

Trade-off Between Complexity and Energy in Quantum Phase Estimation

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Quantifying the energetic cost of implementing quantum operations is essential for assessing the efficiency and scalability of quantum sensing and information-processing technologies. Here, we introduce a framework for analyzing the interplay between complexity and energy cost of quantum processes. In particular, we apply our framework to a sequential quantum phase estimation protocol, where a phase of physical significance is encoded in a quantum channel. The channel is applied to a probe state repeatedly until the probe is measured and the outcome leads to an estimate on the phase. We establish a trade-off relation between the total energy cost of the protocol and the number of times the channel is applied (complexity), while reaching a desired estimation precision. A sweet spot is located where the two quantities are co-optimized. The principles of our analysis can be adapted to benchmark the energetic requirements in other quantum protocols and devices.

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I. INTRODUCTION

Quantum protocols are known to outperform their classical counterparts for various tasks. This so-called *quantum advantage* relies on different characteristics available in the quantum realm, such as coherent superpositions [1] and entanglement of quantum states [2,3]. Identifying and characterizing the *resources* enabling quantum advantages is a major goal of quantum information science, often studied under the umbrella of quantum resource theory [4]. The advantage itself may manifest in a variety of forms and can be quantified by means of different figures of merit, depending on the task at hand. In particular, by quantum advantage in *complexity* we mean that, while reaching the same goal, the total number of elementary operations performed by a quantum protocol is smaller than that by a classical one. For example, in quantum computation [5,6], Grover's algorithm [7] achieves a quadratic speed-up for the searching problem, and, more remarkably, Shor's algorithm [8] is able to attain an exponential speed-up for integer factorization. Both are relevant to important computational fields such

as cryptography [9]. Exponential speed-up is also possible for Hamiltonian simulation [10,11]—following from the idea of simulating physical systems through quantum computers as originally raised by Feynman [12]. Meanwhile, in quantum metrology, there exists a quadratic speed-up to reach the desired estimation precision, termed the Heisenberg limit [13–15]. This can help improve performance in a wide range of applications, including gravitational sensing [16], biological imaging [17], and timing [18].

Despite their promises, the above quantum advantages typically only hold under a noise-free setting. In realistic experimental situations, the target system inevitably interacts with its surroundings, either the experimental device or the inaccessible environment, causing decoherence in its evolution [19]. The resulting noisy quantum operation deviates from the desired one and the *implementation error* generally leads to a larger complexity, sometimes even losing the quantum advantage: under commonly encountered noise models, Refs. [20–23] show that Grover's algorithm and Shor's algorithm, respectively, can only attain partial complexity advantage if the noise is weak enough, while Refs. [24–26] indicate that the metrological advantage is reduced to a constant factor unless specific structures of the noise are assumed.

In order to retain the ideal quantum complexity, various error mitigation techniques have been designed [24–31], and the resulting noisy intermediate-scale quantum technologies [32] represent the state-of-the-art progress made toward commercialization. Generally, to reduce the implementation error we can increase the power of the coupled

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device, or exert external controls to detect and correct the noise. Either way, complexity reduction comes with extra energy cost and a competition takes place between a smaller number of operations and larger energy cost per operation. In Refs. [33–35], the *total energy cost* of quantum protocols is treated as another important quantity that one wants to optimize for both scientific interest and near term realization. From this perspective, since the optimal complexity—corresponding to zero implementation error—can be expensive, some finite error may indeed be preferred for an energy optimization task.

Motivated by the quest to explore more concretely the aforementioned trade-off between the key resources of gate complexity and energy cost, we study the balance between the two in the context of *quantum metrology*. A typical metrological task consists of a pair $(\mathcal{G}_\phi, \text{Strat})$, where $\mathcal{G}_\phi$ is an operation that encodes the parameter ϕ to be estimated and *Strat* refers to strategies adopted to extract the parameter. The complexity here is represented by the number of times $\mathcal{G}_\phi$ is queried. A strategy consists of three main components: state preparation, intermediate controls (including error mitigation procedures), and measurements. References [36–39] have studied the cost of these components, while in most cases $\mathcal{G}_\phi$ is assumed to be the result of a free evolution and so does not incur additional cost. In this work, we will introduce scenarios where energy is required to initiate the evolution. The form of $\mathcal{G}_\phi$ then depends on its implementation energy, and the larger this cost, the smaller the number of iterations of $\mathcal{G}_\phi$ is required to reach a desired estimation precision, leading to a *complexity-energy trade-off relation*. The corresponding implementation error is quantified by a distance between $\mathcal{G}_\phi$ and its ideal form. The main finding of this work exposes a critical error level for complexity-energy co-optimization, beyond which the saving in one factor is overwhelmed by the overhead in the other. We also expect that the combination of our results and the established complementary ones can lead to a more complete energy benchmarking in quantum sensing and metrology protocols.

The overall flow of the paper is as follows: Sec. II introduces the key concepts behind the trade-off relation that are expected to apply qualitatively to all relevant protocols, Sec. III describes the basics of the quantum phase estimation protocol in metrology and its connection with the trade-off framework, while Sec. IV dives deeper into this connection by quantitatively analyzing a full sequential phase estimation protocol carried out on an optical platform and drawing general insights on the interplay between complexity and energy cost in quantum processes. The paper is concluded by a summary and outlook in Sec. V.

II. THE TRADE-OFF FRAMEWORK

Let us begin with a qualitative description of the origin of the trade-off. Suppose we have a fixed objective and a

protocol to reach it. The protocol takes energy to be implemented and involves a certain notion of complexity. Both can depend on various factors, with the implementation quality being the focus of this work. In general, the better the protocol is implemented, the objective can be achieved with a smaller complexity, but at the expense of a larger energy cost. The optimal quality level can be determined accordingly.

To formalize the notion of complexity clearly, we focus on quantum tasks with the following ideal structure:

$$\rho_0 \xrightarrow{U^N} \rho_N (\times Q_N). \quad (1)$$

In words, an input state ρ_0 undergoes a sequence of N identical unitary transformations U . Desired information accumulates in the final state ρ_N and is often extracted through classical postprocessing of measurement results. The whole sequence is repeated independently for Q_N rounds in order to gain a set amount of total information. This set goal can be represented by a fixed constraint, denoted as *Con*, whose exact form depends on the task. The total number of gates implemented is $Q_N \cdot N$ and optimizing over N with respect to the fixed goal gives the (gate) *complexity* C of the protocol:

$$C := \min_N (Q_N \cdot N) |_{\text{Con}}. \quad (2)$$

In terms of the energy cost, the implementation of each gate is assumed to be independent and identically distributed (i.i.d.); therefore, we can denote E as the *energy cost* for constructing each U -block. The *total energy cost*, denoted as R , is then

$$R = C \times E.$$

However, perfect implementation can be affected by both experimental and fundamental limitations. The actual implemented gates will not be unitary in general and an error dependence needs to be added to the quantities defined so far:

$$R(\epsilon) = C(\epsilon) \times E(\epsilon), \quad (3)$$

where ϵ is some error parameter quantifying deviation from the desired unitary gate. A larger error (i.e., lower implementation quality) tends to increase the complexity [20–26, 40–42], as intuitively less information is encoded in each implemented gate, while reducing the energy cost per gate [43, 44].

We are interested in scenarios where such a trade-off leads to an initially decreasing total energy cost $R(\epsilon)$ as a function of the error ϵ , as illustrated in Fig. 1. Note that if ϵ —and hence $E(\epsilon)$ —is fixed, the total energy cost becomes equivalent to the gate complexity as a resource quantifier. It is thus precisely through the dynamic

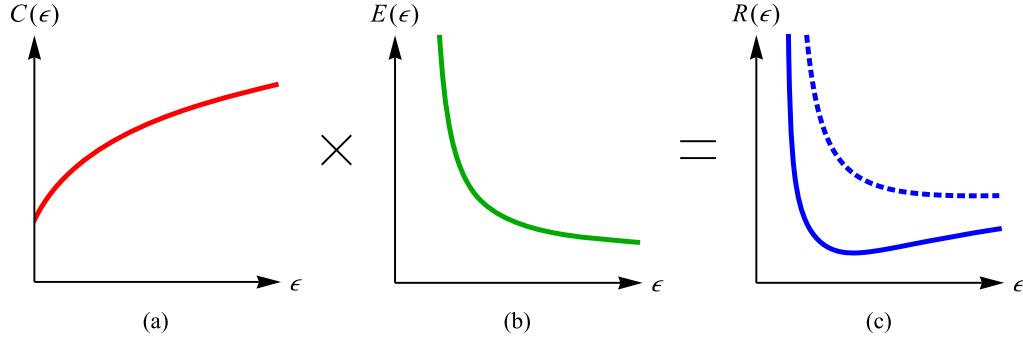


FIG. 1. A qualitative plot of the key variables characterizing a quantum protocol as in Eq. (3). Here ϵ is a generic quantifier for the implementation error of each elementary unitary gate, with $\epsilon \rightarrow 0$ corresponding to the ideal limit of perfect implementation. Competition takes place between (a) growing gate complexity $C(\epsilon)$ and (b) decreasing energy cost $E(\epsilon)$ per gate as ϵ increases. (c) The resulting total energy cost $R(\epsilon)$ may (solid curve) or may not (dashed curve) have a global minimum at finite ϵ .

interplay between the two, as mediated by a variable implementation error, that we can introduce a novel resource analysis on the quantum protocol at hand. In particular, minimizing the complexity is not equivalent to minimizing the total energy cost $R(\epsilon)$ with respect to ϵ . We also emphasize that the objective here is on optimizing the quantum protocol itself, rather than determining if a quantum over classical energetic advantage exists; for works on the latter, we refer to Refs. [45–47].

For a similar purpose to ours (see also the earlier analysis in Ref. [37]), Ref. [33] brings up the necessity to build a framework within which costs of quantum protocols can be analyzed and compared in a universal fashion. The proposed framework is dubbed metric-noise-resource (MNR) [34]: a metric is chosen to assess the performance, the effect of noise is taken into account, and the total resource cost is evaluated. A notion of efficiency is correspondingly defined as

$$\text{Efficiency } \eta = \frac{\text{Metric}}{\text{Resource}}. \quad (4)$$

For us, the constraint can be quantified by a suitably chosen metric, on which both resource cost (taken to be energy here) and complexity depend. Then for a fixed constraint, or equivalently a target value of the metric, we can minimize the total energy cost over ϵ to determine the optimal energy efficiency η . We shall return to this connection in Sec. IV D, after the energy analysis is carried out for a quantum phase estimation protocol performed on an optical platform, with the quantum Fisher information acting as the metric.

Finally, note that the i.i.d. assumptions underlying Eq. (1) may be relaxed. Quantum circuits used to perform quantum information processing tasks typically consist of unitary gates selected from a restricted set $\{U_1, \dots, U_n\}$ —for example, composition of single-qubit Pauli gates and CNOT gates leads to universal quantum computation [5,48], while in quantum simulation tasks one only has access to a

limited set of control Hamiltonians [10,49]. Each elementary gate can then be characterized by its own implementation error ϵ_i , gate complexity $C_i(\epsilon_i)$, and energy cost $E_i(\epsilon_i)$, and the quantities in Eq. (3) are extended to

$$C = \sum_{i=1}^n C_i(\epsilon_i); \quad R = \sum_{i=1}^n C_i(\epsilon_i) \times E_i(\epsilon_i). \quad (5)$$

Optimization on the two types of resource is now over a set of error parameters for each elementary gate. Furthermore, one can introduce implementation schemes where energy is shared or recycled across multiple gates, resulting in spatiotemporal correlations among them. There is no more simple product form for computing the total energy cost, and we may only define an effective mean energy cost of implementing each gate $\bar{E} = R/C$. These generalizations would cover a wider range of practical schemes; nevertheless, our simpler yet illustrative i.i.d. case will turn out to provide already valuable insight on the desired complexity-energy trade-off relation.

III. QUANTUM PHASE ESTIMATION

A. Quantum Fisher information and complexity

A quantum metrology task can often be recast in the form of quantum phase estimation (QPE). In general, a phase ϕ is encoded in some quantum operations which are applied to a probe state and by measuring the probe, we can obtain an estimate on the phase. The precision of the estimation can be quantified by the quantum Fisher information (QFI), which captures geometrically the rate of change of the probe state under an infinitesimal variation of the phase parameter [50]. To compute the QFI, we first introduce the classical Fisher information (CFI): for a positive operator-valued measurement (POVM) defined by the set of operators $\mathbf{M} = \{M_i\}$ and a probe state ρ_ϕ parametrized by the phase ϕ , the CFI is

$$F_c[\mathbf{M}](\phi) = \sum_i \text{Tr}(\rho_\phi M_i) l_i^2, \\ l_i = \partial_\phi \log \text{Tr}(\rho_\phi M_i) = \frac{\text{Tr}(\dot{\rho}_\phi M_i)}{\text{Tr}(\rho_\phi M_i)}, \quad (6)$$

where the dot denotes the derivative over ϕ .

