$, which is not full rank. Two facts hold:

$$\psi^{S \otimes \mathbb{1}} = \frac{1}{2^n} \sum_{s \otimes \mathbb{1}_{n'-n} \in S \otimes \mathbb{1}} s \otimes \mathbb{1}_{n'-n} \in \text{STAB}_{n'}, \quad (\text{A13})$$

$$\langle P_i \rangle_{\psi^S} = \langle P_i \otimes \mathbb{1}_{n'-n} \rangle_{\psi^{S \otimes \mathbb{1}}}, \quad \forall i \in \{1, 2, \dots, m\}. \quad (\text{A14})$$

Equation (A14) follows from Eq. (A13), and the first equation follows from the fact that $\psi^{S \otimes \mathbb{1}}$ is the (proportionally normalized) projector into the subspace of states in $\mathbb{C}^{2 \otimes n'}$, which can be written as uniform distribution over its pure stabilizer basis [56]. By Eq. (A14), it follows that $\Pi_{\mathcal{M}} \vec{\psi}^S = \Pi_{\mathcal{M}} \vec{\psi}^{S \otimes \mathbb{1}}$, thus showing that $\Pi_{\mathcal{M}} \vec{\psi} \in \text{STAB}_{n'}(\mathcal{M} \otimes \mathbb{1})$, and therefore, $\text{STAB}_n(\mathcal{M}) \subseteq \text{STAB}_{n'}(\mathcal{M} \otimes \mathbb{1})$.

Now, consider the other way around. Let $\Pi_{\mathcal{M}} \vec{\psi}' \in \text{STAB}_{n'}(\mathcal{M} \otimes \mathbb{1})$. Then, it admits a decomposition:

$$\Pi_{\mathcal{M}} \vec{\psi}' = \sum_{S' \in \text{stab}_{n'}} p_{S'} \Pi_{\mathcal{M}} \vec{\psi}^{S'}. \quad (\text{A15})$$

Now, consider a tensor splitting $\mathbb{C}^{2 \otimes n'} \cong \mathbb{C}^{2 \otimes n} \otimes \mathbb{C}^{2 \otimes (n'-n)}$. For any $S' \in \text{stab}_{n'}$, we can construct the following stabilizer group:

$$S'|_n = \{s' \in S' | s' = s \otimes \mathbb{1}_{n'-n}\} \in \text{stab}_n, \quad (\text{A16})$$

meaning that it acts nontrivially just on the first tensor factor. Note that imposing this support condition on $s' \in S'$ is the same as imposing $\text{tr}_{n'-n}(s') \neq 0$, due to Paulis being traceless. Finally, we note that:

$$\begin{aligned} \langle P_i \otimes \mathbb{1}_{n'-n} \rangle_{\psi^{S'}} &= \frac{1}{2^n} \sum_{s' \in S'} \text{tr}(P_i \otimes \mathbb{1}_{n'-n} s') \\ &= \frac{1}{2^n} \sum_{s \in S'|_n} \text{tr}(P_i s) = \langle P_i \rangle_{\psi^{S'|_n}}, \end{aligned} \quad (\text{A17})$$

since, for stabilizers in $S' \setminus S'|_n$ will have vanishing contribution. Therefore, this shows that $\Pi_{\mathcal{M}} \vec{\psi}^{S'} = \Pi_{\mathcal{M}} \vec{\psi}^{S'|_n}$, showing the inclusion $\Pi_{\mathcal{M}} \vec{\psi}' \in \text{STAB}_n(\mathcal{M})$, and therefore, $\text{STAB}_{n'}(\mathcal{M} \otimes \mathbb{1}) \subseteq \text{STAB}_n(\mathcal{M})$. ■

2. Proof of Lemma 2

Let us go one-by-one:

(1) The inequality follows as, given any $\mathbf{x} \in \mathbb{R}^{|\text{stab}(\mathcal{M})|}$,

$$\|\mathbf{x}\|_1 = \sum_{j=1}^{|\text{stab}(\mathcal{M})|} |x_j| \geq \left| \sum_{j=1}^{|\text{stab}(\mathcal{M})|} x_j \right| = 1, \quad (\text{A18})$$

by the constraint. If there is a stabilizer state $\phi \in \text{STAB}_n$ agreeing with ψ on all Paulis of $\mathcal{M}$, it follows that $\text{RoM}_{\mathcal{M}}(\psi) = \text{RoM}_{\mathcal{M}}(\phi) = \|\mathbf{x}\|_1 = \sum_j x_j = 1$, since ϕ is a stabilizer mixture. On the other hand, note that the reverse triangle inequality above is tight iff all $x_j \geq 0$. However, if that is the case, note that we have:

$$\Pi_{\mathcal{M}} \vec{\psi} = \Pi_{\mathcal{M}} \left(\sum_{j=1}^{|\text{stab}(\mathcal{M})|} \sum_{S \in \text{stab}_n} x_j p_S^j \vec{\psi}^S \right) \equiv \Pi_{\mathcal{M}} \vec{\phi}, \quad (\text{A19})$$

where we defined $\phi = \sum_{j,S} x_j p_S^j \psi^S \in \text{STAB}_n$, since $\mathbf{x} \in \Delta_{|\text{stab}(\mathcal{M})|}$ and $\mathbf{p}^j \in \Delta_{|\text{stab}_n|}$ for all $j \in [|\text{stab}(\mathcal{M})|]$.

(2) If $\mathcal{E}$ is a TP stabilizer operation:

$$\mathcal{E}(\psi^S) = \sum_{S' \in \text{stab}_n} p_{S'}^S \psi^{S'}, \quad \mathbf{p}^S \in \Delta_{|\text{stab}_n|}. \quad (\text{A20})$$

Hence, given a stabilizer pseudomixture of $\psi = \sum_{S \in \text{stab}_n} x_S \psi^S$, we have that $\Pi_{\mathcal{M}} \mathcal{E}(\psi) = \sum_{S,S'} p_{S'}^S x_S \Pi_{\mathcal{M}} \vec{\psi}^{S'}$, thus, using that the

$\mathcal{M}$ -correlations of pure stabilizer states are in $\text{conv}(\text{stab}(\mathcal{M}))$, we have, by the bound of Proposition 13:

$$\text{RoM}_{\mathcal{M}}(\mathcal{E}(\psi)) \leq \min_{\mathbf{x}} \|\mathbf{x}\|_1 = \text{RoM}_{\mathcal{M}}(\psi). \quad (\text{A21})$$

- (3) Submultiplicativity follows if one writes the stabilizer pseudomixture of the parts:

$$\psi = \sum_{S \in \text{stab}_n} x_S \psi^S; \quad \rho = \sum_{S \in \text{stab}_{n'}} y_{S'} \psi^{S'}, \quad (\text{A22})$$

and noting that $\psi \otimes \rho = \sum_{S, S'} x_S y_{S'} \psi^S \otimes \psi^{S'}$ defines itself a pseudomixture of the product. Consider the decomposition $\Pi_{\mathcal{M} \otimes \mathcal{N}} \psi \otimes \rho = \sum_{S, S'} x_S y_{S'} \Pi_{\mathcal{M} \otimes \mathcal{N}} \psi^S \otimes \psi^{S'} = \sum_{S, S'} x_S y_{S'} \Pi_{\mathcal{M}} \tilde{\psi}^S \otimes \Pi_{\mathcal{N}} \tilde{\psi}^{S'}$, where the last tensor product is of the Pauli coefficient vectors themselves. It is clear that these vectors are, not necessarily extremal, points in the reduced polytope; then, applying Proposition 13 we arrive at:

$$\begin{aligned} \text{RoM}_{\mathcal{M} \otimes \mathcal{N}}(\psi \otimes \rho) &\leq \min_{\mathbf{x}, \mathbf{y}} \sum_{S, S'} |x_S| |y_{S'}| \\ &= \min_{\mathbf{x}, \mathbf{y}} \|\mathbf{x}\|_1 \|\mathbf{y}\|_1 \\ &\leq \text{RoM}_{\mathcal{M}}(\psi) \text{RoM}_{\mathcal{N}}(\rho), \end{aligned} \quad (\text{A23})$$

since we can always choose $\mathbf{x}, \mathbf{y}$ attaining the minimum in the individual LP optimization for each factor. The proof for convexity follows by the same strategy: consider $\Pi_{\mathcal{M}} \tilde{\psi}_j = \sum_S x_{S,j} \Pi_{\mathcal{M}} \tilde{\psi}^S$, which will give an upper bound:

$$\begin{aligned} \text{RoM}_{\mathcal{M}}\left(\sum_j p_j \psi_j\right) &\leq \left\| \sum_j p_j \mathbf{x}_j \right\|_1 \\ &\leq \sum_j |p_j| \|\mathbf{x}_j\|_1, \end{aligned} \quad (\text{A24})$$

yielding $\text{RoM}_{\mathcal{M}}(\sum_j p_j \psi_j) \leq \sum_j |p_j| \text{RoM}_{\mathcal{M}}(\psi_j)$ upon optimization.

