

Emergent Gauge Flux in Mixed QED₃ with Flavor Chemical Potential: Implications for Magnetized U(1) Dirac Spin Liquids

Chuang Chen^{1,2}, Urban F. P. Seifert,³ Kexin Feng^{1,2}, Oleg A. Starykh⁴, Leon Balents^{5,6,7} and Zi Yang Meng^{1,2,*}

¹*Department of Physics and HK Institute of Quantum Science and Technology,
The University of Hong Kong, Pokfulam Road, Hong Kong SAR, China*

²*State Key Laboratory of Optical Quantum Materials, The University of Hong Kong,
Pokfulam Road, Hong Kong SAR, China*

³*Institute for Theoretical Physics, University of Cologne, Zùlpicher Strasse 77a, 50937 Cologne, Germany*

⁴*Department of Physics and Astronomy, University of Utah, Salt Lake City, Utah 84112, USA*

⁵*Kavli Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, USA*

⁶*French American Center for Theoretical Science,*

CNRS, KITP, Santa Barbara, California 93106-4030, USA

⁷*Canadian Institute for Advanced Research, Toronto, Ontario, Canada*



(Received 21 August 2025; revised 6 August 2026; accepted 28 September 2026; published 30 October 2026)

We design a lattice model of a “mixed” U(1) gauge field coupled to fermions with a flavor chemical potential and solve it with large-scale determinant quantum Monte Carlo simulations. In this mixed construction, dynamical magnetic monopole events are suppressed in the temporal-continuum limit. For zero flavor chemical potential, the model realizes three-dimensional quantum electrodynamics (QED₃) which has been argued to describe the ground state and low-energy excitations of the Dirac spin liquid phase of quantum antiferromagnets. At finite flavor chemical potential, which corresponds to a Zeeman field perturbing the Dirac spin liquid, we find a “chiral flux” phase which is characterized by the generation of a finite mean emergent gauge flux and, accordingly, the formation of relativistic Landau levels for the Dirac fermions. We use field-theory arguments to argue that in this state a U(1)_m magnetic symmetry is spontaneously broken, leading to a gapless photon mode. Because of spin-flux attachment this mode would contribute to the low-energy longitudinal spin response. We numerically compute longitudinal and transverse spin structure factors which match our continuum and lattice mean-field theory predictions. In a different region of the phase diagram, strong fluctuations of the emergent gauge field give rise to an antiferromagnetically ordered state with gapped Dirac fermions coexisting with a deconfined gauge field. We also find an interesting intermediate phase where the chiral flux phase and the antiferromagnetic phase coexist. We argue that our results pave the way to testable predictions for magnetized Dirac spin liquids in frustrated quantum antiferromagnets.

DOI: [10.1103/yb9y-3t91](https://doi.org/10.1103/yb9y-3t91)

Subject Areas: Computational Physics,
Condensed Matter Physics,
Strongly Correlated Materials

I. INTRODUCTION

A. Motivation

The emergence of nontrivial quantum field theories in condensed matter systems is a remarkable phenomenon. Among the most interesting are strongly interacting gapless field theories with enhanced (generalized) symmetries.

Three-dimensional (two space plus one time) quantum electrodynamics, or QED₃, is a striking example of such a field theory. Composed of N two-component Dirac spinors interacting with a U(1) gauge field, QED₃ is known to realize a conformal field theory (CFT) for sufficiently large N (it is commonly believed that $N \geq 4$ is sufficient while we caution that distinguishing genuine conformal behavior and pseudocriticality in numerical simulations is challenging) [1–4]. This CFT contains a rich collection of scaling operators including fermion bilinears [SU(N) flavor currents and mass terms] and monopole operators, which generate U(1) magnetic fluxes, and which carry nontrivial representations of flavor. As a relatively simple strongly interacting gauge and conformal field theory, QED₃ is a test case for nonperturbative methods in the high-energy

*Contact author: zymeng@hku.hk

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

community. In condensed matter, QED₃ has been argued to arise as a ground state of certain quantum antiferromagnets, and in this context is known as a Dirac spin liquid (DSL) [5–7]. In the DSL, the U(1) gauge field and fermions are *emergent*: The microscopic formulation begins at a lattice level with only a spin-1/2 direct product Hilbert space. The appearance of QED₃ in such a situation is striking and fascinating, worthy of detailed study and verification.

In our opinion, the existence of QED₃ as a stable phase of matter (i.e., robust to symmetry-allowed perturbations) is well established in some circumstances [this presupposes that a UV symmetry-allowed SO(6) rank-2 symmetric tensor operator is irrelevant; recent conformal-bootstrap studies have analyzed $N = 4$ QED₃ under spectral assumptions concerning this operator [4,8], but do not by themselves establish its irrelevance]. From the field-theory perspective, basic properties of QED₃ are understood, e.g., the set of primary fields, rough determinations of their scaling dimensions [3,4,8,9], and how microscopic (UV) symmetries are implemented [10,11], allowing for a symmetry-based analysis of perturbations [12–14]. However, more detailed properties such as the computation of multi-point correlation functions, and renormalization group flows under various perturbations, are as yet unknown.

From the condensed matter side, the DSLs have been proposed and investigated in spin-1/2 Heisenberg models on kagome [7,15–18] and triangular [11,19–26] lattices by a variety of analytical and numerical techniques. Yet, the understanding of their physical response functions remains limited [27–30].

In this paper, we extend the understanding of QED₃ and the DSL by exploring the effect of a particularly important physical perturbation: an external magnetic field in the condensed matter realization, which couples to the spins via Zeeman interaction. From the QED₃ perspective, this field appears as a flavor chemical potential. In experiments on quantum magnets, the study of the phase diagram enriched by the field axis is a routine and powerful way to probe the physics. In QED₃, we will see that the flavor chemical potential enjoys an intriguing interplay with the emergent flux, and may, for the DSL, make the emergent structure of the system more apparent. In order to decouple the complicated physics of magnetic monopoles from that of the emergent orbital flux and relativistic Landau levels of fractionalized spinons, which is the focus of our study, we propose to investigate a “mixed” lattice gauge theory that suppresses magnetic monopole events when one takes the continuum limit along the temporal direction. We stress that within this mixed theory a magnetic flux through spatial plaquette remains well defined. All our simulations reported below adhere to this central idea. Our simulations therefore isolate the field-induced flux physics in a controlled setting, establishing the chiral flux (CF) phase and its characteristic signatures in the dynamical spin response within the mixed theory while disentangling them from the

additional effects of dynamical monopoles. How these monopole effects reshape the resulting phase in a fully compact gauge theory, as is relevant to microscopic spin models, remains an important question for future work.

Our investigation is inspired by work of Ran *et al.* [31] who considered the response to a Zeeman field in a DSL on a kagome lattice, using a parton mean-field approach and a variational wave function. It is also related to recent work discussing spontaneous symmetry breaking in QED₃ in zero field for $N = 2$, for which there is believed not to be a stable CFT [32]. Yet at the same time, recent lattice model quantum Monte Carlo simulations of QED₃ have shown signatures of DSL and its phase transitions to confined phases at the spatial scale of 20×20 [33–35].

B. Model

Here we carry out a numerically exact determinant quantum Monte Carlo (DQMC) study of a lattice model for QED₃ derived from the one in Refs. [33,34] on a cubic space-time lattice (square spatial lattice). The model is defined by the action

$$S = \sum_{i,n} \left[\psi_i^\dagger(\tau_n) [\psi_i(\tau_n) - \psi_i(\tau_{n-1})] - \frac{1}{2} B \psi_i^\dagger(\tau_n) \sigma^z \psi_i(\tau_n) \right] - t \sum_{\langle ij \rangle, n} \left[e^{ia_{ij}(\tau_n)} \psi_i^\dagger(\tau_n) \psi_j(\tau_n) + \text{H.c.} \right] + \frac{1}{J} \sum_{\langle ij \rangle, n} \left[a_{ij}(\tau_n) - a_{ij}(\tau_{n-1}) \right]^2, \quad (1)$$

where $\psi_i = \begin{pmatrix} \psi_{i,\uparrow} \\ \psi_{i,\downarrow} \end{pmatrix}$ is a fermionic spinor, a_{ij} is a bosonic scalar and it acts as a U(1) gauge field, $i, j \in [1, L^2]$ labels sites of the square lattice, and $\langle ij \rangle$ is the nearest neighbor bond, with the simulated spatial system size $L = 6, 8, 10, \dots, 20$. The temporal system size is $N_\tau = [1/(T\Delta\tau)] = 20L$, where $\tau_n \in [0, \beta = (1/T) = 2L]$ labels the discrete imaginary time with step $\Delta\tau = 0.1$ in our setting and $n \in [0, N_\tau]$ is its integer index, as shown Fig. 1(a). The partition function $Z = \text{Tr}\{e^{-S}\}$ is obtained by tracing out all fermion and boson degrees of freedom; see Eq. (2).

The model in Eq. (1) is a somewhat unconventional variant of the gauge theory. In particular, while the fermion terms are fully periodic under a 2π shift of $a_{ij}(\tau_n)$ on any single bond at any single imaginary time τ_n , the “Maxwell” term in the last line of Eq. (1) lacks this periodicity. Thus the action lacks such periodicity and instead has a *subsystem* periodicity: It is invariant under the shift $a_{ij}(\tau_n) \rightarrow a_{ij}(\tau_n) + 2\pi m_{ij}$ with any set of integers m_{ij} which are time independent. This choice was made in order to be as close as possible to the noncompact theory (which lacks any periodicity) while still allowing the model to dynamically choose a nontrivial flux through spatial slices (horizontal surfaces in space-time defined by $\tau_n = \text{const}$). We refer to

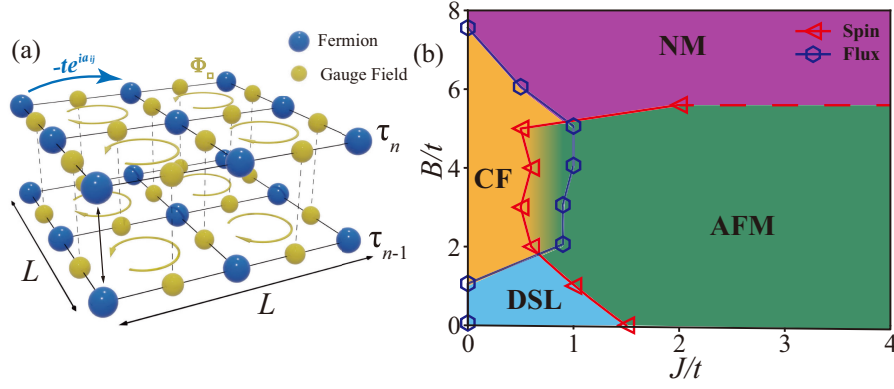


FIG. 1. Lattice model and DQMC phase diagram. (a) Fermions are on the sites of cubic lattice represented by filled blue balls while the gauge fields are on the bonds of the lattice represented by filled yellow balls. The gauge field on the temporal direction is fixed to 0 (denoted by the dashed lines). Fermions hop between nearest neighbor sites with a phase $e^{\pm ia_{ij}}$. Each plaquette is attached to a flux $\Phi_{\square}$, computed from gauge field on the four bonds. Black arrows indicate spatial and temporal directions, with lattice constant 1 for spatial and 0.1 for temporal directions. τ_n and τ_{n-1} are adjacent temporal layers. (b) Phase diagram obtained from DQMC simulation. The red line is determined from spin correlation ratio r_{AFM} while the blue line is from flux Binder cumulant U_{flux} . Right of red line is the AFM phase which breaks the $U(1)_f$ symmetry of spin rotation when $B \neq 0$. Left of blue line is chiral flux state (CF), in which the $\mathcal{T}$ symmetry of flux of the gauge field is broken. There is a nonmagnetic (NM) phase at the top, Dirac spin liquid (DSL) phase at the bottom corner, and a possible coexisting CF-AFM phase, represented by a color gradient between CF and AFM phases. Note that the finite extent of the DSL phase for $B > 0$ is due to the nonzero temperature accessible in our simulations. We expect the DSL to give way to the CF phase for infinitesimal Zeeman fields $B > 0$ in the zero-temperature thermodynamic limit.

our model as the “mixed” compact-noncompact theory, or simply the “mixed” theory.

It is well known that the compact (fully periodic) theory includes “magnetic monopoles”: pointlike defects in space-time (instantons) associated with nonzero quantized values of magnetic three-flux integrated over a closed space-time surface. These defects are dynamical degrees of freedom and to be summed over in a partition function for a fully compact $U(1)$ gauge theory. The critical phases of QED₃ can suffer instabilities driven by monopole proliferation. When exactly such instability occurs depends upon the phase in question and subtle lattice physics due to Berry phase effects [10]. Our simulation has been tailored to suppress monopoles in order to evade such instabilities.

The behavior of monopoles in the mixed theory is somewhat subtle. As we show explicitly in an analytically tractable model in Supplemental Material (SM) [36], the theory Eq. (1) still admits monopole excitations owing to the periodic (in a_{ij}) fermion-gauge coupling, albeit with a highly constrained mobility (i.e., quantized magnetic flux can only penetrate surfaces perpendicular to the temporal direction). As a result, taking the continuum limit along the temporal direction suppresses these monopole excitations, so that the mixed theory behaves as a noncompact gauge theory, *except* that the net flux through a full spatial slice is free and energetically chosen by the model. Our numerical simulations are carried out well into this limit, which corresponds to a highly anisotropic space-time lattice (with $N_\tau \gg N$). Therefore, in our simulation we expect stable phases with both free and strongly interacting photons. We remark that, even when monopoles are suppressed, one may

consider monopole operators as nondynamical “insertions” defined by modifying the partition function appropriately.

Our analysis of the DQMC results rests on an understanding of the symmetries of the problem. In this paper, we follow modern conventions and only refer to symmetries of those operations which act nontrivially on physical states, i.e., on gauge-invariant observables. In particular, gauge invariance does not qualify as a symmetry, but rather reflects a constraint on the Hilbert space. We list important *physical* symmetries of the lattice model Eq. (1) which are relevant to our analysis.

- (1) Considering the model with n flavors of fermions (with flavor index $\alpha = 1 \dots n$), in zero applied Zeeman field $B = 0$, there is an $SU(n)$ flavor symmetry, which we denote $SU(n)_f$. Here we focus on $n = 2$. This symmetry is lowered upon application of a finite Zeeman field $B \neq 0$ [explicitly breaking $SU(2)_f$ symmetry] to $U(1)_f$, associated with rotations around the field axis in flavor space, which acts on Dirac fermions as $U(1)_f: \psi_i(\tau_n) \rightarrow e^{-i\alpha\sigma^z} \psi_i(\tau_n)$ with $\alpha \in [0, 2\pi)$ and leaves the $U(1)$ gauge field invariant.
- (2) In the time-continuum limit, as discussed above, there is a conserved magnetic flux through the spatial slices of the lattice. This may be written $\Phi(\tau_n) = \sum_{(r_x, r_y)} \text{mod} [(\nabla \times a)_{(r_x, r_y, \tau_n)}, 2\pi]$ and the conservation means that $\Phi(\tau_n)$ is independent of τ_n for finite-action configurations in the Monte Carlo procedure in the continuum time limit. This conserved flux defines a $U(1)_m$ magnetic symmetry on

quantum states; i.e., the conserved flux is the symmetry generator for the $U(1)_m$ symmetry. The definition of the symmetry generator is the way a symmetry is defined in quantum mechanics. In a phase in which fermions are gapped, this symmetry can be made manifest through an explicit duality transformation of the mixed theory, which is presented in Supplemental Material [36]. This reveals that the spontaneous breaking of this symmetry manifests in a linearly dispersing “dual” Goldstone mode, i.e., a photon.

- (3) There is a discrete time-reversal invariance $\mathcal{T}$ [$\psi_i(\tau) \rightarrow \psi_i^\dagger(-\tau)$, $\psi_i^\dagger(\tau) \rightarrow -\psi_i(-\tau)$, $a_{ij}(\tau) \rightarrow -a_{ij}(-\tau)$] which is a symmetry of Eq. (1) also at finite Zeeman fields $B \neq 0$. Hence, $\mathcal{T}$ is distinct from the usual time-reversal symmetry for spin-1/2 particles, the latter being broken *explicitly* by finite Zeeman fields $B \neq 0$.
- (4) The model further enjoys spatial translation symmetries (along with point group symmetries, which are not of immediate relevance for the remainder of our work). Translations along $\hat{x}$ act as $T_x: \psi_i(\tau) \rightarrow \psi_{T_x(i)}(\tau)$ and $a_{ij}(\tau_n) \rightarrow a_{T_x(i), T_x(j)}(\tau_n)$ with $T_x(i) \equiv i + \hat{x}$, and analogously for T_y .