The QFI is achieved by optimizing the CFI over all possible POVMs:

$$F_q(\phi) = \max_{\mathbf{M}} F_c[\mathbf{M}](\phi).$$

It turns out that the optimal measurements $\mathbf{M}$ are projectors onto the eigenspaces of the symmetric logarithmic derivative (SLD) operator Λ_ϕ , which satisfies the equation $\partial_\phi \rho_\phi = \frac{1}{2}(\Lambda_\phi \rho_\phi + \rho_\phi \Lambda_\phi)$. Suppose the same procedure is repeated independently for Q times, by the additive nature of the QFI for product states [51], the total QFI becomes $QF_q(\phi)$, and the quantum Cramér-Rao bound [52] states that

$$\text{Var}[\hat{\phi}] \geq \frac{1}{QF_q(\phi)}, \quad (7)$$

where $\hat{\phi}$ is an unbiased estimator of ϕ based on the Q measurement outcomes and $\text{Var}[\hat{\phi}]$ is its variance. Note that the bound is only achievable in the asymptotic limit of large $Q \gg 1$ by employing adaptive estimation procedures for determining the optimal measurement [53,54].

Within this framework we study the sequential strategy of the type described in Eq. (1) as inspired by Refs. [55–57]: the phase ϕ is ideally imprinted via a unitary phase shift operator $\exp(-i\phi H/2)$, where H is a control Hamiltonian. The operator functions as an oracle and each time we make a query it is repeatedly applied to the probe state, which will carry accumulating information about the phase. Finally the state is measured, yielding an estimate on the phase. If the oracle is applied N times, we denote the QFI of the final state by F_N . In the rest of this work we will treat $N \geq 1$ as a continuous variable for simplicity [57,58].

To determine the complexity, we quantify Con in Eq. (2) by demanding a target value, denoted as δ^2 , of the quantity on the right hand side of (7), so that δ^2 lower bounds the estimator variance. Naively, the number of independent repetitions of an N -step sequence needed to reach the set goal is

$$q_N = \frac{1}{\delta^2 \cdot F_N}, \quad (8)$$

and so the complexity can be derived as

$$c = \min_N (q_N \cdot N) |_{\delta^2} \\ = \frac{1}{\delta^2} \cdot \min_N \frac{N}{F_N} \\ = \frac{1}{\delta^2} \cdot \frac{N_{\text{opt}}}{F_{N_{\text{opt}}}}, \quad (9)$$

where N_{opt} is the optimal step for the minimization and $|_{\delta^2}$ is a shorthand for the constraint $|_{[QF_q(\phi)]^{-1} \leq \delta^2}$.

However, Eq. (9) must be taken with a pinch of salt, and c will be referred to as the *raw complexity*; the discrepancy arises due to the fact that q_N as calculated by Eq. (8) is not necessarily an integer and hence implicitly assumes F_N to be linear over N , while Q in (7) is an integer. In Appendix A, we elaborate on the general behavior of F_N and how its nonlinearity calls for a correction to c . To address this discrepancy, the *true complexity* of the QPE protocol will be determined as follows.

Suppose an N -step sequence is first repeated fully for $Q_N = \lfloor q_N \rfloor$ times, $\lfloor q_N \rfloor$ being the integer part of q_N , followed by a final sequence with $N_0 \leq N$ steps. By additivity of the QFI, the overall QFI from the $(Q_N + 1)$ sequences is $Q_N F_N + F_{N_0}$. Thus, to meet the constraint set by δ^2 , N_0 must be such that

$$F_{N_0} = \frac{1}{\delta^2} - Q_N F_N. \quad (10)$$

The true complexity is then given by

$$C = \min_N (NQ_N + N_0) |_{\delta^2}. \quad (11)$$

The minimization will be computed numerically and the minimal point is anticipated to be near $N \approx N_{\text{opt}}$. As seen shortly and qualitatively explained in Appendix A, the true complexity can be well approximated by the raw one when the implementation error is large enough.

B. Modeling the gate implementation error

To incorporate the implementation error, we assume that the (ideally unitary) quantum channels imprinting the phase in each round of the QPE protocol are in fact generated by a random Hamiltonian. For a qubit system, this amounts to taking $H_{\mathbf{n}} = \mathbf{n} \cdot \boldsymbol{\sigma}$, where $\mathbf{n}$ is a unit Bloch vector sampled from some probability distribution $p(\mathbf{n})$ and $\boldsymbol{\sigma}$ is the vector of Pauli operators. In the ideal limit, $p(\mathbf{n})$ is a Dirac delta function at $\mathbf{n}_0$, where $\mathbf{n}_0$ is the Bloch vector of the desired control Hamiltonian H . Both the error and the QFI, $F_N \equiv F_N(p; \phi)$, depend on the distribution. We will use a semicolon to separate free parameters that affect the implementation error. The resulting noisy gate (channel) is

$$\mathcal{G}_{p;\phi}(\cdot) = \int_{|\mathbf{n}|=1} e^{-i\phi \mathbf{n} \cdot \boldsymbol{\sigma}/2}(\cdot) e^{i\phi \mathbf{n} \cdot \boldsymbol{\sigma}/2} p(\mathbf{n}) d\mathbf{n}. \quad (12)$$

Correspondingly, the channel performs a random rotation on the Bloch sphere. Let $\mathbf{s}$ be the Bloch vector of the probe qubit. It is transformed by the implemented gate as

$$\mathbf{s} \rightarrow \mathbf{G}_{p;\phi} \mathbf{s}, \quad \mathbf{G}_{p;\phi} = \int_{|\mathbf{n}|=1} \mathbf{R}_{\mathbf{n}}(\phi) p(\mathbf{n}) d\mathbf{n}, \quad (13)$$

where $\mathbf{R}_{\mathbf{n}}(\phi)$ represents the rotation around $\mathbf{n}$ by an angle ϕ .

C. Computing complexity: An example

As an example of the model introduced in the previous two subsections, we first consider the following setup, studied by Ref. [56]. The initial state is set to be $\rho_0 = |\psi_0\rangle\langle\psi_0|$, $|\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and the desired Hamiltonian is $H = \sigma_z$. The random Bloch vector is sampled from the von Mises-Fisher distribution [59], $p_\kappa(\theta) = \frac{\kappa e^{\kappa \cos \theta}}{4\pi \sinh \kappa}$, where θ is the azimuthal angle, and κ is the concentration parameter. $H_{\mathbf{n}}$ becomes uniformly distributed as $\kappa \rightarrow 0$, and is sharply peaked at σ_z as $\kappa \rightarrow \infty$. This distribution can be seen as the counterpart of a Gaussian over the Bloch sphere and the implementation error can be represented by $\epsilon \sim 1/\kappa$. Furthermore, since $p_\kappa(\theta)$ has axial symmetry around z , the resulting

quantum channel is phase-covariant (commuting with σ_z) and the transformation matrix (13) can be expressed in the form of [60]

$$\mathbf{G}_{\kappa;\phi} = \begin{bmatrix} \lambda_\perp(\kappa;\phi) \cos \phi & -\lambda_\perp(\kappa;\phi) \sin \phi & 0 \\ \lambda_\perp(\kappa;\phi) \sin \phi & \lambda_\perp(\kappa;\phi) \cos \phi & 0 \\ 0 & 0 & \lambda_\parallel(\kappa;\phi) \end{bmatrix}. \quad (14)$$

Using the formula for $F_N(\kappa;\phi)$ derived in Appendix C of Ref. [56], we compute the complexity (11) and plot it in Fig. 2(b). Note that the step that maximizes the QFI is not $N_{\text{opt}}(\kappa;\phi)$ for the raw complexity (9): the quantity to be maximized here is F_N/N (plotted in Fig. 2(a)) rather than F_N . The latter is shown to be well approximated by

$$F_N(\kappa;\phi) \approx N^2 \lambda_\perp^{2N-2} [\lambda_\perp^2 + (\partial_\phi \lambda_\perp)^2], \quad (15)$$

so that F_N and F_N/N attain their maximum at around $-\log(\lambda_\perp)^{-1}$ and $N_{\text{opt}} = -[2 \log(\lambda_\perp)]^{-1}$, respectively.

With the *a priori* probability distribution comes the lack of knowledge on physical details on how the phase-encoding channel is achieved. Consequently, only an educated guess can be made on the energy cost of the implementation. In the next section, we restart with another more practical example and approach the QPE

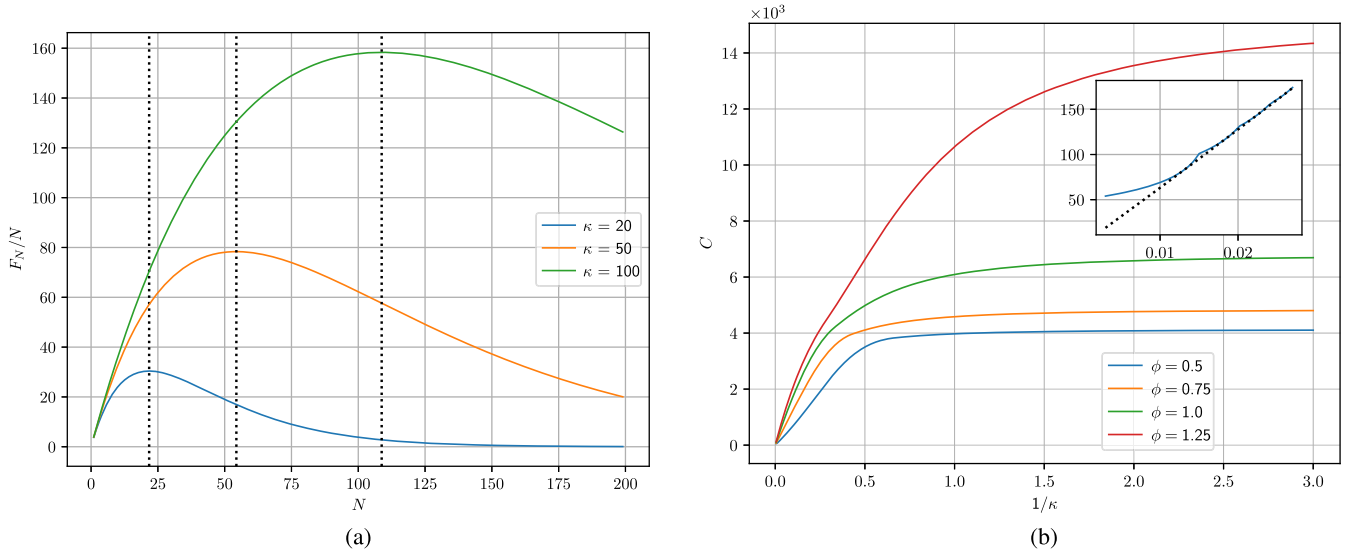


FIG. 2. (a) Plots of the ratio between the QFI and the steps number, F_N/N , for different values of the concentration parameter of the von Mises-Fisher distribution κ and a fixed phase $\phi = 0.5$ to be estimated. Observe the nonmonotonic behavior predicted by Eq. (15): the ratio grows linearly with N first, reaches its maximum and decays exponentially afterward. As indicated by the vertical dotted lines, the optimal step N_{opt} is well approximated by $-[2 \log(\lambda_\perp)]^{-1}$. (b) The resulting complexity (11) is plotted against $1/\kappa$ (representing the implementation error ϵ) for the desired lower bound on the estimator variance, $\delta^2 = 10^{-4}$, implied by Eq. (7). Observe the growing pattern as anticipated by Fig. 1(a). The inset shows an enlargement at small $1/\kappa$ for $\phi = 0.25$. Each tooth of the zigzag pattern corresponds to a region that applies the same number of complete sequences Q_N . The dotted line corresponds to the raw complexity computed by Eq. (9). Notice that the raw complexity vanishes in the ideal limit, and approximates the true complexity better as $1/\kappa$ grows, as discussed in Appendix A.

protocol from its physical foundation to realize a comprehensive resource analysis.

IV. CONSTRUCTION FROM FIRST PRINCIPLES

In this section, we adopt the sequential procedure from Sec. III for a physically motivated task based on light-matter interaction. A coherent laser pulse is used to drive a probe qubit, and the unknown field-qubit coupling strength g appears as the rotation angle of the induced unitary gate. Repeated applications of this gate sequentially encode the parameter g in the probe state, allowing the coupling strength to be estimated through the QPE protocol.

The microscopic construction enables both the implementation error and the associated energy cost to be derived from the same physical model, rather than postulated as in Sec. III C. For each application of the phase-encoding gate, the field is initialized from the vacuum in a coherent state, then allowed to interact with the probe qubit, and subsequently discarded. The field is therefore not reused between successive gates, so no field memory is retained [61], resulting in a Markovian process. Accordingly, the energy cost of each gate is the initialization energy required to prepare the corresponding laser pulse, evaluated as the increase in the field energy relative to its vacuum state. The effect of the implementation error on the complexity is also derived. Combining both factors gives us the total energy cost of the estimation protocol. For completeness, we also include the cost of state preparation and measurement that take place at the beginning and end of each sequence, respectively. Figure 7 at the end of this section exhibits the overall circuit structure, while Table I below makes clear the meaning of each parameter to be defined in the following subsections.

