

Dynamics of Quantum Chiral Solitons

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We introduce a nonperturbative framework for quantizing chiral solitons in interacting quantum spin chains. This approach provides a direct lattice extension of the well-established S -duality between the sine-Gordon and Thirring models, thereby bridging the gap between continuum dualities and their lattice counterparts. By constructing the quantum chiral-soliton operators explicitly, we show how their unconventional dynamics appear in the excitation spectrum and correlation functions across the full Brillouin zone. A key result is that the dominant soliton tunneling amplitude alternates in sign, $\text{sgn}(t_{1+}) = (-1)^{2S+1}$, sharply distinguishing half-odd-integer from integer spin chains. We further identify characteristic signatures of these chiral excitations in the dynamical spin structure factor, demonstrating their visibility in inelastic neutron scattering. Our results open a route to experimentally probing non-perturbative features of dual quantum field theories in condensed-matter settings.

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I. INTRODUCTION

Topological excitations play a central role across many areas of physics, from nonlinear wave equations to quantum field theory and condensed-matter systems. Classically, solitons are localized, particlelike textures whose proliferation generically produces a periodic soliton crystal [1,2]. Quantum mechanics fundamentally alters this picture. Zero-point motion can melt the classical crystal into a quantum liquid, as happens in the quantum spin-1/2 chain considered in this work. This result raises the following questions: How do topological textures become genuine quantum quasiparticles, and how do quantum fluctuations reshape the phases that emerge from their condensation?

Continuum approaches based on the sine-Gordon model and its dual massive Thirring theory provide an elegant

long-wavelength description of quantum solitons, where solitons map to fermionic excitations under S -duality [3]. However, such treatments rely on gradient expansions and are not designed to yield quantitative, momentum-resolved predictions for microscopic lattice magnets or their experimentally measured response functions. In particular, they do not directly establish how topological excitations manifest in the dynamical spin structure factor $\tilde{S}(q, \omega)$ probed by inelastic neutron scattering (INS). Magnetic chains provide an especially fertile ground for exploring soliton physics in quantum materials [1,4–8].

In this work, we develop a fully quantum, nonperturbative, lattice-resolved framework for chiral solitons in a ferromagnetic spin-1/2 chain with Dzyaloshinskii-Moriya interactions and a magnetic field applied parallel to the easy plane. Our approach constructs momentum-resolved many-body wave functions for quantum chiral solitons directly from the microscopic Hamiltonian, without invoking any small expansion parameter, such as $1/S$ or the continuum ratio a/ℓ_s (where a denotes the lattice spacing and ℓ_s the soliton size). Consequently, the theory remains quantitatively controlled even in the strongly quantum regime, including the physically relevant case $S = 1/2$, where soliton cores extend over only a few lattice spacings. Within this framework, solitons emerge as bona fide quantum

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quasiparticles with a well-defined dispersion across the full Brillouin zone, a calculable spectral weight in $\tilde{S}(q, \omega)$, and a systematically derived description of their interactions.

As the magnetic field approaches the saturation value $H = H_c$ from above, quantum chiral solitons become the lowest-energy excitations over a finite field window $H_c < H < H^*$. At $H = H_c$, the soliton chemical potential vanishes, while the conventional magnon mode remains gapped. The persistence of a finite magnon gap implies that interactions between solitons, mediated by virtual magnon excitations, are short ranged and decay exponentially with distance. In this dilute regime, topology and quantum fluctuations reorganize the excitation hierarchy of the system: Topological solitons, rather than spin-wave modes, constitute the relevant low-energy degrees of freedom.

Through a Jordan-Wigner transformation, these excitations admit a fermionic description. Microscopically, we extend the sine-Gordon S -duality to the lattice by constructing localized soliton creation and annihilation operators directly from the spin Hamiltonian. A Wannierization procedure yields orthonormal quantum soliton states, systematically incorporating quantum fluctuations and capturing the tunneling processes that endow solitons with mobility, thereby generating a well-defined and nontrivial dispersion relation. Remarkably, the dominant tunneling amplitude exhibits a spin-parity effect, alternating in sign between half-odd-integer and integer spin chains. The resulting dynamics is described by an effective, weakly coupled, tight-binding fermionic Hamiltonian with a nontrivial dispersion relation. Near $H = H_c$, the short-range interactions are irrelevant in the renormalization-group sense, rendering the critical point a free-fermion fixed point. At finite soliton density, however, the residual interactions become marginally relevant, driving the system into a gapless Tomonaga-Luttinger liquid—the quantum-melted counterpart of the classical soliton lattice phase.

A central question is whether these quantum solitons can be directly observed above H_c and how their dynamics intertwine with the gapped magnons. Because INS measures the dynamical spin structure factor $\tilde{S}(q, \omega)$, a purely topological excitation would carry negligible spectral weight unless it hybridizes with a single magnon mode. Our lattice-resolved framework demonstrates that such visibility is enabled by hybridization between chiral solitons and single-magnon modes. This mixing transfers spectral weight to the soliton branch, thereby rendering it experimentally accessible.

The hybridization becomes particularly pronounced for $H \gtrsim H^*$, where the soliton and magnon dispersions approach each other and cross. In this regime, level repulsion enhances the overlap between the two sectors, producing a characteristic, field-dependent redistribution of spectral weight. The resulting intensity of the soliton branch in $\tilde{S}(q, \omega)$ provides a direct and quantitative measure of soliton-magnon hybridization. The predicted

dispersion and spectral-weight redistribution are quantitatively benchmarked against density matrix renormalization group (DMRG) and matrix product state (MPS) simulations, showing excellent agreement. Together, these results establish a concrete experimental detection mechanism for quantum chiral solitons and a lattice-level connection between topological quasiparticles and measurable dynamical response functions.

More broadly, the procedure introduced here for promoting classical topological textures to quantum quasiparticles directly at the lattice level provides a systematic route toward deriving effective low-energy theories of interacting topological defects. While demonstrated in one dimension, the framework does not rely on integrability or continuum approximations and can, in principle, be extended to higher-dimensional systems, including skyrmion-hosting magnets [9–13]. In this perspective, quantum liquids may emerge as the quantum-melted phases of interacting topological textures, offering an alternative microscopic pathway toward strongly correlated topological matter.

II. MODEL AND CLASSICAL LIMIT

The simplest model that supports quantum chiral solitons is provided by the one-dimensional spin- $\frac{1}{2}$ ferromagnetic Heisenberg chain of length L , oriented along the x axis and subject to a Dzyaloshinskii-Moriya (DM) interaction characterized by a vector $\mathbf{D} = D\hat{z}$. An external magnetic field $\mathbf{H} = H\hat{x}$ is applied perpendicular to the DM vector. The Hamiltonian of the system is

$$\hat{H}_S = - \sum_{j=1}^L \left[J \hat{\mathbf{S}}_j \cdot \hat{\mathbf{S}}_{j+1} + D \left(\hat{S}_j^x \hat{S}_{j+1}^y - \hat{S}_j^y \hat{S}_{j+1}^x \right) + H \hat{S}_j^x \right], \quad (1)$$

where $J > 0$ and $H = g\mu_B B$, with μ_B the Bohr magneton and g the gyromagnetic factor. It is important to note that the magnetic field is applied transverse to the $U(1)$ -invariant axis set by the DM interaction. Consequently, the full Hamiltonian has only discrete symmetries.

We begin by summarizing key facts about the classical limit of Eq. (1), $S \rightarrow \infty$. It is obtained by replacing the spin operators with their expectation values over $SU(2)$ spin coherent states $|\Omega(\theta, \varphi)\rangle$, which represent points on the unit sphere S^2 parametrized by the polar and azimuthal angles θ and φ . In other words, we substitute $\hat{\mathbf{S}}_j \rightarrow S\mathbf{n}_j = S(\sin\theta_j \cos\varphi_j, \sin\theta_j \sin\varphi_j, \cos\theta_j)$. Since the Dzyaloshinskii-Moriya interaction favors spin alignment in the plane perpendicular to $\hat{z}$, the polar angle can be fixed to $\theta = \pi/2$, leading to a configuration of the form $S\mathbf{n}_j = S(\cos\varphi_j, \sin\varphi_j, 0)$. For small pitch angles, $|\varphi_{j+1} - \varphi_j| \ll 1$, it is possible to take the long-wavelength (continuum) limit of the lattice model. In this regime, the system is effectively described by a static chiral

sine-Gordon model for the angular field variable $\varphi = \varphi(x)$, with the Hamiltonian density given by

$$\mathcal{H}_{sG} = JS^2a \left[\frac{1}{2}(\partial_x \varphi)^2 - q_0 \partial_x \varphi + m^2(1 - \cos \varphi) \right]. \quad (2)$$

Here, $q_0 \equiv D/(Ja)$ is the wave number of the zero-field spiral, a the lattice parameter, $m^2 \equiv H/(JSa^2)$, and the full Hamiltonian is given by $H_{sG} = \int dx \mathcal{H}$. A constant m^2 is added to make the energy of the uniform vacuum state equal to zero. The Hamiltonian functional in Eq. (2) is minimized by solving for the extreme condition $\delta \mathcal{H} = 0$,

$$\frac{d}{dx} \frac{\partial \mathcal{H}}{\partial (\partial_x \varphi)} - \frac{\partial \mathcal{H}}{\partial \varphi} = 0. \quad (3)$$

The variational equation is satisfied by the solutions of the static sine-Gordon equation,

$$\frac{d^2 \varphi}{dx^2} = m^2 \sin \varphi. \quad (4)$$

Clearly, in the absence of a magnetic field ($m = 0$), the ground state is a chiral helix [4] given by $\varphi(x) = q_0 x + \varphi(0)$, with the sign of q_0 determining the helix chirality.

When an external magnetic field is applied ($m \neq 0$), soliton solutions appear. They represent a compromise between the exchange and Dzyaloshinskii-Moriya interactions, which favor a uniform spiral, and the transverse magnetic field, which tends to align the spins along a fixed direction within the spiral's polarization plane. These competing tendencies are balanced by the soliton configuration. The single soliton ($\tau = +1$) and antisoliton ($\tau = -1$) solutions are described by

$$\varphi_\tau(x) = \tau 4 \arctan[e^{m(x-x_0)}] = -2\tau i \ln \frac{1 + ie^{m(x-x_0)}}{1 - ie^{m(x-x_0)}}. \quad (5)$$

The corresponding soliton spin configuration is shown in Fig. 1, while its field profile, $\varphi_{\tau=+1}(x)$, is plotted in Fig. 2(b). Solitons have lower energy than antisolitons due to the presence of the DM term in Eq. (1). To verify that our definition is consistent with this condition, consider a spin at site i pointing along the x axis and another spin at site j rotated around the z axis. The corresponding DM energy

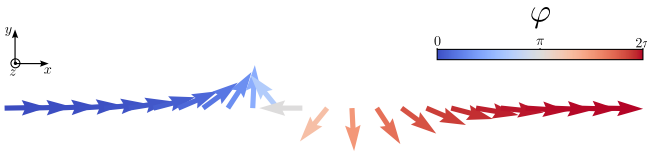


FIG. 1. Profile of the classical chiral soliton. Arrow directions indicate spin orientation, while their color encodes the value of the angle φ .

contribution is around $-\sin \varphi$. For the excitation to lower the energy, we require $\varphi > 0$, which corresponds to a counterclockwise rotation (with φ measured from the x axis under the standard convention). Since our operator produces precisely this type of rotation, we conclude that it generates excitations with the energetically preferred chirality.

The size of the soliton is determined by the magnetization ΔM^x it carries with respect to the fully polarized (FP) state,

$$\Delta M^x = S \int_{-\infty}^{\infty} [1 - \cos \varphi_\tau(x)] dx = \frac{4S}{m}. \quad (6)$$

The characteristic length is therefore $\ell_s(H) = \Delta M^x/S = 4/m(H)$, making explicit its dependence on the external magnetic field H .

The energy of a single soliton is also easy to calculate, $\mathcal{E}_1 = JS^2a(8m - 2\pi q_0)$. Therefore, the energetic cost of the soliton vanishes at the quantum critical point H_c [1]:

$$H_c = \left(\frac{\pi D}{4aJ} \right)^2 JSa^2. \quad (7)$$

For $H < H_c$, the solitons proliferate and form an ordered chiral soliton lattice (CSL). This phase emerges from the competition between the energetic gain of soliton condensation and the repulsive intersoliton interaction, which decays exponentially with the distance between solitons [2]. The chiral soliton lattice [1] is described by

$$\varphi(x) = 2\text{am}(\bar{x}) + \pi, \quad (8)$$

where $\bar{x} = (m/\kappa)x$ and am denotes Jacobi's amplitude function with elliptic modulus $0 < \kappa < 1$. By substituting Eq. (8) into the spin components and applying trigonometric identities, we obtain $S\mathbf{n} = S(2\text{sn}^2(\bar{x}) - 1, -2\text{sn}^2(\bar{x})\text{cn}^2(\bar{x}), 0)$. For sufficiently strong external fields, the FP state ($\varphi = 0$) is recovered as the lowest-energy configuration. The magnetization along the field direction is given by

$$M^x = S \left(\frac{2}{\kappa^2} - \frac{2E}{\kappa^2 K} - 1 \right), \quad (9)$$

where $K \equiv K(\kappa)$ and $E \equiv E(\kappa)$ are the complete elliptic integrals of the first and second kinds, respectively [1]. The elliptic modulus κ is related to the external magnetic field through $\kappa = [(4E)/(\pi q_0)] \sqrt{H/JSa^2}$ [1]. As shown in Fig. 2(a), the magnetization curve exhibits a distinctive profile; most notably, it evolves continuously with the applied field approaching saturation with an infinite derivative. The magnetization is directly related to the number of solitons because the “addition” of a single soliton to the FP state reduces the total magnetization by a finite amount ΔM^x . The continuity of the magnetization curve

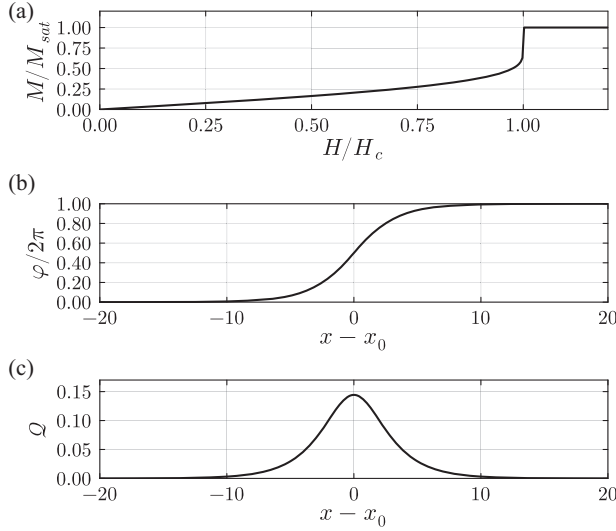


FIG. 2. (a) Field dependence of the magnetization. (b) Spatial profile of the soliton solution $\varphi_+(x)$ in units of 2π . (c) Spatial profile of the topological charge density $Q(x)$ associated with a single soliton at the critical field $H = H_c$.

thus indicates that the density of chiral solitons vanishes as the quantum critical point is approached from below, $H \rightarrow H_c^-$, in agreement with the soliton-condensation mechanism: the single-soliton energy vanishes continuously at $H = H_c$ [Eq. (7)].

This continuous evolution stands in sharp contrast to other topological spin systems—such as conventional skyrmion crystals—where the field-driven transition from a topological to a polarized phase is first order, and the density of topological solitons cannot be reduced continuously [9–13].

In the present case, the field-induced commensurate-incommensurate transition in the classical model is continuous and reflects the gradual suppression of the soliton density as the magnetic field increases. In the quantum spin-1/2 chain, this transition is replaced by a continuous quantum phase transition between a Tomonaga-Luttinger liquid and a gapped, fully polarized phase (see Ref. [14] and Appendix A).

The defining feature of the soliton is its topological structure. The angle $\varphi_\tau(x)$ winds by 2π as x goes over the chain from $-\infty$ to $+\infty$, as Fig. 2(b) illustrates. Mathematically, in the continuum limit, the projection of the spin configuration onto the xy plane defines a mapping $f: S^1 \rightarrow S^1$. The base manifold S^1 corresponds to the spatial coordinate x of the chain with periodic boundary conditions (PBC), while the target manifold, also S^1 , represents the possible orientations of the spin projection in the xy plane. The net soliton charge $\mathcal{C}$ can therefore be identified with the topological degree of this mapping, i.e., the winding number, which is classified by the fundamental homotopy group $\pi_1(S^1) \cong \mathbb{Z}$ [15,16].

The total topological charge is

$$\mathcal{C}_\tau \equiv \int_{-\infty}^{\infty} \mathcal{Q}_\tau(x) dx = \int_{-\infty}^{\infty} \frac{\partial_x \varphi_\tau(x)}{2\pi} dx, \quad (10)$$

where the integrand $\mathcal{Q}_\tau(x)$ defines topological charge density. It is immediately apparent that the term $q_0 \partial_x \varphi$ in the Hamiltonian density, Eq. (2), arising from the DM interaction can be interpreted as a chemical potential coupled to the topological charge $\mathcal{C}$. Individual soliton or antisoliton solutions, Eq. (5), are characterized by the integer charge $\mathcal{C}_\tau = \tau$.

It is important to emphasize here that solitons are *not* the ferromagnetic domain walls, or φ^4 kinks [17–20], which are classified by a different homotopy $\pi_0(\mathbb{Z}_2) \cong \mathbb{Z}_2$ [15,16].

Returning to the lattice problem, the underlying physics remains essentially the same as in the continuum limit [1]. On a discrete chain, the winding number associated with a given spin configuration can be computed through a geodesic interpolation,

$$\mathcal{C} \equiv \frac{1}{2\pi} \sum_{i=1}^L \arcsin [\hat{x} \cdot (\mathbf{S}_i \times \mathbf{S}_{i+1})], \quad (11)$$

where the identification $L + 1 \equiv 1$ enforces periodic boundary conditions, $\mathbf{S}_{L+1} = \mathbf{S}_1$. These boundary conditions are essential for $\mathcal{C}$ to be well defined since the base manifold must be topologically equivalent to S^1 . From this point onward, we assume periodic boundary conditions throughout. The sign of the topological charge carried by each soliton is fixed by the chirality imposed by the Dzyaloshinskii-Moriya interaction, and the total number of solitons in the system is simply given by $N_s = |\mathcal{C}|$.

As stated in the Introduction, the central goal of this work is to explore the physics of the commensurate-incommensurate transition near the quantum critical point, with particular emphasis on the character of the low-energy excitations. For $H > H_c$, the system is fully polarized, and the classical elementary excitations correspond to spin waves with a gapped dispersion centered at $k = 0$, as presented in Fig. 3. The magnon dispersion is given by the familiar expression

$$\omega(k) = -J[\cos(ka) - 1] + H. \quad (12)$$

At the critical field $H = H_c$, the spin-wave spectrum remains gapped, with a gap $\Delta E = H_c$. This exact statement, proven in Appendix B, once again reflects the fact that the commensurate-incommensurate transition is driven by the condensation of chiral solitons, and not of magnons, as in conventional magnetically ordered systems [21].

In the classical limit, the resulting CSL and the spiral magnetic state represent distinct types of order. However, once quantum fluctuations are included, both phases evolve

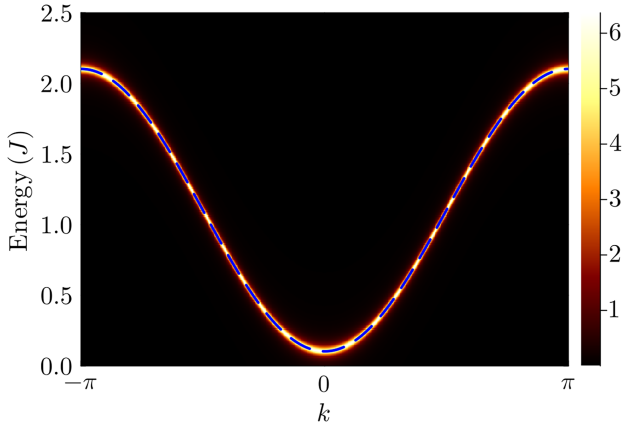


FIG. 3. Dynamical spin structure factor and magnon dispersion (blue dashed line) obtained from linear spin wave theory (LSWT) calculations. The spectral intensities were computed using Sunny.jl library (v0.7) [22].

into a Tomonaga-Luttinger liquid (TLL), and the distinction between them becomes less pronounced. As we show below, these two scenarios can nevertheless be clearly discriminated in the quantum regime by analyzing the nature of the low-energy excitations for $H \gtrsim H_c$. In this regime, the QCP marks the softening of a chiral soliton mode that drives the transition between the TLL and the fully polarized phase.

We are now in a position to specify the parameter regime of our study. Since our objective is to investigate emergent solitonic physics, we focus on configurations in which solitons extend over several lattice sites, ensuring a clear distinction from magnons, which remain localized on the scale of a single site. In other words, we target the mesoscopic regime, defined by $a < \ell_s \ll L$, where a is the lattice spacing and ℓ_s is the soliton size. For comparison, the semiclassical regime—considered a limiting case of the mesoscopic one—corresponds to $a \ll \ell_s \ll L$, where quantum corrections are perturbative and the soliton mass remains much larger than the magnon mass. To access the mesoscopic regime in our numerical simulations, we set $J = 1$ and $D = 1/\sqrt{3}$. This choice allows us to study system sizes that are both computationally tractable and large enough to capture the essential solitonic behavior. The qualitative features discussed in the following are robust in a wide range of D values.

