

Quantum Batteries: A Nonreciprocal Thermal Edge

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Nonreciprocal quantum batteries (QBs) leverage engineered environments to impose unidirectional energy flow, achieving a fourfold enhancement in energy storage over conventional designs. In this work, we establish thermal-reservoir engineering as a powerful tool for controlling and optimizing the performance of QBs. We demonstrate that a thermal-reservoir-driven nonreciprocal QB achieves significantly enhanced charging, surpassing its vacuum-reservoir counterpart. Crucially, the nonreciprocal ergotropy is thermally invariant and coincides with the vacuum-reservoir energy, despite the temperature-dependent rise in total stored energy. This reveals a “thermally shielded” work potential, where nonreciprocity effectively decouples energy storage from energy quality, ensuring robust work extraction despite the thermal environment. Consequently, engineered thermal reservoirs emerge as a key resource for developing high-performance, noise-resilient quantum batteries.

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I. INTRODUCTION

Quantum batteries (QBs) have emerged as a promising class of energy storage devices that leverage quantum-mechanical principles to surpass the fundamental limitations of their classical counterparts [1–21]. Their potential lies in harnessing quantum effects to achieve transformative performance metrics, including ultrafast charging [2, 22], enhanced extractable work (ergotropy) [3], and minimal dissipative loss [23]. Recent theoretical and experimental advances have demonstrated remarkable progress toward these goals, including high ergotropy [3–5], faster charging [24,25], and nearly lossless energy dynamics [23]. A diverse landscape of QB models has been explored, ranging from many-body systems [26] and Dicke QBs [27,28] to implementations using ultracold atoms [29], Sachdev-Ye-Kitaev models [30], and Tavis-Cummings cavities [4]. A deeper understanding and improved control of these platforms have been driven by innovative strategies spanning intelligent control techniques, such as reinforcement learning [28] and optimal-control protocols [31,32], and the exploitation of physical effects such as non-Markovian dynamics [33], wireless charging [14], and weak measurements [34]. Concurrently, a key quantum

advantage, the collective, quadratic scaling of charging power, has been firmly established [5,30,35] and validated in experimental platforms including superconducting qubits [36] and quantum optical systems [27,37]. More recently, the nascent paradigm of nonreciprocal QBs [15, 16] creates a distinct performance advantage by enforcing a unidirectional flow of energy, offering a new route to combat energy leakage and enhance efficiency.

Quantum nonreciprocity, engineered to break time-reversal symmetry and enforce unidirectional propagation, has emerged as a powerful resource in quantum technologies [38–47]. This capability underpins critical functionalities such as directional amplification [48–50], quantum information routing [40], and the stabilization of topologically protected states [51]. Nonreciprocity can be engineered through a diverse suite of physical mechanisms, including reservoir engineering [15,48–53], the magneto-optic effect [54], nonlinear optics [55], quantum squeezing [56], non-Hermitian engineering [57], and chiral or topological photonics [46]. These distinct strategies enable nonreciprocal functionality across a wide range of spectral regimes and platforms, from integrated photonic circuits to quantum communication systems. This foundational progress has culminated in the development of key devices such as optical isolators [40], circulators [38], unidirectional waveguides [44,58], and directional amplifiers [59], with recent extensions into emerging domains such as QB charging [15,16]. Pushing the boundaries further, a groundbreaking frontier involves harnessing dynamical phase transitions in nonreciprocal many-body systems, such as the nonreciprocal Dicke model [60], opening

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new pathways for controlling quantum states and energy transfer.

Recent breakthroughs in nonreciprocal QBs [15,16] have enabled unprecedented spatiotemporal control of energy transfer, heralding new opportunities for quantum technologies. However, the practical implementation of these devices is hindered by a fundamental obstacle: the vulnerability of quantum coherence to decoherence in realistic environments and the thermal passivity of the QBs [61–63]. This manifests as thermal excitations that compete with coherent dynamics, populating the system and fundamentally limiting its performance. Most critically, a thermal reservoir drives the QB toward a passive state [13], from which no work can be extracted via cyclic unitary operations. This irreversible loss of ergotropy ultimately constrains the practical viability of quantum energy storage technologies. This fundamental challenge motivates our investigation into the role of thermally engineered reservoirs in governing the performance of nonreciprocal QBs. We propose a platform comprising two single-mode harmonic oscillators, designated as a charger and a battery, coupled to a shared thermal reservoir. The architecture leverages reservoir engineering to break reciprocity, establishing a unidirectional coherent coupling from the charger to the battery. Importantly, we demonstrate that this thermal-reservoir-driven design achieves significantly enhanced charging performance, surpassing the capabilities of its vacuum-reservoir counterpart. Remarkably, although the total stored energy increases, the extractable work remains identical to the vacuum-reservoir value. This reveals a “thermally shielded” work potential, where nonreciprocity actively protects the QB’s maximum extractable work from environmental noise. Our approach to nonreciprocity is fundamentally rooted in the theory of cascaded quantum systems [64–66], a framework for engineering unidirectional coupling between quantum subsystems. This formalism is not only versatile enough to describe our system but also extensible, for instance, to include amplifying interactions [48].

The quest for stable open QBs is a battle for quantum control, fought on three distinct fronts. The first employs intrinsic control, utilizing adiabatic protocols [67,68] and optimal control [69], to fortify the battery’s state from within. The second leverages extrinsic control through measurement-based feedback [70], Floquet engineering [71], and environment engineering [15,16] to actively suppress decoherence by reshaping the battery’s interaction with its surroundings. The final front addresses the thermodynamic challenge, confronting the fundamental limits imposed by temperature itself [72].

Research has established a fundamental dichotomy between closed and open QB paradigms, defined by their operational goals, physical constraints, and performance characteristics. For instance, Sathe *et al.* [73] demonstrated that no universal charging protocol exists for closed

systems, whereas such protocols can be constructed in open settings. This requires coupling the battery to an external system, such as an auxiliary device or a thermal bath. Related studies by Canzio *et al.* [74] further explored that while open-system QBs can retain a quantum speed advantage, their practical design requires careful management of switching costs, interaction timing, and detuning, with performance characteristics critically dependent on the underlying physical model. Our work extends this line of inquiry by investigating how engineered thermal reservoirs can be harnessed to enforce nonreciprocal energy flow and protect ergotropy storage, thereby bridging concepts of open-system control with practical QB design.

Nonreciprocal phenomena have become a central focus in quantum science and technology, driving extensive theoretical and experimental research. Their implementation is actively being explored across a variety of platforms, including microwave quantum circuits [75], on-chip photonic devices [54], and optoelectromechanical architectures [76–78]. This interest is motivated both by the pursuit of efficient optical and microwave isolators and by the exploration of novel nonreciprocal effects. These techniques enable the realization of unidirectional energy flow and are readily compatible with current experimental capabilities.

The structure of our paper is as follows. Sec. II presents the theoretical framework for the QB model and defines the figures of merit, introducing the Lindblad master equation that governs the system’s dynamics. Subsequently, Sec. III analyzes the charging behavior across different dynamical regimes, with a focus on the thermal effects on QB’s performance. The last section summarizes our conclusions.

II. QUANTUM BATTERY MODEL AND FIGURES OF MERIT

We establish a theoretical framework for a QB charging process, as shown in Fig. 1. The system is composed of two single-mode harmonic oscillators: a charger (mode a with frequency ω_a and decay rate κ_a) and a battery (mode b with frequency ω_b and decay rate κ_b). Importantly, the battery and charger are coupled identically to a common thermal reservoir, with a coupling strength denoted by Γ . We emphasize that the shared-reservoir coupling considered here constitutes a convenient implementation of nonreciprocal dynamics, rather than a necessary condition [48,49]. Moreover, the charger is driven by a classical field with amplitude $\mathcal{A}$ and frequency ω_0 , whereas the battery is initialized in its ground state. Crucially, the charging dynamics are governed by two distinct interactions: a coherent, uniform complex-valued coupling of strength J between the charger and battery, and a dissipative interaction of strength Γ , mediated by the engineered reservoir. Furthermore, we analyze the charging dynamics in the Markovian limit for the engineered reservoir,

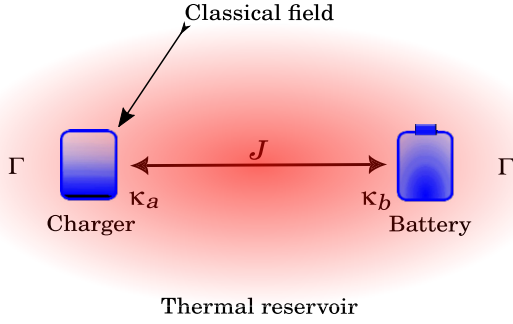


FIG. 1. Schematic of the quantum battery model under a thermal-reservoir engineering. The model comprises a charger and a battery, modeled as single-mode harmonic oscillators with frequencies ω_a and ω_b , and local decay rates κ_a and κ_b , respectively. The charger is driven by a classical field of amplitude $\mathcal{A}$ and frequency ω_0 . The two modes are coherently coupled with strength J and are simultaneously connected to a shared thermal reservoir via a nonlocal dissipative interaction of strength Γ .

which requires that the reservoir-induced damping rate is comparable to the dissipative rates of the charger and battery.

