

Robustness of Real-Space Topology in Moiré Systems

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The appearance of fractional Chern insulators in moiré systems can be rationalized by the presence of a fictitious magnetic field associated with the spatial texture of layer-resolved electronic wave functions. Here, we present a systematic study of real-space topology and the associated fictitious magnetic fields in moiré systems. We first show that at the level of individual Bloch wave functions, the real-space Chern number, akin to a Pontryagin index, is a fragile marker. It generically vanishes except for specific limits where the Bloch functions exhibit fine-tuned zeros within the unit cell, such as the chiral limit of twisted bilayer graphene (TBG) or the adiabatic regime of twisted homobilayer transition metal dichalcogenides (TMDs). We then show that these limitations do not apply to textures associated with ensembles of Bloch wave functions, such as entire bands or the ensemble of states at a given energy. The Chern number of these textures defines a robust topological index protected by a spectral gap. We find that symmetries constrain it to be nonzero for both twisted TMDs and TBG across all twist angles and levels of corrugation, which can be verified in scanning tunneling microscopy measurements. By projection to the band ensemble texture, a single-component Hamiltonian under fictitious magnetic field emerges in broad regime, enabling a direct comparison of multicomponent bands to Landau-level wave functions. We also study real-space topology within the topological heavy fermion model of TBG, finding that the real-space topological features are supported only by the light c electrons.

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I. INTRODUCTION

Moiré materials are now well established as a platform to study correlated electronic phenomena. Two prime examples are twisted bilayer graphene (TBG) [1] and twisted bilayer transition metal dichalcogenides (TMDs) [2], hosting correlated insulating as well as superconducting phases [3–18]. Moreover, three moiré systems—rhombohedral multilayer graphene [19–21], twisted bilayer MoTe_2 [22–27], and TBG above 5 T [28]—have been demonstrated to feature the fractional quantum anomalous Hall effect [29–34].

The emergence of these fractional quantum Hall states at zero magnetic field [29–34] suggests the presence of a mechanism that generates a fictitious magnetic field experienced by the electrons. Support for this intuitive expectation has been provided for twisted bilayer TMDs

within the so-called adiabatic approximation [35–37], where a fictitious magnetic field arises from the geometric phase associated with the spatial rotation of the layer spinor. Experimentally observed in TMDs [38,39], such a layer-skyrmion texture has also been predicted in chiral TBG (and other multilayer graphene systems) [40] and is expected to arise quite generally in ideal Chern bands of multicomponent systems. In the adiabatic limit of both TMDs and chiral TBG, the two-component spinor multiplies a Landau-level wave function [41–46], whose associated effective magnetic field precisely compensates for the skyrmion texture in a sense that we will make precise.

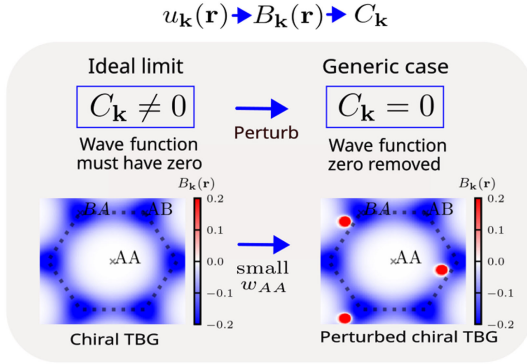
The texture-induced fictitious magnetic field provides a useful intuitive understanding of the properties of moiré systems. However, realistic experimental settings are typically beyond these idealized limits. In this paper, we explore the robustness of the layer skyrmions away from these limits and establish a general method to quantify the real-space topology in multicomponent (layer or sublattice) Bloch bands. We propose a real-space Chern number obtained from an ensemble of wave functions and show that it is both robust and nontrivial for a wide range of parameters in both TBG and TMD systems, implying a generic fictitious magnetic field in these systems.

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We first quantify the skyrmion winding by constructing a real-space Chern number for individual Bloch wave functions. However, we find that this Chern number generically vanishes at zero applied magnetic field. Only in certain special situations such as high-symmetry momenta or the ideal Chern-band and adiabatic limits, this real-space Chern number differs from zero. In particular, a nonzero Chern number requires that the multicomponent Bloch wave function exhibits zeros in the unit cell. Since the zeros are generically lifted by perturbations, this occurs only in fine-tuned situations, as illustrated in Fig. 1(a).

