

Dynamical α -Rényi Entropies of Local Hamiltonians Grow at Most Linearly in Time

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We consider a generic one-dimensional spin system of length L , arbitrarily large, with strictly local interactions, for example, nearest neighbor, and prove that the dynamical α -Rényi entropies, $0 < \alpha \leq 1$, of an initial product state grow at most linearly in time. This result arises from a general relation among dynamical α -Rényi entropies and Lieb-Robinson bounds. We extend our bound on the dynamical generation of entropy to systems with exponential decay of interactions, for values of α close enough to 1, and, moreover, to initial pure states with low entanglement, of the order of $\log L$, that are typically represented by critical states. We establish that low entanglement states have an efficient matrix product state representation that persists at least up to times of the order of $\log L$. The main technical tools are the Lieb-Robinson bounds, to locally approximate the dynamics of the spin chain, a strict upper bound of Audenaert on α -Rényi entropies, and a bound on their concavity. Such a bound, that we provide in an appendix, can be of independent interest.

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I. INTRODUCTION

α -Rényi entropies are generalized quantum entropies that are good quantifiers of entanglement [1–4]. A list of their properties can be found in Ref. [5]. Early works on the characterization of spin systems' ground state with Rényi entropies are Refs. [6,7].

Experimental breakthroughs [8–10] have shown how quantum entropies can be detected and measured, making them not only crucial quantities to characterize the behavior of many-body systems from a theoretical point of view, but also susceptible to experimental tests.

The scaling of α -Rényi entropies, $0 < \alpha < 1$, of the reduced density matrix of a state, of a one-dimensional system, have been shown to determine whether such a state could be efficiently represented by a matrix product state (MPS) [11,12]. In higher-dimensional systems, instead, a nonvolume law for the Rényi entropy does not imply an efficient MPS representation [13].

The variation of the von Neumann entropy of the reduced density of a state evolved in time according to the unitary evolution of a system has attracted a lot of attention starting from the small incremental entangling and small

incremental mixing conjectures attributed by Bravyi in Ref. [14] to Kitaev. See Ref. [14] also for early references on this topic. A recent review covering these themes is Ref. [15]. These conjectures roughly say that the rate of increase in time of the von Neumann entropy and the entropy of mixing are upper bounded by a constant independent from the overall system's size. Audenaert [16], Van Acoleyen, Mariën, and Verstraete [17], and Mariën *et al.* [18] have given proofs of such conjectures. In particular, as a consequence of the (at most) linear growth in time of the entanglement entropy, the authors of Refs. [17,18] were able to prove, using the formalism of the quasiadiabatic continuation [19–23], that states within the same phase of matter have the same scaling of the entanglement entropy, implying the stability of the area law in dimension larger than one. A phase of matter in this context is identified by the set of gapped eigenstates of local Hamiltonians that can be smoothly connected. Recently, a concept of quantum phase has emerged in the more general context of Lindbladian evolution [24,25]. It is well known that the area law for the one-dimensional gapped state has already been proven directly [26,27]. Recently, proofs have also been given for two-dimensional frustration-free systems [28,29]. In general, a proof for higher dimensions is still lacking; nevertheless, under additional mild conditions, Masanes has proven a low entropy law in Ref. [30]. Area laws have also been shown to follow from exponential decay of corrections [31,32]. A rigorous analysis of entanglement rates for α -Rényi entropy with $\alpha > 1$ has been given in Ref. [33]; moreover, for these entropies, the authors of Ref. [34] have shown the

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generation of entanglement, from a specific class of initial product states for the dynamics of the “self-dual” kicked Ising chains, exactly.

Calabrese and Cardy pioneered the use of field theoretic methods for the evaluation of entanglement entropies and their dynamics in spin systems [35–37], providing in Ref. [36] also the solution for the transverse Ising model. Other approaches in the context of integrable systems are Ref. [38] and, for α -Rényi entropy with $\alpha > 1$, Ref. [39].

Symmetries can affect the dynamical behavior of entropies; the authors of Refs. [40–42] have pointed out subballistic growth for diffusive systems.

In our work, we are able to upper bound, as a function of time and α , the variation of the α -Rényi entropy, with $0 < \alpha \leq 1$, of the partial trace of the time evolution of a generic product state, when the time evolution is induced by a nearest-neighbor Hamiltonian $H = \sum_{j=-L}^{L-1} H_{j,j+1}$ in one dimension. This corresponds to a global quench where the ground state of the initial Hamiltonian is a product state. With r the local Hilbert space dimension, the bound reads

$$\Delta S_\alpha(t) \leq K r^{(1/\alpha)-1} \max_j \{ \|H_{j,j+1}\| \} t \log r + K' \quad (1)$$

with both K and K' constants of the order of 1. This is the first rigorous result, as far as we know, on the dynamical evolution of α -Rényi entropy, with $0 < \alpha < 1$. The upper bound on the von Neumann entropy follows as the $\alpha \rightarrow 1$ limit.

The upper bound (1) is obtained from the following general relation among α -Rényi entropies, with $0 < \alpha \leq 1$, and Lieb-Robinson (L-R) bounds. This is the most important result of our paper together with Eq. (1) that is an application of Theorem 1. A further application of it has been obtained for the case of local Hamiltonian systems with slow dynamics (many-body localized, for example), as quantified by a Lieb-Robinson bound giving rise to a logarithmic light cone, in our work in Ref. [43], where we have proven a $\log t$ behavior of $\Delta S_\alpha(t)$.

Theorem 1 (entanglement generation, informal). For a one-dimensional lattice spin system, with local Hilbert space dimension equal to r , with a Hamiltonian as in Eq. (5) and Lieb-Robinson bound as quantified by $\Delta_k(t)$ in Eq. (3), the generation of entanglement as given by the α -Rényi entropy, $0 < \alpha \leq 1$, starting from a product state ρ , being $l \geq 0$ a variational parameter upon which Eq. (2) must be minimized given the explicit expression of $\Delta_k(t)$, is upper bounded by

$$\begin{aligned} \Delta S_\alpha(t) &:= S_\alpha(\text{Tr}_{[1,L]}\rho(t)) - S_\alpha(\text{Tr}_{[1,L]}\rho) \\ &\leq \frac{1}{1-\alpha} \sum_{k=l+1}^L \log \left[1 - \alpha \int_0^t ds \Delta_{k-1}(s) \right. \\ &\quad \left. + (r^{k+1} - 1)^{1-\alpha} \left(\int_0^t ds \Delta_{k-1}(s) \right)^\alpha \right] \\ &\quad + (l+1) \log r. \end{aligned} \quad (2)$$

$\Delta_k(t)$ is upper bounded by

$$\Delta_k(t) \leq \int_0^t ds \| [H_{k,k+1} + H_{-k-1,-k}, e^{is(H_{\Lambda_k} - H_{[0,1]})} H_{[0,1]} e^{-is(H_{\Lambda_k} - H_{[0,1]})}] \|, \quad (3)$$

and $H_{\Lambda_j} := \sum_{k=-j}^{j-1} H_{k,k+1}$.

Proof. The reader will be guided from Eqs. (5)–(36) to the proof of Eq. (2). ■

When the shape of the Lieb-Robinson bound follows uniquely from the locality of the Hamiltonian, as explained, for example, in Appendix B, Eq. (2) implies Eq. (1). A more restrictive Lieb-Robinson bound that implies, for example, a logarithm light cone, gives rise from Eq. (2) to a dynamical generation of entropy that at long times follows a $\log t$ law, as we have proven in Ref. [43].

We are also able to generalize Eq. (1) (see Sec. VI), to the evolution of an initial pure state with α -Rényi entropy up to $O(\log L)$; in this case, the constant K' becomes of the order of the initial entropy of the state. Independently from the existence of an energy gap, the bound (1) shows that a state of a one-dimensional local system with interactions decaying fast enough, that has an efficient MPS representation, continues upon time evolution to have an efficient MPS representation up to times at least of the order of $\log L$.

The linear dependence on t of the upper bound on $\Delta S_\alpha(t)$ that we have established is the best possible for large t . In fact, in Ref. [44], a lower bound on the dynamical generation, starting from a product state, of the entanglement entropy for the Ising model was proven: $\Delta S(t) \geq [4/(3\pi)]t - \frac{1}{2} \ln t - 1$. The Ising model falls into the class of models, one-dimensional nearest neighbor, that we are considering here. $\Delta S(t)$ denotes the variation of the von Neumann entropy. α -Rényi entropies, with $0 < \alpha < 1$, are upper bounds of the von Neumann entropy; therefore, they cannot grow slower than t for large t .

The bound (1) is easy to generalize to the case of a k -neighbors Hamiltonian. We explicitly consider the case of interactions decreasing exponentially fast in Sec. V, proving with equation (68) an upper bound linear in t as Eq. (1) but only for values of α sufficiently close, from below, to 1, according to Eq. (66).

It should be stressed that, for pure states $\rho = |\Phi\rangle\langle\Phi|$, $|\Phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$, the $\alpha = 1/2$ Rényi entropy (of the partial trace) coincides with the logarithmic negativity $E_{\mathcal{N}}(\rho)$ [45]:

$$E_{\mathcal{N}}(\rho) := \log_2 \|\rho^{T_1}\|_1 = S_{1/2}(\text{Tr}_{\mathcal{H}_1}\rho). \quad (4)$$

In Appendix I, we show this equality following Refs. [45,46]. The logarithmic negativity has been studied in the context of many-body physics with field theoretical methods [47–49], including its dynamics [50,51].

Our work is organized as follows: In Sec. II, we present the physical setting and discuss how the unitary dynamics is approximated making use of Lieb-Robinson bounds. Here, we follow the approach employed in Refs. [52,53]. A short proof of the Lieb-Robinson bound, following Ref. [54], is in Appendix B. In Sec. III, we present the upper bound on α -Rényi entropies, $0 < \alpha < 1$, proven by Audenaert in Ref. [55] and discuss how it reduces to the more familiar Fannes-Audenaert-Petz bound [55,56] on the von Neumann entropy in the limit $\alpha \rightarrow 1$. In Sec. IV, we set up and sum up the series that gives rise to the upper bound on the Rényi entropies (1), discussing how the faster-than-exponential decrease of the spatial part of the Lieb-Robinson bound of a strictly local Hamiltonian enables the summation of the series for all $0 < \alpha \leq 1$, for arbitrarily large one-dimensional systems. We eventually state the final bound on the dynamical variation of the Rényi entropies (1). In Sec. V, we extend our bound to systems with exponential decay of interactions for values of $0 < \alpha \leq 1$ close enough to 1. We then show in Sec. VI the generalization of the theory to pure states with entanglement of the order of $\log L$. In Sec. VII, we discuss possible extensions of our work and relations with other results in the literature. A set of appendixes closes the paper by collecting proofs and technicalities.

II. PHYSICAL SETTING

The physical setting is that of a one-dimensional spin chain with sites in the interval $[-L, L]$. The local Hilbert space is $\mathbb{C}^r$. The Hamiltonian is a sum of nearest-neighbor terms:

$$H = \sum_{j=-L}^{L-1} H_{j,j+1}. \quad (5)$$

$H_{j,j+1}$ is a shorthand for $\mathbb{1}_r \otimes \cdots \otimes H_{j,j+1} \otimes \cdots \otimes \mathbb{1}_r$, to make clear that the support of $H_{j,j+1}$ is on the sites $\{j, j+1\}$. We define $J := \max_j \{\|H_{j,j+1}\|\} < \infty$, with $\|\cdot\|$ denoting the operatorial norm, that is, the maximum singular value. Our final goal is the evaluation of the variation, associated to time evolution, of the α -Rényi entropies, with $0 < \alpha \leq 1$, of a reduced density matrix. Namely, we want to upper bound as a function of time:

$$\Delta S_\alpha(t) := |S_\alpha(\text{Tr}_{[1,L]}\rho(t)) - S_\alpha(\text{Tr}_{[1,L]}\rho)|. \quad (6)$$

$\text{Tr}_{[1,L]}$ denotes partial tracing on the Hilbert space $\otimes_{j=1}^L \mathcal{H}_j$, with $\forall j, \mathcal{H}_j = \mathbb{C}^r$, while $\rho(t)$ is the evolution, at time t of the state ρ , $\rho(t) := e^{-itH}\rho e^{itH}$. We rewrite the operator of unitary evolution e^{-itH} , following the approach of Refs. [52,53], to put in evidence that the term $H_{0,1}$ of the Hamiltonian (5) is responsible for the “spread” of entanglement among the two halves of the system. We define

$$V(t) := e^{it(H_{[-L,0]} + H_{[1,L]})} e^{-itH} = e^{it(H - H_{[0,1]})} e^{-itH}. \quad (7)$$

$H_{[-L,0]} := \sum_{j=-L}^{-1} H_{j,j+1}$ and $H_{[1,L]} := \sum_{j=1}^{L-1} H_{j,j+1}$ collect the terms in the Hamiltonian (5) with support contained, respectively, in the intervals $[-L, 0]$ and $[1, L]$. The term $H_{0,1}$ has instead support on both these intervals.

Our theory holds with H as in Eq. (5) with open boundary conditions. If periodic boundary conditions were adopted instead, two terms of the Hamiltonian, say, $H_{0,1}$ and $H_{L,-L}$, would have overlapping supports with the two halves of the system after a bipartition.

It is easy to verify that replacing e^{-itH} with $V(t)$ leaves $\Delta S_\alpha(t)$ in Eq. (6) invariant. In fact,

$$S_\alpha(\text{Tr}_{[1,L]} V(t) \rho V^*(t)) \quad (8)$$

$$= S_\alpha(\text{Tr}_{[1,L]} e^{it(H_{[-L,0]} + H_{[1,L]})} e^{-itH} \rho e^{itH} e^{-it(H_{[-L,0]} + H_{[1,L]})}) \quad (9)$$

$$= S_\alpha(\text{Tr}_{[1,L]} e^{itH_{[-L,0]}} e^{itH_{[1,L]}} e^{-itH} \rho e^{itH} e^{-itH_{[1,L]}} e^{-itH_{[-L,0]}}) \quad (10)$$

$$= S_\alpha[e^{itH_{[-L,0]}} (\text{Tr}_{[1,L]} e^{itH_{[1,L]}} e^{-itH} \rho e^{itH} e^{-itH_{[1,L]}}) e^{-itH_{[-L,0]}}] \quad (11)$$

$$= S_\alpha(\text{Tr}_{[1,L]} e^{itH_{[1,L]}} e^{-itH} \rho e^{itH} e^{-itH_{[1,L]}}) \quad (12)$$

$$= S_\alpha(\text{Tr}_{[1,L]} e^{-itH} \rho e^{itH}). \quad (13)$$

In Eq. (10), we have used the fact that $H_{[-L,0]}$ and $H_{[1,L]}$ have disjoint supports and, therefore, they commute; in Eq. (11) that $H_{[-L,0]}$ is unaffected by the partial trace because it is not supported on the Hilbert space where the partial trace is acting upon; in Eq. (12) that the Rényi entropy of a state is invariant by unitary conjugation of that state; and, finally, in Eq. (13), we have used the fact that $H_{[1,L]}$ has support contained in the Hilbert space that is traced out, and, therefore, the cyclic property of the trace applies.

The operator $V(t)$ in Eq. (7) also equals

$$V(t) = \mathcal{T} \left[\exp \left(-i \int_0^t ds e^{is(H_{[-L,0]} + H_{[1,L]})} H_{[0,1]} \right. \right. \\ \left. \left. \times e^{-is(H_{[-L,0]} + H_{[1,L]})} \right) \right]. \quad (14)$$

This is the so-called interaction picture. $\mathcal{T}(\cdot)$ denotes time ordering, according to the rule that the latest time goes on the left. Equation (14) is proven, showing that, as Eq. (7) is a solution of the differential equation

$$i \frac{d}{dt} V(t) = e^{it(H - H_{[0,1]})} H_{[0,1]} e^{-it(H - H_{[0,1]})} V(t) \quad (15)$$

with a unique solution (once the initial condition $V(0) = \mathbb{1}$ is set), therefore they coincide. The interaction picture (14)

already provides the hint that the support of $V(t)$ expands starting from the origin of the system, $x = 0$, following the evolution of $H_{[0,1]}$, that, captured by the Lieb-Robinson bound of the system, is at most linear in time.

III. AUDENAERT BOUND ON α -RÉNYI ENTROPIES, $0 < \alpha < 1$

In Ref. [55], Audenaert presents a sharp form of the Fannes bound on the difference among the von Neumann entropy of a pair of states (density matrices) ρ and σ . The same bound also appears in the book of Petz [57], Theorem 3.8, where the proof is attributed to Csiszár.

As a by-product, Audenaert was able to obtain in Ref. [55] a sharp bound on the difference among the α -Rényi entropies, with $0 < \alpha < 1$, of ρ and σ . In what follows, all the logarithms are in base 2. The α -Rényi entropies, with $\alpha \neq 1$, are defined as

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log \text{Tr} \rho^\alpha. \quad (16)$$

It holds that $S_\alpha(\cdot) \leq S_\beta(\cdot)$ with $\alpha \geq \beta$. This means that our attention is focused on quantum entropies that upper bound the von Neumann entropy, S , that is, the $\alpha \rightarrow 1$ limit of the α -Rényi entropies:

$$S(\rho) = -\text{Tr} \rho \log \rho. \quad (17)$$

Denoting λ_j the eigenvalues of ρ in Eq. (16), this fact can be easily seen:

$$\begin{aligned} S_\alpha(\rho) &= \frac{1}{1-\alpha} \log \sum_j \lambda_j^\alpha \\ &= \frac{1}{1-\alpha} \left(\log \sum_j \lambda_j^\alpha \Big|_{\alpha=1} + \left(\frac{\partial}{\partial \alpha} \log \sum_j \lambda_j^\alpha \right) \Big|_{\alpha=1} (\alpha-1) \right. \\ &\quad \left. + o(\alpha-1) \right). \end{aligned} \quad (18)$$

With $\{[\partial/(\partial\alpha)] \log \sum_j \lambda_j^\alpha\} \Big|_{\alpha=1} = \sum_j \lambda_j \log \lambda_j$ and taking $\lim_{\alpha \rightarrow 1}$, the result follows. With $T := \frac{1}{2} \|\rho - \sigma\|_1$, the trace distance among the two states ρ and σ , and d the dimension of the Hilbert space where the states are acting upon, the bound of Audenaert reads as Eq. (A3) in Ref. [55]:

$$|S_\alpha(\rho) - S_\alpha(\sigma)| \leq \frac{1}{1-\alpha} \log [(1-T)^\alpha + (d-1)^{1-\alpha} T^\alpha]. \quad (20)$$

This inequality is sharp, in the sense that, for every value of T , $0 \leq T \leq 1$, there is a pair of states, at distance T , that saturates the bound. These are given by

$$\begin{aligned} \rho &= \text{Diag}(1, 0, \dots, 0), \\ \sigma &= \text{Diag}\left(1-T, \frac{T}{d-1}, \dots, \frac{T}{d-1}\right). \end{aligned} \quad (21)$$

The inverse is also true, namely, that these are the only states that saturate the bound [58]. We see that with $\alpha = 0$ the upper bound (20) becomes T independent and, therefore, trivial, being always equal to $\log d$, that is, the maximal value of the α -Rényi entropy $\forall \alpha \geq 0$. The $\alpha = 0$ Rényi entropy is also called Hartley, or max, entropy [5]. In what follows, we explicitly assume $0 < \alpha \leq 1$, with $\alpha = 1$ taken as a limit. The states (21) play the same role also in the Audenaert-Fannes-Petz upper bound on the difference of von Neumann entropy:

$$|S(\rho) - S(\sigma)| \leq T \log(d-1) + H_2(T, 1-T). \quad (22)$$

$H_2(T, 1-T)$ denotes the binary Shannon entropy: $H_2(T, 1-T) := -T \log T - (1-T) \log(1-T)$. It can be easily shown that the bound (22) is the limit for $\alpha \rightarrow 1$ of the bound (20). The Rényi entropies, with the von Neumann entropy being the continuation at $\alpha = 1$, are decreasing in α for all positive α . This can be shown, for example, using the concavity of the logarithm and the Jensen inequality. The upper bound (20) is also decreasing with $0 \leq \alpha < 1$; we explicitly show this in Appendix A.