A. Energy bookkeeping

Before proceeding, we need to establish a consistent operational definition of energy cost, which will be applied throughout the remainder of the paper. Each building block

of the QPE protocol can be described by an initial quantum state ρ of the system under consideration, assumed to be provided at no cost, followed by a target unitary operation V , after which only energy-conserving operations O are allowed (including unitary evolution, discarding subsystems, and measurements). The full QPE circuit is constructed through concatenation of such building blocks: the output quantum state of one block is treated as the free input state for the next one.

We define the energy cost of a block as the change in the system energy induced by V ,

$$E = \text{Tr}[H_0 (V\rho V^\dagger - \rho)], \quad (16)$$

where H_0 denotes the system's Hamiltonian. This definition follows directly from energy conservation [44]: any change in the system energy must be supplied by an external energy storage device, referred to as a battery. Accordingly, E quantifies the energy exchanged with the battery per application of V .

Operationally, V can be realized by coupling the system to a battery state ρ_B via an energy-conserving joint unitary $\tilde{V}$ satisfying $[\tilde{V}, H_B \otimes I_0 + I_B \otimes H_0] = 0$ (where H_B denotes the battery's Hamiltonian), followed by decoupling [43,44,62–66]:

$$\text{Tr}_B[\tilde{V}(\rho_B \otimes \rho)\tilde{V}^\dagger] \approx V\rho V^\dagger. \quad (17)$$

Following [39,44], we implicitly assume access to a battery with sufficiently large energy storage to implement all V operations across the protocol [67].

We shall adopt the above as a bookkeeping convention throughout: different components of the protocol correspond to different choices of ρ and V , but the defining principle of E remains unchanged. Figure 3 illustrates this energy accounting, and Table II summarizes the corresponding objects for each building block of the QPE protocol. This convention isolates the fundamental implementation cost associated with non-energy-conserving operations, leaving aside additional thermodynamic or

TABLE I. Summary of parameters relevant to our model.

Variable parameters	
$\bar{m}$	Number of photons spent per gate
δ^2	Lower bound on the estimator variance as implied by (7)
M_s	Number of cooling qubits per sequence during state preparation
M_m	Number of cooling qubits per sequence during measurement
Fixed parameters	
g	Field-qubit coupling strength (the “phase” to be estimated)
T_0	Temperature of the free qubits used for cooling
ω_0	Transition frequency of the system qubit
ω_1	Transition frequency of the pointer qubit
ω	Frequency of the electromagnetic field

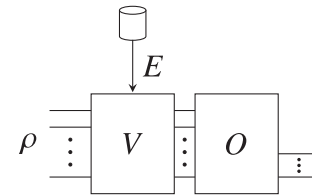


FIG. 3. An elementary building block used to quantify the energy cost: a free initial state ρ undergoes a non-energy-conserving unitary V , followed by energy-conserving operations collectively represented by O . The energy cost E is counted as the energy change of the state caused by V , supplied by a battery. The quantum output leg of O is treated as the initial state for the next elementary block.

TABLE II. Summary of the initial state ρ , the non-energy-conserving unitary operation V , and the subsequent energy-conserving operations O in each building block of the QPE protocol, as illustrated in Fig. 3. Here S, P label the system probe qubit and pointer qubit, respectively, ρ_{vac} is the vacuum state of the EM field, and ρ_β is the thermal state of the cooling auxiliary qubits, as introduced in the following subsections.

Building block	ρ	V	O
Implementation of n th gate	$\rho_{n-1}^{(S)} \otimes \rho_{\text{vac}}$	Displacement operator on the field	Field-qubit free evolution through the coupling (18) Discarding the postevolution field state
State preparation	ρ_β	Cooling unitary	Discarding the auxiliary qubits
Measurement	$\rho_N^{(S)} \otimes 0\rangle\langle 0 ^{(P)}$	CNOT between system and pointer qubits	Measurement on the pointer qubit Discarding the probe qubit

hardware overheads. These are briefly considered in Appendix C for the protocol at hand.

B. Gate implementation

Let the target system qubit (probe) be governed by the Hamiltonian $-\hbar\omega_0\sigma_z/2$, where ω_0 is the transition frequency of the two-level system. The qubit is coupled to a resonant monochromatic electromagnetic (EM) field of frequency ω ($\approx\omega_0$) traveling in the z direction. Suppose the rotating wave approximation [68] holds, $|\omega_0 - \omega| \ll \omega_0 + \omega$. In the interaction picture, the system is then governed by a time-independent Hamiltonian [69,70]:

$$H_S = \hbar g k_0 (E|1\rangle\langle 0| + E^*|0\rangle\langle 1|) = \hbar g k_0 E_0 (\mathbf{n} \cdot \boldsymbol{\sigma}), \quad (18)$$

where k_0 is a unit quantity (time^{-1}) to keep the coupling constant g dimensionless, $\mathbf{n} = [\cos\theta, \sin\theta, 0]^T$, and $E = E_0 e^{i\theta}$ is the amplitude of the field with phase θ . Physically, the parameter g reflects the magnitude of the qubit dipole moment. To estimate it, we set the initial state to be $\rho_0 = |0\rangle\langle 0|$ and aim to implement the unitary gate

$$U_g = e^{-ig\sigma_x/2}.$$

Classically, this is achieved by tuning $\theta = 0$ and $2k_0 E_0 t = 1$, t being the total evolution time. Quantum fluctuation of the field, however, sets a fundamental limit on the implementation quality: semiclassically, E_0 and θ can be treated as random variables with means $\bar{E}_0$ and 0, respectively, and the evolution time is $t = 1/(2k_0 \bar{E}_0)$. The gate implemented is then sampled from the set of unitary operators consisting of

$$e^{-iH_S t/\hbar} = \exp\left\{-ig\frac{E_0}{\bar{E}_0}\mathbf{n} \cdot \boldsymbol{\sigma}/2\right\} = \exp\left\{-ig\sqrt{\frac{\bar{m}}{m}}\mathbf{n} \cdot \boldsymbol{\sigma}/2\right\}, \quad (19)$$

where m is the (coherent) photon number of the EM field, treated as a random variable with mean $\bar{m}$. The ideal limit corresponds to $\bar{m} \rightarrow \infty$ when U_g is exactly applied to the qubit and the error may thus be characterized by $\epsilon \sim 1/\bar{m}$.

For small $\bar{m}$, quantum statistics becomes significant and we have to resort to a full quantum treatment [71,72], where random variables are further replaced with quantum operators.

Appendix B shows that in the semiclassical regime the QFI is well approximated by

$$F_N \equiv F_N(\bar{m}; g) \approx N^2 r^{2N},$$

$$r \approx 1 - \frac{\Delta(g)}{2\bar{m}}, \quad \Delta(g) = \frac{g^2 + 1 - \cos(g)}{4}, \quad (20)$$

provided $\bar{m} \gtrsim 100$, $g^2/\bar{m} \ll 1$, and the EM field is in its coherent state, representing lasers. Geometrically, the transformation on the yz plane of the Bloch sphere by the implemented channel (13) is reduced approximately to a rotation of angle g (corresponding to the desired unitary U_g) combined with a shrinking of factor r , matching the phase-covariant channel (14). The optimal step for the raw complexity (9) and the corresponding number of repetitions (8) are

$$N_{\text{opt}} \equiv N_{\text{opt}}(\bar{m}; g) = -\frac{1}{2\log(r)} \approx -\frac{1}{2\log(1 - \frac{\Delta(g)}{2\bar{m}})} \approx \frac{\bar{m}}{\Delta(g)};$$

$$q_{N_{\text{opt}}} \equiv q_{N_{\text{opt}}}(\bar{m}; g, \delta^2) = \frac{1}{\delta^2 \cdot F_{N_{\text{opt}}}}$$

$$\approx \frac{1}{\delta^2} \cdot \left(\frac{\Delta(g)}{\bar{m}}\right)^2 \cdot \left(1 - \frac{\Delta(g)}{2\bar{m}}\right)^{-\frac{2\bar{m}}{\Delta(g)}} \approx \frac{e}{\delta^2} \cdot \left(\frac{\Delta(g)}{\bar{m}}\right)^2. \quad (21)$$

The raw complexity can be computed as

$$c(\bar{m}; g, \delta^2) = q_{N_{\text{opt}}} \cdot N_{\text{opt}} \approx \frac{e}{\delta^2} \cdot \frac{\Delta(g)}{\bar{m}}. \quad (22)$$

This is a good approximation of the true complexity (11), $C(\bar{m}; g, \delta^2)$, for large enough $1/\bar{m}$, as observed in the inset of Fig. 4(a).

Moreover, we now have knowledge of the physical details of the implementation and hence its energy cost. To construct the n th phase-encoding channel in a sequence,

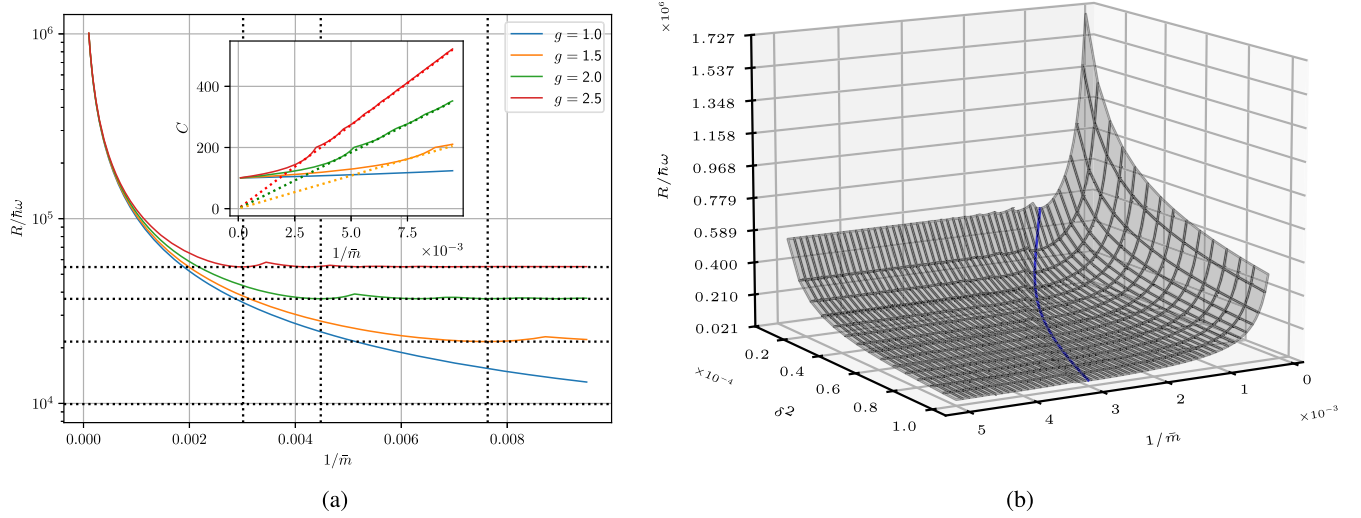


FIG. 4. (a) Plots of the total energy cost R (23), with the desired lower bound on the estimator variance set to $\delta^2 = 10^{-4}$, and the implementation error ϵ represented by $1/\bar{m}$, i.e., the inverse of the number of photons spent on implementing each phase-encoding gate. As reflected by the y-axis label, R is computed in units of the photonic energy of the EM field, $\hbar\omega$, and so numerically equals the total number of photons spent in implementing all the gates. The range of $\bar{m}$ is such that the condition after Eq. (20) is met for our chosen values of the unknown coupling g , so that the QFI is well approximated by its leading order term. The vertical and horizontal dotted lines locate the saturation point (24) where the energy plateau starts. This matches the behavior indicated by the dashed curve in Fig. 1(c). Solid and dotted lines in the inset are the corresponding true and raw complexities (22), respectively. As already observed in Fig. 2(b) and predicted in Appendix A, the latter approximates the former better as the error increases. (b) The total energy plot in (a) is repeated for various δ^2 and fixed $g = 2.5$. The blue solid curve depicts the evolution of the saturation point as both $\bar{m}$ and δ^2 vary.

we follow the established procedure in Fig. 3. As anticipated in Table II, the initial state ρ here is the tensor product of the probe state after the $(n-1)$ th step $\rho_{n-1}^{(S)}$ and the vacuum state ρ_{vac} of the EM field, the non-energy-conserving unitary V is the displacement operator, $\mathcal{D}(\sqrt{\bar{m}})$, on the field that creates the coherent state of mean photon number $\bar{m}$, and the ensuing energy-conserving operations O consist of the free evolution through the field-qubit coupling (18) followed by a complete discarding of the field state at the end of the evolution. The output probe state $\rho_n^{(S)}$ then acts as part of the free initial state for the $(n+1)$ th step, or as the premeasurement state $\rho_N^{(S)}$ after the final step. More precisely, the quantities entering in the energy-defining Eq. (16) are

$$\rho = \rho_{n-1}^{(S)} \otimes \rho_{\text{vac}}, \quad V = I_2 \otimes \mathcal{D}(\sqrt{\bar{m}}),$$

$$H_0 = -\hbar\omega_0 \frac{\sigma_z}{2} \otimes \mathcal{I}^{(F)} + \mathcal{I}^{(S)} \otimes \hbar\omega \hat{n},$$

where $\hat{n}$ is the number operator on the field, and $\mathcal{I}^{(S)}, \mathcal{I}^{(F)}$ are the identity operators on the system qubit and the field, respectively. Since V acts trivially on the probe, the probe contribution to the energy change vanishes, while the energy cost of the displacement operator, as computed by Eq. (16), is the energy difference between the vacuum and the coherent state, $E = \bar{m}\hbar\omega$, supplied by an external battery as in Fig. 3. For convenience, we will set the photonic energy to be unit, $\hbar\omega = 1$, and so $E(\bar{m}) = \bar{m} \sim 1/\epsilon$.