(a) Real-space topology of individual wave functions



(b) Real-space topology of wave function ensembles

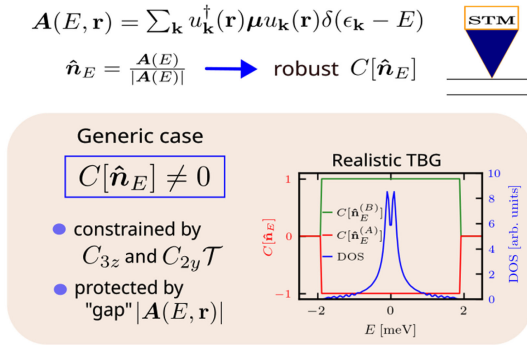


FIG. 1. (a) Illustration of real-space topology of individual wave functions at a given momentum. The real-space Chern number $C_{\mathbf{k}}$ for a spinor wave function $u_{\mathbf{k}}(\mathbf{r})$ is obtained by integrating the real-space Berry curvature $B_{\mathbf{k}}$ over the unit cell. At zero applied magnetic field, the real-space Chern number is nonzero only in fine-tuned cases, requiring the full spinor wave function to vanish identically at some point in the unit cell. Infinitesimal perturbations generically remove this zero and cause the real-space Chern number $C_{\mathbf{k}}$ to vanish. The left-hand inset shows the real-space Berry curvature $B_{\mathbf{k}}$ for a wave function in the chiral limit, integrating to $C_{\mathbf{k}} = -1$. The right-hand inset shows the real-space Berry curvature for a perturbed chiral TBG, leading to the vanishing of the Chern number. (b) Real-space topology of wave function ensembles as probed in scanning tunneling microscopy. In contrast to (a), the real-space Chern number is robust in this case, and protected by symmetries for both TBG and twisted TMDs. Inset shows the energy-dependent sublattice-projected Chern numbers $C[\hat{n}_E^{(A)}]$ and $C[\hat{n}_E^{(B)}]$ for a realistic model of TBG.

We find that the situation changes markedly if one considers textures formed by ensembles of wave functions, for example, within a given band or at a given energy. The texture is defined by a spatially varying three-dimensional vector [Fig. 1(b)], with the topology remaining robust as long as the vector does not vanish. At zero applied flux, this texture can exhibit a nonzero real-space Chern number without fine-tuning, with spatial symmetries imposing general constraints on it. In particular, we demonstrate that both twisted TMDs and sublattice-projected TBG always possess a nontrivial real-space topology as a consequence of their C_{3z} and $C_{2y}\mathcal{T}$ symmetries. Our proposed ensemble real-space texture for bands allows one to directly associate a well-defined fictitious magnetic field to it, forming a complementary characterization beyond conventional band geometry. Crucially, this association does not place any conditions on the underlying Hamiltonian and allows a direct mapping to Landau levels to be established, making it a versatile method to rationalize the emergence of the fractional quantum anomalous Hall effect in moiré materials beyond ideal limits.

We demonstrate this approach by applying it to TBG, finding that the chiral-limit topological texture is present even at realistic corrugation. For a realistic model of twisted bilayer MoTe_2 at $\theta = 2.1^\circ$ [47,48], we find a robust well-defined texture for the relevant band, even though the adiabatic approximation [35–37] breaks down in that model. We show that robust textures allow the multi-component problem to be naturally mapped to a single-component problem with a magnetic field defined by the texture, allowing a natural interpretation for the experimentally observed fractional Chern insulators (FCIs) in terms of Landau levels. We find large Landau-level overlaps for the two topmost bands in twisted bilayer MoTe_2 at $\theta = 2.1^\circ$.

Nontrivial fixed-energy textures are observable in scanning tunneling experiments, and our results for TMDs explain recent experimental observations [38,39]. Remarkably, we find that the same phenomenology is expected for TBG when considering sublattice-projected textures. Furthermore, we find that layer-projected textures in TBG, which are better accessible in STM experiments, are typically also nontrivial provided the $C_{2z}\mathcal{T}$ symmetry is broken, for instance, by a sublattice term.

We also study real-space textures in the presence of an integer number of flux quanta Φ threading the unit cell. We find that in this case, generically, the real-space Chern number equals the applied flux, $C_{\mathbf{k}} = \Phi$. We show that close to ideal limits, this result arises from localized skyrmions that are generated when overall wave function zeros are removed.

Finally, we examine the real-space topology of TBG in the context of the topological heavy fermion (THF) model [49–51]. We find that the Wannier states localized around the AA sites, accounting for most of the central bands,

do not exhibit any real-space topology. Instead, the non-trivial topology originates from the itinerant electrons, which are concentrated in the AB and BA regions where the ensemble texture exhibits significant winding.

This paper is structured as follows. In Sec. II, we present a general theory of real-space topology of individual multicomponent wave functions. In Sec. III, we introduce Landau-level-like wave functions, which feature nonzero real-space Chern numbers. In Sec. IV, we show that the real-space Chern number generically vanishes for wave functions at zero applied magnetic field, and discuss twisted bilayer TMDs and TBG for illustration. In Sec. V, we introduce a theory of real-space topology of ensembles of wave functions, and in Sec. VI, we demonstrate its robustness on twisted bilayer TMDs and TBG. In Sec. VII, we show how band ensemble textures allow multicomponent bands to be understood in terms of single-component wave functions experiencing a fictitious magnetic field. In Sec. VIII, we discuss the textures of TBG in the context of the heavy fermion model. We conclude with a discussion in Sec. IX.

II. REAL-SPACE TEXTURES OF WAVE FUNCTIONS

Consider a general $N \times N$ continuum Hamiltonian in two spatial dimensions, where the N components typically correspond to layer or sublattice degrees of freedom. We consider a situation where the unit cell is threaded by an integer number of Φ flux quanta, so that the system retains its original lattice periodicity with lattice vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and the eigenstates are (magnetic) translation eigenstates labeled by a band index n , a flavor (spin and/or valley) index λ , and a momentum $\mathbf{k}$ in the Brillouin zone. While below we apply this framework to single-particle models, it applies equally well to many-body models where interactions are treated within the mean-field approximation.