A comparison among the bounds (20) and (22) shows that in the first one the trace distance among states is coupled with the exponential of the number of qudits in the systems (that is proportional to the dimension of the Hilbert space) and, instead, in the latter one is coupled with the number of qudits. This means that it is harder to obtain an upper bound, for the dynamically generated entanglement, that is system-size independent when the $0 \leq \alpha < 1$ Rényi entropies are concerned.

Both the bounds (20) and (22) are increasing in T only with $T \in [0, 1 - 1/d]$. With $T = 1 - 1/d$, the maximum value is reached that for both entropies is equal to $\log d$. As a consequence, since it is usually possible to obtain only upper bounds on the trace distance, the following slight modification of the bound (20), that is given in Theorem 3.1 in Ref. [59], is useful. With $T \leq R \leq 1$, it is

$$|S_\alpha(\rho) - S_\alpha(\sigma)| \leq \begin{cases} \frac{1}{1-\alpha} \log[(1-R)^\alpha + (d-1)^{1-\alpha} R^\alpha] & \text{with } R \leq 1 - \frac{1}{d}, \\ \log d, & \text{with } R \geq 1 - \frac{1}{d}. \end{cases} \quad (23)$$

In Eq. (23), ΔS_α is increasing in R . For the analogous resetting of the bound given in Eq. (22), we refer to lemma 1 of Ref. [56], and Ref. [59]. Recently, a new upper bound on the variation of von Neumann entropy that makes use of the trace distance and the norm distance among states has appeared [60].

A further upper bound to Eqs. (20) and (23), that is not tight but increasing in R , for all $0 \leq R \leq 1$, is

$$|S_\alpha(\rho) - S_\alpha(\sigma)| \leq \frac{1}{1-\alpha} \log[1 - \alpha R + (d-1)^{1-\alpha} R^\alpha]. \quad (24)$$

This follows from $(1-u)^\alpha \leq (1-\alpha u)$ for $u \leq 1$ and $0 < \alpha < 1$, and the fact that the logarithm is an increasing function.

An important remark about the bounds (20) and (22) is that their derivatives in the trace distance, when evaluated at $T = 0$, diverge. It means that increasing T , from $T = 0$, where the two states coincide, the difference in entropy increases at a diverging rate. For the von Neumann entropy and the α -Rényi entropy, with $\alpha > 1$, it has been established [17,18,61] that, for local Hamiltonians systems, the dynamically generated entropy actually grows at a finite rate from $t = 0$.

In this work, we are interested in the dynamical generation of entropy over any time interval, proving that with strictly local Hamiltonians is always independent from the system's size for all $\alpha \in (0, 1]$, while when interactions decay exponentially there always exists, for any finite system, a set of $\alpha \leq 1$ close to 1 such that the scaling is system-size independent.

IV. UPPER BOUNDING $\Delta S_\alpha(t)$ FOR A NEAREST-NEIGHBOR HAMILTONIAN

In this section, we make use of the following notation: $\Lambda_j := [-j, j]$, V_{Λ_j} is the V evolution associated to the Hamiltonian $H_{[-j,j]} := \sum_{k=-j}^{j-1} H_{k,k+1}$, namely,

$$\begin{aligned} V_{\Lambda_j}(t) &:= e^{it(H_{[-j,0]} + H_{[1,j]})} e^{-itH_{[-j,j]}} \\ &= e^{it(H_{[-j,j]} - H_{[0,1]})} e^{-itH_{[-j,j]}}. \end{aligned} \quad (25)$$

According to Eq. (25), it is $V_{\Lambda_L}(t) := V(t)$.

We have already defined, in Eq. (6), $\Delta S_\alpha(t) := |S_\alpha(\text{Tr}_{[1,L]}\rho(t)) - S_\alpha(\text{Tr}_{[1,L]}\rho)|$ as the object of our investigation. We now set an upper bound to $\Delta S_\alpha(t)$ in the form of a telescopic sum:

$$\begin{aligned} \Delta S_\alpha(t) &= |S_\alpha(\text{Tr}_{[1,L]}V(t)\rho V^*(t)) - S_\alpha(\text{Tr}_{[1,L]}\rho)| \\ &= |S_\alpha(\text{Tr}_{[1,L]}V_{\Lambda_L}(t)\rho V_{\Lambda_L}^*(t)) \\ &\quad - S_\alpha\left(\text{Tr}_{[1,L]}V_{\Lambda_{L-1}}(t)\rho V_{\Lambda_{L-1}}^*(t)\right) \\ &\quad + S_\alpha\left(\text{Tr}_{[1,L]}V_{\Lambda_{L-1}}(t)\rho V_{\Lambda_{L-1}}^*(t)\right) \\ &\quad - \dots + S_\alpha(\text{Tr}_{[1,L]}V_{\Lambda_1}(t)\rho V_{\Lambda_1}^*(t)) - S_\alpha(\text{Tr}_{[1,L]}\rho)| \end{aligned} \quad (26)$$

$$\begin{aligned} &\leq \sum_{k=L+1}^L |S_\alpha(\text{Tr}_{[1,L]}V_{\Lambda_k}(t)\rho V_{\Lambda_k}^*(t)) \\ &\quad - S_\alpha\left(\text{Tr}_{[1,L]}V_{\Lambda_{k-1}}(t)\rho V_{\Lambda_{k-1}}^*(t)\right)| \\ &\quad + |S_\alpha(\text{Tr}_{[1,L]}V_{\Lambda_1}(t)\rho V_{\Lambda_1}^*(t)) - S_\alpha(\text{Tr}_{[1,L]}\rho)|. \end{aligned} \quad (27)$$

At this point, a crucial observation follows from the assumption of ρ being a product state:

$$\rho = \bigotimes_{j=-L}^L \rho_j \quad (28)$$

with $\rho_j: \mathbb{C}^r \rightarrow \mathbb{C}^r$. This implies that each $\text{Tr}_{[1,L]}V_{\Lambda_k}(t) \times \rho V_{\Lambda_k}^*(t)$ factorizes: The unitary V_{Λ_k} is supported on the interval $[-k, k]$; with a slight abuse of notation, we write $V_{\Lambda_k} = V_{\Lambda_k} \otimes \mathbb{1}_{[-k,k]^c}$, where $[-k, k]^c$ denotes the complement of $[-k, k]$ in $[-L, L]$. Moreover, with

$$\rho_{[-k,k]} := \bigotimes_{j=-k}^k \rho_j, \quad (29)$$

we have that

$$\begin{aligned} &\text{Tr}_{[1,L]}(V_{\Lambda_k}\rho V_{\Lambda_k}^*) \\ &= \text{Tr}_{[1,k]}\text{Tr}_{[k+1,L]}(V_{\Lambda_k}\rho_{[-k,k]}V_{\Lambda_k}^* \otimes \rho_{[-k,k]^c}) \\ &= \text{Tr}_{[1,k]}(V_{\Lambda_k}\rho_{[-k,k]}V_{\Lambda_k}^*) \otimes \text{Tr}_{[k+1,L]}\rho_{[-k,k]^c}. \end{aligned} \quad (30)$$

It follows that

$$\begin{aligned} &|S_\alpha(\text{Tr}_{[1,L]}V_{\Lambda_k}(t)\rho V_{\Lambda_k}^*(t)) \\ &\quad - S_\alpha\left(\text{Tr}_{[1,L]}V_{\Lambda_{k-1}}(t)\rho V_{\Lambda_{k-1}}^*(t)\right)| \end{aligned} \quad (31)$$

$$\begin{aligned} &= |S_\alpha(\text{Tr}_{[1,k]}(V_{\Lambda_k}\rho_{[-k,k]}V_{\Lambda_k}^*) \otimes \text{Tr}_{[k+1,L]}\rho_{[-k,k]^c}) \\ &\quad - S_\alpha\left(\text{Tr}_{[1,k]}\left(V_{\Lambda_{k-1}}\rho_{[-k,k]}V_{\Lambda_{k-1}}^*\right) \otimes \text{Tr}_{[k+1,L]}\rho_{[-k,k]^c}\right)| \end{aligned} \quad (32)$$

$$= |S_\alpha(\text{Tr}_{[1,k]}(V_{\Lambda_k} \rho_{[-k,k]} V_{\Lambda_k}^*)) - S_\alpha\left(\text{Tr}_{[1,k]}\left(V_{\Lambda_{k-1}} \rho_{[-k,k]} V_{\Lambda_{k-1}}^*\right)\right)|. \quad (33)$$

In Eq. (32), we have embedded $V_{\Lambda_{k-1}}$ in the Hilbert space over the interval $[-k, k]$ with a tensorization with the identity that means (with a slight abuse of notation): $V_{\Lambda_{k-1}} = \mathbb{1}_r \otimes V_{\Lambda_{k-1}} \otimes \mathbb{1}_r$. In Eq. (33), we have used the additivity of Rényi entropies: $S_\alpha(\rho \otimes \sigma) = S_\alpha(\rho) + S_\alpha(\sigma)$. We then realize that the density matrices in Eq. (33) are supported on the Hilbert space over the interval $[-k, 0]$ that has dimensionality r^{k+1} . We define

$$T_k(t) := \frac{1}{2} \left\| \text{Tr}_{[1,k]}(V_{\Lambda_k} \rho_{[-k,k]} V_{\Lambda_k}^*) - \text{Tr}_{[1,k]}\left(V_{\Lambda_{k-1}} \rho_{[-k,k]} V_{\Lambda_{k-1}}^*\right) \right\|_1. \quad (34)$$

Using the upper bound (24) for each term of the sum (27) and taking into account Eq. (33), we have that

$$\Delta S_\alpha(t) \leq \frac{1}{1-\alpha} \sum_{k=l+1}^L \log \left[1 - \alpha T_k(t) + (r^{k+1} - 1)^{1-\alpha} T_k(t)^\alpha \right] + (l+1) \log r. \quad (35)$$

We have upper bounded the last term in Eq. (27), $|S_\alpha(\text{Tr}_{[1,L]} \rho_l(t)) - S_\alpha(\text{Tr}_{[1,L]} \rho)|$, with the trivial bound for density matrices supported on a Hilbert space of dimension r^{l+1} , that is, $(l+1) \log r$. This choice will be justified *a posteriori* after the minimization of Eq. (27) with respect to l .

The relation among the trace distance $T_k(t)$ and $\Delta_{k-1}(t)$, as given in Eq. (C14), namely,

$$T_k(t) \leq \int_0^{|t|} ds \Delta_{k-1}(s), \quad (36)$$

inserted into Eq. (35), leads to Eq. (2) of Theorem 1.

We now proceed to prove Eq. (1), taking into account the bound $\Delta_k(t') \leq [(4Jt')^k/k!]$ arising from a generic nearest-neighbor one-dimensional spin Hamiltonian (5), as shown in Appendix B.

The upper bound on the trace distance that is developed in Appendix C, being $J := \max_j \{\|H_{j,j+1}\|\}$ and the rescaled time $t' := 4Jt$, reads

$$T_k(t) \leq \frac{1}{2} \|V_{\Lambda_k} \rho_{[-k,k]} V_{\Lambda_k}^* - V_{\Lambda_{k-1}} \rho_{[-k,k]} V_{\Lambda_{k-1}}^*\|_1 \leq \frac{1}{4} \frac{(4Jt)^k}{k!} = \frac{1}{4} \frac{t'^k}{k!}. \quad (37)$$

We have used the fact that, for a generic matrix A defined on a bipartite Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, it holds $\|\text{Tr}_{\mathcal{H}_2} A\|_1 \leq \|A\|_1$, with $j \in \{1, 2\}$; as a reference see, for example, Eqs. (7) and (17) in Ref. [62]. It also follows from

$$\frac{1}{\dim(\mathcal{H}_1)} \mathbb{1}_{\mathcal{H}_1} \otimes (\text{Tr}_{\mathcal{H}_1} A) = \int_W W^* A W d_{\text{Haar}} W \quad (38)$$

with the integral above performed according to the Haar measure over all the unitaries W supported on $\mathcal{H}_1$. There is a generalization to all completely positive trace-preserving maps $\mathcal{N}$, namely, $\|\mathcal{N}(\rho) - \mathcal{N}(\sigma)\|_1 \leq \|\rho - \sigma\|_1$.

The faster than exponential decrease in k of the upper bound (37) on $T_k(t)$ ensures that, when coupled in Eq. (35) with $(r^{k+1} - 1)^{1-\alpha}$, the sum in Eq. (35) will converge. We consider the $L \rightarrow \infty$ limit. Inserting Eq. (37) into Eq. (35) and defining $u_k(t') := \frac{1}{4} [(et')/k]^k \geq \frac{1}{4} (t'^k/k!)$ [see the brief discussion in Appendix D on $u_k(t')$], we get

$$\Delta S_\alpha(t) \leq \frac{1}{1-\alpha} \sum_{k=l+1}^L \log \left[1 - \alpha u_k(t') + r^{(k+1)(1-\alpha)} u_k(t')^\alpha \right] + (l+1) \log r. \quad (39)$$

In Appendix E it is proven that Eq. (39), at a fixed t' , has a unique minimum with respect to l . A positive integer l that minimizes Eq. (39) satisfies the following pair of conditions. Denoting $f(l, t')$ the rhs of Eq. (39), the minimum is reached for that value of l such that $f(l, t')$ is smaller of both $f(l+1, t')$ and $f(l-1, t')$:

$$\begin{cases} f(l, t') \leq f(l+1, t'), \\ f(l, t') \leq f(l-1, t'). \end{cases} \quad (40)$$

These equations must be thought at a fixed α and at a fixed time t' , with l determined as a function of α and t' . Then, the equations for the minimization of $\Delta S_\alpha(t)$ are

$$\begin{cases} \frac{1}{1-\alpha} \log [1 - \alpha u_{l+1}(t') + r^{(l+2)(1-\alpha)} u_{l+1}(t')^\alpha] \leq \log r, \\ \frac{1}{1-\alpha} \log [1 - \alpha u_l(t') + r^{(l+1)(1-\alpha)} u_l(t')^\alpha] \geq \log r. \end{cases} \quad (41)$$

This implies that, despite that we are not able to solve explicitly the system (41), that can be done numerically, the approximate solution for $u_{l+1}(t')$ that we are going to provide simply gives rise to a further upper bound to Eq. (6). Nevertheless, as discussed after Eq. (1), for a generic local (nearest neighbor in this case) Hamiltonian, the linear dependence in t' of the upper bound to ΔS_α is the best possible.

We choose to provide an approximate solution to Eq. (41) solving, again approximately, with α far enough from 1, the following equation:

$$\frac{1}{1-\alpha} \log \left[r^{(l+1)(1-\alpha)} u_l(t')^\alpha \right] \approx \log r. \quad (42)$$

That leads to

$$u_l(t') \approx r^{-l[(1-\alpha)/\alpha]}, \quad (43)$$

$$\frac{t'^l}{4} \left(\frac{e}{l} \right)^l \approx r^{-l[(1-\alpha)/\alpha]}, \quad (44)$$

$$\frac{t'}{4^{\frac{1}{l}}} \left(\frac{e}{l} \right) \approx r^{-(1-\alpha)/\alpha}, \quad (45)$$

$$e r^{(1-\alpha)/\alpha} t' \approx l 4^{1/l}. \quad (46)$$

It is $1 < 4^{1/l} \leq 4$ with $l \geq 1$. As the approximate solution of Eq. (41), we pick

$$l_\alpha(t') := c r^{(1-\alpha)/\alpha} t' \quad (47)$$

with $c > e$. The reason for $c > e$ in our approach is to ensure exponential decrease in t of the sum in Eq. (39), as discussed in Appendix F. We stress again that, despite Eq. (47) being an approximate solution of Eq. (41), when l is replaced with $l_\alpha(t')$ in Eq. (39), this still provides an upper bound to $\Delta S_\alpha(t)$ because of the uniqueness of the minimum in l of Eq. (39). We remark that l is an integer, while, in general, $l_\alpha(t')$ is not an integer. Nevertheless, Eq. (39) is still well defined with l real; see the change of variable $k \rightarrow n$ in Eq. (F6).

Plugging Eq. (47) into Eq. (39), we expect that for all t' the sum in Eq. (39) will be at most of the order of 1 because of the fast decrease of the $u_k(t')$, at a fixed t' , with $k \geq l_\alpha(t') + 1$. The upper bound to Eq. (39) is performed, in a way that allows us to take the limit $\alpha \rightarrow 1$, in Appendix F. Recollecting that $t' := 4Jt$, this leads to

$$\Delta S_\alpha(t) \leq K r^{(1/\alpha)-1} J t \log r + K' \quad (48)$$

with $K = 4c$ and $K' \leq (1 + \{e/[4c(e-1)]\}) \log r$ (see Appendix F), both of the order of 1. The bound (48) holds in the limit of a system of infinite size $L \rightarrow \infty$, in this case the saturation time for the entropy, namely, the time such that the entropy reaches its maximal value, that is the logarithm of the size of the Hilbert space, is infinite as well; therefore, we can look at the asymptotic behavior at long times of Eq. (48), that is,

$$\lim_{t' \rightarrow \infty} \lim_{L \rightarrow \infty} \frac{\Delta S_\alpha(t')}{t'} \leq c r^{(1/\alpha)-1} \log r. \quad (49)$$

It is interesting to compare our upper bound with the entanglement entropy rate obtained in Ref. [18] in the case of the von Neumann entropy, $\alpha = 1$. Their Eq. (133), being an upper bound on the rate for all times, implies, in their notations,

$$\Delta S_1(t) \leq 2^{\nu+1} c' (\log d) A \sum_r r^{2\nu} \|h(r)\| t. \quad (50)$$

In a one-dimensional system, $\nu = 1$, and for a strictly local Hamiltonian, in Eq. (50), it is $A = 1$, $c' = 2$, and $\sum_r r^{2\nu} \|h(r)\| \leq J$. For comparison, in our notations, this is $\Delta S_1(t) \leq 8(\log r) J t$. Looking at the coefficients of proportionality with t in Eq. (48), with $\alpha = 1$, that is, $4cJ \log r$, it appears that, being $c > e$, we do slightly worse than them. If instead we use the L-R bounds obtained from graph theory [see Eq. (3.51) in Ref. [63]], that allow one to replace $4J$ to $2J$ in Eq. (49), then, in comparison with the results in Ref. [18], we do slightly better.