The total energy cost (3) is then

$$R \equiv R(\bar{m}; g, \delta^2) = C(\bar{m}; g, \delta^2) \cdot \bar{m}. \quad (23)$$

Note that here $E(\epsilon)$ diverges in the ideal limit of a perfect unitary implementation ($\epsilon \rightarrow 0$ or $\bar{m} \rightarrow \infty$) and so the total energy cost $R(\epsilon)$ is expected to decrease initially, matching the qualitative pattern illustrated in Fig. 1. We shall return to this point in Sec. IV D, where more general scenarios are considered.

We plot R in Fig. 4 within the approximation range. Starting from the ideal limit $1/\bar{m} \rightarrow 0$, the total energy cost goes through a sharp drop, then plateaus after a saturation point. Since the complexity increases with $1/\bar{m}$, this point may serve as the *sweet spot* for complexity-energy co-optimization, beyond which energy saving becomes inefficient with respect to complexity overhead. Reading from the graph, the plateau appears around when more than one repetition of the sequence takes place and the sweet spot can be located by setting $q_{N_{\text{opt}}} = 1$ in Eq. (21) and using Eqs. (22) and (23), which leads to

$$\bar{m}_0 \approx \Delta(g) \sqrt{\frac{e}{\delta^2}}, \quad C_0 \approx \sqrt{\frac{e}{\delta^2}}, \quad R_0 \approx \frac{e\Delta(g)}{\delta^2}. \quad (24)$$

From Eq. (22), the product $c \cdot \bar{m} \approx R_0$ does not depend on $\bar{m}$, explaining the flattening up to the leading order.

Two remarks are in order. First, the implementation error of the phase-encoding channel, as modeled in this

section, comes solely from the quantum fluctuation of the EM field. In particular, we treat the field parameters as random variables instead of quantum operators and only extract the leading order effect to allow an analytical derivation. In doing so, we have ignored other sources of noise, including laser inefficiency, decoherence caused by system-environment interaction, and higher-order effects from the field fluctuation, which become significant if the field power is reduced.

Second, a caveat in identifying the sweet spot concerns the attainability of the quantum Cramér-Rao bound (7). Recall that the bound is tight only in the asymptotic limit, requiring in principle many repetitions, whereas the energy-optimal regime identified here involves only a small number of repetitions. Our analysis therefore uses the QFI metric as a benchmark rather than a guarantee of achievable performance. Finite-sample effects may shift the precise location of the optimum, but do not alter the qualitative trade-off between implementation quality and repetition cost. As shown in Appendix D, operating at a smaller photon number per gate $\bar{m}$, in particular in the regime $10^2 \lesssim \bar{m} \lesssim 5 \times 10^2$, improves the consistency between the actual estimator performance and the asymptotic bound.

C. Other sources of cost

1. Complexity and energy minimization

Apart from gate implementation, other components of the protocol also require energy to implement and they may have an impact on the complexity as well. We will next consider a few of these contributions for a more complete energy bookkeeping (see also Appendix C), namely the tasks of state preparation and measurement at the beginning and end of each sequence, respectively. Before proceeding, we first describe qualitatively the difference brought by the additional cost.

The two key quantities involved in the trade-off relation as illustrated in Fig. 1 are the complexity C , or the minimal number of gates needed to reach a set objective (11), and the corresponding total energy cost R . For the latter, we now take into account the additional (or “external”) cost of each sequence other than the one spent on gate implementation, denoted as E_{ext} . Here it is treated as a constant for simplicity, while in general it may depend on parameters that can also affect the complexity, as we will see in Sec. IV C 2. The total energy cost as a function of the sequence steps number N is accordingly modified as

$$R_N \equiv R_N(\bar{m}; g, \delta^2, E_{\text{ext}}) = \begin{cases} \underbrace{(NQ_N + N_0)}_{\text{number of gates}} \times \bar{m} + \underbrace{(Q_N + 1)}_{\text{number of sequences}} \times E_{\text{ext}} & \text{if } N_0 > 0; \\ NQ_N \times \bar{m} + Q_N \times E_{\text{ext}} & \text{if } N_0 = 0. \end{cases} \quad (25)$$

We may denote N_C and N_R as the steps that minimize the total number of gates and the total energy cost, respectively, with the same constraint as in Eq. (11):

$$N_C \equiv N_C(\bar{m}; g, \delta^2) = \arg \min_N (NQ_N + N_0)|_{\delta^2},$$

$$N_R \equiv N_R(\bar{m}; g, \delta^2, E_{\text{ext}}) = \arg \min_N (R_N)|_{\delta^2}. \quad (26)$$

In Sec. IV B, $E_{\text{ext}} = 0$ and the two optimal steps coincide, $N_C = N_R$. However, with the additional energy cost they start to differ and the step optimization becomes dependent on the quantity to be minimized. By construction, optimizing for the total number of gates will result in a lower complexity at the expense of a higher total energy cost,

$$R_{N_C} \geq R_{N_R}, \quad (27)$$

and vice versa, with the equality attained if $E_{\text{ext}} = 0$.

2. State preparation

State preparation often aims to cool a thermal state from its initial temperature to a lower one. In our case, the ideal initial state of the system qubit is $|0\rangle\langle 0|$. However, the third law of thermodynamics implies that no process can reach zero temperature, corresponding to a pure state, with finite resources [73], and so the cooled state can only lie in the vicinity of $|0\rangle\langle 0|$. For the implementation, we adopt the technique of dynamic cooling [74,75]. Consider M_s identical qubits, with one target system and $M_s - 1$ auxiliary qubits. They are governed by the same Hamiltonian, $H_i = -\hbar\omega_0\sigma_z^{(i)}/2$, $i = 1, 2, \dots, M_s$. Their initial state is a product of thermal states with environmental temperature T_0 :

$$\rho_\beta = \bigotimes_{i=1}^{M_s} \left(\frac{e^{-\beta H_i}}{Z(\beta)} \right), \quad (28)$$

where $\beta = 1/k_B T_0$, k_B being Boltzmann's constant, and $Z(\beta)$ is the corresponding partition function. A unitary

operation V_{sp} then acts on all qubits and the auxiliary ones are discarded afterward, leaving the system qubit in a state with new temperature $T < T_0$. It is shown in Ref. [75] that in the low temperature regime, the minimal temperature achievable through cooling is

$$T \approx \frac{2T_0}{M_s}, \quad \text{if } k_B T_0 \ll \hbar\omega_0. \quad (29)$$

The energy cost of state preparation is again interpreted as in Sec. IV A. Here the initial state ρ_β is considered free of cost [76,77], as it is in thermal equilibrium with the environment. The cost W comes from the energy change of the qubits induced by the non-energy-conserving unitary V_{sp} , and the ensuing free operation consists of discarding the auxiliary qubits. Equation (16) thus reads

$$W = \text{Tr} \left[\left(\sum_i H_i \right) (V_{sp} \rho_\beta V_{sp}^\dagger - \rho_\beta) \right]. \quad (30)$$

According to the model of Fig. 3, W can be interpreted as the free energy drawn from an external battery [78] to implement the cooling unitary. Reference [75] shows that W in Eq. (30) is an extensive quantity and the energy cost per qubit in the thermodynamic limit is

$$\bar{w} = \lim_{M_s \rightarrow \infty} \frac{W}{M_s} = \frac{\omega_0}{2\omega} \frac{\tanh(\frac{1}{2\xi})}{e^{\frac{1}{\xi}} + 1}, \quad \xi = \frac{k_B T_0}{\hbar\omega_0}, \quad (31)$$

where the energy unit remains the photonic energy of the EM field. As seen shortly, in practice only a small number of cooling qubits are needed for energy optimization and so the thermodynamic limit does not apply. For small T_0 , we can nevertheless approximate $\bar{w}$ as the energy change of each qubit. The external cost per round for state preparation is then

$$E_{\text{ext}}^{(\text{preparation})} = \bar{w} \cdot M_s. \quad (32)$$

This quantity later turns out to be much smaller than the cost of gate implementation for state-of-the-art technology. In this subsection, we will thus ignore the distinction between complexity and energy minimization introduced in Sec. IV C 1.

After cooling, each sequence now starts with the initial state

$$\rho_0(T) = \frac{e^{-\beta H_i}}{Z(\beta)} = \frac{1 + \gamma(T)}{2} |0\rangle\langle 0| + \frac{1 - \gamma(T)}{2} |1\rangle\langle 1|, \quad (33)$$

where $\gamma(T) = \tanh(\frac{\hbar\omega_0}{2k_B T})$ characterizes the closeness to $|0\rangle\langle 0|$ as the Bloch vector of ρ_0 is $[0, 0, \gamma(T)]^\top$. Derivations in Appendix B imply that the leading correction to the QFI due to the nonideal initial state is a multiplicative factor of $[\gamma(T)]^2$. With Eq. (29) this yields a modification to Eqs. (20)–(22) as

$$\begin{aligned} F_N(\bar{m}; g) &\xrightarrow{\text{state preparation}} F_N(\bar{m}; g, M_s) \approx \left[\gamma\left(\frac{2T_0}{M_s}\right) \right]^2 \cdot F_N(\bar{m}; g); \\ N_{\text{opt}}(\bar{m}; g) &\xrightarrow{\text{state preparation}} N_{\text{opt}}(\bar{m}; g, M_s) \approx N_{\text{opt}}(\bar{m}; g); \\ q_{N_{\text{opt}}}(\bar{m}; g, \delta^2) &\xrightarrow{\text{state preparation}} q_{N_{\text{opt}}}(\bar{m}; g, \delta^2, M_s) \approx \left[\gamma\left(\frac{2T_0}{M_s}\right) \right]^{-2} q_{N_{\text{opt}}}(\bar{m}; g, \delta^2); \\ c(\bar{m}; g, \delta^2) &\xrightarrow{\text{state preparation}} c(\bar{m}; g, \delta^2, M_s) \approx \left[\gamma\left(\frac{2T_0}{M_s}\right) \right]^{-2} c(\bar{m}; g, \delta^2). \end{aligned} \quad (34)$$

The total energy cost is computed through Eq. (25), with N determined by the minimization in (11) and E_{ext} given by Eq. (32). For small $\bar{m}$, this can be approximated through the raw complexity: using Eq. (34) on Eq. (25) leads to,

$$R(\bar{m}; g, \delta^2, M_s) \approx \frac{e}{\delta^2} \cdot \frac{\Delta(g)^2}{\bar{m}^2} \cdot \frac{\bar{w} M_s + \frac{\bar{m}^2}{\Delta(g)}}{\left[\tanh\left(\frac{M_s}{4\xi}\right) \right]^2}. \quad (35)$$

A characteristic value of ξ for contemporary quantum technologies based on techniques such as superconducting and ion trap qubits is approximately 0.2 [79,80]; Eq. (31) then yields $\bar{w} \approx 0.003$, where we take advantage of the

resonance condition $\omega \approx \omega_0$. Within this range, Fig. 5 shows that the total energy cost quickly converges to a minimal value when more cooling qubits are consumed per sequence. From Eq. (35) we anticipate the total cost to eventually increase with M_s —complexity reduction brought by further cooling is overpowered by its energy cost. However, this does not manifest until M_s grows significantly large, since the relative magnitude between the cost of state preparation and gate implementation is as small as

$$\frac{\text{cost of states}}{\text{cost of gates}} \sim \frac{\bar{w} M_s \Delta(g)}{\bar{m}^2} \sim 10^{-7} M_s. \quad (36)$$

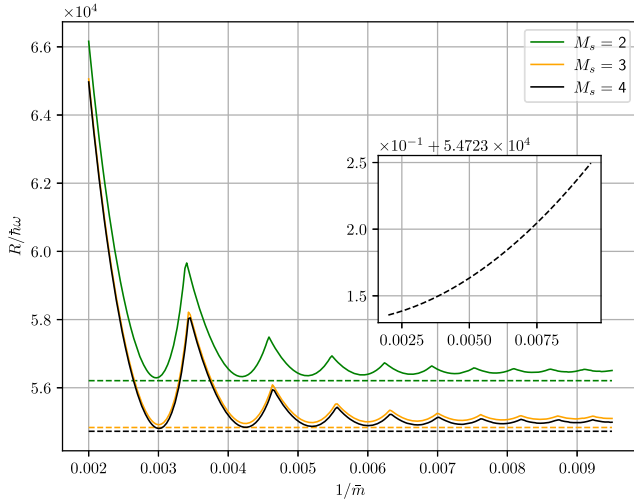


FIG. 5. Plots of the total energy cost (in units of $\hbar\omega$) of implementing the phase-encoding gate (see Fig. 4), now combined with that of state preparation for initializing the probe qubit, with $g = 2.5$, $\xi = 0.2$, $\delta^2 = 10^{-4}$ and the error represented by $1/\bar{m}$. The number of cooling qubits used per sequence, M_s , is also varied to show that a small amount of cooling suffices to approach the ideal limit. Solid lines are computed by Eq. (25), while dashed ones (35) approximate the cost through the raw complexity. Contrary to the plateau in Fig. 4, the total energy cost ends up increasing with larger implementation error as shown in the inset, since a larger error, or smaller $\bar{m}$, leads to more repetitions [see Eq. (21)] and hence a larger cooling cost. This corresponds to the behavior indicated by the solid curve in Fig. 1(c). However, the increment is negligible and only observable by the close-up in the inset as the cost of cooling is much smaller than that of gate implementation.