The system Hamiltonian describing the QB model is

$$\mathcal{H} = \omega_a a^\dagger a + \omega_b b^\dagger b + (J a^\dagger b + J^* b^\dagger a) + \mathcal{A}(e^{i\omega_0 t} a + e^{-i\omega_0 t} a^\dagger), \quad (1)$$

where a and b are the annihilation operators of the charger and battery, respectively. It is worth noting that the dissipative interaction is engineered by coupling both the charger and battery to the common reservoir. Strikingly, this configuration yields an effective dissipative interaction that persists even in the absence of direct reservoir mediation, constituting a non-Hamiltonian contribution to the dynamics.

The nonreciprocal behavior of the QB is driven by the interplay of coherent and reservoir-induced dissipative interactions. The dynamics are governed by a Lindblad master equation for the density matrix ρ [79–83]:

$$\dot{\rho} = -i[\mathcal{H}, \rho] + \sum_{i \in \{a, b\}} \kappa_i \mathcal{L}_i(\rho) + (n_{\text{th}} + 1)\Gamma \mathcal{L}_z(\rho) + n_{\text{th}}\Gamma \mathcal{L}_{z^\dagger}(\rho), \quad (2)$$

In Eq. (2), the first term governs coherent Hamiltonian dynamics, the second describes local cavity losses (κ_a and κ_b), and the third represents the engineered thermal-reservoir coupling. The collective channel is characterized by a damping rate Γ and a mean thermal occupation $n_{\text{th}} = (e^{\omega_v/(k_B T)} - 1)^{-1}$, where ω_v is the reservoir frequency and T its temperature. Notably, a vacuum-reservoir ($T \rightarrow 0$) leads to $n_{\text{th}} \rightarrow 0$. These processes are encapsulated in Lindblad superoperators $\mathcal{L}_x(\rho)$ with $x \in \{a, b, z\}$, defined

as follows:

$$\mathcal{L}_x(\rho) = x\rho x^\dagger - \frac{1}{2}\{x^\dagger x, \rho\}, \quad (3)$$

where $\mathcal{L}_a(\rho)$ and $\mathcal{L}_b(\rho)$ are the Lindblad superoperators modeling the local dissipation of the charger and battery, respectively. The term $\mathcal{L}_z(\rho)$ captures the collective dissipation arising from the coupling of both subsystems to a common engineered reservoir. This is governed by a collective jump operator of the form

$$z = u_a a + u_b b, \quad (4)$$

with complex coefficients u_a and u_b , derived from the cascaded quantum systems formalism [48,64,66,84]. The complex coefficients encode the respective coupling strengths and phases of the charger and battery to the shared reservoir. The interference between these channels underlies the nonreciprocal dynamics, enabling directional energy transfer from the charger to the battery.

To evaluate the effect of thermal reservoirs on the nonreciprocal QBs, we employ two key figures of merit: the energy storage capacity $\mathcal{E}(t)$ and the ergotropy $\mathcal{W}(t)$. These quantify the efficiency of energy transfer from the charger to the battery and the maximum extractable work from the battery, respectively. The energy accumulated into the QB over the charging time t ,

$$\mathcal{E}_b(t) = \text{Tr}[\mathcal{H}_b \rho_b(t)] = \omega \langle b^\dagger b \rangle, \quad (5)$$

where ω denotes the frequency of the harmonic oscillator battery and Tr is the trace over the reduced state (ρ_b) of the battery. The quantity $\mathcal{E}_b(t)$ measures the mean energy stored in the battery at time t . To quantify the useful work capacity of the QB, we employ the concept of ergotropy. It is a central concept in quantum thermodynamics that quantifies the maximum amount of work that can be extracted from a quantum system via a cyclic unitary process and has the form [61]

$$\mathcal{W}_b(t) = \text{Tr}[\mathcal{H}_b \rho_b(t)] - \min_{U_b} \text{Tr}[\mathcal{H}_b U_b \rho_b(t) U_b^\dagger], \quad (6)$$

where $\rho_b(t)$ is the reduced state of the battery at time t , and the minimization in Eq. (6) is over all local unitaries U_b acting on the battery. The second term in Eq. (6) corresponds to the passive energy of the system, the portion of its energy that is inaccessible to any unitary operation U_b . Since the system is Gaussian, its ergotropy can be expressed in terms of operator expectation values of the mode operators as follows [12]:

$$\mathcal{W}_b(t) = \mathcal{E}_b(t) - \mathcal{E}_b(t), \quad (7)$$

where $\mathcal{E}_b(t)$ is the energy of the corresponding passive state of the battery, from which no work can be extracted

cyclically via unitary operations [13]. The energy of the passive state can have the form:

$$\mathcal{E}_b(t) = \omega \left(\frac{\sqrt{\mathcal{D}} - 1}{2} \right), \quad (8)$$

where $\mathcal{D}$ is a dimensionless composite parameter encapsulating the battery's first and second moments. Its explicit form is

$$\mathcal{D} = [1 + 2(\langle b^\dagger b \rangle - |\langle b \rangle|^2)]^2 - 4|\langle bb \rangle - \langle b \rangle^2|^2. \quad (9)$$

For a quantum harmonic oscillator coupled to a zero-temperature Markovian bath (vacuum reservoir), the ergotropy is identical to the coherent part of the energy storage. In this scenario, all second-order moment correlators factorize perfectly into products of first-order moments; namely, $\langle a^\dagger a \rangle = |\langle a \rangle|^2$, $\langle aa \rangle = \langle a \rangle^2$, $\langle b^\dagger b \rangle = |\langle b \rangle|^2$, and $\langle bb \rangle = \langle b \rangle^2$. As a direct consequence, the key quantity $\mathcal{D}$ becomes unity, leading to an exact equality between the battery's internal energy and its ergotropy [$\mathcal{E}_b(t) = \mathcal{W}_b(t)$] [11,12,16]. This ergotropy is fundamentally enabled by the nonreciprocal coupling, which simultaneously suppresses deleterious energy backflow and preserves the quantum coherence essential for maximal work extraction. However, under nonzero incoherent driving, this factorization breaks down, resulting in ergotropy that is different from the total stored energy.

