

Randomized Adiabatic Quantum Linear Solver Algorithm with Optimal Complexity Scaling and Detailed Running Costs

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Solving linear systems of equations is a fundamental problem with a wide variety of applications across many fields of science, and there is increasing effort to develop quantum linear solver algorithms. Subaşı *et al.* [Phys. Rev. Lett. 122, 060504 (2019)] proposed a randomized algorithm inspired by adiabatic quantum computing, based on a sequence of random Hamiltonian simulation steps, with suboptimal scaling in the condition number κ of the linear system and the target error ϵ . Here we go beyond these results in several ways. Firstly, using filtering [Lin and Tong, Quantum 4, 361 (2020)] and Poissonization techniques [Cunningham and Roland, ArXiv:2406.03972 (2024)], the algorithm complexity is improved to the optimal scaling $O(\kappa \log(1/\epsilon))$ —an exponential improvement in ϵ , and a shaving of a $\log \kappa$ scaling factor in κ . Secondly, the algorithm is further modified to achieve constant factor improvements, which are vital as we progress towards hardware implementations on fault-tolerant devices. We introduce a cheaper randomized walk operator method replacing Hamiltonian simulation—which also removes the need for potentially challenging classical precomputations; randomized routines are sampled over optimized random variables; circuit constructions are improved. We obtain a closed formula rigorously upper bounding the expected number of times one needs to apply a block-encoding of the linear system matrix to output a quantum state encoding the solution to the linear system. The upper bound is 837κ at $\epsilon = 10^{-10}$ for Hermitian matrices.

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I. INTRODUCTION

Given an $N \times N$ matrix A and a N -dimensional vector $\mathbf{b}$, quantum linear solver algorithms (QLSAs) are tasked with returning the solution of a linear system $A\mathbf{y} = \mathbf{b}$ encoded as a quantum state. Linear systems are ubiquitous, since many problems admit reductions to them. Notably, many quantum algorithms for linear [1–6] and nonlinear [3,7–12] differential equations rely on linear solvers. Other applications include data fitting [13], scattering [14], machine learning [15,16], and optimization [12,17].

Formally, given $\epsilon > 0$, a QLSA outputs a quantum state ϵ -close to a vector $|y\rangle \propto A^{-1}|b\rangle$, giving the solution vector $\mathbf{y} = A^{-1}\mathbf{b}$ encoded as a quantum state. The cost of the algorithm is given in terms of its *query complexity* Q , i.e., the number of times we need to apply unitaries (“oracles”) for state preparation of $|b\rangle$ and a block-encoding of A ,

namely a unitary encoding A/α for some $\alpha > 0$ in one of its blocks. A worst-case upper bound for the running cost of the algorithm can be given as a function of three parameters: $(\epsilon, \kappa, \alpha)$, where κ is an upper bound on the condition number of the matrix and α is a rescaling constant, which depends on the specific block-encoding construction. Running QLSAs on the early generations of fault-tolerant quantum computers will only be feasible if we are able to (1) bring down their cost and (2) successfully incorporate them within an end-to-end quantum algorithm. The present manuscript focuses on the first problem. We introduce a new randomized QLSA and formally prove its performance is competitive with the state-of-the-art.

After Harrow, Hassidim, and Lloyd (HHL) developed the first QLSA [18], an extensive literature focused on bringing down the *asymptotic scaling* of Q . The original HHL algorithm had an $O(\kappa^2/\epsilon)$ complexity. The dependence on the condition number was almost quadratically improved to $O((\kappa/\epsilon^3)\text{polylog}(\kappa/\epsilon))$ by Ambainis, at the price of a worse error scaling [19]. The poor dependence on ϵ was remedied by Childs *et al.* [20], who obtained an $O(\kappa \text{polylog}(\kappa/\epsilon))$ algorithm, and a similar scaling was

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TABLE I. Comparison of the randomized adiabatic walk method proposed in this work with alternative optimal quantum linear solvers for which explicit, rigorous query complexity bounds have been derived. The values in parenthesis in the third column refer to the special case in which the linear system of equation is Hermitian. The practical performance of all these methods can be expected to be significantly better than what the upper bounds suggest. An initial numerical study of the deterministic adiabatic method and a prior non optimal version of the present algorithm has been done in Ref. [25]. In future work, it would be of interest to perform a detailed numerical study of the present optimal algorithm, and compare it with the other asymptotically optimal algorithms on a range of benchmark problems.

Quantum linear solver	Asymptotic complexity	Explicit upper bound at $(\alpha, \kappa, \epsilon) = (1, 10^6, 10^{-10})$	Phase angle computation
Randomized adiabatic method [22]	$O(\kappa \log(\kappa)/\epsilon)$	1.03×10^{17} (5.14×10^{16})	Yes
Deterministic adiabatic walk method [24]	$O(\kappa \log(1/\epsilon))$	2.35×10^{11} (1.17×10^{11})	No
QSVT reflection and projection [26]	$O(\kappa \log(1/\epsilon))$	8.02×10^7 (8.02×10^7)	Yes
Randomized adiabatic walk method	$O(\kappa \log(1/\epsilon))$	1.72×10^9 (8.61×10^8)	No

also realized in Ref. [21]. A drawback of these works is that they involve a complex “variable-time amplitude-amplification” routine. Using a technique inspired by adiabatic quantum computing, Subaşı *et al.* [22] introduced an adiabatic randomized algorithm that removed the need for such a routine, but with scaling $O(\kappa \log(\kappa)/\epsilon)$, which, however, is exponentially worse in ϵ than Ref. [20]. Lin *et al.* introduced a filtering technique [23], providing an alternative path to an exponentially improved error scaling. This technique was exploited in the QLSA by Costa *et al.* [24], to achieve a claimed optimal $O(\kappa \log(1/\epsilon))$ scaling, with a worst-case constant prefactor to κ upper bounded by 2×10^5 . Numerical studies of the average-case constant prefactors on randomized low-dimensional and low condition number instances ($N \leq 16$, $\kappa \leq 50$) shows the average-case constant prefactors are much lower than the worst-case upper bound [25]. These studies cannot be scaled to (N, κ) relevant in applications, so we need to rely on extrapolation.

Recently, Dalzell [26] proposed an elegant QLSA, that achieves an optimal scaling of $O(\kappa \log(1/\epsilon))$ with much stronger guarantees on the worst-case prefactor to κ , upper bounded by 80. This is the current state-of-the-art for quantum resource estimates with rigorous cost guarantees. The picture has been recently complicated by the introduction of novel techniques that allow improvement in the complexity in the number of calls to the unitary preparing $|b\rangle$ at the price of a small overhead on the number of calls to the block-encoding of A , with constant prefactor analysis yet to be carried out [27]. See Ref. [28] for references to a growing literature on the topic.

In this work we focus on showing that one can obtain state-of-the-art rigorous performance guarantees for randomized adiabatic QLSA. Starting from the randomized adiabatic algorithm in Ref. [22], we apply “Poissonization” [29] with an optimized schedule, filtering techniques [23], and a novel randomized walk operator method that entirely foregoes the need for a Hamiltonian simulation subroutine. Combining these and smaller optimizations at the level of block-encoding constructions, the complexity scaling is

improved from the $O(\kappa/\epsilon \log \kappa)$ of the original proposal [22] to the optimal $O(\kappa \log(1/\epsilon))$, with constant prefactors of about 1734 (867 for Hermitian matrices), a factor of 10–20 from the best available bounds for any QLSA [26]. This puts adiabatic randomized methods on the map of the most competitive QLSA with rigorous non asymptotic cost guarantees. In terms of the broader context and timeliness of our work, we would note the following. Firstly, we expect further improvements can be made on the adiabatic line by exploiting more refined techniques. Secondly, a concrete virtue of our method over prior results is its simplicity of implementation—for example, we do not require the computation of any quantum singular value transform (QSVT) phase factors. This could make it well-suited to near- and mid-term implementations on actual quantum hardware. Thirdly, while there are now a few optimal linear solvers, which can be compared in the worst-case via the constant pre factors, this simple comparison need not hold for particular classes of problems. Ultimately it will be important to compare different linear solvers on important problems of interest, and one method could be more easily tailored to a specific problem at hand (similar to Hamiltonian simulation via Trotterization versus QSVT). Given the elementary ingredients of our algorithm, we expect that it will be well-suited to further optimization and application. In summary, given the very small handful of optimal linear solvers in existence (see Table I for a comparison), we expect that going forward the core method developed here has the ability to both compete with and complement existing methods.

A. Overview of contributions

The high-level goal of fault-tolerant adiabatic algorithms is to encode the normalized solution $|y\rangle$ of the linear system into the nullspace of a Hamiltonian $H(1)$. To obtain $|y\rangle$, we prepare an eigenstate of zero eigenvalue of a simple Hamiltonian $H(0)$ and then change the Hamiltonian to $H(1)$ along a discrete trajectory $H(s_j)$, where each $H(s_j)$ is

constructed from the data A and $|b\rangle$. The scheduling is constructed so that one has rigorous guarantees on the quality of the output.

The original algorithm [22] fixes a set of $s_j \in [0, 1]$, for $j = 1, \dots, q$, and at each point it performs a Hamiltonian simulation routine for a randomized time t_j . The time t_j is sampled from a uniform distribution over an interval wide enough that it approximatively dephases the system with respect to the eigenbasis of $H(s_j)$, up to a controlled error. This sequence of dephasings probabilistically maps the zero eigenstate of $H(s_{j-1})$ into that of $H(s_j)$ at each step $j = 1, \dots, q$. Roughly speaking, the minimal gap of the sequence of Hamiltonians is $O(\kappa)$, leading to $t_j = O(\kappa)$, and $q = O(\epsilon^{-1} \log \kappa)$. Overall, this leads to the $O(\epsilon^{-1} \kappa \log \kappa)$ complexity scaling reported in Ref. [22].

We modify the original algorithm [22] in a number of ways:

- (1) We generate s_j in accordance with a Poisson process with a rate depending on the gap as proposed and discussed in Ref. [29], which leads to $O(\log \kappa)$ in savings. We optimize the rate and tighten the error analysis to achieve constant factor savings.
- (2) Inspired by previous works [30,31], we show that we can entirely forego the Hamiltonian simulation routines. Dephasing is instead achieved by applying a walk operator $W(s_j)$, that is constructed from a block-encoding of $H(s_j)$, a random number of times m . This simplifies the quantum algorithm and reduces its cost. It also removes the need for classical precomputations of phase angles at each j , which is required by state-of-the-art Hamiltonian simulation techniques based on quantum signal processing [32].
- (3) Using results from eigenpath traversal theory [30], we sample m at each s_j from an optimized non uniform random time variable, leading to constant factor savings.
- (4) We prepare a state with a constant error $\epsilon = 1/2$ and then apply filtering techniques [23,24], which probabilistically prepares a state close to the target with exponential savings in ϵ , from $O(1/\epsilon)$ to $O(\log(1/\epsilon))$.

Another difference from Ref. [22] is that we perform a detailed cost analysis, providing analytical, worst-case non asymptotic bounds on the number, Q , of applications of the core unitary block-encodings given α , κ , and ϵ . The following is our main result:

Theorem 1 (Optimal QLSA with explicit counts). Consider a system of linear equations $A\mathbf{y} = \mathbf{b}$, where A is an $N \times N$ -dimensional matrix scaled so that the singular values of A lie in $[1/\kappa, 1]$. Denote by $|b\rangle$ the normalized state that is proportional to $\sum_i b_i |i\rangle$, and by $|A^{-1}b\rangle$ the

normalized state proportional to $A^{-1}|b\rangle$. Assume access to

- (1) A unitary U_A that encodes the matrix A/α in its top-left block, for some constant $\alpha \geq 1$, using a number of ancilla qubits equal to a ,
- (2) An oracle, U_b , preparing $|b\rangle$.

Then, there is a randomized quantum algorithm that outputs a quantum state ϵ -close in 1-norm to $|A^{-1}b\rangle$, using, in the worst case, an expected number of calls Q^* to (controlled) U_A or $U_A^\dagger$ and $2Q^*$ to (controlled) U_b or $U_b^\dagger$, where $Q^* = O(\kappa \log 1/\epsilon)$. Specifically,

$$Q^* \leq 835.4\alpha\kappa + \alpha\kappa \ln \frac{2}{\sqrt{1+\epsilon/4}-1} + 3.$$

The success probability is lower bounded by $1/2 - \epsilon/4$. The algorithm requires $a + 7 + \lceil \log_2 N \rceil$ logical qubits. A factor of 2 to the cost and an ancilla qubit can be saved if A is Hermitian. The expected query complexity including the failure probability is

$$Q = 2Q^*/(1 - \epsilon/2). \quad (1)$$

We highlight that, since the algorithm has a randomized component, for a fixed instance the cost of the algorithm is a random variable. We upper bound the average cost taking the worst case over all $N \times N$ matrices with fixed condition number upper bound, κ . Alternatively, since the single-shot failure probability is $1/2 + \epsilon/4 \approx 1/2$, if we run the simulation a number $\sim \log_2(1/\delta)$ times, we reduce the failure probability to any $\delta > 0$ and obtain a query count $\lceil Q^* \log_2(1/\delta) \rceil$. The parameter α is common to all algorithms invoking a block-encoding of the linear system matrix, and its value depends on the underlying access model to A , whose choice should be tailored to each specific problem. A number of constructions exist for sparse matrices [33–35], which have α equal to the geometric mean of column and row sparsity, and furthermore have $\text{polylog}(N) \log(1/\epsilon)$ gate cost if the position of the non zero elements and their values are efficiently computable [33–35]. Under these assumptions $a = O(\text{polylog}(N))$. Other constructions allow to efficiently block-encode dense matrices with special structure [36, 37]. Finally, note that the above cost does not include the extraction of application-specific relevant information from the solution vector.

B. Overview of the algorithm

We now provide a description of the core components of the algorithm and leave the detailed analysis to the rest of the paper. A summary of the workflow is given at the end of the section.

Step 1: Setting up the problem and constructing the oracles. A linear system $Ay = b$ is given, together with an upper bound, κ , on the condition number of A and an error tolerance, ϵ . Via an appropriate rescaling of the linear system, discussed in Sec. II A, we can always take A to be an $N \times N$ non singular matrix with $\|A\| \leq 1$, where $\|A\|$ is the operator norm of A .

We assume access to two unitaries (the “oracles”) that are at the basis of QLSA:

- (1) A unitary $U_b|0\rangle := |b\rangle \propto \sum_{j=1}^N b_j |j\rangle$, encoding b as a quantum state when applied to a reference state $|0\rangle$.
- (2) A unitary block-encoding of A , i.e., a unitary matrix U_A with the block form

$$U_A = \begin{bmatrix} A/\alpha & * \\ * & * \end{bmatrix}, \quad (2)$$

where $\alpha \geq 1$ without loss of generality and stars indicate any additional matrices consistent with unitarity.

