

Charge-Preserving Operations in Quantum Batteries

André H.A. Malavazi^{1,*}, Borhan Ahmadi^{1,†}, Paweł Horodecki¹, and Pedro R. Dieguez^{2,1,‡}

¹*International Centre for Theory of Quantum Technologies, University of Gdańsk, ul. prof. Marii Janion 4, 80-309 Gdańsk, Poland*

²*Department of Physics, Federal University of Paraná, P.O. Box 19044, 81531-980 Curitiba, Paraná, Brazil*

 (Received 3 November 2025; revised 24 February 2026; accepted 30 March 2026; published 28 April 2026)

Ergotropy provides a fundamental measure of the extractable work from a quantum system and, consequently, of the maximal useful energy, or charge, stored within it. Understanding how this quantity can be manipulated and transformed efficiently is crucial for advancing quantum energy management technologies. Here, we introduce and formalize the concepts of isoergotropic states and ergotropy-preserving operations, which reorganize the internal structure of ergotropy while keeping its total value unchanged. These ideas are illustrated for both discrete (two-level systems) and continuous-variable systems (single-mode Gaussian states). In each case, we show how ergotropy-preserving operations redistribute the respective coherent-incoherent and displacement-squeezing components. We further examine the thermodynamic exchanges accompanying ergotropy-preserving operations, including variations in energy and entropy, and demonstrate that these transformations can be dynamically implemented through standard beam-splitter-type interactions with an auxiliary system. Finally, we discuss the practical implications of isoergotropic states and operations in optimizing charging protocols and mitigating charge loss in open quantum batteries.

DOI: [10.1103/2jtp-jpkn](https://doi.org/10.1103/2jtp-jpkn)

I. INTRODUCTION

Quantum thermal devices designed to perform specific tasks hold significant promise for emerging energy-based technologies. The ability to precisely design and control the energetics across quantum systems has direct applications in a wide range of small-scale quantum devices. Thus, the development of machines capable of harnessing energy fluxes, performing work, and manipulating heat transport, such as quantum heat engines, refrigerators, thermal transistors, rectifiers, etc. [1–9], has been established as one of the main motivations behind current efforts in quantum thermodynamics.

Among such devices, quantum batteries (QBs) have emerged as a promising and powerful paradigm for energy storage. In contrast to its classical counterparts, QBs are capable of allocating a portion of their energy within their quantum degrees of freedom [10–13]. During the last decade, it has motivated a growing interest in various

important aspects of QBs, including the characterization of QB architectures in a wide range of promising physical platforms [14–28], in distinct physical contexts [29–36], and in investigating their intrinsic quantum and collective advantages [37–51]. From a practical perspective, considerable attention has been focused on developing and optimizing charging protocols [52–56], and providing strategies to ensure stable charge retention in the presence of detrimental environmental effects [57–73].

In this context, the charge of a QB is commonly identified by its ergotropy, the operational figure of merit for quantifying the maximal useful extractable energy in closed quantum systems and thus its work capacity [74]. This particular thermodynamic quantity is nonadditive and fundamentally distinct from the internal energy, but closely related to the role that free energy plays [75,76], since it implicitly incorporates the notion that energy extraction requires an external cyclic Hamiltonian control. It is worth mentioning that different notions and extensions of ergotropy have been investigated in interacting subsystems [77] and open quantum systems [78–80].

More recently, ergotropy has been shown to be differently distributed within the multiple internal degrees of freedom of QBs. Notably, it can be generally decomposed into two distinct contributions: an incoherent part, associated with population inversion in the energy eigenbasis, and a coherent part, arising from quantum coherence [81,82]. While the former can be readily identified as a

*Contact author: andrehamalavazi@gmail.com

†Contact author: borhan.ahmadi@ug.edu.pl

‡Contact author: dieguez.pr@gmail.com

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

purely classical contribution, the latter is genuinely quantum, i.e., without any classical analogous. In the context of QBs, coherence has been shown to enhance energy storage [81] and help mitigate ergotropy losses in open quantum batteries [73]. More recently, the coherent ergotropy of a single spin has been experimentally accessible in a nitrogen-vacancy (NV) center platform [83], elucidating its practical relevance for the current state-of-the-art technologies. More importantly, the internal configuration of charge is not unique, and one can distribute it across all the accessible variables. Indeed, the ergotropy stored in arbitrary Gaussian states, for instance, was shown to be divided between independent displacement and squeezing contributions [84].

Different degrees of freedom can store useful energy, so distinct states may carry the same ergotropy (charge) while differing in how that charge is internally encoded. These encodings respond unevenly to control fields and environmental noise, making states with equal charge operationally nonequivalent. This raises a natural question: which encoding is best for a given setting or thermodynamic task? Isoergotropic (ergotropy-preserving) reorganization addresses this question. By redistributing charge among internal degrees of freedom without changing its total amount, one can refine charging protocols, enhance work extraction, propose novel charge stabilization techniques, and prepare batteries that are more robust to specific noise mechanisms—all of which are crucial for realistic, open-system implementations. From a thermodynamic perspective, such operations also enable new cycles whose strokes reshuffle charge internally at a fixed budget. Understanding—and exploiting—the interplay between these internal compositions of ergotropy is therefore both fundamental and practically important for the design of energy-based quantum devices.

In this work, we formalize the set $\tilde{\mathcal{L}}$ of ergotropically equivalent quantum states and the class $\mathcal{K}$ of operations that preserve total ergotropy while redistributing its internal components. Our procedure covers both CPTP channels and nonselective measurements. We then analyze the fundamental trade-offs among internal ergotropy components for two representative battery models: discrete two-level systems (TLSs) and continuous-variable (CV) single-mode Gaussian states. On the implementation side, we show that these ergotropy-preserving operations can be realized dynamically with standard beam-splitter interactions to an auxiliary system, available in current platforms. Finally, we illustrate how isoergotropic states and operations aid practice: they enable the refinement of charging protocols and mitigate charge loss by exploiting an ergotropic counterpart of the Mpemba effect.

The remainder of this paper is structured as follows. In Sec. II we recall the definition of ergotropy and introduce isoergotropic states and ergotropy-preserving operations. In Secs. III and IV, we apply these ideas to quantum

batteries built from discrete and continuous-variable cells, modeled by two-level systems and single-mode Gaussian states, respectively. Each section is self-contained: we identify the corresponding isoergotropic states, specify the ergotropy-preserving operations, and describe how to reorganize ergotropy dynamically within the internal degrees of freedom using a standard beam-splitter coupling. In Sec. V, we illustrate how the formalism developed here applies to relevant scenarios for quantum batteries, including charging and discharging protocols. Finally, Sec. VI summarizes the main conclusions and outlines directions for future work.

II. ERGOTROPY AND ISOERGOTROPIC OPERATIONS

Let us assume a d -dimensional quantum system described by a density matrix $\hat{\rho} = \sum_j^d p_j |j\rangle\langle j| \in \mathcal{L}$ and Hamiltonian $\hat{H} = \sum_k^d E_k |E_k\rangle\langle E_k|$. As a quantum battery, its charge is commonly characterized by the maximum extractable energy through a unitary cyclic Hamiltonian control. This quantity, also known as ergotropy, can be written as [74]

$$\mathcal{R}[\hat{\rho}] := \text{Tr} \left[(\hat{\rho} - \hat{\rho}_p) \hat{H} \right] = \sum_{k,j}^d E_k p_j (|\langle E_k | j \rangle|^2 - \delta_{k,j}), \quad (1)$$

where $\hat{\rho}_p = \sum_i^d p_i |E_i\rangle\langle E_i|$ is the passive state relative to $\hat{\rho}$, and the canonical spectral ordering $p_1 \geq p_2 \geq \dots \geq p_d$ and $E_1 \leq E_2 \leq \dots \leq E_d$ were assumed. Note that Eq. (1) can equivalently be obtained by identifying the unitary $\hat{V}$ that reorders the populations p_j so as to minimize the system's energy, $\hat{\rho} \mapsto \hat{\rho}_p = \hat{V} \hat{\rho} \hat{V}^\dagger$. In general, the total ergotropy of a system may be distributed among distinct internal degrees of freedom. Consequently, different states can possess identical ergotropy—hence the same stored charge—while exhibiting distinct internal configurations. Formally, we introduce the following definition:

Definition 1. Given an active state $\hat{\rho}$ with ergotropy $\bar{\mathcal{R}} = \mathcal{R}[\hat{\rho}]$, the set of *isoergotropic* (*iso- $\mathcal{R}$*) states relative to $\hat{\rho}$ is defined as $\tilde{\mathcal{L}} := \{ \hat{\rho} \in \mathcal{L} \mid \mathcal{R}[\hat{\rho}] = \bar{\mathcal{R}} \}$.

Figure 1 illustrates such states for (a) a two-level system and (b) a single-mode Gaussian state [85]. In panel (a), the colored surfaces on the Bloch sphere, and in panel (b), the colored sheets in the $(|\mu|^2, |\xi|, N)$ space, collect states with the same total ergotropy but different internal compositions of charge. The inset Wigner functions $W(\alpha)$ illustrate distinct Gaussian states with equal total charge and thermal occupation N but different displacement μ and squeezing ξ . Transformations connecting states within a given

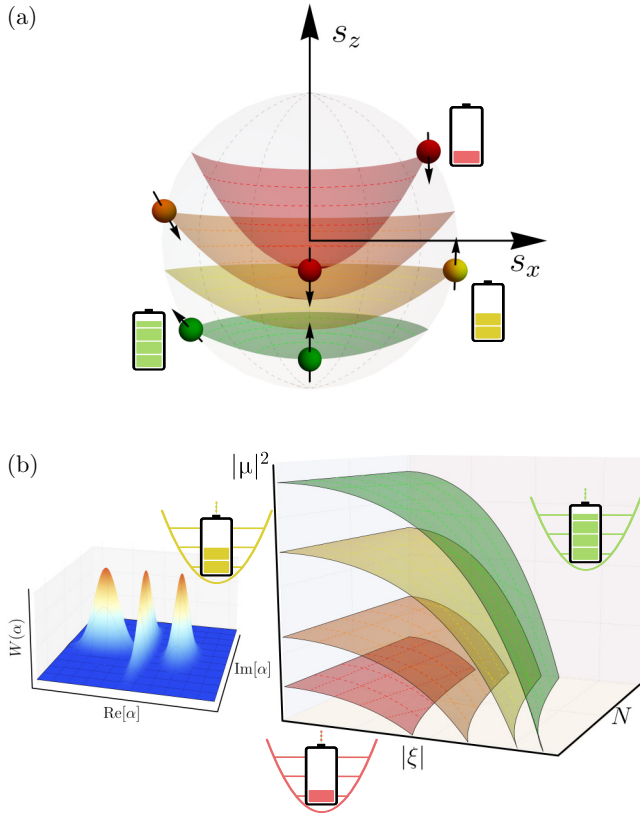


FIG. 1. Isoergotropic states for (a) a single two-level quantum battery and (b) a single-mode Gaussian oscillator. Each colored surface at the (a) Bloch sphere (b) parameter space, corresponds to states with fixed charge but distinct internal configurations of ergotropy $\mathcal{R}$, with red and green being, respectively, low and high charge. The Wigner functions $W(\alpha)$ represent Gaussian states with the same charge and thermal occupation N but different displacement μ and squeezing ξ components of $\mathcal{R}$ [85].

isoergotropic surface correspond to *ergotropy-preserving* operations, which redistribute the internal composition of stored work while keeping its total value invariant. Unitaries that commute with the Hamiltonian, $[\hat{U}, \hat{H}] = 0$ (e.g., free evolution), constitute trivial instances of such operations, leaving the internal configuration unchanged. We therefore give the following definition:

Definition 2. Given a state $\hat{\rho} \in \tilde{\mathcal{L}}$, iso- $\mathcal{R}$ operations are CPTP (measurements) maps satisfying $\mathcal{K} : \tilde{\mathcal{L}} \mapsto \tilde{\mathcal{L}}$.

These operations provide the “free transformations” in a prospective resource theory of ergotropy at a fixed budget. The resource corresponds to the configuration of stored work within a given level set $\tilde{\mathcal{L}}$, while the iso- $\mathcal{R}$ maps $\mathcal{K}$ represent the admissible reshufflings that preserve the total ergotropy. Below, we instantiate this for discrete (TLS) and continuous-variable Gaussian batteries.

We emphasize that isoergotropic operations are defined by the constraint $\mathcal{R}[\mathcal{K}[\hat{\rho}]] = \mathcal{R}[\hat{\rho}]$, whereas the reachability of an entire isoergotropic level set under a physically restricted subclass (e.g., Gaussian-only, LOCC-only, or energy-preserving dilations) is model dependent; in Secs. III and IV we provide constructive connectivity results for the subclasses relevant to our two platforms.

III. TWO-LEVEL SYSTEMS

We begin with a TLS described by the Hamiltonian $\hat{H} = \omega|e\rangle\langle e|$ and by the density operator $\hat{\rho} = (1-p)|g\rangle\langle g| + p|e\rangle\langle e| + \mathcal{C}(e^{i\frac{\theta}{2}}|g\rangle\langle e| + \text{h.c.})/2$, where h.c. denotes the Hermitian conjugate, $\theta \in [0, 4\pi]$ and $\mathcal{C} := \sum_{k \neq j} |\langle k|\hat{\rho}|j\rangle|$, with $\mathcal{C}^2 \in [0, 4p(1-p)]$, is the l_1 norm of coherence in the energy eigenbasis [86].

In general, Eq. (1) for ergotropy can be split into a contribution due to population inversion and a contribution due to coherences (both defined in the Hamiltonian eigenbasis). Define the incoherent state $\hat{\rho}_{\text{diag}} = \sum_k |E_k\rangle\langle E_k| \hat{\rho} |E_k\rangle\langle E_k|$ and set $\mathcal{R}^{\text{inc}}[\hat{\rho}] := \mathcal{R}[\hat{\rho}_{\text{diag}}]$, $\mathcal{R}^{\text{coh}}[\hat{\rho}] := \mathcal{R}[\hat{\rho}] - \mathcal{R}^{\text{inc}}[\hat{\rho}]$. Then the ergotropy is fully parameterized by $(p, \mathcal{C})$: $\mathcal{R}(p, \mathcal{C}) = \mathcal{R}^{\text{inc}}(p) + \mathcal{R}^{\text{coh}}(p, \mathcal{C})$, with [73,81,82]

$$\mathcal{R}^{\text{inc}}(p) = \omega(2p - 1)[1 - \Theta(1/2 - p)], \quad (2)$$

and

$$\mathcal{R}^{\text{coh}}(p, \mathcal{C}) = \frac{\omega}{2} \left(\psi - \sqrt{\psi^2 - \mathcal{C}^2} \right), \quad (3)$$

where $\Theta(x)$ is the Heaviside step function ($\Theta(x) = 1$ for $x \geq 0$, $\Theta(x) = 0$ otherwise), $\mu[\hat{\rho}] := \text{Tr}[\hat{\rho}^2]$ is the purity, and $\psi := \sqrt{2\mu[\hat{\rho}] - 1}$. Both components are independent of the phase θ ; variations of θ thus generate a trivial family of isoergotropic states. Moreover, $\mathcal{R}^{\text{coh}}(p, \mathcal{C}) = 0$ for incoherent states ($\mathcal{C} = 0$), while $\mathcal{R}^{\text{inc}}(p)$ is nonzero if and only if $p > 1/2$ (notice the Heaviside step function guarantees null incoherent ergotropy for passive incoherent states, i.e., $p \leq 1/2$, and positive values for active ones).

A. Isoergotropic states

Given an active incoherent state $\hat{\rho}_{\bar{p}} = (1 - \bar{p})|g\rangle\langle g| + \bar{p}|e\rangle\langle e|$ with $\bar{p} \in]1/2, 1]$ with ergotropy $\mathcal{R}_{\bar{p}} := \mathcal{R}(\bar{p}, 0) = \mathcal{R}^{\text{inc}}(\bar{p})$, the corresponding isoergotropic set is

$$\mathcal{L}_{\bar{p}} := \{ \hat{\rho}_{\bar{p}} \in \mathcal{L} \mid \mathcal{R}[\hat{\rho}_{\bar{p}}(p, \theta)] = \mathcal{R}_{\bar{p}} \}. \quad (4)$$

A convenient parametrization of the states in $\mathcal{L}_{\bar{p}}$ is

$$\hat{\rho}_{\bar{p}}(p, \theta) = \begin{pmatrix} 1-p & \mathcal{C}_{\bar{p}}(p)e^{i\frac{\theta}{2}/2} \\ \mathcal{C}_{\bar{p}}(p)e^{-i\frac{\theta}{2}/2} & p \end{pmatrix}, \quad (5)$$

where $\mathcal{C}_{\bar{p}}^2(p) = 8(\bar{p} - p)P_{\bar{p}}$ and $P_{\bar{p}} \leq p \leq \bar{p}$, with $P_{\bar{p}} = 2\bar{p} - 1$ and $\theta \in [0, 4\pi]$. As extreme states, one has

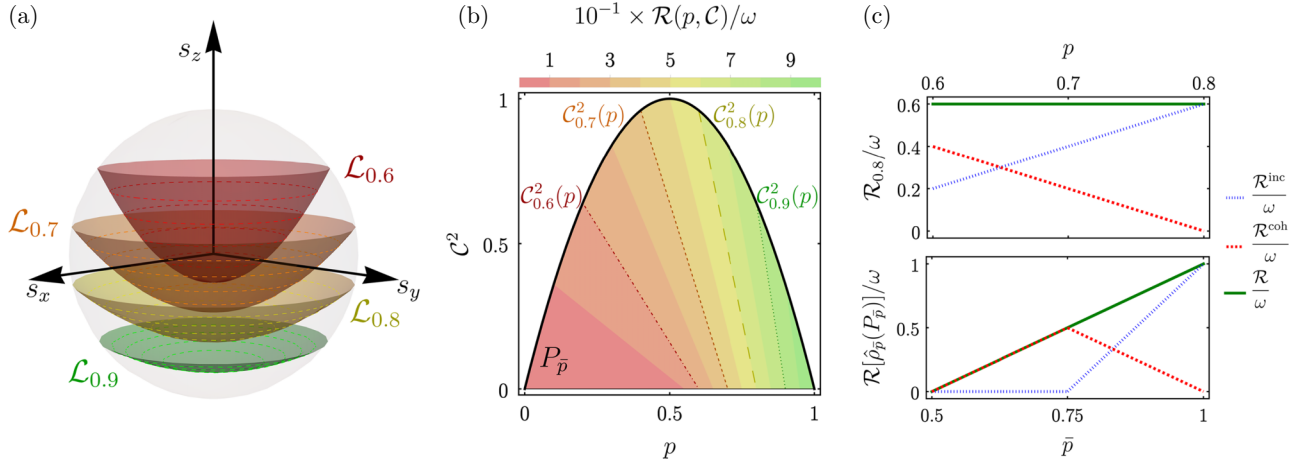


FIG. 2. (a) Isoergotropic surfaces at the Bloch sphere for distinct values of $\bar{p}$. (b) Ergotropy profile in terms of p and $C^2 \in [0, 4p(1-p)]$. The isoergotropic states within $\mathcal{L}_{\bar{p}}$ are represented by the colored (a) surfaces within the Bloch sphere (b) lines $C_{\bar{p}}^2(p) = 8(\bar{p} - p)P_{\bar{p}}$. The black curve in (b) characterizes the isoergotropic pure states $\hat{\rho}_{\bar{p}}(P_{\bar{p}}, \theta) = |\Psi_{\bar{p}, \theta}\rangle\langle\Psi_{\bar{p}, \theta}|$ (surface of the Bloch sphere) relative to the incoherent states $\hat{\rho}_{\bar{p}} = (1-p)|g\rangle\langle g| + p|e\rangle\langle e|$ (at s_z axis). (c) Top: ergotropy distribution between the internal (in)coherent components along the path described by $C_{0.8}^2(p)$. Bottom: ergotropy distribution along the pure states $\hat{\rho}_{\bar{p}}(P_{\bar{p}}, \theta)$ represented by the black curve.