III. CHARGING DYNAMICS

To characterize the thermal reservoir's influence on the nonreciprocal QBs, we derive the equations of motion of the mode operators. Based on the Lindblad master equation, we obtain the following thermal-independent first-order moments for the charger and battery modes:

$$\begin{aligned} \langle \dot{a} \rangle &= - \left(i\omega_a + \frac{\Lambda_a}{2} \right) \langle a \rangle - i \left(J + i\mu \frac{\Gamma}{2} \right) \langle b \rangle - i\mathcal{A}e^{-i\omega_0 t}, \\ \langle \dot{b} \rangle &= - \left(i\omega_b + \frac{\Lambda_b}{2} \right) \langle b \rangle - i \left(J^* + i\mu^* \frac{\Gamma}{2} \right) \langle a \rangle. \end{aligned} \quad (10)$$

where $\mu = -u_a^* u_b$ has magnitude $|\mu| = 1$ for nonzero coefficients. The parameter $\Lambda_i = \Gamma_i + \kappa_i$ with $\Gamma_i = \Gamma|u_i|^2$ for $i \in \{a, b\}$ represents the reservoir-induced damping for the charger and battery. From the Lindblad master equation, we derive the following analytical expressions

for the second-order moments:

$$\begin{aligned} \langle \dot{a^\dagger a} \rangle &= -\Lambda_a \langle a^\dagger a \rangle + 2\text{Im} \left[\left(J + i\mu \frac{\Gamma}{2} \right) \langle a^\dagger b \rangle \right] \\ &\quad - 2\text{Im}[\mathcal{A}e^{i\omega_0 t} \langle a \rangle] + n_{\text{th}} \Gamma_a, \\ \langle \dot{b^\dagger b} \rangle &= -\Lambda_b \langle b^\dagger b \rangle - 2\text{Im} \left[\left(J - i\mu \frac{\Gamma}{2} \right) \langle a^\dagger b \rangle \right] + n_{\text{th}} \Gamma_b, \\ \langle \dot{a^\dagger b} \rangle &= -\frac{\Lambda_a + \Lambda_b}{2} \langle a^\dagger b \rangle - i \left(J^* + i\mu^* \frac{\Gamma}{2} \right) \langle a^\dagger a \rangle \\ &\quad + i \left(J^* - i\mu^* \frac{\Gamma}{2} \right) \langle b^\dagger b \rangle + i\mathcal{A}e^{i\omega_0 t} \langle b \rangle - n_{\text{th}} \mu^* \Gamma, \\ \langle \dot{a^2} \rangle &= -(2i\omega_a + \Lambda_a) \langle a^2 \rangle - 2i \left(J + i\mu \frac{\Gamma}{2} \right) \langle ab \rangle \\ &\quad - 2i\mathcal{A}e^{-i\omega_0 t} \langle a \rangle, \\ \langle \dot{b^2} \rangle &= -(2i\omega_b + \Lambda_b) \langle b^2 \rangle - 2i \left(J^* + i\mu^* \frac{\Gamma}{2} \right) \langle ab \rangle \\ \langle \dot{ab} \rangle &= - \left(i(\omega_a + \omega_b) + \frac{\Lambda_a + \Lambda_b}{2} \right) \langle ab \rangle - i\mathcal{A}e^{-i\omega_0 t} \langle b \rangle \\ &\quad - i \left(J^* + i\mu^* \frac{\Gamma}{2} \right) \langle a^2 \rangle - i \left(J + i\mu \frac{\Gamma}{2} \right) \langle b^2 \rangle. \end{aligned} \quad (11)$$

The engineered reservoir is crucial for establishing nonreciprocal energy transfer between the charger and battery. This condition is mathematically enforced by a precise balancing of the complex coupling between the two systems to a specific value,

$$J = -i\mu \frac{\Gamma}{2}. \quad (12)$$

This directional coupling enforces a nonreciprocal energy transfer, preferentially channeling energy from the charger to the battery while suppressing backflow. Consequently, the charger exerts a causal influence on the battery's dynamics, but not vice versa. The engineered reservoir mediates a nonlocal damping force that couples the charger and battery modes, analogous to a coherent interaction. Crucially, the asymmetry in the coupling coefficients allows us to decouple the charger's dynamics from the battery by enforcing the condition in Eq. (12). This breaks time-reversal symmetry and establishes a unidirectional energy channel in which the charger drives the battery without any back action. Both the charger and the battery are initially prepared in their respective ground states: a natural choice that isolates the effect of the engineered nonreciprocal coupling on ergotropy dynamics. Notably, for this class of initial states, the condition $J = i\mu\Gamma/2$ guarantees that the steady state retains full coherence, independent of the amplitude of the initial charger state, thereby demonstrating robustness within this family.

A. Nonreciprocal charging dynamics

The charging dynamics of the QB system is determined by solving the coupled differential equations with initial conditions $\langle \mathcal{O} \rangle|_{t=0} = 0$, where $\langle \mathcal{O} \rangle$ denotes the expectation values of the mode operators. In the limit of resonance frequency ($\omega_a = \omega_b = \omega_0 = \omega$), the battery's nonreciprocal energy storage is given (see Appendix A for details):

$$\mathcal{E}_b^{\text{nr}}(n_{\text{th}}, t) = \mathcal{E}_b^{\text{nr}}(0, t) + \chi_b^{\text{nr}}(n_{\text{th}}, t), \quad (13)$$

where

$$\begin{aligned} \chi_b^{\text{nr}}(n_{\text{th}}, t) &= \frac{\omega n_{\text{th}}}{\Lambda_b} \left[\Gamma_b (1 - e^{-\Lambda_b t}) - \frac{4\Gamma^2 \kappa_a}{\Lambda_a^2 + \Lambda_a \Lambda_b} - \frac{8\Gamma^2 \Lambda_b}{\Lambda_{ab}^2 e^{\Lambda_a t}} \right. \\ &\quad \times \left. \left\{ \frac{\Gamma_a}{2\Lambda_a} + \frac{(\Gamma_a - \Lambda_{ab})e^{\Lambda_{ab}t}}{2\Lambda_b} - \frac{(2\Gamma_a - \Lambda_{ab})e^{\frac{\Lambda_{ab}t}{2}}}{\Lambda_a + \Lambda_b} \right\} \right], \end{aligned} \quad (14)$$

quantifies the thermal contribution to the battery's energy capacity. The asymmetry parameter $\Lambda_{ab} = \Lambda_a - \Lambda_b$ quantifies the imbalance between the total individual damping rates of the charger and battery. Equation (13) describes the total energy storage, $\mathcal{E}_b^{\text{nr}}(n_{\text{th}}, t)$, under thermal-reservoir engineering. This expression reveals that the thermal contributions to the nonreciprocal charging dynamics depend crucially on the damping rates and the mean thermal occupation of the bath. Structurally, it decomposes into two physically distinct parts: a thermal component (the second term) and the nonreciprocal energy $\mathcal{E}_b^{\text{nr}}(n_{\text{th}} = 0, t)$ stored under an engineered vacuum reservoir, which has the form [15,16]

$$\mathcal{E}_b^{\text{nr}}(0, t) = \omega \left[\frac{4\Gamma \mathcal{A}}{\Lambda_{ab}} \left(\frac{1 - e^{-\Lambda_a t/2}}{\Lambda_a} - \frac{1 - e^{-\Lambda_b t/2}}{\Lambda_b} \right) \right]^2. \quad (15)$$

One can straightforwardly demonstrate that Eq. (13) simplifies to Eq. (15) for a vacuum-reservoir engineering, as reported in the literature [15,16]. Under symmetric dissipative coupling ($\Gamma_a = \Gamma_b = \Gamma$, and $\kappa_a = \kappa_b = \kappa$), the nonreciprocal energy $\mathcal{E}_b^{\text{nr}}(0, t)$ simplifies to [15,16]

$$\mathcal{E}_b^{\text{nr}}(0, t) = \frac{16\omega\Gamma^2 \mathcal{A}^2}{\Lambda^4} \left[1 - \left(1 + \frac{\Lambda t}{2} \right) e^{-\Lambda t/2} \right]^2. \quad (16)$$

Our analysis, however, reveals that the analytical expression for energy dynamics under symmetric dissipative rates [15, Eq. (6)] is mathematically ill-defined, which may potentially leading to misinterpretation of the system's dynamics. To resolve this, we regularize the symmetric limit by introducing an infinitesimal asymmetry between

the dissipative rates Λ_a and Λ_b by setting $\Lambda_a = \Lambda + \sigma$ and $\Lambda_b = \Lambda$. The corrected energy is then defined through the well-defined limit:

$$\mathcal{E}_b^{\text{nr}}(n_{\text{th}}, t) = \lim_{\sigma \rightarrow 0} \mathcal{E}_b^{\text{nr}}(n_{\text{th}}, t) \Big|_{\Lambda_a = \Lambda + \sigma, \Lambda_b = \Lambda}. \quad (17)$$

Evaluating this limit yields the physically consistent result given by Eq. (16). Consequently, the corrected energy expression in Eq. (16) differs fundamentally from Eq. (6) of Ref. [15].