We summarize the key aspects of the mixed gauge theory, and in particular the consequence of the non-periodicity in a_{ij} of the time-derivative Maxwell term. For nonzero Trotterization, monopoles are allowed, but they are very strongly suppressed with diminishing time discretization, and become fully absent in the time-continuum limit. In this limit, a perfectly conserved magnetic flux Φ defines a $U(1)_m$ symmetry.

We emphasize that the lattice model of Eq. (1) is a form of QED₃ in that it describes fermions interacting with a bosonic $U(1)$ gauge field, but it is not a continuum model. Standard continuum QED₃ contains N species of two-component Dirac fermions coupled to such a gauge field. In a certain regime such a theory arises as a continuum limit of Eq. (1) (see below). Because of fermion doubling, the low-energy continuum theory has $N = 2n$ Dirac fermions and the $SU(n)_f$ lattice symmetry is enlarged to an emergent $SU(N)_f = SU(2n) = SU(4)_f$ symmetry. However, we should keep in mind that residual effects originating from lattice corrections to the continuum limit preserve only the $SU(2)_f$ subgroup. Accordingly, we focus on the robust exact symmetries of the lattice model to characterize the system.

C. Summary of results

Aided by these symmetries, our DQMC results, obtained on finite systems of dimensions $\beta \times L \times L$, as explained above, are summarized in the low-temperature phase diagram of Fig. 1(b), spanned by the axes of gauge fluctuation J/t and magnetic field B/t . To characterize

the symmetries and their breaking, we introduce order parameters. There are two manifest order parameters: $N^+ = (-1)^i \langle \psi_i^\dagger \sigma^+ \psi_i \rangle$, which describes spontaneous breaking of the $U(1)_f$ flavor symmetry to form antiferromagnetic (AFM) spin order in the XY plane, and $\chi = \langle \sin \Phi_\square \rangle$, a chirality, which describes broken $\mathcal{T}$ symmetry. Here $\Phi_\square = \nabla \times a$ is the flux through a spatial plaquette, given by the lattice curl of the gauge field. A third order parameter, which is hidden, characterizes the $U(1)_m$ magnetic symmetry: This is a monopole operator $\mathcal{M}^+$, which creates a 2π flux in the gauge field. We do not currently measure $\mathcal{M}^+$ in our simulations, but its presence can be deduced from an understanding of the corresponding Goldstone mode: The Goldstone mode of the $U(1)_m$ symmetry is the *photon* of the gauge field.

We observe three manifestly symmetry-broken phases as visible in the phase diagram Fig. 1: an antiferromagnetic phase, in which N^+ is nonzero but $\chi = 0$, a chiral flux phase in which $\chi \neq 0$ but $N^+ = 0$, and an overlapping chiral AFM phase in which both N^+ and χ are nonzero. We also find two symmetric phases: a Dirac spin liquid phase and a polarized phase at strong field in which the spins are nearly fully aligned with the field (full polarization is not possible at finite temperatures where our simulations are done). In all the phases except the DSL, the $U(1)_m$ symmetry is spontaneously broken, i.e., $\mathcal{M}^+ \neq 0$, witnessed by the presence of a linearly dispersing free-photon mode. We provide an overview of various phases of the model and the symmetries that they preserve and/or spontaneously break in Table I.

In addition to the phase diagram, we explore the magnetic spectra of the gauge-invariant fermion bilinears—the spin operators in the condensed matter language—of QED₃. Universal features of these spectra provide signatures of fractionalization and emergent gauge fields in quantum magnets proximate to a DSL phase. We are particularly interested in uncovering structures that are unexpected without spin liquid physics, and which might be striking aspects to seek experimentally. In that regard, we focus particularly on the chiral phase in which the

TABLE I. Phases and their (spontaneously broken) symmetries. Here, a check mark indicates that a symmetry is preserved in a phase while an “x” character indicates the spontaneous breaking of a symmetry. We use “ $U(1)_m + U(1)_f$ ” to denote a diagonal subgroup of $U(1)_m \times U(1)_f$; see also the discussion in Sec. II C 2.

	$U(1)_f$	$U(1)_m$	$U(1)_m + U(1)_f$	$\mathcal{T}$	T_x, T_y
DSL	✓	✓	✓	✓	✓
CF	✗	✗	✓	✗	✓
AFM	✗	✗	✗	✓	✗
AFM + CF	✗	✗	✗	✗	✗
NM	✓	✗	✗	✓	✓

emergence of Landau-Hofstadter bands that result from the fractional flux in the gauge field may be reflected in the spin correlations. We present DQMC results for the longitudinal and transverse dynamical spin susceptibilities, and demonstrate a strong correlation between these correlations and a mean-field result which precisely describes fermions in Hofstadter bands. We observe a clear splitting of spectral weight into multiple features that can be understood from the effective enlargement of the magnetic unit cell of fermions experiencing a fractional flux per plaquette, despite the lack of translational symmetry breaking in the system. In some cases features reminiscent of the Landau-level nature of spinon bands are visible in the numerically obtained magnetic spectra. Our DQMC results for the longitudinal spin structure factor further show signatures of a gapless mode as the possible manifestation of the gapless photon, originating from gauge field fluctuations and therefore a manifestly beyond-mean-field result.

D. Guide to the paper

The paper is organized as follows. Section II lays out the lattice implementation of the mixed QED₃ model and its DQMC numerical algorithm (Sec. II A), and we further explain the Monte Carlo update scheme of the gauge field to overcome its slow dynamics and anisotropic space-time gauge choice (Sec. II B). Section II C, on the other hand, provides the theoretical discussion of the different phases (DSL, CF, AFM) in the phase diagram at the continuum limit with random phase approximation (RPA). Section III presents the phase diagram. Section III A provides the lattice mean-field analysis at $J = 0$, while Sec. III B determines the phase boundaries using DQMC simulations at finite J and B . Section IV focuses on the data and analysis of the magnetic spectra in the CF and AFM states, starting from the field-theoretical calculation (Sec. IV A), the lattice mean-field calculation (Sec. IV B), and more importantly, to the DQMC simulation results and the discussion of similarity and difference between the numerically unbiased solution and the mean-field analysis (Sec. IV C). Section V provides the discussions of the results and makes connection with potential DSL and CF phases in frustrated magnets and their experimental detections.

Sections I and II of the Supplemental Material (SM) [36] describe the magnetic spectra calculated using lattice mean-field and low-energy field-theory approaches, respectively. Section III reviews Larmor's theorem and its application to the magnetic spectra at the Γ point. Section IV presents flux-correlation results for the pure gauge theory and for the DSL and AFM phases. Section V explains the z-direction flux insertion and global update scheme used in the DQMC simulations. Section VI discusses the U(1)_m symmetry and the duality map. Section VII compares the mean-field and DQMC magnetic spectra at $B/t = 2$. Section VIII presents a finite-size analysis of the order parameters.

II. METHODS AND THEORETICAL EXPECTATIONS

A. QED₃ with lattice DQMC

As mentioned in Sec. I, we consider the QED₃ theory on the cubic lattice, with the action in Eq. (1). The mixed nature of the gauge coupling penalizes monopoles more than in a compact gauge theory, and indeed they are parametrically suppressed in the temporal continuum limit. In that regime we expect our mixed gauge theory to act like a noncompact gauge theory. We discard the K term used in prior work (Ref. [33]), because it adds an undesirable additional pinning field favoring π flux in the gauge field, masking the emergence of non- π -flux states with nonzero Zeeman field.

In the DQMC study of the action above, the partition function takes the form

$$Z = \int D(a, \bar{\psi}, \psi) e^{-(S_a + S_f)} = \int Da e^{-S_a} \text{Tr}_{\psi} [e^{-S_f}], \quad (2)$$

where S_a is the pure gauge field action [the last line in Eq. (1)] and S_f contains all the remaining contributions which are quadratic in fermions. The gauge field a_{ij} is an unconstrained continuous variable living on the nearest spatial bonds of cubic lattice as shown in Fig. 1(a). The quadratic fermion ψ_i can be traced out to give

$$\begin{aligned} \text{Tr}_{\psi} [e^{-S_f}] &= \prod_{\alpha} \left[\det \left(\mathbf{I} + \prod_{n=1}^{N_{\tau}} \mathbf{B}_{\tau_n, \alpha} \right) \right] \\ &= \prod_{\alpha} \det \mathbf{M}_{\alpha}, \end{aligned} \quad (3)$$

where $\mathbf{B}_{\tau, \alpha} = e^{-\mathbf{V}_{\tau, \alpha}}$ is the exponential of fermion-gauge coupling matrix $\mathbf{V}_{\tau, \alpha}$, whose elements are determined by gauge field $-te^{ia_{ij}(\tau)}$ and Zeeman term. $\alpha = \uparrow, \downarrow$ in our case, and the matrix

$$\mathbf{M}_{\alpha} = \begin{pmatrix} \mathbf{1} & 0 & 0 & \cdots & 0 & \mathbf{B}_{\tau_{N_{\tau}}, \alpha} \\ -\mathbf{B}_{\tau_1, \alpha} & \mathbf{1} & 0 & \cdots & 0 & 0 \\ 0 & -\mathbf{B}_{\tau_2, \alpha} & \mathbf{1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{1} & 0 \\ 0 & 0 & 0 & \cdots & -\mathbf{B}_{\tau_{N_{\tau}-1}, \alpha} & \mathbf{1} \end{pmatrix}$$

is of the size of space-time volume $NN_{\tau} \times NN_{\tau}$. We simulate the system size of $N = L \times L = 6 \times 6, 8 \times 8, 10 \times 10, \dots, 16 \times 16, 20 \times 20$, with $N_{\tau} = 20L$ ($\beta = (1/T) = 2L$ and $\Delta\tau = 0.1$) to ensure the low temperature. For the case without Zeeman field, it can be shown that $\det \mathbf{M}_{\alpha} \in \mathbb{R}$ and $\det \mathbf{M}_{\uparrow} = \det \mathbf{M}_{\downarrow}$, thus sign-problem-free for Monte Carlo simulation [33,38]. With

nonzero Zeeman field, $\det \mathbf{M}_\alpha$ is complex. However, from particle-hole symmetry between the two spin flavors, $\det \mathbf{M}_\uparrow = \det \mathbf{M}_\downarrow^\dagger$ is guaranteed so that the full weight remains sign-problem-free as well. Thus the partition function Eq. (2) is amenable to DQMC method [39–41].

B. Update scheme in DQMC simulations

In a nonzero Zeeman field $B > 0$, the preferred configurations of the gauge field have a flux $\Phi_\square$ different from π per plaquette at the mean-field level [7]. Beyond mean field, the flux on each plaquette remains a gauge-invariant observable and the energy of the system depends on the flux configuration. We expect in the DQMC simulation the probability distribution of the flux will thus generally deviate from one centered at π . Therefore, a Monte Carlo update method that changes the flux directly would be highly desirable from the perspective of efficiency. To this end, we combine a local update of the gauge field on each bond $a_{ij}(\tau_n) \rightarrow a_{ij}(\tau_n) + \delta a$ with a global update that can change the flux directly in the entire lattice.

To perform the space-time global update, based on the z -direction flux insertion technique [42,43], one can update the gauge field $a_{ij}(\tau_n)$ for all spatial bonds and imaginary time layers simultaneously with the following scheme,

$$a_{ij}(\tau_n) \rightarrow a_{ij}(\tau_n) + a_{o,ij}, \quad (4)$$

with $a_{o,ij}$ the introduced extra orbital field. In the Landau gauge, $a_{o,ij}$ for bonds in the x direction is $-(2\pi y N_\Phi)/L^2$. While for bonds in the y direction, depending on the location, $a_{o,ij} = 0$ for bonds away from boundary and $a_{o,ij} = [(2\pi x N_\Phi)/L]$ for those cross boundary in y direction. x, y in the expression is the position of the bond in unit of lattice constant and N_Φ is an integer. For detailed derivation of the Landau gauge on the lattice with z -direction flux insertion, please refer to Sec. V of the SM [36].

At each global update, N_Φ is randomly chosen and we apply the same $a_{o,ij}$ (according to the random N_Φ and the Landau gauge) for all imaginary time layers. If the update is accepted, a uniform flux $2\pi N_\Phi/L^2$ is inserted into the model. In DQMC simulations, such an update requires the computation of determinant in Eq. (3) from scratch and it is expensive. We thus choose to perform a combination of a sweep of Metropolis local updates plus 4 times of such global updates, which is defined as one complete sweep in DQMC simulation. In this way, the number of global updates does not scale with system size. As mentioned above, global update helps to quickly evolve to the desired flux sector and local update will explore the whole phase space ergodically.

C. Continuum limit and random phase approximation

To provide a framework to understand the numerical results, we discuss the continuum limit of Eq. (1) and

analyze it in the random phase approximation, allowing for the possibility of spontaneous AFM order. To obtain the continuum limit, we assume first small Zeeman field $B \ll 1$, so that the flux $\Phi_\square \approx \pi$. We then take $a_{i,i+\mu}(\tau_n) = \bar{a}_{i,i+\mu} + A_\mu(x_i, y_i, \tau_n)$, where $\mu = x, y$ and $\bar{a}_{i,i+\mu}$ is a c -number background gauge field representing the π flux, $\nabla \times \bar{a} = \pi$, and $A_\mu(x, y, \tau)$ is a slowly varying continuum field describing small deviations from π flux. When $J \lesssim 1$, we may assume the fluctuations of A_μ are weak, and that the fermionic action may be approximated by the low-energy form near zero frequency and the zero energy Dirac points of the π -flux problem.

For concreteness, we take the gauge $\bar{a}_{i,i+x} = 0$ and $\bar{a}_{i,i+y} = \pi x_i$, which makes the hopping on the y -oriented bonds with odd x_i to be negative relative to other hoppings. This choice doubles the unit cell along the x direction. Then the lattice fermion $\psi_{i,\sigma}$ (with σ a spin index) can be decomposed into slow-varying continuum fields $\Psi_{s,v,\sigma}(x, y, \tau)$ with $s = \text{mod}(x_i, 2) = 0, 1$ a sublattice index, $v = 0, 1$ a valley index, as follows:

$$\begin{aligned} \psi_{i,\sigma}(\tau_n) &\sim \sum_{v=0,1} [i(-1)^v]^{x_i+y_i} \Psi_{\text{mod}(x_i,2),v,\sigma}(x_i, y_i, \tau_n), \\ \psi_{i,\sigma}^\dagger(\tau_n) &\sim \sum_{v=0,1} [-i(-1)^v]^{x_i+y_i} \Psi_{\text{mod}(x_i,2),v,\sigma}^\dagger(x_i, y_i, \tau_n). \end{aligned} \quad (5)$$

We insert this into the action and gradient expand assuming slow variations of A_μ and Ψ , so that the spatial part of the fermion action [the hopping term in Eq. (1)] turns into the sum of two valley contributions $\propto -it(-1)^v \times \int d\tau d^2x \Psi_{s,v,\sigma}^\dagger (\tau_{s,s'}^x \partial_x + \tau_{s,s'}^z \partial_y) \Psi_{s',v,\sigma}$. This is followed by the transformations $\Psi_{s,1,\sigma} \rightarrow \tau_{ss'}^y \Psi_{s',1,\sigma}$, $\Psi_{s,0,\sigma}^\dagger \rightarrow \bar{\Psi}_{s',0,\sigma} \tau_{s's}^y$, which are not unitary but allowed since $\bar{\Psi}$ and Ψ are independent in the path integral, and the “renaming” of the Grassman integration field in the valley $v = 1$ as $\Psi_{s,1,\sigma}^\dagger \rightarrow \bar{\Psi}_{s,1,\sigma}$. The result is the continuum action $S = \int d\tau d^2x \mathcal{L}$, with

$$\mathcal{L} = \sum_{\mu=0}^2 \bar{\Psi} \gamma^\mu v_\mu (\partial_\mu + iA_\mu) \Psi - \frac{b}{2} \bar{\Psi} \gamma^0 \sigma^z \Psi + \frac{1}{g} \sum_{\mu=x,y} (\partial_\tau A_\mu)^2, \quad (6)$$

where $v_0 = 1$, $v_1 = v_2 \equiv v = 2t/\Delta\tau$, $b = B/\Delta\tau$, and $g = J/\Delta\tau$. We can let $\gamma^0 = \tau^y$, $\gamma^1 = \tau^z$, $\gamma^2 = -\tau^x$, where the Pauli matrices τ are defined to act in the sublattice space, σ acts in the spin space, and we suppressed all the sublattice, valley, and spin indices.