the mixed reference state $\hat{\rho}_{\bar{p}}$ (by construction) for $p = \bar{p}$ and its relative isoergotropic pure state $\hat{\rho}_{\bar{p}}(P_{\bar{p}}, \theta) = |\Psi_{\bar{p}, \theta}\rangle\langle\Psi_{\bar{p}, \theta}|$ for $p = P_{\bar{p}}$, with $|\Psi_{\bar{p}, \theta}\rangle = \sqrt{2(1-\bar{p})}|g\rangle + e^{-i\theta/2}\sqrt{2\bar{p}-1}|e\rangle$. Thus, for each fixed $\bar{p}$, the total ergotropy is redistributed between the incoherent and coherent parts, such that

$$\mathcal{R}_{\bar{p}} = \mathcal{R}^{\text{inc}}(p) + \mathcal{R}^{\text{coh}}(p, C_{\bar{p}}), \quad (6)$$

while remaining constant. Figure 2(a) shows the Bloch sphere representation, where $s_k = \text{Tr}[\hat{\sigma}_k \hat{\rho}_{\bar{p}}(p)]$ with $k = x, y, z$, and the isoergotropic surfaces generated by different values of $\bar{p}$. For each $\bar{p}$, the incoherent states $\hat{\rho}_{\bar{p}}$ lies on the s_z axis, while the isoergotropic pure state $\hat{\rho}_{\bar{p}}(P_{\bar{p}}, \theta)$ lies on the surface of the sphere. Figure 2(b) displays the ergotropy profile on the $p - C^2$ surface; the colored curves correspond to the same isoergotropic sets as in (a), and the black curve at $p = P_{\bar{p}}$ marks the pure states. Finally, Fig. 2(c) depicts how the internal ergotropy splits into coherent and incoherent components along two representative paths: (top) $C_{0.8}^2(p)$ and (bottom) $P_{\bar{p}}$.

1. Thermodynamics

Having identified the isoergotropic set $\mathcal{L}_{\bar{p}}$, we now examine how key thermodynamic quantities vary along it. In this sense, note that, despite having the same ergotropy, different iso- $\mathcal{R}$ states necessarily contain different internal energies, unambiguously identified as [87]

$$\langle \hat{H} \rangle = \text{Tr}[\hat{H} \hat{\rho}_{\bar{p}}(p, \theta)] = \omega p. \quad (7)$$

So, any iso- $\mathcal{R}$ transformation $\hat{\rho}_{\bar{p}}(p, \theta) \mapsto \hat{\rho}_{\bar{p}}(p', \theta')$ induces the following energetic change:

$$\Delta U \equiv \Delta \langle H \rangle = \omega(p' - p). \quad (8)$$

If the Hamiltonian is fixed (no explicit time dependence), no work is performed $W = \text{Tr}[\rho \Delta H] = 0$ [88,89], and the first law of thermodynamics $\Delta U = Q + W$ reduces to

$$Q = \omega(p' - p). \quad (9)$$

Therefore, moving between the iso- $\mathcal{R}$ states requires the supply or release of heat. Notice that, for any $\bar{p}$, states closer to the center of the Bloch sphere are more energetic, i.e., p closer to $\bar{p}$ and the z axis. Given the invariance of ergotropy, this also implies the proportion of energy stored as charge decreases, that is, $\mathcal{R}/\langle \hat{H} \rangle < 1$ for $p < P_{\bar{p}}$. To illustrate this, Fig. 3(a) shows the ratio between charge and internal energy for all values of p and $\bar{p}$. Additionally, Fig. 3(b) depicts the internal energy vs entropy diagram for all isoergotropic states. The von Neumann entropy is written as

$$S_{vN}[\hat{\rho}_{\bar{p}}(p, \theta)] = \mathcal{H}_2(2\bar{p} - p), \quad (10)$$

with $\mathcal{H}_2(x) = -x \ln x - (1-x) \ln(1-x)$ being the binary entropy function, varies across the isoergotropic surfaces. The blue lines highlight the mixed states along the z axis with $p = \bar{p}$, while the black dashed lines show the region with pure states $p = P_{\bar{p}}$ at the surface of the Bloch sphere, with $S_{vN}[\hat{\rho}_{\bar{p}}(P_{\bar{p}}, \theta)] = 0$. Consequently, there are no non-trivial operations that simultaneously conserve ergotropy and are isentropic or energy preserving: along $\mathcal{L}_{\bar{p}}$, for fixed

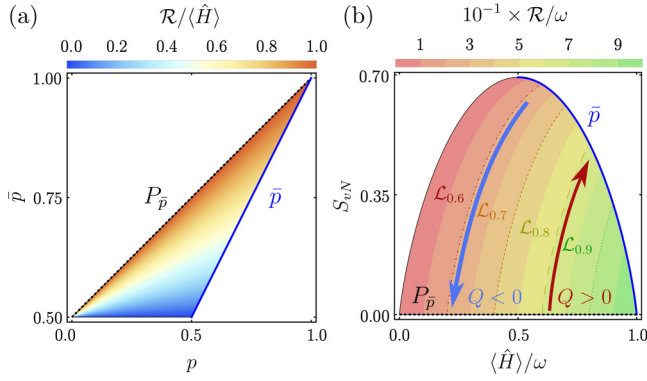


FIG. 3. (a) Ratio $\mathcal{R}/\langle\hat{H}\rangle = (2\bar{p} - 1)/p$ on the $(p, \bar{p})$ plane. For each $\bar{p}$, the ratio decreases monotonically with p from 1 at the pure point $p = P_{\bar{p}}$ to $(2\bar{p} - 1)/\bar{p}$ at the incoherent point $p = \bar{p}$. (b) Energy-entropy diagram on $\mathcal{L}_{\bar{p}}$: $\langle\hat{H}\rangle = \omega p$ and $S_{vN}(p) = \mathcal{H}_2(2\bar{p} - p)$ increase monotonically with p . Arrows indicate heat $Q = \omega(p' - p)$ absorbed (right, red) or released (left, blue) when moving along $\mathcal{L}_{\bar{p}}$. Black dashed: $p = P_{\bar{p}}$ (pure, $S_{vN} = 0$); blue solid: $p = \bar{p}$ (incoherent, $S_{vN} = \mathcal{H}_2(\bar{p})$).

$\hat{H}$, nontrivial iso- $\mathcal{R}$ transformations necessarily involve heat exchange and entropy change; they are not implementable by system-only unitaries, but arise naturally as thermal operations or as reduced dynamics of an energy-conserving global unitary.

B. Ergotropy preserving operations

It is clear that the trade-off between coherent and incoherent components of ergotropy within different states of $\mathcal{L}_{\bar{p}}$ implies that iso- $\mathcal{R}$ operations are, in general, neither incoherent nor purity-preserving (and, as noted above, not entropy-preserving either). In fact, incoherent operations preserve only the incoherent part of ergotropy. Given a state $\hat{\rho}_{\bar{p}}(p, \theta) \in \mathcal{L}_{\bar{p}}$, we call iso- $\mathcal{R}$ operations the CPTP maps (or measurement instruments) that satisfy

$$\mathcal{K} : \mathcal{L}_{\bar{p}} \longrightarrow \mathcal{L}_{\bar{p}}, \quad \hat{\rho}_{\bar{p}}(p', \theta') = \mathcal{K}[\hat{\rho}_{\bar{p}}(p, \theta)], \quad (11)$$

with $p' \in [P_{\bar{p}}, \bar{p}]$. Let $\hat{\rho}_{\bar{p}}(p', \theta') = (1 - p_e) |\psi_g\rangle\langle\psi_g| + p_e |\psi_e\rangle\langle\psi_e|$ be the spectral decomposition of the target state, where $p_e = 2\bar{p} - p'$ and $\{|\psi_g\rangle, |\psi_e\rangle\}$ are its eigenvectors. A convenient Kraus representation $\{\hat{\mathcal{K}}_j\}$, with $\sum_{j=1}^4 \hat{\mathcal{K}}_j^\dagger \hat{\mathcal{K}}_j = \hat{\mathbb{I}}$ of a channel $\mathcal{K}_{\bar{p}, p', \theta'}$ that connects $\hat{\rho}_{\bar{p}}(p, \theta) \rightarrow \hat{\rho}_{\bar{p}}(p', \theta')$ is

$$\begin{aligned} \hat{\mathcal{K}}_1 &= \sqrt{p_e} |\psi_e\rangle\langle\psi_e|, & \hat{\mathcal{K}}_2 &= \sqrt{p_e} |\psi_e\rangle\langle\psi_g|, \\ \hat{\mathcal{K}}_3 &= \sqrt{1 - p_e} |\psi_g\rangle\langle\psi_g|, & \hat{\mathcal{K}}_4 &= \sqrt{1 - p_e} |\psi_g\rangle\langle\psi_e|, \end{aligned} \quad (12)$$

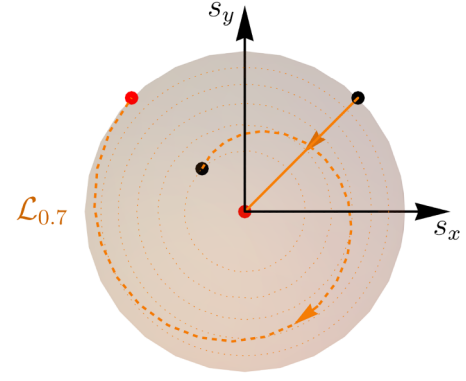


FIG. 4. Isoergotropic operations. The continuous and dashed lines indicate two distinct ergotropy-preserving trajectories along $\mathcal{L}_{0.7}$ from the black to the red points.

such that

$$\hat{\rho}_{\bar{p}}(p', \theta') = \sum_{j=1}^4 \hat{\mathcal{K}}_j \hat{\rho}_{\bar{p}}(p, \theta) \hat{\mathcal{K}}_j^\dagger. \quad (13)$$

Equation (12) has the structure of the generalized amplitude damping channel (GADC) [90]. Equivalently, the same map can be realized as a qubit thermal channel (QTC, or thermal attenuator) [91],

$$\mathcal{E}_{\bar{p}, p', \theta'}[\hat{\rho}] = \text{Tr}_{\text{Aux}} \left[\hat{U} (\hat{\rho} \otimes \hat{\tau}_{\bar{p}}) \hat{U}^\dagger \right], \quad (14)$$

with $\hat{\tau}_{\bar{p}} \equiv \hat{\rho}_{\bar{p}}(p', \theta')$ being an auxiliary state and

$$\hat{U} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (15)$$

a SWAP gate. We highlight the isoergotropic map itself explicitly depends on $\bar{p}$. To illustrate iso- $\mathcal{R}$ operations, Fig. 4 shows two distinct ergotropy-preserving trajectories along $\mathcal{L}_{0.7}$ generated from Eq. (12). The black (red) dots mark the initial (final) states. By choosing the parameters appropriately, any path on the Bloch sphere within $\mathcal{L}_{0.7}$ can be implemented through sequential application of iso- $\mathcal{R}$ maps.

1. Iso- $\mathcal{R}$ realization via POVM

Iso- $\mathcal{R}$ operations can also be realized through selective measurements (POVM elements). A single selective measurement is sufficient to realize an isoergotropic operation that maps an incoherent two-level system active state to any other state within the same isoergotropic family.

Indeed, consider the incoherent active state

$$\hat{\rho}_{\bar{p}} = (1 - \bar{p}) |g\rangle \langle g| + \bar{p} |e\rangle \langle e|, \quad (16)$$

and its corresponding isoergotropic family $\mathcal{L}_{\bar{p}}$. When the desired target is a *pure* member of $\mathcal{L}_{\bar{p}}$, a rank-one selective measurement suffices. Let the single measurement operator be

$$\hat{M} = |\phi_{\text{out}}\rangle \langle \phi_{\text{in}}|, \quad (17)$$

which, acting on $\hat{\rho}_{\bar{p}}$, yields the (unnormalized) postmeasurement state

$$\hat{M} \hat{\rho}_{\bar{p}} \hat{M}^\dagger = \langle \phi_{\text{in}} | \hat{\rho}_{\bar{p}} | \phi_{\text{in}} \rangle |\phi_{\text{out}}\rangle \langle \phi_{\text{out}}|. \quad (18)$$

After renormalization, the output is pure, such that

$$\hat{\rho}' = |\phi_{\text{out}}\rangle \langle \phi_{\text{out}}|. \quad (19)$$

To remain within $\mathcal{L}_{\bar{p}}$, the output state $|\phi_{\text{out}}\rangle$ must be the normalized eigenvector of the target isoergotropic element,

$$|\phi_{\text{out}}\rangle = |\Psi_{\bar{p}}(p, \theta)\rangle = \frac{\left(\frac{C_{\bar{p}}(p)}{2p} e^{i\theta/2} |g\rangle + |e\rangle\right)}{\sqrt{1 + \frac{C_{\bar{p}}^2(p)}{4p^2}}}, \quad (20)$$

so that $\hat{\rho}' = \hat{\rho}_{\bar{p}}(p, \theta)$ is a pure isoergotropic state.

The success probability of this transformation is

$$\mathcal{P}_{\text{succ}}(p) = \langle \phi_{\text{in}} | \hat{\rho}_{\bar{p}} | \phi_{\text{in}} \rangle, \quad (21)$$

which is maximized by choosing $|\phi_{\text{in}}\rangle$ as the eigenvector of $\hat{\rho}_{\bar{p}}$ associated with its largest eigenvalue, i.e. $|e\rangle$. Hence, the optimal Kraus operator reads

$$\hat{M}_{\text{opt}}(p, \theta) = |\Psi_{\bar{p}}(p, \theta)\rangle \langle e|, \quad (22)$$

and the maximal success probability follows as

$$\mathcal{P}_{\text{succ}}^{\text{max}}(p) = \bar{p}. \quad (23)$$

For a general *mixed* target in $\mathcal{L}_{\bar{p}}$, a rank-one operator is no longer sufficient, as it always yields a pure output. Nevertheless, a single selective measurement with a higher-rank Kraus operator can achieve the transformation

in one step. Specifically, consider the ansatz

$$\hat{M} = \sqrt{q} \sqrt{\hat{\rho}_{\bar{p}}(p', \theta')} \hat{\rho}_{\bar{p}}^{-1/2}, \quad (24)$$

with $q \in [0, 1]$ chosen such that $\hat{M}^\dagger \hat{M} \leq \hat{\mathbb{I}}$. Indeed, defining

$$\hat{A} := \hat{\rho}_{\bar{p}}^{-1/2} \hat{\rho}_{\bar{p}}(p', \theta') \hat{\rho}_{\bar{p}}^{-1/2}, \quad (25)$$

which is positive. The necessary and sufficient condition for $\hat{M}^\dagger \hat{M} \leq \hat{\mathbb{I}}$ is

$$q \lambda_{\text{max}}(\hat{A}) \leq 1. \quad (26)$$

By construction,

$$\hat{M} \hat{\rho}_{\bar{p}} \hat{M}^\dagger = q \hat{\rho}_{\bar{p}}(p', \theta'), \quad (27)$$

and the normalized postmeasurement state (upon the success outcome) is precisely the mixed isoergotropic element $\hat{\rho}_{\bar{p}}(p', \theta')$.

For the present incoherent input, one may compute A explicitly

$$\hat{A} = \begin{pmatrix} \frac{1-p'}{1-\bar{p}} & \frac{C_{\bar{p}}(p')}{2\sqrt{(1-\bar{p})\bar{p}}} e^{i\theta'/2} \\ \frac{C_{\bar{p}}(p')}{2\sqrt{(1-\bar{p})\bar{p}}} e^{-i\theta'/2} & \frac{p'}{\bar{p}} \end{pmatrix}. \quad (28)$$

An explicit form for the achievable success probability is

$$q_{\text{max}} = \frac{2}{T + \sqrt{T^2 - 4D}}, \quad (29)$$

where

$$T = \frac{1-p'}{1-\bar{p}} + \frac{p'}{\bar{p}}, \quad D = \frac{(1-p')p' - \left(\frac{C_{\bar{p}}(p')}{2}\right)^2}{(1-\bar{p})\bar{p}}. \quad (30)$$

Finally, if one is allowed to reset the input to $\hat{\rho}_{\bar{p}}$ after any failed attempt, the trials become independent and identical with per-trial success probability q_{max} . In that operational setting the cumulative success probability after N attempts is the geometric (Bernoulli) form

$$\mathcal{P}_{\text{succ}}^{(N)} = 1 - (1 - q_{\text{max}})^N, \quad (31)$$

which tends to unity as $N \rightarrow \infty$. Thus, a single selective measurement suffices to generate *any* state—pure or mixed—within the isoergotropic family $\mathcal{L}_{\bar{p}}$ starting from the incoherent active state $\hat{\rho}_{\bar{p}}$.