Under identical dissipation, the thermal component of the total energy $\chi(n_{\text{th}}, t)$ (14) simplifies to the following form:

$$\chi_b^{\text{nr}}(n_{\text{th}}, t) = \frac{\omega n_{\text{th}} \Gamma^2 t}{\Lambda^2 e^{\Lambda t}} \left[2\kappa - \Gamma \Lambda t - \frac{\Gamma^2 + \kappa^2}{\Gamma \Lambda t} (1 - e^{\Lambda t}) \right]. \quad (18)$$

The total energy capacity of the battery under symmetric dissipation, defined as $\lim_{t \rightarrow \infty} \mathcal{E}_b^{\text{nr}}(n_{\text{th}}, t)$ converges to

$$\mathcal{E}_b^{\text{nr}}(n_{\text{th}}, t \rightarrow \infty) = \frac{16\omega\Gamma^2 \mathcal{A}^2}{\Lambda^4} + \frac{\omega n_{\text{th}} \Gamma (\Gamma^2 + \kappa^2)}{\Lambda^3}. \quad (19)$$

The thermal contributions to the nonreciprocal energy of the charger has the form

$$\mathcal{E}_a(n_{\text{th}}, t) = \mathcal{E}_a(n_{\text{th}} = 0, t) + \frac{\omega n_{\text{th}} \Gamma_a}{\Lambda_a} (1 - e^{-\Lambda_a t}), \quad (20)$$

where

$$\mathcal{E}_a(n_{\text{th}} = 0, t) = \omega \left[\frac{2\mathcal{A}}{\Lambda_a} (1 - e^{-\Lambda_a t/2}) \right]^2, \quad (21)$$

is the nonreciprocal charger's energy in the vacuum-reservoir limit. Crucially, the thermal nonreciprocal charger's energy converges to

$$\mathcal{E}_a(n_{\text{th}}, t \rightarrow \infty) = 4\omega \mathcal{A}^2 / \Lambda_a^2 + \omega n_{\text{th}} \Gamma_a / \Lambda_a, \quad (22)$$

in the steady-state limit. Figure 2(a) illustrates the charging dynamics of the nonreciprocal thermal QB under symmetric dissipation rates ($\kappa_a = \kappa_b = \kappa$, $\Gamma_a = \Gamma_b = \Gamma$). Notably, the model reduces to the vacuum-reservoir case for $n_{\text{th}} = 0$, confirming its internal consistency. In this limit, the battery's stored energy is purely coherent and, thus, entirely extractable as ergotropy, i.e., $\mathcal{W}_b^{\text{nr}}(n_{\text{th}} = 0, t) = \mathcal{E}_b^{\text{nr}}(n_{\text{th}} = 0, t)$. The charger's energy (bold curve) and energy stored in the battery (dashed curves) through interaction with a thermal reservoir reach a significantly higher value compared with the case with a vacuum reservoir. The energy storage in the battery is enhanced through its interaction with the thermal reservoir, which acts as a source of additional thermal energy. This influx of energy increases the

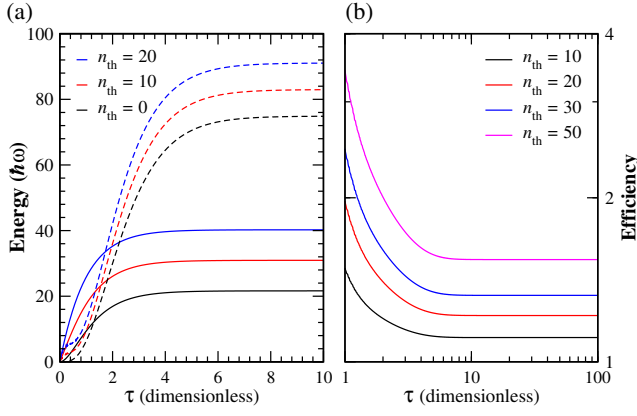


FIG. 2. (a) Energy dynamics (units of $\hbar\omega$ with $\hbar = 1$) of the charger (solid curve) and battery (dashed curve) as a function of the scaled time, $\tau = Jt$ (dimensionless), for $n_{\text{th}} = 0$ (black curves), 10 (red curves), and 20 (blue curves) in the nonreciprocal regime. The parameters are set $\Lambda_a = \Lambda_b = \Lambda$, with identical dissipation rates $\Gamma_a = \Gamma_b = \Gamma = 0.04\omega$, identical coupling losses $\kappa_a = \kappa_b = 0.003\omega$, $\mathcal{A} = 0.1\omega$, and a nonreciprocal coupling strength of $|J| = \Gamma/2$. (b) (Log-log scale) Thermal effect on the battery's energy storage efficiency, $\mathcal{E}_b^{\text{nr}}(n_{\text{th}}, \tau)/\mathcal{E}_b^{\text{nr}}(n_{\text{th}} = 0, \tau)$, versus τ for different values of the thermal occupation number n_{th} .

battery's capacity. Furthermore, the nonreciprocal nature of the system facilitates a more efficient directional energy transfer, preferentially channeling energy into the battery and, thus, optimizing storage in the presence of the thermal reservoir. Most importantly, in the steady-state limit ($\tau \rightarrow \infty$), the energies of the battery and charger converge to Eqs. (19) and (22), respectively. To quantify the role of temperature on the energy storage capacity of the battery, we evaluate the ratio, $\mathcal{E}_b^{\text{nr}}(n_{\text{th}}, \tau)/\mathcal{E}_b^{\text{nr}}(n_{\text{th}} = 0, \tau)$, across various thermal occupation numbers as depicted in Fig. 2(b). Our analysis reveals that the efficiency approaches unity in the vacuum-reservoir limit $n_{\text{th}} = 0$ and exceeds one for any nonzero thermal excitation $n_{\text{th}} > 1$.

Furthermore, our analysis reveals that the thermal reservoir preferentially enhances the battery's and charger's energy, as shown in Fig. 3(a). The energy data for both components are well-described by linear fits, which yield a steeper slope for the charger compared to the battery. This differential enhancement predicts that the charger's energy will exceed the battery's beyond a specific threshold of thermal occupation number, inducing a crossover in their energy dynamics, as given by

$$n_{\text{th}} = \frac{2\mathcal{A}^2(4\Gamma^2 - \Lambda^2)}{\Gamma^2\Lambda\kappa}. \quad (23)$$

Figure 3(b) asymptotic demonstrates the efficiency, defined as the energy ratio, $\mathcal{E}_b^{\text{nr}}(n_{\text{th}}, t \rightarrow \infty)/\mathcal{E}_a(n_{\text{th}}, t \rightarrow \infty)$, clearly indicating that beyond the specific thermal threshold (23), the charger's energy surpasses the battery's

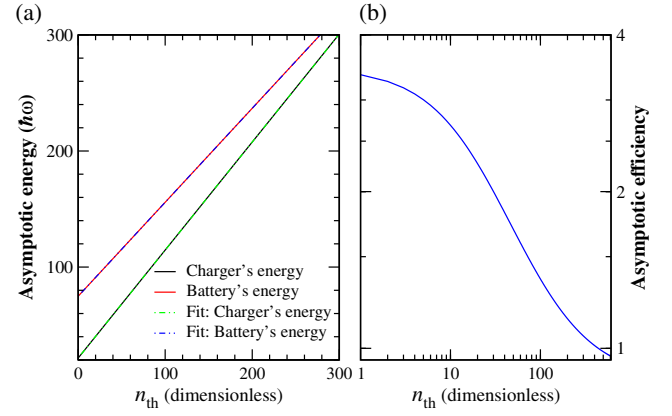


FIG. 3. (a) Asymptotic energy dynamics (units of $\hbar\omega$) of the charger and battery as a function of thermal occupation number in the nonreciprocal regime for symmetric damping rates $\Gamma = \Gamma_a = \Gamma_b = 0.04\omega$, $\kappa_a = \kappa_b = 0.003\omega$, and $\mathcal{A} = 0.1\omega$ with coupling strength $|J| = \Gamma/2$. Linear regression of the energies yielded a steeper slope for the charger compared to the battery. This indicates that the charger's energy will exceed the battery's energy beyond a specific threshold. (b) The asymptotic efficiency, $\mathcal{E}_b^{\text{nr}}(n_{\text{th}}, t \rightarrow \infty)/\mathcal{E}_a(n_{\text{th}}, t \rightarrow \infty)$, as a function of n_{th} .

energy. This threshold signifies the point where thermal energy overwhelms the coherent work-storage mechanism. The nonreciprocal design successfully protects the battery from this thermal noise, but it cannot prevent the charger from becoming thermally populated. Thus, while the battery remains a useful quantum work source, the overall energy partitioning shifts in favor of the thermally flooded charger.