Equation (6) describes a continuum Dirac theory with $2 \times 2 = 4$ flavors arising from the spin and valley, coupled to the continuum U(1) gauge field A_μ , which is precisely QED₃ with the gauge choice $A_\tau = A_0 = 0$.

Using the same transformations, we can also obtain the Néel order parameter,

$$N^+ = \psi_i^\dagger \sigma^+ \psi_i (-1)^{x_i+y_i} \sim \bar{\Psi} \sigma^+ \mu^x \Psi, \quad (7)$$

where $\sigma^+ = (\sigma^x + i\sigma^y)/2$ and we introduced the μ Pauli matrices acting in the valley space.

Such continuum theory is capable of describing the quantum phases close to the zero field DSL, which includes the DSL itself, the CF phase, and the AFM. We discuss each in turn, with an eye to the predictions for various correlation functions and spectral properties to be tested in the DQMC simulations in Secs. III and IV.

1. DSL

The DSL occurs for $b = 0$, in which case this is precisely noncompact QED₃ without any applied potential. This has been analyzed extensively and is believed to describe a scale-invariant conformal field theory. Consequently, power-law behavior is expected for all gauge-invariant observables. Because of fermion doubling in DSL phase, the CFT is expected to have emergent SU(4) [$\sim$ SO(6) up to a sign] symmetry, which enlarges the microscopic SU(2) spin symmetry and some discrete operations in the space group. In the CFT of SU(4) QED₃, the set of primary fields with low scaling dimensions are as follows.

- (i) The set of SU(4) conserved currents $J_\mu^a = \bar{\Psi} \gamma^\mu T^a \Psi$, where μ is a space-time index and a ranges over the 15 generators T^a of SU(4). Like all conserved currents in a $(2+1)$ -dimensional CFT, the currents have the exact scaling dimension $\Delta_J = 2$.
- (ii) Fermion bilinears or mass terms. A singlet $M_s = \bar{\Psi} \Psi$ and a set of adjoints $M_a = \bar{\Psi} T_a \Psi$, where $a = 1 \dots 15$ range over the SU(4) generators. *A priori*, these two sets have independent scaling dimensions, Δ_s and Δ_{adj} , respectively. An estimate from Ref. [3] is $\Delta_s \approx 2.3$, $\Delta_{\text{adj}} \in (1.4, 1.7)$.
- (iii) A set of 6 monopole operators $\mathcal{M}_q$ (and their conjugates $\mathcal{M}_q^\dagger$), which form an antisymmetric tensor representation of SU(4), or a real vector representation of SO(6); hence, $q = 1, \dots, 6$. The estimated scaling dimension from the large N_f expansion [9] is $\Delta_M \approx 1.0$.

These scaling operators and their composites (products) and descendants (derivatives) appear in the long-distance correlations of lattice quantities. For example, we expect that the DSL should exhibit flux-flux correlations dictated by $\Delta_J = 2$, which implies that, for example, the long-time correlations of the local flux should behave as $\langle \Phi_\square(\tau) \Phi_\square(0) \rangle \sim \tau^{-4}$. Note that this is different from the result in nonconformal phases described below in which the flux correlations arise from the Goldstone mode of spontaneously broken $U(1)_m$ symmetry, and correspond to that of a free-photon theory with $\langle \Phi_\square(\tau) \Phi_\square(0) \rangle \sim \tau^{-3}$ (our

numerical results for these temporal flux correlation functions are discussed in Sec. IV of the SM [36]).

The spin correlations, i.e., the dynamic structure factor, in the DSL probe CFT operators, which can appear in the continuum limit of individual spin operators. In this *non-compact* theory, the spin operators (like all operators we consider) conserve flux, and hence monopoles cannot appear in their expansion. Rather, the fermion bilinears $M_{a>0}$ are expected to dominate. One of these operators (see below in the discussion of the AFM phase) corresponds to the Néel field, so that the staggered spin correlations should show power-law behavior with a decay exponent $2\Delta_{\text{adj}} \in (2.8, 3.5)$. Our DQMC results indeed reveal the consistent power-law behavior in spin correlation with $2\Delta = 3.2(1)$, as shown in Fig. 10(b) below. In fact, previous QMC simulation for finite sizes of 20×20 have found similar power-law decays of the spin and dimer operators, see Fig. 4 in Ref. [33].

It is possible in principle to measure monopole correlations by explicitly including a monopole-antimonopole insertion in the partition function, but we have not done this in the current simulation as it requires a significant technical development. In the DSL phase, this correlation is expected to decay with a power law of $2\Delta_M \approx 2$.

2. CF phase

In the CF phase, a spontaneous flux $\langle \nabla \times A \rangle \neq 0$ develops. Any nonzero flux causes the Dirac cones to split into Landau levels with energies $\epsilon_n = \omega_c \sqrt{|n|} \text{sgn}(n)$, with $n \in \mathbb{Z}$, and $\omega_c = \sqrt{2|\phi|}v$ the cyclotron energy when the average flux is ϕ . The latter is determined by the condition that the 0th Landau levels for both valleys are full of up spin fermions and empty of down. This fixes the flux to density relation, $\langle \nabla \times A \rangle = \phi = \pm \pi \langle n_\uparrow - n_\downarrow \rangle$. Note that a finite flux breaks time-reversal symmetry $\mathcal{T}$, with a spontaneously chosen sign. The Zeeman field accordingly acts as an opposite Fermi level for the two spin polarizations, lying between the 0th and the $n = \pm 1$ Landau levels for the up and down spins, respectively. Hence, $\epsilon_{F\alpha} = \alpha b/2 = 0.62\alpha\omega_c$ for $\alpha = \pm 1$ corresponding to up and down spins, respectively. This relation between the Zeeman field b and the spontaneous orbital flux ϕ , and hence ω_c , follows from the mean-field analysis in Ref. [31].

We proceed to analyze the effect of this mean-field flux by analyzing the fluctuations of the gauge field and the system's response. Accordingly, we write $A_\mu = \bar{A}_\mu + a_\mu^c$, where $\nabla \times \bar{A} = \phi$, and a_μ^c describes the fluctuations of the internal “charge” gauge field. To probe the response, we furthermore add a *probe* spin gauge field a_μ^s which couples to the conserved U(1) *spin* three-current of the fermions. This field is fictitious but will be used as an infinitesimal source to generate correlation functions and to characterize the spin response. Note that an infinitesimal change of Zeeman field $b \rightarrow b + \delta b$ is equivalent to including a small

time component of the spin gauge field, $a_0^s = i\delta b$. Observe that a_0^s couples to the density of up or down spin fermions, $\alpha = \pm 1$, with opposite signs.

The full Lagrangian including these fluctuations can therefore be written,

$$\mathcal{L} = \frac{1}{g}(\partial_\tau a_j^c)^2 + \sum_{v=0}^1 \sum_{\alpha=\pm 1} \mathcal{L}_v^{\text{DLL}} \left(a_\mu^c + \frac{\alpha}{2} a_\mu^s, \epsilon_F = \alpha \frac{b}{2} \right), \quad (8)$$

where $\mathcal{L}_v^{\text{DLL}}(a_\mu, \epsilon_F)$ is the Lagrangian for valley v of Dirac fermion Landau levels coupled to a total gauge field a_μ and with Fermi level ϵ_F .

Now we proceed to carry out an RPA treatment, integrating out the fermions to quadratic order in the gauge fields. Since each spin and valley of fermion is decoupled, their contributions can be added. Each contributes a standard effective action for a system of fully filled and empty Landau levels, consisting of a *leading* Chern-Simons term whose coefficient is $1/(4\pi)$ times the Hall conductivity of those fermions, and a *subleading* Maxwell term, representing the polarizability of the fermions [44]. The Hall conductivity of each valley of up or down fermion is $\pm 1/2$, capturing the change of Hall conductivity by one unit for occupying or emptying a Landau level, and accounting for particle-hole symmetry.

Consequently, we obtain

$$\mathcal{L}_{\text{eff}} = \sum_{\alpha=\pm 1} i \frac{\alpha}{4\pi} \epsilon^{\mu\nu\lambda} \left(a_\mu^c + \frac{\alpha}{2} a_\mu^s \right) \partial_\nu \left(a_\lambda^c + \frac{\alpha}{2} a_\lambda^s \right) + \frac{1}{g} (\partial_\tau a_j^c)^2 + \frac{1}{2\tilde{g}} \sum_{\alpha=\pm 1} \left(f_{\mu\nu}^c + \frac{\alpha}{2} f_{\mu\nu}^s \right)^2, \quad (9)$$

where $f_{\mu\nu}^\sigma = \partial_\mu a_\nu^\sigma - \partial_\nu a_\mu^\sigma$, and the *quantized* Chern-Simons terms occupy the first line (note that these terms appear with a factor of i in our Euclidean field theory), and the cutoff dependent and unquantized Maxwell terms the second.

Expanding and regrouping the terms, we find

$$\mathcal{L}_{\text{eff}} = i \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} a_\mu^s \partial_\nu a_\lambda^c + \frac{1}{\tilde{g}} (f_{\mu\nu}^c)^2 + \frac{1}{4\tilde{g}} (f_{\mu\nu}^s)^2, \quad (10)$$

consisting of a *mixed* Chern-Simons term for spin and charge, and Maxwell contributions, describing renormalization of gauge field charges and velocities that depend on the regularization scheme [45].

Equation (10) is short but encodes several important conclusions, which we now discuss.

Gapless mode. First set the probe field $a^s = 0$. Then Eq. (10) becomes just a Maxwell term for the fluctuating gauge field. This describes a massless photon, i.e., a single branch of linearly dispersing mode $\omega = c|k|$ with c the

speed of “light.” This can be regarded as simply arising from the original gauge field in the lattice model, and is protected by gauge symmetry. We discuss the alternative view as a Goldstone mode of spontaneously broken $U(1)_m$ symmetry below.

From this effective action, one can calculate the flux-flux correlations, $\langle \Phi_\square(\tau) \Phi_\square(0) \rangle \sim \tau^{-3}$. This behavior, different from the DSL, is characteristic of the Goldstone mode phase. An explicit demonstration of this conclusion on the lattice is given in Sec. IV of the SM [36].

Low-energy spectral weight. The gapless photon also appears in the correlation of S^z operators. To see this, note that the time component of the probe field a_0^s couples to the conserved spin density $S^z = \frac{1}{2} \bar{\Psi} \gamma^0 \sigma^z \Psi$. Hence the derivative $\delta \mathcal{L}_{\text{eff}} / \delta(\delta b)$ gives the representation of the spin operator in the effective free-photon theory. This gives $S^z \sim \nabla \times a^c$; i.e., the low-energy spin correlations are identical to those of the emergent magnetic flux. Calculating the latter, one obtains

$$\langle S^z S^z \rangle_{\omega, \mathbf{q}} \sim \frac{1}{2} \chi c q \delta(\omega - c q), \quad (11)$$

for small ω, q , where $\chi = (1/N) \sum_i \partial \langle S_i^z \rangle / \partial B > 0$ is the susceptibility. Here we used the *compressible* nature of the CF phase and the fluctuation-dissipation theorem to determine the prefactor, since χ is given by the limit $\omega \rightarrow 0$ followed by $q \rightarrow 0$ of the longitudinal dynamical susceptibility. We present our DQMC results for the longitudinal susceptibility, and analyze them in this context, in Sec. IV C.

Monopole condensate. In the language of generalized symmetry, the gapless photon should be viewed as a Goldstone mode of a spontaneously broken $U(1)_m$ symmetry. However, there are *two* microscopic $U(1)$ symmetries in the model, and only a single Goldstone mode. This means there must remain an unbroken $U(1)$ subgroup of the original $U(1)_m \times U(1)_f$ symmetry. This statement is also seemingly evident from Eq. (8), which is diagonal in spin σ so still invariant under $U(1)_f$ rotations.

To precisely identify the broken symmetry, we would like to write down the corresponding order parameter. Because it should transform under the $U(1)_m$ symmetry, it involves an insertion of 2π flux, which we associate with a monopole operator $\mathcal{M}$. The meaning of $\mathcal{M}_p(\tau)$ is that it creates a flux 2π through plaquette p at time τ .

However, the order parameter is *not* just the “bare” monopole operator $\mathcal{M}$. This can be seen from the mixed Chern-Simons term in Eq. (10), which implies that the charge flux $\nabla \times a^c$ is tied to the spin density S^z (conjugate to a_0^s). Hence, to have a nonzero expectation value, the order parameter must in addition to creating the flux 2π also create the associated change in spin $\Delta S^z = 1$. Consequently, the order parameter for the broken $U(1)$ symmetry is

$$\mathcal{M}_{\text{ord}} = \mathcal{M}S^+, \quad (12)$$

which combines the insertion of flux with a spin flip. From the above arguments, we expect that

$$\langle \mathcal{M}_{\text{ord}} \rangle \neq 0. \quad (13)$$

This operator, as required, breaks $U(1)_m$ symmetry but *also* $U(1)_f$ symmetry. This means that the CF state has a *hidden* breaking of the $U(1)_f$ spin-rotation symmetry. Why is it hidden? It is because the order parameter (and hence the state itself) preserves the combination $\mathcal{M} \rightarrow \mathcal{M}e^{i\chi}$, $S^+ \rightarrow S^+e^{-i\chi}$. The latter corresponds to the residual $U(1)$ subgroup in the CF phase. This ensures that the expectation value of any pure spin operator that changes S^z such as $\langle S_i^\pm \rangle = 0$ vanishes, and there is no antiferromagnetic order. Only composite observables which involve a change of the flux can detect the breaking of the spin-rotation symmetry. Consistent with this observation, the gapless photon does not contribute as an intermediate state in the transverse spin structure factor, which therefore shows a full gap in the CF phase (see also the discussion of gapped transverse susceptibility obtained in our DQMC results as presented in Sec. IV C).

3. AFM

In the AFM phase, the average flux is zero, but spontaneous AFM order has developed. While the AFM order arises from Eq. (6) by the effect of gauge fluctuations, we can model it phenomenologically by introducing an AFM order parameter $N^+(x, y, \tau)$, which weakly fluctuates and couples to the Dirac fermions. Since the system is ordered, it is sufficient to assume $N^+ = |N|e^{i\theta(x, y, \tau)}$. The effective Lagrangian in Eq. (6) is then replaced by

$$\begin{aligned} \mathcal{L} = & \sum_{\mu=0}^2 \bar{\Psi} \gamma^\mu v_\mu (\partial_\mu + iA_\mu) \Psi - \frac{b}{2} \bar{\Psi} \gamma^0 \sigma^z \Psi \\ & - |N| (e^{i\theta} \bar{\Psi} \sigma^- \mu^x \Psi + \text{H.c.}) + \frac{1}{g} \sum_{\mu=x, y} (\partial_\tau A_\mu)^2. \end{aligned} \quad (14)$$

Now we make the unitary transformation $\Psi \rightarrow e^{i\theta\sigma^z/2}\Psi$, $\bar{\Psi} \rightarrow \bar{\Psi}e^{-i\theta\sigma^z/2}$, which is chosen to remove the dependence of the action on a constant phase θ . When this phase depends upon space and time, the result is

$$\begin{aligned} \mathcal{L} \rightarrow & \sum_{\mu=0}^2 \bar{\Psi} \gamma^\mu v_\mu \left(\partial_\mu + iA_\mu + \frac{i}{2} \partial_\mu \theta \sigma^z \right) \Psi - \frac{b}{2} \bar{\Psi} \gamma^0 \sigma^z \Psi \\ & - |N| \bar{\Psi} \sigma^x \mu^x \Psi + \frac{1}{g} \sum_{\mu=x, y} (\partial_\tau A_\mu)^2. \end{aligned} \quad (15)$$

Now we can again perform the RPA, integrating out the fermions to quadratic order in A_μ and $\partial_\mu \theta$. An explicit

calculation is involved due to the need to regularize the Dirac theory, which must be done with care to maintain charge conservation and gauge invariance [45]. Fortunately, the result can be understood on the grounds of symmetries. The general form of the RPA effective action is

$$S_{\text{eff}} = \frac{1}{2} \int d^3x d^3x' \left(A_\mu(x') \partial_\mu \theta(x') \right) \Pi_{\mu\nu}(x-x') \begin{pmatrix} A_\nu(x) \\ \partial_\nu \theta(x) \end{pmatrix}, \quad (16)$$

where here x, x' are space-time coordinates, and $\Pi_{\mu\nu}$ is a generalized polarization tensor.

