

Rotations, Negative Eigenvalues, and Newton Method in Tensor Network Renormalization Group

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In the tensor network approach to statistical physics, properties of the critical point of a 2D lattice model are encoded by a four-legged tensor which is a fixed point of a renormalization group (RG) map. The traditional way to find the fixed-point tensor consists in iterating the RG map after having tuned the temperature to criticality. Here, we develop a different and more direct technique, which solves the fixed-point equation via the Newton method. This is challenging due to the existence of marginal deformations—linear transformations of the coordinate frame, which parametrize a two-dimensional family of fixed points. We address this challenge by including a 90° rotation into the RG map. This flips the sign of the problematic marginal eigenvalues, rendering the fixed point isolated and accessible via the Newton method. We demonstrate the power of this technique via explicit computations for the 2D Ising and three-state Potts models. Using the Gilt-TNR algorithm at bond dimension $\chi = 30$, we find the fixed-point tensors with 10^{-9} accuracy, much higher than what was previously achieved.

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I. INTRODUCTION

In the past 15 years, the use of tensor networks [1] revolutionized renormalization group (RG) studies of two-dimensional statistical lattice models. Tensor RG [2] maps a tensor network representation of the partition function to an equivalent representation involving a smaller number of tensors. By now, there exist many tensor RG algorithms [1,3–11]. They show excellent numerical performance when benchmarked on exactly solvable two-dimensional models such as Ising or three-state Potts, truncating to tensors of finite size. In three dimensions, there is currently no fully successful numerical algorithm [6,11–14]. The analytic theory of tensor RG for infinite-size tensors was initiated in Refs. [16,17] in two dimensions and in Ref. [18] in three dimensions.

Here, we are dealing with tensor RG for two-dimensional models. Our main result is a modification of the existing algorithms which allows one to recover the critical fixed-point tensor in a novel and more efficient way.

Traditionally, one obtains approximate fixed-point tensors by starting from the lattice model at or nearby the exactly known critical temperature and performing a series of 10–20 RG steps, referred below as the “shooting” method. The resulting tensor is close to the fixed point but it is not quite the fixed point, because the irrelevant perturbations have not fully decayed to zero. In this work, we are able to instead obtain the fixed point via a Newton method. The Newton method converges provided that the Jacobian has no eigenvalues 1. Naively, there are always perturbations with eigenvalues 1 in our problem. They are associated with aspect-ratio deformations, which break the rotation invariance of the model while keeping criticality. Our main idea is to change the tensor RG map by including in it a $\pi/2$ rotation of the lattice, which changes the problematic eigenvalues 1 to -1 and allows the Newton method to converge.

Our Newton method setup is general and would apply to any fixed point without marginal scalar deformations. (A notable such exception is the compactified scalar boson theory, arising in the study of the Berezinskii-Kosterlitz-Thouless phase transition.) In this work, we use it to recover the critical isotropic tensor RG fixed points of 2D Ising and three-state Potts models. We achieve accuracy of approximately 10^{-9} in the Hilbert-Schmidt distance, much better than what has been done via the shooting method (see Table I) and limited only by roundoff and truncation errors. The key point, however, is not merely the improvement in numerical accuracy, but that the fixed point can be

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TABLE I. Accuracy of 2D Ising and 3-state Potts fixed-point (FP) determination by the Newton method, compared to previous work using the shooting method. Ising results of Refs. [7,21,22] are reviewed in Sec. III D. There is no prior work quantifying the FP accuracy for Potts.

Algorithm	Model	χ	Ref.	Method	FP accuracy
TNR	Ising	42	[7]	Shooting	10^{-5}
Gilt-HOTRG	Ising	30	[21]	Shooting	10^{-2}
Gilt-HOTRG	Ising	24	[22]	Shooting	3×10^{-4}
Gilt-TNR	Ising	30	This work, Sec. III B	Shooting	5×10^{-5}
Rotating Gilt-TNR	Ising	30	This work, Sec. IV B	Newton	10^{-9}
Rotating Gilt-TNR	Potts	30	This work, Appendix F	Newton	10^{-9}

thus obtained by directly solving the fixed-point equation, something which has not been previously appreciated [19].

Our second result is to validate the method, using the fixed-point tensor to extract physical observables—scaling dimensions of the operators of conformal field theory (CFT) describing the phase transition and comparing with the exact solution. Most tensor RG studies extract the scaling dimensions from the transfer matrix [4] or from the lattice dilatation operator [23,24]. Here, we focus on an alternative technique—using the eigenvalues of the tensor RG map linearized around the fixed point (the Jacobian). This is a standard and general RG technique [25,26], particularly natural in the context of our study. In tensor RG, it has been previously used only a handful of times [21,22]. (The transfer matrix and the lattice dilatation operator will be studied in the companion paper [27].) With the error in fixed-point tensor determination essentially eliminated by the Newton method, the bond dimension truncation error will dominate the error in determination of physical quantities. The main purpose of this paper being conceptual, we are working at a relatively modest bond dimension $\chi = 30$ using a specific tensor RG algorithm Gilt-TNR [11]. We have not tried to compete in accuracy of physical quantities with other tensor RG studies using a higher bond dimension, where the fixed point was approached via the shooting method [7,8,11]. This is postponed to the future.

We encounter two subtle effects when using the Jacobian method.

- (i) The standard relation $\lambda = b^{d-\Delta}$ between the Jacobian eigenvalues and the scaling dimensions, where $b > 1$ is the lattice rescaling factor, are modified by a phase, depending on the spin of the CFT operator under spatial rotations. This extra phase arises due to the $\pi/2$ rotation present in our map. Thus, some eigenvalues change sign and some even become complex.
- (ii) We see that the Jacobian eigenvalues of total derivative operators are not universal and not given by the above formula. This is expected from the CFT point of view, but it may be the first time that this is

seen in a numerical study. We view the observation of this effect as our third main result. We provide a reason why Refs. [21,22] did not see this.

The paper is organized as follows. In the main text, we focus on the 2D Ising model. The three-state Potts case is completely analogous and is treated in Appendix F. We start in Sec. II by describing in general terms our problem, possible stumbling blocks, and how we plan to resolve them. In Sec. III, we illustrate the intuition developed in Sec. II via concrete calculations using Gilt-TNR [11]. This map does not include a rotation; thus, the Jacobian has eigenvalues 1 at the fixed point, and the Newton method is not applicable. We locate the approximate fixed point using the traditional “shooting” method. We extract scaling dimensions of leading CFT quasiprimaries from the Jacobian eigenvalues at the approximate fixed point. We check that the nonquasiprimary Jacobian eigenvalues are nonuniversal, i.e., depend on the RG map. Finally, we present evidence for the existence of a manifold of anisotropic fixed points. We then proceed to describe our main new result in Sec. IV, where we add rotation to the Gilt-TNR map, which turns eigenvalues 1 to -1 . We set up the Newton method and show that it converges. For anisotropic tensors, we exhibit period-2 oscillations of the RG iterates. In Sec. V, we conclude and suggest future research directions.

Our paper contains numerous detailed appendixes which should be useful to anyone wishing to reproduce or improve our results. In Appendix A, we set up the basic conventions concerning tensor RG and symmetries of tensor networks. In Appendix B, we discuss how we initialize our tensor RG flow starting from the nearest-neighbor 2D Ising model, including the anisotropic case. Appendix C provides a detailed description of the Gilt-TNR algorithm [11] used here and of the Gilt-HOTRG algorithm used in Refs. [21,22]. Appendix D describes our continuous and discrete gauge-fixing procedures. In Appendix E, we provide evidence that the Gilt-TNR map is differentiable near the fixed point and optimize the step size used to extract the Jacobian via a finite difference approximation.

Much of our work here is numerical and at an intuitive level of rigor, but we hope that our ideas will also prove useful in the eventual fully rigorous computer-assisted construction of exact tensor network RG fixed points—the program started in Ref. [16].

II. INTUITION FOR RG EIGENVALUES

We start with general remarks about tensor RG as applied to the 2D Ising model. We then explain how CFT helps classify RG eigenvalues. Finally, we discuss when the Newton method can and cannot be used to search for the critical RG fixed point. We see that the Newton method is problematic for nonrotating RG maps which do not preserve rotation symmetry, and we explain how this problem is solved by adding rotation to the RG map.

A. Tensor RG for the 2D Ising

We would like to use tensor RG to study phase transitions and critical points of 2D lattice models. For concreteness, we consider the isotropic 2D Ising model with nearest-neighbor ferromagnetic interactions, on the square lattice. (See Appendix F for the three-state Potts model.) The model is defined by its partition function:

$$Z = \sum_{\sigma_{x,y}=\pm 1} e^{(1/T) \sum_{x,y} (\sigma_{x,y} \sigma_{x+1,y} + \sigma_{x,y} \sigma_{x,y+1})}. \quad (2.1)$$

It has a phase transition at $T = T_c = \{2/[\ln(1 + \sqrt{2})]\}$ separating the ordered and disordered phases. The lattice model is exactly solvable, but the exact solution will not be used here. As a preparation to using tensor RG, we translate the partition function of the model to a tensor network made of a four-legged tensor $A = A_{\text{NN}}(T)$. While there are several ways to do so, probably the simplest way [4] is to rotate the lattice of Ising spins by $\pi/4$ and to define the tensor:

$$\begin{array}{c} \sigma_2 \\ | \\ \textcircled{A} \\ | \\ \sigma_4 \\ \text{---} \sigma_1 \text{---} \sigma_3 \end{array} = e^{\frac{1}{T}(\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_4 + \sigma_4 \sigma_1)}, \quad (2.2)$$

where every index can take values ± 1 , so the bond dimension is 2. The tensor network contraction made out of this tensor A (called tensor network partition function),

$$\begin{array}{c} \dots \\ | \\ \textcircled{A} \text{---} \textcircled{A} \text{---} \textcircled{A} \text{---} \dots \\ | \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ | \\ \textcircled{A} \text{---} \textcircled{A} \text{---} \textcircled{A} \text{---} \dots \\ | \\ \dots \end{array}, \quad (2.3)$$

is then identical to the spin model partition function (2.1); see Appendix B. We are using periodic boundary conditions on the tensor network side. This corresponds to nonstandard boundary conditions on the spin model side. This is not a problem, since our main interest is anyway in the infinite volume limit.

We now briefly explain the main idea of tensor RG (see Ref. [28] and Appendix A. 1). A tensor RG map takes the tensor network contraction (2.3) made of tensor A and finds another tensor $A' = R(A)$ so that the partition function is the same but the size of the network is reduced by some constant integer b (called lattice rescaling factor). For example, a square $N \times N$ network made of A is mapped to a $(N/b) \times (N/b)$ network made of A' with the same periodic boundary conditions and the same value of the partition function. It is convenient to rescale A so that it has unit Hilbert-Schmidt norm:

$$\|A\| = \left(\sum |A_{ijkl}|^2 \right)^{1/2} = 1, \quad (2.4)$$

and similarly A' . Then, the partition function remains the same up to a rescaling.

Furthermore, we assume for now that the map R is nonrotating; i.e., it preserves the orientation of the lattice. This means that if we apply the map to a rectangular tensor network of size $N \times M$, then, after the RG step, the lattice has size $N/b \times M/b$. (While this is very natural, in Sec. II. 5, we find it useful to change this.)

By now, there exist many tensor RG algorithms, which typically involve two steps: (i) disentangling [7], which cleans the network by removing ultrashort-range correlations known as corner-double-line (CDL) tensors [29], and (ii) coarse graining, which reduces the size of the network. In the early days of tensor RG, only coarse graining was used [1,3–6], but state-of-the-art algorithms use both disentangling and coarse graining [7–11]. We rely on the Gilt-TNR algorithm [11], reviewed in Sec. III. 1. For the moment, let us keep the discussion general, applicable to any nonrotating RG map R with disentangling.

The RG flow is initialized by $A^{(0)} = A_{\text{NN}}(T)$ and generated by applying the RG map repeatedly:

$$A^{(n+1)} = R(A^{(n)}), \quad n = 0, 1, 2, \dots \quad (2.5)$$

To preserve the partition function exactly, the bond dimension must be allowed to grow. Tensor RG algorithms truncate the tensor $A^{(n+1)}$ in such a way that the bond dimension never exceeds a finite number χ , a parameter of the algorithm. One would like to take χ as large as possible, within available computational resources. In this paper, we are content to use $\chi = 30$.

It is expected that after many steps this RG evolution will converge to a fixed point, which is a tensor A_{fp} satisfying $R(A_{\text{fp}}) = A_{\text{fp}}$. In fact, we expect three fixed-point tensors:

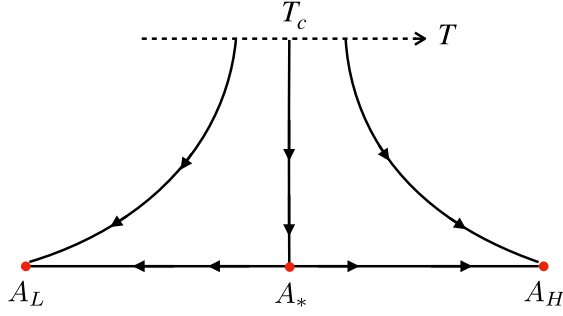


FIG. 1. Expected pattern of tensor RG evolution for the 2D Ising model. The dashed line is the curve of initial tensors corresponding to varying T in $A^{(0)}(T)$.

high-temperature A_H , low-temperature A_L , and critical A_* . The RG evolution starting from $A^{(0)} = A^{(0)}(T)$ should converge to A_H for $T > T_c$, to A_L for $T < T_c$, and to A_* for $T = T_c$ (see Fig. 1). There is much numerical evidence that these expectations can be realized for a tensor RG map with disentangling [7–11].

Tensors A_H and A_L in Fig. 1 are simple explicitly known tensors of bond dimension 1 and 2, respectively. Namely, A_H has a single nonzero tensor element

$$(A_H)_{0000} = 1, \quad (2.6)$$

and tensor A_L is a direct sum of two copies of A_H . See Refs. [16,17] for a rigorous theory of tensor RG in the neighborhood of these two fixed points. The quest of much of the literature, and our focus here, is the much more interesting and complicated critical fixed-point tensor A_* , which is expected to have infinite bond dimension. To give meaning to infinite sums in the tensor network built out of A_* , it is natural to require that it be Hilbert-Schmidt, i.e., have a finite Hilbert-Schmidt norm [30]. A_* is expected to depend on the RG map, although the critical exponents extracted from A_* should be universal. An explicit expression for A_* is unknown, and we would find it surprising if it existed [31]. At present, progress is possible by working at finite χ and computing numerical finite-dimensional approximations to A_* . In the future, one may be able to prove that an exact fixed-point tensor exists in a small neighborhood of numerical approximations, but this is beyond the scope of this work.

According to Fig. 1, to get A_* approximately, one can take the initial tensor $A^{(0)}$ corresponding to $T = T_c$ and do several RG iterations [36]. That is the method used in the prior literature [7–11,21,22], which we refer to as “shooting.” (Often, the fixed-point tensor itself is not exhibited and evidence for reaching the fixed point is presented in the form of gauge-invariant observables reaching limits.) The shooting method is limited due to the fact that the RG map is not a contraction in the neighborhood of A_* . Indeed, the temperature perturbation [38] is relevant, with eigenvalue

$\lambda_t = b^{y_t} > 1$, where $y_t = 1$ (below we recall how this follows from CFT). The number of iterations one can perform is limited by numerical instabilities, and by the residual dependence of T_c on χ . These limitations motivate our main goal—an alternative way to get A_* via the Newton method.

B. CFT expectations for RG eigenvalues

The Newton method will involve the Jacobian ∇R of the RG map. To set things up, we need to have an idea about the eigenvalues and eigenvectors of ∇R . This may be understood with the help of the 2D CFT which describes the critical 2D Ising model [39]. This CFT was exactly solved long ago [41]; in particular, the spectrum of local operators is exactly known. We stress, however, that even rough approximate knowledge of the spectrum would suffice for the discussion below.

Since both the RG fixed point and the CFT describe the same critical state, we expect a broad correspondence between these two descriptions. In particular, perturbations of the RG fixed point A_* should be in one-to-one correspondence with the local translationally invariant perturbations of the CFT, given by integrals of nontrivial (i.e., excluding the unit operator) [42] CFT local operators:

$$\kappa \int d^2x \mathcal{O}(x). \quad (2.7)$$

On the CFT side, the RG transformation corresponds to rescaling distances $x \rightarrow bx$. When we do this, the coupling κ rescales as $\kappa \rightarrow \lambda\kappa$, where

$$\lambda = b^{2-\Delta_{\mathcal{O}}} \quad (2.8)$$

and $\Delta_{\mathcal{O}}$ is the scaling dimension of $\mathcal{O}$. Note that this equation is significant only for operators which are not total derivatives. For $\mathcal{O}$ a total derivative, perturbation Eq. (2.7) vanishes. Hence, the normalization of κ is ambiguous and λ can be anything [43].

In summary, CFT predicts that the RG Jacobian ∇R at the fixed point A_* should have a series of eigenvectors in one-to-one correspondence with nontrivial local CFT operators $\mathcal{O}$. Eigenvalues of perturbations associated with $\mathcal{O}$ which are not total derivatives (such operators are called quasiprimaries) are given by Eq. (2.8). Eigenvalues of total derivative perturbations cannot be predicted by general principles; in particular, they depend on the choice of the RG map.

Quasiprimaries of the Ising CFT are classified as follows [44]. Recall that in the space of local operators we have action of the Virasoro algebra, whose generators are denoted by $L_n, \bar{L}_n$, $n \in \mathbb{Z}$, and of its finite-dimensional subalgebra of global conformal transformations, generated by $L_n, \bar{L}_n$, $n = 0, \pm 1$. Any local operator is characterized by its holomorphic and antiholomorphic weights h and $\bar{h}$

TABLE II. 2D Ising CFT Virasoro primary and quasiprimary operators of scaling dimension $\Delta \leq 5$, with their quantum numbers (see, e.g., Table 8.1 in Ref. [44]). The $\cdots$ stands for a product of lower Virasoro generators of the same total weight as the shown leading terms.

	$\mathbb{Z}_2$	Δ	ℓ
$\mathbb{1}$	+	0	0
$T, \bar{T}$	+	2	± 2
$T\bar{T}$	+	4	0
$TT, \bar{T}\bar{T}$	+	4	± 4
ϵ	+	1	0
$(L_{-4} + \cdots)\epsilon$	+	5	4
σ	-	0.125	0
$(L_{-3} + \cdots)\sigma$	-	3.125	3

defined through equations $L_0\mathcal{O}(0) = h\mathcal{O}(0)$ and $\bar{L}_0\mathcal{O}(0) = \bar{h}\mathcal{O}(0)$, respectively. Linear combinations $L_0 \pm \bar{L}_0$ generating dilatations and rotations, the linear combinations of weights $\Delta = h + \bar{h}$ and $\ell = h - \bar{h}$ give the scaling dimension and spin of a local operator.

One first classifies the Virasoro primary operators $\mathcal{O}(x)$, which are singled out algebraically by the condition $L_n\mathcal{O}(0) = \bar{L}_n\mathcal{O}(0) = 0$, $n \geq 1$, i.e., are annihilated by all positive Virasoro generators. The Ising CFT has three such Virasoro primary operators $\mathbb{1}$, ϵ , and σ ; see Table II. The quasiprimary operators satisfy a weaker constraint $L_n\mathcal{O}(0) = \bar{L}_n\mathcal{O}(0) = 0$, $n = 1$. In particular, all Virasoro primaries are also quasiprimaries. Taking a Virasoro primary and acting on it with sequences of negative Virasoro generators $L_{-n}, \bar{L}_{-n}$, $n < 0$, produces infinitely many additional operators called Virasoro descendants. Other quasiprimaries are appropriate linear combinations of Virasoro descendants which are annihilated by L_1 and $\bar{L}_1$. Note that Virasoro generators $L_{-n}, \bar{L}_{-n}$, $n \geq 2$, must necessarily be involved: Acting with just L_{-1} and $\bar{L}_{-1}$ would not produce a quasiprimary, since these generators generate translations, and applying them is akin to taking a total derivative.

The weights of quasiprimaries can be easily computed from the weights of primaries recalling that L_{-n} increase h by n and $\bar{L}_{-n}$ increases $\bar{h}$ by n . For example, the holomorphic stress-tensor component $T = L_{-2}\mathbb{1}$ has $h = 2$, $\bar{h} = 0$, and analogously the antiholomorphic component $\bar{T}$. These quasiprimaries are present in any CFT.

In Table II, we give the full list of quasiprimaries of scaling dimension $\Delta \leq 5$, along with spin and the global $\mathbb{Z}_2$ symmetry quantum number. It is interesting to note that many candidate quasiprimaries, such as, e.g., $L_{-2}\epsilon + \cdots$ or $L_{-3}\epsilon + \cdots$, do not appear in this table, while the first quasiprimary built on top of ϵ has the form $L_{-4}\epsilon + \cdots$. These missing local operators are referred to as “null descendants.” They happen to vanish

identically in the 2D Ising CFT—a fact closely related to its exact solvability.

Going back to the Jacobian of the RG map at the fixed point, perturbations corresponding to all the operators in Table II should appear among eigenvectors of ∇R , with universal eigenvalues. For example, the eigenvector corresponding to ϵ is the temperature perturbation mentioned above: $\lambda_\epsilon = b^{2-\Delta_\epsilon} = b$.

A second relevant eigenvector corresponds to σ . This eigenvector is, however, $\mathbb{Z}_2$ -odd, and, thus, its existence does not require extra tuning to reach the fixed point. Put another way, in Fig. 1, we were implicitly assuming that the RG map preserves the $\mathbb{Z}_2$ symmetry of the 2D Ising model. Thus, the whole RG evolution happens within the space of $\mathbb{Z}_2$ -even tensors, with all $\mathbb{Z}_2$ -odd perturbations set to zero.

Let us discuss the interplay of tensor RG and symmetry in more detail.

C. Symmetry preserved by the RG map

The 2D Ising model has a global on-site $\mathbb{Z}_2$ (spin flip) symmetry. It is also invariant under spatial symmetry transformations, which in the isotropic case form the group D_4 generated by the axis reflections and $\pi/2$ rotations [45]. Translating the model to a tensor network, one naturally obtains a tensor $A^{(0)}$ which is invariant under all these symmetries. See Appendix A.2 for a recap of symmetries of tensor networks.

Suppose we have a tensor RG map R which preserves symmetry G , which means that the subspace $\mathcal{T}^G$ of G -invariant tensors is invariant under R . Then, if the initial tensor $A^{(0)}$ is G invariant, the whole RG evolution (2.5) happens within the G -invariant subspace $\mathcal{T}^G$. We can also look for a fixed point A_* within $\mathcal{T}^G$. This is convenient for several reasons: first, because it implies a reduction in the number of variables to consider and, second, because we do not have to worry about relevant or marginal directions orthogonal to $\mathcal{T}^G$.

Preserving on-site symmetries such as the Ising spin flip is easy—all tensor RG maps naturally achieve this. This is reflected in Fig. 1, as mentioned at the end of Sec. II.2.