- (4) Note that projecting to a smaller set of coordinates preserves any vector equation; namely, if $\mathcal{M}' \subseteq \mathcal{M}$ then

$$\begin{aligned} \left\{ \mathbf{x} \left| \Pi_{\mathcal{M}} \psi = \sum_{S \in \text{stab}_n} x_S \Pi_{\mathcal{M}} \psi^S \right. \right\} \\ \subseteq \left\{ \mathbf{x} \left| \Pi_{\mathcal{M}'} \psi = \sum_{S \in \text{stab}_n} x_S \Pi_{\mathcal{M}'} \psi^S \right. \right\}. \end{aligned} \quad (\text{A25})$$

Since $\{\Pi_{\mathcal{M}} \psi^S | S \in \text{stab}_n\}$ must contain all extremal vertices of the projected polytope, in addition to some possible redundant non-extremal points, optimizing over both sets above results in the respective reduced robustness of magic. However, minimizing over a larger set results in a lower value:

$$\text{RoM}_{\mathcal{M}}(\psi) \geq \text{RoM}_{\mathcal{M}'}(\psi). \quad (\text{A26})$$

■

3. Proof of Lemma 5

First, we show that all such fully anticommuting sets indeed yield the smallest reduced polytopes, and then we construct representations.

- (1) If $\mathcal{M}$ is fully anticommuting, the frustration graph $G_{\mathcal{M}}$ is the complete graph K_m . It only has singletons on its maximal independent set:

$$I_{\max}(G_{\mathcal{M}}) = \{\{P_i\}\}_{i=1}^m, \quad (\text{A27})$$

since every pair anti-commutes. Then, all bit-string assignments are allowed, $B_{\{P_i\}} = \{\pm\}$, and each vector in the V -representation in Theorem 3 is exactly one of the $2m$ vectors $\{\hat{e}_j^\sigma\}_{j,\sigma}$ in the proof of Proposition 14. Then,

$$\text{STAB}(\mathcal{M}) = \text{conv}\{\hat{e}_j^\sigma\}_{j,\sigma} = B_1(1). \quad (\text{A28})$$

By Proposition 14, it follows that $\text{STAB}(\mathcal{M}) \subseteq \text{STAB}(\mathcal{M}')$ for any length m Pauli set $\mathcal{M}'$.

- (2) It is known [57] that a set of m anti-commuting invertible $D \times D$ matrices should satisfy $m \leq 2 \log(D) + 1$. Since Paulis are invertible, substituting $D = 2^n$ gives us the claimed bound. Hence, given m measurements, at least $\lfloor m/2 \rfloor$ qubits are needed to realize them. To construct an example with $m = 2n + 1$ it suffices to provide a representation of the generators of complex Clifford algebra, that saturates the bound. For $k \in [n]$, define, following the Jordan-Wigner transformation:

$$\gamma_{2k-1} \equiv \prod_{1 \leq j \leq k-1} Z_j X_k, \quad \gamma_{2k} \equiv \prod_{1 \leq j \leq k-1} Z_j Y_k. \quad (\text{A29})$$

Note that $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$, for $i, j \in [2n]$, and are Pauli operators. Letting $\Gamma \equiv \prod_j Z_j$, note also that $\{\Gamma, \gamma_i\} = 0$ for all $i \in [2n]$. Therefore, the set:

$$\mathcal{M}^{\text{anti}} = \{\gamma_1, \gamma_2, \dots, \gamma_{2n}, \Gamma\} \quad (\text{A30})$$

satisfies all the constraints and therefore constitutes the desired example.

■

4. Proof of Lemma 10

We use two standard facts [58]:

(1) Since ν is a 2-design, $\mathbb{E}_{\psi \sim \nu}[\langle P_i \rangle_\psi] = \mathbb{E}_{\psi \sim \text{Haar}}[\langle P_i \rangle_\psi] = 0$,

(2) furthermore, it concentrates, by a standard application of Markov's inequality: $\Pr_{\psi \sim \nu}(|\langle P_i \rangle_\psi| \geq \varepsilon) \leq \Theta(\varepsilon^{-2} 2^{-n})$.

Consider $\vec{v} \equiv (\langle P_1 \rangle_\psi, \langle P_2 \rangle_\psi, \dots, \langle P_m \rangle_\psi)$. We now want to control its fluctuations by $\|\vec{v}\|_\infty = \max(\langle P_1 \rangle_\psi, \langle P_2 \rangle_\psi, \dots, \langle P_m \rangle_\psi)$. By a union bound:

$$\Pr(\|\vec{v}\|_\infty \geq \varepsilon) \leq \Pr\left(\bigcup_{j=1}^m |\langle P_j \rangle_\psi| \geq \varepsilon\right) \leq \sum_{i=1}^m \Pr(|\langle P_i \rangle_\psi| \geq \varepsilon) \leq \Theta\left(\frac{m}{2^n \varepsilon^2}\right) \Rightarrow \Pr(\|\vec{v}\|_\infty \leq \varepsilon) \geq 1 - \Theta\left(\frac{m}{2^n \varepsilon^2}\right). \quad (\text{A31})$$

Then, when this happens, note that:

$$\|\vec{v}\|_1 = \sum_{i=1}^m |\langle P_i \rangle_\psi| \leq m \|\vec{v}\|_\infty = m\varepsilon. \quad (\text{A32})$$

By Proposition 14, we know that if $\|\vec{v}\|_1 \leq 1 \Rightarrow \vec{v} \in \text{STAB}(\mathcal{M})$. Hence, choosing $\varepsilon = 1/m$ yields our desired bound by chaining the events:

$$\Pr(\vec{v} \in \text{STAB}(\mathcal{M})) \geq \Pr(\|\vec{v}\|_1 \leq 1) \geq \Pr\left(\|\vec{v}\|_\infty \leq \frac{1}{m}\right) = 1 - \Theta\left(\frac{m^3}{2^n}\right). \quad (\text{A33})$$

If instead $\nu = \text{Haar}$, we can apply Levy's lemma [58] to find that:

$$\Pr_{\psi \sim \text{Haar}}(|\langle P_i \rangle_\psi| \geq \varepsilon) \leq 2 \exp(-c 2^n \varepsilon^2), \quad (\text{A34})$$

where $c = O(1)$ is a positive constant. Using the same line of reasoning as above, the new failure probability is obtained by substituting $\varepsilon = 1/m$ and multiplying by m due to the 1-norm bound in Eq. (A32), upgrading the previous result by a double-exponential suppression:

$$\Theta\left(\frac{m^3}{2^n}\right) \mapsto \Theta(m 2^{-c' 2^n / m^2}), \quad (\text{A35})$$

where $c' = O(1)$ is again a positive constant. ■

APPENDIX B: PROOF OF THEOREM 3

First, we will show the relation to the maximally commuting subsets defined on Eq. (23), and then do the graph-theoretic interpretation. By Eq. (A3), we know that for any given stabilizer state $\phi = \sum_{S \in \text{stab}_n} p_S \psi^S \in \text{STAB}_n$, its projection has the form:

$$\Pi_{\mathcal{M}} \vec{\phi} = \sum_{\substack{S \in \text{stab}_n \\ S \cap \pm \mathcal{M} \neq \emptyset}} p_S \Pi_{\mathcal{M}} \vec{\psi}^S, \quad (\text{B1})$$

since $\Pi_{\mathcal{M}} \vec{\psi}^S: S \cap \pm \mathcal{M} = \emptyset = \vec{0}$. Now, let us define $S_1 \sim S_2$ in stab_n iff $S_1 \cap \pm \mathcal{M} = S_2 \cap \pm \mathcal{M}$; it is easy to see that this is an equivalence relation. Then, due to Proposition 12, it follows that:

$$\langle P_i \rangle_{\psi^{S_1}} = \langle P_i \rangle_{\psi^{S_2}}, \quad \forall i \in [m] \Rightarrow \Pi_{\mathcal{M}} \vec{\psi}^{S_1} = \Pi_{\mathcal{M}} \vec{\psi}^{S_2}. \quad (\text{B2})$$

Thus, the distinct elements on the convex decomposition of Eq. (B1) are for stabilizer groups that have distinct intersections with the measurement set. We can rewrite Eq. (B1) as

$$\Pi_{\mathcal{M}} \vec{\phi} = \sum_{[S] \in \text{stab}_n|_{\mathcal{M}}} p_{[S]} \vec{v}_{[S]}, \quad \vec{p} \in \Delta_{|\text{stab}_n|_{\mathcal{M}}}, \quad (\text{B3})$$

where $\text{stab}_n|_{\mathcal{M}} \equiv (\text{stab}_n / \sim) \setminus [\emptyset]$ is the set of all equivalence classes except the stabilizers with vanishing intersection with $\mathcal{M}$, $p_{[S]} = \sum_{S' \in [S]} p_{S'}$ and the corresponding vector is

$$(\vec{v}_{[S]})_i = \begin{cases} 1 & \text{if } P_i \in S, \\ -1 & \text{if } -P_i \in S, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{B4})$$

Claim: The correspondence $[S] \in \text{stab}_n|_{\mathcal{M}} \mapsto S \cap \pm \mathcal{M} \in C_{\max}(\mathcal{M})$ is a bijection.

First, by consequence of the equivalence relation, it is well-defined; since S is a stabilizer group, we have $-1 \notin S \cap \pm \mathcal{M}$ and since S is full rank, $S \cap \pm \mathcal{M}$ is maximal, i.e., $S \cap \pm \mathcal{M} \in C_{\max}(\mathcal{M})$.

Given any $Q \in C_{\max}(\mathcal{M})$, since it is a commuting set of Paulis, we can always find a maximal rank stabilizer group $S \supseteq Q$. For any choice of S , we have $S \cap \pm \mathcal{M} = Q$; if this were not the case, say, $S \cap \pm \mathcal{M} \supsetneq Q$, we could take some

$P \in S \cap \pm\mathcal{M} \setminus Q$ and add it to Q , rendering Q not maximal. Thus, the inverse map is well-defined.