III. QUANTIZATION AND S-DUALITY

The profound connection between the bosonic quantum sine-Gordon model and the fermionic Thirring model is based on the identification of the soliton and antisoliton excitations of the quantum sine-Gordon model with the single-particle fermions of the Thirring model. It maps the C-IC transition of the sine-Gordon model to the metal-insulator transition of the Thirring model driven by the chemical potential. We summarize key arguments of this

correspondence here; more details are provided in Appendix A.

The quantum sine-Gordon Hamiltonian density is given by

$$\hat{\mathcal{H}}_{\text{sG}} = \frac{c\kappa}{2} \hat{\Pi}^2 + \frac{c}{2\kappa} (\partial_x \hat{\phi})^2 - \frac{cq_0}{\sqrt{4\pi\kappa}} \partial_x \hat{\phi} - \frac{HS}{a} \cos[\sqrt{4\pi}\hat{\phi}]. \quad (13)$$

Here, $\hat{\Pi}$ denotes the canonical momentum conjugate to the field $\hat{\phi}$, satisfying $[\hat{\phi}(x, t), \hat{\Pi}(x', t)] = i\delta(x - x')$, and the parameter κ is defined in Eq. (A10). Equation Eq. (13) corresponds to the quantum sine-Gordon model describing the dynamics of kink excitations (solitons and antisolitons).

A remarkable discovery by Coleman [23] and Mandelstam [24], and, independently, by Luther and Emery [25], is that Eq. (13) can be exactly mapped to the Thirring model of massive Dirac fermions. The mapping is based on the observation that the exponential operator,

$$\hat{O}(x) = e^{i\zeta \int_{-\infty}^x dx' \hat{\Pi}(x')}, \quad (14)$$

creates a soliton of height ζ centered at x . This result is proven with the help of the operator identity $[\hat{A}, e^{\hat{B}}] = \hat{C}e^{\hat{B}}$, which holds if $\hat{C} = [\hat{A}, \hat{B}]$ commutes with both $\hat{A}$ and $\hat{B}$. Applied to our situation, the identity

$$[\hat{\phi}(y), \hat{O}(x)] = -\zeta \Theta(x - y) \hat{O}(x) \quad (15)$$

means that the action of $\hat{O}(x)$ on an eigenstate $|\varphi\rangle$ of the operator $\hat{\phi}$ changes it to the state $|\varphi'\rangle$ with the eigenvalue $\varphi(y) - \zeta \Theta(x - y)$, which is exactly the kink of height ζ at position x .

This point underlies the bosonization “miracle” [26]. As reviewed in Appendix A, the operator

$$\hat{\psi}_{\pm}(x) = \frac{1}{\sqrt{2\pi\alpha}} \exp \left[\mp i\sqrt{\pi} \left(\hat{\phi}(x) \mp \int_{-\infty}^x dx' \hat{\Pi}(x') \right) \right] e^{-ip_0 x} \quad (16)$$

creates right- and left-moving Dirac fermions with the correct Fermi statistics [24]. Notice that $\hat{\psi}_{\pm}$ contains the exponential soliton operator of Eq. (14) with $\zeta = \sqrt{\pi}$, implying that these fermions are precisely the quantum solitons of the sine-Gordon field $\hat{\phi}$ appearing in Eq. (13).

The remarkable feature of Eq. (16) is the emergence of a finite momentum shift

$$p_0 = \frac{2\pi S}{a},$$

which distinguishes integer- from half-integer-spin chains. For integer S , the soliton dispersion minimum is located at the Brillouin-zone origin because $p_0 = (2\pi/a) \times \text{integer} \equiv 0$. In contrast, for half-integer S , the

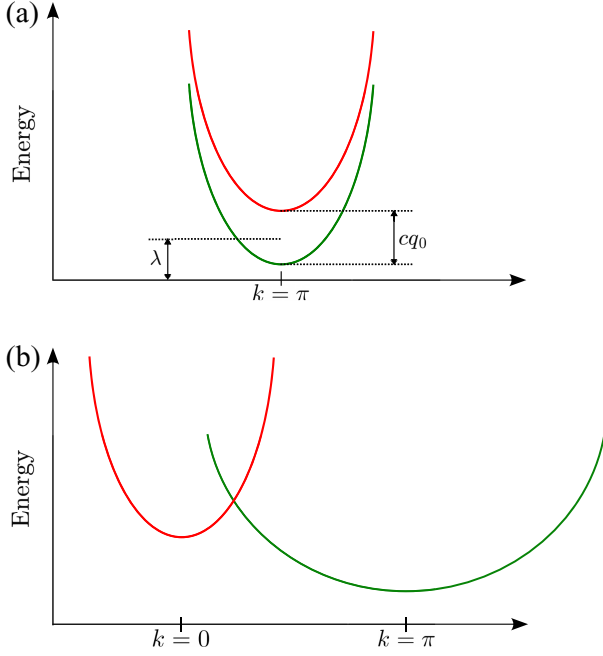


FIG. 4. (a) Schematic representation of the fermionic soliton and antisoliton bands in the massive Thirring model, both featuring minima at $k = \pi$ and possessing identical effective masses. The sine-Gordon vacuum corresponds to the half-filled case, where the chemical potential lies within the band gap [$c|q_0| < 2\lambda$ in Eq. (17)] and energies of solitons and antisolitons are positive. (b) Same as in panel (a) but for the effective lattice model, Eq. (49), where the dispersion minima occur at different momenta ($k = \pi$ for the soliton band and $k = 0$ for the antisoliton band) and the effective masses are not equal (the soliton mass is significantly larger than that of the antisoliton).

soliton band minimum is shifted to the zone boundary, $p_0 \sim \pi/a$. This topological spin-parity effect originates from the Berry-phase term in Eq. (A3) and was previously obtained in the instanton analysis of the quantum sine-Gordon model [27,28]. Our lattice-based calculation of the soliton dispersion, together with the DMRG results presented in the next sections, provides a direct and independent confirmation of this striking prediction.

Subsequent steps of the bosonization technology are well documented [see, e.g., Ref. [26] and Appendix A] and provide a rigorous way to map the sine-Gordon Hamiltonian, Eq. (13), to a massive Thirring model (mT), i.e., a Thirring model with massive Dirac fermions, Eq. (A19).

The fermionic formulation of the problem becomes particularly transparent at the special value $\kappa = 1$, known as the Luther-Emery point [25], where the mapping yields a noninteracting Hamiltonian that can be readily diagonalized. In this limit, the Hamiltonian describes two independent fermionic bands,

$$\hat{H}_{\text{mT}}^{\kappa=1} = \sum_k \epsilon_k (\hat{u}_k^\dagger \hat{u}_k + \hat{h}_k^\dagger \hat{h}_k) - \frac{cq_0}{2} (\hat{u}_k^\dagger \hat{u}_k - \hat{h}_k^\dagger \hat{h}_k), \quad (17)$$

where k is measured from p_0 , $\epsilon_k = \sqrt{c^2 k^2 + \lambda^2}$ is the dispersion relation of solitons and antisolitons [see Fig. 4(a)], $\lambda = HSa/a$ is the mass gap in the fermion spectrum, and positive (negative) q_0 plays the role of chemical potential that adds solitons $\hat{u}_k^\dagger$ (antisolitons $\hat{h}_k^\dagger$) to the system, provided $c|q_0| > 2\lambda$ [14].

It is worth noting the emergence of a $U(1)$ symmetry in the effective low-energy Hamiltonian, Eq. (17), which captures the commensurate-incommensurate transition of the lattice spin model, Eq. (1).

IV. QUANTUM SOLITONS

We now turn to the description of quantum solitons in the lattice model $\hat{H}_S$. Our starting point is an operator that creates a classical soliton—represented as a product state—when acting on the fully polarized reference state. Quantum fluctuations are incorporated through a momentum-space (Bloch-space) orthogonalization procedure, analogous to the construction of Wannier orbitals: The soliton states are first normalized in momentum space and then transformed back to real space. This method produces a set of mutually orthogonal soliton states, enabling us to compute the quantum tunneling of a soliton between different lattice sites. We emphasize that these “Wannierized” soliton states are no longer product states; the orthogonalization procedure necessarily generates entanglement in the spin degrees of freedom.

Within this framework, we can then formulate an effective tight-binding Hamiltonian, which makes explicit how quantum fluctuations generate soliton delocalization and thereby enrich the system’s low-energy dynamics.

Finally, we compare the predictions of this semiclassical framework with the quantum dynamics obtained using time-evolving block decimation (TEBD) techniques and discuss the relevance of these results for potential inelastic neutron scattering experiments.

A. Soliton operator

As shown in Appendix B, the exact ground state for $H \geq H_c$ is the fully polarized configuration along the $\hat{x}$ direction. Since this state contains no solitons, we refer to it as the vacuum state and denote it by

$$|0\rangle \equiv \bigotimes_l |\rightarrow\rangle_l, \quad (18)$$

where $|\rightarrow\rangle_l = (|\uparrow\rangle_l + |\downarrow\rangle_l)/\sqrt{2}$ is the maximal-weight eigenstate of the operator $\hat{S}_l^x$.

The operators that create a classical soliton ($\tau = +1$) or antisoliton ($\tau = -1$) centered at site j on top of the fully polarized ground state are

$$\hat{T}_{j\tau}^\dagger \equiv (\hat{S}_j^z + i\hat{S}_j^y) \prod_{l \neq j} \exp\{-i\varphi_\tau(l-j)\hat{S}_l^z\}. \quad (19)$$

The corresponding annihilation operators are

$$\hat{T}_{j\tau} = \prod_{l \neq j} \exp\{i\varphi_\tau(l-j)\hat{S}_l^z\}(\hat{S}_j^z - i\hat{S}_j^y). \quad (20)$$

As expected, these operators annihilate the vacuum: $\hat{T}_{j\tau}|0\rangle = 0$. In terms of the local spin degrees of freedom, the creation operators $\hat{T}_{j\tau}^\dagger$ rotate the spins around the $\hat{z}$ axis by the angle $\varphi_\tau(x)$ given by Eq. (5), which corresponds to the profile of the single-soliton (or antisoliton) solution of the classical sine-Gordon Hamiltonian, Eq. (2), for $m^2 = H/(JSa^2)$. To properly account for periodic boundary conditions, we define $x \equiv (l-j+L/2) \pmod{L} - L/2$.

As shown in Appendix C, the operators $\{\hat{T}_\tau^\dagger, \hat{T}_\tau, \hat{S}^x\}$ satisfy the following commutation relations:

$$\begin{aligned} [\hat{T}_{j'\tau}, \hat{T}_{j\tau}^\dagger] &= 2\tilde{\delta}_{j'\tau}\hat{S}_j^x, \\ [\hat{T}_{j\tau}^\dagger, \hat{S}_j^x] &= 2\hat{T}_{j\tau}^\dagger, \\ [\hat{T}_{j\tau}, \hat{S}_j^x] &= -2\hat{T}_{j\tau}, \end{aligned} \quad (21)$$

where the function $\tilde{\delta}_{j'\tau}$ plays the role of an approximate Kronecker delta in a coarse-grained description. Equivalently, upon choosing the soliton size as the unit of length, the set $\{\hat{T}_\tau^\dagger, \hat{T}_\tau, \hat{S}^x\}$ provides an approximate realization of the $\mathfrak{su}(2)$ Lie algebra.

As follows from the algebraic structure, the $\hat{T}$ operators obey bosoniclike commutation relations. In a coarse-grained description, solitons can therefore be interpreted as hard-core bosons. Moreover, in the dilute limit, that is, when the soliton density is sufficiently small, this identification becomes exact.

B. Quantum soliton operator: Wannierization of the soliton operator

A drawback of the operators $\hat{T}_{j\tau}$ introduced above is that the single-particle states $\hat{T}_{j\tau}^\dagger|0\rangle = |\varphi_\tau^{(j)}\rangle$ that they generate,

$$|\varphi_\tau^{(j)}\rangle = \bigotimes_l \frac{1}{\sqrt{2}} (e^{-i\varphi_\tau(l-j)/2} |\uparrow\rangle_l + e^{i\varphi_\tau(l-j)/2} |\downarrow\rangle_l), \quad (22)$$

are not mutually orthogonal,

$$\langle\varphi_\tau^{(j)}|\varphi_\tau^{(j')}\rangle = \prod_{l=1}^L \cos\left[\frac{\varphi_\tau(l-j) - \varphi_\tau(l-j')}{2}\right] \neq \delta_{jj'}, \quad (23)$$

for $j, j' = 1, \dots, L$. By using a standard Wannierization procedure, we can construct a new set of single-particle orthonormal states $\{\hat{\mathfrak{T}}_{j\tau}^\dagger|0\rangle\}$ given by

$$\hat{\mathfrak{T}}_{j\tau}^\dagger|0\rangle \equiv \frac{1}{\sqrt{L}} \sum_{k \in \text{1BZ}} e^{-ikj} \frac{\hat{T}_{k\tau}^\dagger}{\|\hat{T}_{k\tau}^\dagger|0\rangle\|} |0\rangle, \quad (24)$$

where

$$\hat{T}_{k\tau}^\dagger = \frac{1}{\sqrt{L}} \sum_{l=1}^L e^{ikl} \hat{T}_{l\tau}^\dagger \quad (25)$$

is the Fourier transform of the soliton operator with $k \in [-\pi, \pi)$. Equation (24) can be rewritten in the form

$$\begin{aligned} \hat{\mathfrak{T}}_{j\tau}^\dagger|0\rangle &= \frac{1}{\sqrt{L}} \sum_k e^{-ikj} \frac{\hat{T}_{k\tau}^\dagger|0\rangle}{\|\hat{T}_{k\tau}^\dagger|0\rangle\|} \\ &= \sum_{l=1}^L \left(\frac{1}{L} \sum_k \frac{e^{-i(j-l)k}}{\|\hat{T}_{k\tau}^\dagger|0\rangle\|} \right) \hat{T}_{l\tau}^\dagger|0\rangle \\ &\equiv \sum_{l=1}^L w_{j-l\tau} \hat{T}_{l\tau}^\dagger|0\rangle. \end{aligned} \quad (26)$$

From the last term, it is evident that the coefficients

$$w_{j-l\tau} = \frac{1}{L} \sum_k \frac{e^{-i(j-l)k}}{\|\hat{T}_{k\tau}^\dagger|0\rangle\|} \quad (27)$$

represent the components of the Wannierized operator in the original basis spanned by the $\hat{T}_{j\tau}^\dagger$ operators.

The norm of the Fourier-transformed soliton operator, $N_{k\tau} \equiv \|\hat{T}_{k\tau}^\dagger|0\rangle\|$, can be evaluated analytically:

$$\begin{aligned} N_{k\tau}^2 &= \frac{1}{L} \sum_{l,l'=1}^L e^{-ik(l-l')} \prod_p \cos\left[\frac{\varphi_\tau(p-l) - \varphi_\tau(p-l')}{2}\right] \\ &= \sum_{l=1}^L e^{-ikl} \prod_p \cos\left[\frac{\varphi_\tau(p) - \varphi_\tau(p-l)}{2}\right], \end{aligned} \quad (28)$$

where, in the second line, we have used translation invariance of the chain (i.e., periodic boundary conditions).

Let us reflect on the previous construction. We began with a well-localized sine-Gordon soliton (or antisoliton) creation operator $\hat{T}_{j\tau}^\dagger$. Through a Wannierization procedure, we constructed a new set of operators as linear combinations of solitons centered at all sites,

$$\hat{\mathfrak{T}}_{j\tau}^\dagger = \sum_{l=1}^L w_{j-l\tau} \hat{T}_{l\tau}^\dagger, \quad (29)$$

with coefficients $w_{j-l\tau}$ that, formally, are the overlaps $w_{j-l\tau} \propto \langle 0|\hat{\mathfrak{T}}_{j\tau}^\dagger \hat{T}_{l\tau}^\dagger|0\rangle$. As shown in Appendix D, these overlaps decay exponentially, $w_{j-j'} \sim e^{-2m|j-j'|}$ for large separation $|j-j'|$.

The Wannier-like operators create a soliton wave packet centered at site j that incorporates quantum delocalization via the generation of entangled single-soliton states that are orthonormal for solitons centered around different sites.

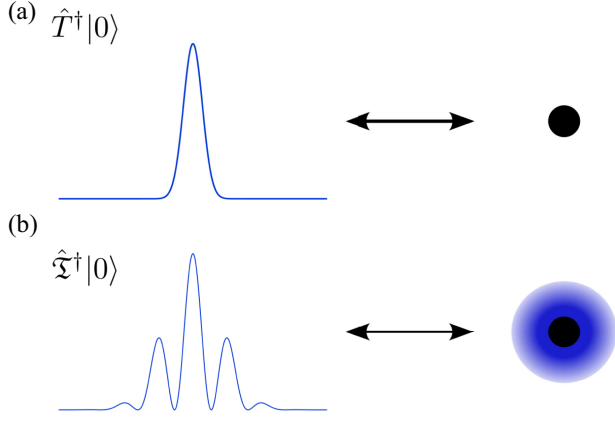


FIG. 5. Schematic representation of the action of the soliton operator (a) and the Wannierized soliton operator (b).

By contrast, the bare states generated by $\hat{T}_{j\tau}^\dagger|0\rangle$ are non-orthogonal product states. The structure of the Wannierized soliton state is sketched in Fig. 5. Henceforth—with a slight abuse of terminology—we refer to these objects as quantum solitons since their strongly localized wave packets still support a quasiparticle interpretation. Equivalently, the Wannierization procedure furnishes a variational construction of the single-soliton states of the spin- $\frac{1}{2}$ chain. Related “quantum skyrmion” operators have been introduced in Refs. [29–31]; similarities and differences with our construction are discussed in Sec. VII.

C. Soliton kinematics

Before addressing the dynamics and deriving an effective model for the solitons, let us first analyze their kinematic properties. The Fourier transform of the Wannierized soliton operator is simply given by

$$\hat{\mathcal{T}}_{k\tau}^\dagger|0\rangle = \frac{1}{\|\hat{T}_{k\tau}^\dagger|0\rangle\|} \hat{T}_{k\tau}^\dagger|0\rangle, \quad (30)$$

that is, the normalized Fourier transform of the classical soliton operator. As expected, the solitons (antisolitons) carry a finite magnetization along the x direction, $\Delta M_{\tau k}^x = M_{\text{FP}}^x - M_{\tau k}^x$, which varies continuously with the momentum k as presented in Fig. 6(a). More interestingly, the solitons carry a finite magnetization component $M_{\tau k}^z$ along the z axis (the DM direction), with a sign set by the sign of momentum [see Fig. 6(b)]. At the special (inversion-symmetric) momenta $k = 0$ and $k = \pi$, however, a mirror symmetry about the midpoint of a bond, which maps $x \rightarrow -x$, $k \rightarrow -k$ and $S^x \rightarrow S^x$, $S^{y,z} \rightarrow -S^{y,z}$, forces this magnetization to vanish. Namely, this symmetry relates $(k, M^z) \leftrightarrow (-k, -M^z)$ and guarantees a symmetric spectrum $E(k) = E(-k)$.

An analogous behavior holds for the antisoliton branch [see Fig. 6(b)], with the opposite sign of the z

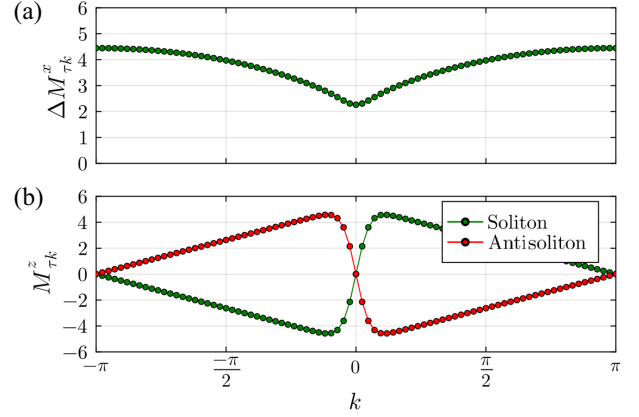


FIG. 6. Field-induced magnetization of the soliton (green) and antisoliton (red). (a) Longitudinal magnetization difference $\Delta M_{\tau k}^x$ as a function of momentum k . Only the soliton branch is shown, as the antisoliton results are identical. (b) Transverse magnetization $M_{\tau k}^z$ along the DM direction. The quantity $M_{\tau k}^z$ changes sign with the momentum and vanishes at $k = 0$ and $k = \pi$. These calculations are at $H = 0.11$.

magnetization at a given k . This momentum-dependent polarization sharply contrasts with single-magnon excitations, whose magnetization along the field axis is fixed and essentially k independent.