3. Measurement

To implement the measurement procedure, we adopt the pointer model [81,82]: the system qubit is coupled to a pointer qubit through a CNOT gate controlled by the system, the pointer is measured with respect to the optimal POVM that maximizes the CFI, reported in Eq. (E1), and the outcome should follow the statistics of the system state. The pointer thus acts as a measuring device. Let it be governed by the Hamiltonian $-\hbar\omega_1\sigma_z/2$, ω_1 being the transition frequency.

For perfect measurement, we want to initialize the pointer in $|0\rangle\langle 0|$. However, again, a pure state requires an infinite amount of resource to prepare, and so does an ideal projective measurement [83]. The resulting nonideal measurement affects the QPE performance [84,85]. In Appendix E, we evaluate the CFI from a mixed pointer state prepared through the dynamic cooling protocol. Despite the imperfection, as observed in Sec. IV C 2, in current setups the energy cost of cooling is negligible compared to that of the phase-encoding gate implementation for energy optimization. We will therefore assume the initial pointer state is $|0\rangle\langle 0|$ for free. For an N -step

sequence the system-pointer state prior to the correlating gate is then of the form $\rho_N^{(S)} \otimes |0\rangle\langle 0|^{(P)}$, where superscripts are used to label the two parts. This is taken as the free initial state in the bookkeeping model of Fig. 3 when formalizing the energy cost, while measurement on the pointer [86] and discarding of the system qubit amount to the free operations at the end. The cost in this case comes from the correlating CNOT gate: the resulting energy increase it causes on the pointer qubit (since the system qubit acts as the control, its energy is not changed) is given by

$$\begin{aligned} E_{\text{ext}}^{(\text{measurement})} &= \text{Tr} \left[\left(\mathcal{I}^{(S)} \otimes \frac{-\omega_1}{2\omega} \sigma_z^{(P)} \right) \right. \\ &\quad \cdot \left(\text{CNOT}(\rho_N^{(S)} \otimes |0\rangle\langle 0|^{(P)}) \text{CNOT}^\dagger \right. \\ &\quad \left. \left. - \rho_N^{(S)} \otimes |0\rangle\langle 0|^{(P)} \right) \right] \\ &= \langle 1 | \rho_N^{(S)} | 1 \rangle \cdot \frac{\omega_1}{\omega}, \end{aligned} \quad (37)$$

in units of the photonic energy. Here $E_{\text{ext}}^{(\text{measurement})}$ depends on the system state just before the CNOT gate is applied. For both simplicity and generality, we will instead use the loose yet constant upper bound on the correlating cost, namely the gap between the two energy levels of the pointer qubit:

$$E_{\text{ext}}^{(\text{measurement})} \cong \frac{\omega_1}{\omega}. \quad (38)$$

Both types of total energy cost in Eq. (27) are plotted in Fig. 6 for a range of $\frac{\omega_1}{\omega}$ chosen for illustrative purpose. Note that they are within the approximation range according to Eq. (36)—replacing ω_0 with ω_1 when computing $\bar{\omega}$ in Eq. (31)—to validate the omission of the state preparation (cooling) cost for the pointer.

The gap between the two costs emphasizes the importance of deciding the optimization priority between complexity and energy, a task closely related to practical constraints. Finally, recall Eq. (24) and its preceding arguments. Due to the additional cost each round, the sweet spot now becomes a turning point of the total energy cost as a function of the implementation error. This characteristic—reminiscent of the qualitative behavior displayed by the solid curve in Fig. 1(c)—further justifies the spot as a feasible candidate for complexity-energy co-optimization. Interestingly, such an optimal working point is uniquely identified regardless of which minimization is chosen for the number of steps, as the gap between R_{N_c} and R_{N_r} only begins to open up at larger error values.

D. Discussion

Figure 7 summarizes the QPE protocol constructed in this section. For a fixed number of photons $\bar{m}$ spent on each gate, we have determined the minimal number of

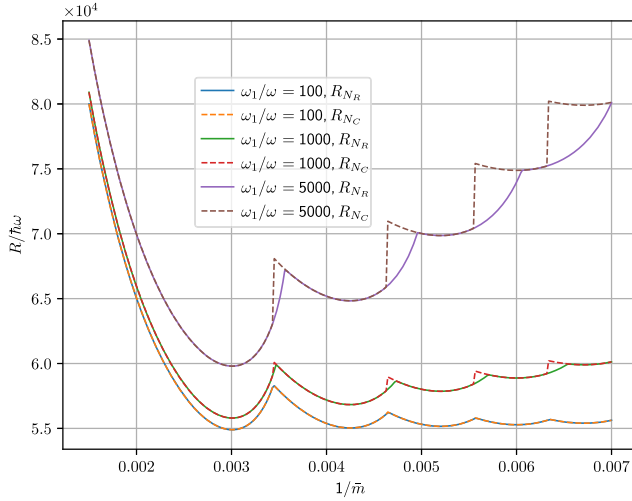


FIG. 6. Plots of the combined total energy cost of measurement and gate implementation (in units of $\hbar\omega$). The fixed parameters are $g = 2.5$, $\delta^2 = 10^{-4}$. Solid and dashed lines correspond to when the total energy cost and the number of gates are minimized, respectively, amounting to R_{N_R} and R_{N_C} ; see Eqs. (25)–(27), with the measurement cost E_{ext} given by Eq. (38), or ω_1/ω . The two costs coincide when they involve the same number of complete sequences, Q_N , since then their measurement costs, $E_{\text{ext}} \cdot (Q_N + 1)$, are the same and so minimizing the total energy cost and only the cost of gate implementation are equivalent. Otherwise, a gap opens up between R_{N_C} and R_{N_R} when the numbers of full sequences for each differ. This happens when the measurement cost of an extra sequence outweighs the cost of the gates it saves for complexity minimization. Observe that the larger E_{ext} and the smaller $\bar{m}$ are, respectively, the larger the gap grows between the two energy costs and the choice regarding which quantity to optimize becomes more important.

gates needed, or the complexity, and their spatial arrangement to reach a desired lower bound on the estimator variance. We have also analyzed the corresponding total energy cost and identified a sweet spot for complexity-energy co-optimization while varying $\bar{m}$.

As mentioned at the end of Sec. II, our energy analysis on the QPE protocol fits naturally into the MNR framework [33,34]. In Eq. (4), we can identify the *metric* as the total QFI of the sequential protocol, constrained by the target bound $1/\delta^2$, while the *resource* is the total energy cost R computed so far. The *noise* in our case comes from the quantum fluctuation of the EM field. Therefore, for each fixed δ^2 , maximizing the efficiency (4), $\eta = \frac{1}{\delta^2 R}$, is equivalent to minimizing the total energy cost.

When only the cost of the sequential gate implementation is factored in as in Sec. IV B, the minimal cost corresponds to the saturation point (24). In this case the efficiency becomes independent of the metric: $\eta = \frac{1}{\delta^2 R_0} = \frac{1}{e\Delta(g)}$. Once the other sources of cost—including but not limited to the ones analyzed in Sec. IV C—kick in,

the total energy cost increases and the metric dependence of the efficiency starts to show. Therefore, within the MNR framework the QPE protocol constructed in this section has an efficiency upper bounded by

$$\eta \leq \frac{1}{e\Delta(g)}. \quad (39)$$

From Eq. (20), $\Delta(g)$ is a monotonically increasing function of g and so, the larger the unknown parameter g , the less energy-efficient the QPE protocol is. The primary objective of our analysis is the joint optimization of $C(\epsilon)$ and $E(\epsilon)$ so as to minimize $R(\epsilon)$ at fixed metrological performance. Equation (39) recasts the resulting optimum within the MNR framework and quantifies the intrinsic efficiency of the quantum implementation considered here. More broadly, the MNR framework can incorporate additional platform-dependent contributions, such as the operation of control equipment and the effects of environmental noise, through the corresponding extensions of the total cost and performance metric. The bound in Eq. (39) therefore provides a fundamental benchmark against which the efficiency of more complete experimental implementations can be assessed.

The energy plateau observed in Fig. 4 motivates us to ask if the asymptotic scaling of the total energy cost with sufficiently large implementation error can be inferred from just a small set of parameters that are well defined for all physical realisations of the QPE protocol, so that qualitative conclusions can be drawn in a universal fashion. This can serve for both theoretical and engineering purposes in designing energy-optimal physical realisations of sequential QPE tasks. Here we undertake a preliminary investigation and determine a condition for the energy plateau to exist.

First, we state our general assumptions, which hold in particular for the optical implementation analyzed in this section:

1. We will stay in the semiclassical regime so that the implementation error ϵ can be treated as a small parameter;
2. Meanwhile, ϵ shall also be large enough such that the raw complexity well approximates the true complexity;
3. Recalling the qualitative form of the QFI in Eq. (A1),

$$F_N = N^2 e^{-f(N,\epsilon)},$$

the exponent f will be assumed to grow no more than polynomially as a function of the number of steps N ;

4. The implementation energy of each phase-encoding gate will be assumed to follow an inverse power law, so that $E(\epsilon) \sim \epsilon^{-\tau}$ for some $\tau > 0$.

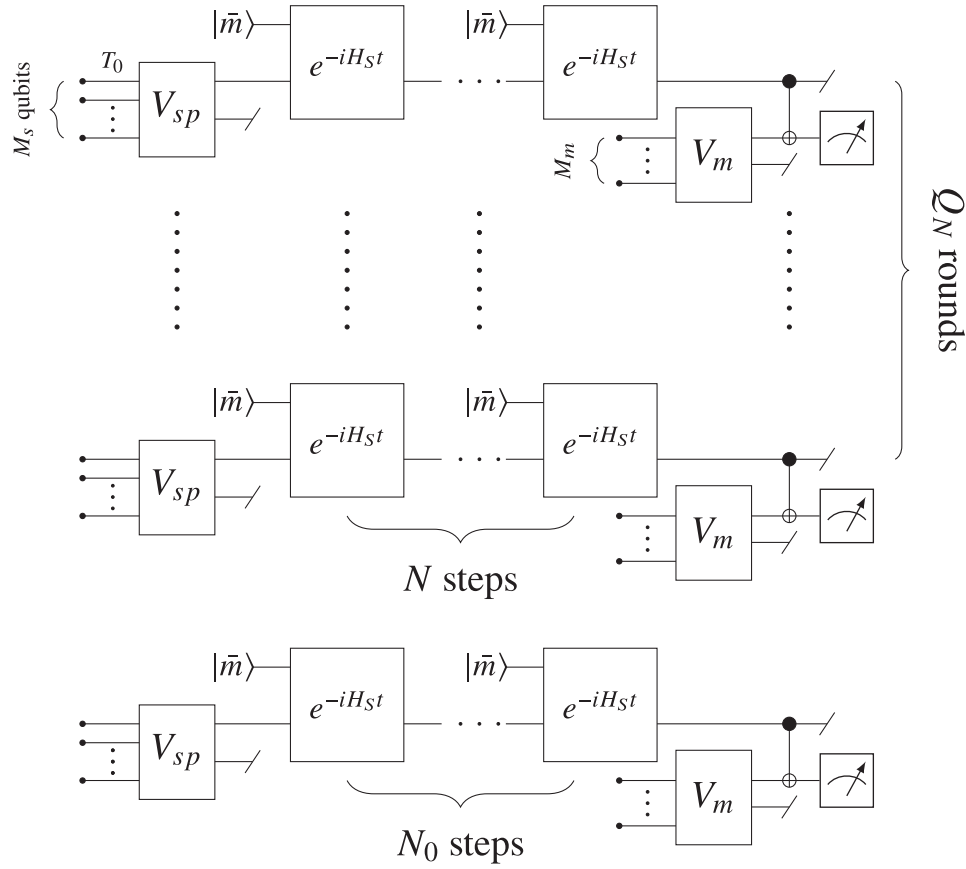


FIG. 7. Full circuit diagram of the QPE protocol for which a trade-off between complexity and energy is established in this work.