On the other hand, it turns out to be hard to implement disentangling while preserving rotations. Available tensor RG maps with disentangling preserve reflections along one axis [7], along both axes [10,46,47], or no spatial symmetry at all [8,9,11,48]. Gilt-TNR [11], which we use in practical calculations, belongs to the last group of algorithms.

For such RG maps, even though the initial tensor respects a spatial symmetry, we cannot impose it on subsequent tensors along the RG evolution. Also, the critical fixed-point tensor cannot be expected to have a spatial symmetry, and indeed our numerical calculations show that it does not. This may seem puzzling, because, of course, the critical point should enjoy the same spatial symmetry as the original tensor and, indeed, a larger one,

because it should be fully $O(2)$ rotationally symmetric. But there is no contradiction: This simply means that the spatial symmetry of the critical point is not manifest.

One may consider this an inconvenience but not necessarily a problem from the practical point of view. And indeed this did not impede the prior tensor RG work, which found the approximate critical fixed-point tensor via shooting from $A^{(0)}(T_c)$. However, for our goal of using the Newton method, this turns out to be a problem, as we now explain.

D. Rotation symmetry breaking and eigenvalues 1

According to Table II, the Jacobian ∇R at the fixed point has two eigenvectors which are marginal, i.e., have eigenvalues 1. These eigenvectors are associated with the CFT stress tensor holomorphic and antiholomorphic components T and $\bar{T}$. In the real stress-tensor component basis, we can associate these eigenvectors with the linear combinations

$$T_{11} - T_{22} \quad \text{and} \quad T_{12}, \quad (2.9)$$

conveniently aligned with the coordinate axes x_1 and x_2 , which we identify with the two axes of the tensor network. (Recall that the stress-tensor trace $T_{11} + T_{22}$ vanishes in a CFT.)

Consider deforming the CFT by the stress-tensor perturbation $\int d^2x h_{\mu\nu} T^{\mu\nu}$ with an x -independent $h_{\mu\nu}$. This deforms the 2D metric: $\delta_{\mu\nu} \rightarrow g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$. Physically, this deforms the rotational symmetry of the critical point, turning circles $|x| = 1$ into ellipses $|x|_g = 1$, where $|x|_g$ is the distance computed with respect to the $g_{\mu\nu}$ metric. For example, the two-point correlation functions in the deformed theory scale as powers of $|x|_g$. The first perturbation in Eq. (2.9) gives an ellipse whose axes are aligned with the coordinate axes, which breaks $\pi/2$ rotations but preserves axis reflections. The second perturbation gives a tilted ellipse and breaks both rotations and reflections. These deformations make sense for infinitesimal and for finite $h_{\mu\nu}$. We call the ellipse $|x|_g = 1$ the anisotropy ellipse. Because of scale invariance, the overall scale of the anisotropy ellipse does not play a role, so it is characterized by two parameters, e.g., the main axes ratio and the tilt angle (see Fig. 2).

Even though the above perturbations break rotations, they preserve criticality, since the correlation length remains infinite. We expect a two-dimensional manifold $\mathcal{M}_2$ of corresponding RG fixed points, in one-to-one correspondence with possible anisotropy ellipses. The critical fixed point A_* in Fig. 1 is but one point of $\mathcal{M}_2$. The complete picture looks like in Fig. 3. The discussed perturbations with eigenvalues 1 span the tangent space of $\mathcal{M}_2$.

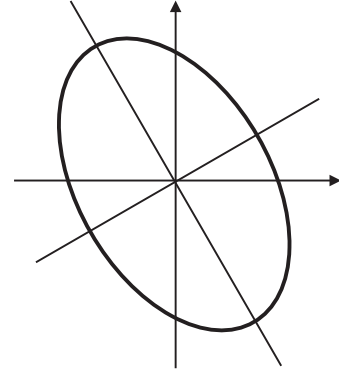


FIG. 2. Anisotropy ellipse parametrizing translationally invariant stress-tensor perturbations of the CFT.

Note that the anisotropy ellipse is a physical quantity which can be associated with any critical tensor, whether it is a fixed point or not. We should just compute two-point correlation functions at large distances and see for which metric g they scale as powers of $|x|_g$. Defined this way, the anisotropy ellipse is an RG invariant on the critical manifold for the kind of RG maps we are considering (nonrotating RG maps changing both lattice dimensions by the same factor b), and all points on the RG trajectory have the same anisotropy ellipse as the initial critical tensor. In the case at hand, the initial tensor represents the nearest-neighbor isotropic Ising model on the square lattice, for which the anisotropy ellipse is, of course, the circle. So, the same will be true for the fixed point A_* which we, therefore,

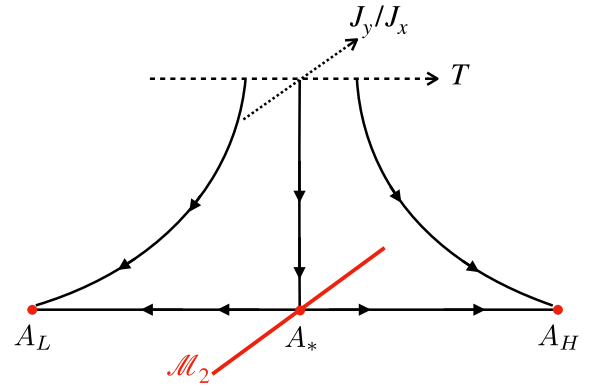


FIG. 3. This figure corrects Fig. 1 to show that we expect a two-dimensional manifold $\mathcal{M}_2$ of fixed-point tensors. (Since two of the three dimensions in the figure represent directions orthogonal to the fixed-point manifold, we had to represent $\mathcal{M}_2$ by a one-dimensional thick red line.) Microscopically, we may consider the anisotropic 2D Ising model with varying J_y/J_x . Tensor RG flows starting from the critical point of this model (dotted line) are attracted to a one-dimensional submanifold of $\mathcal{M}_2$. Adding the next-to-nearest-neighbor interaction, we may cover the whole $\mathcal{M}_2$.

call “isotropic fixed point.” The previous observation explains why RG iterations get attracted to a single fixed point rather than start wandering around the fixed-point manifold $\mathcal{M}_2$.

Now that we know that the fixed point A_* is isotropic, does this help us to find it? The answer would be yes if we had a tensor RG algorithm preserving lattice rotation symmetry. In that case, we could be sure that A_* belongs to a subspace of rotation-symmetric tensors. We could restrict RG evolution to this subspace, within which ∇R would be free of eigenvalues 1. It would have been trivial then to set up the Newton method. But, in the absence of manifest rotation symmetry, the situation is not so rosy. Even though A_* is isotropic, there is nothing manifestly special about A_* as a tensor which could help us to limit the search. We are forced to work in a larger space of all tensors, which contains the above perturbations, and, hence, ∇R will have eigenvalues 1 [49].

Now, it is well known that eigenvalues 1 present a problem for solving a fixed-point equation via the Newton method. At an intuitive level, the problem can be explained as follows. If the Newton method does work, we can rewrite the fixed-point equation in an equivalent form so that the map becomes a contraction in a neighborhood of the solution. A contraction has a unique fixed point. Thus, we conclude that the solution must be isolated. However, we have seen that the solution is not isolated—there is a manifold of fixed points. This contradiction shows that there must be an obstruction to setting up the Newton method. In Sec. II.6, we see the problem more clearly from a formal point of view. But first let us discuss how we will solve the problem and get rid of eigenvalues 1.

E. Adding a rotation to the tensor RG map

In the discussion so far, we assumed (Sec. II.1) that the tensor RG map R is nonrotating; i.e., it preserves the orientation of the underlying lattice. This was needed, e.g., when applying the prediction (2.8) for RG eigenvalues to perturbations associated with CFT operators with spin, like the stress tensor. We saw that such maps have features (eigenvalues 1 and a manifold of fixed points) which are problematic for our goal of using the Newton method. To solve this problem, we propose to consider rotating RG maps, which change the orientation. Concretely, we consider a tensor RG map R° , which is a composition of a nonrotating map R , followed by $\Gamma_{\pi/2}$, rotation of the tensor by $\pi/2$:

$$R^\circ = \Gamma_{\pi/2} R, \quad (2.10)$$

or, graphically,

$$R^\circ(A) = R(A). \quad (2.11)$$

So, the rotating map R° maps a tensor network of size $N \times M$ to one which has size $M/b \times N/b$; see Eq. (A3):

$$N \times M \xrightarrow{R^\circ} M/b \times N/b. \quad (2.12)$$

Adding a rotation to the RG map may appear strange, and indeed recognizing that this is useful is the main idea of our paper [50].

Starting from the isotropic 2D Ising tensor $A^{(0)}(T)$, the RG evolution of R° is expected to show the same pattern of fixed points as that of R in Fig. 1. In particular, the flow from $A^{(0)}(T_c)$ leads to a critical isotropic fixed-point tensor, which we denote A_*° . (Note that $A_*^\circ \neq A_*$, the fixed-point tensor of R , because R does not preserve rotations.) Jacobian eigenvalues around the fixed point A_*° can be computed using the CFT data, up to a change having to do with the rotation. On the CFT side, a spin ℓ operator $\mathcal{O}$ transforms under rotation by angle θ as

$$\mathcal{O} \mapsto e^{i\theta\ell} \mathcal{O}. \quad (2.13)$$

In our case, $\theta = \pi/2$. This implies that Eq. (2.8) for the Jacobian eigenvalues needs to be modified in the case of R° as follows:

$$\lambda = (i)^\ell b^{2-\Delta_\mathcal{O}}. \quad (2.14)$$

Applying this to the CFT stress tensor which has spin ± 2 , we conclude that the eigenvectors of ∇R° corresponding to the stress-tensor components T and $\bar{T}$ have eigenvalues -1 , and not 1 as for ∇R . Relatedly, when extended to anisotropic and not-rotationally invariant tensors, the map R° will not have a two-dimensional manifold of fixed points as R did (Fig. 3). Instead, the isotropic fixed point A_*° will be isolated, and its neighborhood will contain a two-parameter manifold of period-2 orbits of R° .

Eigenvalues-1 pose no problem for the Newton method, and we expect to use the Newton method to find the fixed point A_*° of R° . We stress that the fixed point A_*° of R° is as good as A_* of R for describing the critical point physics. It is only that A_*° can be found using the Newton method, while A_* cannot.

We remark that, in addition to changing the problematic stress-tensor eigenvalues from 1 to -1 , rotation is not expected to have any deleterious effect on other eigenvalues. Before applying rotation, eigenvalues of quasiprimaries were positive real numbers; after rotation, they acquire a spin-dependent phase while keeping the same absolute value. In particular, they become imaginary for odd spin and negative for spin $\ell = 2 \bmod 4$ and remain positive for spin $\ell = 0 \bmod 4$.

A second remark is that the proposed solution of enabling the Newton method by including rotation is not limited to the Ising model. In Appendix F, we show that it works also for the three-state Potts model. Generally, our solution will work as long as the fixed point does not have any scalar operators which are internal symmetry singlets and are marginal, i.e., have dimension 2. Such operators have eigenvalue 1, and, being scalars, their eigenvalue will remain 1 under any rotation [51]. Experience shows that models with marginal scalars are not so frequent, although special models may have them. One such family of models is compactified scalar boson, where marginal scalar perturbation corresponds to changing the compactification radius [44].

F. Newton method

Let us finally set up the Newton method. As mentioned, we are solving the fixed-point equation $R^\circ(A) = A$, which has an isolated solution at A_*° . We rewrite the equation as

$$f(A) = 0, \quad \text{where } f(A) = A - R^\circ(A). \quad (2.15)$$

The Newton method solves the latter equation via iterations [52]

$$A_{(m+1)} = g(A_{(m)}), \quad \text{where } g(A) = A - [J(A)]^{-1} f(A) \quad (2.16)$$

and

$$J(A) = I - \nabla R^\circ(A) \quad (2.17)$$

is the Jacobian of the map f . Note that, since the RG map preserves the subspace of $\mathbb{Z}_2$ -even tensors, here we need only the Jacobian restricted to this subspace [53].

As discussed, $\nabla R^\circ(A_*^\circ)$ does not have eigenvalues 1. This implies that $J(A_*^\circ)$ is invertible and $g(A)$ is well defined in a neighborhood of A_*° [54]. Furthermore, if we compute ∇g and evaluate it at A_*° using $f(A_*^\circ) = 0$, we find

$$\nabla g(A_*^\circ) = 0. \quad (2.18)$$

This implies, in particular, that $g(A)$ is a contraction near A_*° . Therefore, Newton iterations (2.16) converge for a sufficiently good initial approximation.

The Jacobian $J(A)$ entering the definition of $g(A)$ is somewhat expensive to evaluate and invert [55]. Several numerical strategies are available which provide a

sufficiently good approximate inverse Jacobian, whose discussion is postponed to Sec. IV.1 below.

III. RESULTS FOR THE NONROTATING GILT-TNR

We now proceed to concrete tensor RG computations, based on the Gilt-TNR algorithm. We start in this section with several in-depth studies using the original Gilt-TNR algorithm, which is nonrotating. We, thus, are able to compare with the prior work from Refs. [11,21,22] and to test various intuitions about tensor RG from Sec. II, such as the existence of marginal eigenvalues corresponding to the stress-tensor deformations and of a manifold of fixed points corresponding to different anisotropy parameters. We also test the Jacobian method for extracting CFT scaling dimensions. After this thorough warm-up, in the next section, we switch to the rotating Gilt-TNR algorithm and present our main numerical result: Find the fixed point via the Newton method.

Remark 3.1. Before we start, the following comment is in order. In the previous section, when we discussed the conceptual picture, the fixed points A_* or A_*° denoted exact fixed points of the exact nonrotating or rotating RG map. From now on, we are discussing numerical RG maps operating at a finite bond dimension χ . The logic of the previous section still applies to some approximation, but, of course, the fixed points have finite bond dimension and are somewhat different from A_* and A_*° . To stress this, we denote the fixed points as $A_*^{[\chi]}$ and $A_*^{\circ[\chi]}$. Our goal here is to extract these fixed points as precisely as possible, for a finite χ . What happens with $A_*^{[\chi]}$ and $A_*^{\circ[\chi]}$ as $\chi \rightarrow \infty$, whether they converge to A_* or A_*° , is a separate and very interesting question which is beyond the scope of this work. See, however, comments at the end of Sec. III B.

A. Review of the nonrotating Gilt-TNR algorithm

The original, nonrotating, Gilt-TNR RG map [11] is defined as follows. Take a group of four contracted A tensors inside the tensor network (a plaquette). One starts by inserting on the contracted bonds matrices $Q_1, \dots, Q_4$ chosen so that the insertion (i) leaves the contraction intact (approximately) and (ii) eliminates short-range CDL-like correlations around the plaquette. How to choose the Q_i 's is the trademark of the algorithm; see Ref. [11] and Appendix C. Performing a singular value decomposition (SVD) of each Q_i matrix, we get the following result:

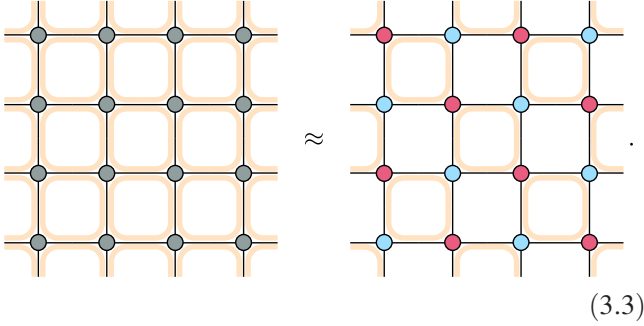
$$A \approx Q_4 \begin{matrix} Q_1 \\ Q_2 \\ Q_3 \end{matrix} \stackrel{\text{svd}}{=} \begin{matrix} \text{Singular Values} \\ \text{Orthogonal Matrices} \end{matrix}, \quad (3.1)$$

where the light-brown shading represents CDL-like correlations.

We now perform the above replacement, with the same $Q_1, \dots, Q_4$, to every second plaquette of the tensor network, in a checkerboard pattern. We then contract the half-tensors with the A 's, defining two tensors B_1 and B_2 as

$$B_1 = \text{diag}(Q_1, Q_2, Q_3, Q_4), \quad B_2 = \text{diag}(Q_2, Q_1, Q_4, Q_3). \quad (3.2)$$

The original tensor network made of A is, thus, mapped to a tensor network made of B_1 and B_2 , which has CDL-like correlations eliminated on every other plaquette:



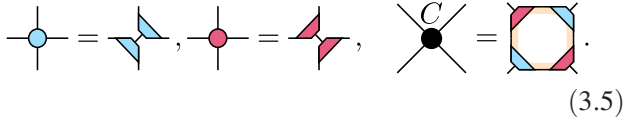
$$(3.3)$$

This finishes the disentangling stage of Gilt-TNR.

The second stage of Gilt-TNR is coarse graining. The default coarse-graining strategy of our code [56] is via two consecutive TRG [1] iterations:

$$\text{TRG}_1: B_1, B_2 \mapsto C, \quad \text{TRG}_2: C \mapsto \tilde{A}. \quad (3.4)$$

Namely, TRG_1 performs SVD decompositions of B_1 and B_2 diagonally and then recombines the SVD pieces into a tensor C , so that CDL-like correlations remaining in Eq. (3.3) are contracted away:



$$(3.5)$$

Analogously, TRG_2 performs an SVD decomposition of C diagonally in two different ways and defines a tensor $\tilde{A}$ by recombining the pieces. The point is that after TRG_1 there are no CDL-like correlations in the networks, and so it makes sense to follow up with TRG_2 right away before any further disentangling [57]. A moment's thought shows that after the two TRG iterations the lattice ends up rotated by $\pi/2$. Since the original Gilt-TNR code [58] aims to be nonrotating, the third and final stage of Gilt-TNR is to define the RG-transformed tensor A' by rotating $\tilde{A}$ back to recover the original orientation.

The map $\mathcal{R}: A \mapsto A'$ is the non-normalized Gilt-TNR map. The corresponding normalized RG map $R(A)$ is

defined by $R(A) = A'/\|A'\|$. The lattice rescaling factor is $b = 2$.

B. Isotropic fixed point by the shooting method

Most of our work in this section is for the 2D Ising model which is isotropic, i.e., has equal nearest-neighbor couplings $J_x = J_y$ (only in Sec. III.5 do we consider the anisotropic case). We transform the partition function of the isotropic model to a tensor network, as outlined in Sec. II.1 and in more detail in Appendix B. The resulting tensor $A_{\text{NN}}(t)$ has bond dimension 2. We parametrize it by the reduced temperature $t = T/T_c$, where T_c is the critical temperature.

We, thus, have the tensor $A^{(0)} = A_{\text{NN}}(t)$ to which we start applying the nonrotating Gilt-TNR algorithm, obtaining a sequence of tensors $A^{(n)}$, $n = 1, 2, \dots$; see Eq. (2.5). Our first goal is to verify the picture from Fig. 1. To see that a tensor $A^{(n)}$ converges to a fixed point, we need to apply gauge fixing after each RG step, as explained below. But as a proxy, to detect fixed-point behavior, it is standard to use gauge-invariant observables. One such set of observables are the singular values of $A^{(n)}$ decomposed along a diagonal:

$$A = U \Lambda V. \quad (3.6)$$

Here, U and V are unitaries and Λ is a diagonal matrix of singular values. In Fig. 4, we plot the singular values,

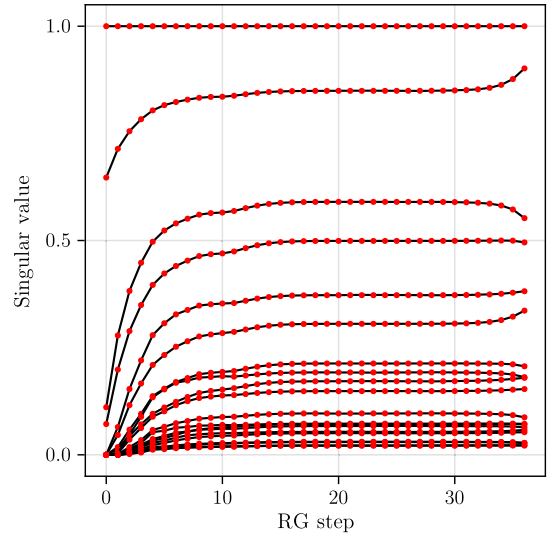


FIG. 4. The RG flow of tensors starting from $A^{(0)}$ corresponding to the Ising model at the reduced temperature $t = 1.000\,011\,004\,3$ and with anisotropy $J_x/J_y = 1$ (i.e., the isotropic tensor), in terms of the 20 largest singular values along the diagonal [Eq. (3.6)] normalized by the first one. Gilt-TNR parameters (see Table V): $\chi = 30$ and $\epsilon_{\text{gilt}} = 6 \times 10^{-6}$.

normalized by the largest singular value, as a function of the RG step. The RG evolution was started at $t = 1.0000110043$, the value found by bisecting [37] up to $\Delta t = 10^{-10}$, to maximize the length of the plateaux visible in Fig. 4, whose presence signals that the RG evolution nears the fixed point, before visibly deviating from it for $n \gtrsim 30$. In the plotted case, t happens to be slightly below the true t_c for the chosen Gilt-TNR parameters, and one can see that the deviation happens in the direction of the low-temperature phase where the fixed-point tensor A_{*L} has two identical singular values, all other ones being equal to zero. For t slightly above t_c , one observes similar plateaux with eventual deviation toward the high-temperature phase where the fixed-point tensor A_{*H} has a single nonzero singular value. Our Fig. 4 compares well, e.g., with the results reported in

- (i) Figure 6 in Ref. [11] using the nonrotating Gilt-TNR;
- (ii) Figure 9 in Ref. [21] for another closely related nonrotating algorithm Gilt-HOTRG;
- (iii) Figure 6, two upper panels, in Ref. [59] for loop-TNR [8] and nuclear norm regularization-TNR.

The plateau in Fig. 4 is a strong indication that the family of tensors $A^{(n)}$ approaches a common gauge equivalence class, so that the gauge-invariant observables do converge. Next, we would like to see this explicitly, by choosing a gauge for each tensor $A^{(n)}$ so that gauge-fixed tensors tend to a limit, in the Hilbert-Schmidt norm. This limit is then a fixed-point tensor. Our gauge-fixing procedure is described in Appendix D. It involves two steps: continuous gauge fixing followed by discrete gauge fixing. Continuous gauge-fixing matrices are obtained from diagonalizing appropriate environments constructed out of $A^{(n)}$. Discrete gauge fixing consists in multiplying by diagonal matrices with ± 1 on the diagonal.

Figure 5 demonstrates the stated convergence of the gauge-fixed RG flow to the fixed point. In this figure, we plot the Hilbert-Schmidt distance between gauge-fixed tensors $A^{(n)}$ and $A^{(n+1)} = R(A^{(n)})$ as a function of the RG step [60]. We see that the distance decreases until $n \sim 23$ and then starts increasing. The decrease and increase of the distance are rather smooth [61], showing that our gauge-fixing procedure works well. The increase of the distance for $n \gtrsim 23$ is roughly consistent with the leading RG eigenvalue being $\lambda = b^{2-\Delta_c} = 2$. This eigenvalue corresponds to the CFT quasiprimary ϵ .