More formally, an (α, a, ϵ_M) -block-encoding of a $2^n \times 2^n$ matrix M is a $2^{a+n} \times 2^{a+n}$ unitary U_M such that

$$U_M |0^a\rangle |\psi_n\rangle = |0^a\rangle \frac{M'}{\alpha} |\psi_n\rangle + |\perp\rangle, \quad (3)$$

where $(\langle 0^a| \otimes I_n)|\perp\rangle = 0$ and $\|M - M'\| \leq \epsilon_M$. Here $|\psi_n\rangle$ denotes an arbitrary n -qubit state, $|0^a\rangle := |0\rangle^{\otimes a}$, and I_n is the identity over n qubits. In other words, U_M encodes in a block (identified by the first a qubits being in state zero) a matrix proportional to M' , which is ϵ_M -close to M .

The central goal of a QLSA is to output a sufficiently good approximation to the quantum state $|y\rangle$, where

$$|y\rangle \propto A^{-1}|b\rangle. \quad (4)$$

The difficulty of the problem is quantified in terms of the *query complexity* Q , i.e., the number of times we need to implement (controlled) U_A or $U_A^\dagger$, U_b or $U_b^\dagger$ in order to realize this output. In this work we shall be problem-agnostic, and assume access to an $(\alpha, a, 0)$ -block-encoding of the linear system matrix A , although the extension to $\epsilon_A > 0$ is straightforward.

Step 2: Hamiltonian encoding and Poisson adiabatic trajectory. Extending the approach from Ref. [22], we drop the assumption that A is Hermitian and encode the solution of the *Hermitian extension* of the linear system as an eigenstate of zero eigenvalue of a particular Hamiltonian. Specifically, we introduce

$$H(s) = |0\rangle\langle 1| \otimes A(s)\Pi + |1\rangle\langle 0| \otimes \Pi A(s), \quad s \in [0, 1], \quad (5)$$

where we have,

$$\begin{aligned} \Pi &:= I - |+, 0, b\rangle\langle +, 0, b| \\ A(s) &:= (1-s)Z \otimes I + sX \otimes \bar{A}, \end{aligned} \quad (6)$$

and

$$\bar{A} = |0\rangle\langle 1| \otimes A + |1\rangle\langle 0| \otimes A^\dagger \quad (7)$$

is the Hermitian extension of A . The linear system associated to $\bar{A}$ is

$$\bar{A}|1\rangle|y\rangle = |0\rangle|b\rangle. \quad (8)$$

Since $\bar{A}^{-1} = |1\rangle\langle 0| \otimes A^{-1} + |0\rangle\langle 1| \otimes (A^\dagger)^{-1}$, using Eq. (4) the solution to the extended linear system is

$$\bar{A}^{-1}|0\rangle|b\rangle = |1\rangle A^{-1}|b\rangle \propto |1, y\rangle. \quad (9)$$

If $N = 2^n$, then $H(s)$ defines a one-parameter family of Hamiltonians on $n + 3$ qubits. Setting

$$|y(s)\rangle \propto |0\rangle \otimes A(s)^{-1}|+, 0, b\rangle, \quad (10)$$

we have that the family of states $\{|y(s)\rangle\}$ are eigenstates in the nullspace of $H(s)$:

$$H(s)|y(s)\rangle = 0, \quad \text{for all } s \in [0, 1]. \quad (11)$$

Moreover, for every s the nullspace of $H(s)$ is two-dimensional, with $\{|y(s)\rangle, |1, +, 0, b\rangle\}$ providing an orthonormal basis. The input $|b\rangle$ and normalized solution $|y\rangle$ to the linear system are encoded in eigenstates of zero eigenvalue of $H(s)$ via

$$|y(0)\rangle = |0, -, 0, b\rangle, \quad |y(1)\rangle = |0, +, 1, y\rangle. \quad (12)$$

The intuition from the quantum adiabatic theorem is that if we start from a preparation of $|y(0)\rangle$ and evolve under the Hamiltonian $H(s)$ while changing the parameter s sufficiently slowly, we output a good approximation to $|y(1)\rangle$. This works even if the nullspace of $H(s)$ is a two-dimensional space spanned by $\{|y(s)\rangle, |1, +, 0, b\rangle\}$, because the evolution under $H(s)$ does not cause any transition between the two orthogonal eigenstates in the nullspace.

More precisely, the adiabatic protocol proceeds in discrete steps as follows: we fix $\gamma \in (0, 1)$, where $1 - \gamma$ encodes the fidelity to which the adiabatic protocol tries to prepare $|y(1)\rangle$. Given κ and γ and a small interval ds around s , we generate a “dephasing event” at s with a probability $\lambda(s)ds$, i.e., according to a Poisson process whose

rate $\lambda(s)$ is larger where the Hamiltonian gap is smaller. We take the general form [29]

$$\lambda(s) = \frac{C(\gamma)}{\Delta(s)^q \Delta_{\min}^{1-q}}, \quad (13)$$

where $\Delta(s)$ is a lower bound on the gap between the zero and non zero energies of $H(s)$ and $\Delta_{\min} = \min_{s \in [0,1]} \Delta(s)$. This is different from Ref. [22], where a set of s_j was pre-determined as a function of κ, γ . We optimize the schedule and find that $q = 1/2$ gives the best upper bounds, so we make this choice from now on. We also find that $C = 68.6$ suffices for $\gamma = 1/2$. We can take [38]

$$\Delta(s) = \sqrt{(1-s)^2 + (s/\kappa)^2}, \quad \Delta_{\min} = (1 + \kappa^2)^{-1/2}.$$

For detailed derivations, see Sec. II B.

Step 3: Dephasing events via the randomized walk method. For each s_j associated to a dephasing event, we want to realize a dephasing in the eigenbasis of $H(s_j)$. More formally, the aim is to effect a quantum channel [30]

$$\mathcal{P}_j(\rho) = P(s_j)\rho P(s_j) + \mathcal{E}_j \circ (I - P(s_j))\rho(I - P(s_j)), \quad (14)$$

where $P(s_j)$ is the projector onto the nullspace of $H(s_j)$ and $\mathcal{E}_j$ is a channel mapping the non zero eigenspaces onto themselves. $\mathcal{P}_j$ acts as a non-selective measurement in the eigenbasis of $H(s_j)$, evolving the instantaneous eigenstate from $|y(s_{j-1})\rangle$ to $|y(s_j)\rangle$ with sufficiently high probability.

The original proposal was to perform at each s_j a randomized Hamiltonian simulation for a time t_j sampled from a uniform random variable [22]. However, this realizes a channel of the form Eq. (14) only approximately and hence leads to overheads that require further analysis [39]. This can be avoided as follows. We can instead sample t_j according to a probability distribution $p_{s_j}(t)$, and correspondingly perform Hamiltonian simulation for time $\alpha_{s_j} t_j$, where α_{s_j} is the block-encoding rescaling factor for $H(s_j)$, which we shall discuss later. A channel of the form Eq. (14) is then realized up to an error given by the maximum of the characteristic function of $p_{s_j}(t)$ evaluated at the non-zero energy eigenvalues of $H(s_j)$ [30]. Therefore, if one uses a $p_{s_j}(t)$ that is bandwidth limited to $[-\Delta(s_j), \Delta(s_j)]$, the error vanishes and we realize a channel of the form $\mathcal{P}_j$ exactly. We need to find a probability distribution with this property that minimizes $\langle t_j \rangle$ [40]. Luckily, this problem has been studied before. Reference [30] proposed $p_{s_j}(t) \propto \text{sinc}^4(\Delta(s_j)t/4)$. Reference [41], in the context of combinatorial optimization algorithms, proposed a numerically optimized polynomial ansatz for the characteristic function of a probability distribution satisfying the properties we need, which returned a 46th-order polynomial (pages 35–36).

Here, we simplify matters by explicitly constructing the probability distribution

$$p_{s_j}(t) \propto \frac{1}{\Delta(s_j)} \left(\frac{J_r(\Delta(s_j)|t|/2)}{\Delta^{r-1}(s_j)|t|^r} \right)^2, \quad (15)$$

where $J_r(z)$ is the Bessel function of the first kind of order $r = 1.165$. This is obtained from an optimized quadratic order characteristic function, and has a value of $\langle t_j \rangle$ within 0.0022% of the proposal in Ref. [41]. It has $\langle |t_j| \rangle = 2.32132/\Delta(s_j)$, and its variance is $9.36238/(\Delta(s_j))^2$.

This leaves us with the requirement to perform, at each s_j , Hamiltonian simulation for a time $\alpha_{s_j} t_j$, with t_j sampled from Eq. (15). The asymptotically best algorithms for Hamiltonian simulation are based on quantum signal processing (QSP) [32] or generalized QSP (GQSP) [42]. One drawback of this approach is that it requires us to compile phase factors for each s_j , involving potentially challenging classical computations. A second drawback is that the technique brings a constant factor overhead of at least $e/2$ in the query count from rigorous non asymptotic estimates of the overhead in the degree of the Jacobi-Anger polynomial approximation of the complex exponential [32].

Here, instead, we replace Hamiltonian simulation by a random walk method, which removes both drawbacks. First, in Sec. II C we present a construction realizing a self-inverse $(\alpha_s, a+2, 0)$ -block-encoding of $H(s)$ with $\alpha_s = (1-s) + \alpha s$ with a single call to U_A , using the linear combination of unitaries (LCU) technique. The LCU construction of Ref. [22] has $\alpha_s = 2(\alpha + 1)$. Since α_s enters linearly in Q , our construction cuts costs by about a factor of 2. Then, we construct the walk operator

$$W(s) = U_{H(s)} \mathcal{Z} U_{H(s)} \mathcal{Z}, \quad (16)$$

where $\mathcal{Z} = (2|0^{a+2}\rangle\langle 0^{a+2}| - I) \otimes I$. This unitary can be written as $W(s) = e^{iH_W(s)}$, for some Hermitian, $H_W(s)$. The target adiabatic protocol is now realized by dephasing relative to the eigenspaces of $H_W(s)$, of which the π -eigenspace encodes the nullspace of $H(s)$. The operator $H_W(s)$ has a gap $\tilde{\Delta}_W(s)$ between π and the remaining eigenvalues, which obeys $\tilde{\Delta}_W(s) \geq 2\Delta(s)/\alpha_s$, as we show in Sec. II E.

At each s_j , we shall apply $W(s)$ a number of times m sampled from the probability distribution

$$p_{s_j}(m) \propto \frac{1}{\tilde{\Delta}_W(s_j)} \left(\frac{J_r(\tilde{\Delta}_W(s_j)|m|/2)}{\tilde{\Delta}_W^{r-1}(s_j)|m|^r} \right)^2, \quad (17)$$

defined over the set $\mathbb{Z}$, and where $r = 1.165$.

We now have all the ingredients. We select s_j according to the ‘‘Poisson random’’ process [with rate in Eq. (13)], and for each s_j we apply the walk operator $W(s_j)$ a number

of times m_j with probability in Eq. (17). For any given realization, we apply

$$W^{m_q}(s_q)W^{m_{q-1}}(s_{q-1}) \cdots W^{m_1}(s_1),$$

on the input state. Averaging over the random variables we get a density operator $\rho(1)$ with

$$\langle y(1)|\rho(1)|y(1)\rangle \geq 1/2. \quad (18)$$

Step 4: Projection on the correct solution. Next, we take the output of the adiabatic protocol and apply an $O(\epsilon)$ -approximate projection onto the nullspace of $H(1)$, with a query cost $O(\kappa \log(1/\epsilon))$.

Here we leverage the filtering results of Ref. [24], which build upon the results in Ref. [23]. These results ensure that we can apply a block-encoding of $P(1)$, the projector onto the nullspace of $H(1)$ (the Hamiltonian at the end of the trajectory), to the state $\rho(1)$ with a circuit involving on average $\left\lceil \alpha\kappa/2 \log\left(\frac{2}{\epsilon_P}\right) \right\rceil$ applications of a block-encoding of $H(1)/\alpha$ or its inverse, and two extra qubits. Since we have access to an $(\alpha, a+2, 0)$ block-encoding of $H(1)$, this gives us an $(\alpha, a+4, \epsilon_P)$ block-encoding of $P(1)$. No complex phase factor pre-computations are required [24]. Counting three extra qubits that we introduced in the constructions, overall we have used seven ancilla qubits.

An appropriate ϵ_P needs to be chosen by combining the errors from the adiabatic stage with those of the filtering stage and the corresponding success probabilities. We do this in Sec. III. With this choice, if we succeed we output an ϵ -approximation to the solution vector $|y(1)\rangle$ (and so $|y\rangle$). Otherwise, we go back to Step 3 and repeat.

This completes an overview of the algorithm. We now move to a detailed analysis.

Workflow summary

Initialization

- (1) Rescale the problem so that $\|A\| \leq 1$.
- (2) Input circuit block-encoding of A/α for some $\alpha > 1$.
- (3) Input state preparation circuit for $|b\rangle$.
- (4) Input error tolerance $\epsilon > 0$.
- (5) Input condition number upper bound κ .

Hamiltonian encoding

- (1) Define Hamiltonian encoding $H(s)$ as in Eq. (5), with block-encoding $U_{H(s)}$ constructed in Sec. II C.
- (2) Generate Poisson points s_j in $[0, 1]$ with rate $\lambda(s)$ according to Eq. (13) with $\gamma = 1/2$, $q = 1/2$.

Quantum algorithm

- (1) Prepare $|y(0)\rangle = |0, -, 0, b\rangle$ [Eq. (12)].
- (2) *Quantum random walk:* for each Poisson point s_j , apply walk operator $W(s)$ in Eq. (16) a number of times m sampled according to $p_{s_j}(m)$ in Eq. (17).
- (3) *Filtering:* Apply quantum circuit plus measurement effecting a projection onto the nullspace of $H(1)$ as described in Sec. II G.
- (4) Repeat till success.

II. ALGORITHMIC DETAILS

A. Bringing the problem into standard form

1. Rescaling

The central aim is to solve $Ay = b$ in its coherent formulation $A|y\rangle \propto |b\rangle$. It is useful to embed the linear system in another one for which the norm of the matrix of coefficients is bounded by 1.

Here we briefly discuss how to do so and why it does not affect the results of our work. Let N_A be an upper bound to the norm of A . Consider the rescaling $A' = A/N_A$. Clearly, $\|A'\| \leq 1$ and so all its singular values lie in the interval $[1/\|(A')^{-1}\|, 1]$. We define $\kappa' = \|(A')^{-1}\|$ so that all singular values of the rescaled matrix are included in $[1/\kappa', 1]$. The parameter κ' is an upper bound on the condition number of the rescaled matrix.