By abuse of notation, we can identify $\vec{v}_Q = \vec{v}_S$ via the aforementioned bijection. Therefore by Eq. (B3):

$$\text{STAB}(\mathcal{M}) = \text{conv}\{\vec{v}_Q\}_{Q \in C_{\max}(\mathcal{M})}. \quad (\text{B5})$$

We can further strengthen this claim: Each $\vec{v}_Q$ is actually an extremal point of $\text{STAB}(\mathcal{M})$, and thus $\text{stab}_n|_{\mathcal{M}}$ also correspond to extremal vertices.

Suppose that $\vec{v}_Q = \sum_R p_R \vec{v}_R$, $p_R > 0$, $\sum_R p_R = 1$, is a convex decomposition with $R \in C_{\max}(\mathcal{M})$. We claim that $Q \subseteq R$ for every R appearing in the sum. Assume, by contradiction, that there exists some R_0 such that $Q \not\subseteq R_0$. Then there is an element $\sigma P_i \in Q \setminus R_0$, with $\sigma \in \{\pm 1\}$. By the definition of $\vec{v}_Q$, this means that $(\vec{v}_Q)_i = \sigma$. On the other hand, $\sigma = (\vec{v}_Q)_i = \sum_R p_R (\vec{v}_R)_i$. Since each $(\vec{v}_R)_i \in \{-1, 0, 1\}$ and the p_R 's form a probability distribution, this equality can hold only if $(\vec{v}_R)_i = \sigma$ for every R with $p_R > 0$. In particular, this must hold for R_0 , which implies $\sigma P_i \in R_0$. This contradicts the choice $\sigma P_i \in Q \setminus R_0$. Therefore $Q \subseteq R$ for all R in the convex decomposition. Since both Q and R are maximal elements of $C_{\max}(\mathcal{M})$, it

follows that $R = Q$ for all terms with positive weight. Hence the convex decomposition is trivial, and $\vec{v}_Q$ is an extremal point.

We now construct a graph theoretic presentation of $C_{\max}(\mathcal{M})$ based on $G_{\mathcal{M}}$. First, we claim that:

$$I(G_{\mathcal{M}}) = \{S \subseteq \mathcal{M} \mid [P_i, P_j] = 0, \forall P_i, P_j \in S\}, \quad (\text{B6})$$

where we denoted $I(G_{\mathcal{M}})$ as the set of independent subsets of $G_{\mathcal{M}}$. Graph theoretically, $S \in I(G_{\mathcal{M}})$ iff its elements do not share an edge in $E(G_{\mathcal{M}})$. For the frustration graph $G_{\mathcal{M}}$, this happens iff all the Paulis commute, since a pair of Paulis either commute or anti-commute, showing the algebraic presentation above. We say that $S \in I(G_{\mathcal{M}})$ is also *maximal* ($S \in I_{\max}(G_{\mathcal{M}})$) if:

$$\forall S' \in I(G_{\mathcal{M}}), \quad S \subseteq S' \Rightarrow S' = S. \quad (\text{B7})$$

Claim: There is a correspondence $Q \in C_{\max}(\mathcal{M}) \longleftrightarrow (S \in I_{\max}, f \in B_S)$, for $S \in \max(G_{\mathcal{M}})$ and $f \in B_S$ [59].

In fact, consider $Q \in C_{\max}(\mathcal{M})$. Since every $P \in Q$ satisfies either $P \in \mathcal{M}$ or $-P \in \mathcal{M}$, we can define $f_Q \in \{\pm 1\}^Q$ with elements:

$$(f_Q)_P = \begin{cases} 1 & \text{if } P \in \mathcal{M}, \\ -1 & \text{if } P \in -\mathcal{M}. \end{cases} \Rightarrow (\vec{v}_Q)_i = \begin{cases} (f_Q)_{P_i} & \text{if } \pm P_i \in Q, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{B8})$$

Thus, there is a subset $S_Q \subseteq \mathcal{M}$ such that we can write:

$$Q = \{(f_Q)_P P\}_{P \in S_Q}. \quad (\text{B9})$$

We note that:

- (i) $f_Q \in B_{S_Q}$, since $-1 \notin \langle Q \rangle$ imposes that, for every $S' \subseteq S_Q$:

$$\prod_{P \in S'} (f_Q)_P P \neq -1. \quad (\text{B10})$$

- (ii) $S_Q \in I_{\max}(G_{\mathcal{M}})$: We know that $S_Q \in I(G_{\mathcal{M}})$, since Q is commuting. Furthermore, if there were $S' \in I(G_{\mathcal{M}})$ with $S \subsetneq S'$, we could add $P_i \in S' \setminus S$ to Q , contradicting its maximality.

So, there is a map $Q \mapsto (S, f)$, for $S \in I_{\max}(G_{\mathcal{M}})$ and $f \in B_S$. We also denote $\vec{v}_{S_Q, f_Q} = \vec{v}_Q$.

We show that we can also reconstruct an element of $C(\mathcal{M})$ from a pair $(S \in I_{\max}, f \in B_S)$. There are two possibilities:

- (i) $B_S \neq \emptyset$: Then, note that, given $f \in B_S$, we can construct:

$$Q_{(S, f)} \equiv \{f_P P\}_{P \in S} \subseteq \pm\mathcal{M}. \quad (\text{B11})$$

Since f satisfies the Pauli product condition, we know that $-1 \notin \langle Q_{(S, f)} \rangle$. Furthermore, since S is maximal, we know that:

$$\nexists P' \in \mathcal{M} \setminus S : [P', S] = 0, \quad (\text{B12})$$

since otherwise $S \subsetneq S \cup \{P'\} \in I(G_{\mathcal{M}})$, contradicting the maximality of S . Note that this condition is robust to signs, since if two Paulis P, P' do commute, so does $[s_1 P, s_2 P'] = 0$ for all $s_1, s_2 \in \{\pm 1\}$. We then conclude that:

$$\nexists P' \in \pm\mathcal{M} \setminus Q_{(S, f)} : [P', Q_{(S, f)}] = 0. \quad (\text{B13})$$

Then, $s \in \langle Q_{(S, f)} \rangle \cap \pm\mathcal{M} \setminus Q_{(S, f)} \subseteq \pm\mathcal{M} \setminus Q_{(S, f)}$ cannot be commuting with $Q_{(S, f)}$, and thus $Q_{(S, f)} \in C_{\max}(\mathcal{M})$. We define on this case:

$$\vec{v}_{S, f} \equiv \vec{v}_{Q_{(S, f)}}. \quad (\text{B14})$$

- (ii) $B_S = \emptyset$: Then there is no consistent stabilizer group among the signed Pauli elements of $S \in I_{\max}(G_{\mathcal{M}})$. We call those cases trivial.

Hence, the set of nontrivial $(S \in I_{\max}(G_{\mathcal{M}}), f \in B_S)$ is in bijection with $C(\mathcal{M})$. Equations (B8) and (B14) define

$\vec{v}_{(S,f)}$ with:

$$(\vec{v}_{(S,f)})_i = \begin{cases} f_{P_i} & \text{if } f_{P_i} P_i \in S \\ 0 & \text{otherwise.} \end{cases}, \quad (\text{B15})$$

and by Eq. (B5) we have the main claim:

$$\begin{aligned} \text{STAB}(\mathcal{M}) &= \text{conv}\{\vec{v}_Q\}_{Q \in C(\mathcal{M})} \\ &= \text{conv}\{\vec{v}_{S,f}\}_{S \in I_{\max}(G_{\mathcal{M}}), f \in B_S}, \end{aligned} \quad (\text{B16})$$

where both lists are of extremal vertices.

$$|\text{stab}(\mathcal{M})| = \sum_{S \in I_{\max}(G_{\mathcal{M}})} |B_S| \leq \sum_{S \in I_{\max}(G_{\mathcal{M}})} 2^{\min\{n, |S|\}} \leq 2^{\min\{n, s_{\max}\}} |I_{\max}(G_{\mathcal{M}})|, \quad (\text{B17})$$

where

$$s_{\max} = \max_{S \in I_{\max}(G_{\mathcal{M}})} |S|. \quad (\text{B18})$$

For the term $|I_{\max}(G_{\mathcal{M}})|$ we also have two separate upper bounds: first, for any graph of m vertices, we know [60]:

$$|I_{\max}(G_{\mathcal{M}})| \leq 3^{m/3}. \quad (\text{B19})$$

On the other hand, as implied by the arguments in Appendix C, every maximal commuting subset S accounts for at least 2 points in $\text{stab}(\mathcal{M})$. Then, it is bounded by the number of stabilizer groups, $2^{(1/2+o(1))n^2}$ [5], accounting for the combined bound

$$|I_{\max}(G_{\mathcal{M}})| \leq 2^{\min\{cm, (1/2+o(1))n^2\}}. \quad (\text{B20})$$

and thus arriving at $|\text{stab}(\mathcal{M})| \leq 2^{\min\{n, m\} + \min\{cm, (1/2+o(1))n^2\}}$ with $c = \log 3/3$. For $m = \Omega(n^2)$, this just reduces to the loose bound $|\text{stab}(\mathcal{M})| \leq |\text{stab}_n|$; however, for $m = O(n)$ we have $|\text{stab}(\mathcal{M})| = 2^{O(m)}$, dropping the explicit dependence on n .