C. Autonomous dynamic ergotropy conversion

We now show how iso- $\mathcal{R}$ operations can be engineered autonomously by coupling a quantum battery (B) to an auxiliary system (A). Consider two TLS with total Hamiltonian $\hat{H} = \hat{H}_B + \hat{H}_A + \hat{V}_{AB}$, where $\hat{H}_{A,B} = \omega_{A,B} |e\rangle\langle e|_{A,B}$ and

$$\hat{V}_{AB} = i\eta (\hat{\sigma}_+^B \otimes \hat{\sigma}_-^A - \hat{\sigma}_-^B \otimes \hat{\sigma}_+^A), \quad (32)$$

with coupling strength η . Let us assume $\omega_B \equiv \omega$ and $\omega_A = \omega + \Delta$, with Δ being the detuning between the two subsystems. If the initial local states are $\hat{\rho}_B(0)$ and $\hat{\rho}_A(0)$, then the joint and local states at later times ($t > 0$) are, respectively, written as $\hat{\rho}_{BA}(t) = \hat{U}(t) (\hat{\rho}_B(0) \otimes \hat{\rho}_A(0)) \hat{U}^\dagger(t)$ and $\hat{\rho}_{B(A)}(t) = \text{Tr}_{A(B)}[\hat{\rho}_{BA}(t)]$, with time-evolution operator

$$\hat{U}(t) = e^{-i(\frac{\Delta}{2} + \omega)t} \begin{pmatrix} e^{i(\frac{\Delta}{2} + \omega)t} & 0 & 0 & 0 \\ 0 & \cos(\chi t) - i\frac{\Delta}{2\chi} \sin(\chi t) & \frac{\eta}{\chi} \sin(\chi t) & 0 \\ 0 & -\frac{\eta}{\chi} \sin(\chi t) & \cos(\chi t) + i\frac{\Delta}{2\chi} \sin(\chi t) & 0 \\ 0 & 0 & 0 & e^{-i(\frac{\Delta}{2} + \omega)t} \end{pmatrix}, \quad (33)$$

where $2\chi = \sqrt{\Delta^2 + 4\eta^2}$. Notice that Eq. (33) has the same swap block structure as Eq. (15). Prepare both systems on the same isoergotropic curve $\mathcal{L}_{\bar{p}}$ with $\hat{\rho}_B(0) = \hat{\rho}_{\bar{p}}(P_{\bar{p}}, \theta)$ and $\hat{\rho}_A(0) = \hat{\rho}_{\bar{p}}(\bar{p}, \phi)$. Then the battery state at time t reads

$$\hat{\rho}_B(t) \equiv \hat{\rho}_{\bar{p}}(p_B(t), \theta(t)) \in \mathcal{L}_{\bar{p}}, \quad (34)$$

where

$$p_B(t) := \frac{\Delta^2 P_{\bar{p}} + 2\eta^2(P_{\bar{p}} + \bar{p}) + 2\eta^2(\bar{p} - 1) \cos(2\chi t)}{4\chi^2},$$

$$\theta(t) := \theta_0 + (\Delta + 2\omega)t + 2 \tan^{-1} \left(-\frac{\Delta \sin(\chi t)}{2\chi \cos(\chi t)} \right), \quad (35)$$

i.e., $\hat{\rho}_B(t)$ remains within the same isoergotropic surface for all t . In this sense, Fig. 5(a) shows the total ergotropy dynamics for the battery (black) and auxiliary (colored) TLS assuming $\bar{p} = 0.8$ for different values of the detuning written in terms of the coupling constant, $\Delta = \alpha\eta$, with $\alpha = 0, 1, 2$. It is clear that the battery remains on $\mathcal{L}_{0.8}$ regardless of the value of Δ , whereas the auxiliary generally oscillates with period $\eta T = 2\pi/\sqrt{\alpha^2 + 4}$ and amplitude depending on the detuning: higher detunings induces lower ergotropy oscillations for subsystem A . Figure 5(b) depicts the internal dynamical ergotropy reconfiguration due to the coupling for $\Delta = \eta$ ($\alpha = 1$): the auxiliary internal ergotropies oscillates, while the battery cyclically converts coherent ergotropy into incoherent ergotropy and back. The increase of $\mathcal{R}_B^{\text{inc}}$ is accompanied by the decrease of $\mathcal{R}_B^{\text{coh}}$ with the inverse behavior for A . More importantly, these continuous oscillations guarantees the battery ergotropy is constant while the net auxiliary charge is not. Notice the first minimum (maximum) of

$\mathcal{R}_B^{\text{coh}}$ ($\mathcal{R}_B^{\text{inc}}$) happens at $\eta T_{\min} := \pi/\sqrt{\alpha^2 + 4}$. Hence, to characterize the role of Δ , panels (c), (d), respectively, display the incoherent and coherent components of ergotropy in terms of α computed at $T_{\min}$ for both subsystems: the maximum internal ergotropy reshuffling is obtained for resonant TLS, i.e., $\Delta = 0$ and $\chi \equiv \eta$. This also implies that high detuning suppresses the internal reconfiguration of ergotropy and the charge oscillation of A .

Although the total ergotropy of the composite is conserved because $[\hat{U}(t), \hat{H}] = 0$, the observed local decrease of auxiliary ergotropy can be seen as the internal energetic cost of the conversion on the battery. It is important to mention that, in realistic settings, the interaction strength η and the detuning Δ can be modulated in time as knobs to tailor this conversion. If only *local* ergotropies are accessible, the full energetic cost comprises both the auxiliary's ergotropy consumption and the control required to modulate these parameters. If *global* access to the composite ergotropy is available, only the external control cost remains relevant.

To better illustrate the autonomous internal ergotropy reconfiguration, Figs. 6(a) and 6(b) show the induced trajectories on the Bloch sphere for resonant ($\Delta = 0$) battery and auxiliary TLS (for several $\bar{p}$ and $\omega = \eta = 1$). Similarly to Fig. 5, panels (c), (d) display the dynamics of total ergotropy and its coherent and incoherent components for both subsystems with $\bar{p} = 0.8$, showcasing the same behavior described earlier. Finally, Fig. 7 further reports (a) ergotropy and (b) the von Neumann entropies S_{vN} and the mutual information $\mathcal{I} = S_{vN}[\hat{\rho}_B] + S_{vN}[\hat{\rho}_A] - S_{vN}[\hat{\rho}_{BA}]$ during the interaction for $\bar{p} = 0.8$ and $\Delta = 0$. The auxiliary's ergotropy oscillations are in resonance with changes in local entropies. The growth of mutual information signals the buildup of correlations, accompanying a decrease (increase) of the battery's coherent ergotropy (the auxiliary's ergotropy).

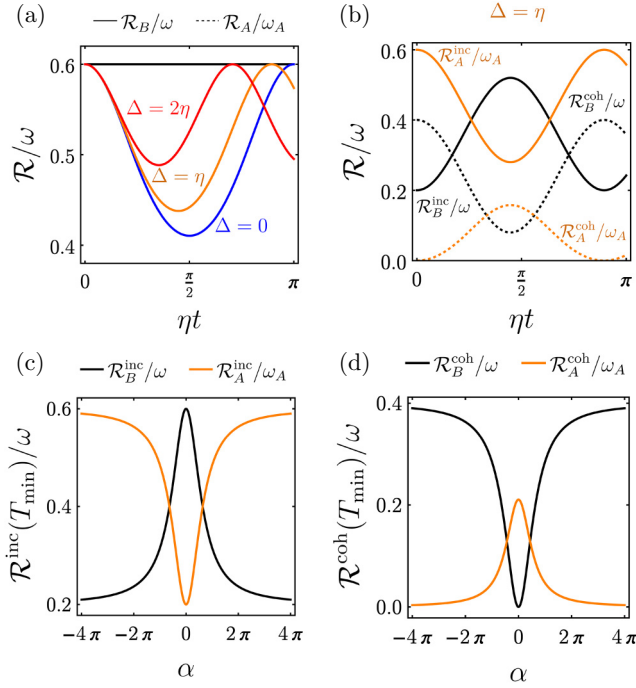


FIG. 5. (a) Battery (black) and auxiliary (colored) TLS ergotropy dynamics induced by $\hat{\mathcal{U}}(t)$, assuming $\hat{\rho}_B(0) = \hat{\rho}_{\bar{p}}(P_{\bar{p}}, \theta)$ and $\hat{\rho}_A(0) = \hat{\rho}_{\bar{p}}(\bar{p}, \phi)$. The ergotropy curves of the auxiliary system A for different values of detuning $\Delta := \omega_A - \omega \equiv \alpha\eta$ are represented. (b) Incoherent (continuous) and coherent (dashed) ergotropy dynamics for B (black) and A (orange) with $\Delta = \eta$ ($\alpha = 1$). Bottom: local (c) incoherent and (d) coherent ergotropy computed at $\eta T_{\min} = \pi/\sqrt{\alpha^2 + 4}$ for distinct values of α . It was assumed $\bar{p} = 0.8$.

D. Extension

The scalability of energy storage is also of fundamental importance for practical applications of quantum batteries. Naturally, one can extend the iso- $\mathcal{R}$ analysis to multicell batteries composed of several TLSs and identify suitable ergotropy-preserving operations implemented by local or global transformations. In this case, the total ergotropy is not only split into the internal degrees-of-freedom of each subsystem but also spread across the different cells.

To illustrate this, let us consider a quantum battery consisting of n uncorrelated and noninteracting cells, such that

$$\hat{\tau}_n = \hat{\rho}_1 \otimes \dots \otimes \hat{\rho}_j \otimes \dots \otimes \hat{\rho}_k \otimes \dots \otimes \hat{\rho}_n, \quad (36)$$

with $\hat{\rho}_k = \hat{\rho}_{\bar{p}_k}(p_k, \theta_k) \in \mathcal{L}_{\bar{p}_k}$ being the state of the k th cell with local ergotropy $\mathcal{R}[\hat{\rho}_k] \equiv \mathcal{R}_{\bar{p}_k}$. It is important to mention that, in general, the global ergotropy $\mathcal{R}[\hat{\tau}_n]$ is nonadditive (even for noninteracting systems), which highlight the nontrivial behavior of this quantity. Now, let us define the set of isoergotropic states relative $\hat{\tau}_n$ as

$$\mathcal{L}_n := \{\hat{\rho} \in \mathcal{L} | \mathcal{R}[\hat{\rho}] = \mathcal{R}[\hat{\tau}_n]\}. \quad (37)$$

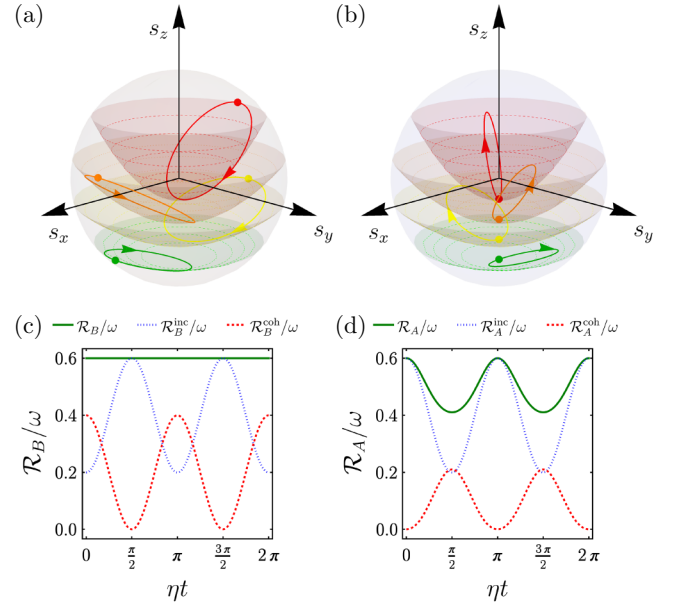


FIG. 6. Top: Bloch sphere representation of the (a) battery and (b) auxiliary resonant ($\Delta = 0$) TLS dynamics induced by $\hat{\mathcal{U}}(t)$, assuming $\hat{\rho}_B(0) = \hat{\rho}_{\bar{p}}(P_{\bar{p}}, \theta)$ and $\hat{\rho}_A(0) = \hat{\rho}_{\bar{p}}(\bar{p}, \phi)$ (colored dots) as initial states, and $\omega = \eta = 1$. The isoergotropic curves $\mathcal{L}_{\bar{p}}$ for different values of $\bar{p}$ are represented. Bottom: ergotropy and internal components dynamics for the (c) battery and (d) auxiliary system for $\bar{p} = 0.8$.

In principle, one can consider maps acting on the whole system or any subset of d cells, with $d < n$. As a representative example for two-cell operations, note that a permutation exchanging the states of cells $j \leftrightarrow k$, such that

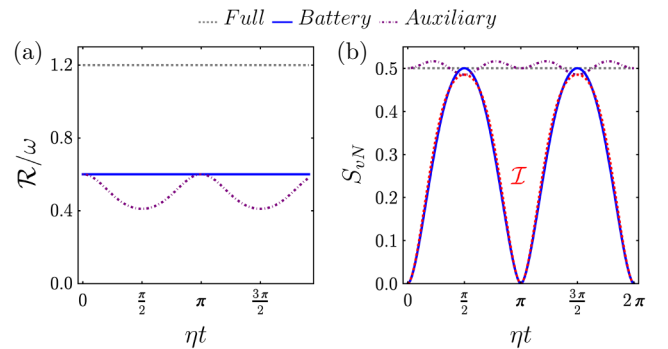


FIG. 7. (a) Charge and (b) von Neumann entropy dynamics induced by $\hat{\mathcal{U}}(t)$ for $\bar{p} = 0.8$ and $\Delta = 0$. The entropy and ergotropies of the full system (computed without the interaction, i.e., $\hat{H}_A + \hat{H}_B$) and the battery are invariant during the time evolution. The ergotropy of the auxiliary system oscillation is accompanied by the conversion of coherent ergotropy to incoherent. Such oscillation is also observed by the local entropies S_{vN} and mutual information $\mathcal{I}$.

$\hat{Q}_j \leftrightarrow \hat{Q}_k$ and described by

$$\hat{V}_{j \leftrightarrow k} \hat{\tau}_n \hat{V}_{j \leftrightarrow k}^\dagger = \hat{Q}_1 \otimes \dots \otimes \hat{Q}_k \otimes \dots \otimes \hat{Q}_j \otimes \dots \otimes \hat{Q}_n, \quad (38)$$

swaps the ergotropy of the pair of cells while maintaining the total ergotropy invariant, i.e., $\mathcal{R}[\hat{V}_{j \leftrightarrow k} \hat{\tau}_n \hat{V}_{j \leftrightarrow k}^\dagger] = \mathcal{R}[\hat{\tau}_n]$. In fact, any permutation of local states, characterized by the map $\mathcal{K}_z[\cdot] := \hat{V}_z \cdot \hat{V}_z^\dagger$ with z being the element of a symmetric group S_n , belongs to a particular subset of iso- $\mathcal{R}$ operations, since

$$\mathcal{K}_z[\hat{\tau}_n] \in \mathcal{L}_n, \quad \forall z. \quad (39)$$

Observe that if all individual cells belongs to the same isoergotropic set $\mathcal{L}_{\bar{p}}$, i.e., $\bar{p}_k = \bar{p}$ for all k , then $\mathcal{K}_z[\cdot]$ is simultaneously locally and globally ergotropy preserving, since both the individual cells and full ergotropies are preserved.

The complexity of this analysis considerably increases if one also consider correlated states, which opens the possibility of finding iso- $\mathcal{R}$ entangled states. As a simple and relevant two-cell example, consider the family of X states with Hamiltonian $\hat{H} = \omega |e\rangle\langle e|_1 \otimes \hat{1}_2 + \hat{1}_1 \otimes \omega |e\rangle\langle e|_2$ and a single scalar parameter $q \in [0, 1]$ [73,92,93]:

$$\hat{\rho}_X(q) = \begin{pmatrix} q/2 & 0 & 0 & q^2 - q/2 \\ 0 & q(1-q) & 0 & 0 \\ 0 & 0 & (1-q)^2 & 0 \\ q^2 - q/2 & 0 & 0 & q/2 \end{pmatrix}. \quad (40)$$

Equation (40) can be obtained as the output of an optimal unitary applied to two Gibbs states that maximizes quantum correlations [93]—including entanglement and discord [92]—and local uncertainty. For this X family, the concurrence is

$$C(q) = \max \left\{ 0, 2q^2 - q - 2(1-q)\sqrt{q(1-q)} \right\}. \quad (41)$$

While entanglement vanishes at a finite critical temperature [92,94,95], other quantum signatures decay gradually with temperature [92,93]. The ergotropy within the bipartition is

$$\mathcal{R}(q) = \omega|1 - 2q|, \quad \mathcal{R}^{\text{inc}}(q) = \omega|(q - 2)(q - \frac{1}{2})|, \quad (42)$$

hence $\mathcal{R}(q) = \mathcal{R}(1 - q)$ for all q . Thus, $\hat{\rho}_X(q)$ and $\hat{\rho}_X(1 - q)$ are isoergotropic, even though their entanglement may differ significantly [73]. Figure 8 (top) highlights the symmetric ergotropy profile and the split between coherent and incoherent contributions; Fig. 8 (bottom) shows

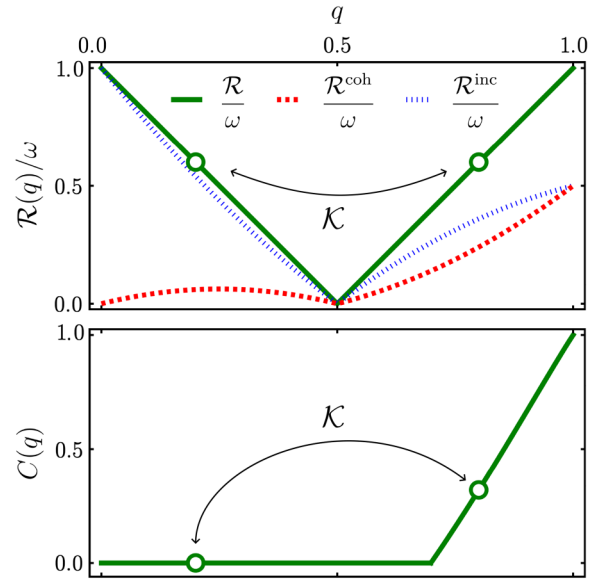


FIG. 8. Top: ergotropy profile as a function of q . The total ergotropy satisfies $\mathcal{R}(q) = \mathcal{R}(1 - q)$ for all q ; the dotted curves show the coherent and incoherent components. Bottom: concurrence $C(q)$. Every uncorrelated state with $0 \leq q \leq 0.3$ has a correlated isoergotropic partner with $0.7 \leq q \leq 1$. The arrow illustrates an isoergotropic operation $\mathcal{K}$ mapping $q \mapsto 1 - q$.

the concurrence, including entanglement sudden death. In particular, every uncorrelated state with $0 \leq q \leq 0.3$ has a correlated isoergotropic partner with $0.7 \leq q \leq 1$. A notable pair is the pure product $\hat{\rho}_X(0)$ and the maximally entangled $\hat{\rho}_X(1)$: both have the same ergotropy, with $\mathcal{R}^{\text{inc}}(1) = \mathcal{R}^{\text{coh}}(1) = \omega/2$. Moreover, one finds that the internal energy is constant for all q while the von Neumann entropy is symmetric about $q = \frac{1}{2}$ (with a maximum at $q = \frac{1}{2}$): $\langle \hat{H} \rangle = \text{Tr}[\hat{H} \hat{\rho}_X(q)] = \omega$ and $S_{vN}(q) = S_{vN}(1 - q)$. Consequently, the fraction of energy stored as charge decreases near $q = 1/2$ and for the isoergotropic transformation $q \mapsto 1 - q$ one has $\Delta U = \Delta S_{vN} = 0$ and $W = -Q$.