In this paper, we assume access to an $(\alpha', a, 0)$ -block-encoding of A' . Note that without loss of generality we can assume $\alpha' \geq 1$. In fact, we can reabsorb $\alpha' < 1$ into a redefinition of the normalization constant N_A . Let us see why: Assume that the previous construction leads to a rescaled A' with a block-encoding $U_{A'}$ with $\alpha' < 1$. This implies that $\|A'\|$ is strictly less than 1. Then define $A'' = A'/\alpha'$. The singular values of the A'' matrix lie in the interval $[1/\kappa'', 1]$, where $\kappa'' = \alpha'\kappa'$. In other words, the condition number upper bound of A'' is a factor of α' smaller than that of A' . Furthermore, access to an $(\alpha', a, 0)$ -block-encoding of A' is equivalent to access to an $(\alpha'', a, 0)$ -block-encoding of A'' , where $\alpha'' = 1$.

In other words, given $U_{A'}$ a block-encoding of A' , with a block-encoding rescaling parameter and condition number upper bound equal to (α', κ') respectively, then we have $U_{A'} = U_{A''}$ a block-encoding for A'' , with new parameters $(1, \alpha'\kappa')$. Since the query cost of QLSA is linear in the block-encoding rescaling and at least linear in the condition number, solving the linear problem for A'' is no more costly than solving the one for A' . Hence, this shows

that we can take without loss of generality the rescaling factor of the block-encoding to be larger or equal to 1.

From now on, for simplicity of notation, we drop primes and assume the rescaling has been done. We have access to U_A which is an $(\alpha, a, 0)$ block-encoding of A , where $\alpha \geq 1$ and $\|A\| \leq 1$.

2. Hermitian extension

We now consider the Hermitian extended linear system as in Eq. (7). We have $\|\bar{A}\| \leq 1$ and so all the singular values of $\bar{A}$ are between $[1/\bar{\kappa}, 1]$, where $\bar{\kappa} := \|\bar{A}^{-1}\|$. Furthermore, $\bar{\kappa} = \|\bar{A}^{-1}\| = \|A^{-1}\| = \kappa$, where we used that A is invertible. Hence, the Hermitian extension does not change the bounds on the singular values. What does change is that we need to block-encode $\bar{A}$ rather than A . As we shall discuss in more detail later, this can be done with at most one call to a block-encoding of A and one call to a block-encoding of $A^\dagger$. If A is Hermitian, the Hermitian extension is not required and a factor of 2 is saved from the overall cost.

B. Properties of $H(s)$

For the results that follow we refer to Ref. [22]. For clarity and ease we report what we need here.

1. The nullspace of $H(s)$

We consider the family of Hamiltonians

$$H(s) = |0\rangle\langle 1| \otimes A(s)\Pi + |1\rangle\langle 0| \otimes \Pi A(s), \quad (19)$$

with

$$\begin{aligned} \Pi &= I - |+, 0, b\rangle\langle +, 0, b| \\ A(s) &= (1-s)Z \otimes I + sX \otimes \bar{A}, \end{aligned} \quad (20)$$

where Z is the Pauli z -matrix. We claim that for all s the nullspace of $H(s)$ is two-dimensional and spanned by the orthonormal states $|y(s)\rangle \propto |0\rangle \otimes A(s)^{-1}|+, 0, b\rangle$ and $|1, +, 0, b\rangle$. Let us prove that this is the case.

Since $H(s)$ are Hermitian, the eigenvalues of $H(s)^2$ are exactly the square of the eigenvalues of $H(s)$ and they have the same eigenvectors. Hence consider

$$H(s)^2 = |0\rangle\langle 0| \otimes A(s)\Pi A(s) + |1\rangle\langle 1| \otimes \Pi A(s)^2 \Pi. \quad (21)$$

From the above, the nullspace of $H(s)$ is spanned by the eigenvectors with zero eigenvalues of the two blocks of $H(s)^2$.

The $|0\rangle$ block is spanned by vectors of the form $|0\rangle \otimes |\psi\rangle$. For this to be an eigenstate with eigenvalue zero we

need

$$(A(s)\Pi A(s))|\psi\rangle = 0. \quad (22)$$

Let us show that $A(s)$ is invertible since A is, by assumption, invertible. Compute

$$\begin{aligned} A^\dagger(s)A(s) &= (1-s)^2 I \otimes I + s^2 (I \otimes \bar{A}^\dagger \bar{A}) \\ &= I \otimes I - s^2 (I \otimes I - I \otimes \bar{A}^\dagger \bar{A}), \end{aligned} \quad (23)$$

where we have used the facts that $\bar{A}$ is Hermitian and Pauli operators anticommute. Using Weyl's inequality, we can lower bound the smallest eigenvalue of $A^\dagger(s)A(s)$ with $(1-s)^2 + (s/\kappa)^2 \geq 1/\kappa^2 > 0$. Hence, $A(s)$ is invertible.

It follows that for every $|\psi\rangle$ there is a vector $|\phi\rangle$ such that $|\psi\rangle = A^{-1}(s)|\phi\rangle$. Then Eq. (22) becomes

$$A(s)\Pi|\phi\rangle = 0, \quad (24)$$

which holds if and only if $|\phi\rangle = |+, 0, b\rangle$. Therefore, for all s we have that $|y(s)\rangle \propto |0\rangle \otimes A^{-1}(s)|+, 0, b\rangle$ is the unique eigenstate of zero eigenvalue of the top block.

We now consider the bottom block. The eigenstates with zero eigenvalue are of the form $|1\rangle \otimes |\psi\rangle$, where $\Pi A(s)^2 \Pi |\psi\rangle = 0$. Choosing $|\psi\rangle = |+, 0, b\rangle$ gives an eigenstate with zero eigenvalue. Again, since $A(s)$ is invertible, we have that this state is unique. In conclusion the nullspace of $H(s)$ is spanned by $|y(s)\rangle \propto |0\rangle \otimes A(s)^{-1}|+, 0, b\rangle$ and $|1, +, 0, b\rangle$.

2. Solution encoding

When $s = 1$, we have that $A(1)^{-1} = X \otimes \bar{A}^{-1}$ and so the associated nullspace vector is given by

$$|y(1)\rangle \propto |0\rangle \otimes (X \otimes \bar{A}^{-1})|+, 0, b\rangle \propto |0, +, 1, y\rangle. \quad (25)$$

The state $|y\rangle \propto A^{-1}|b\rangle$ is the desired solution to the linear system.

3. Gap from zero

Now we consider the gap between the zero eigenvalue and the rest of the eigenvalues in $H(s)$, and show it is lower bounded by

$$\Delta(s) = \sqrt{(1-s)^2 + (s/\kappa)^2}. \quad (26)$$

The gap between the zero eigenvalue and non zero eigenvalues of $H(s)$ is simply the square root of the gap between the zero eigenvalue and non zero eigenvalues for $H(s)^2$. If we set $B(s) = A(s)\Pi$, the top block of $H(s)^2$ reads $B(s)B(s)^\dagger$ and the bottom block reads $B(s)^\dagger B(s)$. The two blocks hence have the same spectrum and we can then focus only on the top block. It follows that the gap from

zero of $H(s)$ is just the square root of the gap from zero of $A(s)\Pi A(s)$.

The gap of $A(s)\Pi A(s)$ was analyzed in the Supplemental Material of Ref. [22] using Weyl's inequalities. One rewrites

$$A(s)\Pi A(s) = A(s)^2 - A(s)\Pi^\perp A(s), \quad (27)$$

where $\Pi^\perp = |+, 0, b\rangle\langle +, 0, b|$. From Weyl's inequality the second smallest eigenvalue of $A(s)\Pi A(s) = A(s)^2 - A(s)\Pi^\perp A(s)$ (which is the gap from zero, since $A(s)\Pi A(s)$ has a zero eigenvalue and is a nonnegative matrix) is lower bounded by the smallest eigenvalue of $A(s)^2$ plus the second smallest eigenvalue of $-A(s)\Pi^\perp A(s)$. We consider each separately.

Note that $A(s)$ has eigenvalues $\pm\sqrt{(1-s)^2 + (s\lambda)^2}$, with λ eigenvalues of A . Recall that $A(s)$ is Hermitian, so the eigenvalues of $A(s)^2$ are of the form $(1-s)^2 + (s\lambda)^2$. Since $1/\kappa$ is a lower bound on the smallest singular value of A by definition of κ , the smallest eigenvalue of $A(s)^2$ is lower bounded by $(1-s)^2 + (s/\kappa)^2$.

Now consider $-A(s)\Pi^\perp A(s)$. This is a non positive, rank-1 matrix, and so the second smallest eigenvalue is zero. We conclude that the gap of $A(s)\Pi A(s)$ is lower bounded by $(1-s)^2 + (s/\kappa)^2$, and so the gap from zero of $H(s)$ is lower bounded by $\Delta(s) = \sqrt{(1-s)^2 + (s/\kappa)^2}$.

4. No transitions between orthogonal components of the nullspace

We also claim that for any fixed s the Hamiltonian never couples the two eigenstates of zero eigenvalue. Specifically, we have that for every $s, s' \in [0, 1]$

$$\begin{aligned} &\langle 1| \langle +, 0, b| H(s) |y(s')\rangle \\ &\propto \langle 1| \langle +, 0, b| (|0\rangle\langle 1| \otimes A(s)\Pi \\ &\quad + |1\rangle\langle 0| \Pi A(s)) |0\rangle A^{-1}(s') |+, 0, b\rangle \\ &= \langle +, 0, b| \Pi A(s) A^{-1}(s') |+, 0, b\rangle = 0. \end{aligned} \quad (28)$$

Therefore, if we initialize the system in the state $|y(0)\rangle$ and perform the sequence of dephasings in the eigenbasis of the Hamiltonian $H(s)$, we will not generate a component along $|1\rangle \otimes |+, 0, b\rangle$.

C. Block-encoding $H(s)$

To construct the block-encoding of $H(s)$ from that of A we rely on the LCU technique. LCU allows us to obtain a $(\|\beta\|_1, \lceil \log_2 k \rceil, 0)$ -block-encoding of $\sum_{i=1}^k \beta_i U_i$, where $\|\beta\|_1 = \sum_i \beta_i$, U_i are unitaries and without loss of generality we assume $\beta_i > 0$ by absorbing a phase in the definition of U_i [32,33]. LCU assumes access to the

controlled unitary

$$V_S = \sum_{i=1}^k |i\rangle\langle i| \otimes U_i, \quad \text{“Select”} \quad (29)$$

and the state preparation unitary

$$V_P |0^{\lceil \log_2 k \rceil}\rangle = \frac{1}{\sqrt{\|\beta\|_1}} \sum_{i=1}^k \sqrt{\beta_i} |i\rangle, \quad \text{“Prepare”}, \quad (30)$$

where we use the shorthand notation $|0^a\rangle := |0\rangle^{\otimes a}$. The claimed block-encoding is then realized as

$$W = (V_P^\dagger \otimes I) V_S (V_P \otimes I). \quad (31)$$

Another general result concerns products of block-encodings. If U_M is an $(\alpha_M, a_M, 0)$ -block-encoding of M and U_N is an $(\alpha_N, a_N, 0)$ -block-encoding of N , then $(I_{a_M} \otimes U_M)(I_{a_N} \otimes U_N)$ is an $(\alpha_M \alpha_N, a_M + a_N, 0)$ -block-encoding of MN (Ref. [32], Sec. 4.4).

Using these results we can show the following, which as discussed improves the LCU decomposition of Ref. [22] by extending the discussion to a setting where A is not necessarily Hermitian and sparse, while cutting the 1-norm by at least a factor of 2.

Lemma 1 (Constructing a block-encoding of $H(s)$). Suppose we access an $(\alpha, a, 0)$ -block-encoding U_A of A and the state preparation unitary U_b with $U_b|0\rangle = |b\rangle$. We can then construct a unitary $U_{H(s)}$ that is an

$$(\alpha_s = 1 - s + \alpha s, a + 2, 0)$$

block-encoding of the Hamiltonian

$$H(s) = |0\rangle\langle 1| \otimes A(s)\Pi + |1\rangle\langle 0| \otimes \Pi A(s), \quad (32)$$

where we have

$$\begin{aligned} \Pi &= I - |+, 0, b\rangle\langle +, 0, b| \\ A(s) &= (1-s)Z \otimes I_{n+1} + sX \otimes \bar{A}, \\ \bar{A} &= |0\rangle\langle 1| \otimes A + |1\rangle\langle 0| \otimes A^\dagger, \end{aligned}$$

with two calls to (controlled) U_A or $U_A^\dagger$ and four calls to U_b or $U_b^\dagger$. Furthermore, the block-encoding is Hermitian, i.e., $U_{H(s)}^\dagger = U_{H(s)}$. If A is a $2^n \times 2^n$ matrix then $U_{H(s)}$ acts on $n + a + 5$ qubits.

Proof. Given a block-encoding U_A of A , $U_A^\dagger$ block-encodes $A^\dagger$. Then

$$U_{\bar{A}} = (|0\rangle\langle 0| \otimes U_A + |1\rangle\langle 1| \otimes U_A^\dagger)(X \otimes I), \quad (33)$$

acts on $n + a + 1$ qubits and gives an $(\alpha, a, 0)$ -block-encoding of $\bar{A}$ involving a single call to controlled U_A and controlled $U_A^\dagger$. Note that $U_{\bar{A}}$ is Hermitian.

Next, we use LCU to obtain a block-encoding of $A(s)$ from the block-encoding of $\tilde{A}$. We do this by considering the expression

$$(1-s)Z \otimes I_{n+a+1} + \alpha s X \otimes U_{\tilde{A}}, \quad (34)$$

which defines an $n+a+2$ qubit operator which is a sum of unitaries and encodes $A(s)$ in the $|0^a\rangle$ block, with zero error. This operator is block-encoded via LCU. LCU requires a “select” unitary

$$V_S = |0\rangle\langle 0| \otimes (Z \otimes I_{n+a+1}) + |1\rangle\langle 1| \otimes (X \otimes U_{\tilde{A}}), \quad (35)$$

and a “prepare” unitary

$$V_P(s)|0\rangle = \frac{1}{\sqrt{1-s+\alpha s}} \left(\sqrt{1-s}|0\rangle + \sqrt{\alpha s}|1\rangle \right). \quad (36)$$

This introduces a single extra qubit to condition on to realize this linear combination, and has a scaling factor given by

$$\|(1-s, \alpha s)\|_1 = 1-s+\alpha s. \quad (37)$$

The LCU unitary is given as

$$U_{A(s)} := (V_P^\dagger(s) \otimes I_{n+a+2}) V_S (V_P(s) \otimes I_{n+a+2}), \quad (38)$$

which is a unitary acting on $n+a+3$ qubits in total and provides a $(1-s+\alpha s, a+1, 0)$ block-encoding of the $n+2$ qubit operator $A(s)$.