D. Effective tight-binding Hamiltonian

Now, assume that the chain hosts a single soliton centered at site j . Since it is a quantum-mechanical soliton, it can tunnel to the neighboring sites. The tunneling or hopping amplitude between sites j and j' is given by the matrix element:

$$t_{jj'\tau} = \langle \Phi_\tau^{(j)} | \hat{H}_S | \Phi_\tau^{(j')} \rangle, \quad (31)$$

with $|\Phi_\tau^{(j)}\rangle = \hat{\mathcal{T}}_{j\tau}^\dagger|0\rangle$. By substituting Eq. (26) into the previous expression, we obtain

$$t_{jj'\tau} = \sum_{l,l'=1}^L (w_{j-l'\tau})^* w_{j'-l\tau} \langle \varphi_\tau^{(l')} | \hat{H}_S | \varphi_\tau^{(l)} \rangle. \quad (32)$$

The explicit form of these hopping amplitudes is obtained by inserting the expressions of the Wannier coefficients $w_{j-l\tau}$ given in Eq. (27) and reordering the factors:

$$t_{jj'\tau} = \frac{1}{L^2} \sum_{k,k'} \frac{e^{-ik'j} e^{-ikj'}}{N_{k\tau} N_{k'\tau}} \sum_{l,l'=1}^L e^{ik'l'} e^{ikl} \langle \varphi_\tau^{(l')} | \hat{H}_S | \varphi_\tau^{(l)} \rangle. \quad (33)$$

Now, we can exploit the translational invariance of $\hat{H}_S$ by defining $\langle \varphi_\tau^{(l')} | \hat{H}_S | \varphi_\tau^{(l)} \rangle \equiv h_\tau(n - n')$, then

$$\begin{aligned} \sum_{l,l'=1}^L e^{ikl} e^{ik'l'} h(l-l') &= \sum_{s=1}^L h_\tau(s) e^{iks} \sum_{s'=1}^L e^{i(k+k')s'} \\ &= L \sum_s h_\tau(s) e^{iks} \delta_{k,-k'}, \end{aligned} \quad (34)$$

to obtain

$$t_{jj'\tau} = \frac{1}{L} \sum_k \frac{e^{ik(j-j')}}{N_{k\tau}^2} \sum_s h_\tau(s) e^{iks} = \frac{1}{L} \sum_k \frac{e^{ik(j-j')}}{N_{k\tau}^2} \tilde{h}_\tau(k), \quad (35)$$

where $\tilde{h}_\tau(k) = \sum_s h_\tau(s) e^{iks}$ is the Fourier transform of $h(s)$. In conclusion,

$$t_{\delta\tau} = \frac{1}{L} \sum_k \frac{\tilde{h}_\tau(k)}{N_{k\tau}^2} e^{ik\delta}, \quad (36)$$

with $\delta = j - j'$, are the hopping amplitudes of the quantum chiral soliton. The next step is to evaluate the matrix elements $h_\tau(s)$. Since the resulting expression is lengthy and offers no direct physical insight, we defer this calculation to Appendix E. As shown there, the matrix elements $h_\tau(s)$ are real, implying that $\tilde{h}_\tau(k) = \tilde{h}_\tau^*(-k)$. This finding in turn implies that the hopping amplitudes $t_{\delta\tau}$ are real.

With these hopping amplitudes, we can now construct an effective tight-binding Hamiltonian for the solitons on the chain. The last necessary ingredient is the associated soliton creation chemical potential, which can be computed as

$$\mu_\tau = -\langle \Phi_\tau^{(j)} | \hat{H}_S | \Phi_\tau^{(j)} \rangle + E_0, \quad (37)$$

where $E_0/L = -J/4 - H/2$ is the ground-state energy density, and j can be any site on the chain due to the translational invariance of the Hamiltonian. The chemical potential as a function of field is presented in Fig. 7(c).

The resulting tight-binding Hamiltonian is given by

$$\hat{H}_{\text{eff}} = \sum_{j,\delta,\tau} (t_{\delta\tau} \hat{b}_{j\tau}^\dagger \hat{b}_{j+\delta\tau} + \text{H.c.}), \quad (38)$$

where the hopping amplitudes $t_{\delta\tau}$ and the chemical potential for each flavor, $\mu_\tau = -t_{0\tau}$, depend on the external field H . Note that the bosonic operators $\hat{b}_\tau$ and $\hat{b}_\tau^\dagger$ also implicitly depend on the field because the soliton size decreases with increasing H .

For periodic boundary conditions ($L + 1 \equiv 1$), the Hamiltonian is diagonalized in reciprocal space,

$$\hat{b}_{k\tau} = \frac{1}{\sqrt{L}} \sum_{l=1}^L e^{ilk} \hat{b}_{l\tau}, \quad (39)$$

to obtain

$$\hat{H}_{\text{eff}} = \sum_{k,\tau} [\omega_k^\tau - \mu_\tau] \hat{b}_{k\tau}^\dagger \hat{b}_{k\tau}, \quad (40)$$

where

$$\omega_k^\tau = \sum_{\delta=1}^n t_{\delta\tau} \cos k\delta. \quad (41)$$

The number n of neighbors retained in the calculation is determined by the decay rate of the hopping amplitudes $|t_{\delta\tau}|$ with the separation δ . We also note that $\omega_k^\tau = \omega_{-k}^\tau$, a direct consequence of the reality of the hopping amplitudes $t_{\delta\tau} = t_{\delta\tau}^*$.

We defer the discussion of the soliton dispersion ω_k^τ to the next section and conclude here by analyzing the dependence of the hopping amplitudes on the external magnetic field. Figures 7(a) and 7(b) show the hopping amplitudes up to fourth neighbors as a function of the field.

For the soliton, the nearest-neighbor hopping amplitude dominates over the entire field range, except within a narrow window around $H = H^* \simeq 0.2$, where it changes sign: $t_{1+} > 0$ for $H < 0.2$ and $t_{1+} < 0$ for $H > 0.2$. This sign reversal shifts the minimum of the soliton dispersion from $k = \pi$ (for $H < 0.2$) to $k = 0$ (for $H > 0.2$). In addition, all hopping amplitudes are strongly suppressed near H^* , leading to an almost flat band. The change of sign of t_{1+} is expected, since the soliton evolves continuously into a magnon as the field increases, and the magnon dispersion attains its minimum at $k = 0$, as shown in Fig. 3.

In contrast, for the antisoliton the nearest-neighbor hopping also dominates but remains negative throughout the entire field range. At higher fields, the nearest-neighbor term essentially becomes the only appreciable contribution for both the soliton and antisoliton, with its magnitude approaching the expected magnon-hopping value in the fully polarized limit.

While our main discussion focused on the $S = \frac{1}{2}$ version of $\hat{H}_S$, Appendix F shows that the lattice formulation for arbitrary S correctly reproduces the integer and half-integer alternation of the soliton-band minimum discussed in Sec. III and reported in Ref. [28] for a particular limit of the model. This alternation is caused by a spin Berry phase [27].

E. Jordan-Wigner transformation

Since the effective model derived in the previous section describes a dilute gas of hard-core bosons in one dimension (1D), it is convenient to employ a generalized Jordan-Wigner transformation [32], which maps hard-core bosons with two flavors ($\tau = \pm$) to constrained ($\hat{c}_{j,+}^\dagger \hat{c}_{j,-}^\dagger = 0$) spin-1/2 fermions:

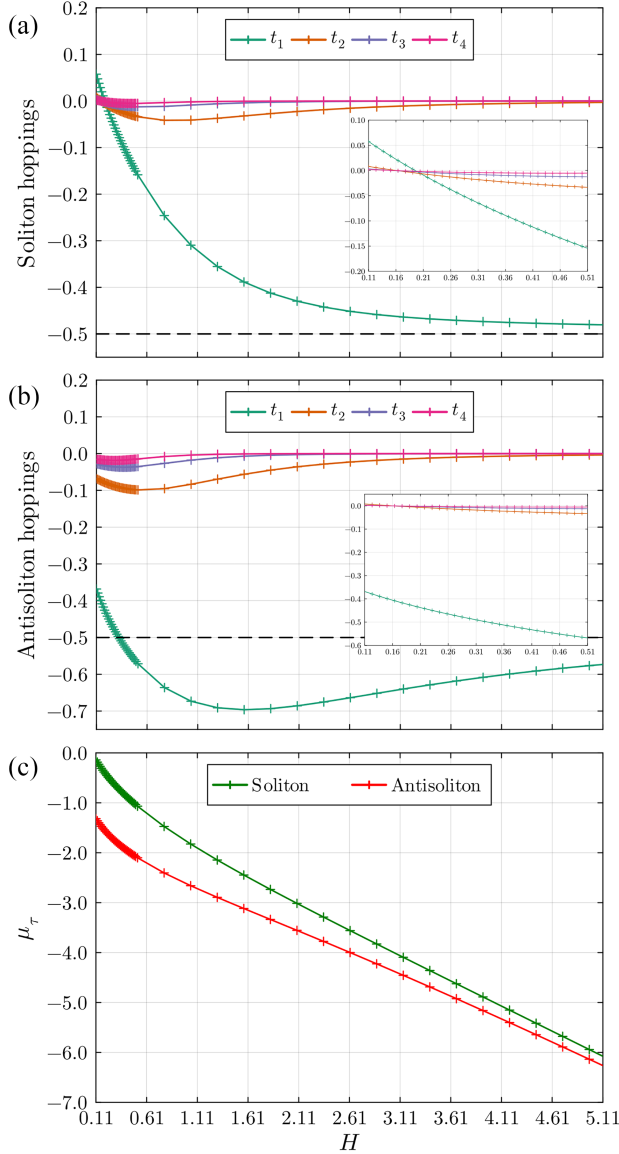


FIG. 7. Soliton (a) and antisoliton (b) hopping amplitudes and (c) chemical potentials as a function of the Zeeman field. The inset in each of the first two panels is a zoom of a small field region. The dashed line in panels (a) and (b) represents the single magnon hopping amplitude.

$$\begin{aligned}\hat{c}_{j,+} &= e^{i\pi \sum_{\ell < j} \hat{n}_{\ell}^{(b)}} \hat{b}_{j,+}, & \hat{c}_{j,-} &= e^{i\pi \sum_{\ell < j} \hat{n}_{\ell}^{(b)}} \hat{b}_{j,-}, \\ \hat{c}_{j,+}^{\dagger} &= \hat{b}_{j,+}^{\dagger} e^{-i\pi \sum_{\ell < j} \hat{n}_{\ell}^{(b)}}, & \hat{c}_{j,-}^{\dagger} &= \hat{b}_{j,-}^{\dagger} e^{-i\pi \sum_{\ell < j} \hat{n}_{\ell}^{(b)}},\end{aligned}$$

where

$$\hat{n}_{\ell}^{(b)} = \sum_{\tau} \hat{b}_{j\tau}^{\dagger} \hat{b}_{j\tau} = \hat{n}_{\ell,+}^{(c)} + \hat{n}_{\ell,-}^{(c)} \equiv \hat{n}_{\ell}^{(c)}, \quad (42)$$

and $\hat{n}_{\ell,\tau}^{(c)} \equiv \hat{c}_{\ell,\tau}^{\dagger} \hat{c}_{\ell,\tau}$. Note that the constraint satisfied by the fermionic operators, $\hat{c}_{j,+}^{\dagger} \hat{c}_{j,-}^{\dagger} = 0$, implies that double-occupied states are excluded from the Hilbert space.

The real-space effective Hamiltonian, Eq. (38), can now be expressed in terms of the fermionic operators:

$$\hat{H}_{\text{eff}} = \sum_{j,\delta,\tau} t_{\delta,\tau} [\hat{c}_{j,\tau}^{\dagger} \hat{S}_{j,\delta} \hat{c}_{j+\delta,\tau} + \text{H.c.}], \quad (43)$$

with

$$\hat{S}_{j,\delta} = \prod_{j \leq \ell < j+\delta} [1 - 2\hat{n}_{\ell}^{(c)}] \quad (44)$$

being the Jordan-Wigner string.

By separating the noninteracting part of $\hat{H}_{\text{eff}}$ from the interaction terms, we obtain

$$\hat{H}_{\text{eff}} = \sum_{j,\delta,\tau} [t_{\delta,\tau} \hat{c}_{j,\tau}^{\dagger} \hat{c}_{j+\delta,\tau} + \text{H.c.}] + \hat{H}_{\text{eff}}^{\text{int}}, \quad (45)$$

where $\hat{H}_{\text{eff}}^{\text{int}}$ corresponds to correlated hopping terms arising from the density operators in Eq. (44). In reciprocal space,

$$\hat{c}_{k,\tau} = \frac{1}{\sqrt{L}} \sum_{l=1}^L e^{ilk} \hat{c}_{l,\tau}, \quad (46)$$

the noninteracting part of $\hat{H}_{\text{eff}}$ becomes diagonal,

$$\hat{H}_{\text{eff}} = \sum_{k,\tau} [\omega_k^{\tau} - \mu_{\tau}] \hat{c}_{k,\tau}^{\dagger} \hat{c}_{k,\tau} + \hat{H}_{\text{eff}}^{\text{int}}, \quad (47)$$

and the connection with Thirring's model with $\kappa = 1$ becomes evident after relabeling operators:

$$\hat{u}_k \equiv \hat{c}_{k,+}, \quad \hat{h}_k \equiv \hat{c}_{k,-}, \quad (48)$$

which leads to

$$\hat{H}_{\text{eff}} = \sum_{k,\tau} [\epsilon_k^{+} - \mu] \hat{u}_k^{\dagger} \hat{u}_k + [\epsilon_k^{-} - \mu] \hat{h}_k^{\dagger} \hat{h}_k + \hat{H}_{\text{eff}}^{\text{int}}, \quad (49)$$

with

$$\mu = \frac{\mu_{+} + \mu_{-}}{2}, \quad \delta\mu = \frac{\mu_{+} - \mu_{-}}{2}, \quad \epsilon_k^{\pm} = \omega_k^{\pm} \mp \delta\mu.$$

Similar to the Thirring model, the noninteracting sector of $\hat{H}_{\text{eff}}$ describes two gapped fermions with the chemical potential determined by the magnetic field. A quantum phase transition occurs when the chemical potential reaches the bottom of the soliton band. Owing to the U(1) invariance of $\hat{H}_{\text{eff}}$, associated with the conservation of soliton charge, the corresponding critical point is a free-fermion fixed point ($\kappa = 1$) characterized by a correlation-length exponent $\nu = 1/2$ and a dynamical exponent $z = 2$, reflecting the quadratic dispersion of the free fermions.

The interaction terms in $\hat{H}_{\text{eff}}^{\text{int}}$ are irrelevant under the renormalization group because the no-double-occupancy constraint forces them to involve spatial derivatives in the continuum limit. Beyond the critical point, however, once the bottom of the conduction band is populated and a finite fermion density develops, these interactions become marginally relevant. The system then enters a Tomonaga-Luttinger liquid phase with Luttinger parameter $\kappa < 1$ in the presence of repulsive interactions.

While the massive Thirring model in Eq. (17) and $\hat{H}_{\text{eff}}$ in Eq. (49) share the same universal low-energy physics, it is nevertheless instructive to examine their qualitative and quantitative differences. Such a comparison clarifies the role of quantum fluctuations and of the higher-order terms in the gradient expansion leading to the sine-Gordon field theory, which is not fully captured by traditional semiclassical treatments. As we will show in the following sections, these nonuniversal aspects play a crucial role in achieving a quantitative description of the soliton dynamics and in guiding the experimental characterization of the phenomena under study.

The first qualitative difference is that, in the massive Thirring model, the soliton and antisoliton have identical effective masses—their parabolic dispersions differ only by a constant vertical shift [Fig. 4(a)]. This property, which is a consequence of neglecting cubic and higher-order terms in the gradient expansion of the DM interaction, is naturally absent in the lattice effective Hamiltonian $\hat{H}_{\text{eff}}$, where the two bands generally acquire distinct curvatures, as shown in Fig. 4(b). Moreover, the minimum of the soliton band can shift to $k = \pi$ because the nearest-neighbor hopping amplitude becomes positive near the saturation field, as illustrated in Fig. 7. A further qualitative distinction is that the effective masses themselves exhibit a pronounced dependence on the applied field, as shown in the same figure.

Beyond these qualitative differences, $\hat{H}_{\text{eff}}$ provides the full soliton and antisoliton dispersions across the entire Brillouin zone. As we discuss in Sec. VI, the field dependence of these dispersions is crucial for identifying experimental signatures of solitons and antisolitons, both in inelastic neutron scattering spectra and in specific-heat measurements.

V. NUMERICAL RESULTS

Throughout this section, we present numerical results for the full quantum model to test and validate the arguments and effective theory developed in the preceding section. Although the classical picture of well-defined solitons in real space is blurred by quantum fluctuations, we show that the essential classical features survive, most notably the continuous field-driven phase transition evident in the magnetization curve. We further demonstrate that, over a finite range of magnetic fields, the lowest-lying excited

states above the ground state are indeed solitons. Building on this idea, we compute their dynamics by evaluating a soliton-soliton dynamical correlator, which, as anticipated, is accurately captured by the soliton effective Hamiltonian constructed earlier. Finally, we discuss the implications of these soliton modes for the full spin-spin dynamical correlator.

All numerical results are obtained using DMRG and TEBD calculations implemented in the ITensor.jl library (v0.9) [33,34] under PBC. Although DMRG with PBC is computationally demanding due to the rapid growth of the bond dimension, open boundary conditions (OBC) would explicitly break translational symmetry and introduce an artificial attractive potential that localizes solitons near the edges. To prevent such artifacts, we perform all simulations on finite rings of manageable size. Our main results correspond to systems with $L = 48$ and $L = 84$ sites, for which no appreciable finite-size effects are observed.

A. Static properties

To assess the correspondence between the quantum soliton solution and the classical result presented in the previous section, we begin by examining the magnetization curve shown in Fig. 8. The agreement between the classical prediction and the quantum numerical data is remarkably good, indicating that the classical picture provides a reliable foundation for understanding the properties of the quantum soliton.

The numerically obtained saturation field, $H_c = 0.105$, is in very good agreement with the classical estimate $H_c = 0.1028$ given in Eq. (7). As expected, the magnetization curve becomes progressively smoother for larger system sizes since the discrete jumps associated with the

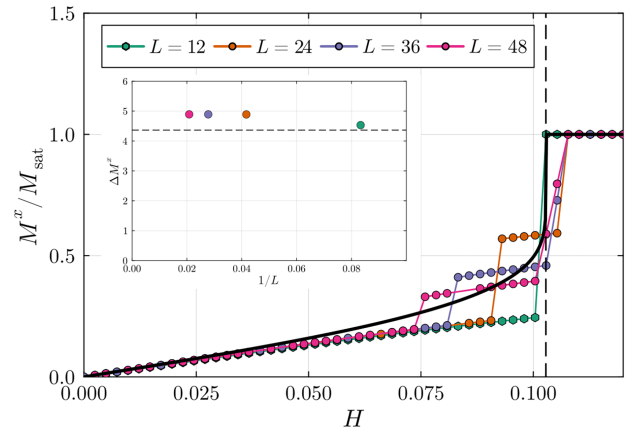


FIG. 8. Magnetization curves obtained from DMRG calculations for different system sizes. The dashed vertical line marks the critical field in the classical continuum limit, while the solid black curve represents the classical and continuum limit. Inset: absolute magnetization jump, ΔM^x , as a function of the inverse system size. The dashed horizontal line shows the continuum-limit prediction for classical solitons.

successive addition of individual solitons diminish in relative magnitude. In the thermodynamic limit, the curve is expected to approach the smooth behavior predicted by the classical continuum solution. An important distinction, however, is that the logarithmic divergence of the slope of the classical magnetization curve as the saturation field is approached from below is replaced, in the quantum case, by a square-root divergence, reflecting the free-fermion fixed point that governs the critical behavior.

The size of the soliton at the critical field can be inferred from the height of the first magnetization jump. From the inset of Fig. 8, we estimate a soliton size $\ell_s \approx 10a$, corresponding to $\Delta M^x \approx 5$, in excellent consistency with the continuum prediction $\Delta M^x \approx 4.36$ obtained from Eq. (6).

While the magnetization curve shows good agreement with classical predictions, it is important to emphasize that, in the chiral-soliton phase, the solitons become fully delocalized. As a consequence, the expectation values of the spin components develop an essentially flat spatial profile along the chain, illustrating how quantum fluctuations melt the classical soliton crystal and restore the translational invariance that is broken in the classical solution.

To further characterize the excitations, we now examine the lowest excited state above the fully saturated phase ($H > H_c$). Figure 9(a) shows the evolution of the energy gap $\Delta = E_1 - E_0$ as a function of the magnetic field for a chain of length $L = 48$, where E_0 and E_1 denote the ground-state and first excited-state energies, respectively.

As shown in Fig. 9(a), the slope of the energy gap Δ changes discontinuously around $h \equiv H - H_c \approx 0.03$, signaling the expected qualitative change in the nature of the lowest-energy excitations. This transition is further corroborated by the magnetization difference $\Delta M_{\text{exc}}^x = M_{\text{FP}}^x - M_1^x$, plotted in Fig. 9(b). For fields $h > 0.03$, the excitations carry $\Delta M_{\text{exc}}^x = \pm 1$, consistent with single-magnon modes. In contrast, for $0 < h < 0.03$, the first

excited state exhibits a magnetization jump of $\Delta M_{\text{exc}}^x \approx 5$, consistent with the soliton size identified earlier.

We also note that the energy gap scales linearly with $H - H_c$, in agreement with the emergent U(1) symmetry of the free-fermion fixed point: $\Delta \propto (H - H_c)^{\nu_z}$ with $\nu = 1/2$ and $z = 2$, as discussed in the previous section.