A. Real-space topology in terms of wave functions

The eigenstates in a given band are expressed as $e^{i\mathbf{k}\cdot\mathbf{r}}u_{\mathbf{k}}(\mathbf{r})$ with the N -component Bloch spinors,

$$u_{\mathbf{k}}(\mathbf{r}) = [u_{\mathbf{k}}^1(\mathbf{r}), u_{\mathbf{k}}^2(\mathbf{r}), \dots, u_{\mathbf{k}}^N(\mathbf{r})]^T, \quad (1)$$

where the band n and flavor index λ are omitted for brevity. We can use standard considerations based on the magnetic translation algebra to define Bloch states in the presence of a uniform magnetic field corresponding to an integer number Φ of flux quanta per unit cell; see Appendix A for details. We use a generalized Landau gauge, which is fully specified by the conditions that (i) $\mathbf{A}$ is linear in the coordinates, (ii) the translation operator along the $\mathbf{a}_1$ direction does not involve an Aharonov-Bohm phase, and (iii) the translation operator in the $\mathbf{a}_2$ direction is a product of commuting exponentials, which depend on

position and momentum, respectively. In this gauge, the Bloch spinors are (quasi)periodic functions over the unit cell with

$$\begin{aligned} u_{\mathbf{k}}(\mathbf{r} + \mathbf{a}_1) &= u_{\mathbf{k}}(\mathbf{r}), \\ u_{\mathbf{k}}(\mathbf{r} + \mathbf{a}_2) &= e^{i\gamma(\mathbf{r})} u_{\mathbf{k}}(\mathbf{r}). \end{aligned} \quad (2)$$

Here, the uniform applied magnetic field leads to the Aharonov-Bohm phase,

$$\gamma(\mathbf{r}) = -\Phi \mathbf{G}_1 \cdot \mathbf{r}, \quad (3)$$

where $\mathbf{G}_1 = 2\pi\{\mathbf{a}_2 \times \hat{\mathbf{z}}/[\hat{\mathbf{z}} \cdot (\mathbf{a}_1 \times \mathbf{a}_2)]\}$ is a reciprocal lattice vector. Importantly, the Aharonov-Bohm phase $\gamma(\mathbf{r})$ introduces Φ additional phase windings of the Bloch functions $u_{\mathbf{k}}(\mathbf{r})$ along the boundary of the unit cell. This follows since

$$\int_{\mathbf{R}+\mathbf{a}_1}^{\mathbf{R}} d\mathbf{r} \cdot \nabla_{\mathbf{r}} \gamma(\mathbf{r}) = 2\pi\Phi, \quad (4)$$

encoding the integer magnetic flux enclosed by the unit cell.

We now consider implications for scalar ($N = 1$) and spinor ($N > 1$) Bloch functions.

- (i) Scalar wave functions can be expressed in terms of modulus and phase factor,

$$u_{\mathbf{k}}(\mathbf{r}) = e^{i\theta_{\mathbf{k}}(\mathbf{r})} |u_{\mathbf{k}}(\mathbf{r})|. \quad (5)$$

Since the Bloch wave function is smooth as a function of $\mathbf{r}$, its phase $\theta_{\mathbf{k}}(\mathbf{r})$ is also smooth and well defined, except at vortex cores where $u_{\mathbf{k}}(\mathbf{r})$ vanishes and around which $\theta_{\mathbf{k}}(\mathbf{r})$ winds by multiples of 2π . The Aharonov-Bohm phase $\gamma(\mathbf{r})$ imposes that $\theta_{\mathbf{k}}(\mathbf{r})$ winds by $2\pi\Phi$ along the unit cell boundary. For nonzero Φ , this implies that $u_{\mathbf{k}}(\mathbf{r})$ must have vortices (and zeros), with a total vorticity of Φ [52]. We note that Landau levels provide a natural basis for describing such wave functions at finite flux [53].

- (ii) For spinor wave functions, we first consider the case where $u_{\mathbf{k}}(\mathbf{r})$ is nonzero everywhere. Then, we can define the normalized spinor,

$$\chi_{\mathbf{k}}(\mathbf{r}) = \frac{u_{\mathbf{k}}(\mathbf{r})}{|u_{\mathbf{k}}(\mathbf{r})|}, \quad (6)$$

and a real-space Berry connection,

$$A_{\mathbf{k}}(\mathbf{r}) = -i\chi_{\mathbf{k}}^\dagger(\mathbf{r}) \nabla_{\mathbf{r}} \chi_{\mathbf{k}}(\mathbf{r}), \quad (7)$$

treating the position $\mathbf{r}$ as a parameter. This Berry connection describes the additional geometric phase obtained by $\chi_{\mathbf{k}}(\mathbf{r})$ during adiabatic variations

of the real-space position $\mathbf{r}$. The associated Berry curvature,

$$B_{\mathbf{k}}(\mathbf{r}) = \nabla_{\mathbf{r}} \times A_{\mathbf{k}}(\mathbf{r}), \quad (8)$$