As a final remark, we recall that, despite the maximal growth of entanglement entropy being linear in time for a generic product state, the authors of Ref. [64] have established that for a system coupled to a large environment the von Neumann entropy rate of the system's reduced density matrix is low with high probability.

V. UPPER BOUNDING $\Delta S_\alpha(t)$ FOR A SYSTEM WITH EXPONENTIAL DECAY OF INTERACTIONS

We extend the results about the dynamical generation of α -Rényi entropy of a strictly local Hamiltonian to the case of an Hamiltonian with exponentially decaying interactions.

We are able to generalize the approach that we set up in Sec. IV for nearest-neighbour Hamiltonians to Hamiltonians that are a sum of terms that are supported on the whole system but with decaying interactions. The scheme that we develop works for interactions with a decay rate that is at least exponential; f denotes such decay.

Denoting $H = \sum_r H_r$ the Hamiltonian, each term H_r is supported on the whole lattice; we refer to H_r as “centered” in r . H_r is an interaction decaying according to the function f , meaning that there exist constants J and ξ that are independent from r , such that with r inside the region X , that for simplicity is assumed connected, it holds that

$$\left\| \frac{1}{2^{|X^c|}} (\text{Tr}_{X^c} H_r) \otimes \mathbb{1}_{X^c} - H_r \right\| \leq J f \left(\frac{\text{dist}(r, X^c)}{\xi} \right). \quad (51)$$

For simplicity, in Eq. (51), we are considering the case of qubits; namely, the local Hilbert space dimension is equal to 2. The physical interpretation of Eq. (51) is

straightforward: The approximation of H_r , in norm, with an operator that is supported on the region X containing r improves enlarging the distance of r from the complement of X . The definition (51) is employed, for example, in Ref. [65]. We observe that if H_r was strictly local with support contained in X , then the lhs of Eq. (51) would be exactly vanishing. Also taking f with compact support, the rhs of Eq. (51) vanishes with X large enough; this corresponds to Hamiltonians with k nearest neighbors, a generalization of the case studied in Sec. IV.

A Hamiltonian with decaying interactions does not provide an immediate way to identify the terms H_r responsible for the spread of entanglement across the two halves of the system that was given for a generic nearest-neighbor Hamiltonian by $H_{[0,1]}$. We introduce a distance b that will be determined by minimization of the upper bound to the dynamical Rényi entropy, such that the terms of H mostly responsible for the connection among the left and right halves of the system are those centered within the distance b from the origin: $\sum_{|r|<b} H_r$. The physical intuition is that b will be in relation with the decaying length ξ of definition (51).

With this in mind, we set up a different telescopic sum than Eq. (26). The first step is to rewrite

$$e^{-itH} = e^{-it(\hat{H}_{[-L,-b]} + \hat{H}_{[b,L]})} e^{it(\hat{H}_{[-L,-b]} + \hat{H}_{[b,L]})} e^{-itH}, \quad (52)$$

where we have introduced, with $1 \leq b \leq j \leq L$,

$$\begin{aligned} \hat{H}_{[b,j]} &:= \frac{1}{2^{(|[1,2j] \cap [1,L])^c|}} \left(\text{Tr}_{([1,2j] \cap [1,L])^c} \sum_{r=b}^j H_r \right) \\ &\otimes \mathbb{1}_{([1,2j] \cap [1,L])^c}. \end{aligned} \quad (53)$$

$\hat{H}_{[b,j]}$ is supported on $[1, 2j] \cap [1, L]$. We also define

$$\begin{aligned} \hat{H}_{[-j,j]} &:= \frac{1}{2^{(|[-2j,2j] \cap [-L,L])^c|}} \left(\text{Tr}_{([-2j,2j] \cap [-L,L])^c} \sum_{r=-j}^j H_r \right) \\ &\otimes \mathbb{1}_{([-2j,2j] \cap [-L,L])^c}. \end{aligned} \quad (54)$$

We notice that $\hat{H}_{[-L,L]} = H$. In Eq. (52), we have put in evidence at the exponent the sum of the terms centered on the left half and right half of the system outside the interval $(-b, b)$ that, as we said, represents the region that we assume to “connect” the left and right halves and be responsible of the entanglement spread.

The first factor of Eq. (52) is not supported across the two halves of the system; therefore, it does not contribute to the entropy. We now look at the second and third factors in Eq. (52). If we would assume that in our lattice we renamed the lattice site $j = 0$ into $j = -1$ —namely, we count the lattice sites of our system only with strictly negative or strictly positive integers (this trick was used, for example,

in Ref. [65])—then, if the Hamiltonian H was a nearest-neighbor Hamiltonian, the V operator that we introduced in Eq. (7), following Refs. [52,53], would coincide with the product of second and third factors in Eq. (52).

We define

$$\hat{V}_{\Lambda_j}(t) := e^{it(\hat{H}_{[-j,-b]} + \hat{H}_{[b,j]})} e^{-it\hat{H}_{[-j,j]}} \quad (55)$$

that is supported on $[-2j, 2j] \cap [-L, L]$. Then,

$$\begin{aligned} \Delta S_\alpha(t) &= |S_\alpha(\text{Tr}_{[1,L]} e^{-itH} \rho e^{itH}) - S_\alpha(\text{Tr}_{[1,L]} \rho)| \\ &= |S_\alpha\left(\text{Tr}_{[1,L]} \hat{V}_{\Lambda_L}(t) \rho \hat{V}_{\Lambda_L}^*(t)\right) - S_\alpha(\text{Tr}_{[1,L]} \rho)| \end{aligned} \quad (56)$$

$$\begin{aligned} &= |S_\alpha\left(\text{Tr}_{[1,L]} \hat{V}_{\Lambda_L}(t) \rho \hat{V}_{\Lambda_L}^*(t)\right) \\ &\quad - S_\alpha\left(\text{Tr}_{[1,L]} \hat{V}_{\Lambda_{L-1}}(t) \rho \hat{V}_{\Lambda_{L-1}}^*(t)\right) \end{aligned} \quad (57)$$

$$+ \cdots + S_\alpha\left(\text{Tr}_{[1,L]} \hat{V}_{\Lambda_1}(t) \rho \hat{V}_{\Lambda_1}^*(t)\right) - S_\alpha(\text{Tr}_{[1,L]} \rho)| \quad (58)$$

$$\begin{aligned} &\leq \sum_{k=l}^{L-1} |S_\alpha\left(\text{Tr}_{[1,L]} \hat{V}_{\Lambda_{k+1}}(t) \rho \hat{V}_{\Lambda_{k+1}}^*(t)\right) \\ &\quad - S_\alpha\left(\text{Tr}_{[1,L]} \hat{V}_{\Lambda_k}(t) \rho \hat{V}_{\Lambda_k}^*(t)\right)| \\ &\quad + |S_\alpha\left(\text{Tr}_{[1,L]} \hat{V}_{\Lambda_l}(t) \rho \hat{V}_{\Lambda_l}^*(t)\right) - S_\alpha(\text{Tr}_{[1,L]} \rho)|. \end{aligned} \quad (59)$$

At this point, we again assume that ρ is a product state; therefore, according to Eqs. (30)–(33), we have

$$\begin{aligned} \Delta S_\alpha(t) &\leq \sum_{k=l}^{L-1} |S_\alpha\left(\text{Tr}_{[1,2(k+1)]} \hat{V}_{\Lambda_{k+1}}(t) \rho_{[-2(k+1), 2(k+1)]} \hat{V}_{\Lambda_{k+1}}^*(t)\right) \\ &\quad - S_\alpha\left(\text{Tr}_{[1,2(k+1)]} \hat{V}_{\Lambda_k}(t) \rho_{[-2(k+1), 2(k+1)]} \hat{V}_{\Lambda_k}^*(t)\right)| \\ &\quad + |S_\alpha\left(\text{Tr}_{[1,2l]} \hat{V}_{\Lambda_l}(t) \rho_{[-2l, 2l]} \hat{V}_{\Lambda_l}^*(t)\right) - S_\alpha(\text{Tr}_{[1,L]} \rho_{[-2l, 2l]})|. \end{aligned} \quad (60)$$

In the above, we have used the fact that $\hat{V}_{\Lambda_k}(t)$ is supported on $[-2k, 2k]$. The dimensional reduction associated with product states allows the application of the Audenaert upper bound (20). We then need to provide an upper bound to the trace distance that according to Appendix G, Eqs. (G52) and (G54), reads

$R_k(t)$

$$:= \frac{1}{2} \|\text{Tr}_{[1,2(k+1)]} \left(\hat{V}_{\Lambda_{k+1}}(t) \rho_{[-2(k+1),2(k+1)]} \hat{V}_{\Lambda_{k+1}}^*(t) \right) - \text{Tr}_{[1,2(k+1)]} \left(\hat{V}_{\Lambda_k}(t) \rho_{[-2(k+1),2(k+1)]} \hat{V}_{\Lambda_k}^*(t) \right)\|_1 \quad (61)$$

$$\leq \frac{1}{2} \|\hat{V}_{\Lambda_{k+1}}(t) \rho_{[-2(k+1),2(k+1)]} \hat{V}_{\Lambda_{k+1}}^*(t) - \hat{V}_{\Lambda_k}(t) \rho_{[-2(k+1),2(k+1)]} \hat{V}_{\Lambda_k}^*(t)\|_1 \\ \leq tJO \left(g(b, J, t) f\left(\frac{k}{4\xi}\right) \right) \quad (62)$$

with $g(b, J, t) := btJ[(e^{v_{LR}t} - 1) + a] + c$, with a and c of $O(1)$, and with v_{LR} the Lieb-Robinson velocity of the Hamiltonian. We can now apply to Eq. (60) the Audenaert bound (20) that reads

$$\Delta S_\alpha(t) \leq \frac{1}{1-\alpha} \sum_{k=l}^{L-1} \log \left[1 - \alpha R_k(t) + (2^{2(k+1)+1} - 1)^{1-\alpha} R_k(t)^\alpha \right] + 2l + 1. \quad (63)$$

We now consider explicitly the case of an Hamiltonian that is a sum of terms with exponential tails $f(k/\xi) = e^{-k/\xi}$. In this case, the upper bound to $\Delta S_\alpha(t)$ is proven in Appendix G and given by Eq. (G62) in the limit $L \rightarrow \infty$. The final result, for $[1/(1 + \frac{1}{8\xi \ln 2})] < \alpha \leq 1$ and with $\beta > \{v_{LR}/[\frac{1}{4\xi} - (2 \ln 2)^{\frac{1-\alpha}{\alpha}}]\}$, is

$$\Delta S_\alpha(t) \leq 2\beta t + A(t) \quad (64)$$

with $\lambda := [\alpha/(4\xi)] - 2 \ln 2(1 - \alpha) > 0$, and $A(t)$ is decreasing exponentially fast in t and given in Eq. (G62) as

$$A(t) := -\frac{\alpha}{1-\alpha} \xi (tJ)^2 e^{v_{LR}t} e^{-\beta t/4\xi} \frac{1}{1 - e^{-1/4\xi}} \\ + \frac{1}{1-\alpha} e^{(3 \ln 2)(1-\alpha)} (\xi(tJ)^2)^\alpha e^{\alpha v_{LR}t} e^{-\lambda \beta t} \frac{1}{1 - e^{-\lambda}} + 1. \quad (65)$$

With

$$\frac{1}{1 + \frac{1}{8\xi \ln 2}} < \alpha \leq 1, \quad (66)$$

this implies, considering the lower bound on β , that

$$\lim_{t \rightarrow \infty} \lim_{L \rightarrow \infty} \frac{\Delta S_\alpha(t)}{t} \leq \frac{c' v_{LR}}{\frac{1}{4\xi} - (2 \ln 2)^{\frac{1-\alpha}{\alpha}}} \quad (67)$$

with $c' > 2$. The limit $\alpha \rightarrow 1$, that provides the von Neumann entropy, is discussed after Eq. (G63), giving

$$\lim_{t \rightarrow \infty} \lim_{L \rightarrow \infty} \frac{\Delta S_1(t)}{t} \leq 4c' v_{LR} \xi. \quad (68)$$

VI. DYNAMICAL α -RÉNYI ENTROPY FOR INITIAL LOW ENTANGLED STATES

In this section, we extend the theory that we have developed in Sec. IV for product states to initial low entangled states. This will allow us to show that, for one-dimensional systems, states with an efficient MPS representation continue to have such efficient representation, upon time evolution, at least up to times of the order of $\log L$.

The $O(\log L)$ scaling for the entanglement entropy can be found for the ground state of critical systems, like the XY model [6,7,66,67], the Ising and XXZ model [35,67,68], and the Hubbard model [69]. Remarkably, Korepin in Ref. [69] also shows, using conformal field theory methods and the results from Ref. [70], that the entanglement entropy is recovered from the thermodynamic entropy as the system's temperature goes to zero. On the other hand, with positive temperature the scaling of the entropy with the system's size is a volume law. The $O(\log L)$ scaling also appears in free fermionic models with a finite Fermi surface [71,72]. For free fermions, the $O(\log L)$ scaling has been proven for all α -Rényi entropies, $\alpha > 0$, in Ref. [73] using previous results of Sobolev.

The main technical tools employed in this section are an important result about the existence of an efficient MPS representation for low entangled states [11,12] in one dimension and an upper bound on the concavity of α -Rényi entropies, $0 < \alpha < 1$, proven using the theory of majorization, in Appendix H; this result can be of independent interest.

The results of this section are based on an assumption about the price paid, in terms of entropy, by replacing a product state with its time evolution; we clearly state this in Assumption 2.

Let us consider a low entangled state vector $|v\rangle \in (\mathbb{C}^2)^{\otimes(2L+1)}$; namely, we assume that $S_\alpha(\text{Tr}_{[1,L]}|v\rangle\langle v|) \leq O(\log L)$. The results of Ref. [11] ensure that a low entangled state $|v\rangle$, together with an additional assumption on the distribution of the tails of the distribution of the Schmidt coefficients of $|v\rangle\langle v|$ [see after Eq. (4) in Ref. [11]], has an efficient MPS representation. This means that the bond dimension (D) of the MPS is of the order of $\text{poly}(L)$. In fact, the rank of the state is upper bounded by D^2 [44,74]. See, for example, Refs. [74,75] for reviews on MPS and related topics. We express this fact by saying that $|v\rangle\langle v|$ has rank of $O[\text{poly}(L)]$ or, equivalently, is the linear combination of $O[\text{poly}(L)]$ terms:

$$|v\rangle\langle v| := \sum_{\substack{j_{-L}, \dots, j_L \\ k_{-L}, \dots, k_L}} c_{(j_{-L}, \dots, j_L)} \bar{c}_{(k_{-L}, \dots, k_L)} |j_{-L}\rangle\langle k_{-L}| \otimes \dots \otimes |j_L\rangle\langle k_L|. \quad (69)$$

The 2^{2L+1} states $\{|j_{-L}\rangle\langle j_{-L}| \otimes \cdots \otimes |j_L\rangle\langle j_L|\}$ in Eq. (69) are all orthonormal.

In Sec. IV, we upper bounded $\Delta S_\alpha(t)$ for initial product states. Restricting our attention to pure product states, it is $S_\alpha(0) = 0$. As we show in Appendix H, Eq. (H12), a convex combination of $O[\text{poly}(L)]$ pure product states has dynamical α -Rényi entropy upper bounded by $K_\alpha t + K' + O(\log L)$:

$$S_\alpha\left(\text{Tr}_{[1,L]}U(t)\left(\sum_j q_j |j\rangle\langle j|\right)U^*(t)\right) \leq K_\alpha t + K' + O(\log L). \quad (70)$$

Equation (70) implies that, for times $t \leq O(\log L)$, the state $U(t)(\sum_j q_j |j\rangle\langle j|)U^*(t)$ has an efficient matrix product operator representation.

Let us consider now a state vector $|v\rangle$ with low entanglement, as above, and make a further assumption on $|v\rangle$ formalized as follows.

Assumption 2. The pure state $|v\rangle\langle v|$, with $|v\rangle \in (\mathbb{C}^2)^{\otimes(2L+1)}$, satisfies

- (i) $S_\alpha(\text{Tr}_{[1,L]}|v\rangle\langle v|) \leq O(\log L)$.
- (ii) For $t \leq O(\log L)$, with $U(t) = e^{-itH}$ and H given in Eq. (5), $|v\rangle$ further satisfies

$$\begin{aligned} S_\alpha\left(\text{Tr}_{[1,L]}\sum_{j_{-L},\dots,j_L}(U(t)|j_{-L}\rangle\langle j_{-L}| \otimes \cdots \otimes |j_L\rangle\langle j_L|U^*(t)) \right. \\ \left. U(t)|v\rangle\langle v|U^*(t)(U(t)|j_{-L}\rangle\langle j_{-L}| \otimes \cdots \otimes |j_L\rangle\langle j_L|U^*(t))\right) \leq \\ S_\alpha\left(\text{Tr}_{[1,L]}\sum_{j_{-L},\dots,j_L}(|j_{-L}\rangle\langle j_{-L}| \otimes \cdots \otimes |j_L\rangle\langle j_L|) \right. \\ \left. U(t)|v\rangle\langle v|U^*(t)(|j_{-L}\rangle\langle j_{-L}| \otimes \cdots \otimes |j_L\rangle\langle j_L|)\right) \\ + O[\text{poly}(\log L)]. \end{aligned} \quad (71)$$

Let us discuss the meaning of this assumption. On the rhs of Eq. (71), we have replaced $U(t)|j_{-L}\rangle\langle j_{-L}| \otimes \cdots \otimes |j_L\rangle\langle j_L|U^*(t)$ with $|j_{-L}\rangle\langle j_{-L}| \otimes \cdots \otimes |j_L\rangle\langle j_L|$, considering this is a product state we know, because of our bound (48), that its entropy increases at most linearly in time, so our reasoning is that overall within a timescale $O(\log L)$ the error in entropy in the lhs of Eq. (71) would be of the same order. The upper bound of Audenaert—Eq. (20)—that we have employed to prove Eq. (48), turns out to be inefficient to try to actually prove (ii) from (i) in Assumption 2 because it requires a too small trace distance. At the same time, Audenaert's bound is designed for two completely generic states, while the pair of states that we are considering in Eq. (71) are in relation with each other.