B. Dynamical spin structure factor

As shown previously, for sufficiently large values of H , the lowest excitations of the system are $S = 1$ magnons. In this regime, we expect to recover the LSWT prediction with a magnon dispersion given by Eq. (12). To test this expectation, we evaluate the real-space, real-time spin-spin correlation function,

$$S^{\alpha\alpha}(j - j', t) = \langle 0 | \hat{S}_j^\alpha(t) \hat{S}_{j'}^\alpha(0) | 0 \rangle, \quad (50)$$

where $|0\rangle$ denotes the fully polarized ground state and $\alpha = x, y, z$. These correlations are computed using the TEBD method, as described in Appendix G. By applying spatial and temporal Fourier transforms, we obtain the dynamical spin structure factor (DSSF), $\tilde{S}(q, \omega) = \sum_\alpha \tilde{S}^\alpha(q, \omega)$, with

$$\tilde{S}^{\alpha\alpha}(k, \omega) = \frac{1}{2\pi L} \sum_{l, l'} \int_{-\infty}^{\infty} dt e^{i[\omega t - k(l - l')]} S^{\alpha\alpha}(l - l', t). \quad (51)$$

The most relevant contributions are the transverse components with $\alpha = y, z$, which correspond to excitations perpendicular to the fully polarized order along the $\hat{x}$ direction. In contrast, the longitudinal component ($\alpha = x$) probes fluctuations parallel to the polarization, which are not expected in the fully polarized state. This expectation is confirmed by our numerical results, which show a negligible response in the longitudinal channel.

In Fig. 10, we present a direct comparison between the DSSF obtained from TEBD and the magnon dispersion predicted by LSWT at $H = 5.0$. The results show excellent agreement, demonstrating that LSWT provides a complete description of the excitation spectrum in this high-field regime.

C. Dynamical soliton structure factor

In Sec. IV, we introduced an effective tight-binding Hamiltonian describing the propagation of a soliton (or antisoliton) excitation over the fully polarized background. Although this construction is based on simple physical arguments, it is crucial to test its validity by confronting its predictions with numerical results. To this end, we define the real-space and real-time soliton-soliton ($\tau = +$) or antisoliton-antisoliton ($\tau = -$) correlation function,

$$\mathcal{T}_\tau(j - j', t) \equiv \langle 0 | \hat{T}_{j\tau}(t) \hat{T}_{j'\tau}^\dagger(0) | 0 \rangle, \quad (52)$$

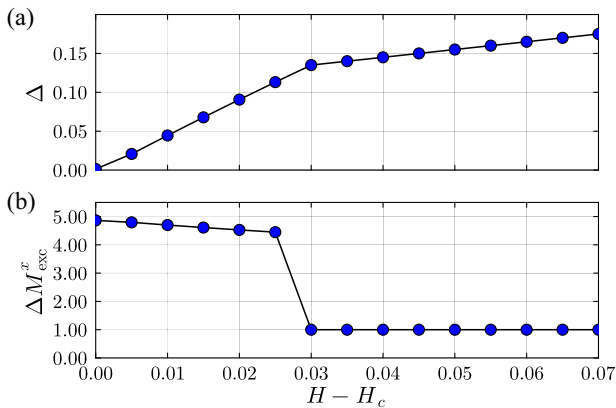


FIG. 9. (a) Energy gap, $\Delta = E_1 - E_0$, and (b) difference between the ground-state magnetization and that of the first excited state, ΔM_{exc}^x , for a chain of $L = 48$ sites, plotted as a function of the external field $H - H_c$. Here, H_c denotes the critical field obtained from DMRG calculations.

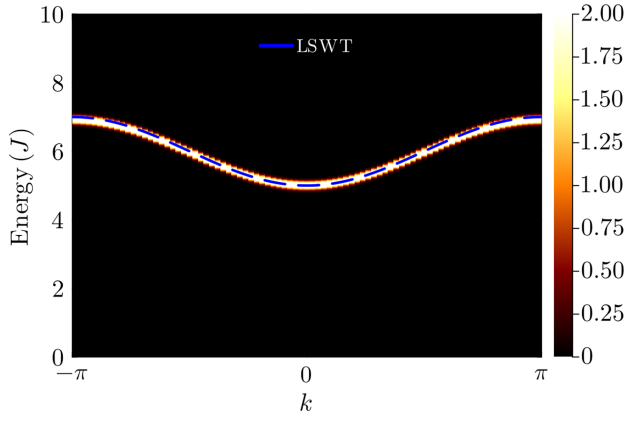


FIG. 10. DSSF for a chain of $L = 48$ and $H = 5 \approx 50H_c$ computed by TEBD. The blue dashed lines show the LSWT magnon dispersion. The time step used for the TEBD calculation is $\delta t = 0.01$ with a truncation error of 10^{-12} for each application of the time evolution operator; the total evolution time is $T_f = 40$ (in units of J^{-1}).

which measures the probability amplitude of finding a soliton at site j and time t , given that a soliton was created at site j' at $t = 0$ on top of the ground state.

The corresponding dynamical soliton or antisoliton spectral function is obtained from the space-time Fourier transform of $\mathcal{T}_\tau(j - j', t)$,

$$\tilde{\mathcal{T}}_\tau(k, \omega) = \frac{1}{2\pi L} \sum_{l, l'} \int_{-\infty}^{\infty} dt e^{i[\omega t - k(l - l')]} \mathcal{T}_\tau(l - l', t), \quad (53)$$

which directly encodes the soliton dispersion and lifetime. This representation provides the natural quantity for comparison with the predictions of the effective tight-binding Hamiltonian.

Here, we employ the original soliton creation operator [see Eq. (19)] rather than its Wannierized form. It is worth noting that using the Wannierized form of the soliton operators in Eq. (52) would only affect the spectral weight of the δ -function peaks in $\tilde{\mathcal{T}}_\tau(k, \omega)$. Since our primary interest lies in extracting the soliton dispersion relation—that is, the frequency of the peak for each value of k —from numerical calculations, and because $\tilde{\mathcal{T}}_\tau(k, \omega)$ is not directly observable, it suffices to define $\mathcal{T}_\tau(j - j', t)$ in terms of the non-Wannierized soliton operators.

The top panel of Fig. 11 presents the numerical results for the dynamical soliton structure factor at several values of the magnetic field, together with the soliton bands predicted by the analytical theory. The agreement is nearly perfect: The predicted soliton bands reproduce the numerically obtained dispersions with high accuracy. The apparent broadening of the numerical spectra originates primarily from the finite frequency resolution imposed by the truncation of the time evolution.

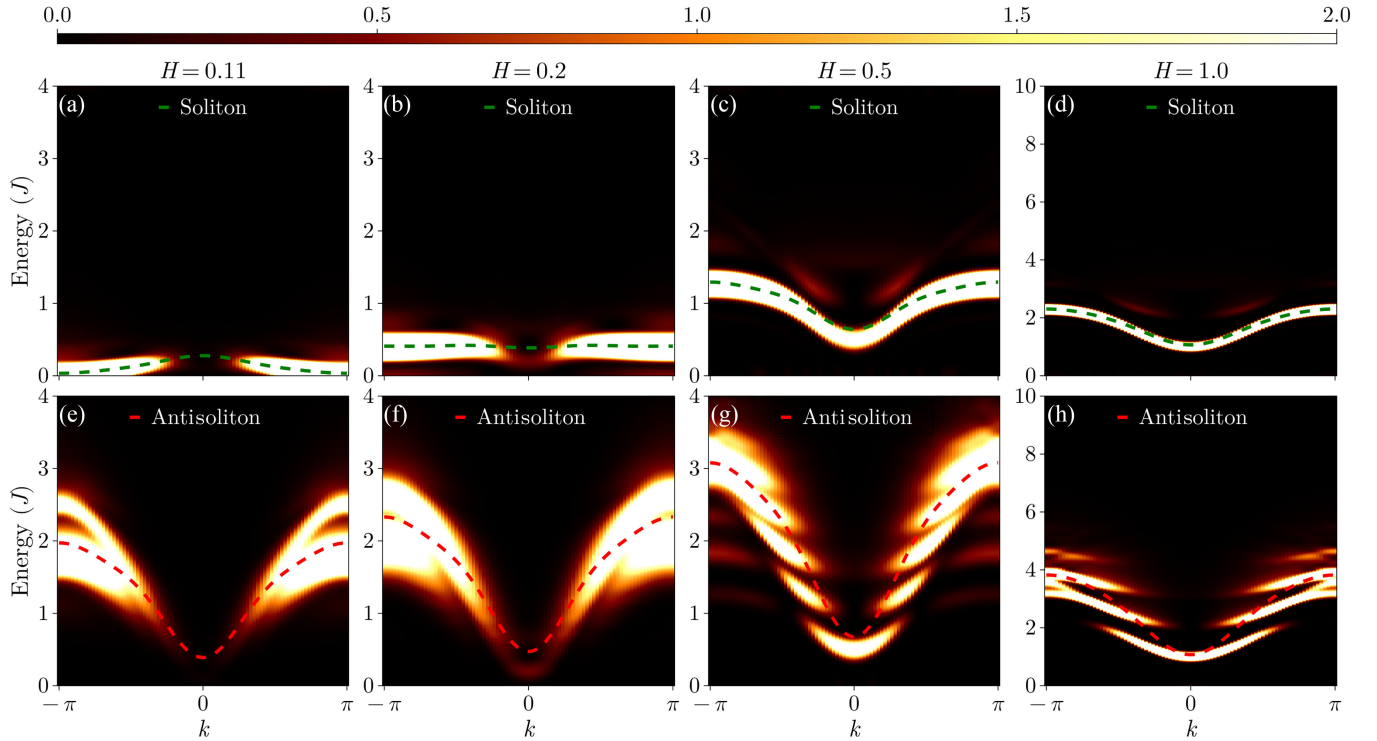


FIG. 11. Dynamical soliton-soliton (top) and antisoliton-antisoliton structure factor (bottom) for a chain of $L = 84$ sites at (a),(e) $H = 0.11$, (b),(f) $H = 0.2$, (c),(g) $H = 0.5$, and (d),(h) $H = 1.0$. Note that the vertical axis scale changes for panels (d) and (h). The TEBD parameters are the same as in Fig. 10.

The bottom panel of Fig. 11 shows the results for the dynamical antisoliton structure factor. In contrast to the soliton case, the spectrum exhibits a more intricate structure. Although the numerically computed intensities broadly follow the analytical antisoliton dispersion, additional modes appear, reconstructing the spectrum and giving rise to richer features. These extra modes are faintly visible in the dynamical soliton structure factor for fields $H = 0.5$ and 1.0 [see Figs. 11(c) and 11(d)]. This behavior provides a clear signature of hybridization between antisoliton (or soliton) modes and other excitations. In particular, because the antisoliton is always the most energetically costly excitation, it tends to strongly overlap with, for instance, the magnon mode, thereby generating the enhanced spectral complexity observed. Far from being a drawback, such mode hybridization offers a pathway to experimentally “illuminate” solitonic excitations, as discussed in the following section.

VI. EXPERIMENTAL SIGNATURES

In the previous section, we validated our effective low-energy theory against numerical calculations, showing that the sine-Gordon model provides an accurate description of the low-energy physics. To summarize, (i) the magnetization curve is well captured by the classical sine-Gordon prediction; (ii) the classical soliton solution offers a meaningful starting point for constructing a “quantum soliton” state; (iii) at the critical field, solitons soften and condense, giving rise to a Tomonaga-Luttinger liquid; and (iv) over a finite field range above saturation, solitons constitute the lowest-energy excitations.

Building on these results, we now identify experimental signatures of chiral solitons in real materials. The central finding is that magnon excitations, directly accessible through INS, acquire a substantial hybridization with the soliton bands over a finite window of magnetic fields, making the solitons observable in INS spectra. In addition, we show that the presence of soliton bands strongly modifies the field dependence of thermodynamic properties, most notably, the specific heat.

A. Magnon-soliton hybridization and inelastic neutron scattering

We begin by summarizing the main results of this section. Figure 12 displays the DSSF, defined in Eq. (51), obtained from TEBD calculations for a chain of $L = 84$ sites at magnetic fields $H = 0.11, 0.2, 0.5$, and 1.0 . As anticipated in the previous sections, the numerical spectra exhibit pronounced deviations from the predictions of linear spin-wave theory (LSWT). The resulting excitation spectrum reveals several distinctive features that we now analyze and relate to the effective theory of quantum solitons developed earlier.

INS probes excitations with $\Delta S^x = \pm 1, 0$. In the present system, the Dzyaloshinskii-Moriya (DM) interaction breaks the $U(1)$ symmetry of $\hat{H}_S$; thus, $[\hat{H}_S, \hat{M}^x] \neq 0$, and the magnetization along the field direction is no longer conserved. Consequently, eigenstates of $\hat{H}_S$ generally involve superpositions of sectors with different magnon numbers. Two notable exceptions are the fully polarized ground state $|0\rangle$ above the saturation field and the zero-momentum single-magnon state $\hat{S}_{k=0}^z|0\rangle$, which remain exact eigenstates with magnetizations $L/2$ and $L/2 - 1$, respectively (see Appendix B).

At sufficiently large magnetic fields, the $U(1)$ symmetry is approximately restored since the energy separation between different magnon-number sectors greatly exceeds the matrix elements of the DM interaction. In this regime, the excitations probed by INS essentially correspond to single magnons throughout the Brillouin zone (see Fig. 10). As the magnetic field is reduced, however, the situation changes qualitatively.

The quantum soliton wave function constructed in Sec. IV can be viewed as a variational state consisting of a coherent superposition of single-magnon and multimagnon bound states, with the finite chirality encoded in the coefficients of the superposition. Although multimagnon bound states incur a larger Zeeman energy cost due to the presence of multiple spin flips, this cost is partially compensated by their mutual binding energy. As a consequence, the bound-state branches are shifted downward in the spectrum. In an intermediate-field regime, the energies of single-magnon and multimagnon bound states approach one another, enabling strong hybridization mediated by the DM interaction. This hybridization transfers spectral weight from the dominant single-magnon modes to the bound states, making the latter directly visible in the DSSF.

We now analyze the excitation spectrum for representative magnetic fields, proceeding from high fields toward the critical field H_c .

1. High field: $H = 1.0 \approx 10H_c$

Two principal bands are clearly visible in this regime [Fig. 12(d)]. The lower-energy branch shows excellent agreement with the soliton dispersion derived in Sec. IV. At this field strength, the classical soliton size—estimated from the magnetization jump relative to the ground state—is $\Delta M^x \approx 1.41$, placing it between the one- and two-magnon bound-state sectors. Although the spatial extent of this excitation becomes comparable to the lattice spacing at such large fields, we retain the term “soliton” to emphasize its continuous evolution into the semiclassical soliton mode as H approaches H_c .

A natural next step is to quantify the hybridization between the two branches. As shown in Appendix H, this analysis yields, within controlled approximations, the following band structure:

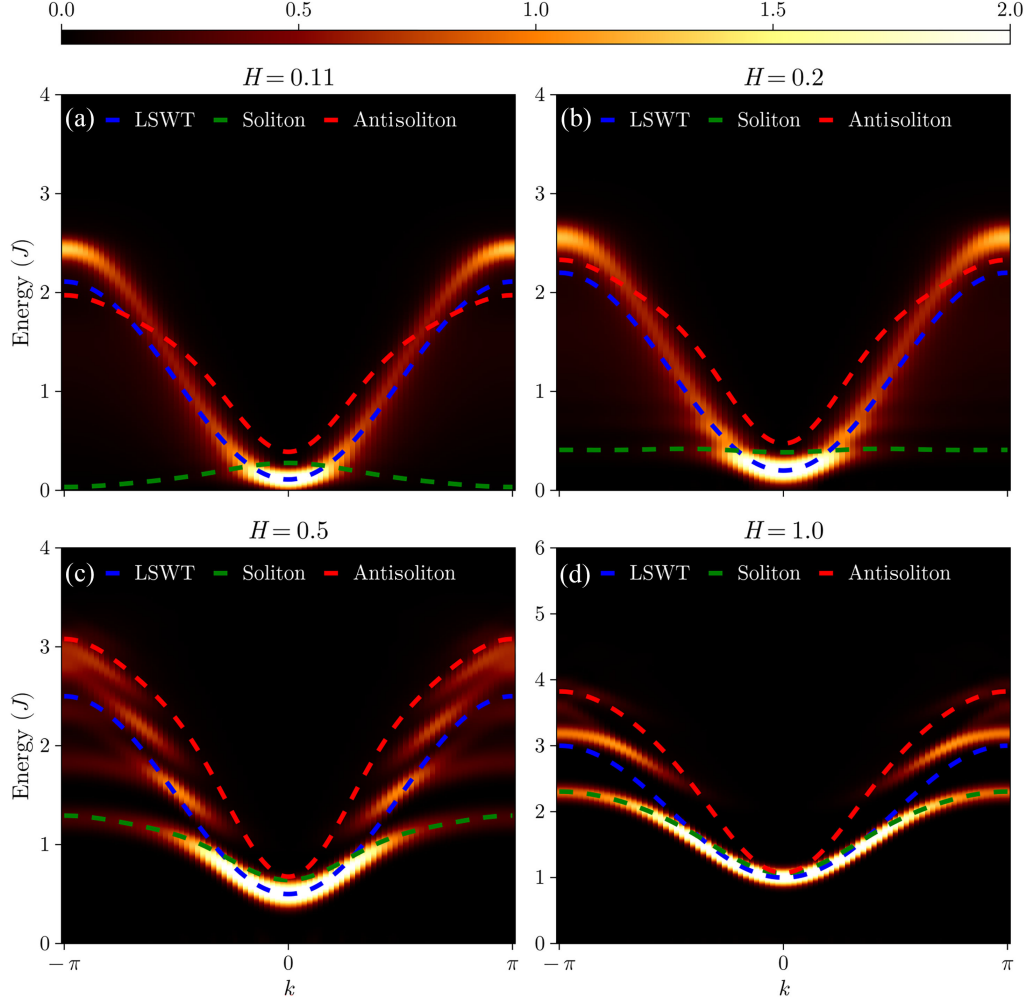


FIG. 12. Dynamical spin structure factor $\tilde{S}(q, \omega)$ for a chain of $L = 84$ sites at (a) $H = 0.11$, (b) $H = 0.2$, (c) $H = 0.5$, and (d) $H = 1.0$. The TEBD parameters are the same as in Fig. 10. Close to the critical field ($H \approx H_c$), hybridization between magnon and soliton modes is negligible because magnons with $k \approx 0$ remain nearly exact eigenstates of the Hamiltonian. At $H \approx 0.2$, incipient magnon-soliton hybridization becomes visible as the two dispersions intersect away from $k = 0$. Hybridization between the magnon band and the multisoliton continuum is also observed at larger momenta. At $H = 0.5$, the full soliton band becomes visible through its hybridization with the magnon mode. At $H = 1$, the soliton and magnon excitations become indistinguishable as the soliton size approaches the lattice spacing.

$$\begin{aligned} E_1(k) &\approx \epsilon_1(k) - \frac{D^2}{H} \sin^4(k/2) \\ E_2(k) &\approx \epsilon_2(k) + \frac{D^2}{H} \sin^4(k/2) \end{aligned} \quad (54)$$

with

$$\epsilon_1(k) = H + 2J \sin^2(k/2), \quad \epsilon_2(k) = 2H + J \sin^2(k/2).$$

As shown in Fig. 13, this simple description reproduces the high-energy band remarkably well while slightly overestimating the energy of the low-energy branch for $k > \pi/2$. The high-energy magenta band corresponds to the lower edge of the two-soliton continuum (green dots), with its spectral weight vanishing as it approaches the two-

magnon continuum (blue dots). As discussed previously, only the magnon excitation is visible in INS at $k = 0$ since $\hat{S}_{k=0}^z|0\rangle$ is an exact eigenstate.