A more stringent figures of metric is the ergotropy, which quantifies the useful energy: the maximum work extractable from the QB, distinguishing it from the total stored energy which may be partially locked in inaccessible forms. By rigorously solving the coupled differential equations (10) and (11) for the expectation values of the mode operators, we find that (see Appendix A for the full derivation)

$$\begin{aligned} \langle bb \rangle &= \langle b \rangle^2, \quad (\text{thermal-independent}); \\ \omega |\langle b \rangle|^2 &= \mathcal{E}_b^{\text{nr}}(n_{\text{th}} = 0, t), \quad (\text{thermal-independent}); \\ \omega \langle b^\dagger b \rangle|_{n_{\text{th}}} &= \mathcal{E}_b^{\text{nr}}(n_{\text{th}} = 0, t) + \chi_b^{\text{nr}}(n_{\text{th}}, t). \end{aligned} \quad (24)$$

Substituting Eq. (24) into Eq. (8) yields the energy of the corresponding passive state,

$$\mathcal{E}_b^{\text{nr}}(n_{\text{th}}, t) = \chi_b^{\text{nr}}(n_{\text{th}}, t). \quad (25)$$

Further substituting Eqs. (13) and (25) into Eq. (7) gives the battery ergotropy,

$$\mathcal{W}_b^{\text{nr}}(n_{\text{th}}, t) = \mathcal{W}_b^{\text{nr}}(n_{\text{th}} = 0, t) = \mathcal{E}_b^{\text{nr}}(n_{\text{th}} = 0, t). \quad (26)$$

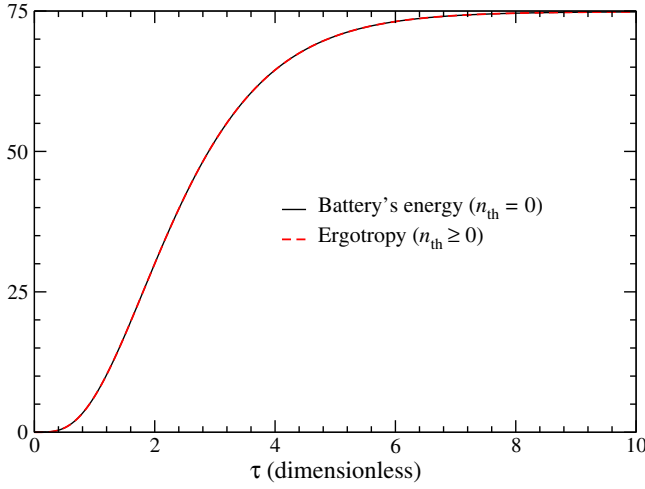


FIG. 4. Dynamics of the total stored energy and ergotropy (in units of $\hbar\omega$, with $\hbar = 1$) for the battery in the nonreciprocal regime as a function of dimensionless scaled time τ . The parameters are chosen as $\Lambda_a = \Lambda_b = \Lambda$, with identical dissipation rates $\Gamma_a = \Gamma_b = \Gamma = 0.04\omega$, identical coupling losses $\kappa_a = \kappa_b = 0.003\omega$, a driving amplitude $\mathcal{A} = 0.1\omega$, and a nonreciprocal coupling strength $|J| = \Gamma/2$.

This result demonstrates that the ergotropy is devoid of any thermal contribution, as it coincides exactly with the energy stored under an engineered vacuum reservoir. Consequently, the nonreciprocal battery operates at the fundamental limit for work extraction in a thermal environment: its extractable work $\mathcal{W}_b^{\text{nr}}(n_{\text{th}}, t)$ saturates the maximum possible value set by the ideal zero-temperature charging dynamics, i.e., $\mathcal{W}_b^{\text{nr}}(n_{\text{th}}, t) = \mathcal{E}_b^{\text{nr}}(n_{\text{th}} = 0, t)$, as illustrated in Fig. 4. This is achieved not by suppressing thermal noise, but by dynamically isolating the coherent, work-capable component of the battery's state from the incoherent thermal population. This temperature independence follows from the fact that thermal noise enters only the occupation number $\langle b^\dagger b \rangle$, while the coherent correlators, responsible for ergotropy, remain unchanged due to the linear, cascaded Lindblad dynamics under the nonreciprocal tuning condition. This constitutes one of the most illuminating findings of our work. The equality between ergotropy and the vacuum-limit stored energy is a direct signature that the nonreciprocal QB operates at its fundamental quantum limit for work extraction. This reveals a powerful principle: tailored dissipation via reservoir engineering can protect quantum performance, making the extractable work robust against thermal noise.

The temperature-dependent increase in total energy resides entirely in the passive (nonextractable) component of the stored energy. Thermal fluctuations populate higher-energy levels incoherently while preserving the passive ordering of populations, thereby precluding the generation of additional extractable work. Since ergotropy quantifies the energy extractable via unitary operations, and thermal

noise neither induces population inversion nor generates useful coherence, the ergotropy remains unchanged. In contrast, the nonreciprocal reservoir determines the non-passive structure of the steady state, which is essentially insensitive to temperature, thereby explaining the robustness of the extractable work against thermal noise.

B. Reciprocal charging dynamics

We now turn to the reciprocal regime ($\Gamma = 0$), where the charging dynamics exhibit characteristics distinctly different from their nonreciprocal counterparts. By solving the coupled differential equations (10) and (11) for the case without a shared reservoir, we derive the exact reciprocal energy dynamics of the battery, $\mathcal{E}_b^r(t)$, as given by [15,16]

$$\mathcal{E}_b^r(t) = \frac{4\omega\mathcal{A}^2\mathcal{C}}{\kappa_a\kappa_b(\mathcal{C}+1)^2} \left[1 - \mathcal{F}(t)e^{-\frac{\kappa_{ab}t}{4}} + \frac{\mathcal{G}(t)e^{-\frac{\kappa_{ab}t}{2}}}{2\Delta^2} \right], \quad (27)$$

with

$$\begin{aligned} \mathcal{F}(t) &= 2 \cosh \frac{\Delta t}{4} + \frac{2\kappa_{ab}}{\Delta} \sinh \frac{\Delta t}{4}, \\ \mathcal{G}(t) &= (\Delta - \kappa_{ab})^2 \cosh \frac{\Delta t}{2} + 2\Delta\kappa_{ab}e^{\frac{\Delta t}{2}} - 4\kappa_a\kappa_b(\mathcal{C}+1). \end{aligned} \quad (28)$$

Here, $\mathcal{C} = 4J^2/\kappa_a\kappa_b$ defines the coherent coupling cooperativity between the charger and battery, $\kappa_{ab} = \kappa_a + \kappa_b$ is the total local dissipation rates, and $\Delta = (|\kappa_a - \kappa_b|^2 - |4J|^2)^{1/2}$. The parameter Δ plays a crucial role, as its nature delineates the dynamical regime: an imaginary value ($\Delta \in \text{Im}$) corresponds to underdamped charging dynamics, whereas a real value ($\Delta \in \text{Re}$) indicates an overdamped regime [15]. Most notably, in the critical damping case ($\Delta = 0$), reciprocal QBs store more energy than their nonreciprocal counterparts when $\Gamma \geq \kappa_b$. This highlights the apparent performance disadvantage, as detailed in the Appendix B. Crucially, the charging dynamics for the reciprocal QB are derived without shared reservoir engineering; consequently, the thermal bath exerts no influence on the QB's performance.