Tightness of $\mathcal{M} = \{X_1, Y_1, Z_1, X_2, \dots, Z_n\}$ follows by recognizing that the corresponding frustration graph is given as

$$G_{\mathcal{M}} = \underbrace{K_3 \sqcup K_3 \sqcup \dots \sqcup K_3}_{m/3 \text{ times}}, \quad (\text{B21})$$

with K_3 being the completely connected graph of 3 vertices. Then, every independent set $S \in I_{\max}(G_{\mathcal{M}})$ can be written as

$$S = \{S_1, S_2, \dots, S_{m/3}\}, \quad (\text{B22})$$

We now move to the size bound. The simplest upper bound one can take for the number of consistent (S, f) pairs is by the following argument; any commuting set of n -qubit Paulis can be extended to a full stabilizer group, which is generated by some set of n Paulis, $\{P_1, \dots, P_n\}$. Then we are free to choose only $f_S(P_i) \in \{\pm 1\}$, as the other Paulis are all products of some combination of the generators. This accounts for at most 2^n possible sign assignments of the extended stabilizer group. On the other hand, the number all sign assignments is $|\{\pm 1\}^S| = 2^{|S|}$. Then we can write:

with $S_i \in \{X_i, Y_i, Z_i\}$ being one of the three independent sets of each K_3 . Hence, we have indeed that $|I_{\max}(G_{\mathcal{M}})| = 3^{m/3}$. Furthermore, note that, for any $S' \subseteq S$, their corresponding Pauli product is also independent, meaning that $\prod_{P \in S'} P \neq \pm 1$, which implies that any $f \in \{\pm 1\}^S$ is also in B_S . Thus, $B_S = \{\pm 1\}^{m/3} = \{\pm 1\}^n$ for any $S \in I_{\max}(G_{\mathcal{M}})$, and we have that by Eq. (B17),

$$\begin{aligned} |\text{stab}(\mathcal{M})| &= \sum_{S \in I_{\max}(G_{\mathcal{M}})} |B_S| = |I_{\max}(G_{\mathcal{M}})| 2^n \\ &= 2^{n + (\log 3/3)m}. \end{aligned} \quad (\text{B23})$$

APPENDIX C: PROOF OF THEOREM 7

We will present the proof by expanding the steps outlined in Algorithm 1. In order to do the detailed analysis, we need to do some preliminaries on Pauli strings. A specific classical representation of $P \in \mathcal{P}_n$ is in terms of its *symplectic representation* [3,4], on which we write:

$$P = \lambda X^{\mathbf{a}} Z^{\mathbf{b}}, \quad (\text{C1})$$

where $\lambda \in \{\pm 1, \pm i\}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{F}_2^n$, and we have used the shorthand notation:

$$X^{\mathbf{a}} \equiv \bigotimes_{i=1}^n X^{a_i}, \quad Z^{\mathbf{b}} \equiv \bigotimes_{i=1}^n Z^{b_i}. \quad (\text{C2})$$

Considering $P_1 = \lambda_1 X^{\mathbf{a}_1} Z^{\mathbf{b}_1}$ and $P_2 = \lambda_2 X^{\mathbf{a}_2} Z^{\mathbf{b}_2}$, they satisfy:

$$P_1 P_2 = (-1)^{\Omega_{1,2}} P_2 P_1, \quad (\text{C3})$$

where $\Omega_{1,2} = \mathbf{a}_1 \cdot \mathbf{b}_2 + \mathbf{a}_2 \cdot \mathbf{b}_1$ is referred to as the symplectic form of the vectors $(\mathbf{a}_1, \mathbf{b}_1)^T \in \mathbb{F}_2^{2n}$ and $(\mathbf{a}_2, \mathbf{b}_2)^T \in \mathbb{F}_2^{2n}$.

Note that storing this bit representation of the Pauli, $P \mapsto (\lambda, \mathbf{a}, \mathbf{b})$ requires $O(n)$ bits, and Pauli multiplication, obtained by considering $(\lambda_1, \mathbf{a}_1, \mathbf{b}_1)$ and $(\lambda_2, \mathbf{a}_2, \mathbf{b}_2)$, one can verify that $P_1 P_2$ is given by the representation $(\lambda_1 \lambda_2 (-1)^{\mathbf{b}_1 \cdot \mathbf{a}_2}, \mathbf{a}_1 + \mathbf{a}_2, \mathbf{b}_1 + \mathbf{b}_2)$, which requires $O(n)$ runtime, and so does checking commutativity of two

Pauli strings, since it can be obtained by addition modulo 2 of the representation of the constituents. We will assume that every single Pauli is stored and manipulated using the symplectic representation. This includes the input, which is given as the following length m array:

$$\mathcal{M} = \{P_1, P_2, \dots, P_m\} \mapsto \{(\lambda_1, \mathbf{a}_1, \mathbf{b}_1), (\lambda_2, \mathbf{a}_2, \mathbf{b}_2), \dots, (\lambda_m, \mathbf{a}_m, \mathbf{b}_m)\}, \quad (\text{C4})$$

stored in $O(mn)$ bits. We now go through the four steps:

- (1) We compute the frustration graph datum by computing its adjacency matrix, which turns out to be the symplectic form itself, which, for $i, j \in [m]$, is given as

$$\Omega_{i,j} \equiv \mathbf{a}_i \cdot \mathbf{b}_j + \mathbf{b}_i \cdot \mathbf{a}_j \in \{0, 1\}, \quad (\text{C5})$$

all of its elements can be computed in $O(m^2 \times n) = O(m^2 n)$ runtime;

- (2) Equipped with a classical representation of the graph $G_{\mathcal{M}}$ by $(\mathcal{M}, [\Omega_{i,j}]_{i,j \in [m]})$, where $\mathcal{M}$ is the list of symplectic representations of the Paulis and the edges are specified by the adjacency matrix. It is known that there is an output-sensitive algorithm [61] that runs in $O(m^3 |I_{\max}(G_{\mathcal{M}})|)$ time and uses $O(m^2)$ space to compute $I_{\max}(G_{\mathcal{M}})$;
- (3) Given $S \in I_{\max}(G_{\mathcal{M}})$, we will find every admissible sign assignment $f \in B_S$. Remember that B_S is the set of all sign assignments that do not incur any product of the type $\prod_{P \in S'} f_P P = -\mathbb{1}$, for all subsets $S' \subset S$.

First, each $S \subseteq \mathcal{M}$ is specified as the list of symplectic representations, as

$$S = \{(\lambda_{S,j}, \mathbf{a}_{S,j}, \mathbf{b}_{S,j})\}_{j=1}^{|S|}, \quad (\text{C6})$$

referring to $(\lambda_{S,j}, \mathbf{a}_{S,j}, \mathbf{b}_{S,j})$ as the symplectic representation of $P_{S,j}$. We can construct the following matrix:

$$\mathbf{M}_S = \begin{pmatrix} \mathbf{a}_{S,1} & \mathbf{a}_{S,2} & \cdots & \mathbf{a}_{S,|S|} \\ \mathbf{b}_{S,1} & \mathbf{b}_{S,2} & \cdots & \mathbf{b}_{S,|S|} \end{pmatrix} \in \mathbb{F}_2^{2n \times |S|}. \quad (\text{C7})$$

Our algorithm is based on the following fact:

Claim: There is a correspondence $\mathbf{c} \in \ker[\mathbf{M}_S] \subseteq \mathbb{F}_2^{|S|} \longleftrightarrow S' \subseteq S : \prod_{P \in S'} P = \pm \mathbb{1}$.

To see why, note that by definition,

$$\mathbf{c} \in \ker[\mathbf{M}_S] \Leftrightarrow (\mathbf{M}_S) \cdot \mathbf{c} = \mathbf{0} \Leftrightarrow \sum_{i=1}^{|S|} \mathbf{c}_i \begin{pmatrix} \mathbf{a}_{S,i} \\ \mathbf{b}_{S,i} \end{pmatrix} = \mathbf{0} \in \mathbb{F}_2^{2n}. \quad (\text{C8})$$

Note that the lhs corresponds to the binary vector of the symplectic representation of $\prod_{j=1}^{|S|} P_j^{\mathbf{c}_j}$, and the rhs is the

representation of the identity, modulo a $\{\pm 1, \pm i\}$ phase. However, since a product of commuting Paulis cannot square to $-\mathbb{1}$, the phase is restricted to ± 1 . Thus:

$$\sum_{i=1}^{|S|} \mathbf{c}_i \begin{pmatrix} \mathbf{a}_{S,i} \\ \mathbf{b}_{S,i} \end{pmatrix} = \mathbf{0} \Leftrightarrow \prod_{j=1}^{|S|} P_j^{\mathbf{c}_j} = \prod_{P \in S'} P = \pm \mathbb{1}, \quad (\text{C9})$$

where we defined $S' \equiv \{P_j \in S \mid \mathbf{c}_j = 1\}$. Consider now $S' \subseteq S$ satisfying the constraint. Define the vector $\mathbf{c} \in \mathbb{F}_2^{|S|}$ by

$$\mathbf{c}_j = 1[P_j \in S'], \quad \forall j \in [|S|], \quad (\text{C10})$$

where the j th component is 1 if $P_j \in S'$ and zero otherwise. Then:

$$\prod_{P \in S'} P = \prod_{j=1}^{|S|} P_j^{\mathbf{c}_j} = \pm \mathbb{1}. \quad (\text{C11})$$

The binary vectors of the corresponding symplectic representation of both sides satisfy:

$$\sum_{i=1}^{|S|} \mathbf{c}_i \begin{pmatrix} \mathbf{a}_{S,i} \\ \mathbf{b}_{S,i} \end{pmatrix} = \mathbf{0} \Leftrightarrow \mathbf{c} \in \ker[\mathbf{M}_S], \quad (\text{C12})$$

showing the claim.