2. Intermediate field: $H = 0.5 \approx 5H_c$

Upon lowering the field [Fig. 12(c)], the excitation spectrum becomes more intricate and displays a richer structure. Nevertheless, the lowest branch remains in excellent agreement with the soliton dispersion predicted by the effective theory. To account for the additional bands, we extend the previous analysis to include hybridization among a larger set of states: the soliton, the antisoliton, the single-magnon excitation, and a composite magnon-soliton state formed by a $k = 0$ magnon and a soliton of momentum k ,

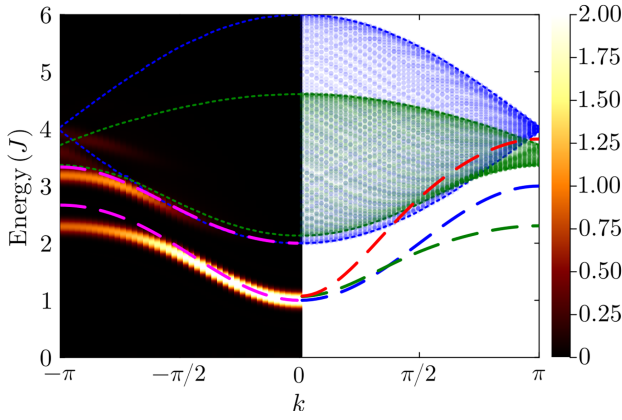


FIG. 13. Left: DSSF showing the bands obtained from Eq. (54) (magenta dashed line). Right: two-magnon continuum (blue dots) and two-soliton continuum (green dots), together with the single-magnon dispersion (blue dashed), soliton dispersion (green dashed), and antisoliton band (red dashed). Results correspond to $H = 1$.

$$|\alpha_k\rangle = \frac{\hat{S}_{k=0}^z \hat{T}_k^\dagger |0\rangle}{\sqrt{\langle 0 | \hat{T}_k (\hat{S}_{k=0}^z)^2 \hat{T}_k^\dagger | 0 \rangle}}. \quad (55)$$

These four states span the low-energy subspace $\mathcal{S}_0$. Because the corresponding variational states are not mutually orthogonal, we construct an orthonormal basis by means of Löwdin symmetric orthogonalization [35,36], which preserves, in a least-squares sense, the geometric structure of the original subspace (see Appendix I). Physically, this procedure allows each orthogonalized state to retain a well-defined dominant character—single soliton, single antisoliton, single magnon, or composite magnon-soliton. The Hamiltonian projected onto this basis then reads

$$h_{jj'}^{\text{multi}}(k) = \langle \psi_j | \hat{H}_S | \psi_{j'} \rangle - E_0 \delta_{jj'}. \quad (56)$$

Diagonalizing this matrix yields the hybridized modes shown in Fig. 14. The dispersive branches are accurately reproduced until they merge into the two-magnon and two-soliton continua. In this field range, all four excitation branches carry observable spectral weight in the DSSF, making both the soliton and antisoliton dispersions experimentally accessible.

3. Near the critical field: $H \gtrsim H_c$

As the field approaches H_c [Figs. 12(a) and 12(b)], the lowest excited state evolves into a coherent superposition of several multimagnon bound states that can be identified with the semiclassical soliton. The soliton gap decreases continuously and vanishes at $H = H_c$, where a finite density of solitons condenses into the ground state, driving

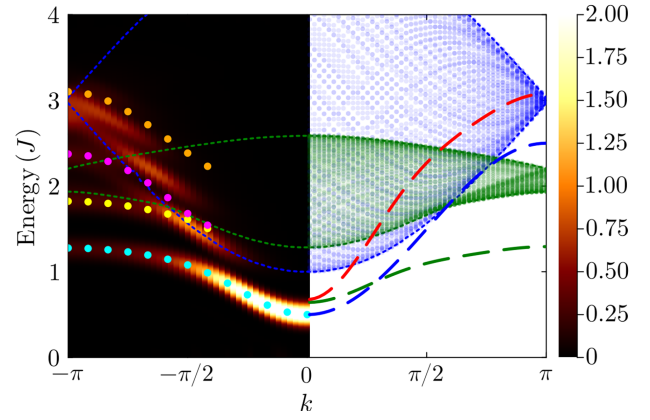


FIG. 14. Left: DSSF with hybridized bands obtained from the projection procedure described in the text. Right: two-magnon continuum (blue dots) and two-soliton continuum (green dots), together with the single-magnon, soliton, and antisoliton dispersions.

the commensurate-to-incommensurate transition into the Tomonaga-Luttinger liquid phase discussed earlier.

From a spectroscopic standpoint, the overlap between the soliton and single-magnon excitations becomes very small near H_c , rendering the soliton mode nearly invisible in INS measurements. Nevertheless, careful inspection of Figs. 12(a) and 12(b) reveals clear solitonic signatures in the excitation spectrum. These features become particularly evident in the constant-momentum cuts shown in Fig. 15, which exhibit well-defined peaks at energies consistent with the soliton dispersion.

4. Character fidelity across the field range

As a final verification of the excitation character across the full field range, we compute the squared overlap between the exact eigenstates and the variational magnon, soliton, and antisoliton states,

$$\mathcal{F}_m = |\langle n | \hat{S}_k^z | 0 \rangle|^2, \quad \mathcal{F}_\tau = |\langle n | \hat{T}_{k\tau}^\dagger | 0 \rangle|^2, \quad (57)$$

where $|n\rangle$ denotes exact eigenstates obtained from exact diagonalization for $L = 12$. The operators $\hat{S}_k^z$ and $\hat{T}_{k\tau}^\dagger$ generate normalized states when acting on $|0\rangle$, so $\mathcal{F} \leq 1$. We refer to these quantities as the fidelity. Figure 16 shows the results for the same field values as above, evaluated at momentum $k = \pi$, together with the longitudinal magnetization ΔM_{exc}^x of the corresponding eigenstates.

At $H = 1$ [Fig. 16(d)], the lowest excitation displays a pronounced solitonic character with fidelity $\mathcal{F}_s \approx 0.95$. Its longitudinal magnetization agrees closely with the value predicted for the variational soliton state. The same mode also contains a substantial magnon component, reflecting the smooth crossover between soliton and magnon excitations at lattice scales. This crossover is precisely what

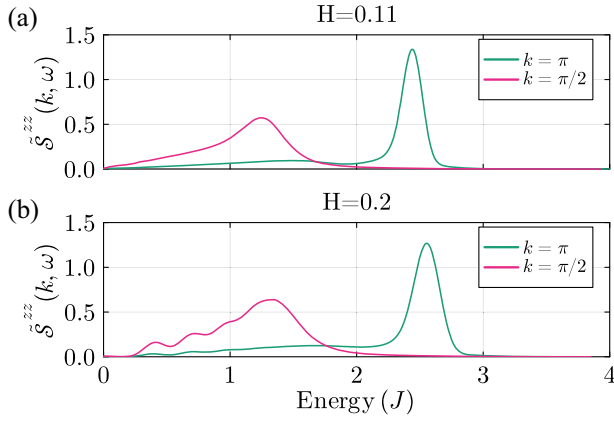


FIG. 15. Constant-momentum cuts of the DSSF $\tilde{S}(k, \omega)$ at $k = \pi$ and $k = \pi/2$ for magnetic fields $H = 0.11$ (a) and $H = 0.2$ (b).

allows INS to detect the soliton mode, as the magnon admixture transfers spectral weight to the $\Delta S^x = \pm 1$ channel.

At $H = 0.5$ [Fig. 16(c)], all four excitation branches are clearly resolved. The intermediate bands are predominantly magnonic, whereas the lowest and highest branches exhibit mainly solitonic and antisolitonic character, respectively. At higher energies, the situation becomes more complex because the magnon and antisoliton modes hybridize strongly with the two-soliton continuum, resulting in a less clearly defined excitation character.

As the field approaches H_c [Figs. 16(a) and 16(b)], the overlap with the single-magnon state decreases progressively, consistent with the soliton becoming increasingly invisible in INS. Nevertheless, the lowest excitation retains its dominant solitonic character throughout the entire field range, and its longitudinal magnetization remains in excellent agreement with the variational prediction.

B. Thermodynamic properties

The thermodynamic consequences of the emergence of solitons as low-energy excitations of the system are no less remarkable. As discussed throughout this work, for fields slightly above the saturation value, solitons constitute the lowest-energy excitations and exhibit a dispersive spectrum. Owing to the relatively flat soliton dispersion in this field range, a Schottky-like anomaly is expected to appear in the specific heat,

$$C_v = \frac{1}{L} \frac{dU}{dT}, \quad (58)$$

where U denotes the internal energy. This low-temperature feature should be followed by a broader magnon-induced contribution at higher energies. At larger fields, the Schottky anomaly is expected to merge with the magnon contribution, resulting in a single broad peak in the specific heat.

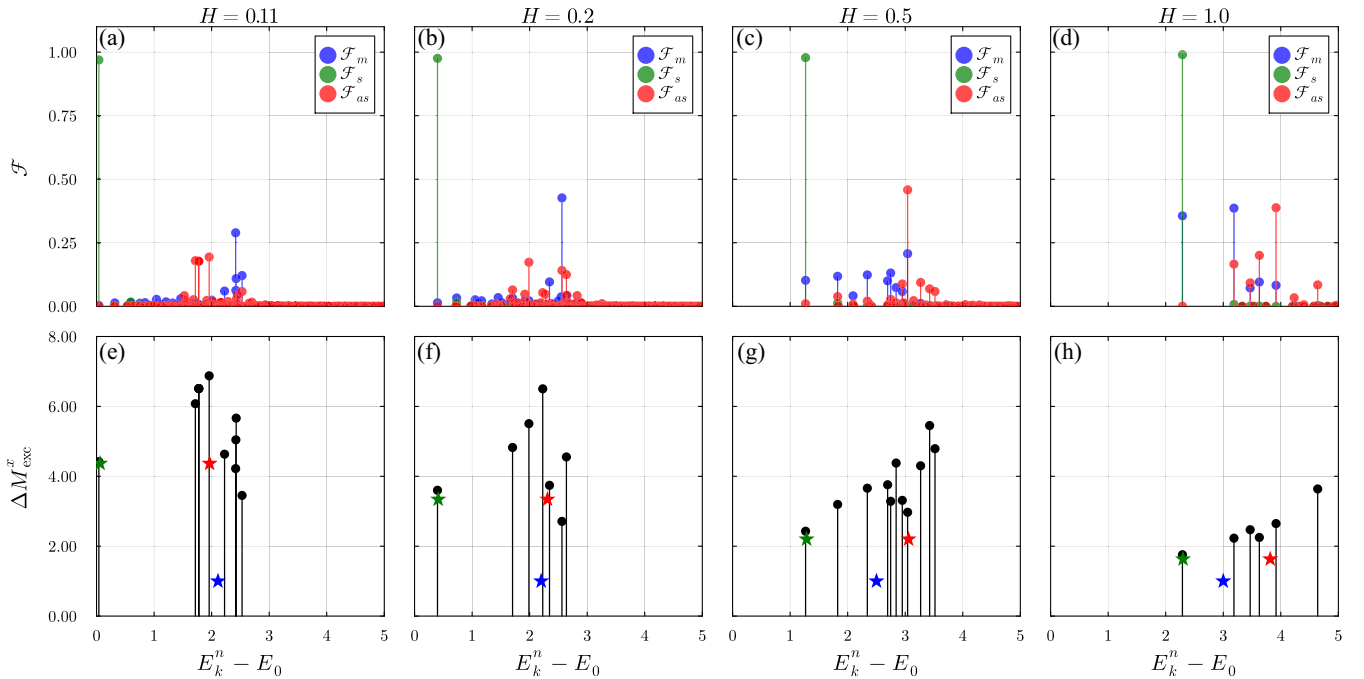


FIG. 16. (a)–(d) Character fidelity, as defined in Eq. (57). (e)–(h) Longitudinal magnetization difference ΔM_{exc}^x of the lowest-lying excitations at momentum $k = \pi$ for different magnetic fields. The stars in the bottom panels indicate the expected energy and magnetization values corresponding to the variational magnon (blue), soliton (green), and antisoliton (red) states.

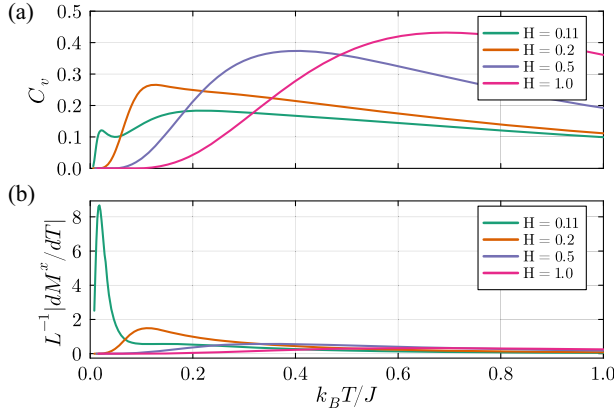


FIG. 17. Temperature dependence of the specific heat (a) and of the absolute value of the temperature derivative of the total magnetization along the field direction (b) for a system with $L = 12$ sites under different external magnetic fields.

Furthermore, because creating a soliton involves a substantial change in the total magnetization, a pronounced response is also expected in the temperature derivative of the magnetization. At the critical field $H = H_c$, the free-fermion fixed point implies a characteristic divergence $L^{-1} dM^x/dT \propto 1/\sqrt{T}$. To test this expectation, we perform exact diagonalization calculations for a system with $L = 12$ sites, and the resulting behavior is shown in Fig. 17.

These observations, together with the magnetization curve shown in Fig. 8, provide clear and accessible signatures of solitonic excitations through relatively simple experimental measurements. Thus, they offer a practical strategy for identifying promising candidate materials for INS studies, where the presence of these excitations can be unambiguously established using the analysis presented in the previous section.

VII. DISCUSSION

This work establishes a fully quantum, lattice-resolved framework for the dynamics and condensation of chiral solitons across the saturation field of a spin- $\frac{1}{2}$ monoaxial chiral helimagnet. The central message is that chiral solitons can be promoted from classical topological textures to microscopic quantum quasiparticles of the lattice Hamiltonian, with a well-defined dispersion over the full Brillouin zone, a systematically derived effective low-energy Hamiltonian in the dilute regime, and, crucially, a quantitative connection to experimentally measured response functions.

A key advantage of the lattice formulation is that it yields direct predictions for the dynamical spin structure factor $\tilde{S}(q, \omega)$ probed by inelastic neutron scattering. Continuum quantum treatments based on the sine-Gordon theory and its dual massive Thirring model provide an elegant long-wavelength description of soliton excitations, but they do not straightforwardly determine how those excitations

appear in $\tilde{S}(q, \omega)$ for a microscopic magnet. By constructing momentum-resolved many-body soliton wave functions and their associated creation operators directly from the spin Hamiltonian, our approach makes the lattice-operator content explicit and therefore allows one to compute not only excitation energies but also their spectral weights. In this way, the theory identifies a concrete detection mechanism for quantum chiral solitons: Soliton-magnon hybridization transfers spectral weight into the soliton sector, rendering solitons visible as sharp features in $\tilde{S}(q, \omega)$ above the saturation field.

From the viewpoint of the low-energy spectrum, the theory reveals how topology and quantum fluctuations reorganize the hierarchy of excitations near the transition. We show that quantum chiral solitons become the lowest-energy excitations over a finite field window $H_c < H < H^*$, while the single-magnon mode remains gapped at $H = H_c$. The persistence of a magnon gap implies that soliton-soliton interactions mediated by virtual magnons are short-ranged and exponentially suppressed in the dilute regime, enabling a controlled effective description. The resulting dual Hamiltonian, obtained directly from the microscopic model, recasts the system as a dilute gas of interacting topological quasiparticles. In the classical limit, the softening of the soliton mode at H_c drives the crystallization into a soliton lattice; in the quantum spin- $\frac{1}{2}$ chain, zero-point motion melts this classical crystal into a gapless Tomonaga-Luttinger liquid, providing a fully quantum realization of soliton condensation.

Our construction also clarifies which aspects of the physics lie beyond the standard sine-Gordon–Thirring paradigm. While the continuum theory predicts symmetric soliton and antisoliton dispersions at leading order, the lattice treatment naturally captures strong asymmetries between the two branches and their distinct momentum dependence across the Brillouin zone. These effects translate directly into measurable consequences through the momentum- and field-dependent redistribution of spectral weight in $\tilde{S}(q, \omega)$. More broadly, the Wannierization-based construction provides a systematic procedure to incorporate quantum fluctuations into topological textures without invoking a gradient expansion or large- S limit, and it exposes lattice-scale processes—such as the tunneling events responsible for soliton mobility and the associated spin-parity effects.

On the experimental side, our results provide concrete guidance for identifying quantum solitons in spectroscopy. The characteristic signatures are (i) a well-defined soliton branch above H_c with finite spectral weight acquired through hybridization with magnons, (ii) a field-tunable level repulsion and intensity transfer near the crossing of magnon and soliton dispersions for $H \gtrsim H^*$, and (iii) a reorganization of the low-energy spectral weight as the system approaches the soliton-condensation point. These predictions are quantitatively benchmarked against DMRG

calculations, demonstrating that the lattice quasiparticle description remains accurate in the strongly quantum regime where soliton cores span only a few lattice spacings.

While the present work focuses on a ferromagnetic spin chain, it is natural to ask about material realizations. Ferromagnetic chiral-chain systems are less common than antiferromagnetic ones, but several candidate platforms are known, including CrNb_3S_6 [37], CsNiF_3 [38–40], and effective $S = \frac{1}{2}$ chain compounds such as CoNb_2O_6 [41] and CsCuCl_3 [42,43]. In CsCuCl_3 , a uniform DM interaction is allowed by the lack of inversion symmetry along the chain direction, although interchain couplings are not negligible. Materials with weaker interchain coupling would provide particularly clean platforms for realizing the soliton-dominated regime described here. Another promising direction is the $S = \frac{1}{2}$ chiral compound YbNi_3Al_9 , which exhibits strong signatures of solitonic physics, although a definitive effective model is still under active development [44,45].

Perhaps the most far-reaching implication of the present work is that it provides a general nonperturbative strategy for deriving effective theories of interacting *topological* quasiparticles directly from microscopic lattice Hamiltonians. Although demonstrated here in one dimension, the framework does not rely on integrability or continuum approximations and is therefore not inherently restricted to 1D, suggesting an alternative route toward topological quantum matter in higher dimensions: Rather than starting from parton or gauge-theory constructions, one may begin with classical spin models whose ground states have well-defined soliton crystals (e.g., skyrmion or meron crystals) and develop a controlled quantum description of their dynamics, interactions, and condensation for the $S = 1/2$ limit. In this perspective, quantum liquids can emerge as the quantum-melted phases of interacting topological textures, providing a microscopic pathway toward chiral quantum spin liquids and related strongly correlated topological phases.

Recent works in $D = 2 + 1$ have explored quantum effects on skyrmion textures using complementary strategies, including the explicit construction of skyrmion creation operators and variational treatments of quantum fluctuations [29–31,46–49]. A central challenge in many classical skyrmion-hosting models is that the field-driven transition between the skyrmion crystal and the polarized state is often strongly first order, which avoids access to a dilute regime where individual topological quasiparticles are well defined and their interactions are relatively weak. Identifying microscopic Hamiltonians or parameter regimes where a continuous, density-tuned transition occurs—analogueous to the commensurate-incommensurate transition studied here—would therefore be especially valuable. In addition, numerical simulations in two dimensions remain substantially more demanding, and finite-size effects can pin textures and obscure intrinsic dynamics.

We expect that the present lattice-level quantum formalism, together with the clear one-dimensional benchmark established here, will motivate further efforts to overcome these obstacles.

In summary, our results provide a unified microscopic description of quantum chiral solitons as observable quasiparticles and of their condensation as a many-body quantum critical phenomenon. We hope that this work will stimulate experimental searches for quantum soliton signatures in spectroscopy and encourage the development of analogous lattice-based, nonperturbative descriptions of topological excitations in higher-dimensional quantum materials.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: SEMICLASSICAL DERIVATION OF THE SINE-GORDON HAMILTONIAN

The classical continuum limit of $\hat{H}_S$ [Eq. (1)] is obtained by parametrizing the spin at site j as $\mathbf{S}_j = S(\sin \theta_j \cos \varphi_j, \sin \theta_j \sin \varphi_j, \cos \theta_j)$ and replacing lattice differences with spatial derivatives to obtain

$$H = \int dx JS^2 a \left(\frac{1}{2} (\partial_x \theta)^2 + \frac{1}{2} \sin^2 \theta (\partial_x \varphi)^2 + q_0 \sin^2 \theta \partial_x \varphi - m^2 \sin \theta \cos \varphi \right), \quad (\text{A1})$$

where a is the lattice spacing, $x = ja$ is the continuous coordinate, $q_0 = D/(Ja)$, and $m^2 = H/(JSa^2)$. It proves convenient to remove the linear derivative term by defining $\tilde{\varphi} = \varphi - q_0 x$ so that

$$H = \int dx JS^2 a \left(\frac{1}{2} (\partial_x \theta)^2 + \frac{1}{2} \sin^2 \theta (\partial_x \tilde{\varphi})^2 - \frac{1}{2} q_0^2 \sin^2 \theta - m^2 \sin \theta \cos(\tilde{\varphi} + q_0 x) \right). \quad (\text{A2})$$

Time dynamics of the continuous fields $\theta, \tilde{\varphi}$ follow from the Landau-Lifshitz equations. An elegant way to incorporate them is via Euler-Lagrange equations of the associated Lagrangian [6,50],

$$L = \int dx \left(\frac{1}{g} (\cos \theta - 1) \dot{\tilde{\varphi}} - \mathcal{H} \right), \quad (\text{A3})$$

where $\mathcal{H}$ is the Hamiltonian density [the integrand of Eq. (A2)] and the dot in the first Berry-phase term denotes the time derivative ∂_t . Fields θ and $\tilde{\varphi}$ depend on x and t , and we abbreviate $1/g = S/a$.

An efficient way to obtain the result is to set $\theta = \pi/2 + \tilde{\theta}$ in Eq. (A3) and expand the Lagrangian up to quadratic terms in $\tilde{\theta}$,

$$L = \int dx \left(-\frac{1}{g} \dot{\tilde{\varphi}} - \frac{1}{g} \tilde{\theta} \dot{\tilde{\varphi}} - JS^2 a \left[\frac{1}{2} q_0^2 \tilde{\theta}^2 + \frac{1}{2} (\partial_x \tilde{\theta})^2 + \frac{1}{2} (\partial_x \tilde{\varphi})^2 - m^2 \cos(\tilde{\varphi} + q_0 x) \right] \right). \quad (\text{A4})$$

For small $|\partial_x \tilde{\theta}| \ll q_0 \tilde{\theta}$, the second term in square brackets can be dropped as well [51].