is invariant under a change of phase, $u_{\mathbf{k}}(\mathbf{r}) \rightarrow e^{i\alpha_{\mathbf{k}}(\mathbf{r})} u_{\mathbf{k}}(\mathbf{r})$. Upon integrating over a unit cell (u.c.), this defines the real-space Chern number,

$$C_{\mathbf{k}} = \frac{1}{2\pi} \int_{\text{u.c.}} d\mathbf{r} B_{\mathbf{k}}(\mathbf{r}) \in \mathbb{Z}, \quad (9)$$

quantifying the skyrmion winding of the spinor wave function. Importantly, the (quasi)periodicity expressed in Eq. (2) fixes the integrated Berry connection along the boundary of the unit cell as

$$\oint_{\partial \text{u.c.}} d\mathbf{r} \cdot A_{\mathbf{k}}(\mathbf{r}) = 2\pi\Phi, \quad (10)$$

corresponding to the Aharonov-Bohm phase when going around the unit cell. Since the spinor $\chi_{\mathbf{k}}(\mathbf{r})$ is a smooth function of $\mathbf{r}$ throughout the unit cell, Stokes's theorem relates the two integrals in Eqs. (9) and (10) and proves that $C_{\mathbf{k}} = \Phi$. The real-space Chern number of wave functions, which are nonzero everywhere, is thus strictly tied to the number of flux quanta per unit cell. In particular, it has to vanish at zero magnetic field.

- (iii) We now consider spinor wave functions that have zeros in the unit cell. Then, the normalized spinor introduced in Eq. (6) is ill defined at the zeros of $u_{\mathbf{k}}(\mathbf{r})$, which prevents a direct use of the Berry connection. Near each zero, the phase of the spinor in Eq. (6) can wind, forming a vortex or antivortex in real space. We explicitly extract the geometric content of the wave function by factoring out these zeros and phase vortices using the decomposition

$$u_{\mathbf{k}}(\mathbf{r}) = \chi_{\mathbf{k}}^{\Phi_1}(\mathbf{r}) \psi_{\mathbf{k}}^{\Phi_2}(\mathbf{r}). \quad (11)$$

Here, $\chi_{\mathbf{k}}^{\Phi_1}$ is a normalized spinor wave function, and $\psi_{\mathbf{k}}^{\Phi_2}$ is a scalar (single-component) wave function with the same norm as $u_{\mathbf{k}}(\mathbf{r})$, or $|\psi_{\mathbf{k}}^{\Phi_2}|^2 = u_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}$, in particular, the same zeros and vortices. The new spinor $\chi_{\mathbf{k}}^{\Phi_1}$ is now a smooth function of $\mathbf{r}$, even at the zeros of $u_{\mathbf{k}}(\mathbf{r})$, and defines a real-space connection as well as a Berry curvature following Eqs. (7) and (8). It can thus be characterized by a real-space Chern number $C_{\mathbf{k}} = \Phi_1$ [54]. From Stokes's theorem, one obtains that the line integral of the Berry connection along the unit cell boundary yields $2\pi\Phi_1$. This implies that the spinor part can be chosen to satisfy the boundary conditions in Eq. (2) with flux Φ_1 . Consequently, the scalar part

$\psi_{\mathbf{k}}^{\Phi_2}$ satisfies the same boundary conditions with the integer flux,

$$\Phi_2 = \Phi - \Phi_1, \quad (12)$$

in order for the Bloch wave function $u_{\mathbf{k}}(\mathbf{r})$ to satisfy Eq. (2) with the applied magnetic flux Φ . For the scalar function $\psi_{\mathbf{k}}^{\Phi_2}$, we return to case (i), according to which the integer Φ_2 determines its total vorticity, corresponding to the sum of the vorticities at all isolated zeros of the function.

Physically, the factorization employed in Eq. (11) thus corresponds to a splitting of the magnetic flux into a skyrmion winding Φ_1 carried by the spinor part and a set of real-space vortices with total vorticity Φ_2 of the scalar part. At zero external flux, we must have $\Phi_1 = -\Phi_2$, implying that the skyrmion texture exactly compensates for the magnetic phase of the scalar part. As we will elaborate below, this structure is precisely the one found in the adiabatic limit of TMDs [37] and in chiral TBG [40,41]. For nonzero external flux Φ , we have that $C_{\mathbf{k}} = \Phi_1 = \Phi - \Phi_2$, so that $C_{\mathbf{k}} \neq \Phi$ is possible if the wave function contains vortices.

B. Formulation in terms of Bloch sphere

For two-component wave functions ($N = 2$), the spinor winding in real space can be equivalently described using the three-dimensional unit vector [55],

$$\hat{\mathbf{n}}_{\mathbf{k}}(\mathbf{r}) = \frac{u_{\mathbf{k}}^{\dagger}(\mathbf{r}) \boldsymbol{\mu} u_{\mathbf{k}}(\mathbf{r})}{u_{\mathbf{k}}^{\dagger}(\mathbf{r}) u_{\mathbf{k}}(\mathbf{r})} = \chi_{\mathbf{k}}^{\dagger}(\mathbf{r}) \boldsymbol{\mu} \chi_{\mathbf{k}}(\mathbf{r}), \quad (13)$$