Lemma 3. With $|v\rangle$ as in Assumption 2, it follows that

$$S_\alpha(\text{Tr}_{[1,L]}U(t)|v\rangle\langle v|U^*(t)) \leq K_\alpha t + K' + O[\text{poly}(\log L)]. \quad (72)$$

Proof.

$$S_\alpha(\text{Tr}_{[1,L]}U(t)|v\rangle\langle v|U^*(t)) \leq \quad (73)$$

$$S_\alpha\left(\text{Tr}_{[1,L]}\sum_{j_{-L},\dots,j_L}(|j_{-L}\rangle\langle j_{-L}| \otimes \cdots \otimes |j_L\rangle\langle j_L|) \right. \\ \left. U(t)|v\rangle\langle v|U^*(t)(|j_{-L}\rangle\langle j_{-L}| \otimes \cdots \otimes |j_L\rangle\langle j_L|)\right) \quad (74)$$

$$\begin{aligned} \leq S_\alpha\left(\text{Tr}_{[1,L]}\sum_{j_{-L},\dots,j_L}U(t)(|j_{-L}\rangle\langle j_{-L}| \otimes \cdots \otimes |j_L\rangle\langle j_L|) \right. \\ \left. |v\rangle\langle v|(|j_{-L}\rangle\langle j_{-L}| \otimes \cdots \otimes |j_L\rangle\langle j_L|)U^*(t)\right) \\ + O[\text{poly}(\log L)] \\ = S_\alpha\left(\text{Tr}_{[1,L]}\sum_{j_{-L},\dots,j_L}|c_{(j_{-L},\dots,j_L)}|^2 \right. \\ \left. U(t)(|j_{-L}\rangle\langle j_{-L}| \otimes \cdots \otimes |j_L\rangle\langle j_L|)U^*(t)\right) \\ + O[\text{poly}(\log L)] \end{aligned} \quad (75)$$

$$\leq K_\alpha t + K' + O[\text{poly}(\log L)]. \quad (76)$$

The inequality in Eq. (73) follows from pinching (see, for example, Sec. II.5 in Ref. [76]) and the fact that pinching with respect to a product basis and partial tracing commute. In Eq. (74), we made use of Assumption 2, and in Eq. (75) we employed the definition (69), where, following the assumption $S_\alpha(\text{Tr}_{[1,L]}|v\rangle\langle v|) \leq O(\log L)$, the number of nonvanishing coefficients $c_{(j_{-L},\dots,j_L)}$ in Eq. (75) is at most of the order of $O[\text{poly}(L)]$. Inequality (76) then follows from Eq. (H12). ■

The conclusion from Eq. (76), and Lemma 2 in Ref. [11], is that a state $|v\rangle$ with an efficient MPS representation will continue to have, under time evolution, such efficient representation for times $t \leq O(\log L)$.

VII. OUTLOOK

In this work, we have made the first step toward a general theory that ensures to obtain from the Lieb-Robinson bounds of local Hamiltonians an upper bound to the dynamical generation of α -Rényi entropies. While here we developed the general theory and explicitly solved the case of systems with linear light cones, in Ref. [43] we present an application to systems with logarithm light

cones, mostly in relation with the many-body-localization phenomenology.

A natural way to extend our theory is to look into higher dimensions and to consider explicitly the Hamiltonians with interactions decreasing slower than exponentially. This would allow one to address the stability of the area law for α -Rényi entropies in local systems using the formalism of the quasiadiabatic continuation, in analogy to what was done for the von Neumann entropy by the authors of Refs. [17,18]. Let us expand this remark.

Mariën *et al.* in Ref. [18] proved that, in any dimension, an area-law state remains area law within the same phase of matter. The definition of phase of matter that they refer to is as follows [21]: Let us consider a (local) Hamiltonian $H(s)$ depending smoothly on a parameter (or set of parameters) with a ground state that remains gapped upon variation of the parameter. A quantum phase is the set of such ground states obtained by varying the parameter.

In one dimension, gapped ground states of local Hamiltonians are area law [26]; therefore, the stability of area law follows. This is true for both the von Neumann entropy and the α -Rényi entropy, with $0 < \alpha < 1$ [77].

In more than one dimension, it is, in general, unknown if gapped ground states are area law despite that Anshu, Arad, and Gosset proved a subvolume law for 2D frustration-free spin systems [28,29].

The mapping from one state to the other in the same phase is given by the so-called quasiadiabatic continuation: It is a unitary map generated by a many-body Hamiltonian $D(s)$ —see below—with subexponential decaying terms, that in Eq. (2.19) in Ref. [22] reads

$$D(s) = \int_{-\infty}^{\infty} dt W_{\gamma}(t) e^{itH(s)} \left(\frac{d}{ds} H(s) \right) e^{-itH(s)} \quad (77)$$

with $W_{\gamma}(t)$ a filter function.

The entanglement rate theory of Mariën *et al.* applied to the quasiadiabatic continuation shows that the difference among the von Neumann entropies of two states in the same phase is $O(1)$ in any dimension.

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APPENDIX A: DECREASE IN α OF THE AUDENAERT UPPER BOUND ON ΔS_{α}

α -Rényi entropies converge to the von Neumann entropy in both the limit $\alpha \uparrow 1$ and $\alpha \downarrow 1$. Therefore, they are

continued for $\alpha = 1$ into the von Neumann entropy. A simple application of the Jensen inequality in relation with the concavity of the log shows that, with $0 < \alpha < 1$, S_{α} is decreasing in α . In the same spirit, we show that the Audenaert upper bound on ΔS_{α} of Eq. (20), and therefore Eq. (23), is decreasing in α :

$$\begin{aligned} & \frac{d}{d\alpha} \left[\frac{1}{1-\alpha} \log((1-T)^{\alpha} + (d-1)^{1-\alpha} T^{\alpha}) \right] \\ &= \frac{1}{(1-\alpha)^2} \log((1-T)^{\alpha} + (d-1)^{1-\alpha} T^{\alpha}) \\ &+ \frac{1}{1-\alpha} \frac{(\log(1-T))(1-T)^{\alpha}}{(1-T)^{\alpha} + (d-1)^{1-\alpha} T^{\alpha}} + \frac{1}{1-\alpha} \\ &\times \frac{-(\log(d-1))(d-1)^{1-\alpha} T^{\alpha} + (d-1)^{1-\alpha} (\log T) T^{\alpha}}{(1-T)^{\alpha} + (d-1)^{1-\alpha} T^{\alpha}} \\ &= \frac{1}{(1-\alpha)^2} \log((1-T)^{\alpha} + (d-1)^{1-\alpha} T^{\alpha}) \\ &+ \frac{1}{(1-\alpha)^2} \frac{(\log(1-T)^{1-\alpha})(1-T)^{\alpha}}{(1-T)^{\alpha} + (d-1)^{1-\alpha} T^{\alpha}} \\ &+ \frac{1}{(1-\alpha)^2} \frac{\log((d-1)^{\alpha-1} T^{1-\alpha})(d-1)^{1-\alpha} T^{\alpha}}{(1-T)^{\alpha} + (d-1)^{1-\alpha} T^{\alpha}}. \quad (A1) \end{aligned}$$

Defining

$$p_1 := \frac{(1-T)^{\alpha}}{(1-T)^{\alpha} + (d-1)^{1-\alpha} T^{\alpha}} \geq 0, \quad (A2)$$

$$p_2 := \frac{(d-1)^{1-\alpha} T^{\alpha}}{(1-T)^{\alpha} + (d-1)^{1-\alpha} T^{\alpha}} \geq 0, \quad (A3)$$

$$x_1 := (1-T)^{1-\alpha}, \quad (A4)$$

$$x_2 := (d-1)^{\alpha-1} T^{1-\alpha}, \quad (A5)$$

we have $p_1 + p_2 = 1$; therefore, since the logarithm is a concave function, we can apply Jensen inequality

$$\sum_j p_j \log x_j \leq \log \left(\sum_j p_j x_j \right) \quad (A6)$$

to the second and third terms in Eq. (A1). Noticing that

$$\sum_j p_j x_j = \frac{1}{(1-T)^{\alpha} + (d-1)^{1-\alpha} T^{\alpha}}, \quad (A7)$$

this results in

$$\frac{d}{d\alpha} \left[\frac{1}{1-\alpha} \log((1-T)^{\alpha} + (d-1)^{1-\alpha} T^{\alpha}) \right] \leq 0. \quad (A8)$$

APPENDIX B: LIEB-ROBINSON BOUNDS FOR STRICTLY LOCAL HAMILTONIANS

The results of this section partially appeared in Refs. [20,54]; we include it because the faster-than-exponential decrease of the spatial part of the Lieb-Robinson bounds in Eq. (B10) is the crucial ingredient for the summation of the series (27). H is as in Eq. (5), and A is any operator supported in $[0, 1]$. The idea that motivates the definition of $\Delta_j(t)$ as below is to quantify how much the evolution in time of an operator is affected by terms in the Hamiltonian that are far away from the support of such operator. We also note that the rhs of Eq. (B1) is as in Eq. (39) in Ref. [78]. The way in which a linear light cone follows from Eq. (B10) was given, for example, in Ref. [79]. The work in Ref. [80] contains the proof that on-site terms in the Hamiltonian cannot increase (but can certainly decrease [81,82]) the Lieb-Robinson velocity.

$$\Delta_j(t) := \|e^{iH_{\Lambda_{j+1}}t} A e^{-iH_{\Lambda_{j+1}}t} - e^{iH_{\Lambda_j}t} A e^{-iH_{\Lambda_j}t}\| \quad (\text{B1})$$

$$= \left\| \int_0^t ds \frac{d}{ds} \left[e^{iH_{\Lambda_{j+1}}(t-s)} \left(e^{iH_{\Lambda_j}s} A e^{-iH_{\Lambda_j}s} \right) e^{-iH_{\Lambda_{j+1}}(t-s)} \right] \right\| \quad (\text{B2})$$

$$= \left\| \int_0^t ds e^{iH_{\Lambda_{j+1}}(t-s)} \left[H_{\Lambda_{j+1}} - H_{\Lambda_j}, e^{iH_{\Lambda_j}s} A e^{-iH_{\Lambda_j}s} \right] e^{-iH_{\Lambda_{j+1}}(t-s)} \right\|. \quad (\text{B3})$$

We notice that $H_{\Lambda_{j+1}} - H_{\Lambda_j} = H_{j,j+1} + H_{-j-1,-j}$ and $H_{\Lambda_{j-1}}$ have disjoint supports, and, hence, they commute; therefore, we can insert $e^{iH_{\Lambda_{j-1}}s} A e^{-iH_{\Lambda_{j-1}}s}$ in the rhs of the commutator in Eq. (B3) provided $[0, 1]$ that is the support of A does not overlap with $[j, j+1]$ and $[-j-1, -j]$ —the smallest j that ensures this to be true is $j = 2$.

$$\Delta_j(t) = \left\| \int_0^t ds e^{iH_{\Lambda_{j+1}}(t-s)} \left[H_{j,j+1} + H_{-j-1,-j}, e^{iH_{\Lambda_j}s} A e^{-iH_{\Lambda_j}s} - e^{iH_{\Lambda_{j-1}}s} A e^{-iH_{\Lambda_{j-1}}s} \right] e^{-iH_{\Lambda_{j+1}}(t-s)} \right\| \quad (\text{B4})$$

$$\leq 4 \max\{\|H_{j,j+1}\|, \|H_{-j-1,-j}\|\} \times \int_0^t ds \|e^{iH_{\Lambda_j}s} A e^{-iH_{\Lambda_j}s} - e^{iH_{\Lambda_{j-1}}s} A e^{-iH_{\Lambda_{j-1}}s}\|. \quad (\text{B5})$$

We recognize that the integrand in Eq. (B5) is $\Delta_{j-1}(t)$, meaning that we have set up a recursive equation. Defining $J := \max_j\{\|H_{j,j+1}\|\}$, Eq. (B5) is rewritten as

$$\Delta_j(t) \leq 4J \int_0^t ds \Delta_{j-1}(s) \quad (\text{B6})$$

$$\leq (4J)^{j-1} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{j-2}} dt_{j-1} \Delta_1(t_{j-1}). \quad (\text{B7})$$

At this point, we observe that, based on Eqs. (B1)–(B3),

$$\Delta_1(t) \leq 4J\|A\|t. \quad (\text{B8})$$

In conclusion,

$$\Delta_j(t) \leq \|A\|(4J)^j \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{j-2}} t_{j-1} dt_{j-1} \leq \|A\|(4J)^j \frac{t^j}{j!}, \quad (\text{B9})$$

that renormalizing time as $t' := 4Jt$, reads

$$\Delta_j(t) \leq \|A\| \frac{t'^j}{j!}. \quad (\text{B10})$$

Following Ref. [79], let us show how a linear light cone emerges from Eq. (B10):

$$\frac{t'^j}{j!} \leq \left(\frac{et'}{j}\right)^j = \exp\left(j \log \frac{et'}{j}\right) \leq e^{et'-j}. \quad (\text{B11})$$

The quantity $v_{LR} := 4eJ$ might be called Lieb-Robinson velocity.

It turns out that an estimate on $\Delta_j(t)$ is equivalent to the usual form of the Lieb-Robinson bounds. Let us sketch this equivalence.

Given A and B operators defined for simplicity on a unidimensional lattice system $\Lambda_L := [-L, L]$, with A supported on $[0, 1]$ and the support of B on $[l+1, l+1+|b|]$, implying $\text{dist}[\text{supp}(A), \text{supp}(B)] = l$, we get

$$\|[A(t), B]\| = \|[A(t) - e^{iH_{\Lambda_l}t} A e^{-iH_{\Lambda_l}t}, B]\| \quad (\text{B12})$$

$$\leq 2\|B\| \|A(t) - e^{iH_{\Lambda_l}t} A e^{-iH_{\Lambda_l}t}\|. \quad (\text{B13})$$

With $A(t) = e^{iH_{\Lambda_L}t} A e^{-iH_{\Lambda_L}t}$, we get

$$A(t) - e^{iH_{\Lambda_l}t} A e^{-iH_{\Lambda_l}t} = e^{iH_{\Lambda_L}t} A e^{-iH_{\Lambda_L}t} - e^{iH_{\Lambda_{l-1}}t} A e^{-iH_{\Lambda_{l-1}}t} \quad (\text{B14})$$

$$+ e^{iH_{\Lambda_{l-1}}t} A e^{-iH_{\Lambda_{l-1}}t} - e^{iH_{\Lambda_{l-2}}t} A e^{-iH_{\Lambda_{l-2}}t} \quad (\text{B15})$$

$$+ \cdots \quad (\text{B16})$$

$$+ e^{iH_{\Lambda_{l+1}}t} A e^{-iH_{\Lambda_{l+1}}t} - e^{iH_{\Lambda_l}t} A e^{-iH_{\Lambda_l}t}. \quad (\text{B17})$$

Then,

$$\|A(t) - e^{iH_{\Lambda_l} t} A e^{-iH_{\Lambda_l} t}\| \leq \|A\| \sum_{j=l}^{\infty} \frac{t'^j}{j!}, \quad (\text{B18})$$

$$\sum_{j=l}^{\infty} \frac{t'^j}{j!} = t'^l \sum_{j=l}^{\infty} \frac{t'^{j-l}}{j!} = t'^l \sum_{k=0}^{\infty} \frac{t'^k}{(k+l)!} \quad (\text{B19})$$

$$= \frac{t'^l}{l!} \sum_{k=0}^{\infty} \frac{t'^k}{\binom{k+l}{k} k!} \leq \frac{t'^l}{l!} \sum_{k=0}^{\infty} \left(\frac{t'}{l+1}\right)^k. \quad (\text{B20})$$

In the last step in Eq. (B20), we have used $\binom{k+l}{k} \geq \frac{(l+1)^k}{k!}$. With $t' < l+1$, it follows that

$$\sum_{j=l}^{\infty} \frac{t'^j}{j!} \leq \frac{t'^l}{l!} \frac{1}{1 - \frac{t'}{l+1}}. \quad (\text{B21})$$

With $t' \leq [(l+1)/2]$, it is $[1/(1 - \frac{t'}{l+1})] \leq 2$; then overall, given that $l := \text{dist}[\text{supp}(A), \text{supp}(B)]$, it holds that

$$\|[A(t), B]\| \leq 4\|A\| \|B\| \frac{t'^l}{l!}. \quad (\text{B22})$$

We stress that the condition $t' \leq [(l+1)/2]$ is not restrictive, in the sense that the trivial bound to $\|[A(t), B]\|$, for generic A and B , is $2\|A\| \|B\|$; then let us determine the time t' such that $(t'^l/l!) = \frac{1}{2}$, that with l large enough is

$$t' = \left(\frac{l!}{2}\right)^{1/l} \leq (el)^{1/l} \frac{l}{e} \leq \frac{l+1}{2}. \quad (\text{B23})$$

To obtain a bound on $\Delta_j(t)$ from a L-R bound is immediate considering Eq. (B3).

We now consider, in one dimension, the more general case of an operator A with support contained in $\Lambda_k := [-k, k]$. Moving the origin of the lattice, this can always be the case. Let us consider $j > k$. To upper bound $\Delta_j(t)$, we repeat the same steps from Eqs. (B1)–(B5), noticing that the insertion in Eq. (B4) of the term $e^{iH_{\Lambda_{j-1}} s} A e^{-iH_{\Lambda_{j-1}} s}$ is allowed provided $j > k$, meaning that the smallest value of j such that this is possible is $j = k+1$. This implies that the iteration continues till under the multiple time integral we have Δ_k :

$$\Delta_j(t) \leq 4J \int_0^t ds \Delta_{j-1}(s) \quad (\text{B24})$$

$$\leq (4J)^{j-k} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{j-k-1}} dt_{j-k} \Delta_k(t_{j-k}). \quad (\text{B25})$$

At this point, we observe that, based on Eqs. (B1)–(B3),

$$\Delta_k(t) \leq 4J \|A\| t. \quad (\text{B26})$$

In conclusion,

$$\begin{aligned} \Delta_j(t) &\leq \|A\| (4J)^{j-k+1} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{j-k-1}} dt_{j-k} t_{j-k} \\ &\quad (\text{B27}) \end{aligned}$$

$$\leq \|A\| (4J)^{j-k+1} \frac{t^{j-k+1}}{(j-k+1)!}. \quad (\text{B28})$$

Equation (B27) reduces to Eq. (B9) in the case $k = 1$. Let us consider an operator B supported at a distance l from A , that we recall has support contained in $[-k, k]$. Repeating the same approach as above, we find that

$$\|[A(t), B]\| \leq 4\|A\| \|B\| \frac{(4Jt)^l}{l!}. \quad (\text{B29})$$

The Lieb-Robinson bound (B29) depends only on the norms of the operators A and B , and J , but not on the size of their supports.