An explicit iso- $\mathcal{R}$ operation connecting the partners can be written as

$$\mathcal{K}[\hat{\rho}_X(q)] = \hat{V} \text{Tr}_{\text{Aux}} \left[\hat{U} (\hat{\rho}_X(q) \otimes \hat{\tau}_{\text{Aux}}) \hat{U}^\dagger \right] \hat{V}^\dagger, \quad (43)$$

where $\hat{U}$ is the $4^2 \times 4^2$ SWAP, the auxiliary state is $\hat{\tau}_{\text{Aux}} = \hat{\tau}_1 \otimes \hat{\tau}_2$ with $\hat{\tau}_k = (1 - q)|g\rangle\langle g|_k + q|e\rangle\langle e|_k$, and $\hat{V} = \hat{v}_2 \hat{v}_1$ with

$$\begin{aligned} \hat{v}_1 &= |gg\rangle\langle gg| + |ge\rangle\langle ge| + |ee\rangle\langle ee| + |eg\rangle\langle eg|, \\ \hat{v}_2 &= |\phi_+\rangle\langle gg| + |ge\rangle\langle ge| + |eg\rangle\langle eg| + |\phi_-\rangle\langle ee|, \end{aligned} \quad (44)$$

and $|\phi_\pm\rangle = (|gg\rangle \pm |ee\rangle)/\sqrt{2}$. This construction realizes the isoergotropic mapping $q \mapsto 1 - q$ by a thermal auxiliary system plus a global unitary and a fixed postprocessing unitary $\hat{V}$. Note that the mapping $q \mapsto 1 - q$ cannot

be implemented deterministically by local operations and classical communications (LOCC) [96] for $q \in (0, 1)$: for $q < \frac{1}{2}$ it generally increases entanglement [e.g., the concurrence of $\hat{\rho}_X(q)$ is smaller than that of $\hat{\rho}_X(1 - q)$], which is forbidden under LOCC; accordingly, our construction employs a thermal ancilla and a global unitary.

In this sense, it is natural to consider isoergotropic transformations according to the LOCC paradigm for work extraction in correlated systems similar to the framework proposed in Ref. [97], where local heat engines can act on each subsystem and classical communication is allowed. In multicell batteries, an operational question is how much of the global ergotropy can be converted into locally extractable work if one restricts to LOCC, without implementing global entangling unitaries across the cells. This question is naturally connected to isoergotropic operations: while isoergotropic operations preserve the total ergotropy and hence move the state within a fixed isoergotropic level set, imposing LOCC further restricts which redistributions within that level set are physically accessible under local control. In particular, even at fixed total ergotropy, nonlocal degrees of freedom may prevent unlocking the work content into local ergotropy.

To formalize this, we define the locally distillable ergotropy

$$W_{\text{LOCC}}[\hat{\rho}] = \sup_{\Lambda \in \text{LOCC}} \sum_k p_k [\mathcal{R}_A[\hat{\rho}_{A|k}] + \mathcal{R}_B[\hat{\rho}_{B|k}]], \quad (45)$$

where Λ denotes an LOCC protocol (local measurements, classical communication, and conditional local unitaries), producing outcomes k with probabilities p_k and conditional local states $\hat{\rho}_{A|k}$ and $\hat{\rho}_{B|k}$. Here $\mathcal{R}_A$ and $\mathcal{R}_B$ denote ergotropy with respect to the local Hamiltonians of the cells, and $\hat{\rho}_A = \text{Tr}_B[\hat{\rho}]$, $\hat{\rho}_B = \text{Tr}_A[\hat{\rho}]$ are the reduced states. By construction, W_{LOCC} satisfies the general bounds

$$\mathcal{R}_A[\hat{\rho}_A] + \mathcal{R}_B[\hat{\rho}_B] \leq W_{\text{LOCC}}[\hat{\rho}] \leq \mathcal{R}[\hat{\rho}], \quad (46)$$

since local unitaries are contained in LOCC, while LOCC is a subset of global operations used in the definition of $\mathcal{R}[\hat{\rho}]$. The associated LOCC ergotropy deficit,

$$\Delta_{\text{LOCC}}[\hat{\rho}] = \mathcal{R}[\hat{\rho}] - W_{\text{LOCC}}[\hat{\rho}] \geq 0, \quad (47)$$

quantifies the portion of work content that is intrinsically nonlocal in the sense that it cannot be unlocked into local ergotropy by LOCC alone, even though it contributes to the total ergotropy that is preserved under global isoergotropic operations. Determining tight characterizations of W_{LOCC} and Δ_{LOCC} is an interesting open direction; a concrete near-term program is to (i) obtain analytic expressions and/or tight bounds for two-qubit X states, (ii) compute numerical bounds for small systems via relaxations of LOCC (e.g., to separable operations), and (iii)

extend the analysis to multimode Gaussian batteries under local Gaussian operations and classical communication.

From an experimental standpoint, implementing LOCC distillation requires high-fidelity local readout, fast feed-forward conditioned on measurement outcomes, and a consistent phase and energy reference to define and calibrate the local work-extraction unitaries. In practice, the full LOCC cycle must be completed within the coherence time of the cells and under realistic constraints such as measurement inefficiency, classical communication latency, and control cross-talk. These considerations clarify when LOCC can be a competitive alternative to global entangling extraction, and they provide a direct bridge to platform-specific implementations (e.g., local projective readout with conditional single-qubit control, or local homodyne detection with classical feedforward for bosonic modes).

From a thermodynamic perspective, the coupling to the auxiliary system implements a heat exchange $Q = \omega(2q - 1)$, while the subsequent unitary $\hat{V}$ performs coherent work $W = -Q$, so that the net energy change vanishes: $\Delta U = Q + W = 0$. Moreover, the initial and final von Neumann entropies coincide, $S_{vN}[\hat{\rho}_X(q)] = S_{vN}[\hat{\rho}_X(1 - q)]$, so the transformation $q \mapsto 1 - q$ is entropy preserving overall. It is worth mentioning that in the SWAP step, the heat exchanged between the composite system and the auxiliary does not go to the correlations inside the composite system since we have no interaction within them: $\langle H \rangle = \langle H_1 \rangle + \langle H_2 \rangle$. This heat is accounted for entirely by changes in the local energies.

As for the local ergotropies, the bipartite state in Eq. (40), the reduced states $\hat{\rho}_k(q) = \text{Tr}_{\bar{k}}[\hat{\rho}_X(q)]$ ($k = 1, 2$; $\bar{k} = 2, 1$) are diagonal with $p_1(q) = 1 - \frac{3}{2}q + q^2$ and $p_2(q) = \frac{3}{2}q - q^2$ being the excited-state populations of the subsystems 1, 2, respectively. Figure 9 plots $\mathcal{R}_1$ (black), $\mathcal{R}_2$ (orange), and $\mathcal{R}$ (green): for $q < \frac{1}{2}$ only cell 2 carries charge; for $q > \frac{1}{2}$ only cell 1 does. The total ergotropy $\mathcal{R}(q) = \omega|1 - 2q| \geq \mathcal{R}_1(q) + \mathcal{R}_2(q)$, the excess being due to correlations and coherences. Under the isoergotropic mapping $q \mapsto 1 - q$, each cell exchanges heat equal to its energy change (no local work), giving

$$Q_1 = \omega \left(q - \frac{1}{2} \right), \quad Q_2 = -Q_1. \quad (48)$$

Hence, for $q > \frac{1}{2}$ cell 1 absorbs heat ($Q_1 > 0$) while cell 2 releases heat ($Q_2 < 0$); the roles reverse for $q < \frac{1}{2}$. The global energy is constant and the total ergotropy is preserved.

IV. BOSONIC GAUSSIAN STATES

Consider a single-mode bosonic field of frequency ω with Hamiltonian $\hat{H} = \omega(\hat{a}^\dagger \hat{a} + \frac{1}{2})$, where $\hat{a}$ and $\hat{a}^\dagger$ satisfy $[\hat{a}, \hat{a}^\dagger] = 1$. Assume the system is in a Gaussian state

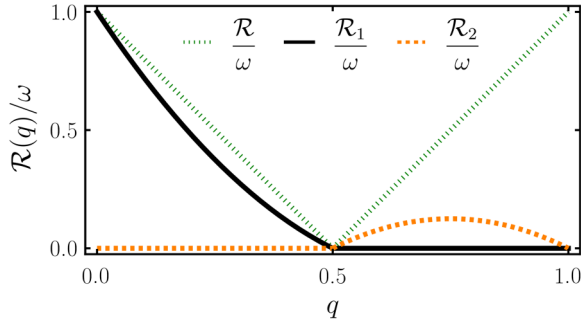


FIG. 9. Ergotropy profile as a function of q for cell 1 (black) and 2 (orange). The total ergotropy is shown in green.

of the form

$$\hat{\rho} = \hat{D}(\mu) \hat{S}(\xi) \hat{\pi}(N) \hat{S}^\dagger(\xi) \hat{D}^\dagger(\mu), \quad (49)$$

where $\hat{\pi}(N)$ is a thermal state with occupation $N = (e^{\beta\omega} - 1)^{-1}$, $\hat{D}(\mu) = \exp(\mu\hat{a}^\dagger - \mu^*\hat{a})$ is the displacement operator, and $\hat{S}(\xi) = \exp[\frac{1}{2}(\xi\hat{a}^{\dagger 2} - \xi^*\hat{a}^2)]$ is the squeezing operator, with complex parameters $\mu = |\mu|e^{i\theta}$ and $\xi = |\xi|e^{i\phi}$. Any single-mode Gaussian state admits the decomposition in Eq. (49) [98]. Equivalently, Gaussian states are fully characterized by first and second moments: the mean vector $\mathbf{d} = (\langle\hat{a}\rangle, \langle\hat{a}^\dagger\rangle)^T = (\mu, \mu^*)^T$ and the covariance matrix $\mathbf{\Theta} = (N + \frac{1}{2})\mathbf{F}(\xi, \phi)$, with

$$\mathbf{F}(\xi, \phi) = \begin{pmatrix} \cosh(2|\xi|) & e^{i\phi} \sinh(2|\xi|) \\ e^{-i\phi} \sinh(2|\xi|) & \cosh(2|\xi|) \end{pmatrix}. \quad (50)$$

A convenient phase-space representation is given by the Wigner function, defined as

$$W(\alpha) = \frac{1}{\pi\sqrt{|\mathbf{\Theta}|}} \exp\left[-\frac{1}{2}(\alpha - \mathbf{d})^\dagger \mathbf{\Theta}^{-1}(\alpha - \mathbf{d})\right], \quad (51)$$

with $\alpha = (\alpha, \alpha^*)$ and $|\mathbf{\Theta}| = \det \mathbf{\Theta}$. Both displacement and squeezing contribute independently to the total ergotropy, so the stored charge can be parameterized by (μ, ξ, N) as $\mathcal{R}(\mu, \xi, N) = \mathcal{R}^d(\mu) + \mathcal{R}^s(\xi, N)$, with [81,84]

$$\begin{aligned} \mathcal{R}^d(\mu) &= \omega \frac{|\mathbf{d}|^2}{2} = \omega |\mu|^2, \\ \mathcal{R}^s(\xi, N) &= \omega \left(\frac{\text{Tr} \mathbf{\Theta}}{2} - \sqrt{|\mathbf{\Theta}|} \right) = \omega \left(N + \frac{1}{2} \right) \\ &\quad \times [\cosh(2|\xi|) - 1]. \end{aligned} \quad (52)$$

Note that N enters only the squeezing contribution. Moreover, Eq. (52) is independent of the phases θ and ϕ , which provides a trivial class of iso- $\mathcal{R}$ states.

A. Isoergotropic states

Given an active Gaussian state of the form $\hat{\rho}_{\bar{\mu}} = \hat{D}(\bar{\mu}) \hat{\pi}(0) \hat{D}^\dagger(\bar{\mu})$, with ergotropy $\mathcal{R}_{\bar{\mu}} := \mathcal{R}(\bar{\mu}, 0, 0) = \mathcal{R}^d(\bar{\mu})$, the set of isoergotropic Gaussian states relative to $\hat{\rho}_{\bar{\mu}}$ is written as

$$\mathcal{L}_{|\bar{\mu}|^2} := \{ \hat{\rho}_{\bar{\mu}} \in \mathcal{L} \mid \mathcal{R}[\hat{\rho}_{\bar{\mu}}(\mu, \xi, N)] = \mathcal{R}_{\bar{\mu}} \}, \quad (53)$$

for $0 \leq |\mu| \leq |\bar{\mu}|$, $|\xi| \geq 0$ and $N \geq 0$. These states can be written as $\hat{\rho}_{\bar{\mu}}(\mu, \xi, N) = \hat{D}(\mu) \hat{S}(\xi) \hat{\pi}(N) \hat{S}^\dagger(\xi) \hat{D}^\dagger(\mu)$, where $\mu = \sqrt{f_{\bar{\mu}}(\xi, N)} e^{i\theta}$ with

$$f_{\bar{\mu}}(\xi, N) = |\bar{\mu}|^2 - \left(N + \frac{1}{2} \right) [\cosh(2|\xi|) - 1], \quad (54)$$

and (ξ, N) satisfying $f_{\bar{\mu}}(\xi, N) \geq 0$. Figure 10 shows the ergotropy profile in terms of parameters (μ, ξ, N) . In (a), the isoergotropic surfaces $\mathcal{L}_1$, $\mathcal{L}_{2.5}$, and $\mathcal{L}_5$ are highlighted. Figure 10(b) shows the surface $N = 1/2$. The dashed lines indicate the surfaces shown in (a). The bottom part of Fig. 10(c) presents the internal ergotropy distribution along the iso-ergotropic curve represented by $f_5(\xi, 1/2)$. The black vertical lines highlight the points where (i) $\mathcal{R}^s = 0$, (ii) $\mathcal{R}^d = \mathcal{R}^s$ and (iii) $\mathcal{R}^d = 0$. The Wigner function $W(\alpha)$ for each of these cases is presented in the plots at the top.

1. Thermodynamics

As in the TLS case, isoergotropic states on the same level set can differ in their thermodynamic properties. For single-mode Gaussian states with Hamiltonian $\hat{H} = \omega(\hat{a}^\dagger \hat{a} + \frac{1}{2})$, the internal energy reads

$$\begin{aligned} \langle \hat{H} \rangle &= \text{Tr} \left[\hat{H} \hat{\rho}_{\bar{\mu}}(\mu, \xi, N) \right] = \mathcal{R}^s(\xi, N) + \mathcal{R}^d(\mu) \\ &\quad + \omega \left(N + \frac{1}{2} \right). \end{aligned} \quad (55)$$

Along an iso- $\mathcal{R}$ manifold $\mathcal{L}_{|\bar{\mu}|^2}$ the total ergotropy $\mathcal{R}_{\bar{\mu}} = \mathcal{R}^s + \mathcal{R}^d$ is fixed, so

$$\langle \hat{H} \rangle = \mathcal{R}_{\bar{\mu}} + \omega \left(N + \frac{1}{2} \right),$$

which is independent of the squeezing parameter ξ . Consequently, for fixed $|\bar{\mu}|^2$ the ratio of charge to energy depends only on the thermal component:

$$\frac{\mathcal{R}_{\bar{\mu}}}{\langle \hat{H} \rangle} = \frac{\mathcal{R}_{\bar{\mu}}}{\mathcal{R}_{\bar{\mu}} + \omega \left(N + \frac{1}{2} \right)}.$$

Figure 11(a) shows this behavior on $\mathcal{L}_5$ as a function of N and ξ ; the green dashed curve marks the zero set of $f_{\bar{\mu}}(\xi, N)$. We quantify mixedness by the Rényi-2 entropy $S_2(\rho) = (1 - 2)^{-1} \ln \text{Tr}[\rho^2]$ [99], for which

$$S_2[\hat{\rho}_{\bar{\mu}}(\mu, \xi, N)] = \ln(1 + 2N), \quad (56)$$

i.e., S_2 depends only on N and is invariant under displacement and squeezing. Figure 11(b) plots the $(\langle \hat{H} \rangle, S_2, \mathcal{R}_{\bar{\mu}})$

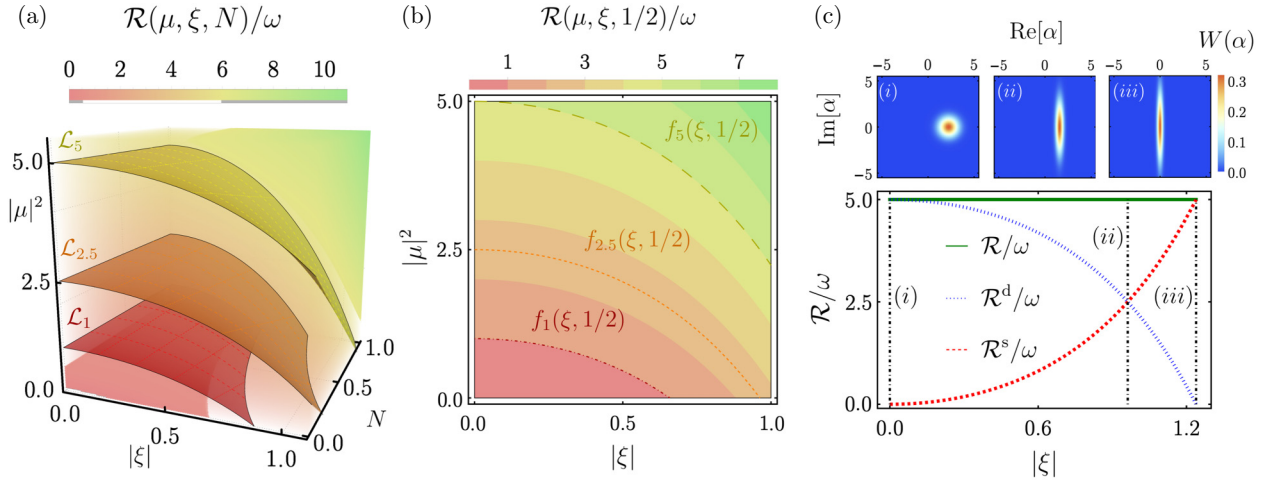


FIG. 10. (a) Ergotropy profile. The isoergotropic surfaces $\mathcal{L}_{|\bar{\mu}|^2}$ are shown for different values of $\bar{\mu}$. (b) Ergotropy profile in terms of $|\mu|^2 - |\xi|$ for $N = 1/2$. The colored lines highlight the iso- $\mathcal{R}$ curves $f_{\bar{\mu}}(\xi, 1/2)$ for the same values shown in (a). (c) Bottom: ergotropy distribution along the isoergotropic curve $f_5(\xi, 1/2)$. The black vertical lines at (i) $|\xi| = 0$, (ii) $|\xi| \approx 0.96$ and (iii) $|\xi| \approx 1.24$ highlight the points where (i) $\mathcal{R}^s = 0$, (ii) $\mathcal{R}^d = \mathcal{R}^s$ and (iii) $\mathcal{R}^d = 0$, respectively. Top: Wigner function $W(\alpha)$ for each of the cases mentioned, assuming $\theta = 0$ and $\phi = \pi$.

diagram for different N and $\bar{\mu}$; black (blue) curves correspond to $N = 0$ ($N = 1$), and the red line highlights $\bar{\mu} = 0$ (no ergotropy: all energy and entropy is thermal). Differentiating Eq. (55) and using $d\mathcal{R}^s + d\mathcal{R}^d = 0$ on $\mathcal{L}_{|\bar{\mu}|^2}$ yields

$$d\langle \hat{H} \rangle = \omega dN.$$

With $\hat{H}$ fixed (no explicit time dependence) the first law gives $W = 0$ and therefore

$$Q = \Delta U = \omega \Delta N.$$

Hence, reshuffling ergotropy between displacement and squeezing at fixed N is thermodynamically neutral on the subsystem:

$$dN = 0 \Rightarrow Q = 0, \quad dS_2 = 0, \quad d\mathcal{R}_{\bar{\mu}} = 0.$$

Only changes in the thermal occupancy N carry heat ($Q = \omega dN$) and entropy [$dS_2 = 2 dN / (1 + 2N)$ under Eq. (56)]. Figure 11(c) shows fixed- N lines on $\mathcal{L}_5$; the yellow curve is an isothermal trajectory ($N = 0.5$). Moving from the black to the red point along this line leaves $\langle \hat{H} \rangle$, S_2 , and $\mathcal{R}_{\bar{\mu}}$ unchanged, while the internal composition $\mathcal{R}^s \leftrightarrow \mathcal{R}^d$ is reconfigured, as illustrated in Fig. 11(d).