As is evident from Eq. (27), the characteristics of the energy dynamics in the reciprocal system is governed by the parameter Δ . To delve deeper, we begin by analyzing the underdamped regime, where Δ is imaginary ($\Delta \in \text{Im}$). This regime is well-established in the literature [15], where oscillatory dynamics in the reciprocal energy were identified for the underdamped regime ($4|J| > |\kappa_a - \kappa_b|$) under identical local damping rates ($\kappa_a = \kappa_b = \kappa$). Our analysis, however, demonstrates that these dynamics are governed by the interplay between coherent coupling and intrinsic dissipation, as illustrated in Fig. 5. Within this underdamped regime, the battery's reciprocal energy [Eq. (27)]

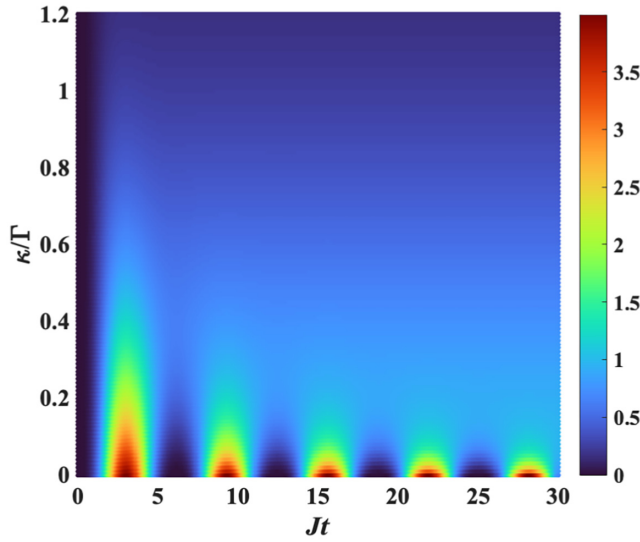


FIG. 5. Phase diagram of the reciprocal charging dynamics (units of $\hbar\omega$) of the QB. The normalized reciprocal battery energy, $\mathcal{E}_b^r(t)/(\omega\mathcal{A}^2/J^2)$, exhibits three dynamical regimes governed by the dimensionless coupling ratio κ/Γ , and the rescaled time Jt (dimensionless), where $\Gamma = 2J$. For $\kappa < \Gamma$, the energy fluctuates at a consistent frequency and damping amplitude in the rescaled time scale, eventually converging to a finite value in the steady-state limit. Conversely, the specified oscillations fade away for $\kappa > \Gamma$.

simplifies to

$$\mathcal{E}_b^r(t) = \mathcal{E}_b^r(t \rightarrow \infty) \left[1 - e^{-\kappa t/2} \left(\cos Jt + \frac{\kappa}{2J} \sin Jt \right) \right]^2, \quad (29)$$

where $\mathcal{E}_b^r(t \rightarrow \infty) = 4\omega\mathcal{A}^2\mathcal{C}/(\kappa(\mathcal{C} + 1))^2$ is the maximum achievable energy obtained in the steady-state limit. The dynamics of the battery's energy are universally governed by the dimensionless ratio κ/Γ , which delineates the following three distinct dynamical underdamped regimes, as summarized in Table I.

- (1) Coherent-dominated regime ($\Gamma \gg \kappa, \kappa/\Gamma \rightarrow 0$): In this regime, one may recover an oscillating behavior of the reciprocal energy, as reported in Ref. [15]. The amplitude of oscillation decays exponentially as $e^{-\kappa t/2}$, but since $\Gamma \gg \kappa$, the decay is

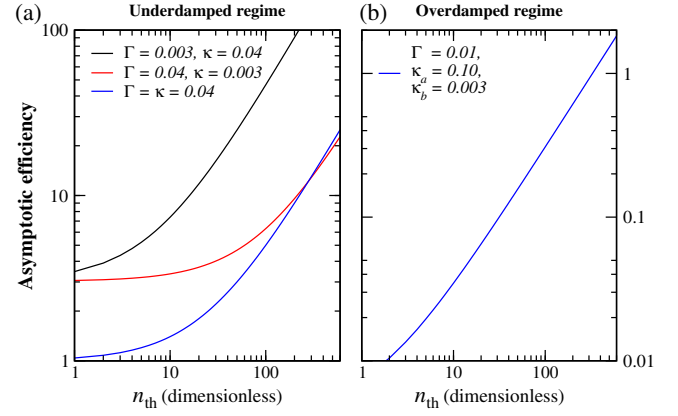


FIG. 6. (Log-log scale) The asymptotic energy ratio (nonreciprocal to reciprocal dimensionless quantity) versus thermal occupation number (dimensionless parameter) for the battery in the (a) underdamped ($4|J| > |\kappa_a - \kappa_b|$) and (b) overdamped ($4|J| < |\kappa_a - \kappa_b|$) regimes, with a fixed coupling strength $|J| = \Gamma/2$ and $\mathcal{A} = 0.1\omega$.

slow, preserving the oscillatory behavior over many cycles.

- (2) Dissipation-dominated regime ($\Gamma \ll \kappa, \kappa/\Gamma \rightarrow \infty$): The energy dynamics exhibit an exponential decay without oscillations.
- (3) Identical coherent-dissipative regime $\Gamma = \kappa$: The energy dynamics, both reciprocal and nonreciprocal tend to converge [$\mathcal{E}_b^r(t) \approx \mathcal{E}_b^{\text{nr}}(n_{\text{th}} = 0, t)$], reflecting the absence of nonreciprocal advantage.

Our analysis identifies a critical limitation: under identical coherent-dissipative conditions and vacuum-reservoir engineering, the steady-state energies of the reciprocal and nonreciprocal configurations converge. The equivalence in asymptotic charging performance indicates no fundamental nonreciprocal advantage. This limitation, however, is overcome by thermal-reservoir engineering: with increasing n_{th} , the nonreciprocal energy rises and exceeds the underdamped reciprocal energy beyond a critical thermal occupation threshold, as shown in Fig. 6(a). The ratio of nonreciprocal to reciprocal energy in the steady-state limit remains greater than unity in both the coherent and dissipation-dominated ($\Gamma \gtrless \kappa$) regimes, indicating a consistent nonreciprocal advantage. In the identical

TABLE I. Reciprocal energy in three distinct underdamped regimes.

Underdamped regime	Condition	Reciprocal energy, $\mathcal{E}_b^r(t)$	Physical Behavior	Nonreciprocal advantage
Coherent-dominated	$\Gamma \gg \kappa$	$\mathcal{E}_b^r(t \rightarrow \infty)(1 - e^{-\kappa t/2} \cos Jt)^2$	Oscillatory decay	Present
Dissipation-dominated	$\Gamma \ll \kappa$	$\mathcal{E}_b^r(t \rightarrow \infty) \left(1 - \frac{\kappa t}{2} e^{-\kappa t/2} \right)^2$	Monotonic decay	Present
Identical coherent-dissipative	$\Gamma = \kappa$	$\mathcal{E}_b^r(t \rightarrow \infty)(1 - \sqrt{2}e^{-Jt} \cos(Jt - \pi/4))^2$	Slow decay	Absent

coherent-dissipative regime ($\Gamma = \kappa$), however, the ratio only surpasses unity for $n_{\text{th}} > 1$, demonstrating a distinct, thermally activated nonreciprocal advantage. Figure 6(b) illustrates the thermal enhancement of QB efficiency through nonreciprocal coupling in the overdamped regime. In the overdamped regime under vacuum-reservoir engineering, the reciprocal charger-battery system demonstrates superior performance. This disadvantage, however, can be reversed by optimizing the nonreciprocal coupling through asymmetric dissipation rates ($\Gamma_a = \zeta\Gamma$ and $\Gamma_b = \Gamma/\zeta$ with $\zeta = \sqrt{\kappa_a/\kappa_b}$), which enables the nonreciprocal mechanism to achieve more efficient energy storage, surpassing the reciprocal case [15]. A more striking result emerges under thermal-reservoir engineering. Even without parameter optimization, the energy ratio $\lim_{t \rightarrow \infty} \mathcal{E}_b^{\text{nr}}/\mathcal{E}_b^r$ increases with the thermal occupation number n_{th} , causing the nonreciprocal energy to surpass the reciprocal energy beyond a critical threshold n_{th} .