Owing to the presence of the gap in the fermion spectrum, the Fourier transform of the polarization must be analytic in frequency and momentum at scales below the gap, allowing a Taylor expansion. Furthermore, charge conservation (which enforces the continuity equation $\partial_\mu j_\mu = 0$ at the operator level) requires that $\partial_\mu \Pi_{\mu\nu}^{AA} = \partial_\nu \Pi_{\mu\nu}^{AA} = 0$, and $\partial_\mu \Pi_{\mu\nu}^{A\theta} = \partial_\nu \Pi_{\mu\nu}^{A\theta} = 0$, where the superscripts A, θ indicate the blocks within the polarization tensor. Assuming, as before, the temporal gauge $A_0 = 0$, writing the most general allowed form of the polarization tensor at lowest order in three-momenta, and Fourier transforming back to space-time, we obtain

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & \frac{1}{4} c_1 F_{0j}^2 + \frac{1}{4} c_2 F_{ij}^2 + \frac{1}{2} c_3 (\partial_0 \theta)^2 + \frac{1}{2} c_4 (\partial_j \theta)^2 \\ & + i c_5 (\partial_i A_i) (\partial_0 \theta), \end{aligned} \quad (17)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field-strength tensor, i, j indicate spatial indices, and 0 is a time index. The constants $c_{1-4} > 0$ for stability. The factor of i in the last term is required by Hermiticity. The Lagrangian Eq. (17) is invariant under the residual gauge transformations $A_j \rightarrow A_j + \partial_j f$ with τ -independent f . Finite frequency excitations of Eq. (17) are represented by the standard transverse mode with speed $v_T = \sqrt{2c_2/c_1}$ and a longitudinal sound mode which mixes θ and the longitudinal part of A , with speed $v_L = \sqrt{(c_4 + 2c_5^2/c_1)/c_3}$.

To illustrate the logic leading to the Lagrangian in Eq. (17), we comment on the last, cross term. Analyticity of $\Pi_{\mu\nu}^{A\theta}$ and charge conservation require that in the momentum space $\Pi_{\mu\nu}^{A\theta}(q) \rightarrow \Pi_{\mu\nu\lambda}^{(1)} q_\lambda$, where the coefficient of q is a tensor of constants antisymmetric in indices μ, λ : $\Pi_{\mu\nu\lambda}^{(1)} = \epsilon_{\mu\lambda\kappa} c_{\nu\kappa}$ and $c_{\nu\kappa}$ are some constants. In space-time, this leads to $c_{\nu\kappa} \epsilon_{\mu\lambda\kappa} \partial_\lambda A_\mu \partial_\nu \theta$. For $c_{\nu\kappa} = \delta_{\nu\kappa}$, this term vanishes. Naively, spatial rotation symmetry and time-reversal symmetry require $c_{01} = c_{02} = 0$, $c_{12} = -c_{21}$, and $c_{11} = c_{22}$. With this form, the $c_{11} = c_{22}$ contribution is a pure boundary term. Then, there remains $c_{00} \neq 0$ term, and the above general form reduces to $F_{12} \partial_0 \theta$ and $F_{i0} \partial_i \theta$ and terms equivalent to these under integration by parts. We

next observe that Eq. (15) is symmetric under paritylike transformation $x \rightarrow -x$, $A_x \rightarrow -A_x$, $\Psi \rightarrow \gamma^x \mu^z \Psi$, and $\bar{\Psi} \rightarrow -\bar{\Psi} \gamma^x \mu^z$, which, however, changes the sign of the flux $F_{12} \rightarrow -F_{12}$ and rules out the first option but preserves the second one, $\propto \partial_i A_i \partial_0 \theta$.

The Lagrangian in Eq. (17) is a coupled Maxwell theory for A_μ and a free massless scalar field theory for θ , which describes two massless linearly dispersing modes. The presence of two such modes indicates that both the $U(1)_m$ and $U(1)_f$ symmetries are spontaneously broken in the AFM phase. The gapless θ mode will appear as a spin-wave-like mode in the *transverse* spin structure factor, with high intensity as it approaches the Bragg peak associated with the AFM order (we show our DQMC results for the transverse magnetic spectra in Sec. IV C).

Because of the presence of the gapless photon, we also expect power-law correlations of the flux in the AFM phase, of the same form as in the CF phase (but distinct from those in the DSL). We refer the reader to our DQMC results for temporal flux correlation functions in Sec. IV of the SM [36]. These power-law flux correlations, together with the gapless spin-wave branch in the structure factor, are signatures of the two broken symmetries of the AFM state.

III. PHASE DIAGRAM

Our numerically obtained phase diagram is presented in Fig. 1(b). At zero magnetic field, there exists a stable Dirac spin liquid phase, which is a nontrivial critical phase in which a configuration with π flux of the gauge field on each plaquette is dynamically selected. Such a small- J DSL phase has also been reported in the compact case in Ref. [33]. There are theoretical arguments which indicate the DSL is unstable in the compact case (see Sec. V). We conclude there are strong finite-size effects which affect the compact model, since for a 20×20 lattice, the expected power-law correlations of the fermion bilinears in spin operators with decay exponent of $2\Delta \sim 3$ (see Sec. II C 1) inside DSL phase have been observed. Similarly, the finite extent of the DSL phase for $B > 0$ in Fig. 1(b) is due to the nonzero temperature accessible in our simulations. We expect the DSL to give way to the CF phase for infinitesimal Zeeman fields in the zero-temperature thermodynamic limit.

At finite B , the flux per plaquette starts to deviate from π and the system enters a chiral flux phase. In this phase the Néel order parameter N^+ remains zero, but the flux per plaquette deviates from π (and 0). Two possible distinct states—with flux in $(0, \pi)$ or in $(\pi, 2\pi)$ —are possible, one of which is spontaneously chosen, resulting in spontaneous breaking of the time-reversal symmetry $\mathcal{T}$. This is characterized by the chiral order parameter $\langle \sin \Phi_\square \rangle$. We stress that $\mathcal{T}$ is broken due to the nontrivial gauge flux in the system: In contrast, a finite Zeeman field or spin polarization at vanishing flux preserves $\mathcal{T}$. As discussed in

Sec. II C 2, there is also a more subtle breaking of $U(1)_m \times U(1)_f$ symmetry, which is reflected in a gapless photon mode and the nonzero susceptibility of the CF phase.

As will be further discussed in Sec. III B, one can define a scale-invariant correlation ratio U_{flux} of the chiral order, and at finite B and small J , one sees that U_{flux} is large while the correlation ratio of the antiferromagnetic phase r_{AFM} is small [meaning no antiferromagnetic long-range order]. While at finite B and large J , the trend is opposite, suggesting the vanishing of the CF order and the establishment of the $U(1)_f$ symmetry breaking in the form of an antiferromagnetic long-range order in the $x - y$ components. To clearly discuss these results, we first provide the mean-field analysis of the phase diagram at $J = 0$ and then discuss the DQMC results at finite J .

A. Mean-field analysis at $J = 0$

The deviation from π flux at nonzero Zeeman field can be demonstrated by a mean-field analysis of the lattice model, which forms a basis for understanding the DQMC results. The mean-field approximation consists of neglecting the dynamics of the gauge field, i.e., assuming $a_{ij}(\tau_n)$ is independent of τ_n . Inspecting the final term in Eq. (1), one sees that this approximation becomes exact in the limit $J \rightarrow 0$, as configurations of the gauge field with any time dependence have infinite action in this limit. The fermionic path integral for a particular time-independent flux configuration gives, then, simply the free-fermion partition function with this flux. At zero temperature ($\beta \rightarrow \infty$), this is $\exp[-\beta E_0(\Phi_\square)]$, where $E_0(\Phi_\square)$ is the fermionic ground state energy with the given flux, and hence the flux becomes (in the thermodynamic and zero temperature limits) constrained to take the value which minimizes the energy. To find the latter, we manually set the flux of each plaquette in Landau gauge and calculate the energies as a sum of the fermion kinetic energy and the Zeeman energy. The input parameters of the mean-field calculation are the Zeeman field B , which acts like an opposite chemical potential for fermion with opposite spin, and the flux sector $\Phi_\square$, which in turn determines the gauge field within Landau gauge, which we choose. We can then construct and diagonalize the fermion hopping matrix, whose eigenvalues, the single-particle energies, are shown in Fig. 2(a) for different flux sectors. The Zeeman field B controls the filling of the fermions, from which we obtain the magnetization $m_z = (1/N) \sum_i \frac{1}{2} (n_{i,\uparrow} - n_{i,\downarrow})$. We then can obtain the total energy of the system as a function of the specific flux sector $\Phi_\square$ and Zeeman field B , from which we extract the flux sector with minimal energy. We observe a significant deviation from π flux for nonzero Zeeman field. The conclusion is consistent with the mean-field analysis in Ref. [31] for a kagome lattice model.

The flux sector corresponding to the minimum energy versus Zeeman field is shown in Fig. 2(c). At zero Zeeman

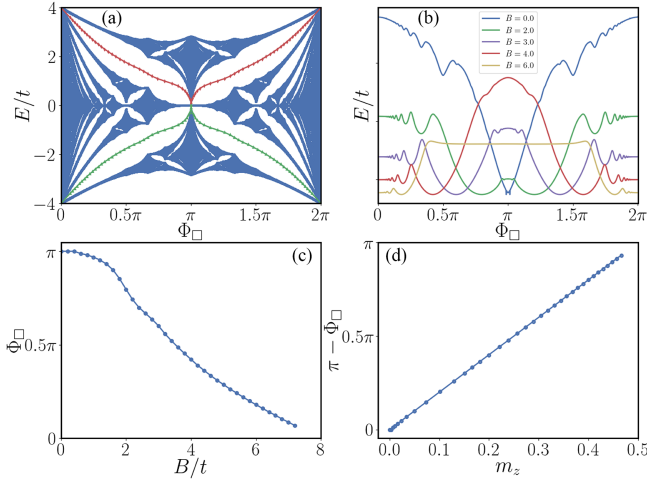


FIG. 2. Lattice mean-field calculations at $L = 32$. (a) Eigenvalues of the fermion hopping matrix at different flux sector from 0 to 2π , known as Hofstadter's butterfly [46]. The red (green) line shows the location of Fermi level for up (down) fermion. (b) Total energy of system versus flux $\Phi_{\square}$ for different values of Zeeman field B . With finite Zeeman field, the flux sectors with the minimal energy deviate from π and are symmetric about π . (c) The flux sector with the minimal total energy versus Zeeman field. The data points are obtained from (b). (d) $\pi - \Phi_{\square}$ vs magnetization m_z , shows linear relation and indicates the induced orbital magnetic field is proportional to magnetization.

field, the system favors the π -flux gauge field arrangement. However, for nonzero Zeeman field, the favorable flux sectors deviate from π flux and are symmetric about π flux from Fig. 2(b) data, indicating broken $\mathcal{T}$ symmetry. By interpreting $\pm B/2$ as an opposite chemical potential for up and down fermions, the $\Phi_{\square}$ vs B/t relation gives the spin-dependent Fermi levels, shown in Fig. 2(a) by red and green lines. In this plot, spin up and down fermions occupy states below the red and green levels, respectively. In Fig. 2(d), we show $\pi - \Phi_{\square}$ vs magnetization m_z , which demonstrates a linear relation. The constant slope indicates that the bands between the two Fermi levels have a fixed total Chern number of 1 for the full range of fields.

B. DQMC simulations with $J \neq 0$

With $J \neq 0$, the fluctuations of the gauge field render the problem in Eq. (1) strongly correlated and one has to rely on the DQMC results. As mentioned in Sec. II A, in our DQMC lattice simulations, we denote the system size of the cubic lattice in Fig. 1(a) by L and β . We scan the phase diagram in the unit of B/t and J/t and scale the inverse temperature $\beta = (1/T) = 2L$ so that we approach zero temperature in the large system limit (in our setting $\Delta\tau = 0.1$ and the integer index in the temporal direction $N_{\tau} = \beta/\Delta\tau = 20L$).

To analyze the symmetry breaking patterns, we consider both chiral flux and antiferromagnetic orders. We first verify that the chiral flux order is uniform and that the

antiferromagnetism is the standard two-sublattice staggered type by considering their corresponding equal time structure factors. For the chiral order, this is

$$S_{\chi}(\mathbf{q}) = \frac{1}{N_{\square}} \sum_{i,j} \langle \sin(\Phi_{\square,i}) \sin(\Phi_{\square,j}) \rangle e^{-i\mathbf{q} \cdot \mathbf{r}_{ij}}, \quad (18)$$

where $N_{\square} = L^2$ is the number of plaquettes. i, j are the position of plaquettes labeled by their lower left-hand site. The maximum of S_{χ} fixes the ordering wave vector, which is at $\mathbf{q} = \Gamma$. For the AFM order, the corresponding (transverse) equal time spin structure factor is

$$S^{\pm}(\mathbf{q}) = \frac{1}{N} \sum_{ij} \langle S_i^{\pm} S_j^{\mp} + \text{H.c.} \rangle e^{-i\mathbf{q} \cdot \mathbf{r}_{ij}}, \quad (19)$$

where $S_i^{+} = c_{i,\uparrow}^{\dagger} c_{i,\downarrow}$, $S_i^{-} = c_{i,\downarrow}^{\dagger} c_{i,\uparrow}$, and $N = L^2$. The maximum structure factor again fixes the ordering wave vector, which is at the $\mathbf{M} = (\pi, \pi)$ point for the AFM and the AFM order parameter discussed in Sec. II C 3, $|N^{+}| = \lim_{N \rightarrow \infty} \sqrt{S^{\pm}(\mathbf{M})/(2N)}$.

With the ordering wave vectors determined, we proceed to find the phase boundaries in the J - B plane by choosing appropriate measures of the two order parameters. An example of data and analysis for a cut at fixed $B/t = 2$ is shown in Fig. 3. For the chiral order [Fig. 3(a)], we measure the Binder cumulant of the sine of the flux on a plaquette. We define $f_{\chi} = (1/N_{\square}) \sum_{\square} \sin(\Phi_{\square})$, and then the Binder cumulant U_{flux} is

$$U_{\text{flux}} = 1 - \frac{\langle f_{\chi}^4 \rangle}{3 \langle f_{\chi}^2 \rangle^2}. \quad (20)$$

Inside the CF phase, $U_{\text{flux}} \rightarrow \frac{2}{3}$, and once the $\mathcal{T}$ symmetry is recovered, for example, by increasing J with fixed B , $U_{\text{flux}} \rightarrow 0$. Hence a crossing is expected in the thermodynamic limit within a plot of U_{flux} along a line crossing the boundary between a phase with chiral order (broken $\mathcal{T}$) and one without.