The Lagrangian form in configuration space, $L = L(\tilde{\varphi}, \dot{\tilde{\varphi}})$, is obtained by integrating out $\tilde{\theta}$. Since the Lagrangian is quadratic in $\tilde{\theta}$, this approach is equivalent to replacing $\tilde{\theta}$ with the solution of the equation of motion:

$$\frac{\delta L}{\delta \tilde{\theta}} = -\frac{1}{g} \dot{\tilde{\varphi}} - JS^2 a q_0^2 \tilde{\theta} = 0, \quad (\text{A5})$$

which gives

$$\tilde{\theta}(x) = -\frac{1}{gJS^2 a} \frac{1}{q_0^2} \dot{\tilde{\varphi}}(x). \quad (\text{A6})$$

The resulting Lagrangian form in configuration space is

$$L = \int dx \left(JS^2 a \left[\frac{1}{2c^2} (\partial_t \tilde{\varphi})^2 - \frac{1}{2} (\partial_x \tilde{\varphi})^2 + m^2 \cos[\tilde{\varphi} + q_0 x] \right] - \frac{1}{g} \partial_t \tilde{\varphi} \right), \quad (\text{A7})$$

with $c = gq_0 JS^2 a = gDS^2$.

Going back to phase space, the sine-Gordon Hamiltonian can now be expressed in terms of the field φ and the conjugate momentum $\tilde{\Pi} = (JS^2 a/c^2) \partial_t \tilde{\varphi} - 1/g$:

$$\tilde{H}_{\text{SG}} = \int dx \mathcal{H}_{\text{SG}}, \quad (\text{A8})$$

where

$$\mathcal{H}_{\text{SG}} = \frac{cK}{2} \left(\tilde{\Pi} + \frac{1}{g} \right)^2 + \frac{c}{2K} (\partial_x \tilde{\varphi})^2 - \frac{HS}{a} \cos[\tilde{\varphi} + q_0 x]. \quad (\text{A9})$$

Here, $K = gq_0$, and we simplified the coefficient of the cosine term using the definition of m^2 . Observe the persistence of a slightly unusual constant term $1/g$ that originates from the linear time-derivative term in Eq. (A3).

Next, we quantize and rescale $\hat{\tilde{\Pi}} = \tilde{\Pi}/\sqrt{4\pi}$, $\hat{\tilde{\varphi}} = \sqrt{4\pi}\tilde{\varphi}$, and $K = 4\pi\kappa$, so that the commutation relation $[\hat{\tilde{\varphi}}(x), \hat{\tilde{\Pi}}(x')] = [\hat{\varphi}(x), \hat{\Pi}(x')] = i\delta(x - x')$ is preserved. In addition, we undo the position-dependent shift of the field $\hat{\tilde{\varphi}}, \hat{\tilde{\varphi}}(x) \rightarrow \hat{\varphi}(x) - q_0 x/\sqrt{4\pi}$, to obtain

$$\hat{\mathcal{H}}_{\text{SG}} = \frac{c\kappa}{2} \left(\hat{\Pi} + \frac{\sqrt{4\pi}}{g} \right)^2 + \frac{c}{2\kappa} (\partial_x \hat{\varphi})^2 - \frac{cq_0}{\sqrt{4\pi\kappa}} \partial_x \hat{\varphi} - \frac{HS}{a} \cos[\sqrt{4\pi}\hat{\varphi}], \quad (\text{A10})$$

We observe that momentum $\hat{\Pi}$ appears in a sum

$$\hat{\Pi} + \frac{\sqrt{4\pi}}{g} = \hat{\Pi} + \Pi_0, \quad (\text{A11})$$

where the “offset” momentum is $\Pi_0 = [(\sqrt{4\pi}S)/a]$.

A remarkable discovery by Coleman [23] and Mandelstam [24], and, independently, by Luther and Emery [25], is that Eq. (A10) can be exactly mapped to the Thirring model of massive Dirac fermions. The mapping is based on the observation that the exponential operator

$$\hat{O}(x) = e^{i\zeta \int_{-\infty}^x dx' \hat{\Pi}(x')} \quad (\text{A12})$$

creates a soliton centered at x . This result is proven with the help of the identity $[\hat{A}, e^{\hat{B}}] = \hat{C}e^{\hat{B}}$ that holds for any two operators $\hat{A}, \hat{B}$ such that their commutator $\hat{C} = [\hat{A}, \hat{B}]$ commutes with both of them, $[\hat{C}, \hat{A}] = [\hat{C}, \hat{B}] = 0$.

The identity

$$[\hat{\varphi}(y), \hat{O}(x)] = i \frac{\delta \hat{O}(x)}{\delta \hat{\Pi}(y)} = -\zeta \Theta(x - y) \hat{O}(x) \quad (\text{A13})$$

means that the action of $\hat{O}(x)$ on an eigenstate $|\varphi\rangle$ of the operator $\hat{\varphi}$ changes it to the state $|\varphi'\rangle$ with the eigenvalue $\varphi(y) - \zeta \Theta(x - y)$, i.e.,

$$\varphi(y) \mapsto \begin{cases} \varphi(y) - \zeta & y < x \\ \varphi(y) & y > x. \end{cases} \quad (\text{A14})$$

In other words, $\hat{O}(x)$ creates a kink of height ζ at position x in the field $\hat{\varphi}(y)$. To describe solitons in Eq. (A10), we need to take $\zeta = 2\pi/\sqrt{4\pi} = \sqrt{\pi}$.

From here on, we make use of the powerful bosonization technique, which is described in vast detail in numerous reviews and textbooks. We choose to follow Ref. [26] and introduce right- and left-moving fermions, describing propagating solitons, via

$$\begin{aligned}\hat{\psi}_+(x) &= \frac{1}{\sqrt{2\pi\alpha}} e^{i\sqrt{4\pi}\hat{\Phi}_+(x)}, \\ \hat{\psi}_-(x) &= \frac{i}{\sqrt{2\pi\alpha}} e^{-i\sqrt{4\pi}\hat{\Phi}_-(x)},\end{aligned}\quad (\text{A15})$$

where the additional factor of i in $\hat{\psi}_-(x)$ is for future convenience. The chiral bosons are defined by

$$\hat{\Phi}_\pm(x) = \frac{1}{2} \left[\hat{\phi}(x) \mp \int_{-\infty}^x dx' \hat{\Pi}(x') \right]. \quad (\text{A16})$$

Equations (A15) and (A16) show that $e^{-i\sqrt{\pi} \int^x dx' \hat{\Pi}(x')}$ is exactly the soliton operator, while the factors $e^{\pm i\sqrt{\pi}\hat{\phi}(x)}$ are needed to enforce the Fermi statistics of the $\hat{\psi}_\pm$ fields [24].

Given that, in our formulation, momentum $\hat{\Pi}$ always appears together with Π_0 , Eq. (A11), we can now make the shift $\hat{\Pi} \rightarrow \hat{\Pi} - \hat{\Pi}_0$ to remove the offset. This shift changes Eq. (A16) by $\hat{\Phi}_\pm \rightarrow \hat{\Phi}_\pm \mp \sqrt{\pi}S(x-L)/a$, where $L \rightarrow -\infty$ is the (constant) contribution from the lower integration limit. Correspondingly, fermion operators $\hat{\psi}_\pm$ in Eq. (A15) change as

$$\hat{\psi}_\pm(x) \rightarrow \hat{\psi}_\pm(x) e^{-i\frac{2\pi S}{a}x}. \quad (\text{A17})$$

Since L/a is an integer, the phase factor $e^{2\pi SL/a} = \pm 1$ can always be removed by a unitary transformation. Equation (A17) signifies the shift of the soliton momentum by $p_0 = 2\pi S/a$ relative to the origin. For the integer spin S , the shift is immaterial since p_0 is then a multiple of the reciprocal lattice momentum. However, in the case of half-integer spin S , the shift $p_0 = \pi/a$ actually describes the soliton that carries the Berry momentum $p_0 = \pi/a$.

We therefore conclude that chiral solitons are distinguished by a nontrivial quantum number, namely, the momentum p_0 . In the integer- S chain, the soliton momentum is zero, $p_0 = 0$, while in the half-integer- S chain, it is given by $p_0 = \pi/a$, the Brillouin zone boundary value. This effect is topological; it originates from the Berry phase in Eq. (A3). This unusual finding agrees with the earlier semiclassical studies, Refs. [27,28].

Subsequent manipulations involve only the slow $\hat{\psi}_\pm(x)$ fields in the right-hand side of Eq. (A17). In particular, in Eq. (A18) below, the fast oscillating factors $e^{\pm i p_0 x}$ are not present at all, which is in complete similarity to the standard bosonization procedure when the fast-oscillating exponentials $e^{\pm i k_F x}$ with Fermi momenta $\pm k_F$ are separated from the slow field $\hat{\psi}_\pm(x)$ describing right- and left-moving

fermions that are bosonized according to the standard rules [14,26].

Lengthy but straightforward calculations [26] lead to the following set of bosonization identities,

$$\begin{aligned}\hat{\psi}_+^\dagger(x)\hat{\psi}_-(x) + \text{H.c.} &= \frac{1}{\pi\alpha} \cos[\sqrt{4\pi}\hat{\phi}(x)], \\ :\hat{\psi}_+^\dagger(x)\hat{\psi}_+(x) &:= \frac{1}{\sqrt{\pi}} \partial_x \hat{\Phi}_+(x), \\ :\hat{\psi}_-^\dagger(x)\hat{\psi}_-(x) &:= \frac{1}{\sqrt{\pi}} \partial_x \hat{\Phi}_-(x), \\ :\hat{\psi}_+^\dagger(x)(-i\partial_x)\hat{\psi}_+(x) &:= (\partial_x \hat{\Phi}_+(x))^2, \\ :\hat{\psi}_-^\dagger(x)(i\partial_x)\hat{\psi}_-(x) &:= (\partial_x \hat{\Phi}_-(x))^2.\end{aligned}\quad (\text{A18})$$

Colons signify normal ordering. These identities allow us to rewrite Eq. (A10) as the massive Thirring model of interacting fermions $\psi_\pm$,

$$\begin{aligned}\hat{\mathcal{H}}_{\text{mT}} &= \frac{c}{2} \left(\kappa + \frac{1}{\kappa} \right) [\hat{\psi}_+^\dagger(x)(-i\partial_x)\hat{\psi}_+(x) + \hat{\psi}_-^\dagger(x)(i\partial_x)\hat{\psi}_-(x)] \\ &\quad + \pi c \left(\kappa - \frac{1}{\kappa} \right) \hat{\psi}_+^\dagger(x)\hat{\psi}_+(x)\hat{\psi}_-^\dagger(x)\hat{\psi}_-(x) \\ &\quad - \frac{cq_0}{2\kappa} [\hat{\psi}_+^\dagger(x)\hat{\psi}_+(x) + \hat{\psi}_-^\dagger(x)\hat{\psi}_-(x)] \\ &\quad - \lambda [\hat{\psi}_+^\dagger(x)\hat{\psi}_-(x) + \hat{\psi}_-^\dagger(x)\hat{\psi}_+(x)]\end{aligned}\quad (\text{A19})$$

and $\lambda = \pi H S \alpha / a$. The first line represents the kinetic energy of Dirac fermions, while the second describes their interaction; notice that this term vanishes for the special value of $\kappa = 1$. The third line shows that incommensuration q_0 plays the role of the fermion chemical potential, while the nonlinear cosine potential now describes backscattering of $\hat{\psi}_\pm$ fermions (the last line). In the field theory language, λ is the mass term, responsible for the gap in the fermion spectrum.

It is worth noting once again that Eq. (A19) is written in terms of the slow $\hat{\psi}_\pm(x)$ fields that do not contain $e^{\pm i p_0 x}$ factors.

The essential physics of Eq. (A19) is understood by analyzing the noninteracting, $\kappa = 1$, (Luther-Emery) point [14]. Here, $\hat{\mathcal{H}}_{\text{mT}}(\kappa = 1)$ is diagonalized by

$$\begin{pmatrix} \hat{\psi}_+(k) \\ \hat{\psi}_-(k) \end{pmatrix} = \begin{pmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{pmatrix} \begin{pmatrix} \hat{u}_k \\ \hat{d}_k \end{pmatrix}, \quad (\text{A20})$$

where

$$\begin{aligned}\alpha_k &= \frac{1}{\sqrt{2}} \left(1 + \frac{c|k|}{\sqrt{c^2 k^2 + \lambda^2}} \right)^{1/2}, \\ \beta_k &= \frac{1}{\sqrt{2}} \left(1 - \frac{c|k|}{\sqrt{c^2 k^2 + \lambda^2}} \right)^{1/2}.\end{aligned}\quad (\text{A21})$$

Note again that, in accordance with our discussion above, k is measured from p_0 . The $\hat{u}_k(\hat{d}_k)$ fermions describe upper (lower) bands with dispersion $\pm\epsilon_k - cq_0/2$, $\epsilon_k = \sqrt{\lambda^2 + c^2k^2}$, representing solitons and antisolitons. As long as the chemical potential $cq_0/2$ is inside the band gap, $q_0 < q_c = 2\lambda/c$, the lower band is completely full and the upper one is completely empty. The density of solitons and antisolitons is zero, which is the commensurate vacuum, or high-field, state with $\langle \partial_x \hat{\phi} \rangle = 0$.

To describe soliton and antisoliton excitations on equal footing, it is convenient to make a particle-hole transformation for the occupied states: $\hat{d}_k \rightarrow \hat{h}_{-k}^\dagger$. The transformed Hamiltonian reads

$$\hat{H}_{\text{mT}}(\kappa = 1) = \sum_k \sqrt{c^2k^2 + \lambda^2} (\hat{u}_k^\dagger \hat{u}_k + \hat{h}_k^\dagger \hat{h}_k) - \frac{cq_0}{2} (\hat{u}_k^\dagger \hat{u}_k - \hat{h}_k^\dagger \hat{h}_k). \quad (\text{A22})$$

Now, $\hat{u}_k(\hat{h}_k)$ describes excitations, which are solitons (antisolitons).

Once the DM interaction becomes strong enough, or the magnetic field is reduced below the critical value, $|q_0| > q_c$, solitons enter the system. Positive q_0 induces solitons, while $q_0 < 0$ brings in antisolitons. The band structure in Eq. (A22) is illustrated in Fig. 4(a).

Let us choose $q_0 > 0$ for concreteness. The fermion density $\rho = k_F/\pi$ determines the Fermi momentum $k_F = \sqrt{q_0^2 - q_c^2}/2$. According to Eq. (A18), the fermion density $\hat{\rho} = \langle \hat{\psi}_+^\dagger(x) \hat{\psi}_+(x) + \hat{\psi}_-^\dagger(x) \hat{\psi}_-(x) \rangle$ is directly proportional to $\partial_x \hat{\phi}$, $\langle \partial_x \hat{\phi} \rangle = \sqrt{\pi} \hat{\rho}$. Thus, $\langle \partial_x \hat{\phi} \rangle = \sqrt{(q_0^2 - q_c^2)/(4\pi)} \sim \sqrt{q_0 - q_c}$. The state with the finite soliton density is the critical Luttinger liquid state of soliton liquid. Classically, this is the soliton lattice state described in Sec. II. Qualitative features of the described commensurate-incommensurate transition from the vacuum to the Luttinger liquid state persist for all values of κ [14].

APPENDIX B: EXACT EIGENSTATES

Let us show that the fully polarized state along the $\hat{x}$ direction is an eigenstate of the Hamiltonian in Eq. (1). The proof for the Heisenberg term is straightforward, as this term is SU(2) invariant; therefore, any fully polarized state is an eigenstate. Similarly, the Zeeman term is trivially satisfied.

We now focus on the Dzyaloshinskii-Moriya (DM) part of the Hamiltonian, which, when rewritten in terms of ladder operators, takes the form

$$\hat{H}_{\text{DM}} = -\frac{D}{2i} \sum_{l=1}^L \hat{h}_{l,l+1}, \quad (\text{B1})$$

with

$$\hat{h}_{j,j+1} = \hat{S}_j^- \hat{S}_{j+1}^+ - \hat{S}_j^+ \hat{S}_{j+1}^-. \quad (\text{B2})$$

The fully polarized state is given by $|\Rightarrow\rangle = (1/2^{L/2}) \otimes_{l=1}^L [|\uparrow\rangle_l + |\downarrow\rangle_l]$, and the local action of $\hat{h}_{l,l+1}$ on this state is

$$\begin{aligned} \hat{h}_{l,l+1} [|\uparrow\rangle_l + |\downarrow\rangle_l] \otimes [|\uparrow\rangle_{l+1} + |\downarrow\rangle_{l+1}] \\ = [|\downarrow\rangle_l \uparrow_{l+1}\rangle - |\uparrow\rangle_l \downarrow_{l+1}\rangle], \end{aligned} \quad (\text{B3})$$

which implies that

$$\begin{aligned} \hat{H}_{\text{DM}} |\Rightarrow\rangle &\propto \sum_{l=1}^L |\dots, [\downarrow_l \uparrow_{l+1} - \uparrow_l \downarrow_{l+1}], \dots\rangle \\ &\propto \sum_{l=1}^L |\dots, \downarrow_l \uparrow_{l+1}, \dots\rangle - |\dots, \uparrow_l \downarrow_{l+1}, \dots\rangle. \end{aligned} \quad (\text{B4})$$

Now, observe that the second term of the pair for index l is $-|\dots, \uparrow_l \downarrow_{l+1}, \dots\rangle$, which is also exactly the same as the first term of the pair with index $l-1$. In other words, every basis configuration of the form $|\dots, \uparrow_l \downarrow_{l+1}, \dots\rangle$ appears twice in the sum: once with a (+) sign coming from $\hat{h}_{l,l+1}$, and once with a (−) sign coming from $\hat{h}_{l-1,l}$. Therefore, the contributions cancel pairwise, and the complete sum vanishes, indicating that $|\Rightarrow\rangle$ is an eigenstate of the DM part of the Hamiltonian with a null eigenvalue. Finally, we have

$$\frac{1}{L} \hat{H}_S |\Rightarrow\rangle = -\left(\frac{J}{4} + \frac{H}{2}\right) |\Rightarrow\rangle. \quad (\text{B5})$$

Now, we can also show that $\hat{S}^z(k=0) |\Rightarrow\rangle$ is an eigenstate of $\hat{H}_S$. Indeed,

$$\hat{S}^z(k=0) = \frac{1}{\sqrt{L}} \sum_l \hat{S}_l^z = \frac{\hat{S}_{\text{tot}}^z}{\sqrt{L}}, \quad (\text{B6})$$

which is proportional to the total z component of the magnetization. From the SU(2) invariance of the Heisenberg term, we have

$$[\hat{H}_J, \hat{S}_{\text{tot}}^z] = 0,$$

and by the U(1) invariance around the z axis of the DM term,

$$[\hat{H}_{\text{DM}}, \hat{S}_{\text{tot}}^z] = 0.$$

Thus, $\hat{S}^z(k=0)$ acting on any eigenstate of $\hat{H}_J + \hat{H}_{\text{DM}}$ also yields an eigenstate.

For the Zeeman contribution,

$$-H \sum_j \hat{S}_j^x = -H \hat{S}_{\text{tot}}^x,$$

the commutator gives

$$[\sqrt{L} \hat{S}^z(k=0), -H \hat{S}_{\text{tot}}^x] = -iH \sqrt{L} \hat{S}_{\text{tot}}^y. \quad (\text{B7})$$

Therefore,

$$\hat{S}_{\text{tot}}^x \hat{S}^z(k=0) | \Rightarrow \rangle = (\hat{S}^z(k=0) \hat{S}_{\text{tot}}^x - i \hat{S}_{\text{tot}}^y) | \Rightarrow \rangle. \quad (\text{B8})$$

From the first term on the rhs, we find

$$\hat{S}^z(k=0) \hat{S}_{\text{tot}}^x | \Rightarrow \rangle = \frac{L}{2} \hat{S}^z(k=0) | \Rightarrow \rangle.$$

To compute the action of $\hat{S}_{\text{tot}}^y$, note that

$$\begin{aligned} i \hat{S}_l^y \frac{1}{\sqrt{2}} (|\uparrow\rangle_l + |\downarrow\rangle_l) &= \frac{1}{2\sqrt{2}} (|\uparrow\rangle_l - |\downarrow\rangle_l) \\ &= \hat{S}_l^z \frac{1}{\sqrt{2}} (|\uparrow\rangle_l + |\downarrow\rangle_l). \end{aligned} \quad (\text{B9})$$

By substituting this into Eq. (B8), we obtain

$$\hat{S}_{\text{tot}}^x \hat{S}^z(k=0) | \Rightarrow \rangle = \left(\frac{L}{2} - 1 \right) \hat{S}^z(k=0) | \Rightarrow \rangle, \quad (\text{B10})$$

which finally proves that $\hat{S}^z(k=0) | \Rightarrow \rangle$ is an eigenvector of $\hat{H}_S$,

$$\hat{H}_S \hat{S}^z(k=0) | \Rightarrow \rangle = - \left(\frac{JL}{4} + H \left(\frac{L}{2} - 1 \right) \right) \hat{S}^z(k=0) | \Rightarrow \rangle. \quad (\text{B11})$$

The energy difference between the fully polarized state and $\hat{S}^z(k=0) | \Rightarrow \rangle$ is simply given by $\Delta E = H$, that is, merely the magnon gap in the fully polarized phase.