On the other hand, the convergence rate for $n \lesssim 23$ roughly agrees with the largest irrelevant RG eigenvalue in the $\mathbb{Z}_2$ -even sector being $\lambda_{\text{irr}} \approx 0.63$ as we see below when studying the Jacobian eigenvalues (Table IV, column “ λ of ∇R ”). This eigenvalue does not correspond to a quasiprimary, the leading irrelevant quasiprimaries being T^2 , $\bar{T}^2$, and $T\bar{T}$ of dimension 4 and eigenvalue 0.25 [62]. Although this eigenvalue corresponds to a total derivative operator, it does control the rate of approach to the fixed point.

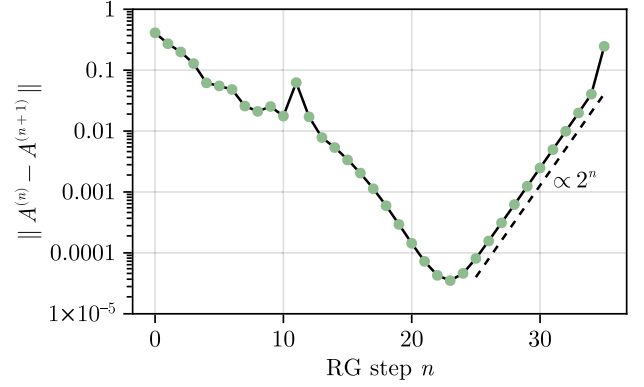


FIG. 5. The Hilbert-Schmidt distance between two subsequent gauge-fixed tensors as a function of the RG step. Gilt-TNR parameters are the same as in Fig. 4.

The following general analysis is instructive. Given the critical temperature to Δt accuracy and the leading irrelevant $\mathbb{Z}_2$ -even eigenvalue λ_{irr} , which is the optimal number of steps n_* for the shooting method, and which accuracy δ_* of approximating the fixed-point tensor may we expect? Basic RG intuition says that, after n steps, the fixed-point tensor is approximated by

$$\delta_n \sim \max[(\lambda_{\text{irr}})^n, (\Delta t)2^n], \quad (3.7)$$

where we use that the relevant $\mathbb{Z}_2$ -even eigenvalue is ≈ 2 . Minimizing this expression over n , we get

$$n_* \sim -\frac{\log_2 \Delta t}{1 - \log_2 \lambda_{\text{irr}}}, \quad \delta_* \sim \Delta t^{1/(1 - \log_2 \lambda_{\text{irr}})}. \quad (3.8)$$

For $\lambda_{\text{irr}} = 0.63$ and $\Delta t = 10^{-10}$, we get $n_* \sim 20$ and $\delta_* \sim 10^{-4}$, in reasonable agreement with Fig. 5.

In Sec. III. 4, we compare our Fig. 5 to the results of Refs. [7,21,22], who also considered the norm difference of two consecutive tensors as a function of the RG step.

Let us discuss next the tensor elements of the approximate fixed-point tensor, defined as the tensor $A^{(n)}$ with $n = n_*$ corresponding to the minimum in the curve in Fig. 5. From the above discussion, we expect that $A^{(n_*)}$ approximates the fixed point $A_*^{[30]}$ of the $\chi = 30$ Gilt-TNR map [66] with accuracy

$$\|A^{(n_*)} - A_*^{[30]}\| \sim 5 \times 10^{-5}. \quad (3.9)$$

It is also interesting to record the distance from the critical nearest-neighbor Ising tensor from which we started the RG evolution:

$$\|A^{(n_*)} - A_{\text{NN}}(1)\| \approx 0.90. \quad (3.10)$$

The first few largest tensor elements of $A^{(n_*)}$ are reported in Table III, column 2. (Column 3 of the same table gives

TABLE III. The first few largest tensor elements of the (approximate) fixed-point tensor. The tensor legs are ordered as left, top, right, and bottom. The leg indices are numbered in the notation $p^\pm$, where $\pm$ is the $\mathbb{Z}_2$ quantum number and $p = 1, 2, 3, \dots$ is the index in the corresponding $\mathbb{Z}_2$ sector on the corresponding leg. Column 1: tensor element indices. Column 2: the approximate fixed-point tensor $A^{(n_*)}$ of the nonrotating Gilt-TNR. Column 3: the approximate fixed-point tensor $A_{(m_*)}$ of the rotating Gilt-TNR, to be discussed in Sec. IV.

$ijkl$	$(A^{(n_*)})_{ijkl}$	$(A_{(m_*)})_{ijkl}$
$1^+1^+1^+1^+$	0.3411	0.3520
$1^+1^+1^-1^-$	0.2107	0.2119
$1^-1^+1^+1^-$	0.2105	0.2027
$1^-1^-1^+1^+$	0.1969	0.2042
$1^+1^-1^-1^+$	0.1971	0.2135
$1^-1^-1^-1^-$	0.1807	0.1829
$1^+1^-1^+1^-$	0.1481	0.1474
$1^-1^+1^-1^+$	0.1438	0.1503
$1^+1^+2^-2^-$	-0.1067	-0.1056
$2^-1^+1^+2^-$	0.1068	0.1024
$1^+2^-2^-1^+$	0.1015	0.1069
$2^-2^-1^+1^+$	-0.1016	-0.1033
$1^-1^+1^-2^+$	0.1048	0.085 93
$2^+1^-1^+1^-$	0.086 48	0.081 52
$1^+1^-2^+1^-$	0.087 05	0.099 59
$1^-2^+1^-1^+$	0.078 21	0.088 98

the approximate fixed-point tensor of the RG map with rotation, to be discussed in Sec. IV. Please ignore them for now.)

We divided the tensor elements of $A^{(n_*)}$ in Table III into several groups. If the fixed-point tensor preserved reflection and/or rotation invariance, tensor elements in each group would be equal, perhaps up to a sign. Instead, we see that the tensor elements within groups are close but not exactly

equal. How should we interpret this? Recall that we started the RG evolution from the isotropic tensor $A^{(0)}$, which is reflection and rotation invariant. However, as we already mentioned, Gilt-TNR does not preserve these symmetries (although it does preserve the spin flip $\mathbb{Z}_2$). That is why the fixed-point tensor in Table III does not respect these symmetries [67].

In Fig. 6, we show absolute values of tensor elements of approximate fixed-point tensors for $\chi = 30$ as well as for $\chi = 10, 20$, for comparison [69]. Absolute values of tensor elements are sorted in descending order and indexed as $|A|_I$, where $I = 1, 2, \dots$ is the position in the sorted list. We then plot $|A|_I$ as a function of I . The left figure shows the first 50 tensor elements, while the right figure gives a global view of all tensor elements, in the log-log scale. The dashed line in this figure corresponds to the slope $|A|_I \sim I^{-1/2}$, which marks the boundary for the finiteness of the Hilbert-Schmidt norm for infinite-dimensional tensors. The curves remain below that line for the values of χ that we studied, but just barely so. We cannot determine from this plot whether or not our finite χ fixed points will converge to a fixed-point tensor with finite Hilbert-Schmidt norm as χ goes to infinity. It may be that the Gilt-TNR algorithm does not have a fixed point with a finite Hilbert-Schmidt norm in the $\chi \rightarrow \infty$ limit. If so, the search for a rigorous infinite-dimensional fixed point will need to employ a different tensor network RG map. We will keep using Gilt-TNR in the rest of this work to demonstrate other ideas such as the effect of rotation and the possibility to set up the Newton method. Those ideas are general and will still apply if in the future we have to change the algorithm. See also Remark 3.1 and Sec. V for further related comments.

C. Jacobian eigenvalues

Now that we provided evidence for the existence of a fixed-point tensor, we would like to explore the RG map Jacobian at the approximate fixed point. First of all, we can

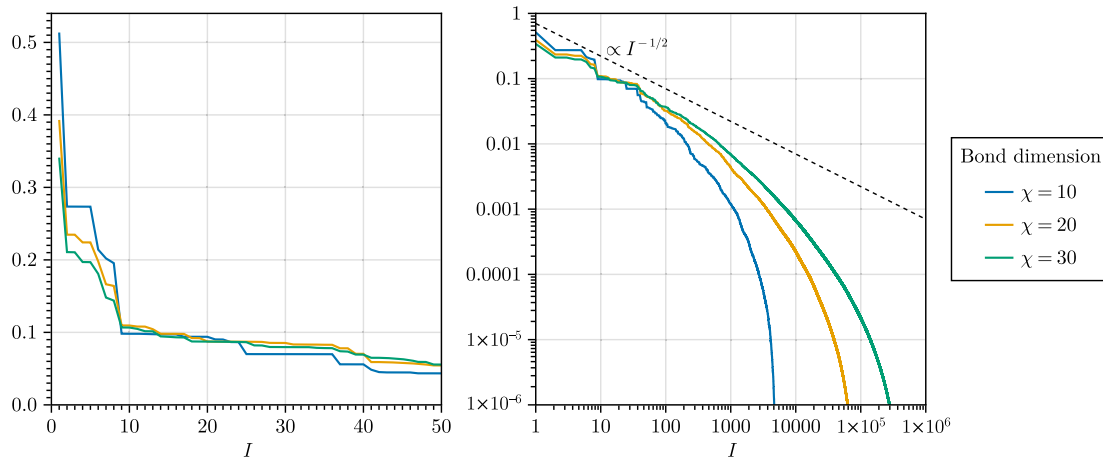


FIG. 6. Absolute values of tensor elements of approximate fixed-point tensors for $\chi = 10, 20, 30$; see the text.

TABLE IV. Column 1: $\mathbb{Z}_2$ quantum number. Column 2,3 (rows with entries in these columns): the first few low-dimension CFT quasiprimaries and their exact eigenvalues. Columns 4 and 5: The first few largest, in absolute value, Jacobian eigenvalues at the approximate fixed point for the nonrotating Gilt-TNR (∇R) and for the rotating Gilt-TNR (∇R°). Rows with no entries in columns 2 and 3 show RG eigenvalues corresponding to derivative interactions, which are not universal.

$\mathbb{Z}_2$	$\mathcal{O}$	$\lambda_{\text{CFT}} = 2^{2-\Delta_{\mathcal{O}}}$	λ of ∇R	λ of ∇R°
+	ϵ	2	1.9996	1.9996
+	T	1	1.0015	-1.0010
+	$\bar{T}$	1	0.9980	-0.9982
+			0.6322	0.7757
+			$0.5941 \pm 0.1195i$	0.6209
			...	...
-	σ	$2^{15/8} \approx 3.668016$	3.6684	3.668 014
-			1.5328	$0.0027 \pm 1.5364i$
-			1.5300	0.8600
-			0.8869	$0.6318 \pm 0.0583i$
			...	...

speak of the Jacobian, as our map is differentiable in a neighborhood of the fixed point. Evidence for this is provided in Appendix E. This statement is less trivial than it may seem. To make Gilt-TNR robustly differentiable, we had to tweak the algorithm a bit. In the original Gilt-TNR (see Appendix C, where this is referred to as “dynamic Q choice”), matrices Q_1 , Q_2 , Q_3 , and Q_4 are chosen iteratively, and the number of iterations is determined at runtime, depending on a certain internal parameter called ϵ_{conv} . For constant ϵ_{conv} , the number of iterations may jump for small variations of tensor A , causing a tiny but noticeable discontinuity (and, hence, nondifferentiability) in $\mathcal{R}(A)$. To avoid this problem, we are using a modified algorithm in which the number of iterations is kept fixed, referred to as “static Q choice” in Appendix C [70].

So, let R be the nonrotating Gilt-TNR RG map with $\chi = 30$ and other parameters as in Fig. 4. Let ∇R be the Jacobian of R at the tensor $A^{(n_*)}$ from Table III. We are interested in the eigenvalues of ∇R .

Note that tensor $A^{(n_*)}$ preserves $\mathbb{Z}_2$ flip symmetry, in the sense that each of its vertical and horizontal indices can be assigned a ± 1 quantum number, such that the tensor remains invariant when acted upon by all these sign flips; i.e., the tensor is $\mathbb{Z}_2$ -even [71].

The full tangent space at $A^{(n_*)}$, on which ∇R acts, splits into a direct sum of subspaces of perturbations which are $\mathbb{Z}_2$ -even and $\mathbb{Z}_2$ -odd. Since R preserves $\mathbb{Z}_2$, ∇R maps $\mathbb{Z}_2$ -even perturbations to $\mathbb{Z}_2$ -even, and similarly for $\mathbb{Z}_2$ -odd. Each eigenvector will be either $\mathbb{Z}_2$ -even or $\mathbb{Z}_2$ -odd.

The first few largest eigenvalues of ∇R at $A^{(n_*)}$ are reported in Table IV, column “ λ of ∇R ”. Since ∇R is a large matrix of size $\chi^4 \times \chi^4$, its full diagonalization would be too time-consuming. We instead compute $O(10)$ largest eigenvalues by the Arnoldi iteration method, implemented in the

JULIA [72,73] package KRYLOVKIT.JL [74]. The Arnoldi method does not require evaluation and storage of the full matrix ∇R . Instead, one provides a function $v \mapsto \nabla R.v$, which computes the directional derivative of R in the direction v . We evaluate this directional derivative via a symmetric finite difference approximation:

$$D_h = \frac{R(A + hv) - R(A - hv)}{2h}, \quad (3.11)$$

where $h = 10^{-4}$ is a small parameter, set at this value to balance truncation and round off errors; see Appendix E [75].

Rows in Table IV which have entries in columns 2 and 3 correspond to CFT quasiprimaries. The corresponding Jacobian eigenvalues can be compared to the CFT predictions in columns 2 and 3, computed via Eq. (2.8). We observe reasonably good agreement, both in the $\mathbb{Z}_2$ -even and in the $\mathbb{Z}_2$ -odd sectors. Note, in particular, the two eigenvalues close to 1 corresponding to the stress-tensor components T and $\bar{T}$.

In the rows in Table IV that do not have entries in columns 2 and 3, we report eigenvalues which do not correspond to quasiprimaries on the CFT side. These rows correspond to derivative operators, whose eigenvalues are not universal, as discussed in Sec. II. 2. For this reason, the CFT entries in these rows are left empty [76].

To demonstrate the latter point, let us compare the Jacobian eigenvalues between R and the rotating Gilt-TNR map R° which is discussed in Sec. IV (Table IV, last column). We see that the quasiprimary eigenvalues change very little, apart from the -1 factor picked up by the T and $\bar{T}$ eigenvalues, as discussed in Sec. II. 5. On the other hand, several eigenvalues in the rows without entries in columns 2 and 3 show significant changes. For example, the leading

irrelevant $\mathbb{Z}_2$ -even eigenvalue λ_{irr} changes from approximately 0.63 to approximately 0.78.

The spectrum of even lower eigenvalues beyond those shown in Table IV becomes rather messy. Unfortunately, we do not have enough precision to compare with the irrelevant quasiprimaries in the $\mathbb{Z}_2$ -even and $\mathbb{Z}_2$ -odd sectors. Their eigenvalues are buried among the nonuniversal eigenvalues corresponding to derivative operators.

D. Comparison to prior shooting method results

Our results in Secs. III B and III C can be compared with the results of Refs. [7,21,22]. This was already partly summarized in Table I (upper part of the table). We give a detailed comparison, divided into two parts: locating the fixed point via the shooting method and computing the Jacobian eigenvalues.

1. Locating the fixed point

In this section, we review past results on the accuracy of approximate fixed points and compare with our results. We emphasize that our goal here is not to compare Gilt-TNR with other algorithms but rather to compare the accuracy of our Newton method for finding the fixed point with past results which use a shooting method to find it.

References [7,21,22] are the only prior works known to us which implemented gauge fixing and plotted the norm difference of two consecutive tensors as a function of the RG step, like our Fig. 5 [77]. References [21,22] used methods closely related to ours, so we discuss them first.

Reference [21] used an algorithm closely related to Gilt-TNR, which we call Gilt-HOTRG, as it uses Gilt for disentangling and HOTRG for coarse graining (see Appendix C.4). It is an algorithm closely related to Gilt-TNR. Several parameter choices were identical for us and for them: $\chi = 30$ and $\epsilon_{\text{gilt}} = 6 \times 10^{-6}$.

Reference [22] also used essentially Gilt-HOTRG with some minor modifications. They use $\chi = 24$ and $\epsilon_{\text{gilt}} = 2 \times 10^{-7}$. They also split an RG step in two and use disentangling only on odd steps. This doubling of the number of steps should be kept in mind when comparing with the plots of Ref. [22].

Reference [21] used gauge fixing similar to the one used by us, while Ref. [22] used a more sophisticated two-step gauge fixing. In the first step, they gauge fix by general invertible matrices to achieve the minimal canonical form (MCF) [78], which minimizes the norm of the gauge-fixed tensor. In the second step, they gauge fix by orthogonal matrices, similarly to Ref. [21] (with some improvements) and to us.

The norm-difference plot in Fig. 3, red curve, in Ref. [22], is rather similar in shape to our Fig. 5, although their minimal distance $\delta \sim 3 \times 10^{-4}$ is larger than our $\delta \sim 5 \times 10^{-5}$. This may be due to them locating the critical temperature not as precisely as us.

On the other hand the norm difference plot in Fig. 9(b) in Ref. [21] reaches a much worse minimal distance $\delta \sim 10^{-2}$ in spite of them locating the critical temperature to the same $\Delta t = 10^{-10}$ accuracy as us. Reference [22] investigated this issue. In their Fig. 13, they reproduce the results of Ref. [21] and note that the rate of approach to the fixed-point tensor is notably slower for the algorithm of Ref. [21] than for their algorithm. In the language of Sec. III. 2, we could say that the algorithm of Ref. [22] has λ_{irr} about the same as our algorithm, while the algorithm of Ref. [21] has λ_{irr} closer to 1. This interpretation is conjectural, since Refs. [21,22] do not report their λ_{irr} .

Therefore, the MCF step in gauge fixing, which is the only significant difference between the algorithms in Refs. [21,22], is responsible for the improvement of the convergence rate to the fixed point in their case. On the other hand, it should be noted that our algorithm is able to reach the same convergence rate without MCF but using Gilt-TNR instead of Gilt-HOTRG. There is a little bit of luck involved here, the leading irrelevant eigenvalue λ_{irr} being nonuniversal.

Finally, let us discuss the earlier Ref. [7]. Using the TNR algorithm that they proposed in that work, they reached approximately 10^{-5} accuracy in a plot like our Fig. 5, using the TNR bond dimension $\chi_{\text{TNR}} = 42$ (see Supplemental Material, Fig. 6 in Ref. [7]). While typically the TNR algorithm needs a smaller bond dimension than Gilt-TNR to achieve the same accuracy [11], here it is the other way around, likely because they did not bisect to get closer to the χ -dependent T_c but started the RG flow at the theoretical T_c . A feature of the plots in Refs. [7,21], possibly due to imperfect gauge fixing, is that they flatten near the minimum, compared to the clean V-shaped plots in our work and in Ref. [22].

2. Jacobian eigenvalues

In Sec. III. 3, we studied the Jacobian of Gilt-TNR close to the critical point. In particular, we reported the leading irrelevant eigenvalue λ_{irr} and observed that the rate of approach to the fixed point is consistent with it. Since Refs. [21,22] do not report the spectrum of the Jacobian of the map used to produce their norm-difference plots, the λ_{irr} test is not possible based on their available data. Such a test would be interesting in the future.

References [21,22] do study the spectrum of a tensor RG map Jacobian (procedure called there “linearized tensor RG”, or “ITRG”) but only for a different RG map, constructed once the (approximate) fixed point is known. This new “frozen” (our term) RG map has the Gilt disentanglers and the HOTRG isometries frozen to their fixed-point values [79].

Reference [21] uses the spectrum of the frozen RG map Jacobian to extract the scaling dimensions of CFT operators with $\Delta \leq 2$ (Table I in Ref. [21]), and Ref. [22] extends this analysis to $\Delta \leq 3$ (Table I in Ref. [22]). Compared to our ∇R spectrum, their accuracy is one order of magnitude

worse for σ and ϵ and is about the same for T and $\bar{T}$. Our better performance for σ and ϵ may be due to our better approximation of the fixed-point tensor.

Their results contain one curious feature: In addition to quasiprimaries σ , ϵ , T , and $\bar{T}$, it contains many derivative operators identified with first and second derivatives of σ and ϵ and first derivatives of T and $\bar{T}$, whose scaling dimensions are correctly reproduced (up to numerical errors) from frozen RG eigenvalues. This is puzzling: We emphasized many times that eigenvalues of derivative operators are not universal and do not have to agree between RG and CFT, and indeed they do not agree for us. So why do they agree for them? This was not discussed in Refs. [21,22].

The answer to this puzzle is instructive. It turns out that the frozen RG map of Refs. [21,22] is very special: Its Jacobian coincides with the lattice dilatation operator. This coincidence, which does not hold for our RG maps R and R° , explains the puzzle, because lattice dilatation operator eigenvalues are universal and reproduce all CFT operator eigenvalues, including total derivatives. It would require too much preparatory work to explain this point in detail here; it will be done in Ref. [27].

We should add that the frozen RG map is defined only once one has already found an approximate fixed point, which can be done only with a full RG map where disentanglers and isometries depend on the tensor. So, while the frozen RG map is an easier map and it has a conceptually simpler Jacobian, it would not be useful for our purposes of setting up a Newton method.

E. Anisotropy: Manifold of fixed points

In the previous sections, we studied the RG flow starting from the isotropic 2D Ising model. We would now like to show the new features which appear when introducing anisotropy. Namely, we consider the nearest-neighbor Ising model with anisotropy parameter $a = J_y/J_x \leq 1$. We transform the partition function of this model to a tensor network as detailed in Appendix B. The resulting tensor has bond dimension 2 and depends on the reduced temperature t and the anisotropy parameter a . Starting from this tensor $A^{(0)} = A_{\text{NN}}(t, a)$, we repeatedly apply the non-rotating Gilt-TNR algorithm to get a sequence $A^{(n)}(t, a)$, $n = 1, 2, \dots$; see Eq. (2.5).

We have two goals in this section: (i) We want to verify Fig. 3, showing that a parametrizes a one-dimensional manifold of fixed points. To our knowledge, this has never been checked using tensor RG. (ii) We want to demonstrate that eigenvalues of ∇R corresponding to quasiprimaries are universal across this manifold and eigenvalues that do not correspond to quasiprimaries can vary.