The authors of the review in Ref. [63] using graph theory [83] are able [see their Eq. (3.51)] to obtain a better scaling with respect to time of the L-R bound that amounts to replace $4J$ with $2J$ in Eq. (B29).

For a more general approach that also applies to any dimension, see the derivation of the Lieb-Robinson bound in Appendix C in Ref. [30].

We conclude this section with two remarks. If the operator A is positive, $A \geq 0$, then the rescaled time becomes $t' = 2Jt$; in fact, we can use $\|A - (\|A\|/2)\mathbb{1}\| = (\|A\|/2)$, from Ref. [84], for a better upper bound. With B a generic operator, we have

$$\|[A, B]\| = \left\| \left[A - \frac{\|A\|}{2} \mathbb{1}, B \right] \right\| \quad (\text{B30})$$

$$\leq 2 \left\| A - \frac{\|A\|}{2} \mathbb{1} \right\| \|B\| = \|A\| \|B\|. \quad (\text{B31})$$

A more general relation is actually true; namely, if $A \geq 0$, denoting $\lambda_{\max}$ and $\lambda_{\min}$ the largest and smallest eigenvalues of A , respectively, then $\|A - \lambda_{\min} \mathbb{1} - [(\lambda_{\max} - \lambda_{\min})/2] \mathbb{1}\| = [(\lambda_{\max} - \lambda_{\min})/2]$. With $\lambda_{\min} = 0$, we recover the previous one.

The second remark regards the extension of Eq. (B10) to any Schatten norm, that, with $|A| := \sqrt{A^* A}$, is defined as

$$\|A\|_p := (\text{Tr}|A|^p)^{1/p}. \quad (\text{B32})$$

Using Hölder's inequality, we have $\|AB\|_p \leq \|A\|_p \|B\|_p$. This implies, with $\text{supp}(A) = [0, 1]$

$$\|e^{iH_{\Lambda_{j+1}}t} A e^{-iH_{\Lambda_{j+1}}t} - e^{iH_{\Lambda_j}t} A e^{-iH_{\Lambda_j}t}\|_p \leq \|A\|_p \frac{t^j}{j!}. \quad (\text{B33})$$

This is relevant when considering quantities like the so-called out-of-time correlators: $\text{Tr}(\sigma A(t) B A(t) B)$. Denoting $\sigma = (1/d)$ the maximally mixed state, with $A = A^*$, $B = B^*$, and $A^2 = B^2 = \mathbb{1}$, and with j the distance among the supports of A and B , we get

$$\frac{1}{d} \| [A(t), B] \|_2^2 := \frac{1}{d} \text{Tr}([A(t), B]^* [A(t), B]) \quad (\text{B34})$$

$$= 2 - \frac{2}{d} \text{Tr}(A(t) B A(t) B) \leq \frac{1}{d} \left(4 \|A\|_2 \frac{t^j}{j!} \right)^2 = \left(4 \frac{t^j}{j!} \right)^2 \quad (\text{B35})$$

The upper bound (B35) follows from (B33) and (B34).

$$\frac{1}{d} \text{Tr}(A(t) B A(t) B) \geq 1 - 8 \left(\frac{t^j}{j!} \right)^2 \geq 1 - \left(8 \left(\frac{et^j}{j} \right)^j \right)^2. \quad (\text{B36})$$

At $t = 0$, A and B commute; moreover, under the assumption $A^2 = B^2 = \mathbb{1}$, it holds that $(1/d) \text{Tr}(ABAB) = (1/d) \text{Tr}(A^2 B^2) = 1$, and then the bound (B36) is saturated. Also,

$$\frac{1}{d} |\text{Tr}(A(t) B A(t) B)| \leq \frac{1}{d} \text{Tr}|A(t) B A(t) B| \quad (\text{B37})$$

$$\leq \frac{1}{d} \|B\| \|A(t) B A(t)\|_1 \leq \frac{1}{d} \|B\| \|A^2(t) B\|_1 \quad (\text{B38})$$

$$\leq \frac{1}{d} \|B\|^2 \|A^2(t)\|_1 = 1. \quad (\text{B39})$$

In Eq. (B37), we have used the fact that, for two matrices C and D such that the product CD is Hermitian, then $\|CD\|_1 \leq \|DC\|_1$, that is, Theorem 8.1 in Ref. [85].

The interpretation of Eq. (B36) is straightforward: The decrease of $\text{Tr}(A(t) B A(t) B)$ signals the increase of the overlap among the supports of $A(t)$ and B starting from a timescale of the order of the distance among the supports of A and B . We stress that with $A \geq 0$ and $B \geq 0$ both positive then $\text{Tr}(A(t) B A(t) B)$ is positive as well; in fact, a product of positive operators is positive.

APPENDIX C: EVALUATING THE TRACE DISTANCE (34)

We now upper bound

$$T_k(t) := \frac{1}{2} \|\text{Tr}_{[1,k]} V_{\Lambda_k}(t) \rho_{[-k,k]} V_{\Lambda_k}^*(t) - \text{Tr}_{[1,k]} V_{\Lambda_{k-1}}(t) \rho_{[-k,k]} V_{\Lambda_{k-1}}^*(t)\|_1 \quad (\text{C1})$$

$$= \frac{1}{2} \|\text{Tr}_{[1,k]} V_{\Lambda_k}(t) \rho V_{\Lambda_k}^*(t) - \text{Tr}_{[1,k]} V_{\Lambda_{k-1}}(t) \rho V_{\Lambda_{k-1}}^*(t)\|_1. \quad (\text{C2})$$

To do so, we employ the following upper bound, where $U(t)$ is a generic time-dependent unitary operator with $U(0) = \mathbb{1}_{\mathcal{H}}$ and σ a generic density matrix:

$$\|U(t) \sigma U^*(t) - \sigma\|_1 = \left\| \int_0^t ds \frac{d}{ds} (U(s) \sigma U^*(s)) \right\|_1 \quad (\text{C3})$$

$$= \left\| \int_0^t ds \left[\left(\frac{d}{ds} U(s) \right) \sigma U^*(s) + U(s) \sigma \left(\frac{d}{ds} U^*(s) \right) \right] \right\|_1 \quad (\text{C4})$$

$$= \left\| \int_0^t ds \left[\left(\frac{d}{ds} U(s) \right) \sigma U^*(s) - U(s) \sigma U^*(s) \left(\frac{d}{ds} U(s) \right) U^*(s) \right] \right\|_1 \quad (\text{C5})$$

$$= \left\| \int_0^t ds U(s) \left[U^*(s) \left(\frac{d}{ds} U(s) \right) \sigma - \sigma U^*(s) \left(\frac{d}{ds} U(s) \right) \right] U^*(s) \right\|_1 \quad (\text{C6})$$

$$\leq \int_0^{|t|} ds \left\| \left[U^*(s) \frac{d}{ds} U(s), \sigma \right] \right\|_1 \quad (\text{C7})$$

$$\leq 2 \int_0^{|t|} ds \left\| U^*(s) \frac{d}{ds} U(s) \right\| \quad (\text{C8})$$

$$\leq 2|t| \sup_{s \in [0, |t|]} \left\| U^*(s) \frac{d}{ds} U(s) \right\|. \quad (\text{C9})$$

We now apply Eq. (C9) to Eq. (C1):

$$\|V_{\Lambda_{k+1}}(t) \rho V_{\Lambda_{k+1}}^*(t) - V_{\Lambda_k}(t) \rho V_{\Lambda_k}^*(t)\|_1 = \|V_{\Lambda_k}^*(t) V_{\Lambda_{k+1}}(t) \rho V_{\Lambda_{k+1}}^*(t) V_{\Lambda_k}(t) - \rho\|_1 \quad (\text{C10})$$

$$\leq 2 \int_{s=0}^{|t|} ds \left\| V_{\Lambda_{k+1}}^*(s) V_{\Lambda_k}(s) \frac{d}{ds} (V_{\Lambda_k}^*(s) V_{\Lambda_{k+1}}(s)) \right\| \quad (\text{C11})$$

$$= 2 \int_{s=0}^{|t|} ds \left\| V_{\Lambda_{k+1}}^*(s) V_{\Lambda_k}(s) \left(\frac{d}{ds} V_{\Lambda_k}^*(s) \right) V_{\Lambda_{k+1}}(s) + V_{\Lambda_{k+1}}^*(s) \frac{d}{ds} V_{\Lambda_{k+1}}(s) \right\|. \quad (\text{C12})$$

Recalling that $V_{\Lambda_k}(s) := e^{is(H_{\Lambda_k} - H_I)} e^{-isH_{\Lambda_k}}$, where we have defined $H_I := H_{[0,1]}$, we have that

$$i \frac{d}{ds} V_{\Lambda_k}(s) = e^{is(H_{\Lambda_k} - H_I)} H_I e^{-is(H_{\Lambda_k} - H_I)} V_{\Lambda_k}(s). \quad (\text{C13})$$

Then,

$$\begin{aligned} T_{k+1}(t) &\leq \int_{s=0}^{|t|} ds \|e^{is(H_{\Lambda_k} - H_I)} H_I e^{-is(H_{\Lambda_k} - H_I)} \\ &\quad - e^{is(H_{\Lambda_{k+1}} - H_I)} H_I e^{-is(H_{\Lambda_{k+1}} - H_I)}\| = \int_{s=0}^{|t|} ds \Delta_k(s) \end{aligned} \quad (\text{C14})$$

$$\leq \int_{s=0}^{|t|} ds \|H_I\| \frac{(4Js)^k}{k!} \leq \frac{1}{4} \frac{t'^{k+1}}{(k+1)!}. \quad (\text{C15})$$

We remark that the dynamics of $\Delta_k(t)$ is here generated by $H_{\Lambda_k} - H_I$, but all the steps that lead to Eq. (B10) are equally valid. To obtain Eq. (C15), we have used Eq. (B10), the definition of $J := \sup\{\|H_{j,j+1}\|\}$, that implies $\|H_I\| \leq J$, and, finally, the definition of $t' := 4Jt$.

Inserting Eq. (C14) into Eq. (35), we obtain Eq. (2).

APPENDIX D: BRIEF DISCUSSION OF $u_k(t)$

We have defined $u_k(t) := \frac{1}{4}[(et)/k]^k \geq \frac{1}{4}(t^k/k!)$. $u_k(t)$ upper bounds the trace distance. With a fixed k , a sufficient condition for $u_k(t) \leq 1$ is given by $t \leq (k/e)$.

It is easy to see that, with a fixed t , $u_k(t)$ is a decreasing function of k ; in fact,

$$\frac{u_{k+1}(t)}{u_k(t)} = \frac{\left(\frac{et}{k+1}\right)^{k+1}}{\left(\frac{et}{k}\right)^k} = \frac{et}{k+1} \left(\frac{k}{k+1}\right)^k < \frac{et}{k+1}. \quad (\text{D1})$$

The condition $t \leq (k/e)$, that implies $u_k(t) < 1$, also ensures that $u_k(t)$ is a decreasing function of k .

APPENDIX E: ALGORITHM FOR THE MINIMIZATION OF THE SUM THAT UPPER BOUNDS $\Delta S_\alpha(t)$

In this section, we discuss the existence and uniqueness of the solution $l_\alpha(t)$ that satisfies the two equations (41). We consider $\alpha \in (0, 1)$; the value $\alpha = 1$ can be dealt with analogously.

- (i) We fix t and consider a value $\bar{l}$ of l such that both the lhs of the equations in Eq. (41) are larger equal than $\log r$; this is certainly the case with $u_l(t) \in [0, 1]$ close enough to 1.
- (ii) We increase l from $\bar{l}$ to $\bar{l} + 1$. We remark the crucial fact that the value taken from the lhs of the second equation is now the value that the lhs of the first equation in Eq. (41) had at the previous step. Therefore, they cannot jump in one step from being

both larger than $\log r$ to be both smaller than $\log r$. Since $u_l(t) \leq 1$ is a decreasing function of l and both the lhs of Eq. (41) are increasing in u with $u_l(t) \in [0, 1]$, they will both decrease going from $\bar{l}$ to $\bar{l} + 1$.

- (iii) We repeat increasing l till the lhs of the first equation becomes smaller than $\log r$. This certainly happens at a certain point; in fact, with u of the order of $r^{-(l+2)[(1-\alpha)/\alpha]}$, the lhs of the first equation in Eq. (41) will be of the order of $\log r$. This is the value of l that minimizes Eq. (35).

APPENDIX F: ESTIMATE OF THE SERIES IN EQ. (39)

We want to estimate the series in Eq. (39), that is copied below Eq. (F1), using $l_\alpha(t') := cr^{(1-\alpha)/\alpha}t'$, with $c > e$ as in Eq. (47), and $u_k(t') := \frac{1}{4}[(et')/k]^k$. The result will be that for every $t' > 0$ and $\alpha \in (0, 1]$ Eq. (F1) is upper bounded by a constant of the order of 1 and is exponentially decreasing in t' :

$$\frac{1}{1-\alpha} \sum_{k=l_\alpha+1}^L \log \left[1 - \alpha u_k(t') + r^{(k+1)(1-\alpha)} u_k(t')^\alpha \right]. \quad (\text{F1})$$

We have already stated and used the fact that the Audenaert upper bound on ΔS_α is increasing in the trace distance. Let us discuss more in detail that the argument of the log in Eq. (F1) is an increasing function of $0 \leq u \leq 1$, at a fixed k :

$$\begin{aligned} 0 &= \frac{\partial}{\partial u} \left[1 - \alpha u + r^{(k+1)(1-\alpha)} u^\alpha \right] \\ &= -\alpha + \alpha r^{(k+1)(1-\alpha)} u^{\alpha-1}. \end{aligned} \quad (\text{F2})$$

That has the solution, corresponding to a maximum, as it is seen by the always negative second derivative, $u = r^{k+1}$. r^{k+1} has minimal value, reached for $k = 1$ [being $k = l_\alpha(t) + 1$, $k = 1$ is given at $t = 0$], equal to r^2 . The minimal value of r is 2; therefore, in the interval $0 \leq u \leq 1$, the rhs of Eq. (F1) is an increasing function of u . It means that by replacing u with an upper bound (smaller than r^2) we still obtain an upper bound, despite the negative term $-\alpha u$ in the argument of the log in Eq. (F1).

Let us change variable in the summation in Eq. (F1), setting $k = l_\alpha + n$; then,

$$u_{l_\alpha+n} = \frac{1}{4} \left(\frac{e}{cr^{(1-\alpha)/\alpha}} \right)^{l_\alpha+n} \left(\frac{l_\alpha}{l_\alpha+n} \right)^{l_\alpha+n}. \quad (\text{F3})$$

The function $[x/(x+n)]^{x+n} = \{1 - [n/(x+n)]\}^{x+n}$ is increasing in x , with $\lim_{x \rightarrow \infty} \{1 - [n/(x+n)]\}^{x+n} = e^{-n}$. With $x = l_\alpha$, the limit $t \rightarrow \infty$ corresponds to $l_\alpha \rightarrow \infty$. Then,

$$u_{l_a+n} \leq \frac{1}{4} \left(\frac{e}{c r^{(1-\alpha)/\alpha}} \right)^{l_a+n} e^{-n} \\ = \frac{1}{4} \left(\frac{e}{c r^{(1-\alpha)/\alpha}} \right)^{l_a} \left(\frac{1}{c r^{(1-\alpha)/\alpha}} \right)^n. \quad (\text{F4})$$

It follows that

$$r^{(l_a+n+1)(1-\alpha)} u_{l_a+n}^\alpha \leq \frac{r^{1-\alpha}}{4^\alpha} \left(\frac{e}{c} \right)^{\alpha l_a} \left(\frac{1}{c^\alpha} \right)^n. \quad (\text{F5})$$

We then upper bound Eq. (F1):

$$\frac{1}{1-\alpha} \sum_{n=1}^{\infty} \log \left[1 - \alpha u_{l_a+n}(t') + r^{(l_a+n+1)(1-\alpha)} u_{l_a+n}(t')^\alpha \right] \quad (\text{F6})$$

$$\leq \frac{1}{1-\alpha} \sum_{n=1}^{\infty} \left[-\alpha \frac{1}{4} \left(\frac{e}{c r^{(1-\alpha)/\alpha}} \right)^{l_a} \left(\frac{1}{c r^{(1-\alpha)/\alpha}} \right)^n \right. \\ \left. + \frac{r^{(1-\alpha)}}{4^\alpha} \left(\frac{e}{c} \right)^{\alpha l_a} \left(\frac{1}{c^\alpha} \right)^n \right] \quad (\text{F7})$$

$$= \frac{1}{1-\alpha} \left[-\alpha \frac{1}{4} \left(\frac{e}{c r^{(1-\alpha)/\alpha}} \right)^{c r^{(1-\alpha)/\alpha} t'} \frac{1}{c r^{(1-\alpha)/\alpha} - 1} \right. \\ \left. + \frac{r^{(1-\alpha)}}{4^\alpha} \left(\frac{e}{c} \right)^{\alpha c r^{(1-\alpha)/\alpha} t'} \frac{1}{c^\alpha - 1} \right]. \quad (\text{F8})$$

With $c > e$, it possible to see that, for all $t > 0$, in the limit $\alpha \rightarrow 0$ the upper bound (F8) tends to zero.

The factor in between square brackets in Eq. (F8) is a differentiable function of α and is vanishing with $\alpha = 1$. This means that the limit $\alpha \rightarrow 1$ of Eq. (F8) equals minus the first derivative in $\alpha = 1$ of such a function. An explicit calculation reveals that the limit $\alpha \rightarrow 1$ of Eq. (F8) is again exponentially decreasing in t .

Denoting $K' - \log r$ the upper bound in Eq. (F8), from Eq. (39) it follows that

$$\Delta S_\alpha(t) \leq l_\alpha(t') \log r + K'. \quad (\text{F9})$$

Then,

$$\Delta S_\alpha(t) \leq c r^{(1/\alpha)-1} t' \log r + K'. \quad (\text{F10})$$

Recollecting that $t' := 4tJ$, we obtain, with $K := 4c$,

$$\Delta S_\alpha(t) \leq K r^{(1/\alpha)-1} J t \log r + K' \quad (\text{F11})$$

as in Eq. (1).