APPENDIX C: $\mathfrak{su}(2)$ ALGEBRA OF SOLITON OPERATORS

This appendix presents the explicit calculations of the commutators involving the soliton and quantum-soliton operators. We begin with the former, evaluating

$$\begin{aligned} \hat{T}_{j\tau} \hat{T}_{j\tau}^\dagger &= \prod_{l \neq j} e^{i\varphi_\tau(l-j) \hat{S}_l^z} (\hat{S}_j^z - i \hat{S}_j^y) (\hat{S}_j^z + i \hat{S}_j^y) \prod_{l \neq j} e^{-i\varphi_\tau(l-j) \hat{S}_l^z} \\ &= \frac{1}{2} + i [\hat{S}_j^z, \hat{S}_j^y] = \frac{1}{2} + \hat{S}_j^x, \\ \hat{T}_{j\tau}^\dagger \hat{T}_{j\tau} &= (\hat{S}_j^z + i \hat{S}_j^y) \prod_{l \neq j} e^{-i\varphi_\tau(l-j) \hat{S}_l^z} \prod_{l \neq j} e^{i\varphi_\tau(l-j) \hat{S}_l^z} (\hat{S}_j^z - i \hat{S}_j^y) \\ &= \frac{1}{2} + i [\hat{S}_j^y, \hat{S}_j^z] = \frac{1}{2} - \hat{S}_j^x, \end{aligned} \quad (\text{C1})$$

which implies that the equal-site commutator is $\hat{T}_{j\tau} \hat{T}_{j\tau}^\dagger - \hat{T}_{j\tau}^\dagger \hat{T}_{j\tau} = 2 \hat{S}_j^x$, while the anticommutator is $\{\hat{T}_{j\tau}, \hat{T}_{j\tau}^\dagger\} = 1$.

For different sites $j' \neq j$, we have

$$\begin{aligned} [\hat{T}_{j'\tau}, \hat{T}_{j\tau}^\dagger] &= (\hat{S}_{j'}^z + i \hat{S}_{j'}^y) (\hat{S}_j^z - i \hat{S}_j^y) \\ &\quad \times \sum_{\sigma=\pm 1} \sigma e^{i\varphi_\tau(j'-j) \hat{S}_{j'}^z} e^{i\varphi_\tau(j-j') \hat{S}_j^z} \\ &= 2i (\hat{S}_{j'}^z + i \hat{S}_{j'}^y) (\hat{S}_j^z - i \hat{S}_j^y) \\ &\quad \times [f_\tau(j' - j) \hat{S}_{j'}^z + f_\tau(j - j') \hat{S}_j^z], \end{aligned} \quad (\text{C2})$$

where the function $f_\tau(x) = \cos \varphi_\tau(x) \sin \varphi_\tau(-x)$ vanishes as x becomes much larger than the size of the chiral soliton (see Fig. 18). Therefore, we can conclude that for all j and j' ,

$$[\hat{T}_{j'\tau}, \hat{T}_{j\tau}^\dagger] = 2 \tilde{\delta}_{j'j\tau} \hat{S}_j^x, \quad (\text{C3})$$

with $\tilde{\delta}_{j'j\tau}$ denoting the coarse-grained version of the Kronecker delta. It is also straightforward to show that

$$\begin{aligned} [\hat{T}_{j\tau}^\dagger, \hat{S}_j^x] &= 2 \hat{T}_{j\tau}^\dagger, \\ [\hat{T}_{j'\tau}, \hat{S}_j^x] &= -2 \hat{T}_{j'\tau}. \end{aligned} \quad (\text{C4})$$

Based on this result, one can construct an approximate $\mathfrak{su}(2)$ Lie algebra generated by $\{\hat{T}_\tau^\dagger, \hat{T}_\tau, \hat{S}^x\}$.

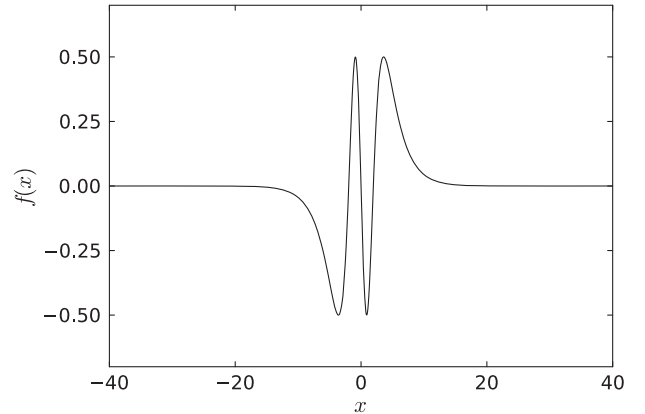


FIG. 18. We show $f(x) = \cos \varphi(x) \sin \varphi(-x)$ as a function of $x = |l - j|$ for $\tau = 1$.

APPENDIX D: LOCALIZATION OF THE QUANTUM SOLITON OPERATOR

We analyze the large-distance behavior of the expectation value $\langle 0 | \hat{\mathfrak{Z}}_{j\tau} \hat{T}_{j'\tau}^\dagger | 0 \rangle$. Using the definition

$$\hat{\mathfrak{Z}}_{j\tau}^\dagger = \sum_p w_{j-p,\tau} \hat{T}_{p\tau}^\dagger, \quad \hat{\mathfrak{Z}}_{j\tau} = \sum_p (w_{j-p,\tau})^* \hat{T}_{p\tau}, \quad (\text{D1})$$

the expectation value can be expressed as

$$\langle 0 | \hat{\mathfrak{Z}}_{j\tau} \hat{T}_{j'\tau}^\dagger | 0 \rangle = \sum_p (w_{j-p,\tau})^* O_{p,j'}, \quad (\text{D2})$$

where

$$O_{p,j'} = \langle 0 | \hat{T}_{p\tau} \hat{T}_{j'\tau}^\dagger | 0 \rangle \quad (\text{D3})$$

is given by Eq. (23). It is convenient to reexpress the overlaps as

$$O_{p,j'} = \sin^2 \frac{\varphi_\tau(j' - p)}{2} \prod_{l \neq p,j'} \cos \left[\frac{\varphi_\tau(l - p) - \varphi_\tau(l - j')}{2} \right], \quad (\text{D4})$$

which follows from the relations $\varphi_\tau(0) = \pi$ and $\varphi_\tau(-x) = 2\pi - \varphi_\tau(x)$. By applying periodic boundary conditions, $x \mapsto \text{mod}(x + L/2, L) - L/2$, both special factors ($l = p$ and $l = j'$) contribute positively because $\cos[(\pi - \varphi_\tau(j' - p))/2] = \sin[\varphi_\tau(j' - p)/2]$.

For large separations $x = |j - j'|$, the soliton profile behaves as

$$\varphi(x) = 2\pi - 4e^{-mx} + \mathcal{O}(e^{-3mx}), \quad (\text{D5})$$

and therefore

$$\sin \frac{\varphi(x)}{2} = \sin(\pi - 2e^{-mx}) \simeq 2e^{-mx}, \quad (\text{D6})$$

which exhibits the expected exponential decay controlled by the mass scale m .

The product of cosines in Eq. (D4) remains of order unity, for sites l far from both centers p and j' , $\varphi(l - p) \approx \varphi(l - j')$ and $\cos[(\varphi(l - p) - \varphi(l - j'))/2] \approx 1$. The only deviations from unity arise from regions of width around $1/m$ near the kink centers, producing a finite, $\mathcal{O}(1)$ prefactor $C(m)$ that does not depend on the distance $|l - p|$.

Combining these results, we obtain, for $x \gg 1$,

$$O_{p,j'} \simeq \pm 4C(m) e^{-2m|j' - p|}. \quad (\text{D7})$$

Hence, $O_{p,j'}$ decays exponentially with rate $2m$. Substituting Eq. (D7) into the definition of the full overlap yields

$$\langle 0 | \hat{\mathfrak{Z}}_{j\tau} \hat{T}_{j'\tau}^\dagger | 0 \rangle \simeq w_{j-j',\tau}^* + 4C(m) \sum_{p \neq j'} w_{j-p,\tau}^* e^{-2m|j' - p|}. \quad (\text{D8})$$

If the Wannier coefficients $w_{j-p,\tau}$ are localized around $p = j$, the sum is dominated by $p \simeq j$, leading to the asymptotic behavior

$$|\langle 0 | \hat{\mathfrak{Z}}_{j\tau} \hat{T}_{j'\tau}^\dagger | 0 \rangle| \propto e^{-2m|j - j'|}. \quad (\text{D9})$$

In other words, the overlap between $\hat{\mathfrak{Z}}_{j\tau}^\dagger | 0 \rangle$ and $\hat{T}_{j'\tau}^\dagger | 0 \rangle$ decays exponentially in the separation $|j - j'|$, with a correlation length $\xi = 1/2m$.

APPENDIX E: MATRIX ELEMENTS FOR HOPPING AMPLITUDES

This appendix presents the calculations for the matrix elements $h_\tau(s)$ defined in the main text in Sec. IV D. The vacuum state corresponds to the fully polarized state along the x direction, given by

$$|0\rangle = \bigotimes_{l=1}^L \frac{1}{\sqrt{2}} [|\uparrow\rangle_l + |\downarrow\rangle_l]. \quad (\text{E1})$$

The action of the soliton (antisoliton) operator centered at site j over the fully polarized state is given by

$$\begin{aligned} |\varphi_\tau^{(j)}\rangle &\equiv \hat{T}_{j\tau}^\dagger |0\rangle = \bigotimes_{l=1}^L \frac{1}{\sqrt{2}} [e^{-i\varphi_\tau(l-j)/2} |\uparrow\rangle_l + e^{i\varphi_\tau(l-j)/2} |\downarrow\rangle_l] \\ &\equiv \bigotimes_{l=1}^L |\mathbf{n}_l[\varphi_\tau(l-j)]\rangle, \end{aligned} \quad (\text{E2})$$

where the last term defines the local coherent state $|\mathbf{n}_l[\varphi_\tau(l-j)]\rangle$ at site l , associated with the soliton (antisoliton) centered at site j . Note that the $\mathbf{n}_l[\varphi_\tau(l-j)]$ unit vectors are perpendicular to the rotation z axis, and $\varphi_\tau(l-j)$ is the corresponding azimuthal angle. The Hamiltonian of Eq. (1) can be decomposed into a sum of bilinear terms acting on pairs of neighboring sites n and $n + 1$, along with the Zeeman term that acts only on site n ,

$$\hat{H}_S = \sum_l \hat{h}_{l,l+1}^{(\text{Bi})} + \sum_l \hat{h}_l^{(\text{Z})}. \quad (\text{E3})$$

Now, the matrix element of the Hamiltonian in the basis of the single soliton can be decomposed as

$$\begin{aligned}
\langle \varphi_\tau^{(j')} | H | \varphi_\tau^{(j)} \rangle &= \sum_l \langle \varphi_\tau^{(j')} | \hat{h}_{l,l+1}^{(\text{Bi})} | \varphi_\tau^{(j)} \rangle + \langle \varphi_\tau^{(j')} | \hat{h}_l^{(Z)} | \varphi_\tau^{(j)} \rangle \\
&= \sum_l \langle \mathbf{n}_{l+1}[\varphi_\tau(l+1-j')] | \langle \mathbf{n}_l[\varphi_\tau(l-j')] | \hat{h}_{l,l+1}^{(\text{Bi})} | \mathbf{n}_l[\varphi_\tau(l-j)] \rangle | \mathbf{n}_{l+1}[\varphi_\tau(l+1-j)] \rangle \prod_{p \neq l, l+1} \mathcal{O}_p^{jj} \\
&\quad + \sum_l \langle \mathbf{n}_l[\varphi_\tau(l-j')] | \hat{h}_l^{(Z)} | \mathbf{n}_l[\varphi_\tau(l-j)] \rangle \prod_{p \neq l} \mathcal{O}_p^{jj},
\end{aligned} \tag{E4}$$

with

$$\mathcal{O}_p^{jj} \equiv \langle \mathbf{n}_p[\varphi_\tau(p-j')] | \mathbf{n}_p[\varphi_\tau(p-j)] \rangle = \cos \left[\frac{\varphi_\tau(p-j') - \varphi_\tau(p-j)}{2} \right]. \tag{E5}$$

We define

$$F_{jj'}^\tau(x) \equiv \prod_{p \neq x} \mathcal{O}_p^{jj} = \prod_{p \neq x} \cos \left[\frac{\varphi_\tau(p-j') - \varphi_\tau(p-j)}{2} \right]. \tag{E6}$$

Now, we compute the matrix elements of the spin operators between two coherent states. Although only the expression for $\hat{S}^x$ is needed, we include the remaining components for completeness:

$$\begin{aligned}
\langle \mathbf{n}_l[\varphi_\tau(l-j')] | \hat{S}_n^x | \mathbf{n}_l[\varphi_\tau(l-j)] \rangle &= \frac{1}{2} \cos \left[\frac{\varphi_\tau(l-j') + \varphi_\tau(l-j)}{2} \right], \\
\langle \mathbf{n}_l[\varphi_\tau(l-j')] | \hat{S}_n^y | \mathbf{n}_l[\varphi_\tau(l-j)] \rangle &= \frac{1}{2} \sin \left[\frac{\varphi_\tau(l-j') + \varphi_\tau(l-j)}{2} \right], \\
\langle \mathbf{n}_l[\varphi_\tau(l-j')] | \hat{S}_n^z | \mathbf{n}_l[\varphi_\tau(l-j)] \rangle &= \frac{i}{2} \sin \left[\frac{\varphi_\tau(l-j') - \varphi_\tau(l-j)}{2} \right].
\end{aligned} \tag{E7}$$

Using this result, the single-site matrix element for the Zeeman part of the Hamiltonian is given by

$$\langle \mathbf{n}_l[\varphi_\tau(l-j')] | \hat{S}_n^x | \mathbf{n}_l[\varphi_\tau(l-j)] \rangle = -\frac{H}{2} \cos \left[\frac{\varphi_\tau(l-j') + \varphi_\tau(l-j)}{2} \right]. \tag{E8}$$

The next step is to compute the matrix elements of the bilinear part of the Hamiltonian,

$$\langle \mathbf{n}_{l+1}[\varphi_\tau(l+1-j')] | \langle \mathbf{n}_l[\varphi_\tau(l-j')] | \hat{h}_{l,l+1}^{(\text{Bi})} | \mathbf{n}_l[\varphi_\tau(l-j)] \rangle | \mathbf{n}_{l+1}[\varphi_\tau(l+1-j)] \rangle. \tag{E9}$$

We note that the product state of two coherent states can be expressed as

$$| \mathbf{n}_l[\varphi_\tau(l-j)] \rangle | \mathbf{n}_{l+1}[\varphi_\tau(l+1-j)] \rangle = \frac{1}{2} \begin{pmatrix} e^{-i\varphi_\tau(l-j)/2} e^{-i\varphi_\tau(l+1-j)/2} \\ e^{-i\varphi_\tau(l-j)/2} e^{i\varphi_\tau(l+1-j)/2} \\ e^{i\varphi_\tau(l-j)/2} e^{-i\varphi_\tau(l+1-j)/2} \\ e^{i\varphi_\tau(l-j)/2} e^{i\varphi_\tau(l+1-j)/2} \end{pmatrix}, \tag{E10}$$

and the matrix representation of the two-site Heisenberg and DM Hamiltonian is given by

$$[\hat{\mathbf{S}}_l \cdot \hat{\mathbf{S}}_{l+1}] = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & -1/4 & 1/2 & 0 \\ 0 & 1/2 & -1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix}, \quad [\hat{S}_l^x \hat{S}_{l+1}^y - \hat{S}_l^y \hat{S}_{l+1}^x] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i/2 & 0 \\ 0 & -i/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{E11})$$

By evaluating the matrix elements and substituting them into Eq. (E4), we obtain

$$\begin{aligned} \langle \varphi_\tau^{(j')} | \hat{H}_S | \varphi_\tau^{(j)} \rangle = & -\frac{J}{8} \sum_n F_{j,j'}^\tau(l, l+1) \cos \left[\frac{\varphi_\tau(l-j') + \varphi_\tau(l+1-j') - \varphi_\tau(l-j) - \varphi_\tau(l+1-j)}{2} \right] \\ & -\frac{J}{4} \sum_l F_{j,j'}^\tau(l, l+1) \cos \left[\frac{\varphi_\tau(l-j') + \varphi_\tau(l-j) - \varphi_\tau(l+1-j') - \varphi_\tau(l+1-j)}{2} \right] \\ & +\frac{J}{8} \sum_l F_{j,j'}^\tau(l, l+1) \cos \left[\frac{\varphi_\tau(l-j') - \varphi_\tau(l+1-j') - \varphi_\tau(l-j) + \varphi_\tau(l+1-j)}{2} \right] \\ & -\frac{D}{4} \sum_l F_{j,j'}^\tau(l, l+1) \sin \left[\frac{\varphi_\tau(l+1-j') - \varphi_\tau(l-j') - \varphi_\tau(l-j) + \varphi_\tau(l+1-j)}{2} \right] \\ & -\frac{H}{2} \sum_l F_{j,j'}^\tau(l) \cos \left[\frac{\varphi_\tau(l-j') + \varphi_\tau(l-j)}{2} \right]. \end{aligned} \quad (\text{E12})$$

The hopping amplitudes can now be evaluated using Eq. (36) of the main text.

APPENDIX F: BERRY-PHASE ORIGIN OF THE INTEGER AND HALF-INTEGER HOPPING SIGN ALTERNATION

This appendix shows that, within our lattice formulation, the minimum of the soliton band lies at $k = \pi$ for half-integer spin chains and at $k = 0$ for integer ones, in agreement with the continuum formulation discussed in Sec. III.

As shown in Appendix E, the soliton centered at site j is represented as a product of ($S = 1/2$) coherent spin states [see Eq. (E2)], a construction that extends straightforwardly to spin- (S) coherent states.

In general, the phase of the overlap between two spin- S coherent states at a given lattice site l is given exactly by the oriented solid angle on the Bloch sphere spanned by their corresponding spin directions and a chosen quantization axis. Keeping the fixed unit vector $\hat{z}$, parallel to the rotation axis of the skyrmion, as the quantization axis, let $|\mathbf{n}_l\rangle$ and $|\mathbf{n}'_l\rangle$ denote two coherent states satisfying

$$\langle \mathbf{n}_l | \hat{\mathbf{S}}_l | \mathbf{n}_l \rangle = S \mathbf{n}_l, \quad \langle \mathbf{n}'_l | \hat{\mathbf{S}}_l | \mathbf{n}'_l \rangle = S \mathbf{n}'_l,$$

where $\hat{\mathbf{S}}_l = (\hat{S}_l^x, \hat{S}_l^y, \hat{S}_l^z)$ is the vector of spin- S operators at site l . Note that this is just a generalization for arbitrary spin S of the $S = 1/2$ coherent states introduced in the previous sections.

The argument of the overlap between a pair of coherent states is

$$\arg(\langle \mathbf{n}_l | \mathbf{n}'_l \rangle) = S \Omega(\hat{z}, \mathbf{n}_l, \mathbf{n}'_l) \pmod{2\pi}, \quad (\text{F1})$$

where $\Omega(\hat{z}, \mathbf{n}_l, \mathbf{n}'_l)$ is the oriented solid angle of the spherical triangle with vertices $\hat{z}$, $\mathbf{n}_l$, and $\mathbf{n}'_l$. For the classical solitons with arbitrary spin S , the vectors $\mathbf{n}_l$ and $\mathbf{n}'_l$ are still *perpendicular* to the rotation $\hat{z}$ axis, implying that we can still parametrize them with the single azimuthal angle φ_l because $\theta_l = \pi/2$. In this case, the overlap between two coherent states with $\mathbf{n}_l$ and $\mathbf{n}'_l$ differing by a small azimuthal shift $\delta\varphi$ satisfies

$$\arg(\langle \mathbf{n}_l | \varphi \rangle \langle \mathbf{n}_l | \varphi + \delta\varphi \rangle) = S \delta\varphi \pmod{2\pi}. \quad (\text{F2})$$

The soliton ($\tau = +$) and antisoliton ($\tau = -$) states centered around the site j can then be expressed as

$$|\varphi_\tau^{(j)}\rangle = \bigotimes_l |\mathbf{n}_l[\varphi_\tau(l-j)]\rangle, \quad (\text{F3})$$

where $\varphi_\tau(l-j)$ is the azimuthal angle of the classical soliton solution. Then, the total overlap between the soliton ($\tau = +$) or antisoliton ($\tau = -$) product states, $|\varphi_\tau^{(j)}\rangle$ and $|\varphi_\tau^{(j+1)}\rangle$ centered at sites j and $j+1$, is

$$\begin{aligned} \zeta \equiv \langle \varphi_\tau^{(j)} | \varphi_\tau^{(j+1)} \rangle &= \prod_l \langle \mathbf{n}_l[\varphi_\tau(l-j)] | \mathbf{n}_l[\varphi_\tau(l-j-1)] \rangle \\ &= |\zeta| e^{iS\Delta\Phi}, \end{aligned} \quad (\text{F4})$$

with

$$\Delta\Phi \equiv \sum_l [\varphi_\tau(l - (j+1)) - \varphi_\tau(l - j)]. \quad (\text{F5})$$

For a chiral soliton or antisoliton with unit winding, $\Delta\Phi = \pm 2\pi$ for $\tau = \pm 1$, we have

$$\zeta = |\zeta| e^{\pm i 2\pi S} = |\zeta| (-1)^{2S}. \quad (\text{F6})$$

Thus, the sign of the soliton overlap alternates between integer and half-integer spin. Note that the real character of ζ makes it independent of the τ index: $\zeta_+ = \zeta_-^* = \zeta_- = \zeta$.