Let us start with the first goal. As discussed in Sec. II. 4, we expect a two-dimensional manifold of RG fixed points for a nonrotating RG map. The critical model corresponds to $t = 1$ in our notation, for all values of a . However, as in

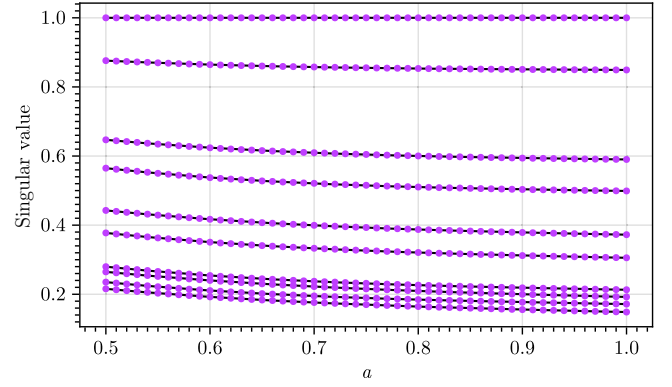


FIG. 7. Dependence of the first few singular values of $A^{(20)}[t \approx t_c(a), a]$ on a . The singular values are obtained via SVD decomposition as in Eq. (3.6). The largest singular value is normalized to 1 as in Fig. 4.

the isotropic case, for finite bond dimension the value of t that gives convergence to the approximate fixed point is not exactly 1 and depends on a . We denote it by $t_c(a)$ [80]. If we start with a critical anisotropic nearest-neighbor Ising model [$t = t_c(a)$], then iterations of the RG map converge to a fixed point that depends on the anisotropy parameter a . To show that different values of a do indeed produce different fixed points, we consider the singular values of the fixed point decomposed along a diagonal as defined in Eq. (3.6). If we plot these singular values as a function of the RG step, we see curves that are qualitatively similar to Fig. 4. In particular, for $t \approx t_c(a)$, there is a range of RG steps where the singular values are almost constant. However, these constant values depend on a . We compute these singular values at the 20th RG step for a ranging from 0.5 to 1.0 and $t \approx t_c(a)$ and plot them as a function of a in Fig. 7. The variation of these values with a shows that, as we vary a , the RG fixed point is varying along a one-dimensional manifold. We expect that by considering a more general initial Ising model with two parameters we could obtain the full two-dimensional manifold of fixed points.

Let us proceed with the second goal. As discussed in Sec. II. 2, the eigenvalues of the RG Jacobian ∇R should be of two types: those that correspond to quasiprimaries in the CFT and those that correspond to total derivatives. The former are given by Eq. (2.8) and so should not depend on which fixed point in the two-dimensional manifold we consider. The latter cannot be predicted from the CFT and may depend on the fixed point at which we compute the Jacobian. In Fig. 8, we plot the first few eigenvalues of ∇R as a function of a . The eigenvalues corresponding to quasiprimaries are shown in red (blue) for the $\mathbb{Z}_2$ -even (-odd) sectors. These eigenvalues are essentially independent of a and fit the CFT predictions of $2^{15/8}$, 2, and 1 very well. The eigenvalues corresponding to total derivatives are shown in light orange (light violet) for the $\mathbb{Z}_2$ -even (-odd)

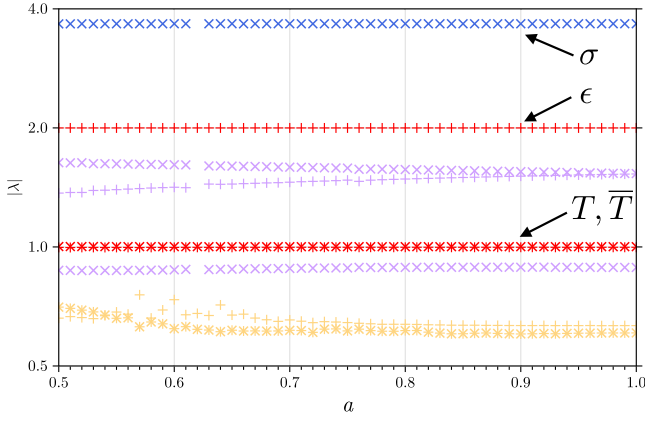


FIG. 8. The first few largest eigenvalues of ∇R computed at $A^{(20)}[t \approx t_c(a), a]$ as a function of the anisotropy parameter a . Note the vertical $\log_2$ scale. We use the following color scheme: red (blue) for $\mathbb{Z}_2$ -even (-odd) quasiprimaries and light orange (light violet) for other $\mathbb{Z}_2$ -even (-odd) eigenvalues. For $a = 1$, the spectrum agrees with the “ λ of ∇R ” column in Table IV.

sectors. For these eigenvalues, one can see significant variation with a . For $a = 1$, the spectrum agrees with the “ λ of ∇R ” column in Table IV [81].

We, thus, see broad agreement with the intuition expounded in Sec. II.

Remark 3.2. In Fig. 8, there are no $\mathbb{Z}_2$ -odd eigenvalues shown for $a = 0.62$. For this value of a , the computation of the $\mathbb{Z}_2$ -odd eigenvalues produced nonsensical results which we do not show. We believe that this is a result of the issues with discrete gauge fixing in the $\mathbb{Z}_2$ -odd sector that are discussed in Appendix D.1.

IV. RESULTS FOR THE ROTATING GILT-TNR

Here, we finally present what we consider our most exciting numerical results. We consider the rotating Gilt-TNR algorithm. We describe how the critical fixed point is found with the Newton method, in agreement with the general discussion in Sec. II. We study the rate of convergence of the Newton method, and we use the fixed point to extract CFT quasiprimary dimensions using the Jacobian. All the above is done in Sec. IV.2 for the isotropic Ising model, and the discussion here parallels Secs. III B and III C for the nonrotating algorithm. (The three-state Potts model is similarly considered in Appendix F.) We consider the anisotropic Ising model in Sec. IV.3, where we exhibit period-2 oscillations for the rotating Gilt-TNR algorithm when starting from an anisotropic critical tensor.

A. Numerical strategies for the inverse Jacobian

We start by describing how to deal numerically with the inverse Jacobian in the Newton iteration map (2.16). We need to solve the linear system

$$J(A)x = b, \quad J(A) \equiv I - \nabla R^\circ(A), \quad (4.1)$$

where $A = A_{(m)}$ and $b = f(A_{(m)})$.

Our code does not store the Jacobian $\nabla R^\circ(A)$ as a matrix but realizes it as a linear operator $v \mapsto \nabla R^\circ(A)v$, computed with a second-order finite difference approximation (see Sec. III.3 for a fully analogous discussion for the nonrotating RG map). We may then solve Eq. (4.1) via an iterative algorithm, GMRES [82] being the standard choice. For unclear reasons, in our numerical tests using KRYLOVKIT.JL [74], GMRES convergence was not always reliable. We, therefore, used a different strategy, which we found numerically stable.

We use the Arnoldi method to find s largest in absolute value eigenvalues of $\nabla R^\circ(A)$ and the corresponding eigenvectors. Let P_s be the orthogonal projector on the subspace spanned by them. We then approximate the Jacobian by

$$J_\approx(A) = I - P_s \nabla R^\circ(A) P_s. \quad (4.2)$$

We take $s \geq 3$, so that the relevant eigenvalue corresponding to the ϵ and the two -1 eigenvalues corresponding to T and $\bar{T}$ are subtracted. The remaining eigenvalues of $\nabla R^\circ(A)$ being less than 1 in absolute value, the approximation is expected to be good. Increasing s further improves the approximation and speeds up convergence, as we see below.

Computing the approximation (4.2) with the Arnoldi method takes a few minutes on a laptop for $s = 10$ and $\chi = 30$. Within this approximation, solving linear system (4.1) is easy, as it amounts to inverting an $s \times s$ matrix. We checked that the results agree closely with GMRES, when the latter converges. We also checked that the accuracy of this approximation and the agreement with GMRES are further improved by including the first few terms of the Neumann series expansion in the difference $J(A) - J_\approx(A)$. This improvement is not used in the results shown below.

To speed up computations, we use one more approximation. Namely, we do not recompute the approximate Jacobian $J(A)$ at every Newton step but compute it just once, for $A = A_{(0)}$. This preserves good local convergence properties of the Newton method.

B. Isotropic 2D Ising fixed point by the Newton method

So let R° be the rotating Gilt-TNR map, constructed by adding a $\pi/2$ rotation to the nonrotating Gilt-TNR. The numerical results below are obtained using the same Gilt parameters as in Sec. III.2: $\chi = 30$ and $\epsilon_{\text{gilt}} = 6 \times 10^{-6}$.

We would like to solve the fixed-point equation for R° via the Newton method. We set up the Newton method as in Sec. II.6 and compute the inverse Jacobian as described in Sec. IV.1. The key ingredients here are the Jacobian ∇R° and its eigendecomposition, which are computed similarly to the nonrotating Gilt-TNR in Sec. III.3.

Ideally, we would like the Newton method to converge to the fixed point when starting from a generic point on the critical manifold, e.g., from the bond dimension-2 tensor $A_{\text{NN}}(t=1)$ obtained when translating the critical isotropic 2D Ising model to the tensor network form (Appendix B). However, it turns out somewhat nontrivial to set up such a faraway convergent Newton method, for reasons discussed in Sec. IV.2.1 below. So in this work we settle for convergence in a smaller domain. Our search for the fixed point consists of two stages.

- (1) Find an approximate fixed-point tensor $A^{(n')}$, obtained by performing n' RG steps starting from $A_{\text{NN}}(t_0)$, where t_0 is near the finite- χ critical temperature $t_c \approx 1$. In our main results, we use $n' = 23$ initial RG steps from $t_0 = 1.000\,013\,131\,20$, obtained by bisecting to $\Delta t = 10^{-10}$ [83]. These parameters can be somewhat relaxed; see Sec. IV.2.1 below.
- (2) Start Newton iterations from $A^{(n')}$, which are then seen to converge to a fixed point.

Denoting $A_{(0)} = A^{(n')}$, the Newton iterations are given by

$$A_{(m+1)} = g_{\approx}(A_{(m)}), \quad n = 0, 1, 2, \dots, \quad (4.3)$$

where $g_{\approx}$ is the approximate Newton map, obtained by replacing the exact Jacobian $J(A)$ by the approximate Jacobian $J_{\approx}$ with projector rank $s \geq 3$ and evaluated at $A_{(0)}$, as described in Sec. IV.1. We see below how various choices of s influence the performance.

Before starting the Newton iterations, it is instructive to consider ∇R° eigenvalues at $A_{(0)}$. Recall (Table IV, column 3) that ∇R had two eigenvalues close to 1 at $A^{(n_*)}$: 1.0015 and 0.9980. In Sec. II.5, we argued that adding the rotation should flip the sign of the two eigenvalues 1, which are associated with the stress-tensor components. Our numerical check fully confirms that intuition: We find that $\nabla R^{\circ}(A_{(0)})$ has precisely two eigenvalues close to -1 (-1.0006 and -0.9985) and no eigenvalues close to 1.

Given this encouraging news, we expect Newton iterations (4.3) to converge, and this is indeed what we observe numerically. Figure 9 demonstrates the rate of convergence, for several projector ranks $s = 3, 9, 26, 54$. In that plot, we show the Hilbert-Schmidt distance $\|A_{(m+1)} - A_{(m)}\|$ as a function of the Newton iteration number $m = 0, 1, 2, \dots$. We see exponential convergence for all values of s , improving for larger s . Recall that the Newton method with the exact Jacobian is expected to converge super-exponentially fast. Using $J_{\approx}$ reduces the convergence to exponential. As usual, the rate of the exponential convergence can be understood in terms of the largest eigenvalue of the Jacobian of the map $g_{\approx}(A)$ at the fixed point. Actually, the convergence rate improvement with s seen in Fig. 9 saturates around the maximal $s = 54$ used in that figure. At the end of this section, we give a more

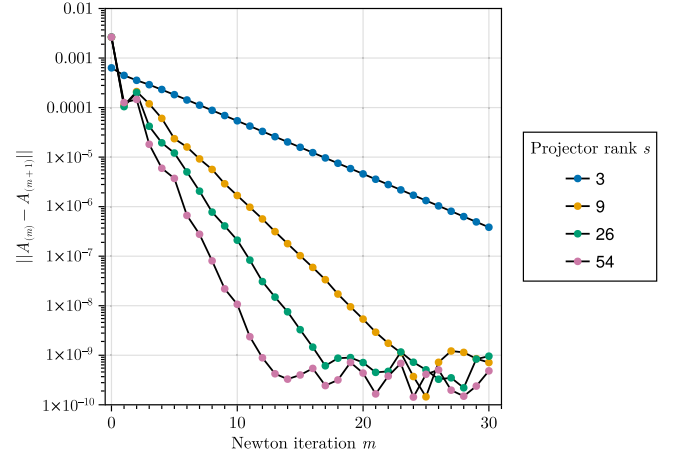


FIG. 9. Convergence of the Newton method for approximations of Jacobian that use different rank s of the P_s projector in Eq. (4.2).

detailed discussion of the convergence rate, to understand why it first improves with s and then saturates, and how to improve it further.

Whatever s , the exponential convergence stops and is replaced by a chaotic behavior when $\|A_{(m+1)} - A_{(m)}\|$ becomes approximately 10^{-9} or a bit smaller (for $s = 3$, this happens outside of the shown range of m). We assign this behavior to roundoff errors. In Appendix E, we provide evidence that Gilt-TNR has roundoff order approximately 10^{-11} . It is not unreasonable that this error is enhanced to approximately 10^{-9} in the Newton method due to additional involved operations, such as evaluating the Jacobian.

All tensors corresponding to points at the bottom in Fig. 9 should approximate the critical tensor equally well. We have checked their pairwise distances and they are all approximately 10^{-9} ; see Fig. 10.

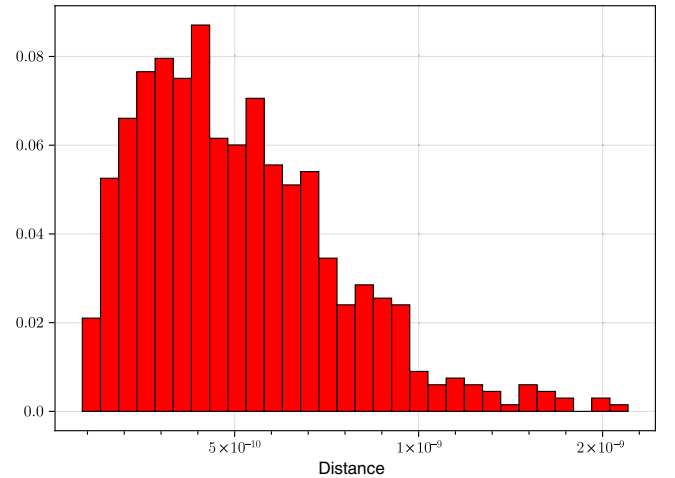


FIG. 10. The histogram of distances between the 37 tensors in Fig. 9 which fall below the 10^{-9} line.

For further numerical tests, we consider the $s = 54$ Newton iteration sequence and pick from it the tensor $A_{(m_*)}$ with $m_* = 14$, which is the last iteration when $\|A_{(m+1)} - A_{(m)}\|$ decreases. From the above discussion, we expect that $A_{(m_*)}$ is an excellent approximation to the (unique) fixed point $A_*^{[30]}$ of rotating $\chi = 30$ Gilt-TNR (for the given Gilt parameters), with error

$$\|A_{(m_*)} - A_*^{[30]}\| \sim 10^{-9}. \quad (4.4)$$

Applying the RG map to $A_{(m_*)}$, we obtain that this shifts the tensor by

$$\|R^\circ(A_{(m_*)}) - A_{(m_*)}\| \approx 3 \times 10^{-10}, \quad (4.5)$$

providing a further check of Eq. (4.4). We also report the distance from the critical nearest-neighbor Ising tensor

$$\|A_{(m_*)} - A_{\text{NN}}(1)\| \approx 0.89, \quad (4.6)$$

as well as the distance from the initial tensor for Newton iterations

$$\|A_{(m_*)} - A_{(0)}\| \approx 3 \times 10^{-3}, \quad (4.7)$$

which gives an initial idea of the size of the convergence domain for our implementation of the Newton method (see Sec. IV.2.1 below).

In Table III, last column, we show the first few tensor elements of $A_{(m_*)}$. We see that they are rather similar, although not identical, to those of $A^{(n_*)}$, the approximate fixed-point tensor of the nonrotating Gilt-TNR. The distance is (see Fig. 11)

$$\|A_{(m_*)} - A^{(n_*)}\| \approx 0.18. \quad (4.8)$$

This is in agreement with our general intuition that, while the critical exponents are universal, RG fixed points are not—different RG maps, in general, have different fixed points.

Just like $A^{(n_*)}$, tensor $A_{(m_*)}$ does not respect reflections and rotations. The general comments in this regard made in Sec. III.2 still apply. We found that the distribution of tensor elements in their absolute magnitude has the same general shape for $A_{(m_*)}$ as for $A^{(n_*)}$, so we do not repeat Fig. 6.

Let us next discuss the eigenvalues of the Jacobian ∇R° at $A_{(m_*)}$. The first few largest eigenvalues are reported in Table IV, last column. The main point here is that the quasiprimary eigenvalues are in good agreement with the CFT and with the observation that T and $\bar{T}$ eigenvalues should flip sign.

Tensor $A_{(m_*)}$ is a much better approximation for the fixed point of rotating Gilt-TNR than $A^{(n_*)}$ was for the

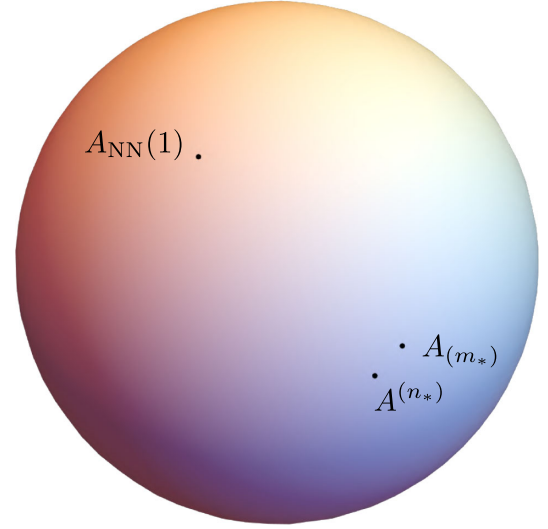


FIG. 11. The geometry on the unit Hilbert-Schmidt sphere corresponding to the distances (3.10), (4.6), and (4.8). The Hilbert-Schmidt sphere has large dimension, but, since there are three tensors involved, the geometry can be shown on a two-dimensional unit sphere as in this figure.

nonrotating Gilt-TNR [10^{-9} vs 5×10^{-5} , Eq. (3.9)]. Does this lead to a better agreement of quasiprimary eigenvalues with CFT? Comparing the ∇R and ∇R° columns in Table IV, we see that σ eigenvalue agreement improves by 2 orders of magnitude, while ϵ , T , and $\bar{T}$ eigenvalues do not improve. Apparently, the error in these eigenvalues is dominated by the finite- χ truncation effects and not by the error in the determination of the finite- χ fixed-point tensors. It is not clear to us why the σ eigenvalue is special in this regard.

As for the nonrotating case, we also see in Table IV a bunch of nonuniversal eigenvalues of ∇R° , associated with derivative interactions. Of some interest here is the leading irrelevant $\mathbb{Z}_2$ -even eigenvalue $\lambda_{\text{irr}} \approx 0.78$, as it would control the rate of approach to the fixed point if the traditional shooting method were used instead of the Newton method; see Remark 4.1 below.

Note that the subleading $\mathbb{Z}_2$ -odd eigenvalues $0.0027 \pm 1.5364i$ of ∇R° in Table IV are close in absolute value to the pair of subleading $\mathbb{Z}_2$ -odd eigenvalues 1.5328 and 1.5300 of ∇R reported in the same table but are almost imaginary rather than real. These eigenvalues are likely associated with the CFT derivative operators $\partial\sigma$ and $\bar{\partial}\sigma$, with the additional phase due to $\pi/2$ rotation of an operator with spin ± 1 . This correspondence between eigenvalues of R and R° would be exact if the fixed points were the same. The fixed points are not exactly the same but close, and it seems that the Jacobian varies sufficiently slowly for the reasoning to be approximately true. Therefore, while the magnitude of derivative operator eigenvalues is not predictable, their relative phase between R and R° may be at least approximately predictable [84]. Similarly, there are

almost imaginary ∇R° eigenvalues in the $\mathbb{Z}_2$ -even sector which may be associated with $\partial\epsilon$ and $\bar{\partial}\epsilon$ and with ∂T and $\bar{\partial}T$ (not shown in Table IV since they belong to the lower part of the spectrum). We will see in Ref. [27] that the same pattern holds rigorously for the eigenvalues of the lattice dilatation operator.

Remark 4.1. It is interesting to discuss what it would take to approach the critical fixed-point tensor with the traditional shooting method as closely as 10^{-9} achieved here with the Newton method. To use the shooting method, we would need to determine the critical temperature sufficiently precisely and then use a sufficiently large number of iterations to approach the fixed point. The final accuracy δ_* is given in terms of the leading irrelevant $\mathbb{Z}_2$ -even eigenvalue λ_{irr} in Eq. (3.8). Using $\lambda_{\text{irr}} \approx 0.63$ for the nonrotating Gilt-TNR map, we find that $\delta_* \sim 10^{-9}$ requires $\Delta t \sim 10^{-23}$, below the approximately 10^{-16} machine precision for the double-precision arithmetic. (For the rotating Gilt-TNR, the required $\Delta t \sim 10^{-34}$ is much smaller due to $\lambda_{\text{irr}} \approx 0.78$ being closer to 1.) Thus, to match the precision level of the Newton method with the shooting method would require multiple precision arithmetic or a tensor RG map with a smaller λ_{irr} . The theoretical lower limit here is $\lambda_{\text{irr}} = 0.25$, corresponding to quasiprimaries T^2 and $\bar{T}^2$ (see Ref. [62]).

Finally, let us come back to the convergence rate in Fig. 9. We checked numerically that, for all s in that figure, the convergence rate is compatible with the largest eigenvalue of the Jacobian of the map $g_\approx$ which is being iterated (as it should be). Here, we would like to provide qualitative understanding of why the convergence rate improves with s and why it eventually saturates, as mentioned above.

From the definition of $g_\approx$, its Jacobian at the fixed point is

$$(\nabla g_\approx)(A_*^\circ) = I - J_\approx^{-1}[I - \nabla R^\circ(A_*^\circ)]. \quad (4.9)$$

We would like to understand the eigenvalues of this matrix. Recall that $J_\approx$ is evaluated not at the fixed point but at $A_{(0)}$, where we start the Newton iterations. Let us consider first an approximation where we neglect this difference:

$$J_\approx \approx I - P_s \nabla R^\circ(A_*^\circ) P_s. \quad (4.10)$$

Then, we have to study the largest eigenvalue of the matrix

$$M_s = I - (I - P_s M P_s)^{-1}(I - M), \quad (4.11)$$

where we denote $M = \nabla R^\circ(A_*^\circ)$ and where P_s is the orthogonal projector on the first s eigenvectors of M . We have the following lemma.

Lemma 4.1. Assume M is diagonalizable with the eigenvalues $\lambda_1, \lambda_2, \dots$ arranged in order of decreasing absolute value. Suppose none of the first s eigenvalues are 1. Then, M_s is diagonalizable with eigenvalues $0, \dots, 0, \lambda_{s+1}, \lambda_{s+2}, \dots$, where eigenvalue 0 has multiplicity s .