For the antiferromagnetic order, we use dimensionless correlation ratio,

$$r_{\text{AFM}} = 1 - \frac{S^{\pm}(\mathbf{q} + \delta\mathbf{q})}{S^{\pm}(\mathbf{q})}, \quad (21)$$

in which $\mathbf{q} = \mathbf{M}$ is the ordering wave vector and $\mathbf{q} + \delta\mathbf{q}$ is the closest adjacent wave vector in momentum space. In the disordered phase, $r_{\text{AFM}} \rightarrow 0$ as the structure is flat in the momentum space, and in the ordered phase $r_{\text{AFM}} \rightarrow 1$, once the Bragg peak at $\mathbf{q}$ is fully developed in the thermodynamic limit. Like for the Binder ratio of CF order, we thus expect a crossing of r_{AFM} on cuts crossing the phase boundary between AFM ordered and disordered regions,

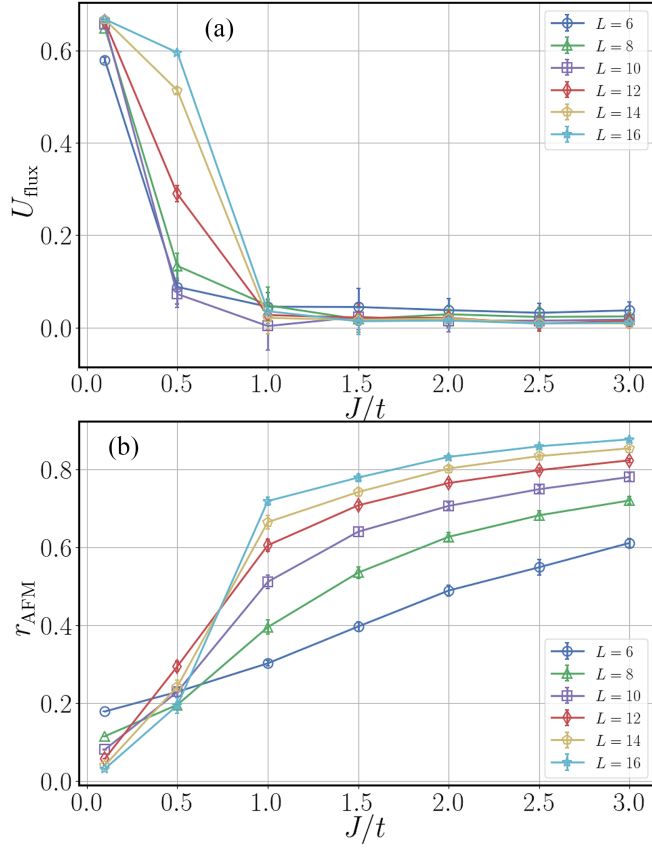


FIG. 3. Determination of the phase boundary in Fig. 1(b) at finite B . Data are at Zeeman field $B/t = 2$ with system sizes $L = 6, 8, \dots, 16$. (a) Critical $J/t \sim 0.9$ determined from flux Binder cumulant U_{flux} . (b) Critical $J/t \sim 0.6$ determined from transverse spin correlation ratio r_{AFM} . The phase boundaries of the CF and AFM phases in Fig. 1(b), at $B/t = 1, 2, 3, \dots, 5$, are determined in this way.

for large system sizes, as shown in Fig. 3(b) for $B/t = 2$ case.

The results of this analysis using multiple cuts are summarized in the phase diagram in Fig. 1(b). We observe substantial regions with CF order, AFM order, and a small overlapping region in which both CF and AFM order coexist. We will present and discuss the magnetic spectra in CF and AFM phases in Sec. IV.

Before doing so, we check the DQMC simulations for evidence of the gauge field dynamics. To this end, we consider the dynamical flux correlation function:

$$C_\chi(\tau) = \frac{1}{N_\square} \sum_{\square} \langle \sin[\Phi_\square(\tau)] \sin[\Phi_\square(0)] \rangle. \quad (22)$$

Figure 4 shows the imaginary time decay of the $C_\chi(\tau)$ at $B/t = 3$ as a function of J . For small $J/t < 1$ (inside CF phase) the decay saturates to a constant at “large” τ , indicating the presence of spontaneous chiral order. At large $J/t > 1$ (inside AFM phase), it decays to a negligible

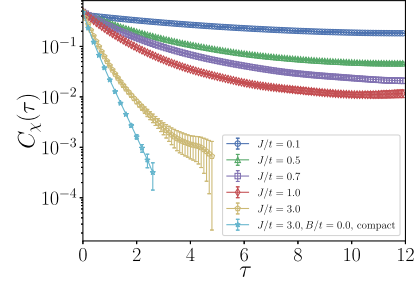


FIG. 4. Dynamical flux correlation function $C_\chi(\tau)$. For system size $L = 12$ and inverse temperature $\beta = 24$, all values of J/t at Zeeman field $B/t = 3$ exhibit clear deviations from exponential decay. This behavior stands in contrast to the compact case at $J/t = 3, B/t = 0$, where the correlation function shows well-defined exponential decay, characteristic of a confined phase [33]. In the mixed case with large $J/t = 3$ and $B/t = 3$, the correlators display persistent curvature in the semilogarithmic plot, indicating a breakdown of exponential scaling and suggesting deconfined gauge field even inside AFM phase.

value. However, even well in the AFM phase, e.g., $J/t = 3$, the form of the decay is clearly subexponential. This indicates power-law correlations consistent with the expected gapless photon mode (see Sec. II C 3). In fact, as shown in Sec. IV of the SM [36], our DQMC simulations find the flux correlation functions inside the AFM phase as $C_\chi(\tau) \sim \tau^{-3}$ and for the DSL phase as $C_\chi(\tau) \sim \tau^{-4}$, consistent with theoretical expectations. For comparison, we also plot the same flux correlator for the *compact* model [33] for $J/t = 3, B/t = 0$ (inside the confined AFM phase). One can clearly see exponential decay in that case, which is due to confinement physics that is absent in the mixed model. While it is difficult to separate the power-law decay from the saturation within the CF phase, a power-law approach to the large-time saturation is also expected in these cases as well. We conclude that through the phase diagram, the gauge field is “deconfined” in the mixed model.

IV. MAGNETIC SPECTRA

A key implication of the generation of a finite flux of the emergent gauge field is that the Dirac fermions experience an orbital magnetic field and form Landau levels [7]. While the single-spinon spectrum is not gauge invariant, one may inspect how the formation of such Landau levels impacts the dynamic spin structure factor, where we consider both the transverse component $S^\pm(\omega, \mathbf{q}) = (1/N) \sum_{ij} \int dt \langle S_i^\pm(t) S_j^\mp(0) + S_i^\mp(t) S_j^\pm(0) \rangle e^{i(\omega t - \mathbf{q} \cdot \mathbf{r}_{ij})}$ as well as the longitudinal component $S^z(\omega, \mathbf{q})$. Formally, in the zero-temperature limit, the dynamic structure factor can be obtained via the fluctuation-dissipation theorem, $S^a(\omega, \mathbf{q}) = -2\Theta(\omega) \text{Im} \chi^a(i\omega \rightarrow \omega + i0^+, \mathbf{q})$ from the dynamical susceptibility (i.e., response) $\chi^a(i\omega, \mathbf{q})$ to an external probe field which couples to the spin density at momentum $\mathbf{q}$ (see also the discussion in Sec. II C). Since the

fermionic spinons carry $S = 1/2$ quantum numbers, this susceptibility can be obtained from dynamical response of the fermionic degrees of freedom. Below, we first infer key qualitative features of the dynamic spin structure factor in a mean-field approximation (both in the continuum limit, where an analytical treatment is feasible, and on the square-lattice geometry), and then present magnetic spectra in the CF phase upon including gauge fluctuations obtained from our DQMC simulations.

A. Continuum field-theory analysis

A continuum field theory allows for insights into the dynamic structure factor near the high-symmetry points Γ , M , and X points. At these points, in the zero-field limit, the magnetic spectrum is dominated by particle-hole excitations on top of the single-particle Dirac cones. Explicitly, these contributions are determined by fermionic bubble diagrams of the form

$$\chi_{\mathbf{Q}}^{\pm}(i\omega, \mathbf{k}) = \text{bubble diagram} + \dots, \quad (23)$$

and similarly for the longitudinal component. Here, $\mathbf{k}$ refers to small momenta close to the high-symmetry points $\mathbf{Q} = \Gamma, X$, and M . The choice of lattice momentum $\mathbf{Q}$ is encoded in the vertices that enter the bubble diagram—these are determined by starting from a microscopic lattice model and making a gradient expansion to read off how microscopic lattice translation or rotation symmetries act on the sublattice and valley indices of the Dirac fermions (we refer the reader to Sec. I of the SM [36] for a lattice expression of this diagram, and Sec. II of the SM [36] for technical details concerning our continuum field theory calculations). Further, in Eq. (23), the dots correspond to corrections arising from interactions with the emergent U(1) gauge field.

We henceforth work on a “bare” level and neglect such interactions with the U(1) gauge field, corresponding to a mean-field approximation. For an effective RPA-level treatment of the fluctuating gauge field, see Sec. II C.

We first give a physical picture for possible excitations that can give rise to poles in the structure factor: Noting the low-energy expressions of the spin operators in terms of spinon fields listed in Eqs. (S11a)–(S11c) of Sec. II of the SM [36], applying the spin-lowering operator $S^- \sim \psi_{\downarrow}^{\dagger} \psi_{\uparrow}$ on top of the ground state with spin-split Landau levels can be seen to create a hole in a filled spin- $\uparrow$ Landau level and create a particle in an otherwise empty spin- $\downarrow$ Landau level. This is illustrated in Fig. 5. Transitions belonging to the same (spin-split) Landau level are in general not allowed (as they are fully occupied or empty for $|n| > 0$), *except* in the 0th Landau level, of which only the spin- $\uparrow$ copy is filled (for $b > 0$). This excitation corresponds to a pole at frequency $\omega = b$ in the transverse contribution to the

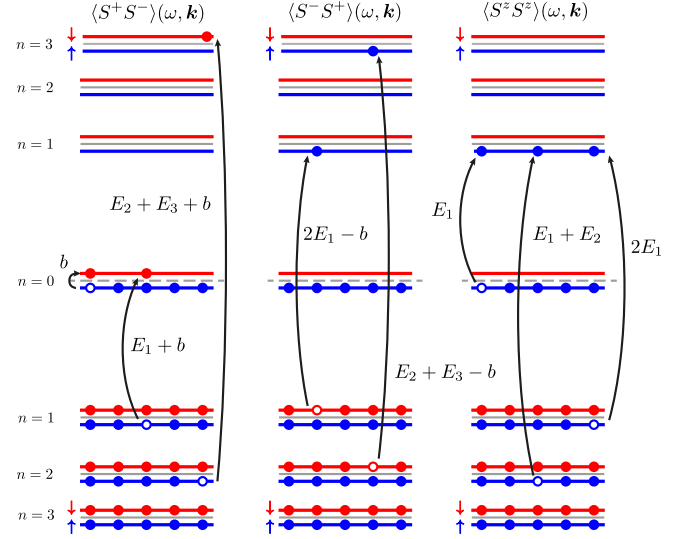


FIG. 5. Illustration of excitations that contribute to the transverse ($\langle S^+ S^- \rangle$ and $\langle S^- S^+ \rangle$) and longitudinal ($\langle S^z S^z \rangle$) channels of magnetic spectra. The finite flux of the gauge field leads to Landau levels of the Dirac fermions, with energies $\pm E_n - ab/2 = \pm \sqrt{2n}/\ell - ab/2$, where the finite Zeeman fields lead to a spin splitting for $\alpha = \uparrow(+1), \downarrow(-1)$ spinons. For the former, spin-flip excitations can occur either within the spin-split $n = 0$ Landau level or between distinct sets of spin-split Landau levels, while the longitudinal channel only receives contributions from same-spin inter-Landau-level transitions.

structure factor. Conversely, now consider the action $S^+ \sim \psi_{\uparrow}^{\dagger} \psi_{\downarrow}$ on top of the ground state with $b \geq 0$. Creating a spin- $\downarrow$ hole and a spin- $\uparrow$ particle is allowed only if they respectively occur in Landau levels $|n| \geq 1$, such that these particle-hole excitations can occur only with energies $\omega = E_n + E_m - b$ with $n, m \geq 1$.

In contrast to the transverse response, the longitudinal susceptibility, and thus the structure factor, will involve excitations that preserve the magnetization. Hence, only inter-Landau-level transition are allowed. These can in general give rise to poles at frequencies $\omega = E_n + E_m$, where $n, m \geq 0$ (but not $n = m = 0$).

With a qualitative understanding of excitations (in the mean-field limit) that can give rise to poles in the transverse and longitudinal structure factors, we now turn to their respective spectral weights at the Γ, M , and X points, where the spinon particle-hole excitations dominate the low-energy response. We obtain the spectral weights via an explicit evaluation of Eq. (23), with technical details provided in Sec. II of the SM [36].

We note the following key predictions from our field-theoretical analysis.

- (1) At the Γ point, only the pole of frequency $\omega = b$ contributes finite spectral weight to the *transverse* structure factor (arising from spin-flip particle-hole excitations in the spin-split 0th Landau level). This is in accordance with Larmor’s theorem: Since

the ground state (at zero field $b = 0$) is SU(2) symmetric, the only dynamic response to the SU(2)-breaking Zeeman field b consists in a transverse spin wave with frequency b , with the spectral weight determined by the system's magnetization m^z [47–49], $\langle S^+ S^- + S^- S^+ \rangle(\omega, \Gamma) = 4\pi m^z \delta(\omega - b)$. We further note that this mode of frequency b also has finite spectral weight in the transverse structure factor at $\mathbf{M}$ (in addition to many other modes), while it is absent in $S^\pm$ at the $\mathbf{X}$ point.

- (2) Turning to the longitudinal structure factor S^{zz} , poles occur at energies proportional to the cyclotron frequency (which is *a priori* independent of the Zeeman energy). We find that $S^{zz}(\Gamma, \omega) = 0$, consistent with Larmor's theorem (see above). We further observe that all low-energy poles at the $\mathbf{X}$ point (i.e., $\omega = E_1, E_1 + E_2, \dots$) generally carry finite weight, while poles with finite weight at the $\mathbf{M}$ point occur at much higher frequencies, i.e., $\omega = 2E_1, 2E_2, \dots$.
- (3) One may further find the structure factor at finite (small) momenta $\mathbf{q}$ relative to these high-symmetry points. We have explicitly obtained $S^\pm(\mathbf{q}, \omega)$, shown in Fig. 6, see also Sec. II of the SM [36] for details: Going away from Γ , other poles (in addition to the Larmor mode) generally acquire finite spectral weight. Here, $\langle S^+ S^- \rangle$ contains poles with frequencies $\omega \geq b$, while $\langle S^- S^+ \rangle$ may contain poles with energies below the Larmor mode. The maximum spectral weight at a given frequency follows a *linear* dispersion $\nu_{\max}(\mathbf{k}) = |\mathbf{k}| + b$ —this is consistent

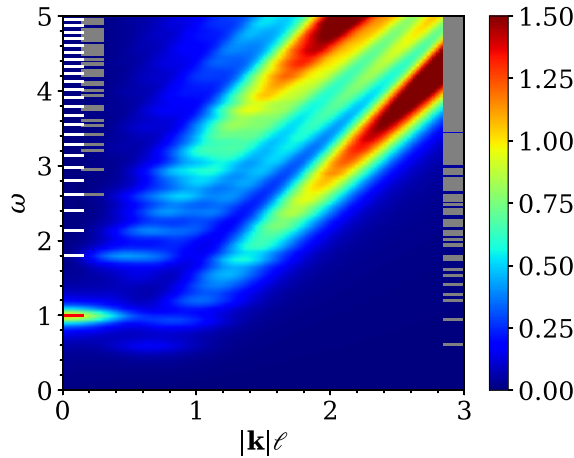


FIG. 6. Transverse structure factor computed in continuum field theory. For $b = 1$, with flux $\Phi = b^2/(2\sqrt{2} \times 0.62)^2 \approx 0.33$ (using $b/2 = 0.62\omega_c$). The white horizontal lines on the left-hand axis denote poles with frequency $\omega = E_n + b$, the gray horizontal lines on the left- and right-hand axes denote poles with $\omega = E_n + E_m + b$ and $\omega = E_n + E_m - b$, respectively. The red line on the left-hand axis indicates the Larmor pole with frequency $\omega = b$.

with continuity arguments wherein going toward the zero-field limit the Landau levels become dense, and linearly dispersing cones emerge in the structure factor (resulting from particle-hole excitations on the spinon Dirac cones).

Below, we compare these insights based on a low-energy theory (assuming a static gauge field) with our numerical results.

B. Lattice mean-field calculations

We now turn to the lattice action in Eq. (1). In the limit of static gauge field, $J = 0$, we perform lattice mean-field calculations (see Sec. I of the SM [36] for details).