To construct a block-encoding of $H(s)$ we can combine the above block-encoding for $A(s)$ with a block-encoding of Π . In particular, we can block-encode $\Pi = I - |+, 0, b\rangle\langle +, 0, b|$ by writing

$$\begin{aligned} \Pi &= (\text{Had} \otimes U_b)(I - |0^{n+2}\rangle\langle 0^{n+2}|)(\text{Had} \otimes U_b^\dagger) \\ &= \frac{1}{2} \left(I + (\text{Had} \otimes U_b) e^{i\pi |0^{n+2}\rangle\langle 0^{n+2}|} (\text{Had} \otimes U_b^\dagger) \right) \end{aligned}$$

as a linear combination of two unitaries

$$U_0 = I, \quad U_1 = (\text{Had} \otimes U_b) e^{i\pi |0^{n+1}\rangle\langle 0^{n+1}|} (\text{Had} \otimes U_b^\dagger),$$

where Had denotes the Hadamard matrix on one qubit, and in the second equality we used $|0^{n+1}\rangle\langle 0^{n+1}| = (I - e^{i\pi |0^{n+1}\rangle\langle 0^{n+1}|})/2$. We can then define the prepare and select unitaries

$$\tilde{V}_P = \text{Had}, \quad \tilde{V}_S = |0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1. \quad (39)$$

The unitary $\tilde{W} = (\tilde{V}_P^\dagger \otimes I_{n+1}) \tilde{V}_S (\tilde{V}_P \otimes I_{n+1})$ is then a $(1, 1, 0)$ -block-encoding of Π , using one call to U_b and one to its inverse, which introduces one additional qubit.

Hence from multiplying the block-encodings, we have that $U_{A(s)} \tilde{W}$ ($\tilde{W} U_{A(s)}$, respectively) is a $(1-s+\alpha s, a+2, 0)$ -block-encoding of $A(s)\Pi$ ($\Pi A(s)$, respectively) acting on $n+a+4$ qubits.

As a final step, note that

$$\begin{aligned} H(s) &= |0\rangle\langle 1| \otimes A(s)\Pi + |1\rangle\langle 0| \otimes \Pi A(s) \\ &= (X \otimes I) [|1\rangle\langle 1| \otimes A(s)\Pi + |0\rangle\langle 0| \otimes \Pi A(s)], \end{aligned}$$

for $s \in [0, 1]$, and so

$$U_{H(s)} := (X \otimes I) [|1\rangle\langle 1| \otimes U_{A(s)} \tilde{W} + |0\rangle\langle 0| \otimes \tilde{W} U_{A(s)}]$$

is a $(1-s+\alpha s, a+2, 0)$ -block-encoding of $H(s)$ acting on a total of $n+a+5$ qubits, which is consistent with $H(s)$ acting on $n+3$ qubits in total. This occurs since we require $a+1$ auxiliary qubits for a block-encoding of $A(s)$ and one additional auxiliary qubit for the block-encoding of the projector. Let us rewrite

$$\begin{aligned} U_{H(s)} &:= (X \otimes I) [|0\rangle\langle 0| \otimes \tilde{W} + |1\rangle\langle 1| \otimes I] U_{A(s)} \\ &\quad \times [|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \tilde{W}], \end{aligned} \quad (40)$$

which shows that $U_{H(s)}$ can be obtained via a single call to $U_{A(s)}$ (i.e., a single call to controlled $U_{\tilde{A}}$, which is one call to controlled U_A and one to controlled $U_A^\dagger$) and two calls to $\tilde{W}$ (i.e., four calls to U_b or $U_b^\dagger$).

Finally, $U_{H(s)}^\dagger = U_{H(s)}$ follows immediately from the fact that $U_{A(s)}^\dagger = U_{A(s)}$ and $\tilde{W}^\dagger = \tilde{W}$.

Since $H(s)$ acts on three more qubits than A , the block-encoding $U_{H(s)}$ acts on $n+a+5$ qubits in total. ■

Given this basic building block, we move on to the details of the randomized part of the algorithm.

D. Construction of a random variable for the randomization scheme

For the randomized protocol we shall need to construct a random variable, $p_s(t)$, for any $s \in [0, 1]$ such that

- The distribution $p_s(t)$ is band-limited: more precisely, its Fourier transform $\chi(\omega)$ has support in $[-\Delta(s), \Delta(s)]$.
- The random variable $T = t$ has $\langle |T| \rangle$ minimized.

To construct $p_s(t)$, recall the definitions of Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$,

$$\mathcal{F}(f)(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} f(t) dt, \quad (41)$$

$$\mathcal{F}^{-1}(g)(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t} g(\omega) d\omega. \quad (42)$$

To construct such a $p_s(t)$ with characteristic function supported only in $[-\Delta(s), \Delta(s)]$ we use the following construction strategy. We fix an ansatz characteristic function $g_s(\omega)$ with $g_s(0) = 1$ and supported only in $[-\Delta(s)/2, \Delta(s)/2]$. Its inverse Fourier transform gives a function $f_s(t)$ which will not in general be positive and is hence not a valid probability density. However we can set $p_s \propto f_s^2$ which, using the convolution theorem, has characteristic function $\chi_s = g_s * g_s$, where $*$ labels convolution. Hence, χ_s is supported in $[-\Delta(s), \Delta(s)]$, p_s is positive and it is a valid probability density once normalized (if it can be normalized). These are the properties we needed to apply the techniques in Ref. [30].

Let us see this strategy in action. Set

$$g_s(\omega) = \left(1 - \frac{4\omega^2}{\Delta(s)^2}\right)^{r-1/2} h\left(\frac{\omega}{\Delta(s)}\right), \quad (43)$$

where we choose $r = 1.165$, $h(x)$ is the step function in $[-1/2, 1/2]$, i.e., $h(x) = 1$ if $|x| \leq 1/2$ and $h(x) = 0$ otherwise (which we constructed by taking a quadratic expression with support in $[-\Delta(s)/2, \Delta(s)/2]$ and satisfying the constraints $g_s(\omega = 0) = 1$, $g_s(\omega = \Delta(s)/2) = 0$). The support of g_s is $[-\Delta(s)/2, \Delta(s)/2]$. The inverse Fourier transform reads

$$\mathcal{F}^{-1}(g_s(\omega)) \propto \frac{J_r(\Delta(s)|t|/2)}{\Delta(s)^{r-1}|t|^r} := f_s(t). \quad (44)$$

Note that $f_s(t)$ is not a valid probability distribution since it can be negative, so we take $p_s(t) = f_s(t)^2/N_s$. It is readily verified that

$$\int_{\mathbb{R}} dp_s(t) = 1, \quad (45)$$

with normalization $0.2379128 \dots \times \Delta(s)$ and therefore $p_s(t)$ is a valid probability density on the real line. The associated characteristic function $\chi_s(\omega)$ is the Fourier transform of this distribution and using the convolution theorem we have

$$\chi_s(\omega) := \int_{-\infty}^{+\infty} e^{i\omega t} p_s(t) dt \propto \mathcal{F}(f_s^2)(\omega) \quad (46)$$

$$= \mathcal{F}(\mathcal{F}^{-1}(g_s)^2)(\omega) \propto (g_s * g_s)(\omega). \quad (47)$$

We therefore have that $\chi_s(\omega)$ only has support in the interval $[-\Delta(s), \Delta(s)]$, and the above $p_s(t)$ is a valid probability density.

Technically to conclude $\sup_{\omega \in [-\Delta(s), \Delta(s)]} |\chi_s(\omega)| = 0$ above we would need $\Delta(s)$ to be a *strict* lower bound to the zero eigenvalue gap of $H(s)$. However, $\Delta(s)$ is only a lower bound to the gap. By continuity one can of course repeat the same construction as the above with a characteristic function g with support in $[(\Delta(s) + \delta)/2, (\Delta(s) - \delta)/2]$ and then make δ arbitrarily small, getting the same results as presented here.

E. Randomization method using the block-encoding walk operator

In this section we introduce a novel randomization method based on the walk operator obtained from block-encoding of the Hamiltonian, as opposed to relying on time evolutions. Start from the block-encoding of $H(s)$:

$$U_{H(s)}|0\rangle|\psi\rangle = |0\rangle\frac{H(s)}{\alpha_s}|\psi\rangle + |\tilde{\perp}(s)\rangle, \quad (48)$$

where $(\langle 0| \otimes \mathbb{I})|\tilde{\perp}(s)\rangle = 0$ and $|\tilde{\perp}(s)\rangle$ is subnormalized. Here $\alpha_s = 1 - s + \alpha s$ is the block-encoding prefactor we obtained in the construction of Lemma 1. Let $|E_i(s)\rangle$ be an eigenstate of $H(s)$ with eigenvalue $E_i(s)$. Then

$$\begin{aligned} U_{H(s)}|0\rangle|E_i(s)\rangle &= \frac{E_i(s)}{\alpha_s}|0\rangle|E_i(s)\rangle + |\tilde{\perp}_i(s)\rangle \\ &= \frac{E_i(s)}{\alpha_s}|0\rangle|E_i(s)\rangle + \sqrt{1 - \left(\frac{E_i(s)}{\alpha_s}\right)^2}|\perp_i(s)\rangle \\ &= \cos(\theta_i(s))|0\rangle|E_i(s)\rangle + \sin(\theta_i(s))|\perp_i(s)\rangle, \end{aligned}$$

where $|\perp_i(s)\rangle$ is now normalized and satisfies

$$(\langle 0| \otimes I)|\perp_i(s)\rangle = 0. \quad (49)$$

Also, we have defined $\theta_i(s) = \arccos(E_i(s)/\alpha_s)$. Since $U_{H(s)}$ is Hermitian (Lemma 1), the span $\{|0\rangle|E_i(s)\rangle, |\perp_i(s)\rangle\}$ is an invariant subspace for $U_{H(s)}$ [33].

Let $\mathcal{Z} = (2|0\rangle\langle 0| - I) \otimes I$. Then it can be shown that the “doubled” walk operator $W(s) = U_{H(s)}\mathcal{Z}U_{H(s)}\mathcal{Z}$ acts in two-dimensional invariant spaces $\mathcal{H}_i := \text{span}\{|0\rangle|E_i(s)\rangle, |\perp_i(s)\rangle\}$ as

$$W(s)|_{\mathcal{H}_i} = \begin{pmatrix} \cos(2\theta_i(s)) & -\sin(2\theta_i(s)) \\ \sin(2\theta_i(s)) & \cos(2\theta_i(s)) \end{pmatrix}. \quad (50)$$

The linear system Hamiltonian $H(s)$ of Eq. (5) has a special spectrum. Its eigenvalues come in pairs of equal magnitude but opposite sign, except for two degenerate eigenvalues that are zero. We will label the positive eigenvalues in an increasing order with positive integers j and the corresponding negative eigenvalues with $-j$. As for the degenerate eigenvalues 0 we use the notation $|E_0^+(s)\rangle := |y(s)\rangle$ and $|E_0^-(s)\rangle := |1\rangle|+, 0, b\rangle$ as basis vectors to span this subspace.

Note that $|E_0^+(1)\rangle$ encodes the solution to the linear systems problem, $|E_0^+(s)\rangle$ is the eigenpath we wish to follow and the orthogonal state $|E_0^-(s)\rangle$ has no s -dependence. We have $E_0^\pm(s) = 0$ and so $\theta_0(s) = \arccos(0) = \pi/2$ and hence $W(s)$ acts as minus identity in both of the two-dimensional subspaces associated with the zero eigenvalues of $H(s)$.

This shows that the “doubled” walk operator leaves the zero energy space (corresponding to $\theta_0 = \pi/2$) invariant.

However, we also need to verify that no other energy eigenspace ends up being associated to the same eigenphase of the walk operator. This requires every other eigenphase of the walk operator to be separated from $2\theta_0 = \pi$ by a finite phase gap.

Since $E_i(s)/\alpha_s \in [-1, 1]$, $\theta_i(s) = \arccos(E_i(s)/\alpha_s) \in (-\pi, \pi]$. Also, for $i \neq 0$, $|E_i(s)|/\alpha_s > \Delta(s)/\alpha_s$, and so

$$\theta_i(s) \in (0, \pi/2 - \delta(s)) \cup (\pi/2 + \delta(s), \pi), \quad (51)$$

where $\pi/2 \mp \delta(s) = \arccos(\pm\Delta(s)/\alpha_s)$, and so $\delta(s) = \pi/2 - \arccos(\Delta(s)/\alpha_s) \geq \Delta(s)/\alpha_s$. This implies that the spectrum of $W(s)$ around the phase π has a gap of

$$\Delta_W(s) = \pi - 2 \arccos(\Delta(s)/\alpha_s) \geq 2\Delta(s)/\alpha_s. \quad (52)$$

Note this is tight only if $\Delta(s) \ll \alpha_s$. We now note that for any $s \in [0, 1]$ we can write $W(s) = e^{iH_W(s)}$ for some (non-unique) Hermitian operator $H_W(s)$. For any such $H_W(s)$ we see that $\Delta_W(s)$ is the gap between the π -eigenvalue of the operator $H_W(s)$ and its other eigenvalues.

The π -eigenspace of $H_W(s)$ is a four-dimensional subspace $\mathcal{H}_W^i$ with basis

$$|\eta_0(s)\rangle = |0\rangle|E_0^+(s)\rangle, \quad (53)$$

$$|\eta_1\rangle = |0\rangle|E_0^-\rangle, \quad (54)$$

$$|\eta_2(s)\rangle = |\perp_0^+(s)\rangle, \quad (55)$$

$$|\eta_3(s)\rangle = |\perp_0^-\rangle, \quad (56)$$

and the walk operator can be written as

$$W(s) = \bigoplus_{i \neq 0} \begin{pmatrix} \cos(2\theta_i(s)) & -\sin(2\theta_i(s)) \\ \sin(2\theta_i(s)) & \cos(2\theta_i(s)) \end{pmatrix}_{\mathcal{H}_i} - \sum_{r=0}^3 |\eta_r(s)\rangle\langle\eta_r(s)|, \quad (57)$$

where $|\perp_0^+(s)\rangle$ is the vector defined in Eq. (49) to be orthogonal to $|E_0^+(s)\rangle$, and similarly $|\perp_0^-\rangle$ for $|E_0^-\rangle$.

Each realization of the randomization method implements a unitary of the form

$$W(s_k)^{m_k} \dots W(s_2)^{m_2} W(s_1)^{m_1}, \quad (58)$$

where k , $\{s_j\}_{j=1}^k$, and $\{m_j\}_{j=1}^k$ are random variable samples, as discussed below.

We could at this point use results from Ref. [30] for the randomization method, but we want to improve further on it, so we will add randomization over the choice of s_j values via Ref. [29]. In Ref. [29], one considers a Poisson process with rate $\lambda(s)$. At each jump point, s_j , a randomized time-evolution with respect to the Hamiltonian $H(s)$

is applied, as originally proposed in Ref. [22], with the evolution time, t_j , chosen from a probability density $p_{s_j}(t)$.