By performing Gaussian elimination, in $O(\min(n, |S|)n|S|)$ time [62], we obtain a list of spanning vectors of its kernel:

$$\mathbf{K} = \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_D \end{pmatrix} \in \mathbb{F}_2^{D \times |S|}, \quad (\text{C13})$$

where $D = \dim \ker[\mathbf{M}_S]$.

By the claim above, we can define a bitstring $\sigma \in \mathbb{F}_2^D$ such that $\prod_{j=1}^{|S|} P_j^{\mathbf{c}_{i,j}} = (-1)^{\sigma_i} \mathbb{1}$ for all $i \in [D]$, where we denote $[\mathbf{c}_i]_j$ to refer to the j th component of the i th spanning vector of Eq. (C13). A second claim is the following fact:

Claim: Let $f \in \{\pm 1\}^S$ be some sign assignment for each Pauli in S . Then, $f \in B_S$ iff:

$$\mathbf{K} \cdot \mathbf{x} = \boldsymbol{\sigma}, \quad (\text{C14})$$

where $f_i = (-1)^{x_i}$, defining $\mathbf{x} \in \mathbb{F}_2^{|S|}$.

To see why, note that membership in B_S is guaranteed if:

$$\prod_{P \in S'} f_P P \neq -\mathbb{1}, \quad \forall S' \subseteq S \Leftrightarrow \prod_{P \in S'} f_P \mathbb{1} \neq - \prod_{P \in S'} P, \quad \forall S' \subseteq S. \quad (\text{C15})$$

Also note that we only need to check this condition for subsets S' such that $\prod_{P \in S'} P \propto \mathbb{1}$, since it is automatically satisfied otherwise. By the correspondence between subsets with trivial product and the kernel of $\mathbf{M}_S$, this corresponds to checking:

$$\prod_{j=1}^{|S|} f_j^{c_j} \mathbb{1} \neq - \prod_{j=1}^{|S|} P_j^{c_j}, \quad \forall \mathbf{c} \in \ker[\mathbf{M}_S] \Leftrightarrow \prod_{j=1}^{|S|} f_j^{[c]_j} \mathbb{1} \neq - \prod_{j=1}^{|S|} P_j^{[c]_j}, \quad \forall i \in [D], \quad (\text{C16})$$

where we have used the fact that the rows of Eq. (C13) span the kernel. In other words, the constraint must be:

$$\prod_{j=1}^{|S|} f_j^{[c]_j} = \sigma_i, \quad \forall i \in [D], \quad (\text{C17})$$

since the lhs is restricted to $\{\pm 1\}$. By the definitions of $\mathbf{x}, \boldsymbol{\sigma}$, the D equations above correspond to:

$$(-1)^{\sum_{j=1}^{|S|} [c]_j x_j} = (-1)^{\sigma_i}, \quad \forall i \in [D] \Leftrightarrow \mathbf{K} \cdot \mathbf{x} = \boldsymbol{\sigma}, \quad (\text{C18})$$

as claimed.

As a consequence, we can count the number of solutions. First, note that if the kernel $\ker[\mathbf{M}_S]$ is empty, i.e., there is no Pauli product proportional to the identity, then *all* sign assignments $f \in \{\pm 1\}^S$ are admissible, accounting for $|B_S| = 2^{|S|}$. If the kernel is not empty, then $\mathbf{K}$ is full rank and consequently it is surjective. Given a particular solution $\mathbf{x}_p : \mathbf{K} \cdot \mathbf{x}_p = \boldsymbol{\sigma}$, the full set of solutions can be written as:

$$B_S \cong \mathbf{x}_p + \ker[\mathbf{K}] \\ = \{\mathbf{x} \in \mathbb{F}_2^{|S|} : \mathbf{x} = \mathbf{x}_p + \mathbf{k}, \mathbf{k} \in \ker[\mathbf{K}]\}, \quad (\text{C19})$$

where the isomorphism is between sets by the relation $\{f_j = (-1)^{x_j}\}_j$. Hence, $|B_S| = 2^{\dim \ker[\mathbf{K}]} = 2^{|S| - \text{rank}[\mathbf{K}]}$,

by the rank-nullity theorem of $\mathbf{K} \in \mathbb{F}_2^{D \times |S|}$. Since the row space of $\mathbf{K}$ is $\ker[\mathbf{M}_S]$, we have that $|B_S| = 2^{|S| - D}$. The rank-nullity theorem for $\mathbf{M}_S \in \mathbb{F}_2^{2n \times |S|}$ implies that $|S| - D = \text{rank}[\mathbf{M}_S]$. We know that $\text{rank}[\mathbf{M}_S] \leq |S|$, since $\mathbf{M}_S$ has $|S|$ columns. Furthermore, its column space forms an isotropic subspace of the symplectic vector space $(\mathbb{F}_2^{2n}, \Omega)$, defined as a subspace with vanishing symplectic form. It is known that such a subspace must have dimension $\leq n$ [3], yielding $\text{rank}[\mathbf{M}_S] \leq n$. Combining the bounds, we get that:

$$|B_S| = 2^{|S| - D} = 2^{\text{rank}[\mathbf{M}_S]} \leq 2^{\min(n, |S|)}. \quad (\text{C20})$$

Given $\mathbf{K}$ obtained through Gaussian elimination, we can now present the algorithm to append all the consistent $\{\vec{v}_{S,f}\}$: (1) Compute $\boldsymbol{\sigma}$ by evaluating the set $\{s_i = \prod_{j=1}^{|S|} P_j^{[c]_j}\}_{i=1}^D$, that can be done in $O(D|S|n) = O(|S|^2 n)$ time, (2) Perform Gaussian elimination on $\mathbf{K}$ to obtain the spanning set of vectors $\ker[\mathbf{K}] = \text{span}_{\mathbb{F}_2}\{\mathbf{c}'_1, \mathbf{c}'_2, \dots, \mathbf{c}'_{|S|-D}\}$, that takes time $O(\min(D, |S|)D|S|) = O(|S|^3)$ and (3) for every $\mathbf{c}' \in \ker[\mathbf{K}]$, compute and append the vector $\vec{v}_{S,f}$ in Eq. (20), that takes $O(2^{|S|-D} \times m) = O(2^{\min(n, |S|)} m)$ time [63]. Summing all up:

$$\underbrace{O(\min(n, |S|)n|S|)}_{\text{Gaussian elimination to obtain } \mathbf{K}} + \underbrace{O(|S|^2 n)}_{\text{Computing the signs } \{s_i\}_i} + \underbrace{O(|S|^3)}_{\text{Gaussian elimination in } \mathbf{K}} + \underbrace{O(2^{\min(n, |S|)} m)}_{\text{Computing the kernel and appending}} \quad (\text{C21})$$

$$= O(2^{\min(n, m)} m + m^3 + m^2 n + \min(m, n) nm), \quad (\text{C22})$$

where on the last line we have used that $|S| \leq m$.

Combining all three steps, we have our algorithm: (1) Compute the frustration graph, (2) list all the maximally independent subsets, and (3) construct all the vectors coming from consistent sign assignments of the independent subsets. Thus, we have, for the runtime:

$$T = \underbrace{O(m^2 n)}_{\text{computing } G_{\mathcal{M}}} + \underbrace{O(m^3 |I_{\max}(G_{\mathcal{M}})|)}_{\text{computing independent subsets}} + \underbrace{|I_{\max}(G_{\mathcal{M}})| \times O(2^{\min(n,m)} m + m^3 + m^2 n + \min(m, n) nm)}_{\text{computing consistent vectors for each } S \in I_{\max}(G_{\mathcal{M}})}, \quad (\text{C23})$$

where we bounded $|S| \leq s_{\max}$, defined on Eq. (B18). This is the full runtime depending on the frustration graph properties. To have an independent upper bound, we use that $s_{\max} \leq m$ and the bound in Eq. (B20). Note that, for $n \rightarrow \infty$ and $m \leq \text{poly}(n)$, the first term that builds the frustration

graph is polynomial and can be dropped in the asymptotic runtime, since the other two are exponential. Furthermore, the last contribution has an extra exponential dependency coming from the sign assignments, making it the single dominant asymptotic term. We thus arrive at our claim:

$$T = O(m 2^{\min(m,n) + \min(cm, (1/2 + o(1))n^2)} + (m^3 + m^2 n + \min(m, n) nm) 2^{\min(cm, (1/2 + o(1))n^2)}), \quad (\text{C24})$$

where $c = \log(3)/3$. ■

APPENDIX D: PROOF OF THEOREM 9

Let us first properly define the CMP as a decision problem:

Definition 3. We define the CMP on n bits as the problem defined by

- (i) Input: A set of probability distributions $\{\vec{p}_1, \vec{p}_2, \dots, \vec{p}_N\}$, with $N \leq \text{poly}(n)$, $\vec{p}_i \in \Delta_{2^{|R_i|}}$, where $R_i \subseteq [n]$, $|R_i| \leq k = O(1)$ for all $i \in [N]$;
- (ii) Decision: Output YES if there exists an n bit probability distribution $\vec{p} \in \Delta_{2^n}$ such that:

$$\sum_{\bar{\mathbf{x}} \in \mathbb{F}_2^{n-|R_i|}} p(\mathbf{x}, \bar{\mathbf{x}}) = p_i(\mathbf{x}), \quad \forall i \in [N], \quad (\text{D1})$$

and NO otherwise.