Our results for the real-frequency transverse and longitudinal spin structure factors are shown along a high-symmetry path in the Brillouin zone in Figs. 7(a) and 8(a),

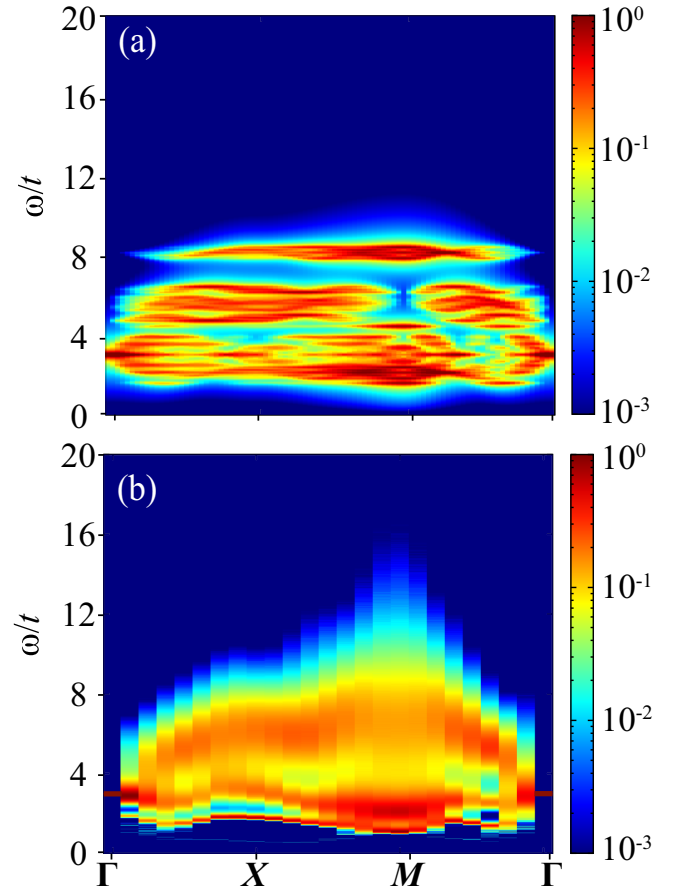


FIG. 7. Transverse magnetic spectra at Zeeman field $B/t = 3$ from lattice mean field and QMC + SAC calculations. (a) $S^\pm(\mathbf{q}, \omega)$ from mean field with lattice size $L = 60$. (b) $S^\pm(\mathbf{q}, \omega)$ from DQMC simulations with $L = 16$, $\beta = 16$, $J/t = 0.1$. Mean-field data show three sets of bands, while spectra from QMC simulations show at least two broad spectral features, with high-energy structure difficult to resolve. Larmor mode at Γ is clearly resolved both in mean-field and QMC data. QMC data also show a reduction of weight at $\mathbf{M}$ point at frequencies corresponding to the second mean-field band set.

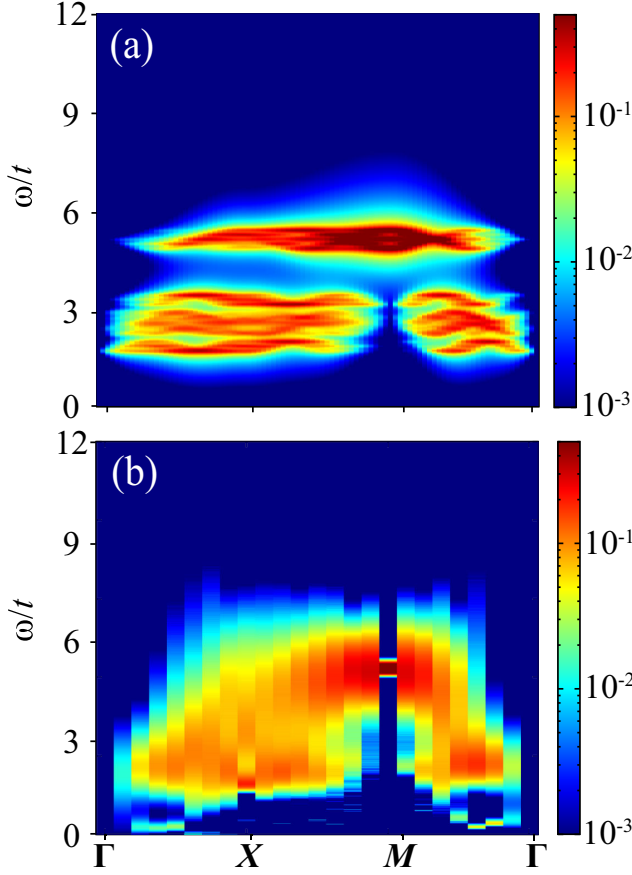


FIG. 8. Longitudinal magnetic spectra at Zeeman field $B/t = 3$ from lattice mean field and QMC + SAC calculations. (a) $S^{zz}(\mathbf{q}, \omega)$ from mean field with lattice size $L = 60$. (b) $S^{zz}(\mathbf{q}, \omega)$ from DQMC simulations with $L = 16, \beta = 16, J/t = 0.1$. QMC and mean-field results are quite consistent, both showing lower- and higher-frequency regions of spectral weight and the absence of spectral weight at $\mathbf{M}$ in the lower-frequency region. Crucially, QMC data show the emergence of a low-energy mode near Γ , possibly consistent with the analytical prediction Eq. (11) in Sec. II C 2 of the gapless photon. Such a mode is absent in the mean-field analysis.

and detailed energy scans at a few representative momenta are shown in Fig. 9.

We first note that the lattice mean-field spectra (both transverse and longitudinal) exhibit a fine structure of many separate nearly dispersionless bands. Equipped with theoretical analysis in Secs. II C 2 and IV A, this discrete spectrum can be directly attributed to inter- or intra-Landau-level transitions. We further observe that the individual bands in the transverse (longitudinal) can be grouped into three (four) sets. We attribute this additional structure to lattice effects that lie beyond our continuum field-theory analysis.

Transverse structure factor. Comparing to our field-theoretic predictions [points (1) and (3) in Sec. IV A], we observe that exactly at the Γ point there is only a single mode (at frequency $\omega = 3t = B$) in the transverse structure factor which carries finite weight—this is precisely the

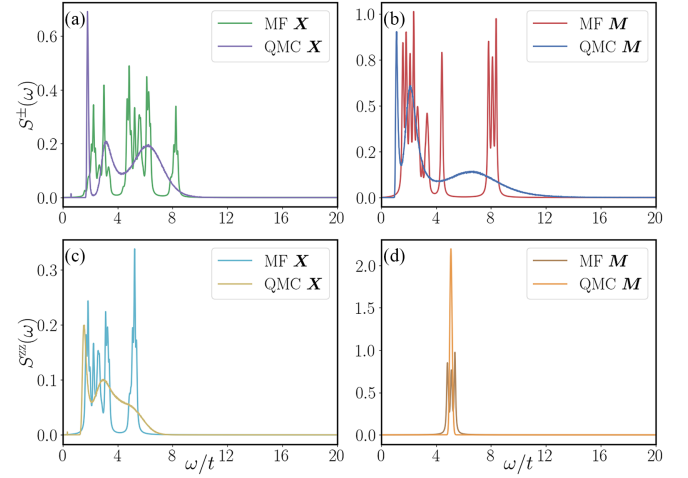


FIG. 9. Transverse and longitudinal magnetic spectra at high-symmetry points. Comparison between QMC and lattice mean-field results for Zeeman field $B/t = 3$ at $\mathbf{X}(\pi, 0)$ and $\mathbf{M}(\pi, \pi)$. (a), (b) for $S^{\pm}(\omega)$. (c), (d) for $S^{zz}(\omega)$. QMC and mean-field results are in better agreement for low frequencies, with the QMC results showing smoother spectra (compared to the mean-field result) at high energies.

Larmor mode. For small momenta near the Γ point, additional modes with finite weight emerge, and we observe that there is a linear scaling for the frequency of the spectra weight onset as a function of momenta $\mathbf{q}$ as a remnant of the linear dispersion of the Dirac fermions in the zero-field case. We further note the low-frequency response at the $\mathbf{M}$ -point features rather intense spectral weight, as visible in Figs. 7(a) and 9(b). This low-energy spectral weight could be an indication of the eventual emergence of a Goldstone magnon at the $\mathbf{M}$ point in the AFM phase, upon increasing the interaction strength.

Longitudinal structure factor. A particularly striking feature of the longitudinal structure factor in Fig. 8(a) at the $\mathbf{M}$ point is the full concentration of spectral weight in the topmost bands, at frequencies $\omega/t \sim 5$ (and a concomitant absence of weight at low and intermediate frequencies). This strong concentration of weight at large frequencies becomes also visible when inspecting the line cuts at the $\mathbf{M}$ point in Fig. 9(d). We argue this to be consistent with our field-theoretic finding that many poles in $S^{zz}(\omega)$ have vanishing spectral weight at the $\mathbf{M}$ point.

C. QMC plus stochastic analytic continuation (SAC) simulations

For finite $J \neq 0$, we obtain the imaginary time correlation function from our DQMC simulations and then perform a stochastic analytic continuation (SAC) to obtain the real-frequency results [50–53]. This QMC + SAC scheme has been applied to a variety of lattice models, including the Dirac fermions under Hubbard-type interaction and the magnetic field [49], producing

reliable spectral properties ranging from magnon and amplitude modes in a magnetically ordered state [54,55], fractionalized excitations in quantum spin liquid and spin ice models [56–58], as well as an emergent Dirac spinon spectrum at deconfined quantum critical points [59] and single-particle spectrum of the angle-tuned Gross-Neveu quantum criticality in twisted bilayer graphene [60].

Qualitative effect of small $J > 0$. In comparing the QMC + SAC spectra (with small nonzero $J/t = 0.1$) to the mean-field spectra (with $J = 0$), we observe two main effects beyond the mean field. First, the non-mean-field spectra are broadened relative to the finely discretized nature of the mean-field spectra associated with inter- or intra-Landau-level excitations. This is expected on general grounds: The interaction with the fluctuating gauge field will yield a broadening (finite lifetimes) of both single-particle spectra as well as the two-particle response functions. Second, in some spectral functions (see below) new collective modes emerge below the support of the mean-field spectra.

However, we note that even in the presence of such broadening, the DQMC structure factors in Figs. 7(b) and 8(b) possess several bandlike high-intensity features that resemble the lattice mean-field spectra in Figs. 7(a) and 8(a). This correspondence is further corroborated by inspecting the line cuts in Fig. 9. We stress that the key qualitative features identified in the continuum field theory and lattice mean-field calculations above persist also at finite interaction strengths.

Transverse structure factor in the CF phase. In this component of the structure factor, the main difference of the mean-field and $J/t = 0.1$ spectra is the broadening. However, broadening is absent, as is seen from Fig. 7(b), at the Γ point, where there is a sharp mode at frequency $\omega = 3t$: This is the Larmor mode, with frequency $\omega = B$ and weight $4\pi m^z$ protected against interactions. Going away from the Γ point, one can further observe a linear dispersion $\omega(\mathbf{k}) \sim |\mathbf{k}|$ of the spectral weight, in line with our theoretical predictions.

Longitudinal structure factor in the CF phase. At the M point, there is a strong concentration of spectral weight in the longitudinal structure factor near $\omega \simeq 5t$, and vanishing spectral weight at all other frequencies [see Figs. 8(b) and 9(d)], in qualitative agreement with our continuum field theory and lattice mean-field calculations. We further comment that such a concentration of spectral weight at high frequencies at the M point is reminiscent of the high-energy single-magnon excitation in the high-field regime of the square-lattice antiferromagnet which emerges upon increasing the interaction strength J/t [61,62].

In the longitudinal structure factor, the $J/t = 0.1$ spectra clearly exhibit a low-energy collective mode near Γ not present in the corresponding mean-field spectra—compare Figs. 8(a) and 8(b). Although the finite-size and finite-temperature effects ($L = \beta = 16$) prevent us from

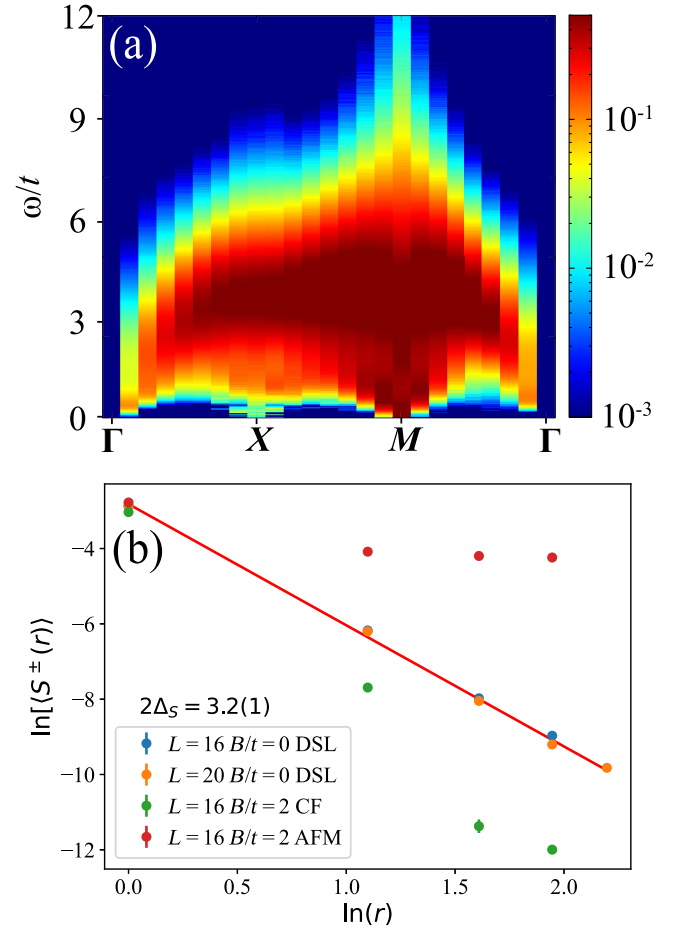


FIG. 10. Magnetic spectra inside the DSL phase and the scaling dimension of fermion bilinear operator. (a) Transverse magnetic spectra $S^\pm(\mathbf{q}, \omega)$ for $J/t = 1$, $B/t = 0$ with system size $L = 16$, $\beta = 16$, mixed case. The model is in the U(1) DSL phase. The spectra show well-known spinon continuum characteristics. (b) Real-space spin correlation $S^\pm(r)$ with r denoting the odd site distance along the x direction. Blue and orange data points are with the same parameters as (a), inside DSL phase. Data exhibit a power-law decay with exponent $2\Delta_S = 3.2(1)$, consistent with the expectation of the scaling dimension of fermion bilinear operators in the CFT of SU(4) QED₃, discussed in Sec. II C 1. Green and red data points are in either CF or AFM phase with $B/t = 2$. Green dots (CF) show fast decay which is consistent with gapped transverse spin spectra, while the red dots (AFM) show long-ranged order.

resolving the predicted linear dispersion quantitatively, the observed low-energy spectral weight near Γ is consistent with, but does not by itself establish, the photon-related response expected from Eq. (11) in Sec. II C 2. Very similar spectra, inside the CF phase at $B/t = 2$, are shown in Sec. VII of the SM [36].