APPENDIX G: ACCOUNTING FOR TAILS OF INTERACTIONS IN THE HAMILTONIAN FOR AN UPPER BOUND OF α -RÉNYI ENTROPY

In this section, we provide the details to obtain the upper bound (61). Applying Eq. (C12), we obtain

$$\|\hat{V}_{\Lambda_{k+1}}(t) \rho \hat{V}_{\Lambda_{k+1}}^*(t) - \hat{V}_{\Lambda_k}(t) \rho \hat{V}_{\Lambda_k}^*(t)\|_1 \leq 2t \sup_{s \in [0, t]} \left\| \hat{V}_{\Lambda_{k+1}}^*(s) \hat{V}_{\Lambda_k}(s) \frac{d}{ds} (\hat{V}_{\Lambda_k}^*(s)) \hat{V}_{\Lambda_{k+1}}(s) + \hat{V}_{\Lambda_{k+1}}^*(s) \frac{d}{ds} \hat{V}_{\Lambda_{k+1}}(s) \right\|. \quad (\text{G1})$$

Recalling that

$$\hat{V}_{\Lambda_k}(t) := e^{it(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} e^{-it\hat{H}_{[-k,k]}}, \quad (\text{G2})$$

it holds that

$$i \frac{d}{ds} \hat{V}_{\Lambda_k}(s) = e^{it(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} \left(\hat{H}_{[-k,k]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]} \right) e^{-it(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} \hat{V}_{\Lambda_k}(s). \quad (\text{G3})$$

Comparing this equation with Eq. (C13), where the generator of V_{Λ_k} was a unitary conjugation of the same Hermitian operator, H_I , for all k , we see that in Eq. (G3) instead we have the operator $\hat{H}_{[-k,k]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]}$ that, despite decaying outside of the interval $[-b, b]$, is supported on $[-2k, 2k]$; therefore, it is k dependent. All together, we have

$$\|\hat{V}_{\Lambda_{k+1}}(t) \rho \hat{V}_{\Lambda_{k+1}}^*(t) - \hat{V}_{\Lambda_k}(t) \rho \hat{V}_{\Lambda_k}^*(t)\|_1 \quad (\text{G4})$$

$$\leq 2t \sup_{s \in [0, t]} \left\| e^{is(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} \left(\hat{H}_{[-k,k]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]} \right) e^{-is(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} \right. \\ \left. - e^{is(\hat{H}_{[-k-1,-b]} + \hat{H}_{[b,k+1]})} \left(\hat{H}_{[-k-1,k+1]} - \hat{H}_{[-k-1,-b]} - \hat{H}_{[b,k+1]} \right) e^{-is(\hat{H}_{[-k-1,-b]} + \hat{H}_{[b,k+1]})} \right\| \quad (\text{G5})$$

$$\begin{aligned}
&\leq 2t \sup_{s \in [0,t]} \|e^{is(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} \left(\hat{H}_{[-k,k]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]} \right) e^{-is(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} \\
&\quad - e^{is(\hat{H}_{[-k-1,-b]} + \hat{H}_{[b,k+1]})} \left(\hat{H}_{[-k,k]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]} \right) e^{-is(\hat{H}_{[-k-1,-b]} + \hat{H}_{[b,k+1]})} \| \\
&\quad + 2t \| \hat{H}_{[-k-1,k+1]} - \hat{H}_{[-k-1,-b]} - \hat{H}_{[b,k+1]} - \left(\hat{H}_{[-k,k]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]} \right) \|. \tag{G6}
\end{aligned}$$

We see that the first operator norm of Eq. (G6) has almost the structure of a $\Delta_k(s)$ as in Appendix B; the difference with Eq. (B1) is that, in the case of a strictly local Hamiltonian, the operator whose Heisenberg evolutions are computed had a fixed support, while in Eq. (G6) it depends, despite only with tails, on k . We recollect that, with r inside the region X , that for simplicity is assumed connected, it holds that

$$\left\| \frac{1}{2^{|X^c|}} (\text{Tr}_{X^c} H_r) \otimes \mathbb{1}_{X^c} - H_r \right\| \leq Jf\left(\frac{\text{dist}(r, X^c)}{\xi}\right). \tag{G7}$$

Let us show that the second operator norm of Eq. (G6) is bounded by $O(JC_\xi f(k/\xi))$, with f denoting the rate of decrease of interactions, as defined in Eq. (51). We start considering $\hat{H}_{[b,k+1]} - \hat{H}_{[b,k]}$. To shorten the equations, we define $\tilde{\text{Tr}}_{[m,n]^c}(A) := [1/(2^{|[m,n]^c|})] \text{Tr}_{[m,n]^c}(A) \otimes \mathbb{1}_{[m,n]^c}$. Let us assume for now $2k < L$; the case $2k \geq L$ is discussed in (G15).

$$\hat{H}_{[b,k+1]} - \hat{H}_{[b,k]} := \tilde{\text{Tr}}_{([1,2(k+1)] \cap [-L,L])^c} \sum_{r=b}^{k+1} H_r - \tilde{\text{Tr}}_{([1,2k] \cap [-L,L])^c} \sum_{r=b}^k H_r \tag{G8}$$

$$= \tilde{\text{Tr}}_{([1,2(k+1)] \cap [-L,L])^c} \left(\mathbb{1} - \tilde{\text{Tr}}_{([1,2k] \cap [-L,L])^c \setminus ([1,2(k+1)] \cap [-L,L])^c} \right) \sum_{r=b}^k H_r + \tilde{\text{Tr}}_{([1,2(k+1)] \cap [-L,L])^c} H_{k+1}. \tag{G9}$$

The norm of the first term in Eq. (G9) is small, of the order of $Jf(k/\xi)$ [see Eq. (G12)], in fact being the complements of the sets evaluated with respect to $[-L, L]$:

$$([1, 2k] \cap [-L, L])^c \setminus ([1, 2(k+1)] \cap [-L, L])^c = [2k+1, 2(k+1)] \cap [-L, L] \tag{G10}$$

where the Hamiltonian term of $\sum_{r=b}^k H_r$ with the closest center to $[2k+1, 2(k+1)] \cap [-L, L]$ is H_k . We also observe that the normalized trace defined above is a projection (see Ref. [65]), namely, $\tilde{\text{Tr}}_{[m,n]^c}(\tilde{\text{Tr}}_{[m,n]^c}(A)) = \tilde{\text{Tr}}_{[m,n]^c}(A)$; therefore, as a superoperator, it has norm equal to 1. This implies that

$$\left\| \tilde{\text{Tr}}_{([1,2(k+1)] \cap [-L,L])^c} \left(\mathbb{1} - \tilde{\text{Tr}}_{([2k+1,2(k+1)] \cap [-L,L])} \right) \sum_{r=b}^k H_r \right\| \leq \left\| \left(\mathbb{1} - \tilde{\text{Tr}}_{([2k+1,2(k+1)] \cap [-L,L])} \right) \sum_{r=b}^k H_r \right\| \tag{G11}$$

$$\leq J \sum_{r=b}^k f\left(\frac{2k+1-r}{\xi}\right) \leq JC_\xi \left(f\left(\frac{k}{\xi}\right) - f\left(\frac{2k-b}{\xi}\right) \right) \tag{G12}$$

with C_ξ a function of ξ . If, for example, the interactions have exponential tails, it is $C_\xi = \xi$; in fact,

$$J \sum_{r=b}^k f\left(\frac{2k+1-r}{\xi}\right) = J \sum_{r=b}^k e^{-(2k+1-r)/\xi} = J \sum_{l=k+1}^{2k+1-b} e^{-l/\xi} = J \sum_{l=0}^{2k+1-b} e^{-l/\xi} - J \sum_{l=0}^k e^{-l/\xi} \tag{G13}$$

$$= J \left(\frac{1 - e^{-(2k+2-b)/\xi}}{1 - e^{-1/\xi}} - \frac{1 - e^{-(k+1)/\xi}}{1 - e^{-1/\xi}} \right) \leq J\xi \left(e^{-k/\xi} - e^{-(2k-b)/\xi} \right). \tag{G14}$$

In the last line, we have used $e^{1/\xi} - 1 \geq (1/\xi)$.

It is important to discuss what happens with $2k \geq L$; in this case, we must recollect that the support, for example, of $\hat{H}_{[b,k]}$ is $[1, 2k] \cap [-L, L]$, and, therefore, in Eq. (G9), we have

$$([1, 2k] \cap [-L, L])^c \setminus ([1, 2(k+1)] \cap [-L, L])^c = [-L, 0] \setminus [-L, 0] = \emptyset. \quad (\text{G15})$$

The partial trace associated with the empty set is the identity; therefore, the term among parentheses in Eq. (G9) is vanishing.

We mention the fact that within this formalism we are able to recover the results for nearest-neighbor Hamiltonians that we obtained in the first part of this paper and actually to generalize those to the case of a k -neighbor Hamiltonian. In fact, with f compactly supported and ξ the size of its support, the rhs of Eq. (G7) is vanishing. More in detail, considering

$$f\left(\frac{j}{\xi}\right) = \delta\left(\left\lfloor \frac{j - \frac{1}{2}}{\xi} \right\rfloor, 0\right), \quad (\text{G16})$$

$\lfloor a \rfloor$ denotes the largest integer such that $\lfloor a \rfloor \leq a$. $\delta(\cdot, \cdot)$ is the Kronecker delta. We take ξ integer, and $\xi \geq 1$. From the assumption $r \in X$ in Eq. (G7) follows that $\text{dist}(r, X^c) \geq 1$; then $j \geq 1$ and integer. According to Eq. (G16), with $j \leq \xi$, it is $f(j/\xi) = 1$, with $j > \xi$, $f(j/\xi) = 0$. This also shows that the value of b , that we have introduced in Eq. (52), for the case of a strictly local Hamiltonian that is the sum of ξ -neighbor terms equals ξ .

We are now ready to upper bound the operator norm of the second term in Eq. (G6). To ease the notation, all the sets are meant to be the intersection with $[-L, L]$:

$$\begin{aligned} & \|\hat{H}_{[-k-1, k+1]} - \hat{H}_{[-k-1, -b]} - \hat{H}_{[b, k+1]} - (\hat{H}_{[-k, k]} - \hat{H}_{[-k, -b]} - \hat{H}_{[b, k]})\| \\ &= \left\| \left(\tilde{\text{Tr}}_{[-2(k+1), 2(k+1)]^c} \left(\mathbb{1} - \tilde{\text{Tr}}_{[-2k, 2k]^c \setminus [-2(k+1), 2(k+1)]^c} \right) \sum_{r=-k}^k H_r \right) + \tilde{\text{Tr}}_{[-2(k+1), 2(k+1)]^c} (H_{-(k+1)} + H_{k+1}) \right. \\ & \quad - \left(\tilde{\text{Tr}}_{[-2(k+1), -1]^c} \left(\mathbb{1} - \tilde{\text{Tr}}_{[-2k, -1]^c \setminus [-2(k+1), -1]^c} \right) \sum_{r=-k}^{-b} H_r \right) - \tilde{\text{Tr}}_{[-2(k+1), -1]^c} H_{-(k+1)} \\ & \quad \left. - \left(\tilde{\text{Tr}}_{[1, 2(k+1)]^c} \left(\mathbb{1} - \tilde{\text{Tr}}_{[1, 2k]^c \setminus [1, 2(k+1)]^c} \right) \sum_{r=b}^k H_r \right) - \tilde{\text{Tr}}_{[1, 2(k+1)]^c} H_{k+1} \right\|. \end{aligned} \quad (\text{G17})$$

Combining together in Eq. (G17) the partial traces on H_{k+1} and $H_{-(k+1)}$, using the same procedure as for Eq. (G9), and employing Eq. (G12), we obtain that Eq. (G17) is upper bounded by $O[J C_\xi f(k/\xi)]$.

More precisely, from Eq. (G17) we have five terms to keep into account.

Term I:

$$\begin{aligned} & \left\| \left(\tilde{\text{Tr}}_{[-2(k+1), 2(k+1)]^c} \left(\mathbb{1} - \tilde{\text{Tr}}_{[-2k, 2k]^c \setminus [-2(k+1), 2(k+1)]^c} \right) \sum_{r=-k}^k H_r \right) \right\| \leq \left\| \left(\mathbb{1} - \tilde{\text{Tr}}_{[-2k-2, -2k-1] \cup [2k+1, 2k+2]} \right) \sum_{r=-k}^k H_r \right\| \\ & \leq \left\| \left(\mathbb{1} - \tilde{\text{Tr}}_{[-2k-2, -2k-1] \cup [2k+1, 2k+2]} \right) \sum_{r=0}^k H_r \right\| + \left\| \left(\mathbb{1} - \tilde{\text{Tr}}_{[-2k-2, -2k-1] \cup [2k+1, 2k+2]} \right) \sum_{r=-k}^0 H_r \right\| \end{aligned} \quad (\text{G18})$$

$$\leq 2J \sum_{r=0}^k f\left(\frac{2k+1-r}{\xi}\right) \leq 2J C_\xi \left(f\left(\frac{k}{\xi}\right) - f\left(\frac{2k}{\xi}\right) \right). \quad (\text{G19})$$

Term II:

$$\begin{aligned} & \left\| \left(\tilde{\text{Tr}}_{[-2(k+1), -1]^c} \left(\mathbb{1} - \tilde{\text{Tr}}_{[-2k, -1]^c \setminus [-2(k+1), -1]^c} \right) \sum_{r=-k}^{-b} H_r \right) \right\| \leq \left\| \left(\mathbb{1} - \tilde{\text{Tr}}_{[-2k-2, -2k-1]} \right) \sum_{r=-k}^{-b} H_r \right\| \\ & \leq J \sum_{r=b}^k f\left(\frac{2k+1-r}{\xi}\right) \leq J C_\xi \left(f\left(\frac{k}{\xi}\right) - f\left(\frac{2k-b}{\xi}\right) \right). \end{aligned} \quad (\text{G20})$$

Term III:

$$\left\| \left(\tilde{\text{Tr}}_{[1,2(k+1)]^c} \left(\mathbb{1} - \tilde{\text{Tr}}_{[1,2k]^c \setminus [1,2(k+1)]^c} \sum_{r=b}^k H_r \right) \right) \right\| \leq JC_\xi \left(f\left(\frac{k}{\xi}\right) - f\left(\frac{2k-b}{\xi}\right) \right). \quad (\text{G21})$$

Term IV:

$$\begin{aligned} \|\tilde{\text{Tr}}_{[-2(k+1),2(k+1)]^c} H_{k+1} - \tilde{\text{Tr}}_{[1,2(k+1)]^c} H_{k+1}\| &= \|\tilde{\text{Tr}}_{[-2(k+1),2(k+1)]^c} \left(\mathbb{1} - \tilde{\text{Tr}}_{[1,2(k+1)]^c \setminus [-2(k+1),2(k+1)]^c} \right) H_{k+1}\| \\ &\leq \left\| \left(\mathbb{1} - \tilde{\text{Tr}}_{[-2(k+1),0]} \right) H_{k+1} \right\| \leq Jf\left(\frac{k+1}{\xi}\right). \end{aligned} \quad (\text{G22})$$

Term V:

$$\|\tilde{\text{Tr}}_{[-2(k+1),2(k+1)]^c} H_{-(k+1)} - \tilde{\text{Tr}}_{[-2(k+1),-1]^c} H_{-(k+1)}\| \leq Jf\left(\frac{k+1}{\xi}\right). \quad (\text{G23})$$

Overall,

$$\text{term I} + \dots + \text{term V} \leq 4JC_\xi \left(f\left(\frac{k}{\xi}\right) - f\left(\frac{2k}{\xi}\right) \right) + 2Jf\left(\frac{k+1}{\xi}\right). \quad (\text{G24})$$

So far, with Eq. (G24) we have upper bounded the second term in Eq. (G6); as mentioned above, the first term is in relation with the object that in Appendix B is called $\Delta_k(t)$. In fact, from the first term in Eq. (G6),

$$\begin{aligned} &\|e^{is(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} \left(\hat{H}_{[-k,k]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]} \right) e^{-is(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} \\ &\quad - e^{is(\hat{H}_{[-k-1,-b]} + \hat{H}_{[b,k+1]})} \left(\hat{H}_{[-k,k]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]} \right) e^{-is(\hat{H}_{[-k-1,-b]} + \hat{H}_{[b,k+1]})}\| \end{aligned} \quad (\text{G25})$$

$$\begin{aligned} &= \left\| \int_0^s du \frac{d}{du} \left[e^{i(s-u)(\hat{H}_{[-k-1,-b]} + \hat{H}_{[b,k+1]})} e^{iu(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} (\hat{H}_{[-k,k]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]}) \right. \right. \\ &\quad \left. \left. \times e^{-iu(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} e^{-i(s-u)(\hat{H}_{[-k-1,-b]} + \hat{H}_{[b,k+1]})} \right] \right\| \end{aligned} \quad (\text{G26})$$

$$\begin{aligned} &\leq \int_0^s du \left\| \left[\hat{H}_{[-k-1,-b]} + \hat{H}_{[b,k+1]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]}, \right. \right. \\ &\quad \left. \left. e^{iu(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} \left(\hat{H}_{[-k,k]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]} \right) e^{-iu(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} \right] \right\|. \end{aligned} \quad (\text{G27})$$

$\hat{H}_{[-k-1,-b]} + \hat{H}_{[b,k+1]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]}$ is “mostly” supported around $\pm(k+1)$ but with tails on $[-2k, 2k]$, whereas $\hat{H}_{[-k,k]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]}$ is “mostly” supported around $[-b, b]$ with tails on $[-2k, 2k]$. The commutator in Eq. (G27) has the structure of a Lieb-Robinson bound, once we have shown that these two operators can be approximated with operators that have supports disjoint and far enough apart to ensure a small upper bound. We now perform such approximations.

Analogously to what was done in Eq. (G17), we have that

$$\|\hat{H}_{[-k-1,-b]} + \hat{H}_{[b,k+1]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]}\| \leq 2JC_\xi \left(f\left(\frac{k}{\xi}\right) - f\left(\frac{2k-b}{\xi}\right) \right) + \|\hat{H}_{k+1} + \hat{H}_{-(k+1)}\| \quad (\text{G28})$$

$$\leq 2JC_\xi \left(f\left(\frac{k}{\xi}\right) - f\left(\frac{2k-b}{\xi}\right) \right) + 2J \leq O(2J). \quad (\text{G29})$$

Equation (G29) bounds the norm of the first entry in the commutator of Eq. (G27).