Here, we instead put forward a new scheme that entirely removes the Hamiltonian simulation routine and directly applies the walk operator $W(s_j)^{m_j}$, where m_j is sampled according to the *discrete* probability

$$p_{s_j}(m) = \frac{1}{\mathcal{N}(s_j)\tilde{\Delta}_W(s_j)} \left(\frac{J_r(\tilde{\Delta}_W(s_j)|m|/2)}{\tilde{\Delta}_W^{r-1}(s_j)|m|^r} \right)^2, \quad (59)$$

with $m \in \mathbb{Z}$ an integer, $\tilde{\Delta}_W(s_j)$ any strict lower bound on $\Delta_W(s_j)$, and $r = 1.165$. Note that

$$W(s_j)^{m_j} = e^{im_j H_W(s_j)}, \quad (60)$$

and so amounts to an evolution under the Hamiltonian $H_W(s_j)$ for an integer-valued time m_j . This then allows us to realize dephasing onto the π -eigenspace of $H_W(s)$ via the randomized method applied to the walk operator.

The characteristic functions of the continuous version of the above discrete probability (where m is extended to $m \in \mathbb{R}$, and the function renormalized accordingly) has support contained in $(-\Delta_W(s_j), \Delta_W(s_j))$ (see Sec II D), with $\Delta_W(s_j) \leq \pi$. By Lemma 4 of Ref. [30], its restriction to the integers [which is the discrete probability in Eq. (59) that we are sampling from] is a well-defined probability distribution with characteristic function contained in $(-\Delta_W(s_j), \Delta_W(s_j))$. Using Theorem 1 of the same paper, the randomized protocol described effects a channel on system plus ancilla of the form

$$\rho \mapsto \bar{P}_W(s_j)\rho\bar{P}_W(s_j) + \bar{\mathcal{C}}_{s_j} \circ (I - \bar{P}_W(s_j))\rho(I - \bar{P}_W(s_j)), \quad (61)$$

where $\bar{P}_W(s_j)$ is the projector onto the subspace $\mathcal{H}_W^i$, and $\bar{\mathcal{C}}_{s_j}$ is a channel that maps the set of states with support entirely in the orthogonal subspace to $\mathcal{H}_W^i$ into itself. The expected number of applications of the walk operator is

$$\sum_{m=-\infty}^{+\infty} |m|p_{s_j}(m) \leq 2.322/\tilde{\Delta}_W(s_j), \quad (62)$$

where the bound has been numerically computed. Each application of the walk operator [or its inverse $W(s)^{-1}$ in the case of negative integer powers] corresponds to two applications of the block-encoding of $H(s)$, and so four applications of the basic unitaries U_A and $U_A^\dagger$. Using $\Delta_W(s) \geq 2\Delta(s)/\alpha(s)$ we can upper bound the expected

query cost of a randomization as

$$\begin{aligned} Q(s_j) &\leq 4 \times 2.322 / \Delta W(s_j) \\ &\leq 2 \times 2.322 \alpha(s_j) / \Delta(s_j). \end{aligned} \quad (63)$$

We now show that such an operation cannot cause transitions from $|\eta_0(s_j)\rangle$ to $|\eta_r(s_{j+1})\rangle$ for $r = 1, 2, 3$. First

$$\begin{aligned} \langle \eta_1(s_{j+1}) | W(s_{j+1}) | \eta_0(s_j) \rangle &= -\langle \eta_1(s_{j+1}) | \eta_0(s_j) \rangle \\ &= -\langle 0|0\rangle \langle E_0^- | E_0^+(s_j) \rangle \\ &= 0. \end{aligned}$$

Similarly for the other two eigenstates we have

$$\begin{aligned} \langle \eta_{2,3}(s_{j+1}) | W(s_{j+1}) | \eta_0(s_j) \rangle &= -\langle \eta_{2,3}(s_{j+1}) | \eta_0(s_j) \rangle \\ &= -\langle \perp_0^\pm(s_{j+1}) | (|0\rangle | E_0^+(s_j) \rangle) \rangle \\ &= 0, \end{aligned}$$

where in the last line we used the fact that $\langle \perp_0^\pm(s_{j+1}) | (|0\rangle \otimes I) = 0$. The fact that no transitions out of $\{|\eta_0(s)\rangle, |\eta_1(s)\rangle\}$ can occur during the randomized evolutions means that if we are promised that we initialize the system at $s = 0$ in the state $|\eta_0(0)\rangle = |0\rangle | E_0^+(0)\rangle$, at every subsequent randomization, then the action of the previous channel has an identical action to the channel

$$\rho \mapsto P(s_j) \rho P(s_j) + \mathcal{C}_{s_j} \circ (I - P(s_j)) \rho (I - P(s_j)), \quad (64)$$

where $P(s_j) = |0\rangle\langle 0| \otimes P_{H(s)}$, with $P_{H(s)}$ the projector onto the zero eigenspace of $H(s)$ and $\mathcal{C}_{s_j}$ is a channel that preserves orthogonality as stated before.

F. Improved Poisson protocol analysis

Following Ref. [29], we set up a stochastic differential equation for the combined randomization over s (locations of the randomization) and m [number of applications of the walk operator at a given location, sampled according to Eq. (59)], where the latter is described by a Poisson process $\mathcal{N}$. Then

$$d\hat{\rho} = (W^m(s) \hat{\rho} W^{\dagger m}(s) - \hat{\rho}) d\mathcal{N}. \quad (65)$$

Denote by $\mathbb{E}_{\mathcal{N},m}$ the average over both $\mathcal{N}$ and the random variable associated to m , and $\mathbb{E}_m$ the average over the random variable associated to m . The Poisson process is

independent of the randomization over m , so

$$\begin{aligned} d\rho &:= \mathbb{E}_{\mathcal{N},m}[d\hat{\rho}] = \mathbb{E}_m[W^m(s) \hat{\rho} W^{\dagger m}(s) - \hat{\rho}] \lambda(s) ds, \\ &= (P(s) \rho P(s) + \mathcal{C}_s \circ (I - P(s)) \rho (I - P(s)) - \rho) \lambda(s) ds, \end{aligned}$$

where we recall that $\lambda(s)$ is the rate of the Poisson process. So, we obtain the differential equation

$$\frac{d\rho}{ds} = (P(s) \rho P(s) + \mathcal{C}_s \circ (P(s)^\perp \rho P(s)^\perp) - \rho) \lambda(s), \quad (66)$$

where $P(s)^\perp = I - P(s)$. We set

$$\rho(0) = |0\rangle\langle 0| \otimes |E_0^+\rangle\langle E_0^+|.$$

Recall that the target output state is $|\eta_0(1)\rangle = |0\rangle | E_0^+(1)\rangle$. The infidelity error γ is given by

$$\begin{aligned} \gamma &= 1 - \text{Tr}[\rho(1)P(1)] = \text{Tr}[\rho(0)P(0)] - \text{Tr}[\rho(1)P(1)] \\ &= -\int_0^1 \frac{d}{ds} (\text{Tr}[\rho(s)P(s)]) ds. \end{aligned} \quad (67)$$

Here, we used the fact that no transition to $|E_0^-\rangle$ happens during randomized evolutions, so we can replace $|E_0^+(s)\rangle\langle E_0^+(s)|$ with $P(s)$.

Let us compute

$$\begin{aligned} \frac{d}{ds} \text{Tr}[\rho(s)P(s)] &= \text{Tr}\left[\frac{d\rho(s)}{ds} P(s)\right] + \text{Tr}\left[\rho(s) \frac{dP(s)}{ds}\right] \\ &\stackrel{(66)}{=} \lambda(s) \text{Tr}[P(s) \mathcal{C}_s \circ (P(s)^\perp \rho P(s)^\perp)] \\ &\quad + \text{Tr}\left[\rho(s) \frac{dP(s)}{ds}\right] \\ &= \text{Tr}\left[\rho(s) \frac{dP(s)}{ds}\right], \end{aligned} \quad (68)$$

where in the last line we used that

$$\text{Tr}[P(s) \mathcal{C}_s \circ (P(s)^\perp \rho P(s)^\perp)] = 0,$$

since $\mathcal{C}_s$ preserves orthogonality.

In Eq. (68), $\rho(s)$ is not known, but we can solve Eq. (66) for $\rho(s)$

$$\rho = -\frac{d\rho/ds}{\lambda} + P\rho P + \mathcal{C} \circ (P^\perp \rho P^\perp), \quad (69)$$

where we dropped the s dependence to avoid notational clutter. Using this in Eq. (68):

$$\begin{aligned} \frac{d}{ds} \text{Tr}[\rho P] &= -\text{Tr}\left[\frac{dP}{ds} \frac{d\rho/ds}{\lambda}\right] + \text{Tr}\left[\frac{dP}{ds} P \rho P\right] \\ &\quad + \text{Tr}\left[\frac{dP}{ds} \mathcal{C}(P^\perp \rho P^\perp)\right] \\ &= -\text{Tr}\left[\frac{dP}{ds} \frac{d\rho/ds}{\lambda}\right]. \end{aligned} \quad (70)$$

The second and third term in the previous equation can be readily seen to vanish. For the second term:

$$\begin{aligned}\frac{dP}{ds} &= \frac{d(PP)}{ds} = \frac{dP}{ds}P + P\frac{dP}{ds} \\ \Rightarrow P\frac{dP}{ds} &= 0 \\ \Rightarrow \text{Tr}\left[\frac{dP}{ds}P\rho P\right] &= 0.\end{aligned}\quad (71)$$

And for the third term:

$$\begin{aligned}\text{Tr}\left[\frac{dP}{ds}\mathcal{C}(P^\perp\rho P^\perp)\right] &= \text{Tr}\left[\left(\frac{dP}{ds}P + P\frac{dP}{ds}\right)\mathcal{C}(P^\perp\rho P^\perp)\right] \\ &= \text{Tr}\left[\frac{dP}{ds}P\mathcal{C}(P^\perp\rho P^\perp)\right] \\ &\quad + \text{Tr}\left[\frac{dP}{ds}\mathcal{C}(P^\perp\rho P^\perp)P\right] = 0,\end{aligned}\quad (72)$$

again since $\mathcal{C}$ preserves orthogonality.

Substituting Eq. (70) in Eq. (67),

$$\begin{aligned}\gamma &= \int_0^1 \text{Tr}\left[\frac{dP}{ds}\frac{d\rho/ds}{\lambda}\right] ds \\ &= \int_0^1 \frac{d}{ds} \left\{ \frac{1}{\lambda} \text{Tr}\left[\frac{dP}{ds}\rho\right] \right\} ds \\ &\quad - \int_0^1 ds \frac{d}{ds} \left(\frac{1}{\lambda} \right) \text{Tr}\left[\frac{dP}{ds}\rho\right] \\ &\quad - \int_0^1 ds \frac{1}{\lambda} \text{Tr}\left[\frac{d^2P}{ds^2}\rho\right] \\ &= \frac{1}{\lambda(1)} \text{Tr}\left[\frac{dP}{ds}(1)\rho(1)\right] - \frac{1}{\lambda(0)} \text{Tr}\left[\frac{dP}{ds}(0)\rho(0)\right] \\ &\quad - \int_0^1 ds \frac{d}{ds} \left(\frac{1}{\lambda} \right) \text{Tr}\left[\frac{dP}{ds}\rho\right] - \int_0^1 ds \frac{1}{\lambda} \text{Tr}\left[\frac{d^2P}{ds^2}\rho\right] \\ &\leq \frac{1}{\lambda(1)} \left\| \frac{dP}{ds}(1) \right\| - \frac{1}{\lambda(0)} \text{Tr}\left[\frac{dP}{ds}(0)\rho(0)\right] \\ &\quad + \int_0^1 ds \left\| \frac{d}{ds} \left(\frac{1}{\lambda} \right) \right\| \left\| \frac{dP}{ds} \right\| + \int_0^1 ds \frac{1}{\lambda} \left\| \frac{d^2P}{ds^2} \right\|. \quad (73)\end{aligned}$$

So far, we have essentially followed the derivations of Ref. [29] in bounding the error γ , albeit applied to a different setup involving random applications of the walk operators. The rest of the derivation improves over the previous results to return better constant factors.

We now simplify Eq. (73) by resolving its terms. Let us start from

$$\begin{aligned}P(s) &= |0\rangle\langle 0| \otimes P_{H(s)} \\ &= |0\rangle\langle 0| \otimes (|E_0^+(s)\rangle\langle E_0^+(s)| + |E_0^-(s)\rangle\langle E_0^-(s)|).\end{aligned}\quad (74)$$

Hence, since $|E_0^-\rangle$ has no s -dependence,

$$\begin{aligned}\frac{dP(s)}{ds} &= |0\rangle\langle 0| \otimes \left(\frac{d}{ds} |E_0^+(s)\rangle\langle E_0^+(s)| + |E_0^+(s)\rangle \frac{d}{ds} \langle E_0^+(s)| \right).\end{aligned}\quad (75)$$

As shown in the proof of Lemma 2 below, by exploiting a phase freedom in the definition of $|E_0^+(s)\rangle$, we can make sure that $\langle E_0^+(s)| (d/ds) |E_0^+(s)\rangle = 0$. Since $\rho(0) = |0\rangle\langle 0| \otimes |E_0^+\rangle\langle E_0^+|$, it follows that the second term in Eq. (73) is zero:

$$\begin{aligned}\gamma &\leq \frac{1}{\lambda(1)} \left\| \frac{dP}{ds}(1) \right\| + \int_0^1 ds \left\| \frac{d}{ds} \left(\frac{1}{\lambda} \right) \right\| \left\| \frac{dP}{ds} \right\| \\ &\quad + \int_0^1 ds \frac{1}{\lambda} \left\| \frac{d^2P}{ds^2} \right\|.\end{aligned}\quad (76)$$

Let us introduce two lemmas, bounding $\|dP/ds\|$ and $\|d^2P/ds^2\|$ in the expression above:

Lemma 2.

$$\left\| \frac{dP}{ds} \right\| \leq \frac{c_1}{\Delta(s)}, \quad (77)$$

where

$$c_1 = \sqrt{\text{Var}_{|E_0^+(s)\rangle} \left[\frac{dH(s)}{ds} \right]} \quad (78)$$

and for an operator M we defined

$$\text{Var}_{|\psi\rangle}[M] = \langle M^2 \rangle - \langle M \rangle^2,$$

with the average taken with respect to $|\psi\rangle$. We find

$$c_1 \leq \sqrt{2}. \quad (79)$$

Proof. From Eq. (74) we have

$$\left\| \frac{dP(s)}{ds} \right\| = \left\| \frac{dP_{H(s)}}{ds} \right\|. \quad (80)$$

Reasoning as in Eq. (75),

$$\frac{dP_{H(s)}}{ds} = \frac{d}{ds} (|E_0^+(s)\rangle\langle E_0^+(s)| + |E_0^+(s)\rangle \frac{d}{ds} \langle E_0^+(s)|). \quad (81)$$

Note that $dP_{H(s)}/ds$ has no support on $|E_0^-\rangle$, so we can ignore this eigenstate. Correspondingly, for simplicity we

use the notation $|E_0^+(s)\rangle \equiv |E_0(s)\rangle$. In this way, all relevant eigenvalues and eigenstates of $H(s)$ are labeled by $E_i, |E_i\rangle$. Since

$$H(s)|E_i(s)\rangle = E_i(s)|E_i(s)\rangle, \quad (82)$$

we have

$$\begin{aligned} \frac{dH(s)}{ds}|E_i(s)\rangle + H(s)\frac{d}{ds}|E_i(s)\rangle \\ = \frac{dE_i(s)}{ds}|E_i(s)\rangle + E_i(s)\frac{d}{ds}|E_i(s)\rangle. \end{aligned} \quad (83)$$

So, for $i \neq 0$,

$$\langle E_0(s)|\frac{d}{ds}|E_i(s)\rangle = \frac{\langle E_0(s)|\frac{dH(s)}{ds}|E_i(s)\rangle}{E_i - E_0} := d_{0i}. \quad (84)$$

Also, $d_{i0} = -d_{0i}^*$.