In the ideal limit of vanishing implementation error, $\epsilon = 0$, we reach the Heisenberg limit of a sequential QPE strategy [87], $F_N \propto N^2$. Without loss of generality, set $f(N, 0) = 0$ so that the constant of proportionality is 1. Then, f admits a power series expansion with respect to $\epsilon \rightarrow 0$ with some leading order $\nu > 0$:

$$f(N, \epsilon) \approx g(N)\epsilon^\nu,$$

for some polynomially-bounded function $g(N)$. The optimal step N_{opt} for the raw complexity (9) is determined by minimizing the quantity $Nf(N, \epsilon)$, leading to

$$N_{\text{opt}} \frac{\partial f(N_{\text{opt}}, \epsilon)}{\partial N} = 1 \longrightarrow N_{\text{opt}} g'(N_{\text{opt}}) \approx \epsilon^{-\nu}.$$

Let $g(N)$ admit a power series expansion with leading order $q > 0$, so that $g(N) \approx c_q N^q$ as $N \rightarrow \infty$. The optimal step and the corresponding raw complexity can then be expressed as

$$N_{\text{opt}} \approx (c_q q \epsilon^\nu)^{-1/q},$$

$$c(\epsilon; \delta^2) = \frac{1}{\delta^2 N_{\text{opt}} e^{-f(N_{\text{opt}}, \epsilon)}} \approx \delta^{-2} (e c_q q \epsilon^\nu)^{1/q}.$$

Within the regime of validity of our assumptions, the total energy cost (3) then behaves as

$$R(\epsilon; \delta^2) = E(\epsilon) \times c(\epsilon; \delta^2) \sim \mathcal{O}\left(\epsilon^{\frac{k}{q} - \tau}\right). \quad (40)$$

Observe that $\nu = q\tau$ and $\nu > q\tau$ correspond to the dashed and solid curve in Fig. 1, respectively, where the total energy cost either eventually plateaus or exhibits a global minimum; on the other hand, $\nu < q\tau$ indicates that the total energy cost vanishes asymptotically, which is incompatible with the stated assumptions.

The example in Sec. IV B has $\epsilon = 1/\bar{m}$, and so, according to Eq. (20), $f(N, \epsilon) = -2N \log(1 - \Delta(g)\epsilon/2)$ with $\nu = 1$, $q = 1$ and $c_q = \Delta(g)$. The energy cost per gate is proportional to the photon number, $\tau = 1$. Thus $q\tau = 1 = \nu$ yields the energy plateau as expected. Including the additional costs associated with state preparation and measurement effectively reduces τ : this breaks the plateau and pushes the total energy (40) in the regime $\nu > q\tau$, leading to the appearance of the turning point as identified in Sec. IV C.

Finally, note that assumption 4 captures the physically relevant situation in which increasingly accurate gates require increasingly energetic control fields. Consequently, the implementation cost diverges as $\epsilon \rightarrow 0$, ruling

out the ideal limit as the energy-optimal operating point and allowing a sweet spot to occur at finite error, as illustrated in Fig. 1. This behavior is not specific to QPE: inverse-power-law relations between control energy and implementation error arise broadly when qubits are driven by EM fields [43,88], including in optical [89,90] and trapped-ion [91,92] platforms.

This interpretation is also consistent with the energy accounting framework introduced in Sec. IV A. The external source prepares the control field, which then supplies the energy required to implement the desired qubit transformation. When the target unitary does not conserve the qubit energy, approaching an ideal implementation generally requires an increasing energetic investment in the control operation [44,67]. Combining this error-energy relation with the corresponding error dependence of the protocol complexity, as in Eq. (5), leads to the same form of complexity-energy co-optimization for a broader class of quantum information-processing tasks.

V. CONCLUDING REMARKS AND OUTLOOK

In this work, we have established a framework for analyzing and quantifying the trade-off between gate complexity and energy cost in quantum processes composed of elementary building blocks. We have applied our framework to the case study of a sequential quantum phase estimation protocol and identified physical conditions for the joint optimization of complexity and energy, given a target metric benchmarking the desired estimation precision. Our analysis rests on a consistent characterization of energy cost across the different components of the protocol, encompassing input probe-state preparation, phase-encoding gate implementation, and output measurement.

Gravitational-wave detection provides a prominent example of quantum-enhanced optical metrology in which energetic and temporal resources must be carefully balanced [93]. Laser interferometers detect the minute relative displacements of suspended mirrors induced by passing gravitational waves, using intense optical fields whose sensitivity is ultimately constrained by quantum noise, including photon shot noise [94–96]. In such settings, increasing the optical energy can improve precision but also raises the energetic cost of operating the detector. Our framework provides a natural starting point for analyzing this type of trade-off between estimation performance, interrogation time, and optical energy expenditure.

As mentioned in the Introduction, all components other than the phase-encoding channel (12) constitute an estimation strategy for the metrology task. A strategy that obeys causal order can be represented by a quantum comb [97–102]. Quantum combs encompass structures more general than the protocol considered here, including temporal and spatial correlations, entanglement among

probe qubits, memory effects across successive gates, and correlated decoherence noise [103]. A particularly important alternative is the parallel strategy, in which several probes interact with the parameter-encoding process simultaneously, employing entanglement among the probes to improve robustness and estimation performance [104,105]. These advantages must be balanced against the additional hardware required to prepare, control, and preserve multipartite entanglement [106,107]. Furthermore, causally indefinite strategies, such as the quantum switch [108], have the potential to surpass conventional precision bounds [109–111]. The energetic cost of quantum-comb strategies for metrology has received comparatively less attention, with Ref. [39] providing one recent development; see also Ref. [112] for a complementary study of work extraction from quantum combs. Combining these results with the framework developed here may help determine the complexity-energy trade-off for both components of a quantum metrology task—the estimation strategy and the parameter-encoding process—and thereby enable a more complete co-optimization.

It is also worth remarking that quantum resources can take forms other than energy. For the QPE protocol, entanglement among the probes can provide the quantum enhancement underlying parallel estimation strategies [87], while Ref. [113] investigates how the quality of phase estimation can be characterized in terms of quantum coherence. A more general framework should therefore be capable of accounting consistently for different types of resource cost. This may be achieved by constructing conversion schemes between resources [114], which could, for example, relate energy expenditure to the generation of entanglement [115–117] or coherence [118,119], or by introducing task-dependent hybrid cost functions.

Finally, the building-block principle developed in this work may be applied to other quantum tasks with structures analogous to Eq. (1). Whenever a protocol consists of repeated elementary units, reducing the energy invested in each unit will generally increase its implementation error and, consequently, the number of units required to achieve a prescribed performance. In Sec. IV D, we obtained a qualitative understanding of this trade-off for QPE without relying on the details of the specific physical implementation. The essential ingredient for extending the analysis is a metric $\mathcal{M}(N, \epsilon)$ that depends on the number of elementary units N and their implementation error ϵ , and that quantitatively assesses the performance of the protocol. For the examples discussed in Sec. I, this metric may be the QFI in quantum metrology as adopted in this work, the overlap between the final and target states in Grover’s algorithm [7], or a suitable norm distance between the implemented and desired evolutions in Hamiltonian simulation [10]. The reasoning developed here can therefore be adapted to a broad range of quantum-information-processing tasks.

A comprehensive analysis on practical complexity-energy trade-off relations for existing and upcoming quantum technology primitives will be pivotal for their sustainable development and widespread impact.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: RAW VERSUS TRUE COMPLEXITY

In Sec. III A, we introduce the QFI as a metric quantifying the estimation precision of the QPE protocol. In general, in noisy quantum metrology [24,120] the QFI can be empirically expressed as

$$F_N = N^2 e^{-f(N,\epsilon)} \quad (\text{A1})$$

for some function $f > 0$ that grows with both N and ϵ , characterizing the information-destroying effect of the error. In the ideal limit, $\epsilon = 0$, we have $f(N, 0) \sim 1$ and we attain the Heisenberg limit for the sequential QPE protocol, $F_N \propto N^2$ [57,87]. However, in the presence of noise, this scaling often cannot be realized in practice. In particular, for $f \sim \log(N)$ we reach the standard quantum limit, $F_N \sim N$, while, for even stronger implementation error, the QFI eventually decays as N grows, and adding more steps becomes detrimental for the estimation precision. The examples we study in this work correspond to the latter case; see Figs. 2(a) and 9.

With the general qualitative form (A1), the difference between the raw and the true complexity becomes clearer to see. For N -step sequences, the raw complexity (9) is computed assuming the last partial sequence consists of $N \cdot \{q_N\}$ steps, where $\{q_N\}$ is the fractional part of q_N , while for the true complexity (11), the last sequence contains N_0 steps that satisfy the condition (10). In general, $F_{N\{q_N\}} \neq F_{N_0}$ unless F_N is linear over N , or when q_N is an integer (and so $N_0 = \{q_N\} = 0$). More precisely, we have

$$\frac{N\{q_N\}}{N} = \frac{F_{N_0}}{F_N} = \frac{N_0^2 e^{-f(N_0,\epsilon)}}{N^2 e^{-f(N,\epsilon)}} > \frac{N_0^2}{N^2},$$

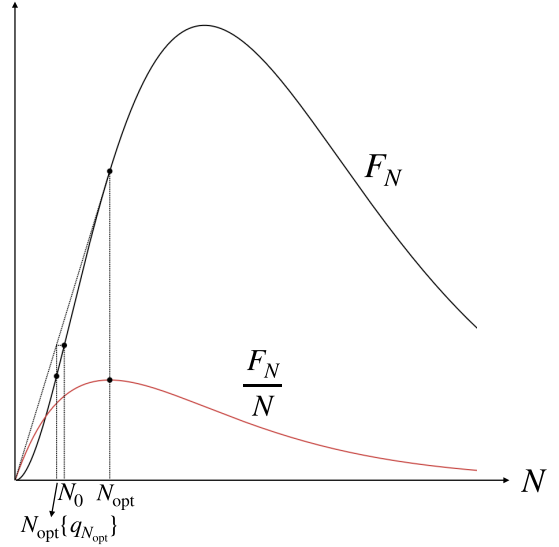


FIG. 8. Typical graph of the QFI F_N and its ratio with N for an asymptotically decaying QFI with respect to the number of steps N in one sequence. Initially, F_N increases quadratically while F_N/N increases linearly. Note that they attain their maximum at different values of N ; in particular, N_{opt} maximizes F_N/N and leads to the raw complexity in Eq. (9). In the computation there, $N_{\text{opt}}\{q_{N_{\text{opt}}}\} < N_{\text{opt}}$ is taken to be the number of steps in the last sequence, while N_0 given by Eq. (10) is instead the true number of steps needed to reach the desired QFI. The discrepancy between the two is caused by the nonlinearity of F_N and is expected to decrease as the implementation error grows.

and a rough upper bound on the difference between the actual length of the final sequence needed to reach the desired total QFI and the one as computed by the raw complexity is $N_0 - N\{q_N\} < N_0 - N_0^2/N \leq N/4$. These quantities are demonstrated in Fig. 8 for a typical F_N with an eventual decay. Intuitively, as ϵ increases, the decaying of both F_N and F_N/N becomes faster and the turning point of the latter, N_{opt} , decreases. As the difference between the raw and the true complexity is upper bounded by roughly $N_{\text{opt}}/4$, and both supposedly increase with increasing error, we anticipate the true complexity to be well approximated by the raw complexity for large enough implementation error ϵ . This is confirmed by the observation made in Figs. 2(b) and 4.