The elementary proof is given below. Hence, the predicted convergence rate is λ_{s+1} . For $s = 3$ this agrees well with Fig. 9, but for larger s the convergence in Fig. 9 is slower. We assign this discrepancy to the breakdown of the approximation (4.10). We observed that reevaluating $J_\approx$ after a few Newton iterations brings the convergence rate closer to the predicted value. In particular, with $J_\approx$ evaluated at $A_{(0)}$, convergence rate saturates around $s = 54$, but after reevaluation convergence keeps improving even for higher s .

Proof of Lemma 4.1. The simple case is when the eigenvectors of M are orthogonal. Then, M_s expressed in the basis of eigenvectors is diagonal with

$$0, \dots, 0, \lambda_{s+1}, \lambda_{s+2}, \dots \quad (4.12)$$

on the diagonal, proving the lemma. In the general case, we work in the basis obtained by the Gram-Schmidt process from the eigenvectors of M . In this basis, all the involved matrices are upper triangular with eigenvalues on the diagonal. In particular, M_s is upper triangular with Eq. (4.12) on the diagonal. ■

1. Can we increase the convergence domain of the Newton method?

The Newton method described above has a somewhat larger local domain of convergence than what is given by Eq. (4.7). One can start the Newton iterations from $A_{(0)} = A^{(n')}$ corresponding to a smaller number n' of initial RG steps. One can also relax the accuracy Δt of the initial critical temperature. Our tests show that $\Delta t = 10^{-7}$ and $n' = 15$ is already enough for the subsequent Newton iterations to converge. This corresponds to the distance 2.4×10^{-2} between the fixed-point tensor and the initial tensor of the Newton iterations, an order of magnitude larger than Eq. (4.7).

At the moment, we are unable to further increase the Newton method convergence domain. The following simple changes of the algorithm do not help: Reevaluate $J_\approx$ after every Newton step; use larger s ; apply the inverse Jacobian via the GMRES algorithm rather than via our approximate procedure; using “damping,” i.e., decreased step size if the full Newton step does not lead to a decrease in the error $\|R^\circ(A) - A\|$. Having examined this issue closely, we traced it to the lack of differentiability of the RG map in a larger domain, due to the discrete gauge-fixing ambiguity. Our discrete gauge-fixing procedure, described in Appendix D, consists in choosing a set I_+ of tensor elements and making them all positive by discrete sign flips on the individual legs. This stops being differentiable when one of the tensor elements crosses zero. We observed that this does tend to happen when one considers the RG map in a sufficiently large domain. We are hopeful that, if this problem is resolved, the Newton method convergence domain could be much increased, all the way to include

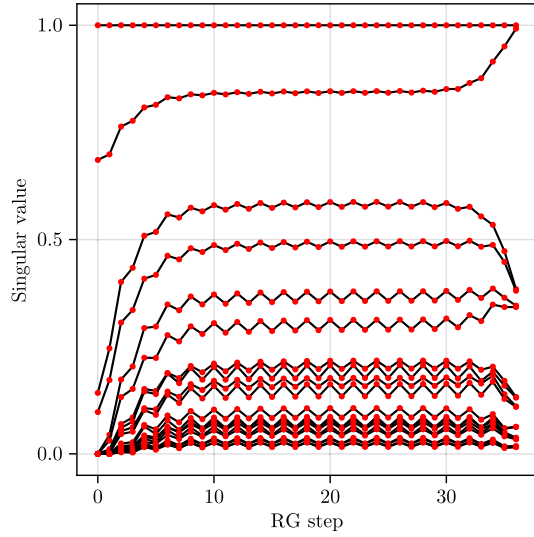


FIG. 12. Period-2 behavior for the rotating RG map. The figure shows the flow of tensors starting from $A^{(0)}$ corresponding to the Ising model at the reduced temperature $t = 1.000\,009\,485\,7$ and with anisotropy $a = 0.8$. As in Fig. 4, we plot the 20 largest singular values along the diagonal normalized by the first one.

the critical nearest-neighbor (NN) Ising tensor $A_{\text{NN}}(1)$. Of course, for a faraway convergence, one will have to recompute the inverse Jacobian at every RG step, at least until one gets sufficiently close to the fixed point.

C. Anisotropy: Period-2 oscillations

Recall that for the nonrotating RG map the eigenvalues of the Jacobian corresponding to the stress-tensor components T and $\bar{T}$ are 1, and we expect a two-dimensional manifold of fixed points for this map. (A one-dimensional slice of this manifold was studied in Sec. III. 5.) On the contrary, for the rotating RG map these eigenvalues are -1 , and we expect an isolated fixed point embedded in a two-dimensional manifold of period-2 orbits (see Sec. II. 5). We can see this period-2 behavior by studying the flow of the singular values of A across the diagonal as we did in Fig. 4 for the nonrotating RG map. We start with an initial A representing the anisotropic Ising model with $a = 0.8$. The flow of the resulting singular values shown in Fig. 12 clearly shows period-2 behavior. Figure 12 shows only a single value of a , but we observed that the amplitude of the oscillations increases for larger $|a - 1|$. These results confirm our intuition about the structure of the RG flow.

V. SUMMARY AND OPEN PROBLEMS

In this paper, we showed that the fixed-point equation of tensor RG can be solved via the Newton method. This needs a judicious modification of the RG map, to include a $\pi/2$ rotation of the lattice. The rationale is to avoid eigenvalue 1 deformations of the fixed point, which are problematic for the Newton method. The corresponding

eigenvectors are associated with the stress-tensor CFT deformations, which have spins $\ell = \pm 2$. The $\theta = \pi/2$ rotation turns their eigenvalues from 1 into $e^{i\theta\ell} = -1$, eliminating the problem.

Our numerical tests of this general idea used Gilt-TNR [11], one of several known tensor RG algorithms which implement the disentangling procedure necessary to solve the CDL problem. Working at bond dimension $\chi = 30$, the Newton method was shown to converge to the fixed point in the Hilbert-Schmidt distance, with accuracy limited only by roundoff errors. To our knowledge, this is the first time that the Newton method was used to find the critical tensor RG fixed point. Working in double-precision arithmetic, we achieved the accuracy $\delta \sim 10^{-9}$. This is much better than what was previously achieved with the traditional shooting method of recovering the fixed point, where one fine-tunes the relevant temperature perturbation and removes the irrelevant perturbations by repeated RG steps ($\delta \sim 10^{-2}$ [21]; $\delta \sim 3 \times 10^{-4}$ [22]; $\delta \sim 5 \times 10^{-5}$ in our Fig. 5). We emphasize that this improvement is a result of using our Newton method rather than the usual shooting method, not an artifact of Gilt-TNR. We expect our method for finding the fixed point will produce similar improvement for other algorithms.

We conclude by listing several open problems raised by our work.

- (1) Can one solve the discrete gauge-fixing issue mentioned in Sec. IV. 2. 1 and obtain a globally differentiable tensor RG map, for which the Newton method convergence domain could be increased beyond what is achieved here, perhaps to include the critical NN Ising tensor $A_{\text{NN}}(1)$?
- (2) It would be interesting to apply the Newton search to locate and study fixed points corresponding to other universality classes, beyond the Ising and three-state Potts considered in our work, such as the following.
 - (i) The tricritical Ising fixed point, initializing the algorithm by a tensor corresponding to the tricritical point of the Blume-Capel model, whose location is known with good accuracy from lattice Monte Carlo studies [85]. The tricritical fixed point having two relevant symmetric deformations, its study would be more challenging using the shooting method but should pose no problem to the Newton method.
 - (ii) The Lee-Yang fixed point corresponding to the Ising model in a critical imaginary magnetic field [86]. This model is not amenable to Monte Carlo studies due to the sign problem. See Ref. [87] for a previous tensor RG study using the shooting method.
 - (iii) Fixed points which do not correspond to exactly solved CFTs, unlike all fixed points mentioned so far. One example is the coupled three-state Potts model [88].

- (3) In this paper, we relied on Gilt-TNR, with static Q choice (Appendix C.2) to ensure differentiability. Can the Newton method be set up for other tensor RG algorithms with disentangling, e.g., for TNR [7], combining them with $\pi/2$ rotation? Our logic about the rotation getting rid of eigenvalues 1 should be robust. But is TNR differentiable, or even continuous, in spite of the iterative optimization of disentanglers which may be trapped in a local minimum (Ref. [70])?
- (4) In this paper, we argued on general grounds and provided numerical evidence that the RG map Jacobian at the fixed point has two kinds of eigenvalues. Eigenvalues associated with quasiprimary CFT operators are universal and predictable from CFT, while eigenvalues of derivative operators are not universal and their values cannot be predicted. Our Table III contains several eigenvalues which do not agree with any CFT quasiprimary, and we associated their eigenvectors with derivative operators. Is there a test via an explicit computation involving the eigenvectors and the fixed-point tensor which can confirm this assignment?
- (5) One somewhat technical issue concerns finding a fully robust gauge-fixing procedure for symmetry-breaking perturbations of the fixed point; see Appendix D.1.
- (6) Why is gauge fixing with orthogonal gauge transformations enough for a tensor RG map to have a fixed point? This has a satisfactory explanation for maps preserving reflection symmetry (Appendix D.3), but it is puzzling why this works for many maps used in the literature and in our work, which do not preserve it. In Appendix D.3.2, we tried to explain this conceptual puzzle via an argument appealing to a “phantom” RG map which has not been provided. Can one either exhibit this map or find another argument which does not postulate its existence?
- (7) Are there practical tensor RG maps which manifestly preserve rotations while solving the CDL problem? One such map of unclear practicality is mentioned in Ref. [48] (it acts on a set of three tensors and has a larger than usual associated set of gauge symmetries). If that map, or any other rotation-invariant map, were available, one could completely circumvent the eigenvalues 1 problem, implementing the Newton method directly within the subspace of rotation-invariant tensors, as discussed in Sec. II.4.
- (8) (See also Sec. III and the end of Sec. III B.) In this paper, we studied the fixed-point tensors for a fixed value of $\chi = 30$ (and sometimes also for $\chi = 10, 20$). This was enough to defend our main points. However, as discussed in Appendix C.3, higher bond dimensions up to $\chi = 120$ used in Ref. [11]

may well be accessible, provided that the Jacobian is evaluated via the automatic differentiation instead of finite differences used here. One could then get evidence for or against the hypothesis that the fixed-point tensors converge as $\chi \rightarrow \infty$ to a limiting tensor (which is then necessarily Hilbert-Schmidt). This hypothesis may or may not be true for the Gilt-TNR map, but hopefully there exists a tensor RG map for which it is true.

- (9) Finally, our work is also an important stepping stone toward proving the existence of the critical tensor RG fixed points at the mathematical level of rigor, the program started in Refs. [16,17]. For this dream to be realized, the tensor RG fixed-point equation has to be reformulated as a fixed-point equation for a map which is a contraction in a neighborhood in an appropriate Banach space of infinite-dimensional tensors. In this paper, we have found a numerical realization of a baby version of this program in a finite-dimensional space of truncated tensors.

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APPENDIX A: TENSOR CONVENTIONS

1. Tensor RG

Let A be a four-legged tensor of bond dimension χ . Consider a periodic tensor network contraction made out of A arranged in an $N \times M$ grid, denoted by

$$Z(A, N \times M), \quad (\text{A1})$$

and called the partition function of the tensor network. We assume that it represents the partition function of a lattice model of spins with periodic boundary conditions on the $N \times M$ lattice with short-range interactions, such as Ising or three-state Potts [90].

We normalize A to have $\|A\| = 1$, where $\|A\|$ is the Hilbert-Schmidt norm.

Tensor RG maps considered in this paper fall into two categories: (i) nonrotating and (ii) involving rotating the

network by $\pi/2$ (“rotating”). Nonrotating tensor RG maps an $N \times M$ tensor network (A1) made of A to an $N/b \times M/b$ tensor network made of another tensor $A' = \mathcal{R}(A)$ so that the partition function remains invariant:

$$Z(A, N \times M) = Z(A', N/b \times M/b) \quad (\text{nonrotating}). \quad (\text{A2})$$

On the other hand, for tensor RG maps with a $\pi/2$ rotation, invariance of the partition function is written as

$$Z(A, N \times M) = Z(A', M/b \times N/b) \quad (\text{rotating}). \quad (\text{A3})$$

There are, of course, many different tensor RG maps, differing in how A' is obtained from A . Some concrete maps were discussed in the main text. Here, we keep things general.

In general, A' is not normalized. Multiplying the tensor of which the tensor network is made by a constant factor C is a relatively minor change, since such a factor can always be factored out from the partition function. It changes the partition function by C^N , where N is the number of tensors in the network. However, this does not change the phase of the microscopic model that the network represents. To discuss phases and fixed points, we have to normalize tensors somehow. We define the normalized RG map by $R(A) = A' / \|A'\|$, so that the RG flow happens in the space of unit-norm tensors. We most often work with $R(A)$, which is the most important part of the non-normalized RG map $\mathcal{R}$. We sometimes abuse the terminology and refer to R as the RG map, dropping “normalized.” An RG fixed point for the map R is a tensor A_* satisfying $R(A_*) = A_*$.

2. Tensor network symmetries

A symmetry transformation of a tensor network is a faithful linear action of a group G on the space of tensors A :

$$T_g: A \mapsto A' \quad (g \in G), \quad (\text{A4})$$

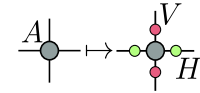
which preserves the partition function of a tensor network built of A :

$$Z(A, N \times M) = Z(T_g A, N \times M). \quad (\text{A5})$$

We say that the tensor network built out of a particular tensor A_0 respects this symmetry, if A_0 is invariant under this action: $T_g A_0 = A_0$ for any $g \in G$.

We consider on-site (global) symmetries and spatial symmetries. An on-site symmetry transformation

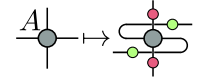
multiplies tensor A by matrices on each leg. For example, an on-site $\mathbb{Z}_2$ acts as



$$A \mapsto V \otimes H \quad (\text{A6})$$

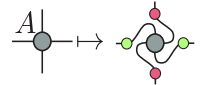
for some matrices H and V such that $H^2 = V^2 = I$.

A spatial symmetry permutes the legs of a tensor and simultaneously multiplies by matrices on each leg. Spatial symmetries mentioned in this paper, on the square lattice, involve a combination of axis reflections and rotations, forming the group D_4 or one of its subgroups $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z}_4$. For example, the horizontal reflection acts as



$$A \mapsto V \otimes H \quad (\text{A7})$$

while rotation by $\pi/2$ acts as

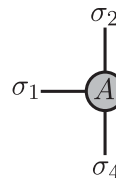


$$A \mapsto V \otimes H \quad (\text{A8})$$

The matrices multiplying the legs must be chosen to get a representation of the spatial symmetry group. Note that they are not, in general, equal to the identity.

APPENDIX B: INITIAL TENSORS FOR TENSOR RG EVOLUTION

Here, we describe the initial tensor for the RG evolution used in our studies, in both the isotropic and anisotropic case. There is more than one way to represent the NN Ising partition function as the contraction of a tensor network. The method we use in this paper is the same as used in Ref. [4]. In this method, the lattice for the Ising model is rotated by $\pi/4$ with respect to the tensor network. In Fig. 13, the sites for the Ising spins are represented by black dots. The coupling constants on the bonds between these spins are equal to J_y for the red bonds and J_x for the green bonds. Note that the Ising spins are located at the midpoints of the bonds in the tensor network. Each leg in the tensor network takes on only the values -1 and $+1$, corresponding to the values of the Ising spin. The contraction of the tensor network is then equal [91] to the NN Ising partition function if we define the components of the tensor A by



$$A_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} = \exp((J_y \sigma_1 \sigma_2 + J_x \sigma_2 \sigma_3 + J_y \sigma_3 \sigma_4 + J_x \sigma_4 \sigma_1)/T). \quad (\text{B1})$$

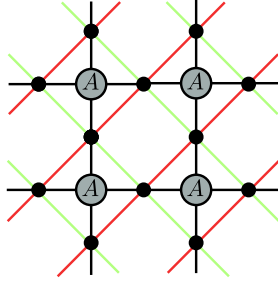


FIG. 13. The Ising spins (black dots) are located at the midpoints of the bonds in the tensor network. The coupling constants for the Ising model are J_y for the red bonds and J_x for the green bonds.

The final step in the definition of the initial A is to apply a gauge transformation that transforms the states $|+\rangle$ and $|-\rangle$ to the $\mathbb{Z}_2$ -even and -odd states, respectively, given by

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \quad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle). \quad (\text{B2})$$

We define $a = J_y/J_x$. For the isotropic case, we set $J_x = J_y = 1$ and then define the reduced temperature by $t = T/T_c$ so that $t = 1$ is the critical point. For the anisotropic case, we set $J_y = a$ and $J_x = 1$ and again let $t = T/T_c$, where now the critical temperature depends on a [92]. As in the isotropic case, the critical point is given by $t = 1$. Only the parameters t and a are used in our discussions of the numerical results. In the numerical results, the critical point is not exactly at $t = 1$, since the bond dimension is finite.

Since the Ising lattice is rotated with respect to the tensor network, in the anisotropic case, the initial A is not invariant under horizontal or vertical reflections. However, it is invariant under the composition of these two reflections.

The initial tensor, which we normalize to have unit Hilbert-Schmidt norm, is denoted $A_{\text{NN}}(t, a)$, or $A_{\text{NN}}(t)$ if $a = 1$.

Another method for representing the Ising partition function as the contraction of a tensor network can be found in Refs. [6,11]. In this approach, the Ising lattice is not rotated with respect to the tensor network.

APPENDIX C: MORE DETAILS ABOUT GILT-TNR

We reviewed the general structure of the nonrotating Gilt-TNR algorithm in Sec. III. 1, and here we provide some more details.

The basic procedure for choosing $Q = Q_1$ (see Sec. IV. B in Ref. [11] and Appendix A in Ref. [21]) consists of four steps.

- (i) Consider the singular value decomposition of an environment shown in Fig. 14(a), built out of four A tensors contracted over three out of four bonds

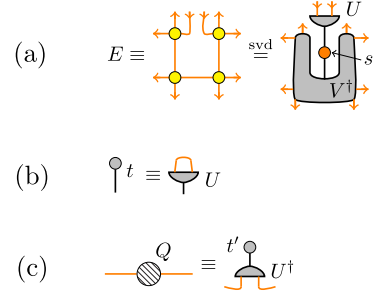


FIG. 14. Defining the matrix $Q \equiv Q_1$: (a) environment and singular value decomposition; (b) vector t ; (c) matrix Q . Figures from Ref. [21].

around the plaquette, omitting the bond into which Q_1 will be inserted.

- (ii) Form vector t by contracting two ingoing legs of U [Fig. 14(b)].
- (iii) Perform soft truncation of vector t , defining vector t' by

$$t'_i = t_i \frac{s_i^2}{s_i^2 + \epsilon_{\text{gilt}}^2}, \quad (\text{C1})$$

where s_i are the singular values in Fig. 14(a) and ϵ_{gilt} is one of the three parameters of the Gilt-TNR algorithm (Table V). Basically, this equation sets to near zero the components of t'_i corresponding to $s_i \ll \epsilon_{\text{gilt}}$.

- (iv) Finally, define matrix Q from tensor $U^\dagger$ and the vector t' as in Fig. 14(c).

The rationale why this construction is a good way to filter short-range correlations from the plaquette was given in Ref. [11].

In practice, however, the above description is just the first step in an iterative procedure of determining Q_1 , Q_2 , Q_3 , and Q_4 . We need two versions of the algorithm, which differ slightly but essentially in how this is implemented.

- (i) *Dynamic Q choice*—That is the one implemented in the original code [58].
- (ii) *Static Q choice*—This is our modification, essential for ensuring differentiability.

1. Dynamic Q choice

This is only outlined in Sec. IV. B in Ref. [11]. Here, we give a more detailed description, which corresponds to

TABLE V. Parameters of Gilt-TNR. Results in this paper use $\epsilon_{\text{conv}} = 10^{-2}$.

χ	Tensor bond dimension
ϵ_{gilt}	Soft truncation parameter in Eq. (C1)
ϵ_{conv}	Convergence parameter; see Eqs. (C2) and (C4)

what is actually done in Ref. [58]. The dynamic algorithm depends on a parameter ϵ_{conv} , whose default value is 10^{-2} ; it appears in Eq. (C2) below.

We start by determining $Q = Q_1$. To do that, one carries out steps 1–9 for $k = 0, 1, 2, \dots, k_{\text{max}}$. Here, k_{max} is determined dynamically during the run of the algorithm, and it depends on ϵ_{conv} as explained below. Smaller ϵ_{conv} lead to larger k_{max} .

- (1) Start with the environment $M^{(k)}$ and the matrix $Q^{(k)}$. At the zero step, $M^{(0)} = EE^\dagger$, where E is in Fig. 14(a) and $Q^{(0)} = 1$. By forming this environment out of E , one avoids having to compute matrix V in Fig. 14(a), which is not needed in the algorithm.
- (2) Do approximate SVD of $Q^{(k)} \approx (Q^{(k)})_L S_{Q^{(k)}} (Q^{(k)})_R$. Here, all the singular values smaller than $10^{-3}\epsilon_{\text{gilt}}$ are thrown away. The factor 10^{-3} is hard coded in Ref. [58].
- (3) Define $M^{(k+1)}$ by absorbing $(Q^{(k)})_L \sqrt{S_{Q^{(k)}}}$ and $\sqrt{S_{Q^{(k)}}} (Q^{(k)})_R$ into $M^{(k)}$.
- (4) By the eigenvalue decomposition of $M^{(k+1)}$ determine $U^{(k+1)}$ and $S^{(k+1)}$.
- (5) Form the vector t .
- (6) Form the vector t' .
- (7) Define $Q^{(k+1)}$.
- (8) Do approximate SVD of $Q^{(k+1)} \approx Q_L^{(k+1)} S_{Q^{(k+1)}} Q_R^{(k+1)}$. Again, all the singular values smaller than $10^{-3}\epsilon_{\text{gilt}}$ are thrown away.
- (9) Check if

$$\max_i |(S_{Q^{(k+1)}})_i - 1| < \epsilon_{\text{conv}}. \quad (\text{C2})$$

If this condition does not hold, then go to step 1 with $k \mapsto k + 1$. Otherwise, we set $k_{\text{max}} = k$ (this is just to store the total number of carried out iterations) and go to the next step. The rationale for condition (C2) is that we would like Eq. (C3) below to converge.

- (10) Define Q via $Q^{(0)}, \dots, Q^{(k_{\text{max}})}$ by combining all absorbed pieces in the appropriate order:

$$Q = \prod_{k=0}^{k_{\text{max}}} \left(Q_L^{(k)} \sqrt{S_{Q^{(k)}}} \right) \prod_{k=k_{\text{max}}}^0 \left(\sqrt{S_{Q^{(k)}}} Q_R^{(k)} \right). \quad (\text{C3})$$

- (11) Define B_1 and B_2 tensors by splitting Q (using approximate SVD, throwing away eigenvalues $< 10^{-3}\epsilon_{\text{gilt}}$ again) and absorbing its halves into A .