where $\boldsymbol{\mu} = (\mu_x, \mu_y, \mu_z)$ is the vector of Pauli matrices and the superscript Φ_1 is omitted for simplicity. Vector fields $\hat{\mathbf{n}}(\mathbf{r})$ on the Bloch sphere can be characterized by their Pontryagin density,

$$B[\hat{\mathbf{n}}](\mathbf{r}) = \frac{1}{2} \hat{\mathbf{n}}(\mathbf{r}) \cdot \partial_x \hat{\mathbf{n}}(\mathbf{r}) \times \partial_y \hat{\mathbf{n}}(\mathbf{r}), \quad (14)$$

and the associated index,

$$C[\hat{\mathbf{n}}] = \frac{1}{4\pi} \int_{\text{u.c.}} d\mathbf{r} \hat{\mathbf{n}}(\mathbf{r}) \cdot \partial_x \hat{\mathbf{n}}(\mathbf{r}) \times \partial_y \hat{\mathbf{n}}(\mathbf{r}). \quad (15)$$

If we compute the Pontryagin density for $\hat{\mathbf{n}} = \hat{\mathbf{n}}_{\mathbf{k}}$ as defined in Eq. (13), we recover the Berry curvature in Eq. (8),

$$B[\hat{\mathbf{n}}_{\mathbf{k}}](\mathbf{r}) = B_{\mathbf{k}}(\mathbf{r}), \quad (16)$$

which also implies

$$C[\hat{\mathbf{n}}_{\mathbf{k}}] = C_{\mathbf{k}} = \Phi_1 \quad (17)$$

for the Pontryagin index. The advantage of this formulation is that it allows the real-space Chern number to be defined also for vector fields that do not arise out of individual wave functions. We will return to this possibility in Sec. V, where we consider textures of wave function ensembles.

The vector-field formulation naturally extends to multi-component wave functions with $N > 2$ using the properties of the $\mathfrak{su}(N)$ Lie algebra [56,57]. One defines a $N(N+1)/2$ -dimensional vector $\hat{\mathbf{n}}_{\mathbf{k}}(\mathbf{r})|_a = \chi_{\mathbf{k}}^\dagger(\mathbf{r})\lambda_a\chi_{\mathbf{k}}(\mathbf{r})$, where λ_a are the generators of $\mathfrak{su}(N)$. The real-space Chern number is then defined using Eq. (15) with a suitable generalization of the triple product to the $\mathfrak{su}(N)$ case [56,57].

Physical insight into $N > 2$ component wave functions can be gleaned by projecting onto specific components, obtaining effective two-component wave functions, and permitting an interpretation in terms of the Bloch sphere as defined above. For instance, $N = 4$ systems comprising layer and sublattice can be projected on a single sublattice to obtain a two-component spinor. We will later perform this projection for TBG.

III. LANDAU-LEVEL-LIKE WAVE FUNCTIONS

We illustrate the decomposition of the Bloch wave function into spinor and scalar components by two examples. The first example considers twisted bilayer transition metal dichalcogenides within the adiabatic approximation. The second corresponds to the abstract class of ideal wave functions, exemplified by twisted bilayer graphene in the chiral limit. In both cases, the scalar part experiences an effective magnetic field and can be expressed in terms of Landau levels.

A. Twisted bilayer TMDs

Hole-doped twisted bilayer TMDs (tTMDs) can be understood in terms of parabolically dispersing holes moving in a layer-Zeeman field denoted $\Delta(\mathbf{r})$ and a scalar potential $\Delta_0(\mathbf{r})$. In the K valley, the Hamiltonian reads

$$H_{\text{tTMD}}^K = -\frac{\hbar^2}{2m^*} \begin{pmatrix} \mathbf{k}^2 & 0 \\ 0 & (\mathbf{k} - \mathbf{q}_1)^2 \end{pmatrix} + \Delta(\mathbf{r}) \cdot \boldsymbol{\mu} + \Delta_0(\mathbf{r})\mu_0, \quad (18)$$

where m^* is the effective mass of the holes, $\mu_{0,x,y,z}$ denotes the Pauli matrices in layer space, and $\mathbf{k} = -i\nabla_{\mathbf{r}}$ the spatial derivative. Moreover, $\mathbf{q}_1 = [0, 4\pi\theta/(3a_0)]$ is the momentum offset between the valence band maxima of the two layers, with a_0 the lattice constant and θ the twist angle.

The layer-Zeeman field can have a nonzero real-space Pontryagin index $C[\hat{\mathbf{n}}_\Delta]$ of the unit vector

$$\hat{\mathbf{n}}_\Delta(\mathbf{r}) = \Delta(\mathbf{r})/|\Delta(\mathbf{r})|. \quad (19)$$