APPENDIX B: COMPUTATION OF THE QUANTUM FISHER INFORMATION F_N

To evaluate the QFI resulting from the implementation scheme in Sec. IV B, we first compute the action of the imperfect quantum channel. Recall Eqs. (12) and (13). Write a general qubit state in its Bloch representation, $\rho = \frac{1}{2}(\mathbf{I}_2 + \mathbf{s} \cdot \boldsymbol{\sigma})$, $\mathbf{s}$ being the Bloch vector. From Eq. (19), the action of the implemented quantum channel on the state can be expressed as

$$\mathcal{G}_{\bar{m};g}(\rho) = \frac{\mathbf{I}_2}{2} + \frac{1}{2} \underbrace{\left(\int_0^{2\pi} \int_0^\infty \mathbf{R}_{\mathbf{n}} \left(g \sqrt{\frac{\bar{m}}{\bar{m}}} \right) p(\theta) q(m) dm d\theta \right)}_{:= \mathbf{G}_{\bar{m};g}} \mathbf{s} \cdot \boldsymbol{\sigma},$$

where $\mathbf{n} = [\cos \theta, \sin \theta, 0]^\top$ and $p(\theta)$, $q(m)$ are the probability distributions followed by the phase and the photon number, respectively. The rotation matrix can be found through Rodrigues' formula: in general,

$$\mathbf{R}_{\mathbf{n}}(\alpha) = \mathbf{I}_3 + (\sin \alpha) \mathbf{N} + (1 - \cos \alpha) \mathbf{N}^2, \quad \mathbf{N} = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}.$$

To assist with analytical derivation, we assume that the average photon number $\bar{m}$ is large enough ($\gtrsim 100$) such that $p(\theta)$ and $q(m)$ are well approximated by normal distributions:

$$q(m) = \frac{1}{\sqrt{2\pi\sigma_m^2}} e^{-\frac{1}{2}\left(\frac{m-\bar{m}}{\sigma_m}\right)^2}, \quad p(\theta) = \frac{1}{\sqrt{2\pi\sigma_\theta^2}} e^{-\frac{1}{2}\left(\frac{\theta}{\sigma_\theta}\right)^2},$$

σ_m and σ_θ being the corresponding variances. With these the integral can be computed to be

$$\mathbf{G}_{\bar{m};g} = \mathbf{I}_3 + \eta^{\frac{1}{2}} B \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + (1-A) \begin{bmatrix} -\frac{1-\eta}{2} & 0 & 0 \\ 0 & -\frac{1+\eta}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \eta = e^{-2\sigma_\theta^2}, \quad (\text{B1})$$

where

$$A = \frac{\bar{m}}{\sqrt{2\pi\sigma_m^2}} \int_{-1}^\infty \cos(g\sqrt{1+t}) e^{-\frac{1}{2}\left(\frac{\bar{m}}{\sigma_m}\right)^2 t^2} dt, \quad B = \frac{\bar{m}}{\sqrt{2\pi\sigma_m^2}} \int_{-1}^\infty \sin(g\sqrt{1+t}) e^{-\frac{1}{2}\left(\frac{\bar{m}}{\sigma_m}\right)^2 t^2} dt.$$

In our case, the Bloch vector of the initial state, $|0\rangle\langle 0|$, is $\mathbf{s}_0 = [0, 0, 1]^\top$. Since $\mathbf{G}_{\bar{m};g}$ acts irreducibly on the yz plane, we may restrict dynamics to this subspace. The restricted Bloch vector after each step evolves as

$$\begin{aligned} \mathbf{s}_N &= (\mathbf{G}_{\bar{m};g})^N \mathbf{s}_0 \\ &= \begin{bmatrix} 1 - \frac{1}{2}(1-A)(1+\eta) & -\eta^{\frac{1}{2}} B \\ \eta^{\frac{1}{2}} B & A \end{bmatrix}^N \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{r^{N-1}}{\sin \alpha} \begin{bmatrix} -\eta^{\frac{1}{2}} B \sin(N\alpha) \\ A \sin(N\alpha) - r \sin((N-1)\alpha) \end{bmatrix}, \end{aligned} \quad (\text{B2})$$

where the two eigenvalues of $\mathbf{G}_{\bar{m};g}$ are expressed as

$$r e^{\pm i\alpha} = \frac{1}{4} \left(1 - \eta + A(\eta + 3) \pm i \sqrt{16\eta^{\frac{1}{2}} B^2 - (1-A)^2(1-\eta)^2} \right).$$

The state after the N th step is $\rho_N = \frac{1}{2}(\mathbf{I}_2 + \mathbf{s}_N \cdot \boldsymbol{\sigma})$. From its Bloch representation, the QFI after the N th step with respect to the parameter g can be readily calculated (see, for example, Ref. [121]):

$$F_N = \begin{cases} |\partial_g \mathbf{s}_N|^2 + \frac{(\mathbf{s}_N \cdot \partial_g \mathbf{s}_N)^2}{1 - |\mathbf{s}_N|^2}, & |\mathbf{s}_N| < 1; \\ |\partial_g \mathbf{s}_N|^2, & |\mathbf{s}_N| = 1. \end{cases} \quad (\text{B3})$$

To continue the computation, we assume further that the light field has photon statistics corresponding to either a Poisson (coherent) or sub-Poisson distribution, such that $\sigma_m = k_m \sqrt{\bar{m}}$, $\sigma_\theta = \frac{k_\theta}{2\sqrt{\bar{m}}}$ with $k_m \sim \mathcal{O}(1)$. Then, for $\bar{m} \gg g^2$, various terms may be approximated as

$$A \approx \sqrt{\frac{\bar{m}}{2\pi k_m^2}} \int_{-\infty}^{\infty} \cos\left(g\left(1 + \frac{t}{2}\right)\right) e^{-\frac{\bar{m}t^2}{2k_m^2}} dt = e^{-\frac{(gk_m)^2}{8\bar{m}}} \cos(g) \approx \left(1 - \frac{(gk_m)^2}{8\bar{m}}\right) \cos(g);$$

$$B \approx \left(1 - \frac{(gk_m)^2}{8\bar{m}}\right) \sin(g); \quad \eta \approx 1 - \frac{k_\theta^2}{2\bar{m}},$$

leading to

$$\alpha \approx g, r \approx 1 - \frac{\Delta(g)}{2\bar{m}}, \quad \Delta(g) = \frac{k_m^2 g^2 + k_\theta^2 (1 - \cos(g))}{4}.$$

Similarly, the Bloch vector (B2) and its derivative are

$$\mathbf{s}_N = r^N \left(\begin{bmatrix} 0 - \sin(Ng) \\ \cos(Ng) \end{bmatrix} + \mathcal{O}\left(\frac{g^2}{\bar{m}}\right) \right), \quad \partial_g \mathbf{s}_N = r^N \left(-N \begin{bmatrix} 0 \\ \cos(Ng) \\ \sin(Ng) \end{bmatrix} + \mathcal{O}\left(\frac{g}{\bar{m}}\right) \right). \quad (\text{B4})$$

The leading order effect of $\mathbf{G}_{\bar{m};g}$ on the yz plane is a rotation of angle g combined with a shrinking of factor r . This coincides with the phase-covariant channel (14) by identifying r with $\lambda_\perp$ (and replacing the Z basis with X basis). Equation (15) may thus be adopted to estimate the QFI, leading to

$$F_N(\bar{m}; g) = N^2 r^{2N} \left(1 + \mathcal{O}\left(\frac{g}{\bar{m}}\right) \right) \approx N^2 \left(1 - \frac{\Delta(g)}{2\bar{m}} \right)^{2N}. \quad (\text{B5})$$

The merit of the approximation is confirmed by Fig. 9 for coherent states.

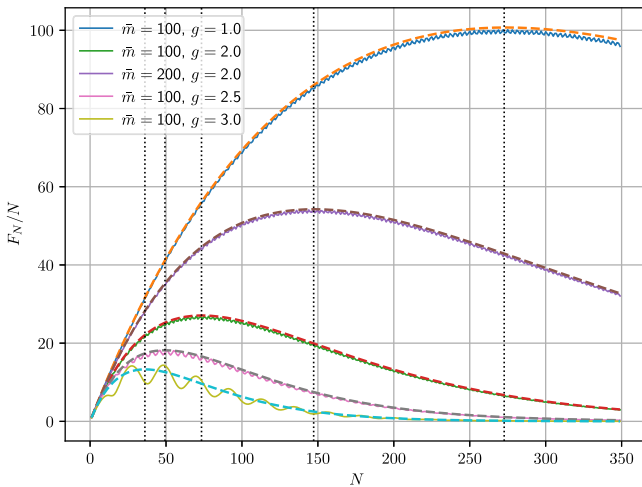


FIG. 9. Plots of F_N/N for various $\bar{m}$ and g , when the field is in coherent states. Solid lines are the exact results using Eqs. (B2) and (B3) and the dashed ones are the approximated ones (B5). The vertical dotted lines approximate the optimal step $N_{\text{opt}} = -[2 \log(r)]^{-1}$. As expected, the percentage error becomes more negligible as g decreases and $\bar{m}$ increases.

Finally, the number-phase uncertainty relation states that $\sigma_m \sigma_\theta \geq \frac{1}{2}$ and so $k_m k_\theta \geq 1$ [122]. This implies the lower bound,

$$\Delta(g) \geq \frac{k_m k_\theta g \sqrt{1 - \cos(g)}}{2} \geq \frac{g \sqrt{1 - \cos(g)}}{2}, \quad (\text{B6})$$

with equality attained at $k_m = g^{-1/2} (1 - \cos(g))^{1/4} = k_\theta^{-1}$. A squeezed coherent state with real squeezing parameter s has $k_m = e^{-s}$, $k_\theta = e^s$ and so the QFI is maximized at $s = \frac{1}{2} \log\left(\frac{g}{\sqrt{1 - \cos(g)}}\right)$; notice this optimal squeezing level is dependent on the unknown parameter g . That said, in the main text we will set the control field to be in coherent states ($s = 0$ and $k_m = k_\theta = 1$ [123]) due to their near-classical properties and easier experimental preparation.

APPENDIX C: FURTHER REMARKS ON ENERGY COST

1. Cost of discarding subsystems

In Sec. IV A, we set the energy cost E to originate from energy conservation, and for each gate applied to the system qubit in the QPE protocol this cost amounts to $\bar{m} \hbar \omega \approx \bar{m} \hbar \omega_0$. Meanwhile, Landauer's principle [124,125] states that the minimal energy cost of erasing information from one qubit is on the scale of $k_B T_0$, with T_0 being the system temperature. The relative magnitude between the two fundamental energy costs is then

$$\frac{E_{\text{conservation}}}{E_{\text{Landauer}}} \sim \frac{\bar{m}}{\xi}, \quad \xi = \frac{k_B T_0}{\hbar \omega_0}. \quad (\text{C1})$$

Using the value $\xi \approx 0.2$ as quoted in Sec. IV C 2 under typical experimental settings, one sees that, for the range of photon number $\bar{m} \gtrsim 100$ considered in our semiclassical approximation on the EM field, the energy cost from

conservation as defined in Eq. (16) is at least several orders of magnitude larger than the fundamental thermodynamic erasure cost arising from Landauer's principle, justifying the omission of the latter in our analysis. This in particular validates our assumption in Table II that discarding ancillary qubits can be regarded as a free operation. As per the fundamental thermodynamic cost of discarding the field state after implementing each phase-encoding channel, note that the backaction on the field due to the field-qubit interaction (18) is small in our semiclassical regime, meaning that the postevolution field state is still very close to a pure state (with vanishing temperature), and its discarding cost is consequently assumed to be negligible as well compared to the total energy of the field. In the following subsection, we briefly discuss how reusing or recycling the postevolution field state may incur further energy savings.

2. Energy saving strategies

Here we qualitatively describe two possible generalizations to reduce the total energy cost, both of common practice. First, as shown in Appendix B, $\Delta(g)$ may be reduced according to Eq. (B6) if squeezing is introduced to the EM field. For a fixed $\bar{m}$, this leads to a smaller complexity [see Eq. (22)], at the cost of extra squeezing energy per gate (the energy of a squeezed coherent state with real squeezing is $\bar{m} + \sinh^2(s)$). An energy-optimal squeezing level can be accordingly determined.

The energy cost can be further saved through field recycling. After the implementation of each channel, the field state is slightly entangled with the system qubit and its final (mixed) state will marginally deviate from the initial (pure) state, leading to larger implementation error for the next iteration if we reuse the final field state as it is. To circumvent such backaction—which, albeit small in each gate implementation, can accumulate significantly throughout the full QPE protocol—in the main text we completely reset the field after each phase-encoding gate is implemented. Note that this may also be the result of an operational constraint, where each query to the agent in charge of implementing the gate must be made independently and hence the implementation error at each step is uncorrelated from the previous history. Instead, we can imagine that some restoring procedure may be allowed to recycle the same field for the next step, hence saving the reinitialisation energy of the field. For example, as studied in Refs. [88,126], consider a squeezed coherent state after interacting with the system qubit and slightly perturbed by the backaction. The original field state can be well restored by sequentially interacting the field with an additional ancilla qubit for some carefully tailored duration, through the free interaction (18). The more times the interaction is repeated, the closer the final field state will be to the original squeezed coherent state, and the smaller the implementation error will be for the next step, at the cost

of larger ancilla overhead. This leads to another complexity versus (recycling) energy trade-off relation. The improvement may also be enhanced if monitoring is allowed on the field [127,128]. A precise treatment on the energy reduction brought by more sophisticated field manipulation procedures is left to a later work.

3. Error mitigation controls

Although we aim to keep the current QPE protocol simple to demonstrate the qualitative behavior of its energy cost from basic building blocks, it is interesting to investigate whether (in addition to varying the number of photons spent per gate) adding intermediate control operations on the probe system, as illustrated in Fig. 10, may help improve the QPE performance.