IV. CONCLUSION

This work has demonstrated the pivotal role of reservoir engineering in optimizing the charging and storage efficiency of nonreciprocal QBs. By tailoring specific reservoir properties, we achieved a significant enhancement in energy storage capacity compared with the vacuum-reservoir counterparts. Specifically, the thermal-reservoir engineering enhanced the battery's energy storage by providing an external source of thermal energy. This effect is amplified by the system's nonreciprocity, which facilitates more efficient unidirectional energy transfer into the battery. Furthermore, our analysis of the ergotropy under a thermal reservoir reveals that nonreciprocity inherently protects the battery's work-extraction capability. Strikingly, thermal noise contributes exclusively to the passive component of the energy, while the nonreciprocal reservoir fixes the nonpassive structure of the steady state, thereby rendering the ergotropy insensitive to the bath temperature. Consequently, the ergotropy is temperature-independent and exactly equals the energy stored under vacuum conditions, demonstrating that the maximum extractable work is protected against thermal degradation.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: NONRECIPROCAL CHARGING DYNAMICS

This appendix details the analytical derivation of the charging dynamics (stored energy and ergotropy) for the thermal-reservoir-assisted nonreciprocal QB [Eq. (15)]. Substituting the nonreciprocal condition $J = -i\mu\Gamma/2$ into the coupled differential equations (10) and (11) yields

$$\begin{aligned}\dot{\langle a \rangle} &= -\left(i\omega + \frac{\Lambda_a}{2}\right) \langle a \rangle - i\mathcal{A}e^{-i\omega t}, \\ \dot{\langle b \rangle} &= -\left(i\omega + \frac{\Lambda_b}{2}\right) \langle b \rangle + \mu^*\Gamma \langle a \rangle, \\ \dot{\langle a^\dagger a \rangle} &= -\Lambda_a \langle a^\dagger a \rangle - 2\text{Im}[\mathcal{A}e^{i\omega t} \langle a \rangle] + n_{\text{th}}\Gamma_a, \\ \dot{\langle b^\dagger b \rangle} &= -\Lambda_b \langle b^\dagger b \rangle + 2\text{Im}[\mu\Gamma \langle a^\dagger b \rangle] + n_{\text{th}}\Gamma_b, \\ \dot{\langle a^\dagger b \rangle} &= -\frac{\Lambda_a + \Lambda_b}{2} \langle a^\dagger b \rangle + \mu^*\Gamma \langle a^\dagger a \rangle + i\mathcal{A}e^{i\omega t} \langle b \rangle \\ &\quad - n_{\text{th}}\mu^*\Gamma \\ \dot{\langle a^2 \rangle} &= -(2i\omega + \Lambda_a) \langle a^2 \rangle - 2i\mathcal{A}e^{-i\omega t} \langle a \rangle, \\ \dot{\langle b^2 \rangle} &= -(2i\omega + \Lambda_b) \langle b^2 \rangle + 2\mu^*\Gamma \langle ab \rangle, \\ \dot{\langle ab \rangle} &= -\left(2i\omega + \frac{\Lambda_a + \Lambda_b}{2}\right) \langle ab \rangle + \mu^*\Gamma \langle a^\dagger a \rangle \\ &\quad + i\mathcal{A}e^{i\omega t} \langle b \rangle.\end{aligned}\quad (\text{A1})$$

The complex coupling breaks time-reversal symmetry, enabling unidirectional energy transfer between the charger and the battery. This asymmetry is manifest in the system's dynamics: while the expectation values of the battery's observables are influenced by the charger, the charger's observables evolve independently of the battery. Consequently, the dynamics of the charger mode, $\langle a \rangle$, are governed by a first-order differential equation whose solution is of the form

$$\langle a \rangle = \frac{2i\mathcal{A}}{\Lambda_a e^{i\omega t}} (e^{-\Lambda_a t/2} - 1), \quad (\text{A2})$$

at resonance frequency $\omega_0 = \omega$. By plugging in the value of $\langle a \rangle$ in the first cumulant of the battery observable $\langle b \rangle$, we obtain

$$\langle b \rangle = \frac{4i\Gamma\mathcal{A}\mu^*}{\Lambda_{ab}e^{i\omega t}} \left(\frac{1 - e^{-\Lambda_b t/2}}{\Lambda_b} - \frac{1 - e^{-\Lambda_a t/2}}{\Lambda_a} \right), \quad (\text{A3})$$

where $\Lambda_{ab} = \Lambda_a - \Lambda_b$ is the difference of the total individual damping rates of the charger and battery. The energy dynamics of the charger can be obtained by putting the value of $\langle a \rangle$ in the second cumulant of the charger mode

$\langle a^\dagger a \rangle$. The second cumulant of the charger mode becomes,

$$\langle a^\dagger a \rangle = \langle a^\dagger a \rangle|_{n_{\text{th}}=0} + \frac{n_{\text{th}}\Gamma_a}{\Lambda_a}(1 - e^{-\Lambda_a t/2}). \quad (\text{A4})$$

Here, the second term represents the thermal contribution to the charger's excitation. The corresponding energy stored in the reservoir-assisted charger, given by $E_a(t) = \omega \langle a^\dagger a \rangle$, is presented in Eq. (20).

To compute the nonreciprocal charging dynamics of the battery, we analyze the time evolution of the operator correlation $\langle a^\dagger b \rangle$. Using Eqs. (A3) and (A4) in $\langle a^\dagger b \rangle$, we obtain

$$\begin{aligned} \langle a^\dagger b \rangle &= \langle a^\dagger b \rangle|_{n_{\text{th}}=0} - \frac{2n_{\text{th}}\Gamma\mu^*}{\Lambda_a(\Lambda_a^2 - \Lambda_b^2)} [\kappa_a\Lambda_{ab} - (\Lambda_a \\ &+ \Lambda_b)\Gamma_a e^{-\Lambda_a t} + (2\Gamma_a - \Lambda_{ab})e^{-\frac{\Lambda_a + \Lambda_b}{2}t}], \end{aligned} \quad (\text{A5})$$

where $\langle a^\dagger b \rangle|_{n_{\text{th}}=0}$ is the expectation value of the $a^\dagger b$ correlator in the vacuum-reservoir limit [16]. Substituting the expression for $\langle a^\dagger b \rangle$ into the equation for $\langle b^\dagger b \rangle$ yields a first-order linear differential equation for the battery population, $\langle b^\dagger b \rangle$. This equation is solved by applying the standard method of integrating factors, with the factor $e^{\Lambda_b t}$, leading to the solution

$$\begin{aligned} \Delta \langle b^\dagger b \rangle &= \frac{n_{\text{th}}}{\Lambda_b} \left[\Gamma_b(1 - e^{-\Lambda_b t}) - \frac{4\Gamma^2\kappa_a}{\Lambda_a^2 + \Lambda_a\Lambda_b} - \frac{8\Gamma^2\Lambda_b}{\Lambda_{ab}^2 e^{\Lambda_a t}} \right. \\ &\times \left. \left(\frac{\Gamma_a}{2\Lambda_a} + \frac{\Gamma_a - \Lambda_{ab}}{2\Lambda_b e^{-\Lambda_{ab} t}} - \frac{2\Gamma_a - \Lambda_{ab}}{\Lambda_a + \Lambda_b} e^{\Lambda_{ab} t/2} \right) \right]. \end{aligned} \quad (\text{A6})$$

Here, $\Delta \langle b^\dagger b \rangle = \langle b^\dagger b \rangle|_{n_{\text{th}}} - \langle b^\dagger b \rangle|_{n_{\text{th}}=0}$. Substituting Eq. (A6) into Eq. (5) yields the energy dynamics of the non-reciprocal reservoir-assisted QBs, as given by Eq. (13). Remarkably, in the vacuum-reservoir limit ($n_{\text{th}} = 0$), Eq. (A6) is simplified to

$$\begin{aligned} \langle b^\dagger b \rangle|_{n_{\text{th}}=0} &= \left[\frac{4\Gamma\mathcal{A}}{\Lambda_{ab}} \left(\frac{1 - e^{-\Lambda_a t/2}}{\Lambda_a} - \frac{1 - e^{-\Lambda_b t/2}}{\Lambda_b} \right) \right]^2, \\ &= |\langle b \rangle|^2. \end{aligned} \quad (\text{A7})$$

To derive an analytical expression for the ergotropy, we solve the differential equations governing the second-order moments: $\langle a^2 \rangle$, $\langle ab \rangle$, and $\langle b^2 \rangle$. The dynamics of the charger mode, in particular, $\langle a^2 \rangle$, are governed by a first-order differential equation whose solution is of the form

$$\langle a^2 \rangle = \frac{2\mathcal{A}^2}{\Lambda_a^2}(1 - e^{-\Lambda_a t/2})^2. \quad (\text{A8})$$