Consider now the DM part of $\hat{H}_S$,

$$\hat{H}_{\text{DM}} = -D \sum_l \hat{h}_l, \quad \hat{h}_l = (\hat{\mathbf{S}}_l \times \hat{\mathbf{S}}_{l+1}) \cdot \hat{\mathbf{e}}_z, \quad (\text{F7})$$

and define

$$h_0 = \langle \varphi_\tau^{(j)} | \hat{H}_{\text{DM}} | \varphi_\tau^{(j)} \rangle, \quad h_1 = \langle \varphi_\tau^{(j)} | \hat{H}_{\text{DM}} | \varphi_\tau^{(j+1)} \rangle. \quad (\text{F8})$$

Using the exact identity for coherent states [52],

$$\frac{\langle \mathbf{n}_l | \hat{S}_l^\alpha | \mathbf{n}'_l \rangle}{\langle \mathbf{n}_l | \mathbf{n}'_l \rangle} = S A^\alpha(\mathbf{n}_l, \mathbf{n}'_l), \quad (\text{F9})$$

where

$$\mathcal{A}(\mathbf{n}_l, \mathbf{n}'_l) = \frac{\mathbf{n}_l + \mathbf{n}'_l - i \mathbf{n}_l \times \mathbf{n}'_l}{1 + \mathbf{n}_l \cdot \mathbf{n}'_l}, \quad (\text{F10})$$

one finds that

$$\begin{aligned} \frac{h_1^\tau}{\zeta} &= -DS^2 \sum_l (\mathcal{A}_l^\tau \times \mathcal{A}_{l+1}^\tau)_z, \\ h_0^\tau &= -DS^2 \sum_l (\mathbf{n}_l [\varphi_\tau(l-j)] \times \mathbf{n}_{l+1} [\varphi_\tau(l+1-j)])_z, \end{aligned} \quad (\text{F11})$$

with

$$\mathcal{A}_l^\tau \equiv \mathcal{A}(\mathbf{n}_l [\varphi_\tau(l-j)], \mathbf{n}_l [\varphi_\tau(l-j-1)]), \quad (\text{F12})$$

To find the sign of the nearest-neighbor soliton hopping, we introduce the difference

$$\delta E_\tau \equiv \frac{\langle \varphi_\tau^{(j)} | \hat{H}_{\text{DM}} | \varphi_\tau^{(j+1)} \rangle}{\langle \varphi_\tau^{(j)} | \varphi_\tau^{(j+1)} \rangle} - \langle \varphi_\tau^{(j)} | \hat{H}_{\text{DM}} | \varphi_\tau^{(j)} \rangle. \quad (\text{F13})$$

This quantity is real because there is a symmetry transformation that exchanges the solitons centered at sites j and $j+1$. According to Eq. (D7), the overlaps $\langle \varphi_\tau^{(j)} | \varphi_\tau^{(j+n)} \rangle = \mathcal{O}(\zeta^n)$, which implies that, to leading order in ζ , the “Wannierization” of the soliton state gives

$$|\Phi_\tau^{(j)}\rangle = |\varphi_\tau^{(j)}\rangle - \frac{\zeta}{2} (|\varphi_\tau^{(j+1)}\rangle + |\varphi_\tau^{(j-1)}\rangle) + \mathcal{O}(\zeta^2), \quad (\text{F14})$$

where the coefficients of the expansion result from imposing the orthogonality condition $\langle \Phi_\tau^{(j)} | \Phi_\tau^{(j')} \rangle = \delta_{jj'}$. Then, the DM-induced hopping matrix element is

$$t_{\text{DM}}^\tau = \langle \Phi_\tau^{(j)} | \hat{H}_{\text{DM}} | \Phi_\tau^{(j+1)} \rangle = \zeta \delta E_\tau + \mathcal{O}(\zeta^2). \quad (\text{F15})$$

Hence, the sign of t_{DM} is determined by the Berry-phase factor in ζ and the real prefactor δE_τ .

Using Eq. (F6) in Eq. (F15), we obtain

$$t_{\text{DM}}^\tau = |\zeta| (-1)^{2S} \delta E_\tau, \quad (\text{F16})$$

which implies

$$\text{sgn}(t_{\text{DM}}^\tau) = \begin{cases} +\text{sgn}(\delta E_\tau) & S \text{ integer} \\ -\text{sgn}(\delta E_\tau) & S \text{ half-integer} \end{cases} \quad (\text{F17})$$

The remaining task is to determine the sign of δE_τ . To this end, we expand to second order in

$$d_l \equiv \varphi_+(l-j-1) - \varphi_+(l-j). \quad (\text{F18})$$

The coherent-state connection expands as

$$\mathcal{A}_l^\tau = \mathbf{n}_l^{(j\tau)} + \tau \tan \frac{d_l}{2} (\hat{\mathbf{z}} \times \mathbf{n}_l^{(j)} - i \hat{\mathbf{z}}), \quad (\text{F19})$$

where we have used $\varphi_+ = -\varphi_-$ and $\mathbf{n}_l^{(j\tau)} \equiv \mathbf{n}_{l+1}[\varphi_\tau(l+1-j)]$. This equation leads to

$$\begin{aligned} (\mathcal{A}_l^\tau \times \mathcal{A}_{l+1}^\tau)_z &= (\mathbf{n}_l^{(j\tau)} \times \mathbf{n}_{l+1}^{(j\tau)})_z - \tau \tan \frac{d_l}{2} \tan \frac{d_{l+1}}{2} \sin d_{l+1} \\ &\quad + \tau \left(\tan \frac{d_{l+1}}{2} - \tan \frac{d_l}{2} \right) \cos d_{l+1}. \end{aligned} \quad (\text{F20})$$

Substituting this expansion into Eq. (F13) gives

$$\begin{aligned} \delta E_\tau &= \tau DS^2 \sum_l \tan \frac{d_l}{2} \tan \frac{d_{l+1}}{2} \sin d_{l+1} \\ &\quad - \tau DS^2 \sum_l \left(\tan \frac{d_{l+1}}{2} - \tan \frac{d_l}{2} \right) \cos d_{l+1}. \end{aligned} \quad (\text{F21})$$

For smooth soliton profiles, Eq. (F18) can be Taylor expanded. In this limit, the second term is parametrically subleading: Its lowest nonvanishing contribution in the gradient expansion arises only at fifth order, whereas the leading term contributes already at third order:

$$\tau \sum_l \left(\tan \frac{d_{l+1}}{2} - \tan \frac{d_l}{2} \right) \cos d_{l+1} \rightarrow \frac{a^4}{4} \int dx \partial_x \varphi_\tau (\partial_x^2 \varphi_\tau)^2.$$

Therefore, to leading order in d_l , we have

$$\delta E_\tau \simeq \tau D S^2 \sum_l \tan \frac{d_l}{2} \tan \frac{d_{l+1}}{2} \sin d_{l+1}, \quad (\text{F22})$$

which, in the long wavelength (continuum) limit, reduces to

$$\delta E_\tau \simeq \frac{a^2 D S^2}{4} \int dx (\partial_x \varphi_\tau(x))^3. \quad (\text{F23})$$

By using the sine-Gordon solution identity $(\partial_x \varphi_\tau)^2 = 2m^2(1 - \cos \varphi_\tau)$, this can be further simplified to

$$\delta E_\tau \simeq -\tau a^2 \pi D^2 S^2 m^2. \quad (\text{F24})$$

We then conclude that $\delta E < 0$ for the soliton solution ($d_l D < 0$), while $\delta E > 0$ ($d_l D > 0$) for the antisoliton solution. Because δE_τ is odd under the transformation $\tau \rightarrow -\tau$, the DM contribution to the hopping amplitude acquires opposite signs for soliton and antisoliton configurations. This result follows from the fact that $\hat{H}_{\text{DM}}$ is odd under spatial inversion about site j , $\hat{X}_j \hat{H}_{\text{DM}} \hat{X}_j = -\hat{H}_{\text{DM}}$, and that this inversion transforms a soliton into an antisoliton, $\hat{X}_j |\varphi_+^{(j)}\rangle = |\varphi_-^{(j)}\rangle$. Thus,

$$\begin{aligned} \langle \varphi_+^{(j)} | \hat{H}_{\text{DM}} | \varphi_+^{(j+1)} \rangle &= \langle \varphi_+^{(j)} | \hat{X}_j^2 \hat{H}_{\text{DM}} \hat{X}_j^2 | \varphi_+^{(j+1)} \rangle \\ &= -\langle \varphi_-^{(j)} | \hat{H}_{\text{DM}} | \varphi_-^{(j+1)} \rangle, \end{aligned} \quad (\text{F25})$$

confirming that the DM-induced hopping amplitudes for solitons and antisolitons differ by a minus sign.

Combined with Eqs. (F16) and (F17), this result yields the integer and half-integer alternation of the DM-induced soliton hopping amplitude. Solitons of the half-integer spin chain have $t_{\text{DM}} > 0$, while for those of the integer one, $t_{\text{DM}} < 0$. Consequently, the Berry-phase structure enforces soliton band minima at $k = 0$ for integer S and at $k = \pi$ for half-integer S . The situation is reversed for antisolitons.

Note that, in this appendix, we have evaluated only the DM contribution to the nearest-neighbor hopping amplitude. This restriction is justified because the DM term yields a contribution whose magnitude exceeds the combined contributions from the Heisenberg and Zeeman terms; consequently, the sign of t_1 is fixed by the DM part computed here. That said, by following an analogous derivation, one can show that the Heisenberg and Zeeman contributions to t_1 also exhibit an alternation between half-integer and integer spin since both are linear in ζ at leading order.

APPENDIX G: DETAILS ABOUT TEBD CALCULATIONS

To compute the dynamical correlation functions, we employ the TEBD method as implemented in the ITensor.jl (v0.9) library [33,34]. The algorithmic steps are as follows:

- (1) Use DMRG to obtain the ground state $|0\rangle$.
- (2) At $t = 0$, apply the spin operator at site j' to the ground state: $|\psi(0)\rangle = \hat{S}_{j'}^\beta |0\rangle$.
- (3) Evolve the state in time with a small time step $\delta t \ll 1$ using TEBD to obtain $|\psi(t = \delta t)\rangle = e^{-i\hat{H}\delta t} |\psi(t = 0)\rangle$.
- (4) Compute $\mathcal{S}_{jj'}^{\alpha\beta}(t = \delta t) = \langle 0 | \hat{S}_j^\alpha |\psi(t = \delta t)\rangle$.
- (5) Continue the time evolution of the initial state by repeating step 3 a total of $n_{\text{steps}} = T_f / \delta t$ times, with T_f the total evolution time.

After completing these steps, a Fourier transformation in both space and time is applied to obtain the dynamical function. However, since the total evolution time T_f is finite ($T_f < \infty$), the Fourier transform of $\mathcal{S}_{jj'}^{\alpha\beta}(t)$ will exhibit nonphysical oscillations due to the truncation in time. To avoid this effect, we multiply the time-dependent correlator by a Parzen function [53],

$$W(x, a) = \begin{cases} 1 - 6|\frac{x}{a}|^2 + 6|\frac{x}{a}|^3 & \text{if } |x| \leq \frac{a}{2} \\ 2(1 - |\frac{x}{a}|^3)^3 & \text{if } \frac{a}{2} < |x| \leq a \\ 0 & \text{if } |x| > a. \end{cases} \quad (\text{G1})$$

Hence, we compute

$$\begin{aligned} \mathcal{S}_{jj'}^{\alpha\beta}(\omega) &= \int_{-T_f}^{T_f} dt e^{i\omega t} \mathcal{S}_{jj'}^{\alpha\beta}(t) W(t, T_f) \\ &= \int_{-\infty}^{\infty} dt e^{i\omega t} \mathcal{S}_{jj'}^{\alpha\beta}(t) W(t, T_f) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' \mathcal{F}[\mathcal{S}_{jj'}^{\alpha\beta}(t)](\omega') \mathcal{F}[W(t, T_f)](\omega - \omega') \\ &= \frac{1}{2\pi} \mathcal{F}[\mathcal{S}_{jj'}^{\alpha\beta}(t)](\omega) * \mathcal{F}[W(t, T_f)](\omega), \end{aligned} \quad (\text{G2})$$

with the convolution kernel

$$\frac{1}{2\pi} \mathcal{F}[W(t, T_f)](\omega) = \frac{96 \sin^4(T_f \omega / 4)}{\pi T_f^4 \omega^4}. \quad (\text{G3})$$

The frequency resolution is determined by the standard error of the distribution, which is equal to $2\sqrt{3}/T_f$.

In addition, several symmetry properties of the correlators can be used to reduce the computational cost.

Summarizing them for the case $\alpha = \beta$ (the one relevant for us), we have

- (1) $\mathcal{S}_{jj'}(t) = \mathcal{S}_{j'j}^*(-t)$ (hermiticity of $\hat{H}$)
- (2) $\mathcal{S}_{jj'}(-t) = \mathcal{S}_{jj'}^*(t)$ [symmetry $\mathcal{T}_R R^z(\pi)$]
- (3) $\mathcal{S}_{jj'}(t) = \mathcal{S}_{j'j}(t)$ (1 and 2)
- (4) $\mathcal{S}_{jj'}(t) = \mathcal{S}(j - j', t)$ (translational invariance)

APPENDIX H: HYBRIDIZATION BETWEEN THE SINGLE-MAGNON AND TWO-MAGNON BOUND STATES IN THE HIGH-FIELD LIMIT

In the high-field limit $H \geq J$, the soliton size becomes of the order of a few lattice spacings. In this case, we can approximately think of it as a few-magnon bound state. Specifically, here we consider the two-magnon bound state and its hybridization with the single-magnon state through the DM interaction, which, as always in our paper, is directed perpendicular to the magnetic field axis. Our consideration of the magnon bound state closely follows Sec. III of the Supplemental Material of Ref. [54], which analyzed the bound magnon pair in the polarized state of the antiferromagnetic chain. This approach is easily adapted to the current case by changing the sign of the exchange $J_1 \rightarrow -J$. Finally, here we assume $D \ll J$ and treat the DM interaction as a perturbation that mixes one- and two-magnon subspaces.

Under these assumptions, the one-magnon state is described by the standard

$$|k\rangle = \frac{1}{\sqrt{L}} \sum_{l=1}^L e^{ikl} \hat{S}_l^- |\uparrow\uparrow\ldots\rangle = \frac{1}{\sqrt{L}} \sum_{l=1}^L e^{ikl} |l\rangle, \quad (\text{H1})$$

where $|l\rangle$ describes the state with down spin at location l and all other spins pointing up. The magnon dispersion is given by the standard $\epsilon_1(k) = H + J(1 - \cos(k)) = H + 2J \sin^2(k/2)$.

The two-magnon bound state with the center-of-mass momentum K is given by [54]

$$|2K\rangle = \sum_{l,l'=1}^L \psi_{ll'} \hat{S}_l^- \hat{S}_{l'}^- |\uparrow\uparrow\ldots\rangle, \\ \psi_{ll'} = \frac{1}{2\sqrt{L}} \sin(K/2) e^{iK(l+l')/2} e^{-\gamma(|l-l'|-1)}, \\ \epsilon_2(K) = 2H + J \sin^2(K/2), \quad (\text{H2})$$

where the bound-state radius $1/\gamma$ is determined by K via $e^{-\gamma} = \cos(K/2)$.

Our goal is to find the overlap $\langle 2K | \hat{H}_{\text{DM}} | k \rangle$. Consider

$$\hat{H}_{\text{DM}} | k \rangle = \frac{D}{2i\sqrt{L}} \sum_{l=1}^L e^{ikl} \sum_{l'=1}^L (\hat{S}_{l'}^z - \hat{S}_{l'+2}^z) (\hat{S}_{l+1}^+ - \hat{S}_{l+1}^-) | l \rangle \\ = \frac{D}{2i\sqrt{L}} \sum_{l=1}^L e^{ikl} (|l, l+1\rangle - |l-1, l\rangle), \quad (\text{H3})$$

where $|l, l'\rangle$ denotes the state with down spins at l and l' . Using Eq. (H2), it is easy to find that

$$\langle 2K | \hat{H}_{\text{DM}} | k \rangle = D \sin^2(k/2) \delta_{k,K}. \quad (\text{H4})$$

Clearly, the hybridization is strongest at the zone boundary, $k = \pi$, and it is absent in the zone center, $k = 0$.

Therefore, the effective Hamiltonian describing the mixing of the one- and two-magnon sectors is given by the 2×2 matrix with diagonal elements $\epsilon_1(k)$ and $\epsilon_2(k)$, and the off-diagonal ones $D \sin^2(k/2)$. The new eigenvalues follow immediately,

$$E(k) = \frac{1}{2} \left[\epsilon_1(k) + \epsilon_2(k) \pm \sqrt{(\epsilon_1(k) - \epsilon_2(k))^2 + 4D^2 \sin^4(k/2)} \right] \\ \approx \begin{cases} \epsilon_2(k) + \frac{D^2}{H} \sin^4(k/2) \\ \epsilon_1(k) - \frac{D^2}{H} \sin^4(k/2), \end{cases}$$

where the last line is obtained in the $H \gg J, D$ limit. It is also clear that the DM interaction couples the two-magnon sector to the three-magnon one, and so on. Thus, one should be able to reconstruct the soliton and antisoliton excitations as multimagnon bound states. At this stage, we refrain from a more general analysis of this interesting problem.

APPENDIX I: LÖWDIN SYMMETRIC ORTHOGONALIZATION

In Sec. VIA, the hybridized band structure at intermediate fields is obtained by projecting the Hamiltonian onto a low-energy subspace $\mathcal{S}_0$ spanned by nonorthogonal variational states. Here, we provide the details of the orthogonalization procedure used in that construction.

Let $\{|\phi_j\rangle\}_{j=1}^N$ denote the set of (in general, nonorthogonal) variational states spanning $\mathcal{S}_0$ —for instance, the soliton, antisoliton, single-magnon, and composite magnon-soliton states introduced in the main text. Their mutual overlaps define the $N \times N$ overlap matrix,

$$S_{jj'} = \langle \phi_j | \phi_{j'} \rangle. \quad (\text{I1})$$

Since the states $\{|\phi_j\rangle\}$ are assumed to be linearly independent, S is a positive-definite Hermitian matrix, and its inverse square root $S^{-1/2}$ is well defined.

The Löwdin symmetric orthogonalization [35,36] constructs an orthonormal basis $\{|\psi_j\rangle\}$ via the transformation

$$|\psi_j\rangle = \sum_{l=1}^N (S^{-1/2})_{jl} |\phi_l\rangle. \quad (\text{I2})$$

By construction, these states satisfy

$$\begin{aligned}\langle\psi_j|\psi_{j'}&= \sum_{l,l'}(S^{-1/2})_{jl}^*(S^{-1/2})_{j'l'}\langle\phi_l|\phi_{l'}\rangle \\ &= (S^{-1/2}SS^{-1/2})_{jj'} = \delta_{jj'}.\end{aligned}\quad (I3)$$

A key property of the Löwdin procedure, which distinguishes it from other orthogonalization methods such as Gram-Schmidt, is that it is symmetric: It treats all basis vectors on equal footing and does not depend on an arbitrary ordering of the states. Moreover, it can be shown [35] that the Löwdin-orthogonalized basis minimizes the total deviation from the original set of vectors in the least-squares sense,

$$\sum_{j=1}^N \|\psi_j - |\phi_j\rangle\|^2 \text{ is minimized,} \quad (I4)$$

among all orthonormal bases of the form $|\psi_j\rangle = \sum_l U_{jl}|\phi_l\rangle$. This property ensures that each orthogonalized state $|\psi_j\rangle$ retains, as closely as possible, the physical character of the corresponding original state $|\phi_j\rangle$.

In practice, we compute $S^{-1/2}$ by diagonalizing the overlap matrix, $S = V\Lambda V^\dagger$, where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$ contains the eigenvalues and V the corresponding eigenvectors, so that

$$S^{-1/2} = V\Lambda^{-1/2}V^\dagger, \quad \Lambda^{-1/2} = \text{diag}(\lambda_1^{-1/2}, \dots, \lambda_N^{-1/2}). \quad (I5)$$

Once the orthonormal basis $\{|\psi_j\rangle\}$ is obtained, the projected Hamiltonian matrix is

$$h_{jj'}^{\text{multi}}(k) = \langle\psi_j|\hat{H}_S|\psi_{j'}\rangle - E_0\delta_{jj'}, \quad (I6)$$

where E_0 is the ground-state energy. The hybridized excitation energies are then given by the eigenvalues of $h^{\text{multi}}(k)$ at each momentum k , and the eigenvectors encode the admixture of solitonic, antisolitonic, and magnonic character in each hybridized mode.

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