Recall the implemented channel, whose action on the Bloch sphere is described by Eq. (B4), can be approximated as the desired unitary operation U_g followed by a bit flip channel with flipping probability $p = \frac{1-r}{2} = \frac{\Delta(g)}{4\bar{m}}$:

$$\mathcal{G}_{\bar{m};g}(\rho) = (1-p)U_g\rho U_g^\dagger + p\sigma_x U_g\rho U_g^\dagger \sigma_x.$$

The goal of the control operations would then be to reduce the bit flip noise.

However, since in our case the noise generator σ_x coincides with the Hamiltonian that generates the signal, it is fundamentally impossible to isolate and suppress the noise. Indeed, results in Ref. [57] indicate that the QFI obtained in Eq. (20) is optimal within our sequential framework. Common error mitigation techniques, such as quantum error correction [28,129–131], feedback controls [26,29], and passive ancillae [24,56,132] cannot help increase the QFI; in other words, for our protocol the best possible control operations $C_1, C_2, \dots$ in Fig. 10 all reduce to the identity. Further improvement is only possible if we introduce entanglement among probes of different sequences. We refrain from doing so since combining energy and entanglement, as two distinct resource quantities, demands more careful bookkeeping of the total resource cost, a point brought up again in the concluding Sec. V.

The aforementioned error mitigation techniques nevertheless become effective when other types of noise take place—such as decoherence arising from system-environment interactions that do not commute with U_g ; common examples include dephasing and depolarization [5]. If the field state is recyclable as described in Appendix 2, the implementation errors at each step are

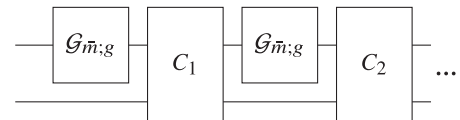


FIG. 10. A typical ancilla-assisted control sequence used to suppress the noise on the system qubit.

correlated through the shared field (as well as other potential sources of memory effects from the environment) and the effective noise becomes non-Markovian, which can be suppressed, e.g., through dynamical decoupling procedures [133,134]. Even within the current scope, when the photon number $\bar{m}$ is small enough, apart from the leading order bit flip noise, noncommuting terms also start to have a significant effect. Intuitively, we anticipate the appearance of yet another complexity versus (control) energy trade-off relation, where the error mitigation control increases the total QFI and hence reduces the complexity, while introducing extra energy cost for its implementation. A more thorough analysis on such an interplay will be explored in a forthcoming work.

APPENDIX D: FINITE-SAMPLE EFFECTS ON ESTIMATOR VARIANCE

The sweet spot discovered in Sec. IV B corresponds to the regime where a single sequence suffices to reach the target total QFI, and hence the desired variance bound δ^2 implied by the quantum Cramér-Rao bound (QCRB) (7).

$$\text{Var}[\hat{g}] \geq \frac{1}{2\Gamma} \int_0^\Gamma dx x \int_{\mu-\frac{\Gamma}{2}}^{\mu+\frac{\Gamma}{2}-x} dg \left[1 - \sqrt{1 - F(\rho_N(g), \rho_N(g+x))} \right], \quad (\text{D1})$$

where $F(\cdot, \cdot)$ is the quantum fidelity between two states and $\rho_N(g)$ is the final state after N steps, for a value g of the parameter to be estimated. For a qubit system, the fidelity between two states with Bloch vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ is

$$F(\rho_1, \rho_2) = \frac{1}{2} \left(1 + \mathbf{s}_1 \cdot \mathbf{s}_2 + \sqrt{(1 - |\mathbf{s}_1|^2)(1 - |\mathbf{s}_2|^2)} \right).$$

In our setting, using Eq. (B4) for the Bloch vector and working in our usual semiclassical regime specified by $\bar{m} \gtrsim 10^2$, we can approximate the shrinking factor with the constant $r \approx 1 - \frac{\Delta(\mu)}{2\bar{m}}$. Under this approximation, the fidelity becomes independent of g and reads

$$F(\rho_N(g), \rho_N(g+x)) \approx 1 - r^{2N} \sin^2\left(\frac{Nx}{2}\right).$$

Substituting into the QZZB (D1), we get

$$\text{Var}[\hat{g}] \gtrsim \frac{1}{2} \int_0^\Gamma dx x \left(1 - \frac{x}{\Gamma} \right) (1 - r^N |\sin(Nx/2)|).$$

For large $N \approx N_{\text{opt}}$, as done in Eq. (21), we use the approximation $r^N \approx e^{-1/2}$ and replace the oscillatory term by its average value, $|\sin(Nx/2)| \rightarrow 2/\pi$, obtaining

While the QFI provides a well-established figure of merit, leading to a convenient and computable constraint when determining the complexity in Eq. (2), it does not automatically reflect the achievable estimation precision in the finite-sample regime. This is because, as remarked in the main text, the QCRB assumes an unbiased estimator from the optimal measurement, which in general only holds asymptotically when the sequence is repeated for a sufficiently large number of times, $Q \gg 1$, and adaptive strategies are implemented [13–15].

To assess the impact of finite-sample effects, one may consider alternative bounds on the estimator variance that do not rely on asymptotic or unbiasedness assumptions, such as the Bayesian generalization of the Cramér-Rao bound based on the van Trees inequality [135]. In what follows, we focus on the quantum Ziv-Zakai bound (QZZB) as a universally valid lower bound that holds for any estimator bias [136] and is therefore applicable even in the single-shot regime $Q \sim 1$. Assuming a uniform prior distribution for g over an interval of width Γ and mean μ , the QZZB reads

$$\text{Var}[\hat{g}] \gtrsim k\Gamma^2, \quad k = \frac{1 - \frac{2}{\pi\sqrt{e}}}{12} \approx 0.05. \quad (\text{D2})$$

This bound provides a finite-sample benchmark that can be compared with the asymptotic QCRB. For the QFI derived in the main text to reflect the achievable estimation precision, we require the QCRB (7) to be at least as informative as the QZZB bound (D2). This is satisfied if, again using the derivation in Eq. (21),

$$\frac{1}{F_{N_{\text{opt}}}} \approx e \left(\frac{\Delta(\mu)}{\bar{m}} \right)^2 \gtrsim k\Gamma^2 \Rightarrow \Gamma \lesssim \frac{14.8}{\bar{m}},$$

for $\mu \approx 2.5$ as adopted in Figs. 4–6. Therefore, a smaller $\bar{m}$ allows for a less stringent demand on the prior knowledge on g , translating into a more informative QCRB in the finite-sample regime. Furthermore, recall that the constraint to determine the complexity (11) is a desired lower bound δ^2 on the estimator variance. Naturally, this bound should be at least tighter than the one given by the prior uniform distribution on g , meaning that $\delta^2 \lesssim \Gamma^2/12$, and thus

$$\bar{m} \lesssim \frac{4.3}{\delta}.$$

For our choice of $\delta^2 = 10^{-4}$ in the main text, this imposes a constraint on the photon number, $\bar{m} \lesssim 5 \cdot 10^2$.

Combining the above considerations, we identify a *Goldilocks regime*

$$10^2 \lesssim \bar{m} \lesssim 5 \cdot 10^2, \quad (\text{D3})$$

in which the assumptions underlying the analysis in the main text—most notably the semiclassical approximation—are satisfied and the QCRB remains an informative benchmark for the achievable estimation precision despite finite-sample effects. Consequently, the sweet spot identified in Sec. IV B yields a physically meaningful resource trade-off.

Overall, the analysis in this section indicates that operating at smaller photon number per gate not only reduces the implementation energy, but also improves the consistency between finite-sample performance and the asymptotic precision benchmark.

APPENDIX E: CLASSICAL FISHER INFORMATION FROM IMPERFECT MEASUREMENTS

Following from the pointer model introduced at the beginning of Sec. IV C 3, suppose the optimal measurement that yields the QFI is represented by the POVM operators $\mathbf{M} = (M_1, M_2)$ and we perform it on the pointer state. If the pointer is initially in the pure state $|0\rangle\langle 0|$, then it can be checked that the corresponding POVM operators on the system qubit are exactly $\mathbf{M}$ as desired. However, the pointer is instead prepared in a thermal state (33) after a dynamic cooling procedure, as described in Sec. IV C 2. For simplicity, we assume the cooling qubits are the same as those during the state preparation stage, although in the main text they may have different initial temperatures and transition frequencies depending on experimental realization. The same measurement procedure now yields the modified POVM operators,

$$\tilde{\mathbf{M}} = (1 - \varepsilon)\mathbf{M} + \varepsilon\mathbf{N}, \quad \mathbf{N} = (\sigma_x M_1 \sigma_x, \sigma_x M_2 \sigma_x), \quad \varepsilon = \frac{1 - \gamma(T)}{2},$$

where $T \approx 2T_0/M_m$ and M_m is the number of qubits consumed to cool the pointer. Consequently, the estimation precision is reflected by the CFI (6) achieved by the nonoptimal $\tilde{\mathbf{M}}$, rather than the QFI achieved by $\mathbf{M}$. To compute the effect of this imperfect measurement, we adopt the results from Ref. [85]. The susceptibility of the CFI with respect to a small disturbance on the POVM operators is defined as

$$\chi[\mathbf{M}, \mathbf{N}] = \lim_{\varepsilon \rightarrow 0} \frac{F_c[\mathbf{M}] - F_c[\tilde{\mathbf{M}}]}{\varepsilon F_c[\mathbf{M}]}.$$

In our case, $F_c[\mathbf{M}] = F_N(\bar{m}; g)$. The CFI can be approximated up to the leading order of the perturbation ε as $F_c[\mathbf{M}](1 - \varepsilon\chi[\mathbf{M}, \mathbf{N}])$. By plugging in Eq. (6) explicitly and using $M_1 + M_2 = N_1 + N_2 = I_2$, $\text{Tr}(\rho_N) = 1$, $\text{Tr}(\dot{\rho}_N) = 0$, this can be simplified to

$$F_N(\bar{m}; g) \xrightarrow{\text{measurement}} F_N(\bar{m}; g, \varepsilon) = F_N(\bar{m}; g) - \left(\frac{\dot{p}^2(q-p)(1-2p)}{p^2(1-p)^2} - \frac{2\dot{p}(\dot{q}-\dot{p})}{p(1-p)} \right) \varepsilon + \mathcal{O}(\varepsilon^2),$$

where $p = \text{Tr}(M_1 \rho_N)$, $q = \text{Tr}(N_1 \rho_N) = \text{Tr}(\sigma_x M_1 \sigma_x \rho_N)$ and $\dot{p} = \text{Tr}(M_1 \partial_g \rho_N)$, $\dot{q} = \text{Tr}(N_1 \partial_g \rho_N)$. To determine the optimal POVM, recall that while extracting the leading order contribution to the QFI (B5), the Bloch vector of the probe state is approximated by Eq. (B4); in particular, $\mathbf{s}_N \cdot \partial_g \mathbf{s}_N \approx 0$. The SLD operator can then be easily guessed as $\Lambda_g = \partial_g \mathbf{s}_N \cdot \boldsymbol{\sigma} = 2\partial_g \rho_N$ (also see Ref. [121]). The POVM operators that maximize the CFI are projectors onto the eigenspaces of Λ_g ,

$$M_i = \frac{\mathbf{I}_2 + \hat{m}_i \cdot \boldsymbol{\sigma}}{2}, \quad \hat{m}_i \approx \pm \frac{\partial_g \mathbf{s}_N}{|\partial_g \mathbf{s}_N|}, \quad (\text{E1})$$

leading to

$$p = \frac{1 + \hat{m}_1 \cdot \mathbf{s}_N}{2} \approx \frac{1}{2}, \quad q = \frac{1 - \hat{m}_1 \cdot \mathbf{s}_N}{2} \approx \frac{1}{2};$$

$$\dot{p} = -\dot{q} = \frac{\hat{m}_1 \cdot \partial_g \mathbf{s}_N}{2} \approx \frac{|\partial_g \mathbf{s}_N|}{2},$$

where we use the fact that $\mathbf{s}_N$ always stays in the yz plane and so a σ_x -rotation merely flips its sign. Therefore, the leading order CFI is

$$F_N(\bar{m}; g, \varepsilon) \approx F_N(\bar{m}; g) - 4|\partial_g \mathbf{s}_N|^2 \varepsilon \approx (1-4\varepsilon)F_N(\bar{m}; g),$$

where the last relation comes from Eq. (B3).

Combined with Eq. (34), the overall modification to the QFI due to nonideal state preparation and measurement is

$$F_N(\bar{m}; g) \xrightarrow{\text{state preparation + measurement}} F_N(\bar{m}; g, M_s, M_m) \approx \left[\gamma \left(\frac{2T_0}{M_s} \right) \right]^2 \left[2\gamma \left(\frac{2T_0}{M_m} \right) - 1 \right] F_N(\bar{m}; g). \quad (\text{E2})$$

The resulting complexity and total energy cost can be derived in the same fashion as in Sec. IV C 2.

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