We can now move to the two reductions:

- (1) $\text{CMP} \leq_{\text{poly}} \text{SMP}$: Given the inputs $\{\vec{p}_1, \vec{p}_2, \dots, \vec{p}_N\}$, note that we can compute the following set of states

$$\left\{ \phi_i \equiv \sum_{\mathbf{x} \in \mathbb{F}_2^{|R_i|}} p_i(\mathbf{x}) |\mathbf{x}\rangle \langle \mathbf{x}| \right\}_{i=1}^N, \quad (\text{D2})$$

in $O(1) \times N \leq \text{poly}(n)$ time. For each bit string $\mathbf{x} = (x_1, \dots, x_{|R_i|}) \in \mathbb{F}_2^{|R_i|}$, the computational-basis state $|\mathbf{x}\rangle$ is a pure stabilizer state. Indeed, it is stabilized by $\langle (-1)^{x_1} Z_1, \dots, (-1)^{x_{|R_i|}} Z_{|R_i|} \rangle$. Therefore $|\mathbf{x}\rangle \langle \mathbf{x}| \in \text{stab}_{|R_i|} \subseteq \text{STAB}_{|R_i|}$. Since $p_i(\mathbf{x})$ is a probability distribution, Eq. (D2) expresses ϕ_i as a convex combination of pure stabilizer states. Hence $\phi_i \in \text{STAB}_{|R_i|}$ for every i , and the collection $\{\phi_i\}_{i=1}^N$ is a valid input for the SMP. We claim that:

$$\exists \phi \in \text{STAB}_n : \text{tr}_{\bar{R}_i} \phi = \phi_i, \quad \forall i \in [N] \Leftrightarrow \exists \vec{p} \in \Delta_{2^n} : \sum_{\bar{\mathbf{x}} \in \mathbb{F}_2^{n-|R_i|}} p(\mathbf{x}, \bar{\mathbf{x}}) = p_i(\mathbf{x}), \quad \forall i \in [N]. \quad (\text{D3})$$

- (a) ($\Leftarrow$): If there is a global probability distribution $\vec{p} \in \Delta_{2^n}$, note that we can construct:

$$\phi \equiv \sum_{\mathbf{x} \in \mathbb{F}_2^n} p(\mathbf{x}) |\mathbf{x}\rangle \langle \mathbf{x}| \in \text{STAB}_n. \quad (\text{D4})$$

And then, given any $i \in [N]$:

$$\text{tr}_{\bar{R}_i} \phi = \sum_{\mathbf{x} \in \mathbb{F}_2^{|R_i|}} \left(\sum_{\bar{\mathbf{x}} \in \mathbb{F}_2^{n-|R_i|}} p(\mathbf{x}, \bar{\mathbf{x}}) \right) |\mathbf{x}\rangle \langle \mathbf{x}| = \sum_{\mathbf{x} \in \mathbb{F}_2^{|R_i|}} p_i(\mathbf{x}) |\mathbf{x}\rangle \langle \mathbf{x}| = \phi_i, \quad (\text{D5})$$

since it correctly marginalizes to $\vec{p}_i$.

- (b) ($\Rightarrow$) If there is a global stabilizer state $\phi \in \text{STAB}_n$, consider the action of the completely dephasing channel on the computational basis, $\mathcal{D}(\cdot)$, that yields:

$$\mathcal{D}(\phi) = \sum_{\mathbf{x} \in \mathbb{F}_2^n} \langle \mathbf{x} | \phi | \mathbf{x} \rangle | \mathbf{x} \rangle \langle \mathbf{x} | \equiv \sum_{\mathbf{x} \in \mathbb{F}_2^n} p(\mathbf{x}) | \mathbf{x} \rangle \langle \mathbf{x} |. \quad (\text{D6})$$

We claim that $\vec{p} = (p(\mathbf{x}))_{\mathbf{x} \in \mathbb{F}_2^n}$ is our desired global probability distribution. From the commutativity of the dephasing channel with partial traces $\mathcal{D}[\text{tr}_X(\cdot)] = \text{tr}_X \mathcal{D}(\cdot)$, we have:

$$\mathcal{D}(\phi_i) = \text{tr}_{\bar{R}_i} \mathcal{D}(\phi), \quad \forall i \in [N] \Leftrightarrow \sum_{\mathbf{x} \in \mathbb{F}_2^{|R_i|}} p_i(\mathbf{x}) | \mathbf{x} \rangle \langle \mathbf{x} | = \sum_{\mathbf{x} \in \mathbb{F}_2^{|R_i|}} \left(\sum_{\bar{\mathbf{x}} \in \mathbb{F}_2^{n-|R_i|}} p(\mathbf{x}, \bar{\mathbf{x}}) \right) | \mathbf{x} \rangle \langle \mathbf{x} |, \quad \forall i \in [N], \quad (\text{D7})$$

which is only true if $\vec{p}$ indeed is a YES instance of the CMP.

Hence, we have shown that under poly-time preprocessing, the decision of Definition 3 reduces to Definition 2.

- (2) $\text{SMP} \leq_{\text{poly}} \text{rSMP}$: Consider:

$$\mathcal{M}_{1,\dots,N} \equiv \bigcup_{i=1}^N \tilde{\mathcal{P}}_{R_i}, \quad (\text{D8})$$

where $\tilde{\mathcal{P}}_{R_i}$ are all the nontrivial $4^{|R_i|} - 1$ Pauli strings supported on R_i . Note that, furthermore:

$$|\mathcal{M}_{1,\dots,N}| \leq \sum_{i=1}^N (4^{|R_i|} - 1) \leq N(4^k - 1) \quad (\text{D9})$$

is polynomially bounded in n , since $N \leq \text{poly}(n)$ and $k = O(1)$. Since each P is supported on some R_i , we can always compute its expectation value on some reduced density matrix, say $\phi_{i(P)}$. Then, we can define:

$$\vec{V}_{1,\dots,N} \equiv (\text{tr}(P \phi_{i(P)}))_{P \in \mathcal{M}_{1,\dots,N}}. \quad (\text{D10})$$

Each expectation value can be computed in $O(1)$ time, and hence $\vec{V}_{1,\dots,N}$ can be obtained in $\text{poly}(n)$ time. We claim that $(\mathcal{M}_{1,\dots,N}, \vec{V}_{1,\dots,N})$ are the desired inputs for the reduced stabilizer membership problem: Let us show that

$$\vec{V}_{1,\dots,N} \in \text{STAB}(\mathcal{M}_{1,\dots,N}) \Leftrightarrow \exists \phi \in \text{STAB}_n: \text{tr}_{\bar{R}_i} \phi = \phi_i, \quad \forall i \in [N]. \quad (\text{D11})$$

- (a) ($\Leftarrow$): If there is a global stabilizer state that marginalizes correctly to the $\{\phi_i\}$, it follows that $\vec{V}_{1,\dots,N} = \Pi_{\mathcal{M}} \vec{\phi} \in \text{STAB}(\mathcal{M})$;
- (b) ($\Rightarrow$) If the vector lies in the reduced polytope, there is a $\phi \in \text{STAB}_n$ such that $\vec{V}_{1,\dots,N} = \Pi_{\mathcal{M}} \vec{\phi}$. However, note that, by definition on Eq. (D8), $\mathcal{M}_{1,\dots,N}$ contains a tomographic complete set of Paulis on all $\{R_i\}$, which fixes all the N reduced density matrices of the state ϕ to be exactly $\{\phi_i\}$.

Again, we have shown that the decision of Definition 1 can be made with classical poly-time by considering Definition 2. ■

APPENDIX E: TRIANGLE CRITERION AS A REDUCED POLYTOPE

In this section we prove that we recover any single triangle criterion with a proper choice of measurement set $\mathcal{M}$; as a by-product, the reduced polytope will include several distinct triangle inequalities corresponding to a family of neighbor stabilizer state triples.

Lemma 15. Let ψ_1, ψ_2, ψ_3 be pure stabilizer states on n qubits satisfying

$$\text{tr}(\psi_i \psi_j) = \frac{1}{2} \quad \text{for all } i \neq j. \quad (\text{E1})$$

Then there exist Pauli operators P_1, P_2, P_3 , and a stabilizer group S of rank $n - 1$ such that

- (1) The stabilizer group of ψ_i is $\langle S \cup \{P_i\} \rangle = S \cup P_i S$,
- (2) the operators P_i pairwise anti-commute:

$$\{P_i, P_j\} = 0 \quad \text{for all } i \neq j. \quad (\text{E2})$$

Moreover, up to a Clifford unitary, the triple $\{\psi_1, \psi_2, \psi_3\}$ is equivalent to the single-qubit eigenstates of the Pauli operators Z , X , and Y .