B. Ergotropy-preserving operations

Restricting to Gaussian states, we likewise restrict to Gaussian iso- $\mathcal{R}$ operations. Given an initial state $\hat{\rho}_{\bar{\mu}}(\mu, \xi, N)$, an iso- $\mathcal{R}$ Gaussian map $\mathcal{K} : \mathcal{L}_{|\bar{\mu}|^2} \rightarrow \mathcal{L}_{|\bar{\mu}|^2}$

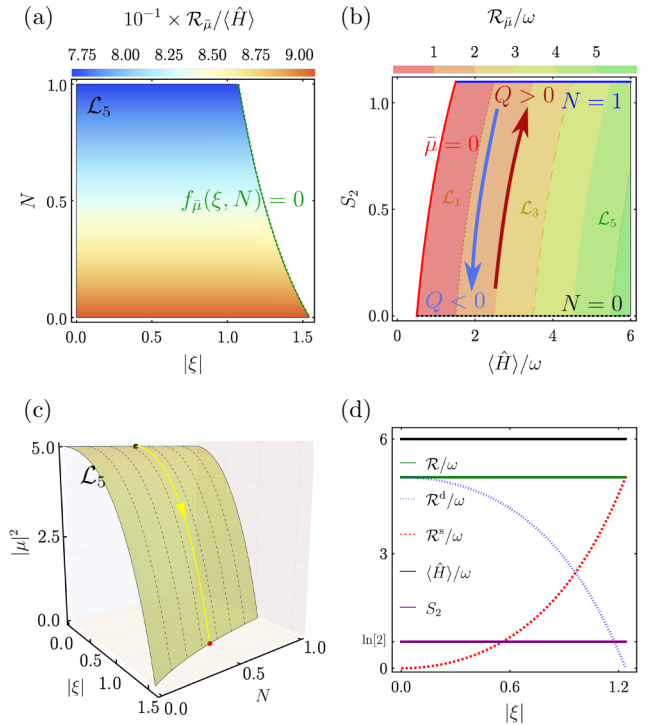


FIG. 11. (a) Charge-energy ratio on $\mathcal{L}_5$: $\mathcal{R}_{\bar{\mu}} / \langle \hat{H} \rangle = \mathcal{R}_{\bar{\mu}} / (\mathcal{R}_{\bar{\mu}} + \omega(N + \frac{1}{2}))$. It decreases with N (purely thermal energy) and is independent of ξ . Green dashed: $f_{\bar{\mu}}(\xi, N) = 0$. (b) Internal energy vs Rényi-2 entropy vs $\mathcal{R}_{\bar{\mu}}$ for several N and $\bar{\mu}$; black (blue): $N = 0$ ($N = 1$). Red: $\bar{\mu} = 0$ (no ergotropy). Arrows indicate heat $Q = \omega \Delta N$ absorbed (right, red) or released (left, blue) when moving along $\mathcal{L}_{|\bar{\mu}|^2}$. (c) Fixed- N lines on $\mathcal{L}_5$; yellow: isothermal path ($N = 0.5$). (d) Along fixed- N paths, $Q = \Delta S_2 = 0$ and $\langle \hat{H} \rangle$ and $\mathcal{R}_{\bar{\mu}}$ are invariant while $\mathcal{R}^s$ and $\mathcal{R}^d$ reshuffle.

produces

$$\hat{\rho}_{\bar{\mu}}(\mu', \xi', N') = \mathcal{K}[\hat{\rho}_{\bar{\mu}}(\mu, \xi, N)], \quad (57)$$

with total ergotropy $\mathcal{R}_{\bar{\mu}}$ unchanged. Gaussian channels are fully specified by their action on the first moments $\mathbf{d}$ and covariance matrix Θ [85,100]. Therefore, ergotropy invariance under $(\mu, \xi, N) \mapsto (\mu', \xi', N')$ is guaranteed by

$$\Theta \mapsto \Theta' = (N' + \frac{1}{2}) \mathbf{F}(\xi', \phi'), \quad \mathbf{d} \mapsto \mathbf{d}' = (\mu', \mu'^*)^T, \quad (58)$$

with the constraint $\mu' = \sqrt{f_{\bar{\mu}}(\xi', N')} e^{i\theta'}$, so that $\mathcal{R}^d(\mu') + \mathcal{R}^s(\xi', N') = \mathcal{R}_{\bar{\mu}}$. Operationally, such a map can be realized as (i) a thermalization step that sets the target occupation N' , followed by (ii) a squeezing $\hat{S}(\xi')$ and (iii) a displacement $\hat{D}(\mu')$. The first step may be implemented by a single-mode thermal attenuator of transmissivity η , for which

$$\mathbf{d} \mapsto \sqrt{\eta} \mathbf{d}, \quad \Theta \mapsto \eta \Theta + (1 - \eta) (N' + \frac{1}{2}) \mathbb{1}_2, \quad (59)$$

while $\hat{S}(\xi')$ updates Θ and $\hat{D}(\mu')$ updates $\mathbf{d}$. Equivalently—and closer to our two-level construction—one can append an auxiliary single-mode isoergotropic Gaussian state $\hat{\tau}_{\text{Aux}} = \hat{\rho}_{\bar{\mu}}(\mu', \xi', N')$ with moments $(\mathbf{d}', \Theta')$, realize a beam splitter-SWAP interaction, and trace out the auxiliary mode [90,91]. Writing the joint first moments $\mathbf{D} = (\mathbf{d}, \mathbf{d}')^T$ and joint covariance $\Xi = \Theta \oplus \Theta'$, the idealized SWAP block acts as

$$\Xi = \begin{pmatrix} \Theta & \mathbf{0} \\ \mathbf{0} & \Theta' \end{pmatrix} \mapsto \hat{U} \Xi \hat{U}^\dagger = \begin{pmatrix} \Theta' & \mathbf{0} \\ \mathbf{0} & \Theta \end{pmatrix}, \quad (60)$$

$$\mathbf{D} = (\mu, \mu^*, \mu', \mu'^*)^T \mapsto \hat{U} \mathbf{D} = (\mu', \mu'^*, \mu, \mu^*)^T,$$

with

$$\hat{U} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (61)$$

which swaps the two modes. Tracing out the auxiliary leaves the target isoergotropic state $\hat{\rho}_{\bar{\mu}}(\mu', \xi', N')$. It is worth highlighting the map itself depends on the parameter $|\bar{\mu}|^2$.

Figure 12 illustrates two different trajectories induced by isoergotropic Gaussian operations along $\mathcal{L}_5$, together with the Wigner functions of the depicted states. The black dot (i) marks the initial state, while the red dots (ii)–(iii) indicate two distinct final states.

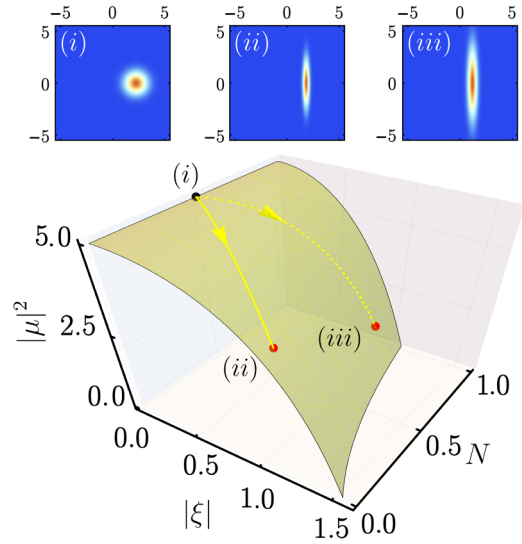


FIG. 12. Isoergotropic Gaussian operations. The continuous and dashed lines indicate two distinct ergotropy-preserving trajectories along $\mathcal{L}_5$ from the black to the red points. It was assumed $\phi = \phi' = \pi$, $\theta = \theta' = 0$ (i) $N = 0.5$, $|\xi| = 0$; (ii) $N' = 0.1$, $|\xi'| = 1$; (iii) $N' = 0.8$, $|\xi'| = 1$.

1. Iso- $\mathcal{R}$ realization via selective Gaussian measurements

Let the input be an arbitrary element of the isoergotropic family

$$\hat{\rho}_{\bar{\mu}}(\mu, \xi, N) = \hat{U}(\mu, \xi) \hat{\pi}(N) \hat{U}^\dagger(\mu, \xi), \quad (62)$$

with $\hat{U}(\mu, \xi) = \hat{D}(\mu) \hat{S}(\xi)$, $\hat{\pi}(N) = \sum_{n \geq 0} p_n(N) |n\rangle \langle n|$ and

$$p_n(N) = \frac{N^n}{(N+1)^{n+1}}. \quad (63)$$

The eigenvectors of $\hat{\rho}_{\bar{\mu}}(\mu, \xi, N)$ are $\hat{U}(\mu, \xi) |n\rangle$ with eigenvalues $p_n(N)$. In particular the principal eigenvalue is

$$\lambda_{\max}(\hat{\rho}_{\bar{\mu}}(\mu, \xi, N)) = p_0(N) = \frac{1}{N+1}. \quad (64)$$

For the isoergotropy reference given by the displaced vacuum, the desired pure target is

$$\hat{\rho}_{\text{tar}} = |\bar{\mu}\rangle \langle \bar{\mu}|, \quad |\bar{\mu}\rangle = \hat{D}(\bar{\mu}) |0\rangle. \quad (65)$$

Choose the rank-one Kraus

$$\hat{M} = |\bar{\mu}\rangle \langle \phi|, \quad (66)$$

where $|\phi\rangle$ is any normalized vector. Acting on the input yields the unnormalized postmeasurement operator

$$\hat{M} \hat{\rho}_{\bar{\mu}}(\mu, \xi, N) \hat{M}^\dagger = \langle \phi | \hat{\rho}_{\bar{\mu}}(\mu, \xi, N) | \phi \rangle |\bar{\mu}\rangle \langle \bar{\mu}|. \quad (67)$$

After renormalization, the conditional postmeasurement state is exactly the pure target:

$$\frac{\hat{M}\hat{\rho}_{\bar{\mu}}\hat{M}^\dagger}{\text{Tr}[\hat{M}\hat{\rho}_{\bar{\mu}}\hat{M}^\dagger]} = |\bar{\mu}\rangle \langle \bar{\mu}|. \quad (68)$$

The success probability for this outcome equals

$$\mathcal{P}_{\text{succ}}(\phi) = \langle \phi | \hat{\rho}_{\bar{\mu}}(\mu, \xi, N) | \phi \rangle. \quad (69)$$

Maximizing over $|\phi\rangle$ yields the principal eigenvalue of the input:

$$\mathcal{P}_{\text{succ}}^{\max} = \lambda_{\max}(\hat{\rho}_{\bar{\mu}}(\mu, \xi, N)) = p_0(N) = \frac{1}{N+1}, \quad (70)$$

and the optimal vector is

$$|\phi_{\text{opt}}\rangle = \hat{U}(\mu, \xi) |0\rangle, \quad (71)$$

so that the optimal Kraus reads

$$\hat{M}_{\text{opt}} = |\bar{\mu}\rangle \langle \phi_{\text{opt}}|. \quad (72)$$

Analogously to the qubit case, this shows that selective measurements can be employed in Gaussian systems to transform one state into another while preserving the ergotropy.

C. Autonomous dynamic ergotropy conversion

Similarly to the TLS case, isoergotropic operations can be performed dynamically and autonomously in a Gaussian quantum battery (B) by coupling it to an auxiliary system (A). Assume two bosonic modes with total Hamiltonian $\hat{H} = \hat{H}_B + \hat{H}_A + \hat{V}_{AB}$, where $\hat{H}_B = \omega_B(\hat{b}^\dagger\hat{b} + 1/2)$, $\hat{H}_A = \omega_A(\hat{a}^\dagger\hat{a} + 1/2)$ and a beam-splitter interaction

$$\hat{V}_{AB} = i\eta(\hat{b}^\dagger \otimes \hat{a} - \hat{b} \otimes \hat{a}^\dagger), \quad (73)$$

with η being the coupling strength. Under these circumstances, the joint unitary dynamics is Gaussian and fully captured by the following Lyapunov equation

$$\begin{aligned} \frac{d}{dt}\mathbf{D}(t) &= \mathbf{W}\mathbf{D}(t), \\ \frac{d}{dt}\mathbf{\Xi}(t) &= \mathbf{W}\mathbf{\Xi}(t) + \mathbf{\Xi}(t)\mathbf{W}^\dagger, \end{aligned} \quad (74)$$

for the first and second moments, with joint mean vector $\mathbf{D}(t) = (\mu_B(t), \mu_B^*(t), \mu_A(t), \mu_A^*(t))^T$, covariance matrix

$\mathbf{\Xi}(t)$ at instant t , and drift matrix

$$\mathbf{W} = \begin{pmatrix} -i\omega_B & 0 & \eta & 0 \\ 0 & i\omega_B & 0 & \eta \\ -\eta & 0 & -i\omega_A & 0 \\ 0 & -\eta & 0 & i\omega_A \end{pmatrix}. \quad (75)$$

Equation (74) describes a beam-splitter rotation between A and B at rate η while both modes freely precess at frequency $\omega_{A,B}$. Formally, the solution of Eq. (74) is given by

$$\mathbf{D}(t) = \mathbf{\Lambda}(t)\mathbf{D}(0), \quad \mathbf{\Xi}(t) = \mathbf{\Lambda}(t)\mathbf{\Xi}(0)\mathbf{\Lambda}^\dagger(t) \quad (76)$$

where $\mathbf{\Lambda}(t) = e^{\mathbf{W}t}$. Assuming $\omega_B \equiv \omega$ and $\omega_A = \omega + \Delta$ with detuning $\Delta \equiv \alpha\eta$, and $\mathbf{\Xi}(0) = \mathbf{\Theta}_B(0) \oplus \mathbf{\Theta}_A(0)$ with

$$\mathbf{\Theta}_K(0) = (N_K(0) + 1/2)\mathbf{F}(\xi, \phi_0), \quad K = B, A \quad (77)$$

one can compute the displacements and covariance matrix analytically at later times: defining $2\chi := \eta\sqrt{\alpha^2 + 4}$, $\lambda_K := \eta(2\mu_K(0) + i\alpha\mu_{K'}(0))/2\chi$ and $g_K := \Delta^2(N_K(0) + 1/2)(2\chi)^{-2}$ with $K' \neq K$, we have

$$\begin{aligned} \mu_B(t) &= e^{-i(\omega + \frac{\Delta}{2})t} (\lambda_A \sin(\chi t) + \mu_B(0) \cos(\chi t)), \\ \mu_A(t) &= e^{-i(\omega + \frac{\Delta}{2})t} (\mu_A(0) \cos(\chi t) - \lambda_B \sin(\chi t)), \end{aligned} \quad (78)$$

and

$$\begin{aligned} \Theta_B(t) &= (N_B(t) + 1/2)\mathbf{F}(\xi, \phi_t) + g_B\tilde{\mathbf{F}}(\xi, \phi_t, \Lambda_t), \\ \Theta_A(t) &= (N_A(t) + 1/2)\mathbf{F}(\xi, \phi_t) + g_A\tilde{\mathbf{F}}(\xi, \phi_t, \Lambda_t^*), \end{aligned} \quad (79)$$

where $\Lambda_t := \cos(2\chi t) + i\frac{2\chi}{\Delta}\sin(2\chi t)$ and

$$\tilde{\mathbf{F}}(\xi, \phi_t, \Lambda_t) = \begin{pmatrix} \cosh(2|\xi|) & e^{i\phi_t}\Lambda_t \sinh(2|\xi|) \\ e^{-i\phi_t}\Lambda_t^* \sinh(2|\xi|) & \cosh(2|\xi|) \end{pmatrix}, \quad (80)$$

with time-dependent phase $\phi_t = \phi_0 - (2\omega + \Delta)t$, and

$$\begin{aligned} N_K(t) &= 2 \frac{(N_K(0) - N_{K'}(0)) \cos(2\chi t)}{(\alpha^2 + 4)} \\ &\quad + 2 \frac{N_K(0) + N_{K'}(0) + 1}{(\alpha^2 + 4)} - \frac{1}{2}. \end{aligned} \quad (81)$$

In general, the local dynamics induced by Eq. (74) are not isoergotropic at all times for fixed parameters. Still, one can achieve iso- $\mathcal{R}$ operations by carefully tuning the relevant subset of parameters. To illustrate this, assume initial isoergotropic Gaussian states $\hat{\rho}_K(0) = \hat{\rho}_{\bar{\mu}}(\mu_K(0), \xi, N_K(0))$, such that

$$\mu_K(0) = \sqrt{f_{\bar{\mu}}(\xi, N_K(0))} e^{i\theta_K}, \quad (82)$$

with $\theta_A = \theta_B + \pi/2$. Figure 13 shows the ergotropy dynamics for the (a) battery and (b) auxiliary system