This ends the procedure for Q_1 . One then proceeds to do the same for matrices Q_3 , Q_2 , and Q_4 , in this order [68]. Of course, when we proceed to Q_3 , we use tensors B_1 and B_2 to construct the relevant environments. Once Q_3 is found, we update B_1 and B_2 and proceed to Q_2 , etc.

Once the Q_a ($a = 1, \dots, 4$) are constructed, we say that we “completed a lap around the plaquette.” We then check

the spectra of $Q_a = Q_{aL} S_{Q_a} Q_{aR}$ (again truncated SVD) and ask if

$$\max_i |(S_{Q_a})_i - 1| < \epsilon_{\text{conv}}. \quad (\text{C4})$$

If this condition does not hold for one or more bonds around the plaquette, we are supposed to run the above cycle again for those bonds, redefining yet again matrices Q_a and tensors B_1 and B_2 . This will be the second, third, etc., lap. In practice, one usually needs two laps.

Let us give a couple of examples of how this works in practice. Both examples are for $\chi = 30$, $\epsilon_{\text{gilt}} = 6 \times 10^{-6}$, and $\epsilon_{\text{conv}} = 10^{-2}$.

- (1) Applied to $A^{(n_*)}$ in Table III, the dynamic algorithm does two laps, with the following number k_{max} of k iterations for Q_1 , Q_3 , Q_2 , and Q_4 :

$$\text{lap 1: } (19, 29, 10, 28); \quad \text{lap 2: } (1, 1, 1, 1). \quad (\text{C5})$$

- (2) Applied to $A_{(0)}$ in Sec. IV. 2, the dynamic algorithm does two laps, with the following number k_{max} of k iterations for Q_1 , Q_3 , Q_2 , and Q_4 :

$$\text{lap 1: } (22, 8, 24, 20); \quad \text{lap 2: } (1, 1, 1, 1). \quad (\text{C6})$$

2. Static Q choice

In the static algorithm, we do not need the ϵ_{conv} parameter. Instead, we specify the “static instruction,” which includes (i) the number of laps and (ii) k_{max} values for Q_1 , Q_3 , Q_2 , and Q_4 for each lap. The checks of conditions (C2) and (C4) of the dynamic algorithm are omitted. The rest of the static algorithm is the same as the dynamic algorithm. This modification ensures that the static algorithm is continuous and differentiable.

The static algorithm is used when we need the RG map to be differentiable, as when the Jacobian is involved. To select a static instruction, we typically run the dynamic algorithm once for a tensor around which we intend to compute the Jacobian.

3. Parameter choices and scaling

In our paper, the bond dimension χ was varied in the range 10–30, and the parameter ϵ_{gilt} was varied from 10^{-4} for $\chi = 10$ to 6×10^{-6} for $\chi = 30$.

We use the dynamic Q choice in calculations which do not need the Jacobian, such as the determinations of the critical temperature via bisection, and Figs. 4, 5, and 12. In all dynamic Q -choice calculations, the parameter ϵ_{conv} is fixed to 10^{-2} (the default value of Ref. [58]).

We use the static Q choice when computing the Jacobian eigenvalues and when using the Newton method which needs the map to be differentiable. When computing the Jacobian eigenvalues at $A^{(n_*)}$ from Table III, we use the

static instruction (C5) obtained by running once the dynamic algorithm at that tensor. In Sec. IV.2, we use the dynamic Q choice for the RG iterations from $A^{(0)}$ to $A^{(n')}$. Then, we switch to the static Q choice with the static instruction (C6) from the dynamic algorithm at $A_{(0)} = A^{(n')}$. We use this static instruction for all subsequent calculations in that section:

- (i) computing the eigenvalue decomposition of $\nabla R^\circ(A_{(0)})$ which enters $J_\approx$;
- (ii) running the Newton method and producing Fig. 9; and
- (iii) computing the Jacobian eigenvalues ∇R° at $A_{(m_*)}$.

In this paper, we limit ourselves to $\chi \leq 30$, as this is enough to demonstrate our main points regarding the applicability of the Newton method. Reference [21] also used $\chi = 30$. In the future, it would be interesting to carry out our analysis for larger χ . (Reference [11] went up to $\chi = 120$.) In the rest of this section, we discuss how various steps of our algorithms scale with χ and which other difficulties may arise when increasing χ .

The computational cost of one Gilt-TNR algorithm step scales as $O(\chi^6)$. Evaluating the first s eigenvalues and eigenvectors of $\nabla R(A)$ at any point A using the Arnoldi method scales as $O(s\chi^6)$ [93]. This operation is required only once when setting up our Newton method with approximate Jacobian, at $A = A_{(0)}$. For $\chi = 30$ and $s = 9$, it takes 14 min on a laptop. One Newton method step costs $O(s^2\chi^4 + \chi^6)$, where χ^6 comes from the Gilt-TNR step and $s^2\chi^4$ from applying $[\mathbb{1} - P_s \nabla R(A_{(0)}) P_s]^{-1}$ to $A - R(A)$. For example, for $\chi = 30$ and $s = 9$, a laptop can perform 30 Newton method steps from $A_{(0)}$, reaching the approximate fixed point, in 4 min (yellow curve in Fig. 9).

The most time-consuming part of our calculations was tuning the critical temperature, which remains necessary, as the Newton algorithm may not converge to the critical point

from a remote starting tensor. Finding the critical temperature using the bisection method with $\Delta t = 10^{-10}$ takes around 80 min on a laptop.

A potential issue for going to larger χ has to do with the error in evaluating the Jacobian via finite differences. As discussed in Appendix E, this error cannot be made arbitrary small due to roundoff errors. For $\chi = 30$, the error for an optimal step size h was quite small (see Fig. 15), at least for derivatives in large eigenvalue directions. However, we observe that the optimal error attainable for the finite difference method increases with χ . Extrapolating from $\chi = 10, 20, 30$, we suspect that for, e.g., $\chi = 60$ it may reach values which will make the finite difference method unusable. Automatic differentiation techniques [94] famously avoid roundoff errors inherent for the finite difference method. We expect that a rewrite of our code using automatic differentiation would allow one to go to higher χ .

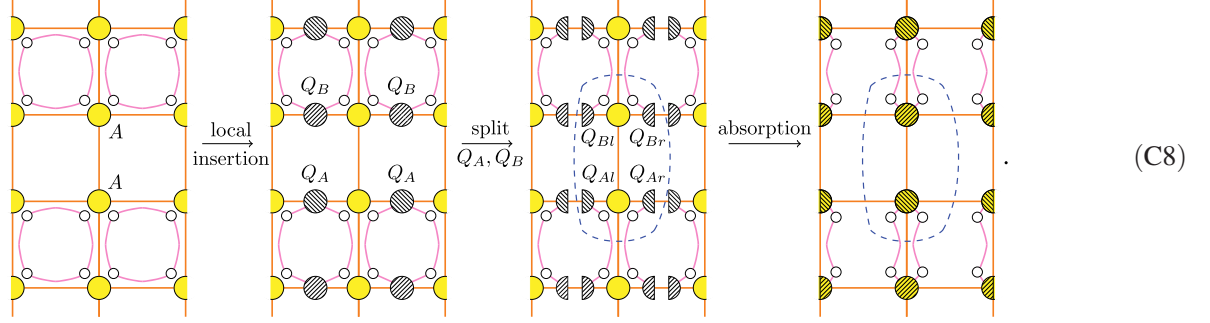
4. Gilt-HOTRG [21]

In this section, we describe the algorithm used in Refs. [21,95], which somewhat differs from Gilt-TNR used in Ref. [11] and our work. We refer to it as Gilt-HOTRG. Having this description is useful to understand and appreciate some details of the results of Ref. [21], such as, for example, the fact that their linearized RG map coincides with the lattice dilatation operator and, hence, its eigenvalues agree with (exponentiated) CFT scaling dimensions even for descendants [27].

As the name suggests, Gilt-HOTRG [21] combines Gilt with HOTRG [6]; i.e., it uses Gilt to filter local correlations and HOTRG for coarse graining, while Gilt-TNR uses TRG for coarse graining. The main equation which defines their RG transformed tensor in terms of the original tensor is [figure from Ref. [21], Eq. (31)]

$$A_c = \text{[Network of } A, Q_B, Q_A, Q'_B, Q'_A \text{ tensors]} \quad (C7)$$

Let us explain this equation further. As a first step, one chooses Gilt matrices Q_B and Q_A , applying the Gilt procedure to horizontal links of a subset of plaquettes (figure from Ref. [21], Fig. 11):



As one can see in Eq. (C8), matrices Q_B and Q_A are then SVD decomposed, and their halves are absorbed into A [this step breaks reflection symmetry, since, in general, $Q_{Br} \neq (Q_{Bl})^t$ and $Q_{Ar} \neq (Q_{Al})^t$]. One then performs HOTRG coarse graining in the vertical direction, using isometries w and $w^\dagger$ in Eq. (C7) [see also Ref. [21], Eq. (A6)].

As a second step, the whole procedure is repeated in the horizontal direction. The final result is an RG map which reduces the lattice size by factor of 2 in both directions.

The authors of Ref. [21] do not mention if they use iterative optimization of Gilt matrices.

Gilt-HOTRG is a nonrotating RG map, so it has eigenvalues 1 around the fixed point. Like Gilt-TNR, it can be accompanied by a rotation, turning eigenvalues 1 to -1 and making the Newton method applicable.

APPENDIX D: GAUGE FIXING

As mentioned in Sec. III. 2, to see that the tensors $A^{(n)}$ converge to a fixed point, we need to apply after each Gilt-TNR step an appropriately chosen gauge transformation. Tensors related by a gauge transformation are physically equivalent, as they yield the same partition function. We need to choose a unique tensor from each gauge equivalence class to discuss the fixed-point tensor. This can be done by employing a gauge-fixing procedure.

A general gauge transformation preserving the partition function would consist in multiplying the tensor on horizontal legs with matrices G_h^{-1} and G_h , where G_h is an invertible $\chi \times \chi$ matrix, and similarly on the vertical legs with G_v^{-1} and G_v . In this work, we use a restricted class of gauge transformation where G_v and G_h are orthogonal matrices.

Why are orthogonal gauge transformations enough? This may be expected by looking at the trajectories of singular values along the diagonal, which show plateaux (Fig. 4). Indeed, singular values are invariant under orthogonal gauge transformations [see Eq. (3.6)], while they would not be invariant under more general gauge transformations with invertible matrices. For further rationale, see Appendix D. 3.

For orthogonal gauge transformations, we have $G_h^{-1} = G_h^T$ and $G_v^{-1} = G_v^T$. Graphically, such a gauge transformation acts on a four-legged tensor A of bond dimension χ as follows:

$$A \mapsto A_G = \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ | \\ \text{---} \text{---} \end{array} \mapsto A_G = \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ | \\ \text{---} \text{---} \end{array} \begin{array}{c} G_v \\ G_h \end{array}, \quad (\text{D1})$$

where $G_v, G_h \in O(\chi)$.

“Gauge fixing” means that we apply the gauge transformation (D1) with some appropriately chosen G_v and G_h . Our gauge-fixing procedure involves continuous gauge fixing followed by discrete gauge fixing. Continuous gauge fixing is done as follows. Consider the following two contractions of two copies of A , called environments:

$$(\text{env}_v(A))_{ij} = \begin{array}{c} i \\ \text{---} \text{---} \text{---} \text{---} \\ | \\ \text{---} \text{---} \end{array} \begin{array}{c} j \\ \text{---} \text{---} \end{array}, \quad (\text{env}_h(A))_{ij} = \begin{array}{c} i \\ \text{---} \text{---} \end{array} \begin{array}{c} j \\ \text{---} \text{---} \end{array}. \quad (\text{D2})$$

The contractions in Eq. (D2) are chosen in such a way that both $\text{env}_v(A)$ and $\text{env}_h(A)$ are symmetric and can be diagonalized by orthogonal matrices. We choose G_v and G_h that diagonalize these environments. Then we perform the gauge transformation (D1). The tensor A_G is the result of continuous gauge fixing.

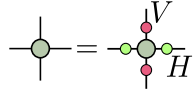
There is some ambiguity in the choice of G_v and G_h . If the spectra of $\text{env}_v(A)$ and $\text{env}_h(A)$ are nondegenerate, as is generically true, then this ambiguity reduces to multiplying $G_v \rightarrow G_v S_v$ and $G_h \rightarrow G_h S_h$, where S_v and S_h are diagonal matrices with ± 1 on the diagonal. In terms of A_G , this amounts to applying an additional gauge transformation:

$$A_G \mapsto A_{GS} = \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ | \\ \text{---} \text{---} \end{array} \begin{array}{c} S_v \\ S_h \end{array}, \quad (\text{D3})$$

Discrete gauge fixing amounts to choosing the matrices S_v and S_h appropriately. The tensor A_{GS} is the final result of our gauge-fixing procedure.

Note that there are some replacements of S_v and S_h which leave A_{GS} invariant:

- (i) $S_v \mapsto -S_v$;
- (ii) $S_h \mapsto -S_h$;
- (iii) the simultaneous replacement $S_v, S_h \mapsto VS_v, HS_h$ provided that, as is the case for us, the tensor A_G enjoys a $\mathbb{Z}_2$ symmetry



$$\text{---} \bigcirc \text{---} = - \text{---} \bigcirc \text{---} \quad (\text{D4})$$

with H and V diagonal matrices with ± 1 on the diagonal [see Eq. (A6)].

We perform discrete gauge fixing by imposing the following gauge-fixing condition on the tensor A_{GS} :

$$(A_{GS})_{ijkl} > 0 \quad \forall (i, j, k, l) \in I_+ \quad (\text{D5})$$

for an appropriately chosen set of tensor components $I_+ \subset \{1, \dots, \chi\}^4$. In words, this says that, after the discrete gauge transformation, a certain subset of (nonzero) tensor components should be positive. Equivalently, Eq. (D5) can be written as

$$(S_v)_i (S_h)_j (S_v)_k (S_h)_l = \text{sgn}[(A_G)_{ijkl}] \quad \forall (i, j, k, l) \in I_+, \quad (\text{D6})$$

where we introduce a shorthand notation $(S_v)_i$ for $(S_v)_{ii}$.

The set I_+ should be not too large [because otherwise Eq. (D6) will not have a solution] and not too small [because otherwise Eq. (D6) will have multiple solutions and the gauge fixing will not be complete]. In other words, I_+ should be such that S_v and S_h satisfying Eq. (D6) are unique (up to the above-mentioned replacements, which do not change A_{GS}).

How do we choose a suitable I_+ and solve Eq. (D6)? Let $I_{\neq 0}$ be the set of all components such that $(A_G)_{ijkl} \neq 0$ for all $(i, j, k, l) \in I_{\neq 0}$. Consider *all* equations of the form

$$(S_v)_i (S_h)_j (S_v)_k (S_h)_l = \text{sgn}[(A_G)_{ijkl}], \quad (\text{D7})$$

where $(i, j, k, l) \in I_{\neq 0}$. Let us parametrize

$$(S_v)_i = (-1)^{v_i}, \quad (S_h)_i = (-1)^{h_i}, \quad \text{sgn}[(A_G)_{ijkl}] = (-1)^{a_{ijkl}}, \quad (\text{D8})$$

where $h_i, v_i, a_{ijkl} \in \{0, 1\}$. Then we rewrite Eq. (D7) as linear equations in modular arithmetic:

$$v_i + h_j + v_k + h_l = a_{ijkl} \pmod{2}. \quad (\text{D9})$$

Note that the number of these equations is much higher than the number of unknowns; thus, we cannot hope to solve all of them (in other words, we cannot hope to find a discrete gauge transformation that makes *all* nonzero elements of A_{GS} positive). Instead, we choose a specific subset $I_+ \subset I_{\neq 0}$ and solve the corresponding subset of equations. To choose I_+ , we first order Eq. (D9) in the order of decreasing $|(A_G)_{ijkl}|$. Then, we form the matrix M so that (ordered) Eq. (D9) can be written as

$$M\vec{x} = \vec{a}, \quad (\text{D10})$$

where $\vec{x}$ is the vector of 2χ unknowns which combines h_i 's and v_i 's. Next, we perform Gaussian elimination on M to identify a maximal independent set of rows, starting with row 1 and then trying subsequent rows one by one and adding them to the set if they are linearly independent from the previously added rows, until no more rows can be added. The resulting set of linearly independent rows is what gives us the set I_+ . We find that the size of I_+ is, as expected, $2\chi - 3$, corresponding to 2χ unknowns minus 3 transformations which were discussed above and which determine the kernel of M .

Note that the choice of I_+ described so far depends only on the ordering of Eq. (D9) in the order of decreasing $|(A_G)_{ijkl}|$ that we chose. Now we take system Eq. (D10), restrict it to rows I_+ , and solve for $\vec{x}$ in terms of $\vec{a}$. The solution is not unique, but, as discussed, different solutions are equivalent, since they give the same A_{GS} . In practice, for this last step, we use a JULIA [72,73] package ABSTRACTALGEBRA.JL [96,97], which includes a linear solver for finite fields [98]. We use this solution to fix the discrete gauge.

From the described procedure, it can be seen that I_+ as defined above remains constant in a small neighborhood U_1 of A in which the ordering of the elements $|(A_G)_{ijkl}|$ which enter into its definition does not change. This is a good feature. Indeed, if I_+ changed as A varied, this could lead to discontinuities in the RG map. We cannot allow such discontinuities: We want our RG map to be continuous and even differentiable. In our final algorithm, we enforce A independence of I_+ in an even larger neighborhood $U_2 \supset U_1$ by first computing I_+ for a tensor $A_{\approx}$ approximating the critical fixed-point tensor and then using exactly the same I_+ everywhere in U_2 . The size of U_2 is determined by the condition that no tensor element $|(A_G)_{ijkl}|$ with $(i, j, k, l) \in I_+$ vanishes in U_2 .

1. Comment about gauge fixing for $\mathbb{Z}_2$ -odd perturbations

The previous discussion solves the gauge-fixing problem with the $\mathbb{Z}_2$ -even subspace to which the fixed-point tensor A_* belongs and within which the RG evolution converging to A_* takes place. For a tensor A not belonging to the

$\mathbb{Z}_2$ -even subspace, $R(A)$ as described up to now would be computed up to a $\mathbb{Z}_2$ symmetry transformation. It is important to fix this ambiguity consistently, to get a differentiable map in a neighborhood of a fixed point [99]. In our algorithm, this is fixed as follows. The potential source of ambiguous signs is the SVD operation $M = U\Lambda V$, since multiplying row i of U by a sign s_i while simultaneously multiplying column i of V by the same sign gives an equally valid SVD. It is hard to control how these signs are chosen in a built-in SVD function. Instead, our algorithm uses a custom-made SVD function which, after calling the built-in SVD, applies the above transformation with

$$s_i = \text{sgn}\left(\sum_j w_j U_{ij}\right), \quad (\text{D11})$$

where w_j are some chosen fixed weights, e.g., $w_j = 1$.

This simple trick, combined with the gauge-fixing procedures described above, is sufficient to get a differentiable map in a neighborhood of A_* , including the $\mathbb{Z}_2$ -odd directions with some rare exceptions. When one considers a family of problems, like in Sec. III. 5, where we varied the anisotropy parameter a , this method fails for a few parameter values. The method can also fail for a few values of a when one varies the step size s in the numerical differentiation. We conjecture this happens because at some step of the algorithm a row of U occurs whose elements nearly sum to zero, so that Eq. (D11) becomes numerically ill defined. In some but not all cases, this problem can be solved by using a different weight vector in Eq. (D11), such as, e.g., $w_j = 1/(1+j)$. Further work is needed to fully understand this problem.

2. Relation with the approach of Ref. [21]

Let us now discuss how our gauge-fixing procedure relates to the procedure used in Ref. [21]. Their RG map, Gilt-HOTRG, was reviewed in Appendix C. 4 (without a gauge-fixing step). It can be seen from the definition of Gilt-HOTRG that if the tensor A is acted upon by an orthogonal gauge transformation, this has no effect on A' . This is because various unitary matrices which enter into SVD decompositions and the HOTRG step transform in such a way that the final result remains unchanged (see Appendix B in Ref. [21] for a discussion). Based on this observation, the authors of Ref. [21] observe that there is no need to perform continuous gauge fixing in their algorithm [100].

Of course, discrete gauge fixing is essential for Ref. [21] as it is for us. Regarding discrete gauge fixing, there are two main differences between our procedure and the procedure in Ref. [21]. The first one is that, instead of the search of I_+ , Ref. [21] makes a particular choice of this subset. While they do not provide exhaustive details, it appears to us that they pick I_+ to contain elements of the form

$(i^+, 1^+, 1^+, 1^+)$, $(1^+, i^+, 1^+, 1^+)$, $(i^-, 1^-, 1^+, 1^+)$, and $(1^-, i^-, 1^+, 1^+)$, where the superscript refers to the $\mathbb{Z}_2$ quantum number. The second difference is that, instead of absolute gauge-fixing condition (D5), Ref. [21] imposes a relative condition:

$$\text{sgn}[(A'_S)_{ijkl}] = \text{sgn}(A_{ijkl}) \quad \forall (i, j, k, l) \in I_+. \quad (\text{D12})$$

The choice of I_+ as above allows one to solve Eq. (D12) easily (so no need to solve a system of linear equations in modular arithmetic, as in our algorithm). Specifically, we can fix $(S_h)_{1^+} = (S_v)_{1^+} = (S_v)_{1^-} = 1$ using the symmetries discussed after Eq. (D3). Then, the solution of Eq. (D12) is

$$\begin{aligned} (S_h)_{i^+} &= \text{sgn}(A'_{i^+1^+1^+1^+})\text{sgn}(A_{i^+1^+1^+1^+}), \\ (S_v)_{i^+} &= \text{sgn}(A'_{1^+i^+1^+1^+})\text{sgn}(A_{1^+i^+1^+1^+}), \\ (S_h)_{i^-} &= \text{sgn}(A'_{i^-1^-1^+1^+})\text{sgn}(A_{i^-1^-1^+1^+}), \\ (S_v)_{i^-} &= \text{sgn}(A'_{1^-i^-1^+1^+})\text{sgn}(A_{1^-i^-1^+1^+})(S_h)_{1^-}. \end{aligned} \quad (\text{D13})$$

In our opinion, our discrete gauge-fixing method is better protected from vanishing tensor elements in I_+ than the procedure in Ref. [21]. In practice, however, both procedures work well and allow one to demonstrate the convergence of the RG flow to the critical tensor (see our Fig. 5 and Fig. 9 in Ref. [21]).

3. Why is orthogonal gauge fixing enough?

We would like to reflect here, nonrigorously, on why gauge fixing via orthogonal gauge transformations was enough to get, in Secs. 3 and 4, RG maps having fixed points.

An RG map is designed to preserve the partition function. When an RG evolution starts from an initial tensor corresponding to the critical temperature, it is reasonable to expect that, with a bit of care and luck [101], a fixed-point tensor will be reached, as long as an appropriate gauge fixing is incorporated. However, we may expect that a gauge fixing with general invertible matrices may be needed, as this is the most general transformation of tensors preserving the partition function. We know of only one work [22] which used such a general gauge fixing, by going to the minimal canonical form [78]. The works which studied the singular values along the diagonal without doing explicit gauge fixing (e.g., Refs. [11,59]) also provide evidence that orthogonal gauge fixing is enough, since general gauge fixing does not leave these singular values invariant. So, while general gauge fixing may be useful, it does not appear indispensable.