For $i = 0$ instead, without loss of generality we can choose

$$\langle E_0(s)|\frac{d}{ds}|E_0(s)\rangle = 0, \quad \forall s. \quad (85)$$

Let us see that this is the case. $|E_0(s)\rangle$ is defined modulo a phase, $|\bar{E}_0(s)\rangle = e^{i\theta(s)}|E_0(s)\rangle$. Therefore,

$$\frac{d}{ds}|\bar{E}_0(s)\rangle = e^{i\theta(s)}i\frac{d\theta(s)}{ds}|E_0(s)\rangle + e^{i\theta(s)}\frac{d}{ds}|E_0(s)\rangle. \quad (86)$$

Hence,

$$\begin{aligned} \langle \bar{E}_0(s)|\frac{d}{ds}|\bar{E}_0(s)\rangle &= e^{-i\theta(s)}\langle E_0(s)|\frac{d}{ds}|\bar{E}_0(s)\rangle \\ &= i\frac{d\theta(s)}{ds} + \langle E_0(s)|\frac{d}{ds}|E_0(s)\rangle. \end{aligned} \quad (87)$$

So we can set

$$\frac{d\theta(s)}{ds} = \langle E_0(s)|i\frac{d}{ds}|E_0(s)\rangle, \quad \theta(0) = 0. \quad (88)$$

We can then choose a phase such that Eq. (85) holds. And now we will drop the bar for simplicity.

Using Eqs. (84) and (85):

$$\begin{aligned} \frac{dP_{H(s)}}{ds} &= \sum_{i \neq 0} \left(\langle E_i(s)|\frac{d}{ds}(|E_0(s)\rangle)\rangle |E_i(s)\rangle \langle E_0(s)| \right. \\ &\quad \left. + \frac{d}{ds}(\langle E_0(s)|)|E_i(s)\rangle |E_0(s)\rangle \langle E_i(s)| \right) \\ &= \sum_{i \neq 0} (-d_{0i}|E_i(s)\rangle \langle E_0(s)| + d_{0i}|E_0(s)\rangle \langle E_i(s)|). \end{aligned}$$

The singular values of the matrix $dP_{H(s)}/ds$ are zero and

$$\begin{aligned} \sqrt{\sum_{i \neq 0} d_{0i}^2} &\leq \frac{1}{\Delta(s)} \sqrt{\sum_{i \neq 0} \left| \langle E_i(s)|\frac{dH(s)}{ds}|E_0(s)\rangle \right|^2} \\ &= \frac{1}{\Delta(s)} \sqrt{\langle E_0(s)|\frac{dH(s)}{ds}(I - |E_0(s)\rangle \langle E_0(s)|)\frac{dH(s)}{ds}|E_0(s)\rangle} \\ &= \frac{1}{\Delta(s)} \sqrt{\text{Var}_{|E_0^+(s)\rangle} \left[\frac{dH(s)}{ds} \right]}, \end{aligned}$$

since $|E_0(s)\rangle = |E_0^+(s)\rangle$ with the previous notation. The result in Eq. (77) follows. For the second part

$$\begin{aligned} \left\| \frac{dP}{ds} \right\| &\leq \frac{1}{\Delta(s)} \sqrt{|\langle E_0^+(s)| \left(\frac{dH(s)}{ds} \right)^2 |E_0^+(s)\rangle|} \\ &\leq \frac{1}{\Delta(s)} \sqrt{|\langle y(s)| (H(1) - H(0))^2 |y(s)\rangle|} \\ &\leq \frac{1}{\Delta(s)} \|H(1) - H(0)\| \\ &\leq \frac{\sqrt{2}}{\Delta(s)}. \end{aligned}$$

To prove the last line, note that

$$\|H(1) - H(0)\| = \|\lvert 0 \rangle \langle 1 \rvert \otimes B + \lvert 1 \rangle \langle 0 \rvert \otimes B^\dagger\| \quad (89)$$

where $B = (Z \otimes I + X \otimes \bar{A})\Pi$. Then,

$$\begin{aligned} (\lvert 0 \rangle \langle 1 \rvert \otimes B + \lvert 1 \rangle \langle 0 \rvert \otimes B^\dagger)^\dagger (\lvert 0 \rangle \langle 1 \rvert \otimes B + \lvert 1 \rangle \langle 0 \rvert \otimes B^\dagger) \\ = \lvert 0 \rangle \langle 0 \rvert \otimes BB^\dagger + \lvert 1 \rangle \langle 1 \rvert \otimes B^\dagger B, \end{aligned}$$

which means that

$$\|H(1) - H(0)\| = \|B\| \leq \|-Z \otimes I + X \otimes \bar{A}\| \quad (90)$$

Now,

$$(-Z \otimes I + X \otimes \bar{A})^\dagger (-Z \otimes I + X \otimes \bar{A}) = I \otimes (I + \bar{A}^2)$$

is a matrix whose eigenvalues are bounded by 2, because $\|\bar{A}^2\| \leq \|\bar{A}\|^2 \leq 1$. Hence $\|-Z \otimes I + X \otimes \bar{A}\| \leq \sqrt{2}$ and so $\|H(1) - H(0)\| \leq \sqrt{2}$ immediately follows. ■

This lemma strengthens the bound in Lemma 3 of Ref. [29], which gave $\|dP/ds\| \leq 2\sqrt{2}/\Delta(s)$.

Lemma 3. Assume $d^2H/ds^2 = 0$. Then

$$\left\| \frac{d^2P}{ds^2} \right\| \leq 6 \frac{\left\| \frac{dH}{ds} \right\|^2}{\Delta^2(s)}. \quad (91)$$

For us, we have that $\|dH/ds\| \leq \sqrt{2}$, and so

$$\left\| \frac{d^2 P}{ds^2} \right\| \leq \frac{12}{\Delta(s)^2}. \quad (92)$$

Proof. Recall that from Eq. (74) $P(s) = |0\rangle\langle 0| \otimes P_{H(s)}$. First write

$$P_{H(s)} = -\frac{1}{2\pi i} \oint_{\Gamma} R_z dz \quad (93)$$

where Γ is a circle of radius $\Delta(s)/2$ in the complex plane with center 0, and the contour integral is taken in the anti clockwise direction. We also have

$$R_z = (zI - H(s))^{-1} \quad (94)$$

being the resolvent. Using $dM^{-1}/ds = -M^{-1}(dM/ds)M^{-1}$, we have $\frac{dR_z}{ds} = R_z(dH/ds)R_z$. With this

$$\begin{aligned} \frac{d^2 P_{H(s)}}{ds^2} &= -\frac{1}{2\pi i} \oint_{\Gamma} \frac{d^2 R_z}{ds^2} dz \\ &= -\frac{1}{2\pi i} \oint_{\Gamma} \left(2R_z \frac{dH}{ds} R_z \frac{dH}{ds} R_z + R_z \frac{d^2 H}{ds^2} R_z \right) dz \\ &= -\frac{1}{2\pi i} \oint_{\Gamma} \left(2R_z \frac{dH}{ds} R_z \frac{dH}{ds} R_z \right) dz \\ &= -2G \left(\frac{dH}{ds}, \frac{dH}{ds} \right), \end{aligned} \quad (95)$$

where we used that in our setting $d^2 H(s)/ds^2 = 0$ and we introduced

$$G(X, Y) = \frac{1}{2\pi i} \oint_{\Gamma} (R_z X R_z Y R_z) dz. \quad (96)$$

The operator G was analyzed in Ref. [43]. In particular, in the blocks defined by the images of $P_{H(s)}$ and $P_{H(s)}^{\perp}$, G has the following structure (proof of Lemma 5 [43]):

$$\begin{aligned} G \left(\frac{dH}{ds}, \frac{dH}{ds} \right) &= P_{H(s)} B_1 P_{H(s)} + P_{H(s)} B_2 P_{H(s)}^{\perp} \\ &\quad - P_{H(s)}^{\perp} B_2 P_{H(s)} - P_{H(s)}^{\perp} B_1 P_{H(s)}^{\perp}, \end{aligned} \quad (97)$$

where $B_1 = \tilde{K}^2$, $B_2 = \tilde{K}\tilde{K} - \tilde{K}K$, $K = dH/ds$ and

$$\tilde{X} := \frac{1}{2\pi i} \oint_{\Gamma} R_z X R_z dz, \quad (98)$$

was called in Ref. [43] the “twiddle operation”. dH/ds is Hermitian, and the twiddle operation preserves Hermiticity. Hence B_1 is Hermitian and B_2 is anti-Hermitian. G is

then Hermitian, so its norm just equals the largest absolute value of the eigenvalues. Compute

$$\begin{aligned} Gv &= GP_{H(s)}v + GP_{H(s)}^{\perp}v \\ &= [P_{H(s)}B_1 - P_{H(s)}^{\perp}B_2]v_1 + [P_{H(s)}B_2 - P_{H(s)}^{\perp}B_1]v_2 \\ &= P_{H(s)}(B_1v_1 + B_2v_2) - P_{H(s)}^{\perp}(B_2v_1 + B_1v_2) \end{aligned} \quad (99)$$

where $v_1 = P_{H(s)}v$, $v_2 = P_{H(s)}^{\perp}v$ and v is any unit vector. This is the sum of two orthogonal vectors, so the norm of Gv is

$$\begin{aligned} \|Gv\|^2 &= \|P_{H(s)}(B_1v_1 + B_2v_2)\|^2 + \|P_{H(s)}^{\perp}(B_2v_1 + B_1v_2)\|^2 \\ &\leq \|(B_1v_1 + B_2v_2)\|^2 + \|(B_2v_1 + B_1v_2)\|^2 \\ &= v_1^{\dagger}B_1^2v_1 + v_2^{\dagger}B_1^2v_2 - v_1^{\dagger}B_2^2v_1 - v_2^{\dagger}B_2^2v_2 \\ &\quad + v_2^{\dagger}(-B_2B_1 + B_1B_2)v_1 + v_1^{\dagger}(B_1B_2 - B_2B_1)v_2. \end{aligned} \quad (100)$$

If $b_1 = \|B_1\|$, $b_2 = \|B_2\|$, $x_1 = \|v_1\|$, $x_2 = \|v_2\|$, we get

$$\|Gv\|^2 \leq b_1^2(x_1^2 + x_2^2) + b_2^2(x_1^2 + x_2^2) + 4b_1b_2x_1x_2. \quad (101)$$

The norm of the matrix is upper bounded by maximizing the last expression over all $x_1 \geq 0$, $x_2 \geq 0$ with $x_1^2 + x_2^2 = 1$. Since $x_1x_2 \leq 1/2$ on the unit circle

$$\|Gv\|^2 \leq (b_1 + b_2)^2. \quad (102)$$

Hence

$$\begin{aligned} \|G\| &\leq \|B_1\| + \|B_2\| \\ &= \|\tilde{K}\|^2 + \left\| -\tilde{K}\tilde{K} + \tilde{K}K \right\|. \end{aligned} \quad (103)$$

Using Lemma 7 in Ref. [43],

$$\|\tilde{K}\| \leq \frac{\|K\|}{\Delta} = \frac{\|dH/ds\|}{\Delta}. \quad (104)$$

Putting all of this together we find

$$\begin{aligned}
\left\| \frac{d^2 P}{ds^2} \right\| &= \left\| \frac{d^2 P_{H(s)}}{ds^2} \right\| \\
&= 2 \|G\| \\
&\leq 2 \left(\left\| \frac{dH}{ds} \right\|^2 + \left\| \frac{dH}{ds} \frac{dH}{ds} \right\| + \left\| \frac{dH}{ds} \frac{dH}{ds} \right\| \right) \\
&\leq 2 \left(\left\| \frac{dH}{ds} \right\|^2 + \frac{\left\| \frac{dH}{ds} \right\| \left\| \frac{dH}{ds} \right\|}{\Delta} + \frac{\left\| \frac{dH}{ds} \right\| \left\| \frac{dH}{ds} \right\|}{\Delta} \right) \\
&\leq 2 \frac{\left\| \frac{dH}{ds} \right\|^2}{\Delta^2} + 4 \frac{\left\| \frac{dH}{ds} \right\| \left\| \frac{dH}{ds} \right\|}{\Delta} \\
&\leq 2 \frac{\left\| \frac{dH}{ds} \right\|^2}{\Delta^2} + 4 \frac{\left\| \frac{dH}{ds} \right\|^2}{\Delta^2} \\
&= 6 \frac{\left\| \frac{dH}{ds} \right\|^2}{\Delta^2}. \tag{105}
\end{aligned}$$

In the proof of Lemma 2 we have seen $\|dH/ds\| \leq \sqrt{2}$, so the final result follows immediately. ■

Note that the bound above strengthens Lemma 3 in Ref. [29], which gave $\|d^2 P/ds\| \leq 16/\Delta^2$.

With these estimates the error bound becomes

$$\begin{aligned}
\gamma &\leq \frac{\sqrt{2}}{|\lambda(1)|\Delta(1)} + \int_0^1 ds \frac{\sqrt{2}}{\Delta(s)} \left| \frac{d}{ds} \left(\frac{1}{\lambda} \right) \right| + \int_0^1 ds \frac{12}{|\lambda|\Delta^2} \\
&= (1) + (2) + (3). \tag{106}
\end{aligned}$$

Now, in our case $\Delta(s) = \sqrt{(1-s)^2 + (s/\kappa)^2}$. Furthermore, we shall take the rate

$$\lambda(s) = \frac{C}{\Delta(s)^q \Delta_{\min}^{1-q}}, \tag{107}$$

where $\Delta_{\min}^{1-q}$ is the minimal gap, which occurs at $s = (1 + 1/\kappa^2)^{-1}$ and equals

$$\Delta_{\min} = \sqrt{\frac{1}{1 + \kappa^2}}. \tag{108}$$

Now,

$$\left| \frac{d}{ds} \frac{1}{\lambda} \right| = \frac{q\Delta^{q-1} | -1 + s + s/\kappa^2 |}{C\Delta} \Delta_{\min}^{1-q} \tag{109}$$

and $\Delta(1) = 1/\kappa$, $\lambda(1) = C\kappa^q/\Delta_{\min}^{1-q}$. So

$$\begin{aligned}
(1) &= \frac{\sqrt{2}\Delta_{\min}^{1-q}}{C\kappa^{q-1}}, \\
(2) &= \int_0^1 ds \frac{\sqrt{2}q\Delta(s)^{q-3}}{C} \Delta_{\min}^{1-q} \left| -1 + s + \frac{s}{\kappa^2} \right|, \\
(3) &= \int_0^1 ds \frac{12\Delta(s)^{q-2} \Delta_{\min}^{1-q}}{C}.
\end{aligned}$$

We find that taking $q = 1/2$ gives the minimal error. We next establish a sufficient value for C to ensure that $\gamma \leq 1/2$.