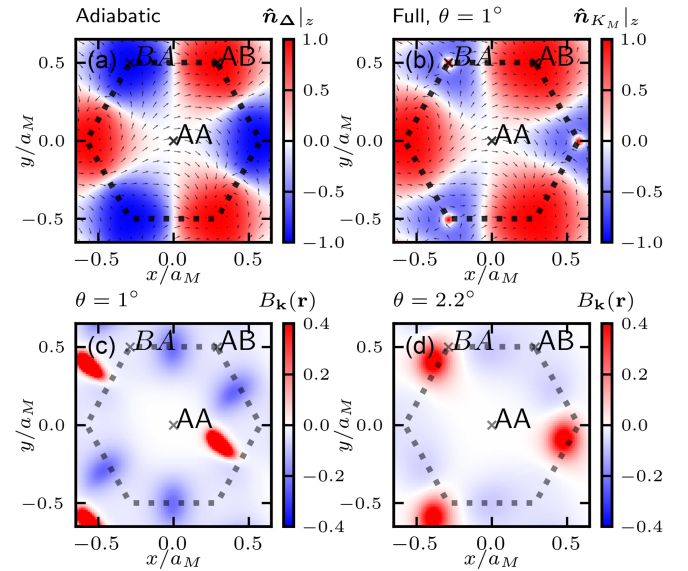


FIG. 2. (a) Unit cell map of the texture $\hat{\mathbf{n}}_\Delta(\mathbf{r}) = \Delta(\mathbf{r})/|\Delta(\mathbf{r})|$ [Eq. (19)]. Color map shows the z component, while the arrows denote the in-plane direction of $\hat{\mathbf{n}}$. (b) Same as (a) but for a texture for the wave function of the topmost band wave function of WSe₂ at $\theta = 1^\circ$, defined in in Eq. (13). (c) Real-space Berry curvature at $\theta = 1^\circ$ at a generic momentum $\mathbf{k} = [-0.2, 0.2]q_1$, with $q_1 = |\mathbf{q}_1|$. (d) Same as (c), but at $\theta = 2.2^\circ$.

For illustration, Fig. 2(a) exhibits the texture corresponding to the K -valley Hamiltonian of WSe₂, using model parameters from Ref. [58] [see Appendix B for explicit forms of $\Delta_0(\mathbf{r})$ and $\Delta(\mathbf{r})$]. This texture has $C[\hat{\mathbf{n}}_\Delta] = -1$, so that the vector $\hat{\mathbf{n}}_\Delta(\mathbf{r})$ covers the entire unit sphere as the unit cell is traversed.

In the adiabatic approach [35–37], one assumes that the layer pseudospin aligns with the local layer-Zeeman field at every $\mathbf{r}$. Focusing on the low-energy subspace, we can then write the wave function as

$$u_{\mathbf{k}}^{\text{Ad}}(\mathbf{r}) = \chi^+(\mathbf{r})\psi_{\mathbf{k}}(\mathbf{r}), \quad (20)$$

where we use the eigenspinors of the local layer-Zeeman field:

$$\Delta(\mathbf{r}) \cdot \boldsymbol{\mu}\chi^\pm(\mathbf{r}) = \pm|\Delta(\mathbf{r})|\chi^\pm(\mathbf{r}). \quad (21)$$

Thus, the spinor inherits a skyrmion winding directly from the winding of the layer-Zeeman field as described by $\hat{\mathbf{n}}_\Delta(\mathbf{r})$. In practice, this approximation holds for sufficiently small twist angles, where the kinetic energy remains perturbative [59,60].

We remark that the same approximation has been used to understand the properties of electrons in a skyrmion lattice [61–63], which can be described using a Hamiltonian such as Eq. (18).

The scalar wave function $\psi_{\mathbf{k}}$ in Eq. (20) is an eigenfunction of the effective Hamiltonian:

$$H_{\text{Adiabatic}}^K = -\frac{[\hbar\mathbf{k} - e\tilde{\mathbf{A}}(\mathbf{r})]^2}{2m^*} + \tilde{V}(\mathbf{r}). \quad (22)$$

Importantly, this Hamiltonian includes a fictitious vector potential. The associated fictitious magnetic field is also determined by the texture of $\hat{\mathbf{n}}_{\Delta}$:

$$\nabla \times \tilde{\mathbf{A}}(\mathbf{r}) = -\frac{\hbar}{e} B[\hat{\mathbf{n}}_{\Delta}](\mathbf{r}). \quad (23)$$

The fictitious field is nonuniform but due to the winding of $\hat{\mathbf{n}}_{\Delta}$, its average corresponds to one flux quantum threading the moiré unit cell. The additional scalar potential is given by $\tilde{V}(\mathbf{r}) = -D(\mathbf{r}) + |\Delta(\mathbf{r})| + \Delta_0(\mathbf{r})$ including the correction,

$$D(\mathbf{r}) = \frac{\hbar^2}{8m^*} \sum_{i=x,y} [\partial_i \hat{\mathbf{n}}_{\Delta}(\mathbf{r})]^2, \quad (24)$$

associated with the quantum geometry.