Structure factor in the U(1) DSL phase. We show the magnetic spectra in π -flux U(1) deconfined phase (mixed theory) at zero Zeeman field ($B = 0$) in Fig. 10(a), which demonstrates well-known continuum characteristics with gapless Dirac cones at the Γ , X , and M points. The spectra

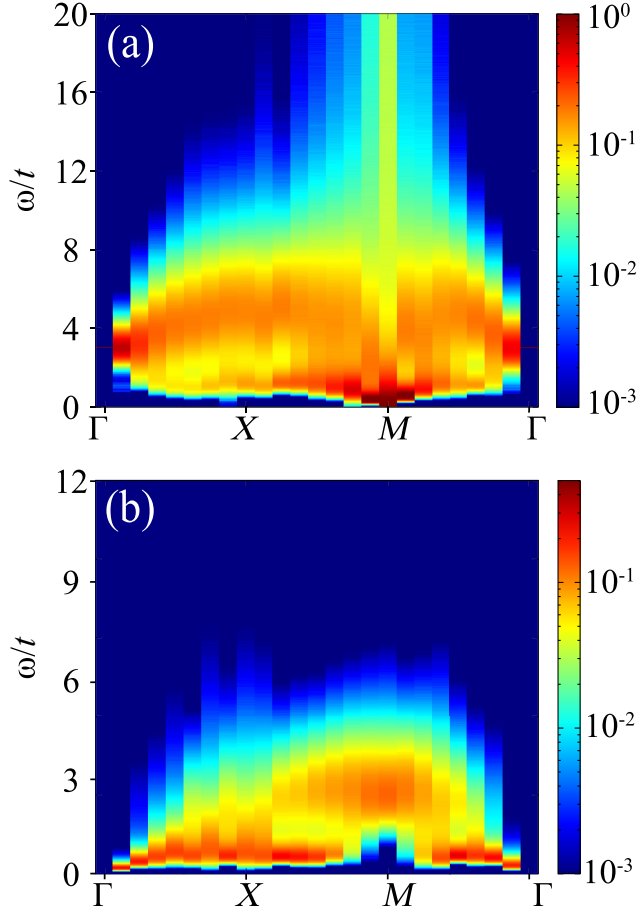


FIG. 11. Transverse and longitudinal magnetic spectra inside the AFM phase. (a) $S^\pm(\mathbf{q}, \omega)$ and (b) $S^{zz}(\mathbf{q}, \omega)$ for $J/t = 3, B/t = 3$ with system size $L = 16, \beta = 16$. At point M of $S^\pm(\mathbf{q}, \omega)$, there is considerable weight at zero frequency, indicating the Goldstone mode. In parallel, $S^{zz}(\mathbf{q}, \omega)$ shows a peak around $\omega = B$ at M , in agreement with the spin-wave expectations for the ordered phase [63].

can be qualitatively reproduced by a simple RPA calculation of the noninteracting structure factor of the π -flux fermion hopping model [59]. However, a quantitative analysis shows the effects of gauge fluctuations. Specifically, we have also computed the real-space spin correlation function inside the DSL phase up to $L = 20$. As shown in Fig. 10(b), our data exhibit a power-law decay with exponent $2\Delta_S = 3.2(1)$, consistent with the expectation of $2\Delta_{\text{adj}} \in (2.8, 3.5)$ for the fermion bilinear operators in the CFT of SU(4) QED₃, discussed in Sec. II C 1. We note that similar power-law spin and dimer correlations with the exponent of $2\Delta \sim 3$ have also been seen in the DSL phase in Ref. [33]. For comparison, real-space correlations inside CF and AFM phase with Zeeman field $B/t = 2$ are shown in Fig. 10. Both display non-power-law decay, with fast decay in CF phase which indicates short-range transverse spin correlations and gapped transverse spectra. In AFM, spin correlations show long-range order.

Magnetic spectra in the AFM phase. When J/t is larger, there can be substantial deviations from the mean-field spectrum. Physically, the probability for fermions to bind or recombine into collective modes becomes large, and such modes may be more dominant in the spectral response. The QMC + SAC results for the transverse and longitudinal spectra in the AFM phase at $J/t = 3$ for $B/t = 3$ are shown in Fig. 11. In Fig. 11(a), one observes spin-wave-like features near M point, with the M point itself hosting a gapless Goldstone mode (see also Sec. II C 3). In Fig. 11(b), the longitudinal spectrum is fully gapped, and in particular there is a high-intensity feature ($\omega/t \approx 3$) at the ordering wave vector M . This feature is similar to that of the Heisenberg square-lattice antiferromagnet in strong Zeeman fields [61,62], which has only localized bosonic states. This is in agreement with a picture of substantial recombination of the fermions into bosonic local excitations.

V. DISCUSSION

In this work, we have studied the response of QED₃ to a flavor chemical potential, which emerges in a condensed matter context as the low-energy theory for the response of a U(1) Dirac spin liquid to an externally applied Zeeman field.

Utilizing large-scale quantum Monte Carlo simulations for the case of a mixed gauge field, we unambiguously establish that for finite flavor chemical potentials (Zeeman field) there exists a stable “chiral flux” phase which is characterized by the emergent gauge field developing a finite net average flux, with the system’s magnetization being proportional to the induced flux, $m^z \propto \pi - \Phi_\square$.

A. Key results and implications

The field-induced generation of *emergent gauge flux* has several key implications.

- (1) The fermionic spectrum of the theory is given by spin-split relativistic Landau-Hofstadter levels and thus becomes gapped. The transverse and longitudinal magnetic structure factors exhibit signatures of the Landau-level spectrum, and we match a number of qualitative features obtained within a continuum field-theory analysis with our numerical results, for example, the presence of a Larmor mode in the transverse magnetic structure factor [48,49].
- (2) Owing to the suppression of monopoles in the time-continuum limit (see Ref. [36] for an explicit demonstration in an analytically tractable model) in the model at hand, the U(1) gauge field remains coherent in the CF phase and features a gapless photon excitation. The effective theory predicts a photon-related low-energy contribution to the longitudinal magnetic structure factor. The QMC + SAC spectra show low-energy spectral weight near Γ

consistent with this expectation; further work is required to resolve the predicted linear dispersion quantitatively. Such a response also distinguishes the strongly correlated nature of the CF phase at finite J , that is beyond the mean-field analysis at $J = 0$.

- (3) In an effective field-theory framework, the CF state can be described by a mixed Chern-Simons term for fluctuations of the emergent gauge field and an external “spin” gauge field, which encodes the attachment of spin S^z to flux of the emergent gauge field. This implies that the CF state is characterized by a composite order parameter $\langle \mathcal{M}S^+ \rangle \neq 0$, where $\mathcal{M}$ is a 2π -gauge flux insertion operator. This is consistent with our numerical observation that the transverse spin spectra show a full gap in the CF phase.

We argue that these results are directly related to the generation of a finite flux in the QED₃ gauge theory. Connecting with quantum magnets, we stipulate that observing such behavior of a quantum spin liquid candidate material in an applied magnetic field would be hard to reconcile with more conventional descriptions (i.e., which do not rely on fractionalization and emergent gauge fields).

B. Effects due to dynamical monopoles

When QED₃ arises as an emergent gauge field in a microscopic spin model, the theory is inevitably *fully* periodic in the gauge field. In general, this allows dynamical *monopole instantons*, which are suppressed in our study. It is known that monopoles carry distinct quantum numbers under different microscopic (lattice-scale) symmetries [10,11], and that their presence may have severe consequences for the stability of U(1) Dirac spin liquids at the lowest energies on different lattices. The similarity of the results in this paper at zero field for the mixed theory to zero-field simulations of the *compact* theory in Refs. [33,64] suggests, however, that such effects may become significant only at very low (and vanishing in some cases) energy scales. We leave a *quantitative analysis* of such very low-energy effects to future (numerical) work. This limitation does not affect the controlled conclusion of the present work: Within the mixed theory, the field-induced chiral flux phase and its characteristic dynamical spin-response signatures are established numerically. The following discussion instead addresses how monopole effects may modify this physics in a fully compact gauge theory.

While abovementioned effects are crucial for understanding the stability of U(1) Dirac spin liquids *in general*, we now turn to the central object of study in this paper, namely, the field-induced chiral flux phase with nonzero gauge flux: Here, the eventually inevitable compactness of the U(1) gauge field, and the associated monopole excitations, will have a particularly prominent effect. We argue that they render the “hidden” order [of joint 2π -flux insertion and simultaneous change of $\Delta S^z = 1$; see also

Sec. II C 2 and in particular Eqs. (12) and (13) therein] to be physical, thus allowing it to be resolved in terms of experimentally accessible local order parameters. To demonstrate this, we posit that one may relate expectation values of observables $\langle \mathcal{O} \rangle_{\text{comp}}$ in the full compact theory (with dynamical monopoles) to expectation values in the mixed theory (absent of monopoles) by means of an expansion in the monopole fugacity λ ; i.e., we deform the action of the mixed theory by 2π -flux insertion operators, $\mathcal{S}_{\text{mix}} \rightarrow \mathcal{S}_{\text{mix}} - \int d^3x [\lambda \mathcal{M} + \lambda^* \mathcal{M}^\dagger]$ (for simplicity, we adopt a continuum formulation, but the argument also applies to discretized space-time). Now, consider the in-plane spin density S^+ as an observable. As argued in Sec. II C 2, in the mixed theory $\langle S^+ \rangle_{\text{mix}} = 0$ vanishes in the chiral flux phase. However, in the compact theory, we observe

$$\begin{aligned} \langle S^+(x) \rangle_{\text{comp}} &= \langle S^+(x) \rangle_{\text{mix}} + \int d^3y \lambda \langle S^+(x) \mathcal{M}(y) \rangle_{\text{mix}} + \cdots \\ &\approx +\lambda \langle S^+(x) \mathcal{M}(x) \rangle_{\text{mix}} \neq 0. \end{aligned} \quad (24)$$

The fugacity expansion (to leading order) thus produces the *composite order parameter* which, within the mixed theory, is finite in the chiral flux phase: Therefore, the presence of dynamical monopoles in the fully compact theory makes the hidden composite order in the chiral flux phase visible as *finite in-plane magnetic order*.

We emphasize that the nature of the in-plane magnetic order, for example, the ordering wave vector, will depend on the UV details of the theory which determine how the microscopic symmetries act on the low-energy degrees of freedom in the QED₃ field theory. For the challenges elucidated above, a controlled study of this monopole-induced effect within numerical simulations of the QED₃ low-energy field theory appears challenging at present. However, from a complementary standpoint, one may construct microscopic (variational) wave functions associated with a chiral flux state by Gutzwiller-projecting Slater determinants of fermionic spinons on a lattice, thereby accounting for aforementioned UV details. A recent analysis [65] of such a family of finite-flux wave functions for triangular-lattice systems has indeed revealed highly pronounced *antiferromagnetic (120°) in-plane spin correlations*, consistent with the general field-theoretic argument provided above.

C. Outlook

We note that recently several candidate materials have been identified that exhibit signatures consistent with a (proximate) U(1) Dirac spin liquid ground state, such as the triangular-lattice antiferromagnets YbZn₂GaO₅ [66,67] and the A-YbSe₂ delafossites [68,69] as well as the kagome antiferromagnet YCu₃(OD)₆Br₂[Br_x(OD)_{1-x}] [18,70–72], where inelastic neutron scattering experiments have found

spectra that are consistent with a Dirac cone filled with a continuum of excitations. In light of our results, it will be highly interesting to scrutinize the behavior of these systems in applied magnetic fields. In particular, we foresee the experimental observation of a gapless magnetic spectral feature related to the photon mode in the longitudinal spin structure factor, possibly via inelastic neutron scattering, to be interesting evidence of the magnetized DSL state. An interesting direction for further study is to investigate what perturbations might lead to different filling of the spinon Landau levels in the CF phase (while maintaining gauge invariance or particle-hole symmetry), and establish possible connections to experimentally observed unusual magnetic oscillations in the YCu₃(OD)₆Br₂[Br_x(OD)_{1-x}] kagome antiferromagnet [73] (see also Ref. [74]).

Going beyond quantum magnets, we stress that our main result, the generation of the chiral flux phase due to a flavor chemical potential, could also be of relevance to other systems with phases (or phase transitions) which have been proposed to be described by ($N \geq 2$) QED₃ [34,35,75–77]. We are hopeful that our results are a key step toward establishing experimentally *testable predictions* that will eventually enable the identification of a fractionalized phase with deconfined gauge fields in a condensed matter system.

Note added. Recently, Ref. [78] appeared, which considers symmetry breaking in QED₃ in a mean background gauge flux (without an externally imposed Zeeman field) and is in agreement with our conclusions in Secs. II C 1 and II C 2 where results overlap. Further, Refs. [79,80] also appeared recently, which study field-induced phases of frustrated triangular-lattice quantum antiferromagnets, to which our results may be applicable. In a similar vein, Ref. [65] studies Gutzwiller-projected finite-flux wave functions for triangular-lattice quantum antiferromagnets. We also note recent progress in hybrid Monte Carlo simulations of the lattice gauge theory (at zero Zeeman field) [64] closely related to the current investigation.

ACKNOWLEDGMENTS

We acknowledge discussions with A. Capelli, J. Knolle, A. Rosch, I. Sodemann, and A. Vainshtein. C. C. and Z. Y. M. acknowledge Y. D. Liao and Y. Qi for collaboration on a previous project concerning magnetized Dirac cones in the Hubbard model [49]. U. F. P. S., O. A. S., L. B., and Z. Y. M. are grateful to the Pollica Physics Centre for hospitality during the stimulating workshop “Exotic quantum matter: from quantum spin liquids to novel field theories.” C. C., K. F., and Z. Y. M. acknowledge the support from the Research Grants Council (RGC) of Hong Kong (Projects No. C7037-22GF, No. 17302223, No. 17301924, No. 17301725), the ANR/RGC Joint Research Scheme sponsored by RGC of Hong Kong and

French National Research Agency (Project No. A_HKU703/22). We thank HPC2021 system under the Information Technology Services at the University of Hong Kong, as well as the Beijing PARATERA Tech CO., Ltd for providing HPC resources that have contributed to the research results reported within this paper. U. F. P. S. acknowledges support from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) via SFB 1238, Project ID No. 277146847, the Emmy Noether Program, Project ID No. 544397233. L. B. is supported by the NSF CMMT program under Grant No. DMR-2419871, and by the Simons Collaboration on Ultra-Quantum Matter, which is a grant from the Simons Foundation (Grant No. 651440).

DATA AVAILABILITY

Selected numerical data used to generate the figures are publicly available in the authors’ GitHub data repository [81]. The complete bin-resolved DQMC outputs and the original imaginary-time correlation functions used for SAC are available from the authors upon reasonable request because of their volume. The DQMC code is not publicly available.

-
- [1] N. Karthik and R. Narayanan, *No evidence for bilinear condensate in parity-invariant three-dimensional QED with massless fermions*, *Phys. Rev. D* **93**, 045020 (2016).
 - [2] S. Giombi, I. R. Klebanov, and G. Tarnopolsky, *Conformal QED_d, F-theorem and the ϵ -expansion*, *J. Phys. A* **49**, 135403 (2016).
 - [3] L. Di Pietro and E. Stamou, *Scaling dimensions in QED₃ from the ϵ -expansion*, *J. High Energy Phys.* **12** (2017) 054.
 - [4] Y.-C. He, J. Rong, and N. Su, *Conformal bootstrap bounds for the U(1) Dirac spin liquid and N = 7 Stiefel liquid*, *SciPost Phys.* **13**, 014 (2022).
 - [5] M. Hermele, T. Senthil, M. P. A. Fisher, P. A. Lee, N. Nagaosa, and X.-G. Wen, *Stability of U(1) spin liquids in two dimensions*, *Phys. Rev. B* **70**, 214437 (2004).
 - [6] M. Hermele, T. Senthil, and M. P. A. Fisher, *Algebraic spin liquid as the mother of many competing orders*, *Phys. Rev. B* **72**, 104404 (2005).
 - [7] Y. Ran, M. Hermele, P. A. Lee, and X.-G. Wen, *Projected-wave-function study of the spin-1/2 Heisenberg model on the kagomé lattice*, *Phys. Rev. Lett.* **98**, 117205 (2007).
 - [8] S. Albayrak, R. S. Erramilli, Z. Li, D. Poland, and Y. Xin, *Bootstrapping $N_f = 4$ conformal QED₃*, *Phys. Rev. D* **105**, 085008 (2022).
 - [9] S. M. Chester and S. S. Pufu, *Towards bootstrapping QED₃*, *J. High Energy Phys.* **08** (2016) 019.
 - [10] X.-Y. Song, C. Wang, A. Vishwanath, and Y.-C. He, *Unifying description of competing orders in two-dimensional quantum magnets*, *Nat. Commun.* **10**, 4254 (2019).
 - [11] A. Wietek, S. Capponi, and A. M. Läuchli, *Quantum electrodynamics in 2 + 1 dimensions as the organizing principle of a triangular lattice antiferromagnet*, *Phys. Rev. X* **14**, 021010 (2024).