For $q = 1/2$ we have that

$$\begin{aligned}
(1) &= \frac{\sqrt{2}\Delta_{\min}^{1/2}\kappa^{1/2}}{C}, \\
(2) &= \frac{\sqrt{2}\Delta_{\min}^{1/2}}{2C} \int_0^1 ds \frac{\left| -1 + s + \frac{s}{\kappa^2} \right|}{\Delta(s)^{5/2}}, \\
(3) &= \frac{12\Delta_{\min}^{1/2}}{C} \int_0^1 ds \frac{1}{\Delta(s)^{3/2}}.
\end{aligned}$$

We first note that $\Delta_{\min} = 1/\sqrt{1 + \kappa^2}$, and for the first term we have

$$(1) = \frac{2}{C} \sqrt{\frac{\kappa}{1 + \kappa^2}} \leq \frac{\sqrt{2}}{C}. \tag{110}$$

The second term can be written as

$$(2) = \frac{\sqrt{2}\Delta_{\min}^{1/2}}{2C} \int_0^1 ds \frac{\left| \frac{d\Delta}{ds} \right|}{\Delta(s)^{3/2}}, \tag{111}$$

however $d\Delta/ds \geq 0$ for $s \in [s_{\min}, 1]$, and $d\Delta/ds < 0$ for $s \in [0, s_{\min})$, where $s_{\min} = \kappa^2/(1 + \kappa^2)$. This implies that

$$\begin{aligned}
(2) &= \frac{\sqrt{2}\Delta_{\min}^{1/2}}{2C} \left(- \int_0^{s_{\min}} ds \frac{\frac{d\Delta}{ds}}{\Delta(s)^{3/2}} + \int_{s_{\min}}^1 ds \frac{\frac{d\Delta}{ds}}{\Delta(s)^{3/2}} \right) \\
&= \frac{\sqrt{2}\Delta_{\min}^{1/2}}{2C} \left(2 \int_0^{s_{\min}} ds \frac{d(\Delta^{-1/2})}{ds} - 2 \int_{s_{\min}}^1 ds \frac{d(\Delta^{-1/2})}{ds} \right) \\
&= \frac{1}{\sqrt{2}C} \frac{1}{(1 + \kappa^2)^{1/4}} (4(1 + \kappa^2)^{1/4} - 2(1 + \sqrt{\kappa})) \\
&\leq \frac{\sqrt{2}}{C}. \tag{112}
\end{aligned}$$

Finally, the third term is given by

$$(3) = \frac{12\Delta_{\min}^{1/2}}{C} \int_0^1 ds \frac{1}{((1-s)^2 + (s/\kappa)^2)^{3/4}}. \tag{113}$$

We want to upper bound the integral

$$I_\kappa = \int_0^1 ds \frac{1}{((1-s)^2 + (s/\kappa)^2)^{3/4}}. \quad (114)$$

First we perform a change of variables

$$\tilde{s} := \frac{1 + \kappa^2}{\kappa}(s - s_{\min}). \quad (115)$$

to obtain

$$I_\kappa = \frac{\kappa}{(\kappa^2 + 1)^{1/4}} \int_{-\kappa}^{1/\kappa} d\tilde{s} \frac{1}{(1 + \tilde{s}^2)^{3/4}}. \quad (116)$$

The integral on the right-hand side can be split into two integrals over $[-\kappa, 0]$ and $[0, 1/\kappa]$. Changing variables $v = 1/u$ we can rewrite the first integral as

$$\int_{-\kappa}^0 d\tilde{s} \frac{1}{(1 + \tilde{s}^2)^{3/4}} = \int_{1/\kappa}^\infty dv v^{-1/2} \frac{1}{(1 + v^2)^{3/4}} \quad (117)$$

$$\leq \int_{1/\kappa}^\infty dv \frac{1}{(1 + v^2)^{3/4}}, \quad (118)$$

where for the inequality we used the fact that $v^{-1/2} < 1$ in the domain of integration. Combining the integrals over both domains we obtain

$$\int_{-\kappa}^{1/\kappa} d\tilde{s} \frac{1}{(1 + \tilde{s}^2)^{3/4}} \leq \int_0^\infty d\tilde{s} \frac{1}{(1 + \tilde{s}^2)^{3/4}}, \quad (119)$$

$$= \frac{2\sqrt{\pi}\Gamma(5/4)}{\Gamma(3/4)}, \quad (120)$$

where $\Gamma(x)$ is the Gamma function. Putting this back into Eq. (116) we obtain

$$I_\kappa \leq \left(\frac{2\sqrt{\pi}\Gamma(5/4)}{\Gamma(3/4)} \right) \frac{\kappa}{(\kappa^2 + 1)^{1/4}} \kappa^{1/2}, \quad (121)$$

which then implies

$$(3) \leq \frac{24\sqrt{\pi}\Gamma(5/4)}{C\Gamma(3/4)}. \quad (122)$$

Therefore, we have

$$\gamma \leq (1) + (2) + (3) \leq \frac{1}{C} \left(2\sqrt{2} + \frac{24\sqrt{\pi}\Gamma(5/4)}{\Gamma(3/4)} \right). \quad (123)$$

Choosing $C \geq 68.59$ then implies that $\gamma \leq 1/2$.

The overall average query cost is then [recall Eqs. (63) and (107)]

$$\begin{aligned} \int_0^1 \lambda(s) Q(s) ds &\leq 2 \times 2.322\alpha \int_0^1 \frac{\lambda(s)}{\Delta(s)} ds \\ &\leq 318.6\alpha(1 + \kappa^2)^{1/4} \int_0^1 \frac{1}{[(1-s)^2 + (s/\kappa)^2]^{3/4}}, \end{aligned}$$

where we use that $\alpha \geq \alpha(s)$ for all s (not using this inequality gives little to no improvement). Using Eq. (121)

$$\int_0^1 \lambda(s) Q(s) ds \leq 318.6 \frac{2\sqrt{\pi}\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})} \alpha \kappa \approx 835.4\alpha\kappa. \quad (124)$$

G. Filtering cost

We next consider applying $P(1)$ to the output state of the adiabatic component of the protocol, where $P(1)$ is the projector onto the nullspace of the Hamiltonian $H(1)$. A QSP implementation was given in Ref. [23], but it requires phase factor precomputations.

We expand and formalize the analysis of Ref. [24], proving stronger guarantees for their algorithm.

Lemma 4 (Filtering cost). Let U_H be an $(\alpha_H, m_H, 0)$ block encoding of a Hermitian operator H , let P denote the projection onto the nullspace of H and Δ a lower bound on the gap between zero and the nearest non zero eigenvalue. Then we can realize a $(1, m_H + 2, \epsilon_P)$ block-encoding of P with a number of calls to $U_H, U_H^\dagger$ equal to

$$l_P = \left\lceil \frac{\alpha_H}{\Delta} \ln \frac{2}{\epsilon_P} + 2 \right\rceil, \quad (125)$$

with no QSP phase factor precomputation required.

Proof. We consider a circuit PREP that prepares a “window” state $|\psi\rangle$, given as

$$|0\rangle \mapsto |\psi\rangle := \sum_{j=0}^l \sqrt{\psi_j} |j\rangle. \quad (126)$$

The SELECT is

$$\sum_{j=0}^l |j\rangle\langle j| \otimes W_2^j, \quad (127)$$

where W_2 is the walk operator

$$W_2 = U_H^\dagger \mathcal{Z} U_H \mathcal{Z}, \quad (128)$$

and $\mathcal{Z} = (2|0\rangle\langle 0| - I) \otimes I$ where $|0\rangle$ labels the block-encoding space. The overall protocol then involves the

standard $\text{PREP}^\dagger \circ \text{SELECT} \circ \text{PREP}$ routine. The Hermitian operator H is assumed to have eigendecomposition

$$H = \sum_{m,s} \lambda_m |\varphi_{m,s}\rangle \langle \varphi_{m,s}|, \quad (129)$$

where $\{|\varphi_m\rangle\}$ is a normalized basis of eigenvectors of H and s is a degeneracy index. We label the nullspace by $m = 0$, so that $\lambda_{0,s} = 0$.

The measurement operator induced on the system by finding the ancillas where the window state was prepared in the zero state is

$$R' = (\langle \psi | \otimes I) \text{SELECT}(|\psi\rangle \otimes I) \quad (130)$$

$$= \sum_{j=0}^l \psi_j W_2^j. \quad (131)$$

The walk operator can be written as (see Sec. II E)

$$W_2 = \sum_{m,s} e^{i2\phi_m} |\xi_{m,s}^+\rangle \langle \xi_{m,s}^+| + e^{-i2\phi_m} |\xi_{m,s}^-\rangle \langle \xi_{m,s}^-|, \quad (132)$$

where

$$|\xi_{m,s}^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle|\varphi_{m,s}\rangle \mp i|\perp_{m,s}\rangle), \quad (133)$$

$$\phi_m = \arccos(\lambda_m/\alpha). \quad (134)$$

Let us define $\tilde{\psi}(\zeta) := \sum_{j=0}^l \psi_j e^{i\zeta j}$. Then

$$R' = \sum_m \tilde{\psi}(2\phi_m) \Pi_m^+ + \tilde{\psi}(-2\phi_m) \Pi_m^-, \quad (135)$$

with $\Pi_m^\pm = \sum_s |\xi_{m,s}^\pm\rangle \langle \xi_{m,s}^\pm|$. We also have that

$$R = (\langle 0 | \otimes I) R' (|0\rangle \otimes I) \quad (136)$$

$$= \sum_m \frac{1}{2} (\tilde{\psi}(2\phi_m) + \tilde{\psi}(-2\phi_m)) P_m, \quad (137)$$

where $P_m = \sum_s |\varphi_{m,s}\rangle \langle \varphi_{m,s}|$. For the nullspace $m = 0$ we have

$$2\phi_0 = 2 \arccos(0) = \pi. \quad (138)$$

If we require $\tilde{\psi}(\zeta = \pm\pi) = 1$, it follows that

$$\|R - P\| \leq \frac{1}{2} \max_{m \neq 0} |\tilde{\psi}(2\phi_m) + \tilde{\psi}(-2\phi_m)|. \quad (139)$$

Following the same reasoning seen in Sec. II E, ϕ_m takes values in $[0, \pi]$, with a gap around $\phi_0 = \pi/2$ of at least

Δ/α . So

$$\|R - P\| \leq \frac{1}{2} \max_{\zeta \notin [\pi-2\Delta/\alpha_H, \pi+2\Delta/\alpha_H]} |\tilde{\psi}(\zeta) + \tilde{\psi}(-\zeta)|. \quad (140)$$

Following Ref. [24], we choose ψ_j to provide a Dolph-Chebyshev window function, with Fourier transform

$$\tilde{\psi}(2\theta_m) = \epsilon_P T_l[\beta \cos(2\theta_m - \pi)], \quad (141)$$

where $\theta_m = \pi/2(1 + (m/l))$, with $m = -l, \dots, l$ and $T_l(x)$ are Chebyshev polynomials of the first kind. Here β is the trade-off parameter between main lobe and side lobe weights. We fix it as

$$\beta = \cosh\left(\frac{1}{l} \cosh^{-1}\left(\frac{1}{\epsilon_P}\right)\right), \quad (142)$$

where ϵ_P is the required attenuation. Then, with this choice we have, as required above,

$$\tilde{\psi}(\zeta = \pm\pi) = \epsilon_P T_l[\beta] = 1. \quad (143)$$

Furthermore, we want to set the width of the peak to coincide with the phase gap, which equals $2\Delta/\alpha_H$.

Since $T_l(1) = 1$, we set the relation between β and Δ to be

$$\beta \cos(2\Delta/\alpha_H) = 1. \quad (144)$$

Combining the above equations gives

$$l = \left\lceil \frac{\cosh^{-1}(1/\epsilon_P)}{\cosh^{-1}(1/(\cos(2\Delta/\alpha_H)))} \right\rceil \leq \frac{\alpha_H}{2\Delta} \ln \frac{2}{\epsilon_P} + 1. \quad (145)$$

With this choice,

$$\max_{\zeta \notin [\pi-2\Delta/\alpha_H, \pi+2\Delta/\alpha_H]} |\tilde{\psi}(\pm\zeta)| \leq \epsilon_P, \quad (146)$$

and so $\|R - P\| \leq \epsilon_P$.

However, each call to the walk operator involves one call to U_H and one call to $U_H^\dagger$ and so $l_P = 2l \leq (\alpha_H/\Delta) \ln(2/\epsilon_P) + 2$, as claimed. The circuit implementation of the above is the same as in Ref. [24], using a just-in-time unary encoding of the controlled walk operators, where the uncompute ($\text{PREP}^\dagger$) of the window function state is inverted in its controls. The orderings of these uncompute control operations are then changed so that at most two qubits are ever required coherently in the control register. ■

As highlighted in Ref. [24], the use of the just-in-time method also means that mid-circuit measurements can

reduce circuit costs by terminating the circuit earlier than a pure post selection circuit.

Here we apply this with $H = H(1)$, $\alpha_H = \alpha$, $\Delta = \Delta(1) = 1/\kappa$. What is more, a factor of 2 is saved by flagging a failure early.

H. Total qubit count

We now specify the total number of logical qubits needed for the algorithm. The following auxiliary qubits are required by the algorithm:

- (a) A total of a auxiliary qubits for the block-encoding U_A of the matrix A .
- (b) A single qubit for the Hermitian extension $A \rightarrow \tilde{A}$.
- (c) A single qubit for $\tilde{A} \rightarrow A(s)$.
- (d) A single qubit for $A(s) \rightarrow H(s)$.
- (e) Two extra qubits for block-encoding $H(s) \rightarrow U_{H(s)}$.
- (f) Two extra qubits for performing the eigenspace filtering.

This gives a total number of auxiliary qubits being $a + 7$. Therefore, with A acting on n qubits, the total number of logical qubits required for the algorithm is

$$n_L = n + a + 7, \quad (147)$$

as claimed in the main text.