Thus, the wave function Eq. (20) in the adiabatic approximation has precisely the structure of Eq. (11). The spinor winding of $\chi_+(\mathbf{r})$ gives $\Phi_1 = -1$, while the scalar wave function has $\Phi_2 = 1$ and can be decomposed into Landau-level wave functions, $\mathbf{u}_{n,\mathbf{k}}^{\Phi=1}$ with Landau-level index n and wave vector $\mathbf{k}$,

$$\psi_{\mathbf{k}}^{\Phi_2=1}(\mathbf{r}) = \sum_{n=0}^{+\infty} c_n^{\mathbf{k}} \mathbf{u}_{n,\mathbf{k}}^{\Phi=1}(\mathbf{r}), \quad (25)$$

which are eigenstates of the magnetic translations across the moiré unit cell [64]. Consequently, the fictitious magnetic field exactly compensates for the spinor winding, consistent with the absence of an applied magnetic field, $\Phi = 0$. We note that in this adiabatic approximation, $\hat{\mathbf{n}}_{\Delta}(\mathbf{r})$ of Eq. (19) and $\hat{\mathbf{n}}_{\mathbf{k}}(\mathbf{r})$ of Eq. (13) coincide, since the spinor part of the wave function precisely follows the texture of $\Delta(\mathbf{r})$.

B. Ideal bands

Ideal Chern bands are believed to be well suited for hosting fractional Chern insulator phases [42,44,45,53,65–69] allowing for the construction of Laughlin-like trial states [44]. At zero applied magnetic field $\Phi = 0$, their wave functions possess an exact Landau-level representation [42],

$$u_{\mathbf{k}}^{\text{Ideal}}(\mathbf{r}) = N_{\mathbf{k}} \mathcal{B}(\mathbf{r}) \mathbf{u}_{0,\mathbf{k}}^{\Phi=1}(\mathbf{r}), \quad (26)$$

and carry a momentum-space Chern number of 1. Here, $\mathbf{u}_{0,\mathbf{k}}^{\Phi=1}(\mathbf{r})$ is the zeroth Landau-level wave function at

momentum $\mathbf{k}$ for a magnetic field of one flux quantum per unit cell, $N_{\mathbf{k}}$ is a normalization constant, and $\mathcal{B}(\mathbf{r})$ is a scalar or vector function. Note that the lowest-Landau-level wave function $\mathbf{u}_{0,\mathbf{k}}^{\Phi=1}(\mathbf{r})$ is quasiperiodic, satisfying Eq. (2) with $\Phi = 1$. The Landau-level quasiperiodicity of $\mathbf{u}_{0,\mathbf{k}}^{\Phi=1}(\mathbf{r})$ implies that $\mathcal{B}(\mathbf{r})$ is necessarily also quasiperiodic, with an effective flux $\Phi = -1$. For certain models with double Dirac cones [70,71], $\mathcal{B}(\mathbf{r})$ may have a single component and behave as a Landau level with negative flux, compensating the flux of $\mathbf{u}_{0,\mathbf{k}}^{\Phi=1}$. However, for single-Dirac models, $\mathcal{B}(\mathbf{r})$ is nonvanishing everywhere and necessarily possesses $N > 1$ components. We prove this result in Appendix C. Then, the negative flux $\Phi = -1$ arises as the geometric phase induced by a real-space vector winding. In this case, the ideal-band wave function of Eq. (26) reproduces the form of Eq. (11) with the identification,

$$\psi_{\mathbf{k}}^{\Phi_2=1}(\mathbf{r}) = N_{\mathbf{k}} |\mathcal{B}(\mathbf{r})| \mathbf{u}_{0,\mathbf{k}}^{\Phi=1}(\mathbf{r}), \quad (27)$$

$$\chi^{\Phi_1=-1}(\mathbf{r}) = \mathcal{B}(\mathbf{r})/|\mathcal{B}(\mathbf{r})|, \quad (28)$$

and therefore possesses the real-space Chern number $C_{\mathbf{k}} = -1$ for all $\mathbf{k}$. For TBG, the decomposition also applies [72] to a nonzero magnetic field, $\Phi \neq 0$: The spinor part $\chi^{\Phi_1=-1}$ is unchanged, while the scalar part evolves as $N_{\mathbf{k}} |\mathcal{B}(\mathbf{r})| \mathbf{u}_{0,\mathbf{k}}^{1+\Phi}(\mathbf{r})$, where the lowest-Landau-level wave function carries the flux $\Phi_2 = 1 + \Phi$. Ideal Chern bands with a momentum Chern number of -1 exhibit the same decomposition with the opposite values $\Phi_1 = 1$ and $\Phi_2 = -1$.

An important example of ideal bands is twisted bilayer graphene in the chiral limit. The K -valley Hamiltonian of twisted bilayer graphene is

$$H_{\text{TBG}}^K = \begin{pmatrix} v_F \mathbf{k} \cdot \boldsymbol{\sigma} & T(\mathbf{r}) \\ T^\dagger(\mathbf{r}) & v_F (\mathbf{k} - \mathbf{q}_1) \cdot \boldsymbol{\sigma} \end{pmatrix}, \quad (29)$$

where v_F is the Dirac velocity of monolayer graphene and the tunneling matrix is

$$T(\mathbf{r}) = w_{AA} \sigma_0 (1 + e^{i\mathbf{g}_5 \cdot \mathbf{r}} + e^{i\mathbf{g}_6 \cdot \mathbf{r}}) + w_{AB} \sigma_x (1 + e^{2\pi i/3 \sigma_z} e^{i\mathbf{g}_5 \cdot \mathbf{r}} + e^{4\pi i/3 \sigma_z} e^{i\mathbf{g}_6 \cdot \mathbf{r}}). \quad (30)$$

Here, w_{AA} (w_{AB}) is the intrasublattice (intersublattice) tunneling and the moiré reciprocal vectors $\mathbf{g}_j$ are defined as the $(j-1)$ th counterclockwise C_{6z} rotations of $\mathbf{g}_1 = [4\pi\theta/(\sqrt{3}a_0), 0]$.