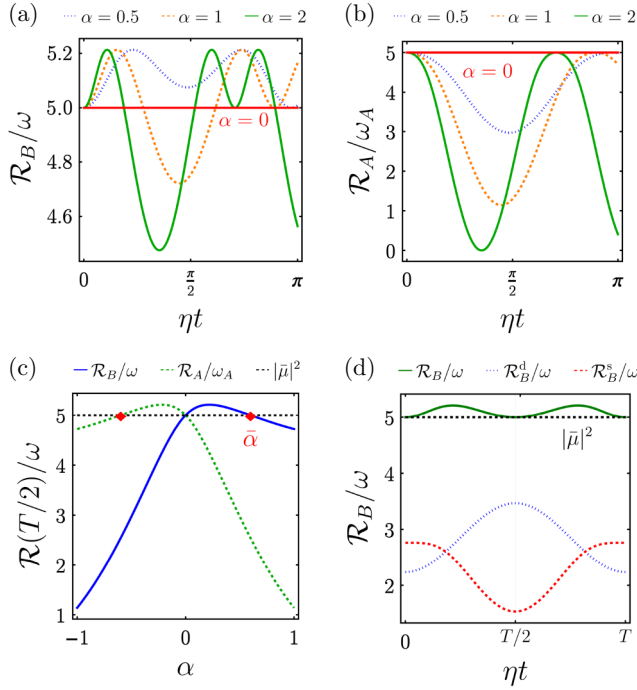


FIG. 13. Ergotropy dynamics of the (a) battery and (b) auxiliary Gaussian states for $\alpha = 0, 0.5, 1, 2$, assuming $\hat{\rho}_K(0) = \hat{\rho}_{\bar{\mu}}(\mu_K(0), \xi, N_K(0))$ with $K = B, A$ as initial states. (c) Ergotropy α at half period $T/2$. The red diamonds represent the detuning $\bar{\alpha}$ where the battery (approximately equal to 0.6) or the auxiliary (approximately equal to -0.6) system returns to the initial isoergotropic curve $\mathcal{L}_{|\bar{\mu}|^2}$. (d) Internal ergotropy dynamics for $\bar{\alpha}$. At $t = T/2$ one can achieve an iso- $\mathcal{R}$ operation, with the displacement and squeezing components being reshuffled. It was assumed $|\bar{\mu}|^2 = 5$, $N_A(0) = N_B(0) = 0.5$, and $|\xi| = 1$.

for different values of detuning α , assuming $|\bar{\mu}|^2 = 5$, $N_A(0) = N_B(0) = 0.5$, $|\xi| = 1$. We observe that both ergotropies (i) are invariant for $\alpha = 0$ (ii) oscillate with period $\eta T = 2\pi/\sqrt{\alpha^2 + 4}$; however, in the latter case, $\mathcal{R}_B$ can still cross the initial isoergotropic surface at earlier times, depending on α . In this sense, Fig. 13(c) depicts the ergotropies as a function of α computed at half period $T/2$, showing that with detuning $\bar{\alpha} \approx 0.6$ (≈ -0.6) the battery(auxiliary system) returns to $\mathcal{L}_5$ while the auxiliary(battery) state remains outside the surface. Finally, Fig. 13(d) presents the internal ergotropy dynamics for $\bar{\alpha} \approx 0.6$: while the dynamics is not isoergotropic during all time, for $t = T/2$ one can achieve an iso- $\mathcal{R}$ operation. Initially, the charge is mainly concentrated in the squeezing component. After half of a period, one observes an internal ergotropy reshuffle, leading to $\mathcal{R}_B^d > \mathcal{R}_B^s$.

We saw earlier that for resonant modes ($\alpha = 0$), the local ergotropies remain invariant. In this case, one has $\chi = \eta$, $g_K = 0$ and $\lambda_K = \mu_K(0)$, resulting in both battery and auxiliary systems remaining within the same initial

isoergotropic surface with fixed ξ , i.e.,

$$\hat{\rho}_{B(A)}(t) \equiv \hat{\rho}_{\bar{\mu}}(\mu_{B(A)}(t), \xi, N_{B(A)}(t)) \in \mathcal{L}_{\bar{\mu}^2}. \quad (83)$$

Figure 14 shows the dynamics of the (a) battery and (b) auxiliary system, assuming $|\bar{\mu}|^2 = 5$, $N_A(0) = 0$, $N_B(0) = 0.8$, and fixed $|\xi| = 1$. The initial state of B (A) is represented by the yellow (orange) dots. It is clear that, during the dynamics, both systems change within $\mathcal{L}_5$. While both local ergotropies are constant, the displacement and squeezing components oscillate with period $\eta T = \pi$. The former changes due to $\mu_K(t)$, while the latter is due to $N_K(t)$. Moreover, Fig. 14(e) illustrates the Wigner function at different instants of time, for $\theta_B = 0$, $\phi_B = \pi$, and $\omega = \eta = 1$. For $t = 0$, most of the battery's charge is stored in the squeezing component. In $t = \pi/4$, one observes $\mathcal{R}_B^d = \mathcal{R}_B^s$, until the ergotropy is transferred to the displacement component. The auxiliary system follows the opposite behavior. Notice that such a process has no internal cost of ergotropy, i.e., both the total and local charges are invariant, as shown by Fig. 15(a). Thus, the only cost to realize such internal reorganization is due to the interaction term. To evaluate how the correlations change, Fig. 15(b) depicts how the Rényi-2 entropies and mutual information change during the dynamics.

V. APPLICATIONS

To highlight the potential significance of understanding the set of isoergotropic states, in the following, we will examine two relevant scenarios for quantum batteries: charging and discharging.

A. Charging

Charging a quantum battery—i.e., increasing its ergotropy—amounts to transforming a passive state into an active one. In the simplest setting this can be done by coupling the storage to a semiclassical charger for a fixed duration $\mathcal{T}$. This direct charging protocol [31] is modeled by a time-dependent Hamiltonian $\hat{H}(t) = \hat{H}_B + \hat{V}(t)$, with $\hat{H}_B$ the battery Hamiltonian and $\hat{V}(t)$ the charging drive, switched on only for $0 < t < \mathcal{T}$ ($\hat{V}(t) = 0$ otherwise). The evolution is unitary, $\dot{\hat{\rho}}(t) = -i[\hat{H}(t), \hat{\rho}(t)]$, and supplies both energy $\langle \hat{H}_B \rangle$ and ergotropy $\mathcal{R}$ to the battery. A natural performance goal is to maximize the average charging power under realistic constraints. We define

$$\langle P \rangle := \frac{\langle W \rangle}{\mathcal{T}}, \quad \langle W \rangle = \text{Tr} \left[\hat{H}_B (\hat{\rho}(\mathcal{T}) - \hat{\rho}(0)) \right], \quad (84)$$

and assume a bounded driving strength, $\|\hat{H}(t)\| \leq \varepsilon$. For a single TLS with $\hat{H}_B = |e\rangle\langle e|$ and Bloch representation $\hat{\rho}(t) = (\hat{1} + \mathbf{s}(t) \cdot \boldsymbol{\sigma})/2$, $\mathbf{s}(t) = s(\sin \alpha_t \cos \varphi_t, \sin \alpha_t \sin \varphi_t,$

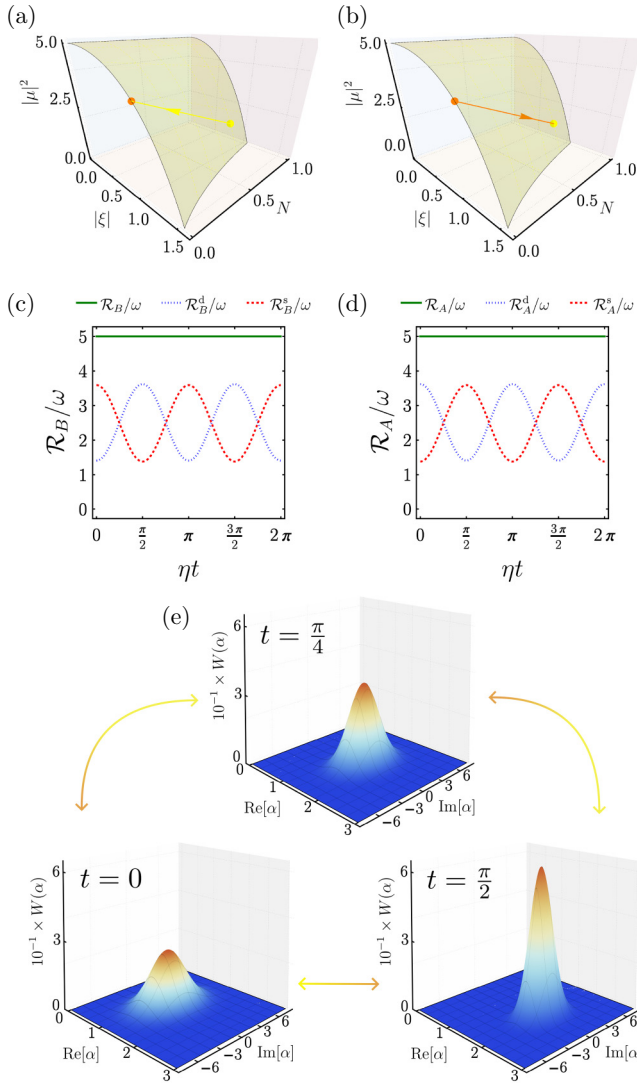


FIG. 14. Top: parameter space representation of the (a) battery and (b) auxiliary Gaussian states dynamics induced by Eq. (78) along the isoergotropic surface $\mathcal{L}_5$, assuming $\hat{\rho}_K(0) = \hat{\rho}_\mu(\mu_K(0), \xi, N_K(0))$ with $K = B, A$ (colored dots) as initial states, and $|\bar{\mu}|^2 = 5$, $N_A(0) = 0$, $N_B(0) = 0.8$ and $|\xi| = 1$. Middle: ergotropy and internal components dynamics for the (c) battery and (d) auxiliary system for the same parameters. Bottom: (e) Wigner function isoergotropic dynamics for the battery at instants $t = 0, \pi/4, \pi/2$, assuming $\theta_B = 0$, $\phi_B = \pi$ and $\omega = \eta = 1$.

$\cos \alpha_t)^T$, the optimal protocol with respect to the power under the norm bound is [39]

$$\hat{H}(t) = -\frac{\varepsilon}{2} \sin \varphi_t \hat{\sigma}_x + \frac{\varepsilon}{2} \cos \varphi_t \hat{\sigma}_y, \quad (85)$$

which drives

$$\varphi_t = \varphi_0, \quad \alpha_t = \alpha_0 + \varepsilon t.$$

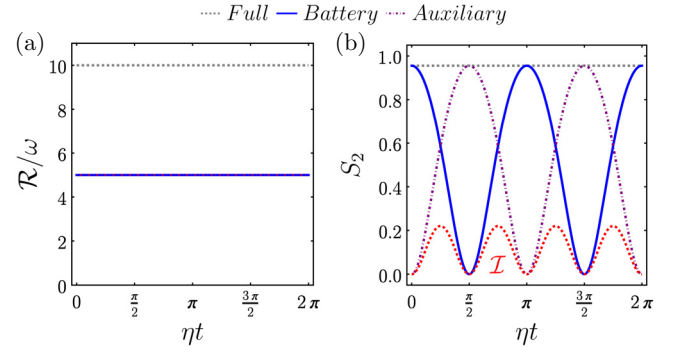


FIG. 15. (a) Charge and (b) Rényi-2 entropy dynamics induced by Eq. (74). The full (computed without the interaction) and local ergotropies are invariant during the time evolution. The local Rényi-2 entropies and mutual information $\mathcal{I}$ oscillate during the internal conversion of ergotropy.

Starting from a passive (uncharged) state on the north pole, $\alpha_0 = 0$ and $\mathbf{s}(0) = s_0(0, 0, 1)^T$ with $s_0 \in [0, 1]$, the average power evaluates to $\langle P \rangle = s_0 [1 - \cos(\varepsilon T)] / 2T$. Maximizing $\langle P \rangle$ over T yields

$$\cos(\varepsilon T) + \varepsilon T \sin(\varepsilon T) = 1, \quad (86)$$

whose solution gives $\alpha_T = \varepsilon T \simeq 0.74\pi$. Thus the set of active states attainable with optimal power forms a cone on the Bloch sphere,

$$\mathbf{s}_{\text{Opt}} = s (\sin \alpha_T \cos \varphi, \sin \alpha_T \sin \varphi, \cos \alpha_T)^T,$$

with $s \in [0, 1]$, $\varphi \in [0, 2\pi]$. Figure 16 shows this optimal set together with several isoergotropic surfaces. Their intersections (solid colored curves) are parameterized by $\bar{s} = P_{\bar{p}} \csc^2(\alpha_T/2)$ for different $\mathcal{L}_{\bar{p}}$. Consequently, for an initially uncharged battery the isoergotropic family reachable at maximum power is

$$\bar{p} = \frac{1}{2} \left(1 + \frac{s_0}{\csc^2(\alpha_T/2)} \right), \quad (87)$$

and the corresponding states $\hat{\rho}_{\bar{p}}(p, \theta)$ are given by

$$p = \frac{1}{2} (1 - \bar{s} \cos \alpha_T), \quad (88)$$

as indicated by the colored dots (initial states) and dashed lines (driven trajectories) in Fig. 16.

B. Open QB and discharging

External dissipation and decoherence degrade the performance of quantum devices in general [101–105]. Mitigating ergotropy leakage is therefore essential for realistic quantum batteries [71, 73]. The ability to reshuffle the internal composition of ergotropy, developed above, enables

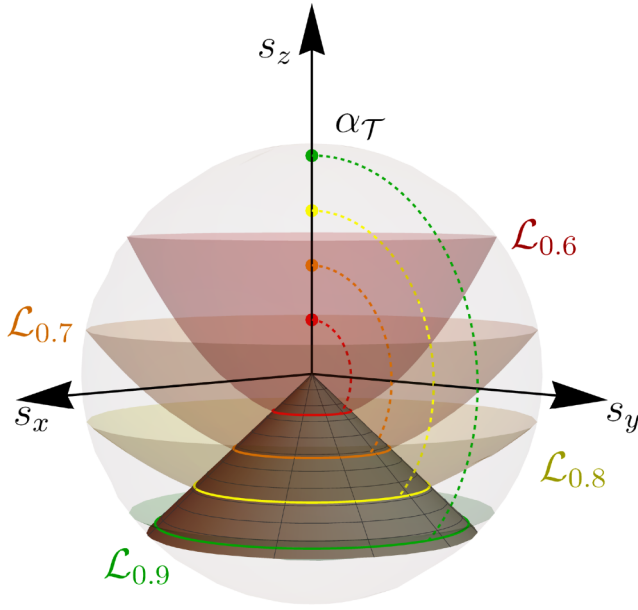


FIG. 16. Isoergotropic surfaces and optimal-power charging. Starting from passive (uncharged) states, the cone marks all states reachable at maximal average power in the direct protocol (see Ref. [39] for more details); it corresponds to the fixed polar angle $\alpha_T = \varepsilon T \simeq 0.74\pi$. Colored dots and dashed arcs illustrate representative charging trajectories. The circular arcs are the intersections between the cone and the iso- $\mathcal{R}$ surfaces $\mathcal{L}_{\bar{p}}$; points on an intersection are optimal states $\hat{\rho}_{\bar{p}}(p, \theta)$ with $p = \frac{1}{2}(1 - \bar{s} \cos \alpha_T)$ and $\bar{s} = P_{\bar{p}} \csc^2(\alpha_T/2)$.

the preparation of states that are more robust against environmental noise. To illustrate this, we couple the battery weakly to a thermal reservoir at inverse temperature β . The local dynamics of a TLS and of a single-mode Gaussian state (GS) are described by the GKLS master equation ($\hbar = k_B = 1$) [106]:

$$\begin{aligned} \text{TLS: } \dot{\hat{\rho}} &= -i[\hat{H}, \hat{\rho}] + \gamma(1 - \bar{n})\mathcal{D}_{\rho}[\hat{\sigma}_+] + \gamma\bar{n}\mathcal{D}_{\rho}[\hat{\sigma}_-], \\ \text{GS: } \dot{\hat{\rho}} &= -i[\hat{H}, \hat{\rho}] + \gamma(1 + \bar{n})\mathcal{D}_{\rho}[\hat{a}] + \gamma\bar{n}\mathcal{D}_{\rho}[\hat{a}^\dagger], \end{aligned} \quad (89)$$

where γ is the decay rate, $\bar{n} = (e^{\beta\omega} \pm 1)^{-1}$ is the fermionic (bosonic) occupation number for the TLS (GS), and $\mathcal{D}_{\rho}[\hat{o}] = \hat{o}\hat{\rho}\hat{o}^\dagger - \{\hat{o}^\dagger\hat{o}, \hat{\rho}\}/2$. Equation (89) relaxes each system to the Gibbs state at temperature $1/\beta$, discharging the battery. Crucially, the coherent and incoherent (displacement and squeezing) parts of ergotropy decay differently, and the observed discharge reflects their interplay.

Regarding the TLS system at time t the state has the form $\hat{\rho}(t) = (1 - p_t)|g\rangle\langle g| + p_t|e\rangle\langle e| + C_t(e^{i\frac{\theta}{2}}|g\rangle\langle e| + h.c.)/2$, with populations and coherence evolving independently as $p_t = (p_0 - \bar{n})e^{-\gamma t} + \bar{n}$ and $C_t^2 = C_0^2 e^{-\gamma t}$. Thus $p_\infty = \bar{n}$ and $C_\infty = 0$. The (in)coherent components of the ergotropy can be computed using Eqs. (2) and

(3). Interestingly, as observed by a previous work [73], despite the monotonically decreasing behavior of the population and coherence, nonintuitive ergotropy dynamics can appear due to coherence, i.e., depending on the initial conditions, the coherent ergotropy becomes nonmonotonic in time. More specifically, the maximum of $\mathcal{R}^{\text{coh}}$ is reached for $p = 1/2$, thus if $p_0 > 1/2$, one observes a first increase of the coherent ergotropy until $p_{\tau_{1/2}} = 1/2$ before asymptotically approaching zero, where $\tau_{1/2} := \gamma^{-1} \ln[(p_0 - \bar{n})(1/2 - \bar{n})^{-1}]$ is the time required for the incoherent state become passive. Figures 17(a)–17(c) show, for all initial states on $\mathcal{L}_{0.8}$, that the total and incoherent charges decay monotonically, while the coherent part can exhibit a transient increase. Panel 17(a) marks the half-life $T_{1/2}$ defined by $\mathcal{R}(T_{1/2}) = \mathcal{R}(0)/2$. States with larger initial coherence have longer $T_{1/2}$ despite sharing the same initial charge. Figure 17(d) depicting the half-life time in terms of $\bar{p}$ and $P_{\bar{p}} \leq p \leq \bar{p}$ confirms this trend across different iso- $\mathcal{R}$ surfaces: within a given $\mathcal{L}_{\bar{p}}$, coherent (near-pure) states on the Bloch-sphere surface are the most resilient to environment-induced dissipation.