Let us assume that we are given an RG map R which has a fixed-point tensor A_* , up to gauge fixing with general invertible matrices:

$$R(A_*) = \begin{array}{c} G_v^{-1} \\ | \\ G_h^{-1} A_* G_h \\ | \\ G_v \end{array} . \quad (\text{D14})$$

(So that if we redefine R by including gauge fixing into it, A_* would become a fixed point.)

Our strategy is as follows. First, we discuss tensors and RG maps satisfying some additional assumptions, namely, the presence of reflection symmetry. Under those assumptions, we show that if Eq. (D14) holds, then a similar equation holds with orthogonal gauge-fixing matrices; i.e., orthogonal gauge fixing is enough. Then, we consider the case of weakly broken reflection symmetry, as is the case for the maps in Secs. 3 and 4.

a. With reflection symmetry

Suppose that the following conditions hold in addition to Eq. (D14).

- (i) Tensor A_* respects horizontal reflection symmetry in the form

$$\begin{array}{c} A_* \\ | \\ \text{---} \end{array} = \begin{array}{c} T_v \\ | \\ A_* \\ | \\ T_v \end{array} , \quad (\text{D15})$$

where T_v is a diagonal matrix with ± 1 on the diagonal. Note that this is not the most general form of reflection symmetry compared to Eq. (A7), as we do not include matrices on the horizontal legs.

$$\begin{array}{c} T_v \\ | \\ R_h^T P_h^{-1} A_* G_v \\ | \\ T_v \end{array} \begin{array}{c} P_h R_h \\ | \\ \text{---} \end{array} \stackrel{(4)}{=} \begin{array}{c} T_v \\ | \\ R(A_*) \\ | \\ T_v \end{array} \stackrel{(3)}{=} \begin{array}{c} \text{---} \\ | \\ R(A_*) \\ | \\ \text{---} \end{array} \stackrel{(1)}{=} \begin{array}{c} G_v \\ | \\ A_* P_h R_h \\ | \\ G_v^{-1} \end{array} \stackrel{(2)}{=} \begin{array}{c} G_v \\ | \\ R_h^T P_h A_* \\ | \\ G_v^{-1} \end{array} \begin{array}{c} P_h^{-1} R_h \\ | \\ \text{---} \end{array} . \quad (\text{D17})$$

We read it from the center toward the sides. In (1), we use Eq. (D14) replacing $G_h = P_h R_h$. In (2), we use reflection symmetry of A_* . In (3), we use reflection symmetry of $R(A_*)$. In (4), we use Eq. (D14). Now compare the rightmost and leftmost term in Eq. (D17). Using that $[G_v, T_v] = 0$ on the vertical legs and multiplying by $P_h^{-1} R_h^{-T}$ and $R_h^{-1} P_h$ on the horizontal legs, we obtain

$$\begin{array}{c} P_h^{-2} A_* P_h^2 \\ | \\ \text{---} \end{array} = \begin{array}{c} A_* \\ | \\ \text{---} \end{array} . \quad (\text{D18})$$

This equation means that A_* commutes with P_h^2 , when multiplied on horizontal legs, for any values of vertical

- (ii) Tensor A_* also respects vertical reflection symmetry in the form of equation analogous to Eq. (D15) where crossing of vertical legs is equivalent to multiplying horizontal legs by T_h .
- (iii) Tensor $R(A_*)$ also respects both horizontal and vertical reflection symmetry, in the form of equations like Eq. (D15) [with A_* replaced by $R(A_*)$] and of its vertical analog, and with exactly the same T_v and T_h .
- (iv) G_v and G_h in Eq. (D14) commute with T_v and T_h , respectively. This is very natural. The eigenvalues of T_v and T_h have the meaning of the reflection quantum numbers of the basis elements of the vertical and horizontal bond Hilbert spaces, respectively. We expect that gauge transformations may mix only vectors having the same quantum numbers (both reflections and $\mathbb{Z}_2$). For reflections, this implies $[G_v, T_v] = [G_h, T_h] = 0$.

Under the above assumptions, we can prove that an analog of Eq. (D14) holds with orthogonal gauge-fixing matrices.

Proof. Consider the polar decomposition of G_h and G_v of the form

$$G_h = P_h R_h, \quad G_v = P_v R_v, \quad (\text{D16})$$

where R_h and R_v are orthogonal and P_h and P_v are symmetric positive matrices. Note that, since in this paper we work with real tensors, we assume that G_h and G_v are real.

Now consider the following string of equations:

indices. By standard arguments, it then also commutes with $f(P_h^2)$ for any f . In particular, choosing $f(x) = \sqrt{x}$, we obtain that A_* commutes with P_h .

Repeating the above argument in the vertical direction, we obtain that A_* also commutes with P_v when multiplied on vertical legs. Combining the two properties, we obtain that the rhs of Eq. (D14), in fact, equals

$$\begin{array}{c} R_v \\ | \\ R_h^T A_* R_h \\ | \\ R_v^T \end{array} . \quad (\text{D19})$$

Thus, we obtain that $R(A_*)$ equals A_* up to an orthogonal gauge transformation. ■

Remark D.1. That orthogonal gauge fixing should be enough to get a fixed point for RG maps preserving reflection symmetry was stated without proof in, e.g., Refs. [13,28]. We do not know a place where this statement was proven before our work.

b. Without reflection symmetry

The above result would be good for us if our RG maps preserved reflections. Unfortunately, they do not. They do not even preserve reflections in a weaker form [Eq. (A7)] with matrices on both horizontal and vertical legs. Breaking of reflections occurs for two reasons. First, the matrices Q_1 , Q_2 , Q_3 , and Q_4 are chosen consecutively in Eq. (3.1). Second, when these matrices are SVD decomposed in the rhs of Eq. (3.1), the left and the right parts of the decompositions are not transposes of each other, unless the Q_i 's are positive definite, which is not generally the case.

We note in passing that the RG map in Ref. [11] (which is the same as our RG map without rotation) and the RG maps in Ref. [21] also break reflections [102]. Why then is orthogonal gauge fixing enough for all these maps, including ours, to exhibit a fixed point?

We advance here one possible reason, which has to do with our maps breaking reflections only weakly. Consider first the rotating RG map from Sec. IV. It has a fixed point attainable by the Newton method. If the map preserved reflections, the existence of a fixed point with orthogonal gauge fixing would not be surprising given the result from the previous subsection. We know that the reflections are broken, but rather weakly, as can be seen from the fact that the fixed-point tensor elements in Table III, column 3, which are related by reflections do differ but not too much. Given this, we may speculate that our map is perhaps close to some “phantom” RG map which does preserve reflections (and, hence, has a fixed point with orthogonal gauge fixing). The existence of a fixed point for our map would then be natural, given that the Jacobians of our map and of the phantom RG map are close and that it is the Jacobian which enters into the Newton method. The nonrigorous part of this argument is that we have not proved the existence of the phantom RG map.

As to the RG map without rotation, we can try to think along the same lines (with Table III, column 2 instead of column 3), up to one further subtlety. Namely, as discussed many times in this paper, the map without rotation has eigenvalues 1 due to the existence of aspect-ratio-changing deformations of the fixed point. Consider the set of critical tensors corresponding to aspect ratio 1 (rotation symmetry). These are the tensors for which the anisotropy ellipse is a circle. It is a nonlinear submanifold of the space of all tensors, over which we have little control apart from its existence. The RG map preserves the criticality and the aspect ratio and, hence, acts within this submanifold

(see Sec. II.4). Within the submanifold, there are no eigenvalues 1, and the above argument may apply.

We stress again that the just given arguments are highly nonrigorous. Unfortunately, we do not have better arguments for the moment.

APPENDIX E: GILT-TNR DIFFERENTIABILITY

Let R be the Gilt-TNR map described in Sec. III 1 followed by the gauge-fixing procedure from Appendix D. In this appendix, we explore the smoothness properties of this RG map. In particular, we provide evidence that the map R is thrice differentiable in a neighborhood of the critical fixed-point tensor A_* .

Let A be an approximation of A_* and v be a four-legged tensor with the same bond dimension. We can try to evaluate the derivative of R at A in the direction v using the symmetric finite difference approximation. This approximation depends on the step size h and is given by Eq. (3.11), which we copy here:

$$D_h = \frac{R(A + hv) - R(A - hv)}{2h}. \quad (\text{E1})$$

Without loss of generality, we assume that v is unit normalized.

Standard considerations (see Sec. V.7 in Ref. [103]) suggest that we have

$$D_h = (\nabla R) \cdot v + \epsilon_t + \epsilon_r, \quad (\text{E2})$$

where ϵ_t is the truncation error (from truncating the Taylor series) and ϵ_r is the roundoff error due to accumulated numerical errors in the evaluation of $R(A \pm sv)$. For R thrice differentiable in a neighborhood of A_* , we may Taylor-expand $R(A \pm hv)$, up to h^3 terms given the differentiability assumption. The $O(h^0)$ and $O(h^2)$ terms canceling when using the symmetric difference, we obtain $\epsilon_t \sim h^2$. On the other hand, ϵ_r can be estimated as $\epsilon_r \sim \epsilon_R/h$, where ϵ_R is the roundoff error in evaluation of R . We, thus, expect

$$D_h = (\nabla R)(A) \cdot v + O(h^2) + O(\epsilon_R/h). \quad (\text{E3})$$

Note that the $O(h^2)$ truncation error is expected to scale smoothly with h , with overall size which depends on the direction of v , while the $O(\epsilon_R/h)$ roundoff error is expected to be random but of a magnitude which is largely v independent.

Our first goal is to test this equation. We do it by computing D_h for a discrete sequence of values of h , $h_q \rightarrow 0$ geometrically and monitoring the differences $D_{h_q} - D_{h_{q+1}}$. Figure 15 shows the result of this test for Gilt-TNR parameters: $\chi = 30$ and $\epsilon_{\text{gilt}} = 6 \times 10^{-6}$, for $A = A^{(n_*)}$ from Table III, and for two choices of a

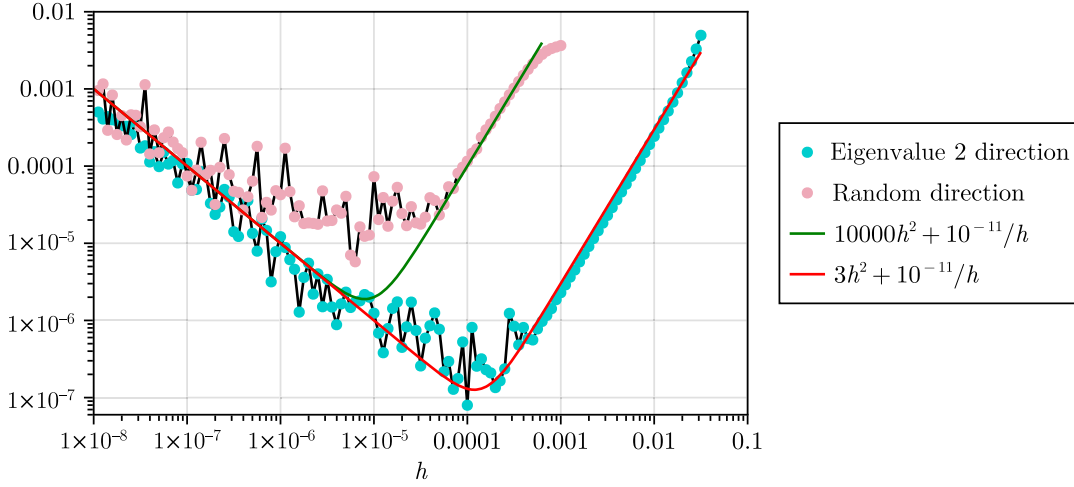


FIG. 15. The result of the differentiability test for Gilt-TNR at the approximate fixed point for $\chi = 30$ given in Table III. The green and red lines illustrate the qualitative behavior of asymptotics and serve to guide the eye. The coefficients for the h^2 and $1/h$ terms in both lines have been fixed manually to get a visually reasonable agreement and not as a result of a fit. The 10^{-11} coefficient of the $1/h$ term gives an idea of the roundoff error of Gilt-TNR.

unit-normalized direction v . The first direction is along the eigenvector of ∇R with the largest $\mathbb{Z}_2$ -even eigenvalue, $\lambda \approx 2$ [104]. The second direction is a random tensor. (We repeated this test for several random v 's, with similar results.) On the vertical axis in Fig. 15, we show the difference $\|D_{h_q} - D_{h_{q+1}}\|$ as a function of $h_q = 10^{-3-0.05q}$.

Let us discuss what we observe in Fig. 15. Consider first the test result for the random direction. In this case, D_h converges as $O(h^2)$ for $7 \times 10^{-5} < h < 5 \times 10^{-4}$. For $h > 5 \times 10^{-4}$, the points deviate from $O(h^2)$, indicating that higher-order terms become significant. For $h < 7 \times 10^{-5}$, the points exhibit chaotic behavior until around $h = 10^{-6}$, where they start aligning with the $O(\epsilon_R/h)$ line as predicted. The poor agreement of the test result with Eq. (E3) for $10^{-6} < h < 7 \times 10^{-5}$ may be due to the fact that the random vector v has a large overlap with the lower part of the ∇R spectrum, which is perhaps more sensitive to rounding errors. As to the test result for the eigenvalue 2 direction, it agrees with Eq. (E3) very well over the full range, except for $h > 0.03$, where points start deviating from the $O(h^2)$ line, which again must be due to higher-order corrections.

We consider these results to agree well with Eq. (E3), providing evidence that our RG map is at least thrice differentiable near the fixed point. Note that the coefficients of the $1/h$ chaotic behavior at small h are roughly similar in Fig. 15 for both directions, in agreement with Eq. (E3) for $\epsilon_R \sim 10^{-11}$. On the other hand, the coefficients of the h^2 truncation error are vastly different, which suggests that the third derivative along a random direction is much larger than along a typical direction corresponding to low-lying eigenvalues.

We also use Fig. 15 to optimize the choice of s in Eq. (E1). We choose $h = 10^{-4}$ for our computations, because (i) this is very close to the optimal spot for the eigenvalue 2 direction, where $O(h^2) + O(\epsilon_R/h)$ error is minimal, and (ii) the derivative in the random direction has effectively converged at $h = 10^{-4}$, and rounding errors have not yet spoiled the result. Based solely on the result for the random direction, a better choice could appear to be $h \approx 6 \times 10^{-5}$ where, despite the noise, $\|D_{h_q} - D_{h_{q+1}}\|$ takes the smallest values. However, we believe $h = 10^{-4}$ is more appropriate, as we are primarily concerned with derivatives in the directions of large eigenvalue vectors of ∇R .

It may be possible to improve the Jacobian evaluation by using automatic differentiation (as mentioned in Appendix C.3), which would remove the ϵ_R/h error, replacing it by $O(\epsilon_R)$. Or one could use a finite difference approximation of higher order, which still has ϵ_R/h error but reduces it since a larger step h can be chosen for the same accuracy. In this paper, we have not pursued these, because the error in the fixed-point determination that we achieved was 10^{-9} , already quite close to the error $\epsilon_R \sim 10^{-11}$ of a single RG step. Thus, we believe that single RG step error largely determines the final accuracy, and optimizing Jacobian evaluation is not likely to be worthwhile at this stage (but may be useful in the future).

APPENDIX F: THE 2D THREE-STATE POTTS MODEL FIXED POINT VIA THE NEWTON METHOD

Prior to this work, a tensor-RG study of the 2D three-state Potts model was provided in Ref. [105], where the

loop-TNR [8] algorithm was used to determine critical temperature, central charge, and scaling dimensions of the critical theory to a good precision. Here, we are interested in slightly different questions. This section is dedicated to finding the RG critical fixed point and studying the RG map linearization around it. We begin with a brief description of the three-state Potts model and its tensor network representation. Then, we demonstrate the result of the Newton method and examine the upper part of the spectrum of the RG map's Jacobian at the critical fixed point to verify consistency with the CFT predictions.

1. Three-state Potts model tensor network representation

The 2D three-state Potts model is defined by the following partition function:

$$Z = \sum_{s_{x,y}=\pm 1,0} e^{(1/T) \sum_{x,y} \cos((2\pi/3)(s_{x+1,y}-s_{x,y})) + \cos((2\pi/3)(s_{x,y+1}-s_{x,y}))}.$$

$$s_1 \text{---} \textcircled{A} \text{---} s_3 = e^{\frac{1}{T} \left(\cos\left(\frac{2\pi}{3}s_{12}\right) + \cos\left(\frac{2\pi}{3}s_{23}\right) + \cos\left(\frac{2\pi}{3}s_{34}\right) + \cos\left(\frac{2\pi}{3}s_{41}\right) \right)}.$$

Here, $s_{ij} = s_i - s_j$ denotes the difference between neighboring spins.

The three-state Potts model exhibits the $S_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ symmetry generated by the transformations

$$\mathbb{Z}_3: s \mapsto s + 1 \pmod 3; \quad (\text{F3})$$

$$\mathbb{Z}_2: s \mapsto -s \pmod 3. \quad (\text{F4})$$

Let η be the generator of the $\mathbb{Z}_3$ subgroup of S_3 acting on the index space of the tensor A defined in Eq. (F2):

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (\text{F5})$$

In our numerical analysis, we express A in the orthonormal basis of η 's eigenvectors $|0\rangle$, $|1\rangle$, and $|2\rangle$, corresponding to eigenvalues $e^{2\pi i q/3}$ with $q = 0, 1, 2$, respectively. We denote the resulting normalized tensor by $A_{3P}(t)$, where $t = T/T_c$ is the reduced temperature [106].

2. The Newton method results

The numerical results presented in this section are obtained using the following Gilt parameters: $\chi = 30$ and $\epsilon_{\text{gilt}} = 3 \times 10^{-5}$.

Before applying the Newton method, one needs a sufficiently accurate initial tensor $A_{(0)}$. In particular, $A_{(0)}$ should be close enough to the exact fixed-point tensor so

that they both belong to one connected region where the RG map is differentiable. To obtain $A_{(0)}$, we employ the shooting method, which yielded the finite- χ critical temperature $t_c^{[30]} \approx 0.999\,906\,314\,9$ at bond dimension $\chi = 30$.

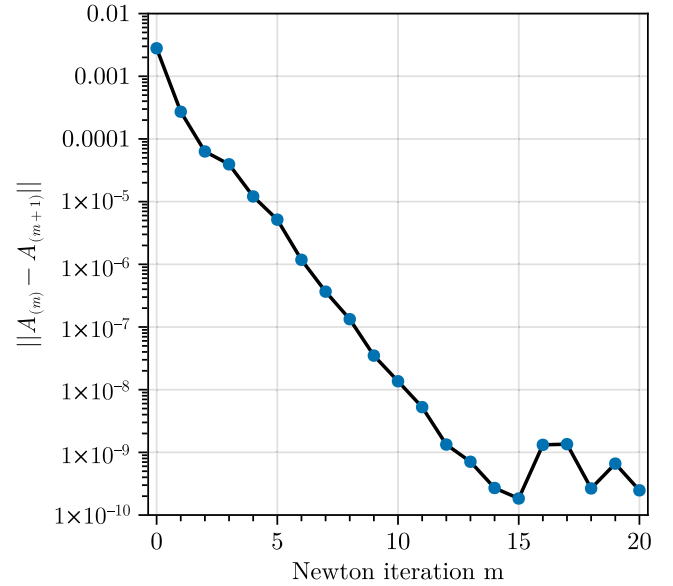


FIG. 16. Convergence of the Newton method for the three-state Potts model. Gilt parameters: $\chi = 30$ and $\epsilon_{\text{gilt}} = 3 \times 10^{-5}$. The trajectory is initiated at the tensor $A_{(0)}$ whose distance to the exact fixed-point tensor (for $\chi = 30$) can be estimated as $\sim 3 \times 10^{-3}$.

TABLE VI. 2D three-state Potts CFT Virasoro primary and quasiprimary $\mathbb{Z}_3$ -invariant operators of scaling dimension $\Delta \leq 2$, with their quantum numbers (see, e.g., Ref. [44]).

	1	ϵ	$\Phi, \bar{\Phi}$	$T, \bar{T}$
Δ	0	4/5	9/5	2
ℓ	0	0	± 1	± 2

Running the RG trajectory starting at $A_{3P}(t \approx t_c^{[30]})$ for 20 steps, we get tensor $A_{(0)}$ that approximates the fixed-point tensor with accuracy $\sim 5 \times 10^{-3}$.

Figure 16 shows the convergence of the Newton method initiated at $A_{(0)}$. For this plot, we use the projector P_s with rank $s = 16$ [see Eq. (4.2)]. We see exponential convergence similar to the Ising case. All tensors in the lower portion in Fig. 16 should equally well approximate the exact fixed point. For further computations, we choose tensor $A_{(m_*)}$ with $m_* = 15$ as our approximation of the critical fixed point, which is the last iteration where $\|A_{m+1} - A_m\|$ decreases.

A direct method to quantify the quality of our solution $A_{(m_*)}$ is to check how much the RG step changes it. Applying R° , we get

$$\|R^\circ(A_{(m_*)}) - A_{(m_*)}\| \sim 10^{-13}, \quad (\text{F6})$$

which confirms that we have converged to a solution of the RG equation.

To conclude this section, let us demonstrate that the relevant and marginal part of the Jacobian spectrum agrees with the CFT expectations. We do not consider the irrelevant part of the spectrum, as it is flooded with nonuniversal eigenvalues coming from total derivative operators (see the discussion in Sec. II.2). Our map acts in the $\mathbb{Z}_3$ -invariant space of tensors. Table VI lists the relevant and marginal $\mathbb{Z}_3$ -invariant quasiprimaries of the three-state Potts model.

The Jacobian eigenvalues should be related to the scaling dimensions and spins of nontrivial quasiprimaries (i.e., excluding 1) from Table VI via Eq. (2.14). Table VII demonstrates agreement with this prediction. However, the relative difference between the CFT predictions and the Jacobian eigenvalues is approximately an order of magnitude higher than in the Ising case. Let us comment on this.

We observe that the accuracy of physical quantities improves when ϵ_{gilt} goes down. This is due to the error introduced to the network by Gilt matrices Q becoming smaller for smaller ϵ_{gilt} (see Appendix C). However, as one decreases ϵ_{gilt} to 0, the effectiveness of Gilt at filtering out CDL tensors decreases, and eventually the RG map will not have a fixed point. Also, recall that Gilt-TNR consists of two steps: Gilt followed by TRG. The Gilt step reduces the bond dimension of the network. Smaller ϵ_{gilt} would lead to

TABLE VII. Columns 1 and 2: relevant and marginal CFT quasiprimaries ($\mathbb{Z}_3$ invariant sector) and their exact eigenvalues. Column 3: the first few largest, in absolute value, Jacobian eigenvalues at the approximate fixed point for the rotating Gilt-TNR (∇R°). Column 4: relative differences between the CFT predictions and the Jacobian eigenvalues. It is a bit hard to compare our accuracy to Ref. [105], where the results are presented as a graph and not in numerical form, but it appears comparable.