Proof. A standard property of pure stabilizer states that if two stabilizer states ϕ and φ satisfy $0 < |\langle \phi | \varphi \rangle| < 1$, then [64,65]:

$$|\langle \phi | \varphi \rangle|^2 = 2^{-k}; \quad k \in \{1, 2, \dots, n\}, \quad (\text{E3})$$

and furthermore, there exists a Clifford unitary C mapping them to stabilizer states whose nontrivial overlap is supported on exactly k logical qubits. That is,

$$\begin{aligned} C|\phi\rangle &= |0\rangle^{\otimes k} \otimes |0\rangle^{\otimes(n-k)}; \\ C|\psi\rangle &= \underbrace{|\psi'\rangle}_{\in \mathbb{C}^{2^{\otimes k}}} \otimes |0\rangle^{\otimes(n-k)}, \end{aligned} \quad (\text{E4})$$

where ψ' is another stabilizer state with $\langle 0|^{\otimes k} |\psi'\rangle = 2^{-k/2}$. To show this, let C_1 be the Clifford such that $C_1|\phi\rangle = |0\rangle^{\otimes n}$. Then, we have:

$$|\langle 0|^{\otimes n} C_1 |\psi\rangle|^2 = |\langle \phi | \psi \rangle|^2 = 2^{-k}. \quad (\text{E5})$$

Since $C_1|\psi\rangle$ is a pure stabilizer state, it has the following form for its computational basis expansion [66]:

$$C_1|\psi\rangle = \frac{1}{\sqrt{2^{\dim(V)}}} \sum_{\mathbf{x} \in V} \psi_{\mathbf{x}} |\mathbf{x}\rangle, \quad (\text{E6})$$

with $V \subseteq \mathbb{F}_2^n$ being a binary affine subspace and $\psi_{\mathbf{x}} \in \{\pm 1, \pm i\}$. By Eq. (E5), we find that $\dim(V) = k$. Thus, by Gaussian elimination, one can find a Clifford unitary C_2 composed of CNOTS such that [67]:

$$C_2 C_1 |\psi\rangle = |\psi'\rangle \otimes |0\rangle^{\otimes(n-k)}, \quad (\text{E7})$$

with $|\psi'\rangle$ another stabilizer state with the correct $2^{-k/2}$ overlap with $|0\rangle^{\otimes k}$. Since C_2 is defined from a decomposition of cNOTs, it stabilizes $|0\rangle^{\otimes n} = C_1|\phi\rangle$. Defining $C \equiv C_2 C_1$ gives us our desired Clifford unitary on Eq. (E4).

Applying this fact to ψ_1 and ψ_2 for $k = 1$, there exists a Clifford unitary U such that

$$\begin{aligned} \tilde{\psi}_1 &\equiv U\psi_1 U^\dagger = |0\rangle\langle 0| \otimes \tau; \\ \tilde{\psi}_2 &\equiv U\psi_2 U^\dagger = |+\rangle\langle +| \otimes \tau, \end{aligned} \quad (\text{E8})$$

with $\tau = |0\rangle\langle 0|^{\otimes(n-1)}$. Defining also $\tilde{\psi}_3 \equiv U\psi_3 U^\dagger$, note that $\text{Tr}(\tilde{\psi}_i \tilde{\psi}_j) = 1/2$ for $i \neq j$. Because $\tilde{\psi}_3$ is itself a

stabilizer state and the overlap $1/2$ is the maximal non-trivial overlap between distinct stabilizer states, it follows that ψ_3 must share the same $(n-1)$ -qubit stabilizer component $\tau = |0\rangle\langle 0|^{\otimes(n-k)}$. Hence,

$$\tilde{\psi}_3 = |\chi\rangle\langle \chi| \otimes \tau \quad (\text{E9})$$

for some single-qubit stabilizer state $|\chi\rangle$ satisfying

$$|\langle 0 | \chi \rangle|^2 = |\langle + | \chi \rangle|^2 = \frac{1}{2}. \quad (\text{E10})$$

Since the only solutions are $|y_{\pm}\rangle = (|0\rangle \pm i|1\rangle)/\sqrt{2}$, we have:

$$|\chi\rangle = |y_{\pm}\rangle := \frac{|0\rangle \pm i|1\rangle}{\sqrt{2}}, \quad (\text{E11})$$

namely a Y -eigenstate. Consequently, up to the Clifford unitary U , the three stabilizer states are precisely

$$|0\rangle, \quad |+\rangle, \quad |y_{\pm}\rangle, \quad (\text{E12})$$

which are eigenstates of the Pauli operators

$$Z, \quad X, \quad \pm Y, \quad (\text{E13})$$

respectively. We may therefore define

$$P_1 = U^\dagger Z_1 U, \quad P_2 = U^\dagger X_1 U, \quad P_3 = U^\dagger Y_1 U, \quad (\text{E14})$$

where X_1, Y_1, Z_1 denote the Pauli operators acting on the first qubit. Since Clifford conjugation preserves the Pauli group and preserves commutation relations, the operators P_i are Pauli strings satisfying

$$\{P_i, P_j\} = 0 \quad (i \neq j), \quad (\text{E15})$$

and each ψ_i is an eigenstate of P_i . For completeness, the common stabilizer group S is built from the image of the generators of the stabilizer group of τ under conjugation by U :

$$S = \langle U^\dagger Z_2 U, U^\dagger Z_3 U, \dots, U^\dagger Z_n U \rangle, \quad (\text{E16})$$

and thus $\{P_i, U^\dagger Z_2 U, U^\dagger Z_3 U, \dots, U^\dagger Z_n U\}$ is a list of generators for the stabilizer group of ψ_i . ■

We now have a compact description of the relevant Pauli operators corresponding to a single neighbor triple; a natural question is the following: if we choose the measuring set $\mathcal{M}$ to contain exactly the three stabilizer groups associated with a neighboring triple, does the reduced polytope include the triangle criterion as a facet? We answer this question in the affirmative and also characterize all the other facets:

Theorem 16. Let ψ_i be three neighboring stabilizer states, with stabilizer groups $S_i := \langle S \cup \{P_i\} \rangle$, and consider the measuring set $\mathcal{M} = S_1 \cup S_2 \cup S_3$. The facets of the reduced polytope are exactly the triangle criterion for all possible consistent choices of changing signs in the generators of S and independent choices for the signs of P_i .

Before we prove the theorem, let us fix some notation. Given a mixed state ρ and some $P \in \mathcal{M}$, let

$$r_P = \text{Tr}(P\rho), \quad (\text{E17})$$

be its Pauli coordinates. We will define another natural set of coordinates.

Let S^* denote the character group of S . For each character $\chi: S \rightarrow \{\pm 1\}$, define the stabilizer projector

$$\Pi_\chi = \frac{1}{|S|} \sum_{Q \in S} \chi(Q) Q. \quad (\text{E18})$$

We introduce syndrome coordinates by

$$p_\chi = \text{Tr}(\Pi_\chi \rho) = \frac{1}{|S|} \sum_{Q \in S} \chi(Q) r_Q, \quad (\text{E19})$$

and

$$q_{\chi, P_i} = \text{Tr}(\Pi_\chi P_i \rho) = \frac{1}{|S|} \sum_{Q \in S} \chi(Q) r_{QP_i}. \quad (\text{E20})$$

Since this is just the Fourier transform over the Abelian group S , it is invertible. So we can express the original coordinates in terms of the syndrome coordinates:

$$r_Q = \sum_{\chi \in S^*} \chi(Q) p_\chi, \quad r_{QP_i} = \sum_{\chi \in S^*} \chi(Q) q_{\chi, P_i}. \quad (\text{E21})$$

In particular,

$$\sum_{\chi \in S^*} p_\chi = r_I = \text{tr}(\rho) = 1. \quad (\text{E22})$$

Proposition 17. In the syndrome-logical coordinates, the reduced stabilizer polytope $\text{STAB}(\mathcal{M})$ is given by

$$\text{STAB}(\mathcal{M}) = \left\{ (p_\chi, q_{\chi, P_1}, q_{\chi, P_2}, q_{\chi, P_3})_{\chi \in S^*} : \sum_{\chi} p_\chi = 1, \right. \quad (\text{E23})$$

$$\left. |q_{\chi, P_1}| + |q_{\chi, P_2}| + |q_{\chi, P_3}| \leq p_\chi \quad \forall \chi \in S^* \right\}. \quad (\text{E24})$$

Proof. First, take any stabilizer mixture ρ . Measuring the commuting stabilizer context S and postselecting the syndrome corresponding to χ is a stabilizer operation. Therefore, whenever $p_\chi > 0$, the normalized logical state in that syndrome sector is a one-qubit stabilizer mixture. Namely, its Bloch vector is

$$\left(\frac{q_{\chi, P_1}}{p_\chi}, \frac{q_{\chi, P_2}}{p_\chi}, \frac{q_{\chi, P_3}}{p_\chi} \right). \quad (\text{E25})$$

The one-qubit stabilizer polytope is the octahedron, so

$$\left| \frac{q_{\chi, P_1}}{p_\chi} \right| + \left| \frac{q_{\chi, P_2}}{p_\chi} \right| + \left| \frac{q_{\chi, P_3}}{p_\chi} \right| \leq 1. \quad (\text{E26})$$

Multiplying by $p_\chi \geq 0$ gives the inequalities in Eq. (E23). If $\text{Tr}(\Pi_\chi \rho) \equiv p_\chi = 0$, positivity of the projection $\Pi_\chi \rho \Pi_\chi$ implies it is actually the zero matrix, and $q_{\chi, P_1} = q_{\chi, P_2} = q_{\chi, P_3} = 0$, so the same inequality holds.

We now justify the converse construction in Eq. (E23). The only point that has to be checked is that the different syndrome sectors do not interfere with each other.