Remarkably, the second cumulant of the charger mode, $\langle a^2 \rangle$, is temperature-independent and factorizes exactly

into $\langle a \rangle^2$. By plugging in the value of $\langle a^2 \rangle$ and $\langle b \rangle$ in the time evolution of the operator correlation $\langle ab \rangle$, we obtain

$$\langle ab \rangle = \frac{8\Gamma\mu^*\mathcal{A}^2(1 - e^{-\frac{\Lambda_a t}{2}})}{\Lambda_a\Lambda_{ab}e^{2i\omega t}} \left[\frac{1 - e^{-\frac{\Lambda_a t}{2}}}{\Lambda_a} - \frac{1 - e^{-\frac{\Lambda_b t}{2}}}{\Lambda_b} \right]. \quad (\text{A9})$$

Notably, the second cumulant $\langle ab \rangle$ exhibits exact factorization into $\langle a \rangle \langle b \rangle$ and is entirely independent of temperature, as confirmed by the solutions for the mode operators in Eqs. (A2) and (A3). Substituting the value of $\langle ab \rangle$ into the equation for the second cumulant of the battery mode, $\langle b^2 \rangle$, yields a solution of the form

$$\langle b^2 \rangle = - \left[\frac{4\Gamma\mathcal{A}\mu^*}{\Lambda_{ab}e^{i\omega t}} \left(\frac{1 - e^{-\Lambda_b t/2}}{\Lambda_b} - \frac{1 - e^{-\Lambda_a t/2}}{\Lambda_a} \right) \right]^2. \quad (\text{A10})$$

The second cumulant of the battery mode, $\langle b^2 \rangle$, is temperature-independent and factorizes exactly into $\langle b \rangle^2$ [cf. Eq. (A3)].

It is crucial to emphasize that the second cumulant $\langle b^\dagger b \rangle$ is temperature-dependent and does not generally factorize into $\langle b^\dagger \rangle \langle b \rangle$. However, under vacuum-reservoir conditions, this cumulant exhibits perfect factorization, yielding $\langle b^\dagger b \rangle = |\langle b \rangle|^2$ and $\langle bb \rangle = \langle b \rangle^2$. This leads directly to a unit dimensionless parameter $\mathcal{D} = 1$ and a zero passive state energy ($\mathcal{E}_b^{\text{nr}} = 0$). Consequently, the battery maintains an equal amount of stored energy and ergotropy, such that

$$\mathcal{W}_b^{\text{nr}}(n_{\text{th}} = 0, t) = \mathcal{E}_b^{\text{nr}}(n_{\text{th}} = 0, t). \quad (\text{A11})$$

In contrast, under a nonzero incoherent driving, the factorization of the battery's second cumulant is violated ($\langle b^\dagger b \rangle \neq |\langle b \rangle|^2$), fundamentally altering the relationship between ergotropy and total stored energy [61].

APPENDIX B: CRITICALLY DAMPED CHARGING DYNAMICS ($\Delta = 0$)

The critical damping regime corresponds to $\Delta = 0$, which occurs when the dissipative asymmetry balances the coherent coupling, $|\kappa_a - \kappa_b| = |4J|$ with $J = \Gamma/2$. Assuming $\kappa_a > \kappa_b$ and $J > 0$ and identical local dissipation rates $\Gamma_a = \Gamma_b = \Gamma$, the reciprocal battery energy [Eq. (27)] simplifies to

$$\mathcal{E}_b^r(0, t) = \frac{4\omega\Gamma^2\mathcal{A}^2}{\Lambda_b^4} \left[1 - \left(1 + \frac{\Lambda_b t}{2} \right) e^{-\Lambda_b t/2} \right]^2, \quad (\text{B1})$$

where $\Lambda_b = \Gamma + \kappa_b$ is the total dissipation rate of the battery. The steady-state energy, reached as $t \rightarrow \infty$, is

$$\mathcal{E}_b^r(0, t \rightarrow \infty) = \frac{4\omega\Gamma^2\mathcal{A}^2}{\Lambda_b^4}. \quad (\text{B2})$$

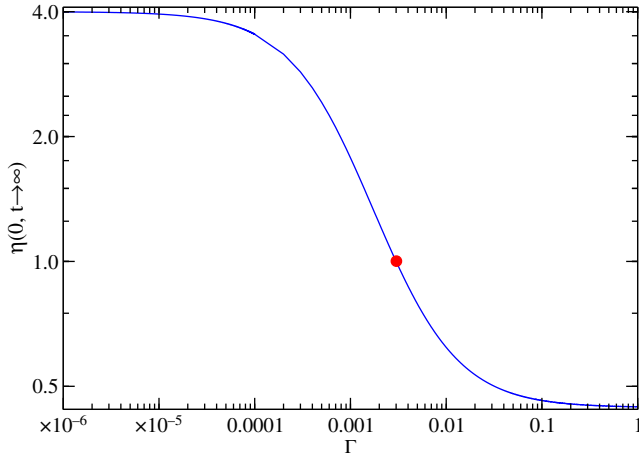


FIG. 7. (Log-log scale) Asymptotic efficiency $\eta(0, t \rightarrow \infty)$ (nonreciprocal-to-reciprocal energy ratio) versus the local dissipation rate Γ for the critical damping condition $\kappa_a = \kappa_b + 4J$ with $\kappa_b = 0.003\omega$ and $J = \Gamma/2$. The red circle marks the crossover point $\Gamma = \kappa_b$, where the charging dynamics shifts from a coherent-dominant regime ($\Gamma < \kappa_b$) to a dissipation-dominant regime ($\Gamma > \kappa_b$).

In this critically damped limit, the system exhibits the fastest possible monotonic relaxation without oscillations, optimizing the charging time by avoiding both underdamped coherent back action. This inverse quartic scaling $\mathcal{E}_b^r(0, t) \propto \Lambda_b^{-4}$, highlights a strong dependence of the battery's storage capacity on dissipation: minimizing Λ_b (the total battery damping) dramatically enhances the maximum storable energy.

The performance of the nonreciprocal charging protocol can be quantified by comparing the battery's stored energy with that achieved in a conventional reciprocal charger-battery system under identical conditions. For the critically damped nonreciprocal case ($\Delta = 0$), the battery energy dynamics simplify to

$$\mathcal{E}_b^{\text{nr}}(0, t) = \frac{4\omega\mathcal{A}^2}{\Lambda_b^2(2\Gamma + \Lambda_b)^2} [2\Gamma - (2\Gamma + \Lambda_b)e^{-\Lambda_b t/2} + \Lambda_b e^{-(2\Gamma + \Lambda_b)t/2}]^2. \quad (\text{B3})$$

The steady-state energy, reached as $t \rightarrow \infty$, is

$$\mathcal{E}_b^{\text{nr}}(0, t \rightarrow \infty) = \frac{16\omega\Gamma^2\mathcal{A}^2}{\Lambda_b^2(2\Gamma + \Lambda_b)^2}. \quad (\text{B4})$$

To benchmark the nonreciprocal advantage, we compare the steady-state battery's stored energy $\mathcal{E}_b^{\text{nr}}(0, t \rightarrow \infty)$ against the energy $\mathcal{E}_b^r(0, t \rightarrow \infty)$ of a reciprocal charger-battery system operating under the same physical constraints. The steady-state charging efficiency, defined as the ratio of the nonreciprocal to reciprocal energies, is

therefore

$$\eta(0, t \rightarrow \infty) = \frac{4(\Gamma + \kappa_b)^2}{(3\Gamma + \kappa_b)^2}. \quad (\text{B5})$$

This efficiency quantifies the energy-storage advantage provided by the nonreciprocal coupling. Notably, $\eta(0, t \rightarrow \infty) > 1$ for all finite κ_b , indicating that the nonreciprocal protocol outperforms the reciprocal benchmark in the steady-state energy stored, as depicted in Fig. 7. Notably, a significant nonreciprocal advantage ($\eta(0, t \rightarrow \infty) > 1$) is observed primarily in the coherent-dominant regime. In the dissipation-dominant regime, the efficiency decreases with Γ , eventually falling below unity. This crossover indicates that the nonreciprocal protocol becomes less efficient than the reciprocal benchmark when dissipative losses overwhelm coherent coupling.

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