To apply the theory of Lieb-Robinson bounds for local Hamiltonians with tails, we need to ensure that the two operators that enter the commutator are far apart; since they are actually terms of the Hamiltonian—namely, they have tails—we need to restrict their supports, provided the error in doing so is small. Let us consider $\hat{H}_{k+1}$ and $\hat{H}_{-(k+1)}$, that have supports $[1, 2(k+1)]$ and $[-2(k+1), -1]$. We define

$$\hat{H}_{k+1} := \tilde{\text{Tr}}_{[\lfloor \frac{3(k+1)}{4} \rfloor, \lfloor \frac{5(k+1)}{4} \rfloor]^c} H_{k+1}, \quad (\text{G30})$$

$$\hat{H}_{-(k+1)} := \tilde{\text{Tr}}_{[\lfloor \frac{-5(k+1)}{4} \rfloor, \lfloor \frac{-3(k+1)}{4} \rfloor]^c} H_{-(k+1)}. \quad (\text{G31})$$

$\lfloor a \rfloor$ denotes the largest integer such that $\lfloor a \rfloor \leq a$. With this choice, the size of the support is approximately $|k|/2$, see Fig. 1. It is easy to see that the error made is upper bounded by $Jf[k/(4\xi)]$; in fact,

$$\hat{H}_{k+1} - \tilde{\text{Tr}}_{[\lfloor \frac{3(k+1)}{4} \rfloor, \lfloor \frac{5(k+1)}{4} \rfloor]^c} H_{k+1} = \left(\tilde{\text{Tr}}_{[1, 2(k+1)]^c} \left(\mathbb{1} - \tilde{\text{Tr}}_{[\lfloor \frac{3(k+1)}{4} \rfloor, \lfloor \frac{5(k+1)}{4} \rfloor]^c \setminus [1, 2(k+1)]^c} \right) H_{k+1} \right) \quad (\text{G32})$$

is such that $[\lfloor 3(k+1)/4 \rfloor, \lfloor 5(k+1)/4 \rfloor]^c \setminus [1, 2(k+1)]^c = [1, \lfloor 3(k+1)/4 \rfloor - 1] \cup [\lfloor 5(k+1)/4 \rfloor, 2(k+1)]$, and, therefore, the distance of $k+1$ from this set is approximately $\lfloor k/4 \rfloor$.

On the other hand,

$$\hat{H}_{[-k, k]} - \hat{H}_{[-k, -b]} - \hat{H}_{[b, k]} \quad (\text{G33})$$

$$\begin{aligned} &= \tilde{\text{Tr}}_{[-2k, 2k]^c} \left(\sum_{r=-k}^{-\lfloor \frac{k}{4} \rfloor - 1} H_r + \sum_{r=-\lfloor \frac{k}{4} \rfloor}^{\lfloor \frac{k}{4} \rfloor} H_r + \sum_{r=\lfloor \frac{k}{4} \rfloor + 1}^k H_r \right) \\ &\quad - \tilde{\text{Tr}}_{[-2k, -1]^c} \left(\sum_{r=-k}^{-\lfloor \frac{k}{4} \rfloor - 1} H_r + \sum_{r=-\lfloor \frac{k}{4} \rfloor}^{-b} H_r \right) - \tilde{\text{Tr}}_{[1, 2k]^c} \left(\sum_{r=b}^{\lfloor \frac{k}{4} \rfloor} H_r + \sum_{r=\lfloor \frac{k}{4} \rfloor + 1}^k H_r \right). \end{aligned} \quad (\text{G34})$$

We pair terms from Eq. (G34) as follows:

$$\left\| \left(\tilde{\text{Tr}}_{[-2k, 2k]^c} - \tilde{\text{Tr}}_{[-2k, -1]^c} \right) \sum_{r=-k}^{-\lfloor \frac{k}{4} \rfloor - 1} H_r \right\| \leq JC_\xi \left(f\left(\frac{k}{4\xi}\right) - f\left(\frac{k}{\xi}\right) \right), \quad (\text{G35})$$

$$\left\| \left(\tilde{\text{Tr}}_{[-2k, 2k]^c} - \tilde{\text{Tr}}_{[1, 2k]^c} \right) \sum_{r=\lfloor \frac{k}{4} \rfloor + 1}^k H_r \right\| \leq JC_\xi \left(f\left(\frac{k}{4\xi}\right) - f\left(\frac{k}{\xi}\right) \right). \quad (\text{G36})$$

It is left $\tilde{\text{Tr}}_{[-2k, 2k]^c} \sum_{r=-\lfloor \frac{k}{4} \rfloor}^{\lfloor \frac{k}{4} \rfloor} H_r - \tilde{\text{Tr}}_{[-2k, -1]^c} \sum_{r=-\lfloor \frac{k}{4} \rfloor}^{-b} H_r - \tilde{\text{Tr}}_{[1, 2k]^c} \sum_{r=b}^{\lfloor \frac{k}{4} \rfloor} H_r$. We replace this operator with the same one but with support restricted to $[-\lfloor k/2 \rfloor, \lfloor k/2 \rfloor]$; we denote this operator as

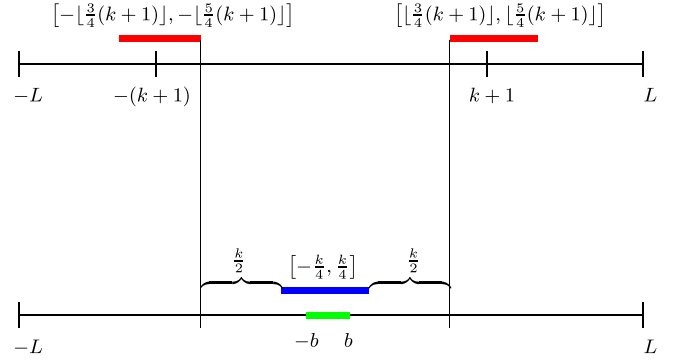


FIG. 1. In red, the regions where the supports of the operators $\hat{H}_{-(k+1)}$ and $\hat{H}_{k+1}$ have been restricted. These operators with restricted support have been denoted $\hat{H}_{k+1}$ and $\hat{H}_{-(k+1)}$. In green, the region of “maximal” support of the operator $\hat{H}_{[-k, k]} - \hat{H}_{[-k, -b]} - \hat{H}_{[b, k]}$ and, in blue, the region $[-(k/4), (k/4)]$ where the support of this operator has been restricted. This operator has been denoted $\hat{H}_{[-b, b]}$. The distance among the red and green regions, that is, the distance among the supports of $\hat{H}_{\pm(k+1)}$ and $\hat{H}_{[-b, b]}$, is approximately equal to $k/2$.

$$\hat{H}_{[-b,b]} := \tilde{\text{Tr}}_{[-\lfloor \frac{k}{4} \rfloor, \lfloor \frac{k}{2} \rfloor]^c} \sum_{r=-\lfloor \frac{k}{4} \rfloor}^{\lfloor \frac{k}{4} \rfloor} H_r - \tilde{\text{Tr}}_{[-\lfloor \frac{k}{2} \rfloor, -1]^c} \sum_{r=-\lfloor \frac{k}{4} \rfloor}^{-b} H_r - \tilde{\text{Tr}}_{[1, \lfloor \frac{k}{2} \rfloor]^c} \sum_{r=b}^{\lfloor \frac{k}{2} \rfloor} H_r. \quad (\text{G37})$$

As shown in Eq. (G38), it follows that

$$\left\| \tilde{\text{Tr}}_{[-2k, 2k]^c} \sum_{r=-\lfloor \frac{k}{4} \rfloor}^{\lfloor \frac{k}{4} \rfloor} H_r - \tilde{\text{Tr}}_{[-2k, -1]^c} \sum_{r=-\lfloor \frac{k}{4} \rfloor}^{-b} H_r - \tilde{\text{Tr}}_{[1, 2k]^c} \sum_{r=b}^{\lfloor \frac{k}{4} \rfloor} H_r - \hat{H}_{[-b,b]} \right\| \quad (\text{G38})$$

$$= \left\| \tilde{\text{Tr}}_{[-2k, 2k]^c} \left(\mathbb{1} - \tilde{\text{Tr}}_{[-\lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor]^c \setminus [-2k, 2k]^c} \right) \sum_{r=-\lfloor \frac{k}{4} \rfloor}^{\lfloor \frac{k}{4} \rfloor} H_r - \tilde{\text{Tr}}_{[-2k, -1]^c} \left(\mathbb{1} - \tilde{\text{Tr}}_{[-\lfloor \frac{k}{2} \rfloor, -1]^c \setminus [-2k, -1]^c} \right) \sum_{r=-\lfloor \frac{k}{4} \rfloor}^{-b} H_r \right. \\ \left. - \tilde{\text{Tr}}_{[1, 2k]^c} \left(\mathbb{1} - \tilde{\text{Tr}}_{[1, \lfloor \frac{k}{2} \rfloor]^c \setminus [1, 2k]^c} \right) \sum_{r=b}^{\lfloor \frac{k}{4} \rfloor} H_r \right\| \quad (\text{G39})$$

$$\leq JC_\xi \left(4f\left(\frac{k}{4\xi}\right) - 2f\left(\frac{k}{2\xi}\right) - 2f\left(\frac{\frac{k}{2}-b}{\xi}\right) \right). \quad (\text{G40})$$

To upper bound the norm of the commutator (Lieb-Robinson bound) in Eq. (G27), we also need to upper bound the norm of each entry of the commutator. The norm of the first entry is provided by Eq. (G29).

The norm of the second entry of the commutator in Eq. (G27) is bounded as follows:

$$\|\hat{H}_{[-k,k]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]}\| := \left\| \tilde{\text{Tr}}_{[-2k, 2k]^c} \sum_{r=-k}^k H_r - \tilde{\text{Tr}}_{[-2k, -1]^c} \sum_{r=-k}^{-b} H_r - \tilde{\text{Tr}}_{[1, 2k]^c} \sum_{r=b}^k H_r \right\| \quad (\text{G41})$$

$$\leq \left\| \tilde{\text{Tr}}_{[-2k, 2k]^c} (\mathbb{1} - \tilde{\text{Tr}}_{[-2k, -1]^c \setminus [-2k, 2k]^c}) \sum_{r=-k}^{-b} H_r \right\| + \left\| \tilde{\text{Tr}}_{[-2k, 2k]^c} (\mathbb{1} - \tilde{\text{Tr}}_{[1, 2k]^c \setminus [-2k, 2k]^c}) \sum_{r=b}^k H_r \right\| + \|\hat{H}_{[-b+1, b-1]}\| \quad (\text{G42})$$

$$\leq \left\| (\mathbb{1} - \tilde{\text{Tr}}_{[-2k, -1]^c \setminus [-2k, 2k]^c}) \sum_{r=-k}^{-b} H_r \right\| + \left\| (\mathbb{1} - \tilde{\text{Tr}}_{[1, 2k]^c \setminus [-2k, 2k]^c}) \sum_{r=b}^k H_r \right\| + (2b-1)J \quad (\text{G43})$$

$$\leq 2J \sum_{r=b}^k f\left(\frac{r}{\xi}\right) + (2b-1)J \leq 2JC_\xi \left(f\left(\frac{b}{\xi}\right) - f\left(\frac{k}{\xi}\right) \right) + (2b-1)J \leq O(2bJ). \quad (\text{G44})$$

Overall, the norm of the commutator in Eq. (G27) is bounded as follows:

$$\left\| \left[\hat{H}_{[-k-1, -b]} + \hat{H}_{[b, k+1]} - \hat{H}_{[-k, -b]} - \hat{H}_{[b, k]} \right], e^{iu(\hat{H}_{[-k, -b]} + \hat{H}_{[b, k]})} \left(\hat{H}_{[-k, k]} - \hat{H}_{[-k, -b]} - \hat{H}_{[b, k]} \right) e^{-iu(\hat{H}_{[-k, -b]} + \hat{H}_{[b, k]})} \right] \right\| \\ \leq \left\| \left[\hat{H}_{k+1} + \hat{H}_{-(k+1)}, e^{iu(\hat{H}_{[-k, -b]} + \hat{H}_{[b, k]})} \left(\hat{H}_{[-k, k]} - \hat{H}_{[-k, -b]} - \hat{H}_{[b, k]} \right) e^{-iu(\hat{H}_{[-k, -b]} + \hat{H}_{[b, k]})} \right] \right\| \\ + 2 \left(2JC_\xi \left(f\left(\frac{k}{\xi}\right) - f\left(\frac{2k-b}{\xi}\right) \right) + 2Jf\left(\frac{k}{4\xi}\right) \right) \left(2JC_\xi \left(f\left(\frac{b}{\xi}\right) - f\left(\frac{k}{\xi}\right) \right) + (2b-1)J \right) \quad (\text{G45})$$

$$\leq \left\| [\hat{H}_{k+1} + \hat{H}_{-(k+1)}, e^{iu(\hat{H}_{[-k, -b]} + \hat{H}_{[b, k]})} \hat{H}_{[-b, b]} e^{-iu(\hat{H}_{[-k, -b]} + \hat{H}_{[b, k]})}] \right\| \quad (\text{G46})$$

$$+ 8O(bJ^2) \left(2C_\xi \left(f\left(\frac{k}{\xi}\right) - f\left(\frac{2k-b}{\xi}\right) \right) + 2f\left(\frac{k}{4\xi}\right) \right) + 4O(C_\xi J^2) \left(6f\left(\frac{k}{4\xi}\right) - 4f\left(\frac{k}{2\xi}\right) - 2f\left(\frac{\frac{k}{2}-b}{\xi}\right) \right) \\ \leq O(bJ^2)(e^{v_{LR}t} - 1)f\left(\frac{k}{2\xi}\right) + J^2 D_\xi. \quad (\text{G47})$$

In the last line, Eq. (G47), we have introduced $D_\xi \leq O\{f[k/(4\xi)]\}$ to shorten the notation. We see that, in comparison to the Lieb-Robinson bound of strictly local operators, there is a term D_ξ that vanishes for large distances, of $O(k)$, among the supports of $\hat{H}_{k+1}$ and $\hat{H}_{[-b,b]}$, as $f[k/(2\xi)]$. We stress that when f has a compact support, as discussed before in Eq. (G17), we recover the case of strictly local Hamiltonians and D_ξ is identically vanishing. The norm of the commutator in Eq. (G46) is upper bounded by a Lieb-Robinson bound for

Hamiltonians with rapidly decaying interactions [see, for example, Eq. (S17) in Ref. [86]]; we denote v_{LR} the corresponding velocity. For a one-dimensional system, the L-R bound in Eq. (S17) in Ref. [86] is independent of the size of the supports of the operators. We notice that the supports of the operators involved in the Lieb-Robinson bound are at a distance of the order of $k/2$.

We are now ready to go back to Eq. (G6), that we copy below, to obtain the bound on the trace distance that we are looking for:

$$\frac{1}{2} \|\hat{V}_{\Lambda_{k+1}}(t) \rho \hat{V}_{\Lambda_{k+1}}^*(t) - \hat{V}_{\Lambda_k}(t) \rho \hat{V}_{\Lambda_k}^*(t)\|_1 \quad (\text{G48})$$

$$\begin{aligned} &\leq t \sup_{s \in [0,t]} \|e^{is(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} \left(\hat{H}_{[-k,k]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]} \right) e^{-is(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} \\ &\quad - e^{is(\hat{H}_{[-k-1,-b]} + \hat{H}_{[b,k+1]})} \left(\hat{H}_{[-k,k]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]} \right) e^{-is(\hat{H}_{[-k-1,-b]} + \hat{H}_{[b,k+1]})} \| \\ &\quad + t \|\hat{H}_{[-k-1,k+1]} - \hat{H}_{[-k-1,-b]} - \hat{H}_{[b,k+1]} - \left(\hat{H}_{[-k,k]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]} \right)\| \end{aligned} \quad (\text{G49})$$

$$\begin{aligned} &\leq t \sup_{s \in [0,t]} \int_0^s du \left\| \left[\hat{H}_{[-k-1,-b]} + \hat{H}_{[b,k+1]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]}, \right. \right. \\ &\quad \left. \left. e^{iu(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} \left(\hat{H}_{[-k,k]} - \hat{H}_{[-k,-b]} - \hat{H}_{[b,k]} \right) e^{-iu(\hat{H}_{[-k,-b]} + \hat{H}_{[b,k]})} \right] \right\| \\ &\quad + 4tJC_\xi \left(f\left(\frac{k}{\xi}\right) - f\left(\frac{2k}{\xi}\right) \right) + tJf\left(\frac{k+1}{\xi}\right) \end{aligned} \quad (\text{G50})$$

$$\leq t^2 O(bJ^2)(e^{v_{LR}t} - 1) f\left(\frac{k}{2\xi}\right) + t^2 J^2 D'_\xi + tJD'_\xi \quad (\text{G51})$$

$$\leq tJO\left(g(b, J, t)f\left(\frac{k}{4\xi}\right)\right). \quad (\text{G52})$$

In Eq. (G51), we have introduced

$$D'_\xi := 4C_\xi \left(f\left(\frac{k}{\xi}\right) - f\left(\frac{2k}{\xi}\right) \right) + f\left(\frac{k+1}{\xi}\right) \leq O\left(f\left(\frac{k}{\xi}\right)\right). \quad (\text{G53})$$

When f has a compact support [see Eq. (G16)], D'_ξ , as D_ξ , vanishes. In Eq. (G52), it is

$$g(b, J, t) \leq btJ((e^{v_{LR}t} - 1) + O(1)) + O(1). \quad (\text{G54})$$

The minimization with respect to b of the trace distance (G52) is straightforward, in fact, g being proportional to b in Eq. (G54) up to corrections of $O(1)$, the minimum is reached for the smallest b that makes the theory consistent. Since we have seen for the strictly local case [see Eq. (G16)] [that $b = \xi$, then we pick $b = \xi$ also in the general case.