As for the GS states, since the dissipators in Eq. (89) are linear in $\hat{a}$ and $\hat{a}^\dagger$, Gaussianity is preserved. The ergotropy at time t is given by Eq. (52) with time-dependent parameters (μ_t, ξ_t, N_t) evolving as [84]

$$\begin{aligned} |\mu_t|^2 &= |\mu_0|^2 e^{-\gamma t}, \\ |\xi_t| &= \frac{1}{2} \cosh^{-1} \left[\frac{\Delta_t + (2N + 1) e^{-\gamma t} \sinh^2(|\xi_0|)}{N_t + 1/2} \right], \\ N_t &= \sqrt{\Delta_t^2 + \frac{(2N + 1)(2\bar{n} + 1)}{e^{\gamma t}(1 - e^{-\gamma t})^{-1}} \sinh^2(|\xi_0|)} - \frac{1}{2}, \end{aligned} \quad (90)$$

with $\Delta_t = 2(N - \bar{n})e^{-\gamma t} + (\bar{n} + 1/2)$. Figures 17(e)–17(g) display the decay of charge for different initial points on $\mathcal{L}_5$ with $N = 0.5$: both the squeezing and displacement contributions decrease monotonically, but the half-life is shorter for larger initial $|\xi|$. This “hotter empties faster” behavior is the ergotropic analogue of the Mpemba effect [84]. Panel 17(h) shows $T_{1/2}$ versus $|\xi|$ for several N , confirming the trend across $\mathcal{L}_5$.

In summary, consistent with Refs. [73,84], tailoring the internal composition of ergotropy—either at preparation (charging) or via iso- $\mathcal{R}$ operations—yields states that retain charge longer under open-system dynamics, providing a practical advantage for quantum batteries in realistic noisy environments.

Many previous works have investigated optimizing quantum-battery performance by engineering the charging and discharging dynamics itself, for instance via time-periodic (Floquet) control to tailor effective generators, or via reservoir engineering to exploit structured

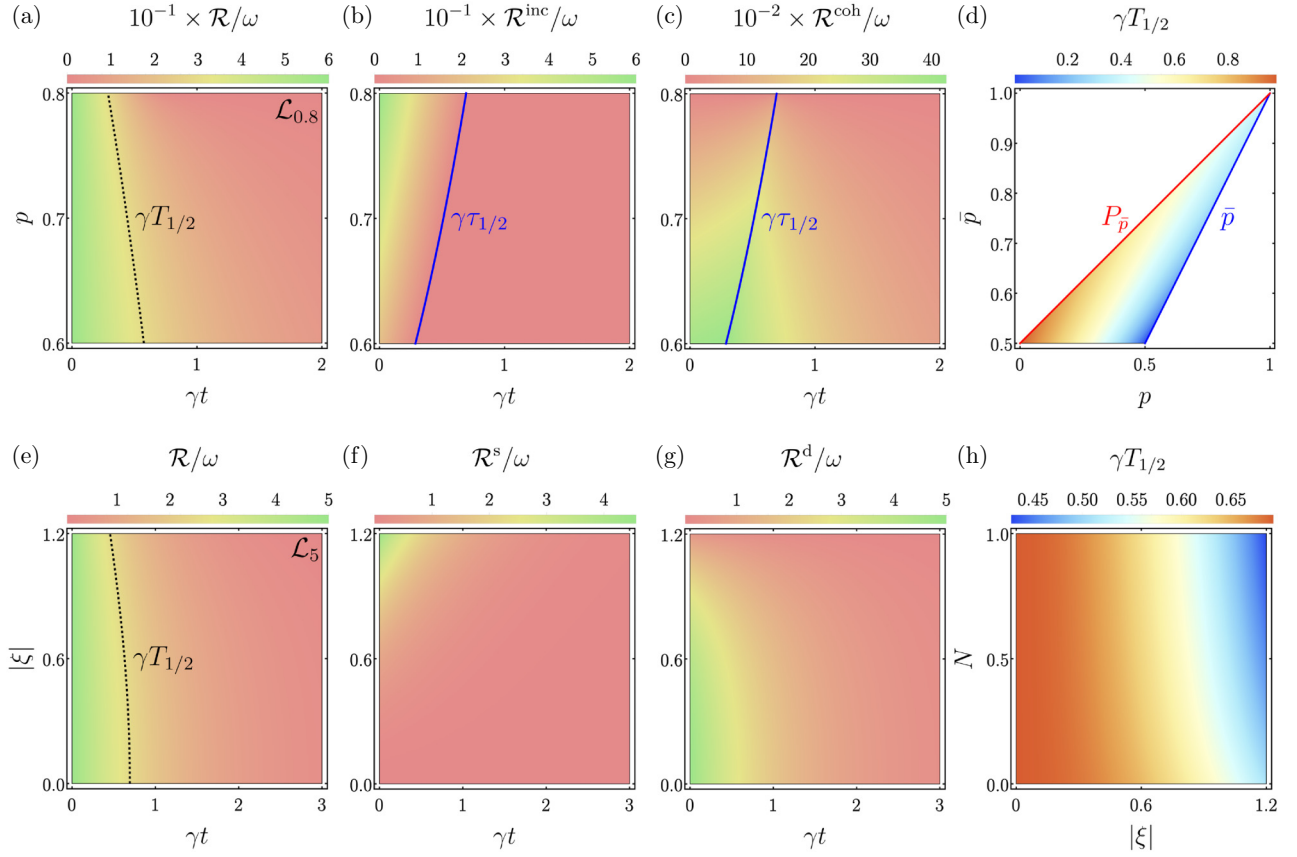


FIG. 17. Ergotropy decay for (a)–(d) TLS (e)–(h) single-mode Gaussian-based open quantum batteries. (a) Ergotropy dynamics for the distinct isoergotropic states along $\mathcal{L}_{0.8}$. The black dashed line highlights the half-life of the battery. The (b) incoherent and (c) coherent components present distinct behavior. The blue continuous line indicates the instant with maximum coherent ergotropy, where $p_{\tau_{1/2}} = 1/2$. As a result, states with coherence possess lower discharging rates. The half-life behavior for all values of $\bar{p}$ is shown in (d). (e) Ergotropy dynamics for the distinct isoergotropic states along $\mathcal{L}_5$ and $N = 0.5$. Both the (f) squeezing and (g) displacement components present a monotonic decay with time. Also, it is clear that states with higher values of $|\xi|$ discharge faster. This behavior is observed for distinct values of N , as shown in (h). It was assumed (a)–(d) $\bar{p} = 0.8$ $\bar{n} = 0.2$, (e)–(h) $N = 0.5$, $\bar{n} = 0.3$, and $|\bar{\mu}|^2 = 5$.

environments that stabilize excitations, suppress degradation, or enable remote energy transfer (see, e.g., Ref. [68] and references therein). A complementary line of work targets robustness by increasing the fraction of ergotropy stored in coherence: Ref. [28] reports that coherent ergotropy can decay more slowly than incoherent ergotropy under self-discharging, and proposes enhancing stability by maximizing the coherent-to-total ergotropy ratio.

Isoergotropic operations address a different, orthogonal layer of optimization: they keep the total ergotropy fixed while reshuffling its internal composition. This makes them naturally composable with the above dynamical strategies. Concretely, (i) after a given charging protocol (including Floquet-optimized or reservoir-engineered schemes), an isoergotropic postprocessing stroke can convert a fragile encoding of ergotropy into a more robust one without changing the stored work content; (ii) during storage in an open system, interleaving isoergotropic

reshuffling provides a direct mechanism to implement the stabilization intuition of Ref. [28] at fixed total ergotropy by reallocating ergotropy toward the components that decay more slowly under the relevant noise; and (iii) in setups where reservoir engineering yields a fixed steady-state ergotropy (or where drive constraints fix the reachable ergotropy), reshuffling can be the dominant performance lever, because improving the delivered work then requires optimizing the encoding rather than increasing the total charge.

These comparisons also clarify when ergotropy reshuffling can outperform conventional methods: in robustness-limited operation (where dissipation rapidly degrades certain components) and in constrained-budget regimes (fixed drive power, fixed cycle time, or fixed steady-state ergotropy), isoergotropic operations can increase the usable work extracted over a discharge window by extending the ergotropy lifetime and/or by steering the charged state into isoergotropic encodings that optimize

the operational figure of merit (power, robustness, or local extractability), without demanding stronger modulation or additional entangling control.

VI. DISCUSSION

A central challenge in the study of QBs lies not only in maximizing their stored energy but also in understanding how this charge is internally structured and can be effectively manipulated. While ergotropy $\mathcal{R}$ provides a fundamental measure of the extractable work from a quantum state, recent insights show that this quantity can be distributed among distinct internal components, classical and quantum degrees of freedom. This internal composition plays a crucial role in determining how the stored charge can be accessed, transformed, or preserved during realistic operations, including efficient charging and work extraction.

In this broader context, we introduced and investigated the set $\mathcal{L}$ of isoergotropic states, defined as states possessing the same total ergotropy but different internal configurations of ergotropy. Although such states are equivalent in terms of their stored charge, they may exhibit markedly distinct physical properties, including internal energy, purity, and entropy. We further explored the class $\mathcal{K}$ of quantum operations—comprising completely positive trace-preserving (CPTP) maps and selective measurements that act within $\mathcal{L}$, preserving total ergotropy while redistributing its internal components. The thermodynamics involved during such processes is also verified, including heat transfer and the possibility of isentropic and internal energy-conserving transformations. As relevant examples, these concepts are illustrated and applied with both discrete and continuous-variable quantum batteries, respectively represented by two-level systems and Gaussian states. While in the former, the charge is stored within the coherent and incoherent components, in the latter, it is independently distributed along the displacement and squeezing ones.

We also demonstrated that dynamical autonomous conversion between the internal components of ergotropy can be achieved by coupling the quantum battery to an auxiliary system. In particular, Eqs. (32) and (73), describing the interactions for the TLS and Gaussian cases, share the structure of a beam-splitter-type coupling. Such interactions enable coherent energy exchange while conserving the total number of excitations in the composite system. These couplings allow the autonomous reconfiguration of ergotropy and the simultaneous realization of isoergotropic transformations within the target systems. Based on the results reported in Ref. [83], we anticipate that these processes could be exploited to selectively assess and extract specific components of the stored charge. Beam-splitter-type interactions are standard tools in quantum optics

and are widely used across various quantum technologies [107–109]. Moreover, such couplings can be readily engineered in both optical and superconducting platforms [110–112]. Therefore, isoergotropic operations can be realistically implemented in quantum batteries using current state-of-the-art technologies.

To this end, we also explored the practical implications of iso- $\mathcal{R}$ states and operations in relevant scenarios. Considering a direct charging protocol for a TLS, we characterized the set of states that achieve optimal charging power and identify the corresponding optimal isoergotropic states within each $\mathcal{L}_{\bar{p}}$, as illustrated by the intersections in Fig. 16. In this way, one can either tune the initial discharged state $\hat{\rho}_0$ to reach a desired iso- $\mathcal{R}$ surface or determine the optimal charged state attainable from a given $\hat{\rho}_0$. Similar analyses can be performed for other optimization criteria and charging strategies. Furthermore, we investigated the robustness of different isoergotropic states under environmental noise. By accounting for the distinct dynamical behavior of the ergotropic components, one can mitigate dissipative charge loss in open quantum batteries—either by charging into suitable isoergotropic states or by dynamically reshuffling the ergotropy between components. As shown in Fig. 17(a), isoergotropic states with coherence exhibit longer half-life times $T_{1/2}$, thus discharging more slowly. A similar advantage is observed in Fig. 17(b), where maximizing the ergotropy stored in the displacement component enhances robustness against dissipation.

The formalism and analyses developed here can be systematically extended to arbitrary quantum battery architectures. Future work should aim to identify additional applications and explicitly incorporate iso- $\mathcal{R}$ operations into practical protocols. For example, optimal charge stabilization strategies can be found through quantum control techniques for different relevant dynamical scenarios [113], including non-Markovian environments. Additionally, they can be utilized in the selective charging and extraction of specific ergotropic components. Such operations could be harnessed to design novel thermodynamic cycles whose strokes internally redistribute charge while maintaining a fixed energy budget. Also, one can study the potential use of non-isoergotropic local operations along with global isoergotropic processes in multicell quantum batteries. In this sense, one can also study and pinpoint the role of nonclassical correlations for quantum energy storage and distribution across multipartite batteries.

A complementary figure of merit to ergotropy is antiergotropy [114], which quantifies the maximum energy that must be injected into a state by cyclic unitaries during the charging leg. Using antiergotropy as a companion metric thus identifies iso- $\mathcal{R}$ encodings that are not only equally charged but also cheaper to refill in realistic, noisy cycles.

Finally, by taking isoergotropic transformations as free operations, one can build a resource theory of ergotropy

in direct analogy with standard quantum resource theories (see the review [115]). Framed this way, ergotropy can be integrated with existing resource theories of coherence [86] and Gaussianity [116], enabling analyses beyond the usual settings. Notably, recent work has designed nonlinear coherent heat machines that convert passive Gaussian states into non-Gaussian active ones [117]. This indicates that nonlinearity cannot be treated as a free resource in any continuous-variable resource theory of ergotropy. Investigating the interplay among these resources—and identifying appropriate monotones and conversion rules—would be valuable, but lies beyond the scope of the present work.

ACKNOWLEDGMENTS

A.H.A.M. acknowledges support from National Science Centre, Poland Grant OPUS-21 (Grant No. 2021/41/B/ST2/03207). B.A. and P.H. acknowledge support from IRA Programme (project no. FENG.02.01-IP.05-0006/23) financed by the FENG program 2021-2027, Priority FENG.02, Measure FENG.02.01., with the support of the FNP. P.R.D. acknowledges support from CNPq/MCTI/FNDTC 22/2024 (Grant No. 446775/2024-0), and NCN Poland, ChistEra-2023/05/Y/ST2/00005 under the project Modern Device Independent Cryptography (MoDIC).

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

- [1] J. Roßnagel, S. T. Dawkins, K. N. Tolazzi, O. Abah, E. Lutz, F. Schmidt-Kaler, and K. Singer, A single-atom heat engine, *Science* **352**, 325 (2016).
- [2] J. P. S. Peterson, T. B. Batalhão, M. Herrera, A. M. Souza, R. S. Sarthour, I. S. Oliveira, and R. M. Serra, Experimental characterization of a spin quantum heat engine, *Phys. Rev. Lett.* **123**, 240601 (2019).
- [3] J. Klatzow, J. N. Becker, P. M. Ledingham, C. Weinzel, K. T. Kaczmarek, D. J. Saunders, J. Nunn, I. A. Walmsley, R. Uzdin, and E. Poem, Experimental demonstration of quantum effects in the operation of microscopic heat engines, *Phys. Rev. Lett.* **122**, 110601 (2019).
- [4] X. Nie, X. Zhu, K. Huang, K. Tang, X. Long, Z. Lin, Y. Tian, C. Qiu, C. Xi, X. Yang *et al.*, Experimental realization of a quantum refrigerator driven by indefinite causal orders, *Phys. Rev. Lett.* **129**, 100603 (2022).
- [5] G. Manzano, G.-L. Giorgi, R. Fazio, and R. Zambrini, Boosting the performance of small autonomous refrigerators via common environmental effects, *New J. Phys.* **21**, 123026 (2019).
- [6] A. H. A. Malavazi, B. Ahmadi, P. Mazurek, and A. Mandarino, Detuning effects for heat-current control in quantum thermal devices, *Phys. Rev. E* **109**, 064146 (2024).
- [7] U. N. Ekanayake, S. D. Gunapala, and M. Premaratne, Engineered common environmental effects on multitransistor systems, *Phys. Rev. B* **107**, 075440 (2023).
- [8] A. Rajapaksha, S. D. Gunapala, and M. Premaratne, Enhanced thermal rectification in coupled qutrit-qubit quantum thermal diode, *APL Quantum* **1**, 046123 (2024).
- [9] P. R. Dieguez, V. F. Lisboa, and R. M. Serra, Thermal devices powered by generalized measurements with indefinite causal order, *Phys. Rev. A* **107**, 012423 (2023).
- [10] R. Alicki and M. Fannes, Entanglement boost for extractable work from ensembles of quantum batteries, *Phys. Rev. E* **87**, 042123 (2013).
- [11] A. Camprose, T. Virgili, F. Lombardi, G. Cerullo, D. Pisignano, and M. Polini, Quantum batteries: A materials science perspective, *Adv. Mater.* **37**, 2415073 (2025).
- [12] F. Campaioli, S. Gherardini, J. Q. Quach, M. Polini, and G. M. Andolina, *Colloquium*: Quantum batteries, *Rev. Mod. Phys.* **96**, 031001 (2024).
- [13] B. Ahmadi, A. B. Ravichandran, P. Mazurek, S. Barzanjeh, and P. Horodecki, Harnessing environmental noise for quantum energy storage, *arXiv:2510.06384*.
- [14] F. H. Kamin, F. T. Tabesh, S. Salimi, F. Kheirandish, and A. C. Santos, Non-Markovian effects on charging and self-discharging process of quantum batteries, *New J. Phys.* **22**, 083007 (2020).
- [15] F. H. Kamin, S. Salimi, and M. B. Arjmandi, Steady-state charging of quantum batteries via dissipative ancillas, *Phys. Rev. A* **109**, 022226 (2024).
- [16] S. Zakavati, F. T. Tabesh, and S. Salimi, Bounds on charging power of open quantum batteries, *Phys. Rev. E* **104**, 054117 (2021).
- [17] S. Zakavati, S. Salimi, and B. Arash, Optimizing the charging of open quantum batteries using long short-term memory-driven reinforcement learning, *arXiv:2504.19840*.
- [18] G. Gemme, M. Grossi, D. Ferraro, S. Vallecorsa, and M. Sassetti, IBM quantum platforms: A quantum battery perspective, *Batteries* **8**, 43 (2022).
- [19] F.-Q. Dou and F.-M. Yang, Superconducting transmon qubit-resonator quantum battery, *Phys. Rev. A* **107**, 023725 (2023).
- [20] F.-M. Yang and F.-Q. Dou, Resonator-qutrit quantum battery, *Phys. Rev. A* **109**, 062432 (2024).
- [21] R.-H. Zheng, W. Ning, Z.-B. Yang, Y. Xia, and S.-B. Zheng, Demonstration of dynamical control of three-level open systems with a superconducting qutrit, *New J. Phys.* **24**, 063031 (2022).
- [22] I. Maillette de Buy Wenniger, S. E. Thomas, M. Maffei, S. C. Wein, M. Pont, N. Belabas, S. Prasad, A. Harouri, A. Lemaître, I. Sagnes, N. Somaschi, A. Auffèves, and P. Senellart, Experimental analysis of energy transfers between a quantum emitter and light fields, *Phys. Rev. Lett.* **131**, 260401 (2023).
- [23] A. Rojo-Francàs, F. Isaule, A. C. Santos, B. Juliá-Díaz, and N. T. Zinner, Stable collective charging of ultracold-atom quantum batteries, *Phys. Rev. A* **110**, 032205 (2024).
- [24] J. Q. Quach, K. E. McGhee, L. Ganzer, D. M. Rouse, B. W. Lovett, E. M. Gauger, J. Keeling, G. Cerullo, D. G. Lidzey,