- [12] Z.-X. Luo, U. F. P. Seifert, and L. Balents, *Twisted bilayer $U(1)$ Dirac spin liquids*, *Phys. Rev. B* **106**, 144437 (2022).
- [13] G. Nambiar, D. Bulmash, and V. Galitski, *Monopole Josephson effects in a Dirac spin liquid*, *Phys. Rev. Res.* **5**, 013169 (2023).
- [14] U. F. P. Seifert, J. Willsher, M. Drescher, F. Pollmann, and J. Knolle, *Spin-Peierls instability of the $U(1)$ Dirac spin liquid*, *Nat. Commun.* **15**, 7110 (2024).
- [15] M. B. Hastings, *Dirac structure, RVB, and Goldstone modes in the kagomé antiferromagnet*, *Phys. Rev. B* **63**, 014413 (2000).
- [16] Y. Iqbal, F. Becca, and D. Poilblanc, *Projected wave function study of $\mathbb{Z}_2$ spin liquids on the kagome lattice for the spin- $\frac{1}{2}$ quantum Heisenberg antiferromagnet*, *Phys. Rev. B* **84**, 020407 (2011).
- [17] Y.-C. He, M. P. Zaletel, M. Oshikawa, and F. Pollmann, *Signatures of Dirac cones in a DMRG study of the kagome Heisenberg model*, *Phys. Rev. X* **7**, 031020 (2017).
- [18] X. Lu, A. Raikos, M. Song, Z. Li, L. Han, S. Li, S. Capponi, Z. Y. Meng, and C. Zhou, *Unraveling the kagome antiferromagnetic 3J model and its materials: An integrated approach*, *arXiv:2609.11745*.
- [19] R. Kaneko, S. Morita, and M. Imada, *Gapless spin-liquid phase in an extended spin $1/2$ triangular Heisenberg model*, *J. Phys. Soc. Jpn.* **83**, 093707 (2014).
- [20] Z. Zhu and S. R. White, *Spin liquid phase of the $S = \frac{1}{2} J_1 - J_2$ Heisenberg model on the triangular lattice*, *Phys. Rev. B* **92**, 041105 (2015).
- [21] W.-J. Hu, S.-S. Gong, W. Zhu, and D. N. Sheng, *Competing spin-liquid states in the spin- $\frac{1}{2}$ Heisenberg model on the triangular lattice*, *Phys. Rev. B* **92**, 140403 (2015).
- [22] Y. Iqbal, W.-J. Hu, R. Thomale, D. Poilblanc, and F. Becca, *Spin liquid nature in the Heisenberg J_1 - J_2 triangular antiferromagnet*, *Phys. Rev. B* **93**, 144411 (2016).
- [23] S.-S. Gong, W. Zhu, J.-X. Zhu, D. N. Sheng, and K. Yang, *Global phase diagram and quantum spin liquids in a spin- $\frac{1}{2}$ triangular antiferromagnet*, *Phys. Rev. B* **96**, 075116 (2017).
- [24] S. N. Saadatmand and I. P. McCulloch, *Detection and characterization of symmetry-broken long-range orders in the spin- $\frac{1}{2}$ triangular Heisenberg model*, *Phys. Rev. B* **96**, 075117 (2017).
- [25] S. Hu, W. Zhu, S. Eggert, and Y.-C. He, *Dirac spin liquid on the spin- $1/2$ triangular Heisenberg antiferromagnet*, *Phys. Rev. Lett.* **123**, 207203 (2019).
- [26] F. Ferrari, J. Willsher, U. F. P. Seifert, R. Valentí, and J. Knolle, *Stability of algebraic spin liquids coupled to quantum phonons*, *Phys. Rev. Res.* **7**, L042053 (2025).
- [27] F. Ferrari and F. Becca, *Dynamical structure factor of the J_1 - J_2 Heisenberg model on the triangular lattice: Magnons, spinons, and gauge fields*, *Phys. Rev. X* **9**, 031026 (2019).
- [28] N. E. Sherman, M. Dupont, and J. E. Moore, *Spectral function of the J_1 - J_2 Heisenberg model on the triangular lattice*, *Phys. Rev. B* **107**, 165146 (2023).
- [29] M. Drescher, L. Vanderstraeten, R. Moessner, and F. Pollmann, *Dynamical signatures of symmetry-broken and liquid phases in an $S = \frac{1}{2}$ Heisenberg antiferromagnet on the triangular lattice*, *Phys. Rev. B* **108**, L220401 (2023).
- [30] J. Willsher and J. Knolle, *Dynamics and stability of $U(1)$ spin liquids beyond mean-field theory: Triangular-lattice J_1 - J_2 Heisenberg model*, *arXiv:2503.13831*.
- [31] Y. Ran, W.-H. Ko, P. A. Lee, and X.-G. Wen, *Spontaneous spin ordering of a Dirac spin liquid in a magnetic field*, *Phys. Rev. Lett.* **102**, 047205 (2009).
- [32] T. T. Dumitrescu, P. Niro, and R. Thorngren, *Symmetry breaking from monopole condensation in QED₃*, *arXiv:2410.05366*.
- [33] X. Y. Xu, Y. Qi, L. Zhang, F. F. Assaad, C. Xu, and Z. Y. Meng, *Monte Carlo study of lattice compact quantum electrodynamics with fermionic matter: The parent state of quantum phases*, *Phys. Rev. X* **9**, 021022 (2019).
- [34] W. Wang, D.-C. Lu, X. Y. Xu, Y.-Z. You, and Z. Y. Meng, *Dynamics of compact quantum electrodynamics at large fermion flavor*, *Phys. Rev. B* **100**, 085123 (2019).
- [35] L. Janssen, W. Wang, M. M. Scherer, Z. Y. Meng, and X. Y. Xu, *Confinement transition in the QED₃-Gross-Neveu-XY universality class*, *Phys. Rev. B* **101**, 235118 (2020).
- [36] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/yb9y-3t91> for details of the lattice mean-field and continuum field-theory calculations, DQMC and SAC methods, and supplementary results, which includes Ref. [37].
- [37] I. S. Gradshteyn, I. M. Ryzhik, D. Zwillinger, and V. Moll, *Table of Integrals, Series, and Products*, 8th ed. (Academic Press, Amsterdam, 2015).
- [38] G. Pan and Z. Y. Meng, *The sign problem in quantum Monte Carlo simulations*, in *Encyclopedia of Condensed Matter Physics (Second Edition)*, edited by T. Chakraborty (Academic Press, Oxford, 2024), pp. 879–893, [10.1016/B978-0-323-90800-9.00095-0](https://doi.org/10.1016/B978-0-323-90800-9.00095-0).
- [39] R. Blankenbecler, D. Scalapino, and R. Sugar, *Monte Carlo calculations of coupled boson-fermion systems. I*, *Phys. Rev. D* **24**, 2278 (1981).
- [40] F. F. Assaad and H. G. Evertz, *World-line and determinantal quantum Monte Carlo methods for spins, phonons and electrons*, in *Computational Many-Particle Physics, Lecture Notes in Physics*, edited by H. Fehske, R. Schneider, and A. Weiße (Springer, Berlin, Heidelberg, 2008), Vol. 739, pp. 277–356, [10.1007/978-3-540-74686-7_10](https://doi.org/10.1007/978-3-540-74686-7_10).
- [41] X. Y. Xu, Z. Hong Liu, G. Pan, Y. Qi, K. Sun, and Z. Y. Meng, *Revealing fermionic quantum criticality from new Monte Carlo techniques*, *J. Phys. Condens. Matter* **31**, 463001 (2019).
- [42] F. F. Assaad, M. Bercx, F. Goth, A. Götz, J. S. Hofmann, E. Huffman, Z. Liu, F. P. Toldin, J. S. E. Portela, and J. Schwab, *The ALF (Algorithms for Lattice Fermions) project release 2.0. Documentation for the auxiliary-field quantum Monte Carlo code*, *SciPost Phys. Codebases* **1** (2022).
- [43] W. Jiang, Y. Liu, A. Klein, Y. Wang, K. Sun, A. V. Chubukov, and Z. Y. Meng, *Monte Carlo study of the pseudogap and superconductivity emerging from quantum magnetic fluctuations*, *Nat. Commun.* **13**, 2655 (2022).
- [44] E. Fradkin, *Field Theories of Condensed Matter Physics*, 2nd ed. (Cambridge University Press, Cambridge, 2013).

- [45] A. Zee, *Quantum Field Theory in a Nutshell*, 2nd ed. (Princeton University Press, Princeton, NJ, 2010).
- [46] D. R. Hofstadter, *Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields*, *Phys. Rev. B* **14**, 2239 (1976).
- [47] L. Balents and O. A. Starykh, *Collective spinon spin wave in a magnetized U(1) spin liquid*, *Phys. Rev. B* **101**, 020401 (2020).
- [48] M. Agarwal, O. A. Starykh, D. A. Pesin, and E. G. Mishchenko, *Collective spin oscillations in a magnetized graphene sheet*, *Phys. Rev. B* **109**, 224302 (2024).
- [49] C. Chen, Y. Da Liao, C. Zhou, G. Pan, Z. Y. Meng, and Y. Qi, *Universal collective Larmor-Silin mode emerging in magnetized correlated Dirac fermions*, *Phys. Rev. B* **110**, L121112 (2024).
- [50] A. W. Sandvik, *Stochastic method for analytic continuation of quantum Monte Carlo data*, *Phys. Rev. B* **57**, 10287 (1998).
- [51] K. S. D. Beach, *Identifying the maximum entropy method as a special limit of stochastic analytic continuation*, *arXiv: cond-mat/0403055*.
- [52] O. F. Syljuåsen, *Using the average spectrum method to extract dynamics from quantum Monte Carlo simulations*, *Phys. Rev. B* **78**, 174429 (2008).
- [53] A. W. Sandvik, *Constrained sampling method for analytic continuation*, *Phys. Rev. E* **94**, 063308 (2016).
- [54] H. Shao, Y. Q. Qin, S. Capponi, S. Chesi, Z. Y. Meng, and A. W. Sandvik, *Nearly deconfined spinon excitations in the square-lattice spin-1/2 Heisenberg antiferromagnet*, *Phys. Rev. X* **7**, 041072 (2017).
- [55] C. Zhou, Z. Yan, H.-Q. Wu, K. Sun, O. A. Starykh, and Z. Y. Meng, *Amplitude mode in quantum magnets via dimensional crossover*, *Phys. Rev. Lett.* **126**, 227201 (2021).
- [56] G.-Y. Sun, Y.-C. Wang, C. Fang, Y. Qi, M. Cheng, and Z. Y. Meng, *Dynamical signature of symmetry fractionalization in frustrated magnets*, *Phys. Rev. Lett.* **121**, 077201 (2018).
- [57] C.-J. Huang, Y. Deng, Y. Wan, and Z. Y. Meng, *Dynamics of topological excitations in a model quantum spin ice*, *Phys. Rev. Lett.* **120**, 167202 (2018).
- [58] Y.-C. Wang, M. Cheng, W. Witczak-Krempa, and Z. Y. Meng, *Fractionalized conductivity and emergent self-duality near topological phase transitions*, *Nat. Commun.* **12**, 5347 (2021).
- [59] N. Ma, G.-Y. Sun, Y.-Z. You, C. Xu, A. Vishwanath, A. W. Sandvik, and Z. Y. Meng, *Dynamical signature of fractionalization at a deconfined quantum critical point*, *Phys. Rev. B* **98**, 174421 (2018).
- [60] C. Huang, N. Parthenios, M. Ulybyshev, X. Zhang, F. F. Assaad, L. Classen, and Z. Y. Meng, *Angle-tuned Gross-Neveu quantum criticality in twisted bilayer graphene*, *Nat. Commun.* **16**, 7176 (2025).
- [61] O. F. Syljuåsen, *Numerical evidence for unstable magnons at high fields in the Heisenberg antiferromagnet on the square lattice*, *Phys. Rev. B* **78**, 180413 (2008).
- [62] A. Lüscher and A. M. Läuchli, *Exact diagonalization study of the antiferromagnetic spin-1/2 Heisenberg model on the square lattice in a magnetic field*, *Phys. Rev. B* **79**, 195102 (2009).
- [63] M. Mourigal, M. E. Zhitomirsky, and A. L. Chernyshev, *Field-induced decay dynamics in square-lattice antiferromagnets*, *Phys. Rev. B* **82**, 144402 (2010).
- [64] K. Feng, C. Chen, and Z. Y. Meng, *Scalable hybrid quantum Monte Carlo simulation of U(1) gauge field coupled to fermions on GPU*, *SciPost Phys.* **20**, 060 (2026).
- [65] W. O. Wang, U. F. P. Seifert, O. A. Starykh, and L. Balents, *Chirality and quasi-long-range order in finite-flux Gutzwiller states for magnetized frustrated magnets*, *Phys. Rev. Lett.* **137**, 076701 (2026).
- [66] R. Bag, S. Xu, N. E. Sherman, L. Yadav, A. I. Kolesnikov, A. A. Podlesnyak, E. S. Choi, I. da Silva, J. E. Moore, and S. Haravifard, *Evidence of Dirac quantum spin liquid in YbZn₂GaO₅*, *Phys. Rev. Lett.* **133**, 266703 (2024).
- [67] H. C. H. Wu, F. L. Pratt, B. M. Huddart, D. Chatterjee, P. A. Goddard, J. Singleton, D. Prabhakaran, and S. J. Blundell, *Spin dynamics in the Dirac U(1) spin liquid YbZn₂GaO₅*, *Phys. Rev. Lett.* **135**, 046704 (2025).
- [68] A. O. Scheie *et al.*, *Proximate spin liquid and fractionalization in the triangular antiferromagnet KYbSe₂*, *Nat. Phys.* **20**, 74 (2024).
- [69] A. O. Scheie, M. Lee, K. Wang, P. Laurell, E. S. Choi, D. Pajerowski, Q. Zhang, J. Ma, H. D. Zhou, S. Lee, C. Huan, S. M. Thomas, M. O. Ajeesh, P. F. S. Rosa, A. Chen, V. S. Zapf, M. Heyl, C. D. Batista, E. Dagotto, J. E. Moore, and D. A. Tennant, *Spectrum and low-temperature bulk properties of triangular quantum spin liquid candidate NaYbSe₂*, *arXiv:2406.17773*.
- [70] Z. Zeng, C. Zhou, H. Zhou, L. Han, R. Chi, K. Li, M. Kofu, K. Nakajima, Y. Wei, W. Zhang, D. G. Mazzone, Z. Y. Meng, and S. Li, *Spectral evidence for Dirac spinons in a kagome lattice antiferromagnet*, *Nat. Phys.* **20**, 1097 (2024).
- [71] L. Han, Z. Zeng, M. Long, M. Song, C. Zhou, B. Liu, M. Kofu, K. Nakajima, P. Steffens, A. Hiess, Z. Y. Meng, Y. Su, and S. Li, *Spin excitations arising from anisotropic Dirac spinons in YCu₃(OD)₆Br₂[Br_{0.33}(OD)_{0.67}]*, *Phys. Rev. B* **112**, 045114 (2025).
- [72] Z. Y. Meng, C. D. Batista, and S. Li, *An integrated theoretical and numerical approach to understand modern experiments on quantum magnetism*, *Nat. Phys.* **22**, 1189 (2026).
- [73] G. Zheng, Y. Zhu, K.-W. Chen, B. Kang, D. Zhang, K. Jenkins, A. Chan, Z. Zeng, A. Xu, O. A. Valenzuela, J. Blawat, J. Singleton, S. Li, P. A. Lee, and L. Li, *Unconventional magnetic oscillations in a kagome Mott insulator*, *Proc. Natl. Acad. Sci. U.S.A.* **122**, e2421390122 (2025).
- [74] I. S. Villadiego, *Pseudoscalar U(1) spin liquids in α -RuCl₃*, *Phys. Rev. B* **104**, 195149 (2021).
- [75] X.-Y. Song, H. Goldman, and L. Fu, *Emergent QED₃ from half-filled flat Chern bands*, *Phys. Rev. B* **108**, 205123 (2023).
- [76] F. Paoletti, D. Guerzi, G. Sangiovanni, U. F. P. Seifert, and E. J. König, *Topologically enabled superconductivity: Possible implications for rhombohedral graphene*, *arXiv:2504.13166*.
- [77] T. Wang, X.-Y. Song, M. P. Zaletel, and T. Senthil, *Emergent QED₃ at the bosonic Laughlin state to superfluid transition*, *arXiv:2507.07611*.
- [78] T. T. Dumitrescu and J. Maldacena, *Comments on QED₃ in a magnetic field*, *arXiv:2508.03532*.

-
- [79] A. Keselman, X. Xu, H. Zhang, C. D. Batista, and O. A. Starykh, *Magnetization plateaux as a roadmap to quantum spin liquids*, *Phys. Rev. Lett.* **137**, 116702 (2026).
- [80] T. Bader, S. Feng, S. Budaraju, F. Becca, J. Knolle, and F. Pollmann, *Triangular J_1 - J_2 Heisenberg antiferromagnet in a magnetic field*, *Phys. Rev. B* **113**, L241114 (2026).
- [81] C. Chen, U. F. P. Seifert, K. Feng, O. A. Starykh, L. Balents, and Z. Y. Meng, *Numerical data for emergent gauge flux in mixed QED₃ with flavor chemical potential*, [Dataset], GitHub repository, https://github.com/baucd/mixed_qed3_cf_data (2026).