We are now ready to upper bound the sum in Eq. (63), that we copy below Eq. (G55), to obtain $\Delta S_\alpha(t)$ of Eq. (64) in the case of a Hamiltonian with exponential decrease of interactions. As done in the case of strictly local Hamiltonians, we need to find out the value of l that minimizes the sum. This will turn out to be dependent on α , v_{LR} , and ξ . We adopt a different, but equivalent, approach

than the one in Appendix F, where we had an approximate value of l from the solution of the minimization condition. Here, we pick up $l = \beta t$ with a β to be determined in such a way to hold a linear increase in t for large t of $\Delta S_\alpha(t)$; we

know in fact that a “slower” increase has been already ruled out by Ref. [44]. With, for large enough t , $R_k(t) \leq \xi(tJ)^2 e^{v_{LR}t} e^{-k/4\xi}$ and

$$\Delta S_\alpha(t) \leq \frac{1}{1-\alpha} \sum_{k=l}^{L-1} \log \left[1 - \alpha R_k(t) + (2^{2(k+1)+1} - 1)^{1-\alpha} R_k(t)^\alpha \right] + 2l + 1 \quad (\text{G55})$$

$$\leq \frac{1}{1-\alpha} \sum_{k=l}^{\infty} \left(-\alpha R_k(t) + 2^{(2k+3)(1-\alpha)} R_k(t)^\alpha \right) + 2\beta t + 1 \quad (\text{G56})$$

$$\leq \frac{1}{1-\alpha} \sum_{n=0}^{\infty} \left(-\alpha R_{l+n}(t) + 2^{(2(l+n)+3)(1-\alpha)} R_{l+n}(t)^\alpha \right) + 2\beta t + 1 \quad (\text{G57})$$

$$\leq \frac{1}{1-\alpha} \sum_{n=0}^{\infty} \left(-\alpha \xi(tJ)^2 e^{v_{LR}t} e^{-(\beta t+n)/4\xi} + e^{\ln 2(2(\beta t+n)+3)(1-\alpha)} (\xi(tJ)^2)^\alpha e^{\alpha v_{LR}t} e^{-\alpha[(\beta t+n)/4\xi]} \right) + 2\beta t + 1 \quad (\text{G58})$$

$$\leq \frac{1}{1-\alpha} \sum_{n=0}^{\infty} \left(-\alpha \xi(tJ)^2 e^{v_{LR}t} e^{-(\beta t+n)/4\xi} + e^{3 \ln 2(1-\alpha)} (\xi(tJ)^2)^\alpha e^{\alpha v_{LR}t} e^{-\lambda(\beta t+n)} \right) + 2\beta t + 1. \quad (\text{G59})$$

In the last step, we have introduced $\lambda := [\alpha/(4\xi)] - 2 \ln 2(1-\alpha)$. We now carry out two important steps. The first is to ensure that the sum over n is finite; this gives a condition on the set of allowed α . The second step is to impose that the sum of the series decreases exponentially in time; this will provide a restriction on the value of β and, therefore, the large t behavior of $\Delta S_\alpha(t)$. From the second sum in Eq. (G59), we get a restriction on the value of α as follows:

$$\lambda > 0 \Rightarrow \frac{\alpha}{4\xi} - 2 \ln 2(1-\alpha) > 0 \Rightarrow \alpha > \frac{1}{1 + \frac{1}{8\xi \ln 2}}. \quad (\text{G60})$$

With $\xi \ll 1$, that corresponds to the Hamiltonians' term becoming almost on site, Eq. (G60) reduces to $\alpha \gtrsim 0$. In the limit of extended interactions instead $\xi \gg 1$, being by assumption $0 < \alpha \leq 1$, Eq. (G60) gives $\alpha \approx 1$.

On the other hand, the restriction on β to ensure that the sum in Eq. (G59) decreases exponentially in t provides

$$\lambda\beta - \alpha v_{LR} > 0 \Rightarrow \beta > \frac{\alpha v_{LR}}{\lambda} > \frac{v_{LR}}{\frac{1}{4\xi} - (2 \ln 2) \frac{1-\alpha}{\alpha}}. \quad (\text{G61})$$

It can be checked that the condition for the exponential decrease in time of the first term of the sum in Eq. (G59) is less restrictive than Eq. (G61).

Overall, making use of the conditions (G60) and (G61), we have that

$$\Delta S_\alpha(t) \leq \frac{1}{1-\alpha} \left[-\alpha \xi(tJ)^2 e^{v_{LR}t} e^{-\beta t/4\xi} \frac{1}{1 - e^{-1/4\xi}} + e^{(3 \ln 2)(1-\alpha)} (\xi(tJ)^2)^\alpha e^{\alpha v_{LR}t} e^{-\lambda\beta t} \frac{1}{1 - e^{-\lambda}} \right] + 1 + 2\beta t. \quad (\text{G62})$$

Considering the lower bound on β (G61), the best upper bound on $\Delta S_\alpha(t)$, with $c' > 2$, reads

$$\lim_{t \rightarrow \infty} \lim_{L \rightarrow \infty} \frac{\Delta S_\alpha(t)}{t} \leq \frac{c' v_{LR}}{\frac{1}{4\xi} - (2 \ln 2) \frac{1-\alpha}{\alpha}}. \quad (\text{G63})$$

The limit $\alpha \rightarrow 1$ gives the von Neumann entropy. The factor proportional to $\frac{1}{1-\alpha}$ in (G62) goes to zero with $\alpha \rightarrow 1$, therefore being differentiable in α , the limit $\alpha \rightarrow 1$, equals to minus the derivative of such factor in $\alpha = 1$. It is immediate to realize that such derivative is exponentially decreasing in t , then it follows:

$$\lim_{t \rightarrow \infty} \lim_{L \rightarrow \infty} \frac{\Delta S_1(t)}{t} \leq 4c' v_{LR} \xi. \quad (\text{G64})$$

This shows that the limits $\alpha \rightarrow 1$ and $\lim_{t \rightarrow \infty}$ of the rate $\lim_{L \rightarrow \infty} \frac{\Delta S_1(t)}{t}$ can be exchanged leading to the same upper bound in (G64), that in fact coincides with the limit $\alpha \rightarrow 1$ of the rhs of (G63).

**APPENDIX H: AN UPPER BOUND
ON THE CONCAVITY OF THE α -RÉNYI
ENTROPIES, $0 < \alpha < 1$**

Lemma 4. Given a convex combination of states $\rho := \sum_{i=1}^N p_i \rho_i$, with $\rho_i: \mathbb{C}^d \rightarrow \mathbb{C}^d$, denoting $\rho_i = \sum_{j=1}^{n_i} \lambda_j^i |e_j^i\rangle\langle e_j^i|$ the spectral decomposition of each ρ_i , for $0 < \alpha < 1$, it holds that

$$\sum_{i=1}^N p_i S_\alpha(\rho_i) \leq S_\alpha(\rho) \leq H_\alpha\{p_i \lambda_j^i\}, \quad (\text{H1})$$

where $H_\alpha\{p_i \lambda_j^i\} := [1/(1-\alpha)] \log \sum_{i=1}^N \sum_{j=1}^{n_i} (p_i \lambda_j^i)^\alpha$ is the α -Rényi entropy of the probability distribution $\{p_i \lambda_j^i\}$. Moreover,

$$H_\alpha\{p_i \lambda_j^i\} \leq H_\alpha\{p_i\} + \max_i \{S_\alpha(\rho_i)\}. \quad (\text{H2})$$

Therefore, if $\{\rho_i\}$ are pure states, it is $S_\alpha(\rho) \leq H_\alpha\{p_i\}$, with the upper bound depending only on the probability distribution $\{p_i\}$.

Proof. We now sketch a proof; for more details, consult Lemma 1.24 (p. 18) and Example 11.12 (p. 178) in Ref. [87]. The first inequality in Eq. (H1) is the concavity of the Rényi entropy with $0 < \alpha < 1$ [5,88]. The proof of the second inequality in Eq. (H1) is a simple application of the theory of majorization; see Chap. II in Ref. [76]. The spectral decomposition of ρ reads $\rho = \sum_{k=1}^m a_k |v_k\rangle\langle v_k|$. Since $\{|v_k\rangle\}$ is an orthonormal set, it follows that $m \leq d$. The set $\{|e_j^i\rangle\}$, that collects the eigenvectors of all $\{\rho_i\}$, is, in general, not orthonormal. In fact, for example, two states ρ_m and ρ_n can share an eigenvector. It follows that, completing the matrix V with entries $V_{j,k}^i := \sqrt{[(p_i \lambda_j^i)/a_k] \langle v_k | e_j^i \rangle}$ to a unitary matrix U , it holds that

$$|e_j^i\rangle = \sum_k U_{j,k}^i |v_k\rangle. \quad (\text{H3})$$

It follows from the definition of V , and the unitarity of U , that

$$p_i \lambda_j^i = \sum_k |U_{j,k}^i|^2 a_k. \quad (\text{H4})$$

The matrix with entries $|U_{j,k}^i|^2$ is a double stochastic matrix, meaning that $\sum_k |U_{j,k}^i|^2 = \sum_{i,j} |U_{j,k}^i|^2 = 1$; this implies—see Theorem II.1.10 (p. 33) in Ref. [76]—that the set $\{p_i \lambda_j^i\}$ is majorized by the set $\{a_k\}$. Each set is supposed to be ordered in a nonincreasing fashion. It then follows from Theorem II.3.1 (p. 40) in Ref. [76], and the concavity of the function x^α , with $0 < \alpha < 1$, that

$$\sum_k a_k^\alpha \leq \sum_{i,j} (p_i \lambda_j^i)^\alpha. \quad (\text{H5})$$

That implies, the logarithm being an increasing function, $S_\alpha(\rho) := H_\alpha\{a_k\} \leq H_\alpha\{p_i \lambda_j^i\}$.

We now prove Eq. (H2):

$$\begin{aligned} H_\alpha\{p_i \lambda_j^i\} &:= \frac{1}{1-\alpha} \log \sum_{i,j} (p_i \lambda_j^i)^\alpha \\ &= \frac{1}{1-\alpha} \log \sum_{i=1}^N p_i^\alpha \sum_j (\lambda_j^i)^\alpha \end{aligned} \quad (\text{H6})$$

$$\begin{aligned} &= \frac{1}{1-\alpha} \log \sum_{i=1}^N p_i^\alpha \exp \left((1-\alpha) \frac{1}{1-\alpha} \log \sum_j (\lambda_j^i)^\alpha \right) \\ &= \frac{1}{1-\alpha} \log \sum_{i=1}^N p_i^\alpha \exp ((1-\alpha) S_\alpha(\rho_i)) \end{aligned} \quad (\text{H7})$$

$$\begin{aligned} &\leq \frac{1}{1-\alpha} \log \sum_{i=1}^N p_i^\alpha \exp \left((1-\alpha) \max_i \{S_\alpha(\rho_i)\} \right) \\ &= H_\alpha\{p_i\} + \max_i \{S_\alpha(\rho_i)\}. \end{aligned} \quad (\text{H8})$$

■

Equation (H1) holds also for the von Neumann entropy, with the Rényi entropy H_α replaced by the Shannon entropy H . In fact, the same proof applies with the function x^α replaced by $-x \log x$ in Eq. (H5). See Eq. (2.3) in Ref. [89], and also Theorem 11.10 (p. 518) in Ref. [90], or Theorem 3.7 (p. 35) in Ref. [57] for alternative proofs. For the von Neumann entropy, we can see that the upper bound takes a more explicit form:

$$S(\rho) \leq H\{p_i\} + \sum_i p_i S(\rho_i). \quad (\text{H9})$$

In fact,

$$\begin{aligned} H\{p_i \lambda_j^i\} &:= - \sum_{i,j} p_i \lambda_j^i \log(p_i \lambda_j^i) \\ &= - \sum_{i,j} p_i \lambda_j^i (\log p_i + \log \lambda_j^i) \end{aligned} \quad (\text{H10})$$

$$\begin{aligned} &= - \sum_i p_i \log p_i - \sum_{i,j} p_i \lambda_j^i \log \lambda_j^i \\ &= H\{p_i\} + \sum_i p_i S(\rho_i). \end{aligned} \quad (\text{H11})$$

This implies that the difference $S(\rho) - \sum_i p_i S(\rho_i) \leq H\{p_i\}$ is upper bounded by a quantity depending on only the probability distribution $\{p_i\}$, but not on other quantities like the states $\{\rho_i\}$ or the dimension of the Hilbert space. The bound (H9) has been improved in Theorem 14 in

Ref. [16], where it also appears an upper bound on the difference of Holevo quantities for different ensembles that is independent from the Hilbert space dimension.

In general, this is not the case for the Rényi entropy. Let us consider the state ρ of Eq. (21) and the maximally mixed state of $\mathbb{C}^d$, $(1/d)\mathbb{1}_d$. It is easy to see that $S_\alpha[p\rho + (1-p)(1/d)\mathbb{1}_d] \approx \log d$, with fixed α and p , both not too close to 1, and $d \gg 1$. Then, $S_\alpha[p\rho + (1-p)(1/d)\mathbb{1}_d] - (1-p)S_\alpha(1/d\mathbb{1}_d) \approx p \log d$, depending on the size of the Hilbert space.

Within the setting of the physical system described in Sec. II and the notations of Lemma 4, let us consider the case where $\max\{S_\alpha(\rho_i)\} \leq O(\log L)$ and $N \leq O(L)$; then, according to Eq. (H2), we have

$$S_\alpha(\rho) \leq \frac{1}{1-\alpha} \log \sum_{i=1}^N p_i^\alpha + O(\log L) \leq O(\log L). \quad (\text{H12})$$

This shows that when convex combinations of few, $O(L)$, low entangled states, $S_\alpha(\rho_i) \leq O(\log L)$, are considered, then $\rho = \sum_i p_i \rho_i$ is still low entangled: $S_\alpha(\rho) \leq O(\log L)$.

APPENDIX I: PROOF OF EQ. (4): THE $\frac{1}{2}$ -RÉNYI ENTROPY (OF THE PARTIAL TRACE) EQUALS THE LOGARITHMIC NEGATIVITY FOR PURE STATES

In this short section, we reproduce the argument in Proposition 8 in Ref. [45] (see also the appendix in Ref. [46]), leading to the proof that the $\frac{1}{2}$ -Rényi entropy equals the logarithmic negativity for pure states.

Given a state ρ defined on the tensor product of finite-dimensional Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$, the partial transpose with respect to the first space ρ^{T_1} has matrix elements such that

$$\langle e_j^1 \otimes e_k^2, \rho^{T_1}(e_l^1 \otimes e_m^2) \rangle = \langle e_l^1 \otimes e_k^2, \rho(e_j^1 \otimes e_m^2) \rangle. \quad (\text{I1})$$

The characterization of separable states by the properties of their partial transpose started with the Peres criterion [91], stating that a necessary condition for separability is that all the eigenvalues of the partial transpose are negative. This is also sufficient, as established in Ref. [92]; see their Remark 2. Vidal and Werner defined the logarithmic negativity in Ref. [45] as

$$E_{\mathcal{N}}(\rho) := \log_2 \|\rho^{T_1}\|_1. \quad (\text{I2})$$

Following Proposition 8 in Ref. [45], let us show that, for a pure state $\rho = |\Phi\rangle\langle\Phi|$, and the Schmidt decomposition of $|\Phi\rangle = \sum_\alpha c_\alpha |e_\alpha^1 \otimes e_\alpha^2\rangle$, with $c_\alpha > 0$, it holds that

$$S_{1/2}(\text{Tr}_{\mathcal{H}_1} \rho) := 2 \log \sum_\alpha c_\alpha = \log_2 \|\rho^{T_1}\|_1. \quad (\text{I3})$$

It is well known that the common nonzero eigenvalues of $\text{Tr}_{\mathcal{H}_1} \rho$ and $\text{Tr}_{\mathcal{H}_2} \rho$ are the set $\{c_\alpha^2\}$:

$$(|\Phi\rangle\langle\Phi|)^{T_1} = \sum_{\alpha,\beta} c_\alpha c_\beta (|e_\alpha^1 \otimes e_\alpha^2\rangle\langle e_\beta^1 \otimes e_\beta^2|)^{T_1}. \quad (\text{I4})$$

It is

$$(|e_\alpha^1 \otimes e_\alpha^2\rangle\langle e_\beta^1 \otimes e_\beta^2|)^{T_1} = |e_\beta^1 \otimes e_\alpha^2\rangle\langle e_\alpha^1 \otimes e_\beta^2|; \quad (\text{I5})$$

in fact,

$$\langle e_\gamma^1 \otimes e_\delta^2 | e_\beta^1 \otimes e_\alpha^2 \rangle \langle e_\alpha^1 \otimes e_\beta^2 | e_\lambda^1 \otimes e_\eta^2 \rangle = \delta_{\gamma,\beta} \delta_{\delta,\alpha} \delta_{\alpha,\lambda} \delta_{\beta,\eta}. \quad (\text{I6})$$

On the other hand,

$$\langle e_\lambda^1 \otimes e_\delta^2 | e_\beta^1 \otimes e_\alpha^2 \rangle \langle e_\beta^1 \otimes e_\alpha^2 | e_\gamma^1 \otimes e_\eta^2 \rangle = \delta_{\lambda,\alpha} \delta_{\delta,\alpha} \delta_{\beta,\gamma} \delta_{\beta,\eta}. \quad (\text{I7})$$

Then,

$$(|\Phi\rangle\langle\Phi|)^{T_1} = \sum_{\alpha,\beta} c_\alpha c_\beta |e_\beta^1 \otimes e_\alpha^2\rangle\langle e_\alpha^1 \otimes e_\beta^2|. \quad (\text{I8})$$

Let us define the operator $F: \mathcal{H}_1 \otimes \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$, such that $F(e_\alpha^1 \otimes e_\beta^2) = (e_\beta^1 \otimes e_\alpha^2)$. It is immediate to verify that F is unitary; in fact, F^{-1} coincides with F , while the adjoint, F^* , of F is such that given any two vectors ψ and ϕ in $\mathcal{H}_1 \otimes \mathcal{H}_2$:

$$\psi = \sum_{\alpha,\beta} c_\alpha c_\beta |e_\alpha^1 \otimes e_\beta^2\rangle, \quad \phi = \sum_{\gamma,\delta} d_\gamma d_\delta |e_\gamma^1 \otimes e_\delta^2\rangle, \quad (\text{I9})$$

$$\langle \phi | F^* | \psi \rangle := \overline{\langle \psi | F | \phi \rangle} \quad (\text{I10})$$

$$= \sum_{\alpha,\beta,\gamma,\delta} c_\alpha c_\beta \overline{d_\gamma d_\delta} \langle e_\alpha^1 \otimes e_\beta^2 | F | e_\gamma^1 \otimes e_\delta^2 \rangle \quad (\text{I11})$$

$$= \left(\sum_\alpha c_\alpha \overline{d_\alpha} \right)^2. \quad (\text{I12})$$

It can be easily checked that $\langle \phi | F | \psi \rangle$ coincides with Eq. (I12). Defining

$$C_1 := \sum_\alpha c_\alpha |e_\alpha^1\rangle\langle e_\alpha^1|, \quad C_2 := \sum_\beta c_\beta |e_\beta^2\rangle\langle e_\beta^2|, \quad (\text{I13})$$

F as defined above is extended $F: \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, in such a way it acts like the identity on “bras.” This implies that $F(C_1 \otimes C_2)$ equals the rhs of Eq. (I8):

$$F(C_1 \otimes C_2) := \sum_{\alpha, \beta} c_\alpha c_\beta |e_\beta^1 \otimes e_\alpha^2\rangle \langle e_\alpha^1 \otimes e_\beta^2|. \quad (\text{I14})$$

This extended F is unitary as well. This follows from the proof above, or it can be checked directly over the space of matrices with complex entries $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, where the adjoint F^* is this time defined, with $A, B \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, as

$$\langle A, F^*(B) \rangle = \overline{\langle B, F(A) \rangle} = \overline{\text{Tr}}. \quad (\text{I15})$$

Finally, we have that

$$\|(|\Phi\rangle\langle\Phi|)^{T_1}\|_1 = \|F(C_1 \otimes C_2)\|_1 = \|C_1 \otimes C_2\|_1 \quad (\text{I16})$$

$$= \|C_1\|_1 \|C_2\|_1 = \left(\sum_\alpha c_\alpha \right)^2 \quad (\text{I17})$$

that proves Eq. (I3).

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