- and T. Virgili, Superabsorption in an organic microcavity: Toward a quantum battery, *Sci. Adv.* **8**, 3160 (2022).
- [25] J. Joshi and T. S. Mahesh, Experimental investigation of a quantum battery using star-topology NMR spin systems, *Phys. Rev. A* **106**, 042601 (2022).
- [26] C. Cruz, M. F. Anka, M. S. Reis, R. Bachelard, and A. C. Santos, Quantum battery based on quantum discord at room temperature, *Quantum Sci. Technol.* **7**, 025020 (2022).
- [27] L. Q. Galvão, A. C. das Neves, M. F. Anka, and C. Cruz, Simulating work extraction in a dinuclear quantum battery using a variational quantum algorithm, *Phys. Rev. E* **111**, 064119 (2025).
- [28] W.-L. Song, J.-L. Wang, B. Zhou, W.-L. Yang, and J.-H. An, Self-discharging mitigated quantum battery, *Phys. Rev. Lett.* **135**, 020405 (2025).
- [29] F. H. Kamin, F. T. Tabesh, S. Salimi, and A. C. Santos, Entanglement, coherence, and charging process of quantum batteries, *Phys. Rev. E* **102**, 052109 (2020).
- [30] B. Ahmadi, P. Mazurek, P. Horodecki, and S. Barzanjeh, Nonreciprocal quantum batteries, *Phys. Rev. Lett.* **132**, 210402 (2024).
- [31] B. Ahmadi, P. Mazurek, S. Barzanjeh, and P. Horodecki, Superoptimal charging of quantum batteries via reservoir engineering: Arbitrary energy transfer unlocked, *Phys. Rev. Appl.* **23**, 024010 (2025).
- [32] F. Barra, K. V. Hovhannisyan, and A. Imparato, Quantum batteries at the verge of a phase transition, *New J. Phys.* **24**, 015003 (2022).
- [33] V. Shaghaghghi, V. Singh, G. Benenti, and D. Rosa, Micro-masers as quantum batteries, *Quantum Sci. Technol.* **7**, 04LT01 (2022).
- [34] V. Shaghaghghi, V. Singh, M. Carrega, D. Rosa, and G. Benenti, Lossy micromaser battery: Almost pure states in the Jaynes–Cummings regime, *Entropy* **25**, 430 (2023).
- [35] M.-L. Hu, T. Gao, and H. Fan, Efficient wireless charging of a quantum battery, *Phys. Rev. A* **111**, 042216 (2025).
- [36] F. T. Tabesh, F. H. Kamin, and S. Salimi, Environment-mediated charging process of quantum batteries, *Phys. Rev. A* **102**, 052223 (2020).
- [37] G. M. Andolina, D. Farina, A. Mari, V. Pellegrini, V. Giovannetti, and M. Polini, Charger-mediated energy transfer in exactly solvable models for quantum batteries, *Phys. Rev. B* **98**, 205423 (2018).
- [38] J. Monsel, M. Fellous-Asiani, B. Huard, and A. Auffèves, The energetic cost of work extraction, *Phys. Rev. Lett.* **124**, 130601 (2020).
- [39] F. C. Binder, S. Vinjanampathy, K. Modi, and J. Goold, Quantacell: powerful charging of quantum batteries, *New J. Phys.* **17**, 075015 (2015).
- [40] Y.-Y. Zhang, T.-R. Yang, L. Fu, and X. Wang, Powerful harmonic charging in a quantum battery, *Phys. Rev. E* **99**, 052106 (2019).
- [41] B. Çakmak, Ergotropy from coherences in an open quantum system, *Phys. Rev. E* **102**, 042111 (2020).
- [42] D. Ferraro, M. Campisi, G. M. Andolina, V. Pellegrini, and M. Polini, High-power collective charging of a solid-state quantum battery, *Phys. Rev. Lett.* **120**, 117702 (2018).
- [43] F. Campaioli, F. A. Pollock, F. C. Binder, L. Céleri, J. Goold, S. Vinjanampathy, and K. Modi, Enhancing the charging power of quantum batteries, *Phys. Rev. Lett.* **118**, 150601 (2017).
- [44] F.-Q. Dou, Y.-J. Wang, and J.-A. Sun, Highly efficient charging and discharging of three-level quantum batteries through shortcuts to adiabaticity, *Front. Phys.* **17**, 31503 (2022).
- [45] F.-Q. Dou, H. Zhou, and J.-A. Sun, Cavity Heisenberg-spin-chain quantum battery, *Phys. Rev. A* **106**, 032212 (2022).
- [46] F.-Q. Dou, Y.-Q. Lu, Y.-J. Wang, and J.-A. Sun, Extended Dicke quantum battery with interatomic interactions and driving field, *Phys. Rev. B* **105**, 115405 (2022).
- [47] J.-Y. Gyhm, D. Šafránek, and D. Rosa, Quantum charging advantage cannot be extensive without global operations, *Phys. Rev. Lett.* **128**, 140501 (2022).
- [48] L. F. C. de Moraes, A. C. Duriez, A. Saguia, A. C. Santos, and M. S. Sarandy, Quantum battery supercharging via counter-diabatic dynamics, *Quantum Sci. Technol.* **9**, 045033 (2024).
- [49] X. Zhang and M. Blaauboer, Enhanced energy transfer in a Dicke quantum battery, *Front. Phys.* **10**, 1097564 (2023).
- [50] J.-Y. Gyhm and U. R. Fischer, Beneficial and detrimental entanglement for quantum battery charging, *AVS Quantum Sci.* **6**, 012001 (2024).
- [51] B. Ahmadi, A. H. Malavazi, P. Mazurek, P. Horodecki, and S. Barzanjeh, Reservoir-engineered exceptional points for quantum energy storage, *arXiv:2511.20569*.
- [52] F. Mazzoncini, V. Cavina, G. M. Andolina, P. A. Erdman, and V. Giovannetti, Optimal control methods for quantum batteries, *Phys. Rev. A* **107**, 032218 (2023).
- [53] R. R. Rodríguez, B. Ahmadi, G. Suárez, P. Mazurek, S. Barzanjeh, and P. Horodecki, Optimal quantum control of charging quantum batteries, *New J. Phys.* **26**, 043004 (2024).
- [54] C.-K. Hu, J. Qiu, P. J. P. Souza, J. Yuan, Y. Zhou, L. Zhang, J. Chu, X. Pan, L. Hu, J. Li, Y. Xu, Y. Zhong, S. Liu, F. Yan, D. Tan, R. Bachelard, C. J. Villas-Boas, A. C. Santos, and D. Yu, Optimal charging of a superconducting quantum battery, *Quantum Sci. Technol.* **7**, 045018 (2022).
- [55] M.-L. Hu, T. Gao, and H. Fan, Enhancing the charging performance of an atomic quantum battery, *Adv. Quantum Technol.* **8**, e00422 (2025).
- [56] I. Beder, D. Ferraro, and P. A. Brandão, Work extraction from a quantum battery charged through an array of coupled cavities, *Phys. Rev. A* **113**, 022210 (2026).
- [57] F. Pirmoradian and K. Mølmer, Aging of a quantum battery, *Phys. Rev. A* **100**, 043833 (2019).
- [58] A. C. Santos, Quantum advantage of two-level batteries in the self-discharging process, *Phys. Rev. E* **103**, 042118 (2021).
- [59] C. A. Downing and M. S. Ukharty, Hyperbolic enhancement of a quantum battery, *Phys. Rev. A* **109**, 052206 (2024).
- [60] A. C. Santos, B. Çakmak, S. Campbell, and N. T. Zinner, Stable adiabatic quantum batteries, *Phys. Rev. E* **100**, 032107 (2019).
- [61] M. T. Mitchison, J. Goold, and J. Prior, Charging a quantum battery with linear feedback control, *Quantum* **5**, 500 (2021).

- [62] J. Liu, D. Segal, and G. Hanna, Loss-free excitonic quantum battery, *J. Phys. Chem. C* **123**, 18303 (2019).
- [63] K. Xu, H.-G. Li, H.-J. Zhu, and W.-M. Liu, Inhibiting the self-discharging process of quantum batteries in non-Markovian noises, *Phys. Rev. E* **109**, 054132 (2024).
- [64] J. Q. Quach and W. J. Munro, Using dark states to charge and stabilize open quantum batteries, *Phys. Rev. Appl.* **14**, 024092 (2020).
- [65] M. Carrega, A. Crescente, D. Ferraro, and M. Sassetti, Dissipative dynamics of an open quantum battery, *New J. Phys.* **22**, 083085 (2020).
- [66] F. Barra, Dissipative charging of a quantum battery, *Phys. Rev. Lett.* **122**, 210601 (2019).
- [67] D. Farina, G. M. Andolina, A. Mari, M. Polini, and V. Giovannetti, Charger-mediated energy transfer for quantum batteries: An open-system approach, *Phys. Rev. B* **99**, 035421 (2019).
- [68] W.-L. Song, H.-B. Liu, B. Zhou, W.-L. Yang, and J.-H. An, Remote charging and degradation suppression for the quantum battery, *Phys. Rev. Lett.* **132**, 090401 (2024).
- [69] S.-Y. Bai and J.-H. An, Floquet engineering to reactivate a dissipative quantum battery, *Phys. Rev. A* **102**, 060201(R) (2020).
- [70] F. Zhao, F.-Q. Dou, and Q. Zhao, Quantum battery of interacting spins with environmental noise, *Phys. Rev. A* **103**, 033715 (2021).
- [71] S. Gherardini, F. Campaioli, F. Caruso, and F. C. Binder, Stabilizing open quantum batteries by sequential measurements, *Phys. Rev. Res.* **2**, 013095 (2020).
- [72] Z.-G. Lu, G. Tian, X.-Y. Lü, and C. Shang, Topological quantum batteries, *Phys. Rev. Lett.* **134**, 180401 (2025).
- [73] A. H. A. Malavazi, R. Sagar, B. Ahmadi, and P. R. Dieguez, Two-time weak-measurement protocol for ergotropy protection in open quantum batteries, *PRX Energy* **4**, 023011 (2025).
- [74] A. E. Allahverdyan, R. Balian, and Th. M. Nieuwenhuizen, Maximal work extraction from finite quantum systems, *Europhys. Lett.* **67**, 565 (2004).
- [75] M. Łobejko, The tight second law inequality for coherent quantum systems and finite-size heat baths, *Nat. Commun.* **12**, 918 (2021).
- [76] T. Biswas, M. Łobejko, P. Mazurek, K. Jałowiecki, and M. Horodecki, Extraction of ergotropy: Free energy bound and application to open cycle engines, *Quantum* **6**, 841 (2022).
- [77] R. Castellano, R. Nery, K. Simonov, and D. Farina, Parallel ergotropy: Maximum work extraction via parallel local unitary operations, *Phys. Rev. A* **111**, 012212 (2025).
- [78] R. Salvia, G. De Palma, and V. Giovannetti, Optimal local work extraction from bipartite quantum systems in the presence of Hamiltonian couplings, *Phys. Rev. A* **107**, 012405 (2023).
- [79] G. Di Bello, D. Farina, D. Jansen, C. A. Perroni, V. Cataudella, and G. De Filippis, Local ergotropy and its fluctuations across a dissipative quantum phase transition, *Quantum Sci. Technol.* **10**, 015049 (2025).
- [80] R. Castellano, D. Farina, V. Giovannetti, and A. Acin, Extended local ergotropy, *Phys. Rev. Lett.* **133**, 150402 (2024).
- [81] G. Francica, F. C. Binder, G. Guarnieri, M. T. Mitchison, J. Goold, and F. Plastina, Quantum coherence and ergotropy, *Phys. Rev. Lett.* **125**, 180603 (2020).
- [82] K. Simonov, S. Roy, T. Guha, Z. Zimborás, and G. Chiribella, Activation of thermal states by coherently controlled thermalization processes, *New J. Phys.* **27**, 074502 (2025).
- [83] Z. Niu, Y. Wu, Y. Wang, X. Rong, and J. Du, Experimental investigation of coherent ergotropy in a single spin system, *Phys. Rev. Lett.* **133**, 180401 (2024).
- [84] I. Medina, O. Culhane, F. C. Binder, G. T. Landi, and J. Goold, Anomalous discharging of quantum batteries: The ergotropic Mpemba effect, *Phys. Rev. Lett.* **134**, 220402 (2025).
- [85] A. Serafini, *Quantum Continuous Variables: A Primer of Theoretical Methods* (CRC Press, Boca Raton, FL, 2017).
- [86] A. Streltsov, G. Adesso, and M. B. Plenio, *Colloquium: Quantum coherence as a resource*, *Rev. Mod. Phys.* **89**, 041003 (2017).
- [87] A. H. A. Malavazi and F. Brito, A Schmidt decomposition approach to quantum thermodynamics, *Entropy* **24**, 1645 (2022).
- [88] R. Alicki, The quantum open system as a model of the heat engine, *J. Phys. A: Math. Gen.* **12**, L103 (1979).
- [89] B. Ahmadi, S. Salimi, and A. S. Khorashad, On the contribution of work or heat in exchanged energy via interaction in open bipartite quantum systems, *Sci. Rep.* **13**, 160 (2023).
- [90] S. Khatri, K. Sharma, and M. M. Wilde, Information-theoretic aspects of the generalized amplitude-damping channel, *Phys. Rev. A* **102**, 012401 (2020).
- [91] M. Rosati, A. Mari, and V. Giovannetti, Narrow bounds for the quantum capacity of thermal attenuators, *Nat. Commun.* **9**, 4339 (2018).
- [92] N. Behzadi, E. Soltani, and E. Faizi, Thermodynamic cost of creating global quantum discord and local quantum uncertainty, *Int. J. Theor. Phys.* **57**, 3207 (2018).
- [93] V. S. Gomes, P. R. Dieguez, and H. M. Vasconcelos, Realism-based nonlocality: Invariance under local unitary operations and asymptotic decay for thermal correlated states, *Physica A* **601**, 127568 (2022).
- [94] M. A. Nielsen, Quantum information theory, *arXiv*: 0011036.
- [95] R. Horodecki, M. Horodecki, and P. Horodecki, Entanglement processing and statistical inference: The Jaynes principle can produce fake entanglement, *Phys. Rev. A* **59**, 1799 (1999).
- [96] M. A. Nielsen, Conditions for a class of entanglement transformations, *Phys. Rev. Lett.* **83**, 436 (1999).
- [97] J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, Thermodynamical approach to quantifying quantum correlations, *Phys. Rev. Lett.* **89**, 180402 (2002).
- [98] G. Adam, Density matrix elements and moments for generalized Gaussian state fields, *J. Mod. Opt.* **42**, 1311 (1995).
- [99] G. Adesso, D. Girolami, and A. Serafini, Measuring Gaussian quantum information and correlations using the Rényi entropy of order 2, *Phys. Rev. Lett.* **109**, 190502 (2012).
- [100] J. B. Brask, Gaussian states and operations—a quick reference, *arXiv*:2102.05748.

- [101] Z. Cai, R. Babbush, S. C. Benjamin, S. Endo, W. J. Huggins, Y. Li, J. R. McClean, and T. E. O’Brien, Quantum error mitigation, *Rev. Mod. Phys.* **95**, 045005 (2023).
- [102] F. Galve, A. Mandarino, M. G. A. Paris, C. Benedetti, and R. Zambrini, Microscopic description for the emergence of collective dissipation in extended quantum systems, *Sci. Rep.* **7**, 42050 (2017).
- [103] J. Etchezarreta Martinez, P. Fuentes, P. M. Crespo, and J. Garcia-Frias, Quantum outage probability for time-varying quantum channels, *Phys. Rev. A* **105**, 012432 (2022).
- [104] O. A. D. Molitor, A. H. A. Malavazi, R. D. Baldijão, A. C. Orthey, I. L. Paiva, and P. R. Diegue, Quantum switch instabilities with an open control, *Commun. Phys.* **7**, 373 (2024).
- [105] S. Aziz, A. H. A. Malavazi, and P. R. Diegue, Revival and instabilities of entanglement in monitoring maps with indefinite causal order, *J. Phys.: Conf. Ser.* **3017**, 012044 (2025).
- [106] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University, Oxford, 2007).
- [107] D. Makarov, Theory for the beam splitter in quantum optics: Quantum entanglement of photons and their statistics, HOM effect, *Mathematics* **10**, 4794 (2022).
- [108] L.-P. Yang and Y. Chang, Quantum beam splitter as a controller of higher-order quantum coherence, *Phys. Rev. A* **110**, 023722 (2024).
- [109] L. O. R. Solak, C. J. Villas-Boas, and D. Z. Rossatto, Beam splitter for dark and bright states of light, *Phys. Rev. A* **111**, 053702 (2025).
- [110] Y. Y. Gao, B. J. Lester, Y. Zhang, C. Wang, S. Rosenblum, L. Frunzio, L. Jiang, S. M. Girvin, and R. J. Schoelkopf, Programmable interference between two microwave quantum memories, *Phys. Rev. X* **8**, 021073 (2018).
- [111] B. J. Chapman, S. J. de Graaf, S. H. Xue, Y. Zhang, J. Teoh, J. C. Curtis, T. Tsunoda, A. Eickbusch, A. P. Read, A. Koottandavida, S. O. Mundhada, L. Frunzio, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, High-on-off-ratio beam-splitter interaction for gates on bosonically encoded qubits, *PRX Quantum* **4**, 020355 (2023).
- [112] Y.-H. Huang, K.-M. Hsieh, F. Aziz, Z. Q. Niu, P. Y. Wen, Y.-T. Cheng, Y.-S. Tsai, J. C. Chen, X. Wang, A. F. Kockum, Z.-R. Lin, Y.-H. Lin, and I.-C. Hoi, Tunable coherent microwave beam splitter and combiner at the single-photon level, *Appl. Phys. Lett.* **127**, 014002 (2025).
- [113] G. Kurizki and A. G. Kofman, *Thermodynamics and Control of Open Quantum Systems* (Cambridge University Press, Cambridge, 2022).
- [114] A. Ali, S. Elghaayda, S. Al-Kuwari, M. I. Hussain, M. T. Rahim, H. Kuniyil, T. Byrnes, J. Q. Quach, M. Mansour, and S. Haddadi, Magnetic dipolar quantum battery with spin-orbit coupling, *Adv. Quantum Technol.* **8**, e00361 (2025).
- [115] E. Chitambar and G. Gour, Quantum resource theories, *Rev. Mod. Phys.* **91**, 025001 (2019).
- [116] L. Lami, B. Regula, X. Wang, R. Nichols, A. Winter, and G. Adesso, Gaussian quantum resource theories, *Phys. Rev. A* **98**, 022335 (2018).
- [117] T. Opatrný, Šimon Bräuer, A. G. Kofman, A. Misra, N. Meher, O. Firstenberg, E. Poem, and G. Kurizki, Nonlinear coherent heat machines, *Sci. Adv.* **9**, 1070 (2023).