$\mathcal{O}$	$\lambda_{\text{CFT}} = i^{\ell_{\mathcal{O}}} 2^{2-\Delta_{\mathcal{O}}}$	λ of ∇R°	Relative difference
ϵ	$2^{6/5} \approx 2.2974$	2.2897	0.3%
Φ	$2^{1/5} i \approx 1.1487i$	$-0.001 + 1.1456i$	0.3%
$\bar{\Phi}$	$-2^{1/5} i \approx -1.1487i$	$-0.001 - 1.1456i$	0.3%
T	-1	-0.9958	0.4%
$\bar{T}$	-1	-0.9686	3.1%

a larger intermediate bond dimension after Gilt, leading to larger errors in the subsequent TRG steps. Thus, for a given bond dimension χ , one should try to choose ϵ_{gilt} as small as possible but such that the errors of the Gilt and TRG steps are balanced and the fixed point still exists. The existence of the fixed point can be tested by examining the shooting method results. A larger bond dimension χ generally allows for smaller ϵ_{gilt} .

For the Potts model with $\chi = 30$, we find $\epsilon_{\text{gilt}} = 3 \times 10^{-5}$ to be a good value. This is, however, much larger than $\epsilon_{\text{gilt}} = 6 \times 10^{-6}$ used for the Ising model. This larger value may explain the lower accuracy of our Potts model results. The reason for Potts model requiring larger ϵ_{gilt} is not clear. We observe, though, an evident correlation—the larger number of symmetry sectors (three in Potts vs two in Ising) requires more thorough disentangling and so larger ϵ_{gilt} .

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- [1] M. Levin and C. P. Nave, *Tensor renormalization group approach to 2D classical lattice models*, *Phys. Rev. Lett.* **99**, 120601 (2007).
 - [2] We use this term inclusively, not limited to the first TRG algorithm proposed in Ref. [1], but to all RG maps manipulating tensor networks, whether or not they involve disentangling steps or only coarse graining.
 - [3] Z. Y. Xie, H. C. Jiang, Q. N. Chen, Z. Y. Weng, and T. Xiang, *Second renormalization of tensor-network states*, *Phys. Rev. Lett.* **103**, 160601 (2009).
 - [4] Z.-C. Gu and X.-G. Wen, *Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order*, *Phys. Rev. B* **80**, 155131 (2009).
 - [5] H. H. Zhao, Z. Y. Xie, Q. N. Chen, Z. C. Wei, J. W. Cai, and T. Xiang, *Renormalization of tensor-network states*, *Phys. Rev. B* **81**, 174411 (2010).
 - [6] Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, and T. Xiang, *Coarse-graining renormalization by higher-order*

- singular value decomposition, *Phys. Rev. B* **86**, 045139 (2012).
- [7] G. Evenbly and G. Vidal, *Tensor network renormalization*, *Phys. Rev. Lett.* **115**, 180405 (2015).
- [8] S. Yang, Z.-C. Gu, and X.-G. Wen, *Loop optimization for tensor network renormalization*, *Phys. Rev. Lett.* **118**, 110504 (2017).
- [9] M. Bal, M. Mariën, J. Haegeman, and F. Verstraete, *Renormalization group flows of Hamiltonians using tensor networks*, *Phys. Rev. Lett.* **118**, 250602 (2017).
- [10] G. Evenbly, *Implicitly disentangled renormalization*, [arXiv:1707.05770](https://arxiv.org/abs/1707.05770).
- [11] M. Hauru, C. Delcamp, and S. Mizera, *Renormalization of tensor networks using graph independent local truncations*, *Phys. Rev. B* **97**, 045111 (2018).
- [12] T. Yamashita and T. Sakurai, *A parallel computing method for the higher order tensor renormalization group*, *Comput. Phys. Commun.* **278**, 108423 (2022).
- [13] X. Lyu and N. Kawashima, *Essential difference between 2D and 3D from the perspective of real-space renormalization group*, [arXiv:2311.05891](https://arxiv.org/abs/2311.05891).
- [14] Although significant progress was recently reported in Ref. [15].
- [15] X. Lyu and N. Kawashima, *Three-dimensional real space renormalization group with well-controlled approximations*, *Phys. Rev. E* **111**, 054140 (2025).
- [16] T. Kennedy and S. Rychkov, *Tensor RG approach to high-temperature fixed point*, *J. Stat. Phys.* **187**, 33 (2022).
- [17] T. Kennedy and S. Rychkov, *Tensor renormalization group at low temperatures: discontinuity fixed point*, *Ann. Henri Poincaré* **25**, 773 (2024).
- [18] N. Ebel, *3D tensor renormalisation group at high temperatures*, *Ann. Henri Poincaré* **26**, 1291 (2024).
- [19] We are aware of only one previous work [20] which tried to apply the Newton method to locate tensor RG fixed points. Their search failed beyond bond dimension $\chi = 8$, for reasons which were not clearly identified at the time but likely due to a combination of not recognizing the importance of eliminating the stress-tensor eigenvalue 1 and working with a tensor RG algorithm without disentangling.
- [20] M. Hinczewski and A. N. Berker, *High-precision thermodynamic and critical properties from tensor renormalization-group flows*, *Phys. Rev. E* **77**, 011104 (2008).
- [21] X. Lyu, RuQing G. Xu, and N. Kawashima, *Scaling dimensions from linearized tensor renormalization group transformations*, *Phys. Rev. Res.* **3**, 023048 (2021).
- [22] W. Guo and T.-C. Wei, *Tensor network methods for extracting conformal field theory data from fixed-point tensors and defect coarse graining*, *Phys. Rev. E* **109**, 034111 (2024).
- [23] This is the term we propose hoping that it will stick, as there is no generally accepted term.
- [24] G. Evenbly and G. Vidal, *Local scale transformations on the lattice with tensor network renormalization*, *Phys. Rev. Lett.* **116**, 040401 (2016).
- [25] K. Wilson and J. B. Kogut, *The renormalization group and the epsilon expansion*, *Phys. Rep.* **12**, 75 (1974).
- [26] F. Wegner, *The critical state, general aspects*, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. Green (Academic, New York, 1976), Vol. 6, pp. 8–126.
- [27] N. Ebel, T. Kennedy, and S. Rychkov, *Transfer matrix and lattice dilatation operator for high-quality fixed points in tensor network renormalization group*, [arXiv:2409.13012](https://arxiv.org/abs/2409.13012) [Phys. Rev. B (to be published)].
- [28] G. Evenbly, *Algorithms for tensor network renormalization*, *Phys. Rev. B* **95**, 045117 (2017).
- [29] M. Levin, *Real space renormalization group and the emergence of topological order*, in *Workshop Topological Quantum Computing*, Institute for Pure and Applied Mathematics, UCLA (2007), <https://www.ipam.ucla.edu/abstract/?tid=6595&pcode=TQC2007>.
- [30] This is because Hilbert-Schmidt tensors can be contracted along one bond: The result is finite and Hilbert-Schmidt. This implies that the partition function of an $N \times M$ tensor network built of Hilbert-Schmidt tensors is finite [17], while this property is not guaranteed for tensors which are not Hilbert-Schmidt.
- [31] Recently, there were claims of exact tensor network RG fixed points constructed in Refs. [32–34]. We find some aspects of those works questionable. In particular, it has not been verified that the proposed exact fixed-point tensors remain Hilbert-Schmidt when the cutoff is removed. See Ref. [35], where we also provide an argument that tensor RG without disentangling cannot have an exact RG fixed point which is Hilbert-Schmidt and corresponds to a nontrivial CFT.
- [32] A. Ueda and M. Yamazaki, *Fixed-point tensor is a four-point function*, [arXiv:2307.02523](https://arxiv.org/abs/2307.02523).
- [33] A. Ueda, *Renormalization group flow and fixed-point in tensor network representations*, Ph.D. thesis, University of Tokyo [arXiv:2401.18068].
- [34] G. Cheng, L. Chen, Z.-C. Gu, and L.-Y. Hung, *Precision reconstruction of rational conformal field theory from exact fixed-point tensor network*, *Phys. Rev. X* **15**, 011073 (2025).
- [35] N. Ebel, T. Kennedy, and S. Rychkov, *Renormalization Group with Tensor Networks: Exact Fixed Points, Hilbert-Schmidt Condition, and Necessity of Disentangling* (to be published).
- [36] The critical temperature T_c may be exactly known if the lattice model is exactly solvable, like the nearest-neighbor 2D Ising. In practical calculations at finite χ , the critical temperature shifts a bit from the exact one. To find the shifted critical temperature, one may use the bisection method; see Ref. [37].
- [37] As mentioned in Ref. [36], the critical temperature shifts from the exact $t_c = 1$ when working at finite bond dimension. To locate the finite- χ critical temperature, one uses the bisection algorithm: (i) Start from a “bracketing interval” $I = [t_l, t_h]$ such that t_l is in the low-temperature phase and t_h is in the high-temperature phase. (ii) Test the midpoint of the interval I —which phase is it in? This is done by running tensor RG for several steps until the tensor starts approaching the high-temperature or lower-temperature fixed points, which can be noticed by monitoring the singular values computed as in Eq. (3.6). (iii) Bisection: Depending on the outcome of (ii), replace the interval I by its lower or upper half I' whose end points

- are in different phases. (iv) Repeat the above three steps until the bracketing interval has length smaller than the required accuracy Δt .
- [38] This standard terminology may be a bit confusing, since temperature is a microscopic variable and cannot be varied directly around A_* . What is meant is that the eigenvector arises from the temperature perturbation of the microscopic tensor $A^{(0)}(T_c)$ after many RG steps.
- [39] For prior discussions of relations between tensor RG and CFT, see, e.g., Refs. [21,22,24,40]. Aspects related to rotation needed for us were not discussed in those references.
- [40] A. Ueda and M. Oshikawa, *Finite-size and finite bond dimension effects of tensor network renormalization*, *Phys. Rev. B* **108**, 024413 (2023).
- [41] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, *Nucl. Phys. B* **241**, 333 (1984).
- [42] Perturbing by the unit operator $\mathcal{O} = \mathbb{1}$ simply rescales the partition function. In tensor RG, this would translate to rescaling A . Since we fixed the normalization of A [Eq. (2.4)], such a perturbation is not present among the ∇R eigenvectors.
- [43] In the standard RG parlance, perturbation by a total derivative is classified as “redundant” [26].
- [44] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory* (Springer, New York, 1997).
- [45] By spatial symmetry, we mean point group symmetry. Lattice translational symmetry is always present and is left implicit. For unequal horizontal and vertical nearest-neighbor couplings $J_x \neq J_y$, spatial symmetry gets reduced to a smaller group. The final spatial symmetry depends on whether the reduction of the spin model to a tensor network is performed rotating by $\pi/4$ not. In this paper, we study the case $J_x \neq J_y$ with $\pi/4$ rotation; see Appendix B.
- [46] M. Hauru, G. Evenbly, W. W. Ho, D. Gaiotto, and G. Vidal, *Topological conformal defects with tensor networks*, *Phys. Rev. B* **94**, 115125 (2016).
- [47] G. Evenbly, Example codes: TNR, <https://www.tensors.net/tnr> (accessed: 2024-03-23).
- [48] The isotropic TNR scheme from Appendix C in Ref. [28] would preserve rotation symmetry. This complicated map acts on networks built of three different tensors. To our knowledge, it has not been implemented in practice.
- [49] If we used a tensor RG algorithm preserving one or both axis reflections, such as Refs. [7,10,46,47], we could project out the second perturbation in Eq. (2.9), but the first one would still be present.
- [50] The earliest tensor RG map [1] did change the orientation; however, the point we make—that this eliminates eigenvalues 1—did not play any role in that pioneering paper, which was focused on other issues.
- [51] We do not know of any interesting model where operators of spin $\ell = 4, 8, \dots$ have dimension 2. In reflection positive models, they should have dimension at least ℓ by the unitarity bounds.
- [52] Note that we use superscripts to number RG iterates (2.5) while subscripts for Newton iterates.
- [53] Below, when extracting the CFT spectrum from the linearized RG map, we also consider the Jacobian in the $\mathbb{Z}_2$ -odd subspace.
- [54] On the contrary, if we try to apply this to R instead of R° , ∇R has eigenvalues 1, J is not invertible, and the Newton method fails.
- [55] In particular, for the bond dimension $\chi = 30$ used in our numerical calculations, just storing full $J(A)$ would require around a terabyte of RAM (assuming we use 64-bit floating point numbers). Such memory requirement is far beyond our computational resources.
- [56] https://github.com/ebelnikola/GILT_TNR_R.
- [57] We thank Clément Delcamp for explaining this point and for passing to us a minimal working version of the original Gilt-TNR code [58] where the described strategy is realized.
- [58] Gilt-TNR, <https://github.com/Gilt-TNR/Gilt-TNR>.
- [59] K. Homma, T. Okubo, and N. Kawashima, *Nuclear norm regularized loop optimization for tensor network*, *Phys. Rev. Res.* **6**, 043102 (2024).
- [60] To compute the distance between tensors $A^{(n)}$ with $n \leq 10$, we applied additional relative discrete gauge fixing (akin to that used in Ref. [21]), as using I_+ evaluated at the tensor approximating the critical one was not enough to bring tensors close to each other (see Appendix D).
- [61] Except for a blip around step 11, where a small irregularity is also seen in the singular values (Fig. 4).
- [62] For the trajectory originating in the nearest-neighbor Ising model, operator $T\bar{T}$ is known to be fine-tuned to zero [40,63–65].
- [63] M. Caselle, M. Hasenbusch, A. Pelissetto, and E. Vicari, *Irrelevant operators in the two-dimensional Ising model*, *J. Phys. A* **35**, 4861 (2002).
- [64] P. Reinicke, *Analytical and non-analytical corrections to finite-size scaling*, *J. Phys. A* **20**, 5325 (1987).
- [65] A. Poghosyan, *Shaping lattice through irrelevant perturbation: Ising model*, *J. High Energy Phys.* **11** (2019) 083.
- [66] See Remark 3.1. Apart from $\chi = 30$, the fixed point $A_*^{[30]}$ also depends, of course, on other parameters of Gilt-TNR such as ϵ_{gilt} , but we leave this dependence implicit.
- [67] One can also see from Table III that the horizontal reflection is broken by a smaller amount than rotations and the vertical reflection. For example, in column 2, we have $A_{1+1+1-} \approx A_{1-1+1-}$, $A_{1-1-1+} \approx A_{1+1-1+}$ (horizontal reflections) hold to a much better accuracy than say $A_{1+1+1-} \approx A_{1-1-1+}$ (vertical reflection). A likely reason for this is given in Ref. [68].
- [68] That Q_1 and Q_3 are chosen before Q_2 and Q_4 is the likely reason for why the horizontal reflection symmetry is broken weaker than the vertical one [67].
- [69] For $\chi = 10, 20$, we follow the same procedure as for $\chi = 30$. First, we find approximate t_c by bisecting to 10^{-10} and looking for the longest plateau in singular values as in Fig. 4. We find $t_{\chi=10} = 1.001\,277\,863\,2$ ($\epsilon_{\text{gilt}} = 10^{-4}$) and $t_{\chi=20} = 1.000\,061\,866\,4$ ($\epsilon_{\text{gilt}} = 2 \times 10^{-5}$). Then, we produce a plot of Hilbert-Schmidt distances between subsequent gauge-fixed tensors, as in Fig. 5. The approximate fixed-point tensor is defined as $A^{(n)}$ corresponding to the minimum of this plot.

- [70] Other tensor RG algorithms involving disentanglers, such as, e.g., TNR [7], also involve iterative optimization of disentanglers, to minimize a cost function expressing a truncation error. It would be interesting to see if those algorithms are, or may be made, continuous or differentiable in a small neighborhood of the critical fixed point. To our knowledge, this has never been tested carefully.
- [71] A tensor A is $\mathbb{Z}_2$ -even or $\mathbb{Z}_2$ -odd if $A_{ijkl}s_i s_j s_k s_l = \pm A_{ijkl}$, where s_i, s_j, s_k , and s_l are the quantum numbers of the four legs and the sign $\pm$ is $+$ for even and $-$ for odd.
- [72] JULIA, <https://julialang.org/>.
- [73] J. Bezanson, A. Edelman, S. Karpinski, and V.B. Shah, *JULIA: A fresh approach to numerical computing*, *SIAM Rev.* **59**, 65 (2017).
- [74] J. Haegeman, KrylovKit (2024), <https://github.com/Jutho/KrylovKit.jl>.
- [75] Computing $O(10)$ largest eigenvalues and the corresponding eigenvectors of ∇R by this method requires $O(100)$ calls of the RG map and for $\chi = 30$ takes a few minutes on a laptop.
- [76] Note that the Jacobian matrix is real but, in general, not symmetric, and so some eigenvalues come in complex-conjugate pairs.
- [77] Other works [9,24] said they could reach an approximate fixed-point tensor by gauge fixing, but without a plot like Fig. 5 the accuracy is hard to know. Reference [59] quantified the rate of convergence to the fixed point by plotting the norm difference of the vectors of singular values for consecutive tensors as a function of the RG step but without performing gauge fixing.
- [78] A. Acuaviva, V. Makam, H. Nieuwboer, D. Pérez-García, F. Sittner, M. Walter, and F. Witteveen, *The minimal canonical form of a tensor network*, in *Proceedings of the 2023 IEEE 64th Annual Symposium on Foundations of Computer Science (FOCS)* (IEEE, New York, 2023), pp. 328–362 [arXiv:2209.14358].
- [79] The change in the RG map can be inferred from the linearization expression [Eq. (40) in Ref. [21]] and is also stated explicitly in Ref. [22] below Eq. (18).
- [80] Approximate values of $t_c(a)$ were obtained by the bisection method as explained after Eq. (3.6) with $\Delta t = 10^{-10}$.
- [81] For all values of a in Fig. 8, there is a pair of complex-conjugate eigenvalues in the lower part of the plot. For most values of a , the pair is the 9th and 10th eigenvalues, but for a few values of a it is the 8th and 9th. For complex eigenvalues, we plot their absolute value. Since the two eigenvalues in this pair have the same absolute value, the result is an “ $\times$ ” and a “ $+$ ” right on top of each other.
- [82] Y. Saad and M.H. Schultz, *GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems*, *SIAM J. Sci. Stat. Comput.* **7**, 856 (1986).
- [83] Tensor $A^{(n')}$ in this section should not be confused with tensor $A^{(n_*)}$ in Sec. III.2. They are different tensors because the RG map is different. Also, $n_* = 23$ in Sec. III.2, which was chosen to minimize $\|R(A^{(n)}) - A^{(n)}\|$, while here we choose $n' = 23$ simply as a sufficiently large number.
- [84] We thank Naoki Kawashima for a discussion.
- [85] J. Zierenberg, N. G. Fytas, M. Weigel, W. Janke, and A. Malakis, *Scaling and universality in the phase diagram of the 2D Blume-Capel model*, *Eur. Phys. J. Special Topics* **226**, 789 (2017).
- [86] M.E. Fisher, *Yang-Lee edge singularity and ϕ^3 field theory*, *Phys. Rev. Lett.* **40**, 1610 (1978).
- [87] S. Basu and V. Oganessian, *Critical correlations of Ising and Yang-Lee critical points from tensor RG*, [arXiv:2204.06763](https://arxiv.org/abs/2204.06763).
- [88] V. Dotsenko, J.L. Jacobsen, M.-A. Lewis, and M. Picco, *Coupled Potts models: Self-duality and fixed point structure*, *Nucl. Phys. B* **546**, 505 (1999).
- [89] G. Evenbly, TensorTrace, <https://www.tensortrace.com/> (accessed: 2024-07-17).
- [90] Sometimes, when the lattice model is translated to a tensor network representation, the spin lattice is rotated by $\pi/4$ with respect to the tensor network, as in Appendix B. The lattice model then has somewhat unusual periodic boundary conditions. This microscopic detail does not play a role in our paper.
- [91] Because of the lattice rotation by $\pi/4$, the equality requires nonstandard periodic boundary conditions on the Ising side, assuming the usual periodic boundary conditions on the tensor network side.
- [92] The critical temperature is given implicitly by $\sinh(2J_y/T_c) \sinh(2J_x/T_c) = 1$.
- [93] The Arnoldi method requires $O(s)$ evaluations of directional derivatives $\nabla R^\circ \cdot v$, each of which takes $O(\chi^6)$ with our finite difference approximation method.
- [94] Wikipedia, Automatic differentiation, https://en.wikipedia.org/wiki/Automatic_differentiation.
- [95] <https://github.com/brucelyu/tensorRGflow/tree/main>.
- [96] C. Fieker, W. Hart, T. Hofmann, and F. Johansson, *Nemo/Hecke: Computer algebra and number theory packages for the JULIA Programming Language*, in *Proceedings of the 2017 ACM on International Symposium on Symbolic and Algebraic Computation, ISSAC '17* (ACM, New York, 2017), pp. 157–164.
- [97] ABSTRACTALGEBRA.JL., <https://github.com/Nemocas/AbstractAlgebra.jl>.
- [98] The linear solver returns one solution and, optionally, the kernel of the matrix, corresponding in our case to vectors that realize symmetries discussed before Eq. (D4).
- [99] In practice, this need arises when evaluating the Jacobian eigenvalues around the fixed point in the $\mathbb{Z}_2$ -odd sector. This requires computing $R(A_* + \delta A)$ for small $\mathbb{Z}_2$ -odd perturbations δA .
- [100] As a matter of fact, this observation also applies to Gilt-TNR used to our work. In fact, we have checked that the fixed point is reached even if the continuous gauge-fixing step is omitted in our algorithm. Nevertheless, we decided to keep the continuous gauge-fixing step, since it is not expensive. Also, having this step prompted us to reflect more deeply why the orthogonal gauge fixing is actually sufficient in our algorithm (Appendix D.3).
- [101] By care we mean, e.g., the need to solve the CDL problem. By luck we mean, e.g., that it is hard to *a priori* exclude such situations as cycles of finite length. For example, discrete dynamical systems are known to generically undergo period-doubling bifurcations.
- [102] We believe the statement to the contrary below Eq. (B5b) in Ref. [21] to be incorrect.

- [103] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes 3rd Edition: The Art of Scientific Computing* (Cambridge University Press, Cambridge, England, 2007).
- [104] We obtain the eigenvector using the Arnoldi method. For this, we approximate $\nabla R.v$ using finite difference formula Eq. (E1) with $h = 10^{-4}$.
- [105] G. Li, K. H. Pai, and Z.-C. Gu, *Tensor-network renormalization approach to the q -state clock model*, [Phys. Rev. Res. **4**, 023159 \(2022\)](#).
- [106] While the basis vectors $|0\rangle$, $|1\rangle$, and $|2\rangle$ are complex, one may show that the tensor $A_{3P}(t)$ is real due to S_3 symmetry of A . The initial tensor is invariant under an S_3 action, and our RG map preserves this symmetry.