I. Combining adiabatic and filtering errors

In this section we determine how the adiabatic error ϵ_{AD} combines with the filtering error ϵ_P . We will repeatedly use the following simple results:

Lemma 5. For any two square matrices A, B ,

$$\|AB\|_1 \leq \min\{\|A\|_1 \|B\|, \|A\| \|B\|_1\}. \quad (148)$$

Proof. From the definition of trace-norm [44],

$$\|AB\|_1 = \max_U |\langle AB, U \rangle|, \quad (149)$$

where the maximization is over all unitaries and $\langle \cdot, \cdot \rangle$ is the Hilbert-Schmidt scalar product. Using Hölder's inequality for Schatten p -norms and their invariance under composition with unitaries we have

$$|\langle AB, U \rangle| = |\langle A, UB^\dagger \rangle| \leq \|A\|_1 \|UB^\dagger\| = \|A\|_1 \|B\|.$$

Similarly, we have that $|\langle A, UB^\dagger \rangle| \leq \|A\| \|UB^\dagger\|_1 = \|A\| \|B\|_1$. Therefore we can take the smaller of the two results as the upper bound on $\|AB\|_1$, which completes the proof. ■

Lemma 6 (Perturbing 1-norm). Assume $\|\tilde{A} - A\| \leq \delta$, $\|\tilde{B} - B\| \leq \delta$. Then for every density operator ρ ,

$$\|\tilde{A}\rho\tilde{B} - A\rho B\|_1 \leq \delta(\|A\| + \|B\|) + \delta^2. \quad (150)$$

Proof. Using Lemma 5,

$$\begin{aligned} \|\tilde{A}\rho\tilde{B} - A\rho B\|_1 &= \|(\tilde{A} - A)\rho(\tilde{B} - B) + (\tilde{A} - A)\rho B + A\rho(\tilde{B} - B)\|_1 \\ &\leq \|(\tilde{A} - A)\rho(\tilde{B} - B)\|_1 + \|(\tilde{A} - A)\rho B\|_1 + \|A\rho(\tilde{B} - B)\|_1 \\ &\leq \|\tilde{A} - A\| \|\tilde{B} - B\| \|\rho\|_1 + \|\tilde{A} - A\| \|B\| \|\rho\|_1 \\ &\quad + \|\tilde{B} - B\| \|A\| \|\rho\|_1 \\ &\leq \delta^2 + \delta(\|A\| + \|B\|). \end{aligned}$$

Which completes the proof. ■

Lemma 7 (Perturbing probabilities). Let $A, \tilde{A}$ be square matrices satisfying $\|A - \tilde{A}\| \leq \delta$ and ρ a density operator. Then

$$\begin{aligned} \text{Tr}(\rho A^\dagger A) - 2\delta\|A\| &\leq \text{Tr}(\rho \tilde{A}^\dagger \tilde{A}) \leq \text{Tr}(\rho A^\dagger A) + 2\delta\|A\| + \delta^2, \end{aligned} \quad (151)$$

Proof. Let $\Delta A = \tilde{A} - A$. Then using Hölder's inequality and submultiplicativity

$$\begin{aligned} \text{Tr}(\rho \tilde{A}^\dagger \tilde{A}) &= \text{Tr}(\rho(A^\dagger + \Delta A^\dagger)(A + \Delta A)) \\ &= \text{Tr}(\rho A^\dagger A) + \text{Tr}(\rho(A^\dagger \Delta A + \Delta A A^\dagger)) \\ &\quad + \text{Tr}(\rho \Delta A^\dagger \Delta A). \end{aligned} \quad (152)$$

The last term is always positive, and therefore we have that

$$\begin{aligned} \text{Tr}(\rho A^\dagger A) - |\text{Tr}(\rho(A^\dagger \Delta A + \Delta A A^\dagger))| &\leq \text{Tr}(\rho \tilde{A}^\dagger \tilde{A}) \\ &\leq \text{Tr}(\rho A^\dagger A) + |\text{Tr}(\rho(A^\dagger \Delta A + \Delta A A^\dagger))| \\ &\quad + \text{Tr}(\rho \Delta A^\dagger \Delta A). \end{aligned} \quad (153)$$

Making use of the Hölder inequality $|\text{Tr}(A^\dagger B)| \leq \|A\|_1 \|B\|$ and submultiplicativity of the operator norm we have that

$$\begin{aligned} \text{Tr}(\rho A^\dagger A) - 2\delta\|A\| &\leq \text{Tr}(\rho \tilde{A}^\dagger \tilde{A}) \leq \text{Tr}(\rho A^\dagger A) + 2\delta\|A\| + \delta^2, \end{aligned} \quad (154)$$

as required. ■

Theorem 2 (Adiabatic and filtering errors). Consider the following protocol:

- (1) Apply the Poisson adiabatic protocol with $\gamma \geq 1/2$, outputting a state ρ .
- (2) Apply an approximation R of $P_{H(1)}$, where $P_{H(1)}$ is the nullspace projector for $H(1)$, as in Lemma 4, and $\|R - P_{H(1)}\| \leq \epsilon_P$.

Let $|y\rangle$ denote the normalized state $\propto A^{-1}|b\rangle$ and $\tilde{\sigma}$ the state prepared on the same registers by the above protocol. We have

$$\|\tilde{\sigma} - |y\rangle\langle y|\|_1 \leq 8\epsilon_P + 4\epsilon_P^2, \quad (155)$$

with success probability p_{succ} satisfying

$$p_{\text{succ}} \geq \frac{1}{2} - 2\epsilon_P. \quad (156)$$

Proof. As we have shown, the nullspace of $H(1)$ is spanned by $|0, +, 1, y\rangle$ and $|1, +, 0, b\rangle$, however ρ has zero overlap with $|1, +, 0, b\rangle$, due to the form of the adiabatic protocol. Therefore, we have that

$$\frac{1}{\text{Tr}(P_{H(1)}\rho)} P_{H(1)}\rho P_{H(1)} = |0, +, 1, y\rangle\langle 0, +, 1, y|. \quad (157)$$

Here, $\text{Tr}(P_{H(1)}\rho) \geq 1/2$, from the choice of adiabatic parameters.

However, for the second step we apply an approximate projector R to ρ to obtain the quantum state

$$\frac{1}{\text{Tr}(R^\dagger R \rho)} R \rho R^\dagger. \quad (158)$$

The success probability of implementing this is $\text{Tr}(R^\dagger R \rho)$, but using Lemma 7 we have that

$$\text{Tr}(R^\dagger R \rho) \geq \text{Tr}(P_{H(1)}\rho) - 2\epsilon_P \geq \frac{1}{2} - 2\epsilon_P. \quad (159)$$

We next bound the L_1 norm error in the output. We have from Lemma 6 that

$$\|R \rho R^\dagger - P_{H(1)}\rho P_{H(1)}\|_1 \leq 2\epsilon_P + \epsilon_P^2. \quad (160)$$

The distance between the output of the algorithm and the target is then bounded as

$$\begin{aligned} & \left\| \frac{R \rho R^\dagger}{\text{Tr}(R \rho R^\dagger)} - \frac{P_{H(1)}\rho P_{H(1)}}{\text{Tr}(P_{H(1)}\rho)} \right\|_1 \\ & \leq \left\| \frac{R \rho R^\dagger}{\text{Tr}(R \rho R^\dagger)} - \frac{R \rho R^\dagger}{\text{Tr}(\rho P_{H(1)})} \right\|_1 \\ & \quad + \left\| \frac{R \rho R^\dagger}{\text{Tr}(\rho P_{H(1)})} - \frac{P_{H(1)}\rho P_{H(1)}}{\text{Tr}(\rho P_{H(1)})} \right\|_1. \end{aligned}$$

We will bound the two terms separately. First,

$$\left\| \frac{R \rho R^\dagger}{\text{Tr}(R \rho R^\dagger)} - \frac{R \rho R^\dagger}{\text{Tr}(\rho P_{H(1)})} \right\|_1 \quad (161)$$

$$\begin{aligned} & = \|R \rho R^\dagger\|_1 \left| \frac{1}{\text{Tr}(R \rho R^\dagger)} - \frac{1}{\text{Tr}(\rho P_{H(1)})} \right| \\ & = \|R \rho R^\dagger\|_1 \frac{|\text{Tr}(R \rho R^\dagger) - \text{Tr}(\rho P_{H(1)})|}{\text{Tr}(R \rho R^\dagger) \text{Tr}(\rho P_{H(1)})} \\ & = \frac{|\text{Tr}(R \rho R^\dagger) - \text{Tr}(\rho P_{H(1)})|}{\text{Tr}(\rho P_{H(1)})}, \end{aligned} \quad (162)$$

where in the last step we used that $\|R \rho R^\dagger\|_1 = \text{Tr}(R \rho R^\dagger)$.

Using Lemma 7, we have seen that inequality Eq. (159) holds. We will apply Lemma 7 again. Inverting the right-hand side inequality one gets

$$\text{Tr}(\rho A^\dagger A) \geq \text{Tr}(\rho \tilde{A}^\dagger \tilde{A}) - 2\delta\|A\| - \delta^2. \quad (163)$$

Setting $A = P_{H(1)}$, $\tilde{A} = R$, $\delta = \epsilon_P$ this gives

$$\text{Tr}(\rho P_{H(1)}) \geq \text{Tr}(R \rho R^\dagger) - 2\epsilon_P - \epsilon_P^2,$$

and so $|\text{Tr}(R \rho R^\dagger) - \text{Tr}(\rho P_{H(1)})| \leq 2\epsilon_P + \epsilon_P^2$.

Putting things together

$$\left\| \frac{R \rho R^\dagger}{\text{Tr}(R \rho R^\dagger)} - \frac{R \rho R^\dagger}{\text{Tr}(\rho P_{H(1)})} \right\|_1 \leq 4\epsilon_P + 2\epsilon_P^2. \quad (164)$$

We also have that

$$\begin{aligned} & \left\| \frac{R \rho R^\dagger}{\text{Tr}(\rho P_{H(1)})} - \frac{P_{H(1)}\rho P_{H(1)}}{\text{Tr}(\rho P_{H(1)})} \right\|_1 \\ & = \frac{\|R \rho R^\dagger - P_{H(1)}\rho P_{H(1)}\|_1}{\text{Tr}(\rho P_{H(1)})} \\ & \leq 4\epsilon_P + 2\epsilon_P^2. \end{aligned} \quad (165)$$

Summing the contributions we conclude

$$\left\| \frac{R \rho R^\dagger}{\text{Tr}(R \rho R^\dagger)} - \frac{P_{H(1)}\rho P_{H(1)}}{\text{Tr}(P_{H(1)}\rho)} \right\|_1 \leq 8\epsilon_P + 4\epsilon_P^2. \quad (166)$$

From this, using Eq. (157) and contractivity of the trace norm under the partial trace:

$$\|\tilde{\sigma} - |y\rangle\langle y|\| \leq 8\epsilon_P + 4\epsilon_P^2, \quad (167)$$

where $\tilde{\sigma} = \text{Tr}_{1,2,3}[R \rho R^\dagger / \text{Tr}(R \rho R^\dagger)]$. ■

We set $8\epsilon_P + 4\epsilon_P^2 = \epsilon$, which gives

$$\epsilon_P = \sqrt{1 + \epsilon/4} - 1. \quad (168)$$

Using this for the filtering step with the gap $\Delta = 1/\kappa$, and $\alpha_{H(1)} = \alpha$, and adding to the expected adiabatic query count upper bound of $835.4\alpha\kappa$, gives a total expected count on success of

$$Q^* \leq 835.4\alpha\kappa + l_P = 835.4\alpha\kappa + \left\lceil \alpha\kappa \ln \frac{2}{\sqrt{1 + \epsilon/4} - 1} + 2 \right\rceil. \quad (169)$$

The success probability is,

$$\begin{aligned} p_{\text{succ}} &\geq \frac{1}{2} - 2(\sqrt{1 + \epsilon/4} - 1) \\ &\geq 1/2 - \epsilon/4 \end{aligned} \quad (170)$$

and so the total expected cost is

$$Q \leq \frac{2Q^*}{1 - \epsilon/2}. \quad (171)$$

Note in the expression for Q^* that if we set $\alpha = 1$, $\kappa = 10^6$ and $\epsilon = 10^{-10}$ we have an adiabatic cost of $\sim 8 \times 10^8$, whereas the cost of filtering is $\sim 1.4 \times 10^6$, so the former dominates. For $\kappa = 10^3$ these two costs become 835 331 and 1387, respectively.

III. SUMMARY AND OUTLOOK

We have developed a quantum linear solver algorithm whose complexity scales optimally in the condition number [as $O(\kappa)$] and in the error (as $O(\log 1/\epsilon)$). We have also provided an extensive analysis giving guaranteed upper bounds on the worst-case non asymptotic query counts.

Our algorithm is constructed combining a range of techniques. We substantially improve on the adiabatic computing-inspired algorithm given in Ref. [22], via modifications that are informed by eigenpath traversal theory [30,41], quantum eigenstate filtering [23,24] (for which we give an expanded analysis), and improved block-encoding construction. We also incorporate a Poissonization technique introduced in Ref. [29], which we optimize and for which we present a tighter analysis based on adiabatic approximation theory [43]. Finally, we introduce a randomized walk operator method into the quantum linear solver, inspired by works on eigenphase traversal via unitaries [30] and previous works where the need for Hamiltonian simulation was removed from quantum phase estimation [31]. This allows us to entirely forego the quantum Hamiltonian simulation subroutine and correspondingly the need for classical phase factor precomputations, which considerably simplifies the algorithmic implementation and compilation.

Our analysis returns, for a non-Hermitian linear system, a query count upper bound at $\epsilon = 10^{-10}$ of 1722κ . This is intermediate between the two other guaranteed worst-case non asymptotic query count upper bounds currently available, which have $234\,562\kappa$ [24] and 80κ [26], as reported in the latter work (see also Table I for a broader comparison of the algorithms). These numbers are important in setting a ceiling to the worst-case costs, but they do not define a ranking between the algorithms. We could complement these studies with numerical explorations, but these are necessarily restricted to low-dimensional problem instances and small condition number, e.g., $N \leq 16$, $\kappa \leq 50$ in Ref. [25].

In terms of further improvements to the algorithm, our choice of sampling the walk operator is certainly not optimal, even if we require perfect dephasing of the nullspace from other eigenspaces. It would be of interest, however, to drop the condition of perfect dephasing and allow additional single-step errors, since cancellations can occur for multiple steps, as shown in Ref. [39]. Leveraging these results could provide non trivial constant prefactor gains.

Another route is to improve query count bounds by exploiting structure about the problem. For example, further information about the distribution of singular values in the linear system matrix would give us a better handle on the eigenvalues of the associated $H(s)$; in turn, this would allow us to sharpen the Poissonization analysis. It would also be of value to explore how the linear-solver algorithm can be tailored to important sub-classes of problems, for example in the context of linear systems arising from discretization of partial differential equations. We leave these directions to future work.

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Authors are listed alphabetically. M.L. conceived the core algorithm, following discussions with Y.S. on variable-time amplitude amplification and adiabatic methods. M.L. optimized the randomized method, aided by analytical analysis from Y.S. M.L. and Y.S. introduced the randomized walk operator method. D.J. and M.L. developed the filtering component and the error propagation analysis. M.L., D.J., and Y.S. computed the analytical cost of the algorithm. M.L. and D.J. wrote the paper and A.S., S.P., Y.S. contributed to reviewing the article. A.S. and S.P. coordinated the collaboration.

DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

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