Claim: The projectors Π_χ are mutually orthogonal:

$$\Pi_\chi \Pi_{\chi'} = \frac{1}{|S|^2} \sum_{P, Q \in S} \chi(P) \chi'(Q) PQ = \delta_{\chi, \chi'} \Pi_\chi. \quad (\text{E27})$$

In fact, writing $R = PQ$, so that $Q = PR$, and using the multiplicativity of characters, we get

$$\Pi_\chi \Pi_{\chi'} = \frac{1}{|S|^2} \sum_{R \in S} \left(\sum_{P \in S} \chi(P) \chi'(P) \right) \chi'(R) R. \quad (\text{E28})$$

By character orthogonality,

$$\sum_{P \in S} \chi(P) \chi'(P) = \begin{cases} |S|, & \chi = \chi', \\ 0, & \chi \neq \chi', \end{cases} \quad (\text{E29})$$

and the claim follows. Thus the projectors Π_χ decompose the Hilbert space into orthogonal syndrome sectors.

Now assume that the coordinates satisfy

$$\sum_{\chi} p_\chi = 1, \quad |q_{\chi, P_1}| + |q_{\chi, P_2}| + |q_{\chi, P_3}| \leq p_\chi \quad \forall \chi. \quad (\text{E30})$$

For each χ with $p_\chi > 0$, define

$$u_\chi = \left(\frac{q_{\chi, P_1}}{p_\chi}, \frac{q_{\chi, P_2}}{p_\chi}, \frac{q_{\chi, P_3}}{p_\chi} \right). \quad (\text{E31})$$

By Eq. (E30), each u_χ belongs to a one-qubit stabilizer octahedron; let $\{e_1, e_2, e_3\}$ be the canonical basis in $\mathbb{R}^3$:

$$u_\chi = \sum_{i=1}^3 \sum_{t=\pm 1} \lambda_{\chi,E,t} t e_i, \quad \lambda_{\chi,E,t} \geq 0, \quad \sum_{E,t} \lambda_{\chi,E,t} = 1. \quad (\text{E32})$$

Hence, the full state corresponding to the syndrome χ can be written as a convex combination of the six one-qubit stabilizer vertices, namely the six n -qubit stabilizer states defined by

$$\omega_{\chi,P}^t = \frac{1}{2} \Pi_\chi (I + tP), \quad E \in \{P_1, P_2, P_3\}, \quad t \in \{\pm 1\}. \quad (\text{E33})$$

These are pure stabilizer states: Π_χ fixes the $n-1$ stabilizers in the common context C , while $I + tE$ fixes the logical Pauli E with eigenvalue t .

Set

$$\rho_\chi = \sum_{E,t} \lambda_{\chi,E,t} \omega_{\chi,E}^t. \quad (\text{E34})$$

Then ρ_χ is a stabilizer mixture supported entirely in the syndrome sector χ . Moreover,

$$\text{Tr}(\Pi_\chi \rho_\chi) = 1, \quad (\text{E35})$$

and

$$\text{Tr}(\Pi_\chi P_1 \rho_\chi) = u_{\chi,P_1}, \quad \text{Tr}(\Pi_\chi P_2 \rho_\chi) = u_{\chi,P_2}, \quad \text{Tr}(\Pi_\chi P_3 \rho_\chi) = u_{\chi,P_3}. \quad (\text{E36})$$

If $p_\chi = 0$, we simply omit that sector from the convex combination.

Finally define

$$\rho = \sum_{\chi: p_\chi > 0} p_\chi \rho_\chi. \quad (\text{E37})$$

Since $p_\chi \geq 0$ and $\sum_\chi p_\chi = 1$, this is a convex combination of stabilizer states. By the orthogonality relation in Eq. (E27), different syndrome sectors do not mix. Therefore, for every χ ,

$$\text{Tr}(\Pi_\chi \rho) = p_\chi, \quad (\text{E38})$$

and

$$\text{Tr}(\Pi_\chi P_1 \rho) = q_{\chi,P_1}, \quad \text{Tr}(\Pi_\chi P_2 \rho) = q_{\chi,P_2}, \quad \text{Tr}(\Pi_\chi P_3 \rho) = q_{\chi,P_3}. \quad (\text{E39})$$

Hence every point satisfying the block-octahedron constraints in Eq. (E30) is realized by a stabilizer mixture. This proves the converse inclusion. ■

The description in Eq. (E23) is a direct sum of subnormalized one-qubit stabilizer octahedra, one for each syndrome χ . Therefore its facets are immediate.

Theorem 18. The facets of $\text{STAB}(\mathcal{M})$ are exactly the inequalities

$$p_\chi + s_{P_1} q_{\chi,P_1} + s_{P_2} q_{\chi,P_2} + s_{P_3} q_{\chi,P_3} \geq 0, \quad (\text{E40})$$

for

$$\chi \in S^*, \quad (s_{P_1}, s_{P_2}, s_{P_3}) \in \{\pm 1\}^3. \quad (\text{E41})$$

Equivalently, in the original Pauli coordinates, the facets are

$$\frac{1}{|S|} \sum_{Q \in S} \chi(Q) [r_Q + s_{P_1} r_{QP_1} + s_{P_2} r_{QP_2} + s_{P_3} r_{QP_3}] \geq 0. \quad (\text{E42})$$

These are precisely the triangle inequalities obtained by considering every possible syndrome χ in the common stabilizer group S , and every sign for the anticommuting triple P_1, P_2, P_3 , accounting for 2^{n+2} facets.

Proof. For each fixed χ , Eq. (E23) imposes the subnormalized octahedron constraint

$$|q_{\chi,P_1}| + |q_{\chi,P_2}| + |q_{\chi,P_3}| \leq p_\chi. \quad (\text{E43})$$

This is equivalent to the eight linear inequalities

$$p_\chi + s_{P_1} q_{\chi,P_1} + s_{P_2} q_{\chi,P_2} + s_{P_3} q_{\chi,P_3} \geq 0, \quad (s_{P_1}, s_{P_2}, s_{P_3}) \in \{\pm 1\}^3. \quad (\text{E44})$$

Together with the affine normalization $\sum_{\chi} p_{\chi} = 1$, these inequalities give the full H-representation of the polytope in Eq. (E23). Let us check that each inequality is facet-defining; since each family of eight inequalities (E44) is uncoupled from the others, we may count the dimension of each syndrome block separately.

The ambient space has dimension $4|S^*| = 2^{n+1}$, and the normalization $\sum_{\chi} p_{\chi} = 1$ reduces the affine dimension to

$$\dim \text{STAB}(\mathcal{M}) = 2^{n+1} - 1. \quad (\text{E45})$$

Now let us compute the dimension of the exposed facets. Fix χ_0 and $s = (s_{P_1}, s_{P_2}, s_{P_3})$. The equality

$$p_{\chi_0} + s_{P_1} q_{\chi_0, P_1} + s_{P_2} q_{\chi_0, P_2} + s_{P_3} q_{\chi_0, P_3} = 0 \quad (\text{E46})$$

defines a facet of the four-dimensional cone

$$\{(p, q_{P_1}, q_{P_2}, q_{P_3}) : |q_{P_1}| + |q_{P_2}| + |q_{P_3}| \leq p\}. \quad (\text{E47})$$

Thus it has dimension 3 in the χ_0 -block. The remaining $|S^*| - 1$ syndrome blocks are unconstrained except for

the global normalization, and contribute $4(|S^*| - 1) - 1$ dimensions. Hence the total dimension of the exposed face is

$$3 + 4(|S^*| - 1) - 1 = 2^{n+1} - 2 = \dim \text{STAB}(\mathcal{M}) - 1. \quad (\text{E48})$$

Therefore each inequality in Eq. (E40) defines a facet. Since Eq. (E44) gives the full H-representation and all its inequalities are irredundant, there are no other facets.

Finally, Eq. (E42) follows from substituting the Fourier definitions in Eqs. (E19) and (E20). To identify it with a triangle inequality, define the stabilizer states

$$\psi_{\chi, P}^t = \frac{1}{2} \Pi_{\chi}(I + tP), \quad P \in \{P_1, P_2, P_3\}, \quad t \in \{\pm 1\}. \quad (\text{E49})$$

These three states are pairwise neighbor stabilizer states whenever the logical Paulis are distinct and anticommute. Moreover,

$$\text{Tr} \left[\left(\psi_{\chi, P_1}^{s_{P_1}} + \psi_{\chi, P_2}^{s_{P_2}} - \psi_{\chi, P_3}^{-s_{P_3}} \right) \rho \right] = \frac{1}{2} (p_{\chi} + s_{P_1} q_{\chi, P_1} + s_{P_2} q_{\chi, P_2} + s_{P_3} q_{\chi, P_3}). \quad (\text{E50})$$

Thus Eq. (E40) is equivalent to the triangle witness

$$W_{\chi, s} = \psi_{\chi, P_1}^{s_{P_1}} + \psi_{\chi, P_2}^{s_{P_2}} - \psi_{\chi, P_3}^{-s_{P_3}}. \quad (\text{E51})$$

This proves that the facets are exactly the triangle inequalities associated with the syndrome χ and the logical octahedron direction s .

Therefore, for the measurement set $\mathcal{M}$, the reduced stabilizer polytope has exactly

$$8|S^*| = 8 \cdot 2^{n-1} = 2^{n+2} \quad (\text{E52})$$

facets, all of which are triangle inequalities. ■

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