In the chiral limit, the intrasublattice tunneling w_{AA} is set to zero, $w_{AA} = 0$ [41], while the intersublattice is $w_{AB} = 110$ meV. At the magic angle $\theta = 1.09^\circ$, one obtains two sublattice-polarized ideal bands at zero energy.

In what follows, we will use an explicit sublattice mass $m_s > 0$ to tune the Hamiltonian of Eq. (29):

$$H_{\text{TBG};m_s}^K = H_{\text{TBG}}^K + m_s \sigma_z. \quad (31)$$

Such a term can arise either due to sublattice-polarizing interactions or an aligned hexagonal boron nitride (hBN) layer. In this model, the two zero-energy sublattice-polarized flat bands are split and have energies $\pm m_s$, allowing them to be studied independently. The two flat bands are ideal bands. The A-polarized band at energy m_s has a momentum-space Chern number of 1 and follows the form of Eq. (26), while the B-polarized band at energy $-m_s$ has opposite Chern numbers and effective fluxes. The real-space Berry curvature of the A band is illustrated in Fig. 1(a) and integrates to $C_k = -1$ over the unit cell. We note that other twisted multilayer configurations also exhibit ideal bands in the chiral limit [73–76].

IV. STABILITY OF REAL-SPACE CHERN NUMBERS FOR INDIVIDUAL WAVE FUNCTIONS

While these two examples exhibit a nontrivial decomposition, with Φ_1 and Φ_2 both being nonzero at vanishing magnetic field ($\Phi = 0$), we will now show that these cases are fine-tuned. Broadly speaking, in the classification of Sec. II, case (iii) is fine-tuned, whereas case (ii) represents the generic situation. Generically, the real-space Chern number is zero, $C_k = 0$, in the absence of a magnetic field. As a direct consequence, the wave functions of twisted bilayer TMDs beyond the adiabatic approximation and twisted bilayer graphene away from the chiral limit both exhibit a vanishing real-space Chern number.

A. Stability to perturbations of the zeros of the wave function in Eq. (11)

A crucial feature of the wave functions in case (iii) of Sec. II is that they vanish at certain points in the unit cell. We therefore consider the stability of the zeros of multi-component wave functions. We write a general perturbed wave function,

$$u_{\mathbf{k}}(\mathbf{r}) = \chi_{\mathbf{k}}^{\Phi_1}(\mathbf{r})\psi_{\mathbf{k}}^{\Phi_2}(\mathbf{r}) + \epsilon \Delta u_{\mathbf{k}}(\mathbf{r}), \quad (32)$$

where $\Delta u_{\mathbf{k}}(\mathbf{r})$ perturbs weakly ($\epsilon \ll 1$) about a wave function with a zero in the unit cell. Importantly, we expect $\Delta u_{\mathbf{k}}(\mathbf{r})$ to be a generic wave function, which is not necessarily aligned with $\chi_{\mathbf{k}}^{\Phi_1}(\mathbf{r})$. We decompose

$$\Delta u_{\mathbf{k}}(\mathbf{r}) = \Delta u_{\mathbf{k}}^{\parallel}(\mathbf{r}) + \Delta u_{\mathbf{k}}^{\perp}(\mathbf{r}) \quad (33)$$

into two orthogonal components, where $\Delta u_{\mathbf{k}}^{\parallel}(\mathbf{r})$ is aligned with the unperturbed spinor $\chi_{\mathbf{k}}^{\Phi_1}(\mathbf{r})$ and $\Delta u_{\mathbf{k}}^{\perp}(\mathbf{r})$ is orthogonal to it at every $\mathbf{r}$. The admixture of $\Delta u_{\mathbf{k}}^{\parallel}(\mathbf{r})$ preserves the

structure of Eq. (11), and therefore also the real-space Chern number, so that we can absorb it into the unperturbed wave function. In contrast, the admixture of $\Delta u_{\mathbf{k}}^{\perp}(\mathbf{r})$ will generically remove any zeros. To see that, consider

$$|u_{\mathbf{k}}(\mathbf{r})|^2 = |\psi_{\mathbf{k}}^{\Phi_2}(\mathbf{r})|^2 + \epsilon^2 |\Delta u_{\mathbf{k}}^{\perp}(\mathbf{r})|^2, \quad (34)$$

where we used the orthogonality property of $\Delta u_{\mathbf{k}}^{\perp}(\mathbf{r})$. The full wave function can only have a zero if both $\psi_{\mathbf{k}}^{\Phi_2}(\mathbf{r})$ and $\Delta u_{\mathbf{k}}^{\perp}(\mathbf{r})$ vanish simultaneously. We do not expect the zeros of a perturbing wave function to coincide with those of the original wave function so that, generically, the zeros of $\psi_{\mathbf{k}}^{\Phi_2}(\mathbf{r})$ will be removed.

Thus, the zeros of the wave function in Eq. (11) are generically lifted under perturbations. Their removal corresponds to a discontinuous transition from case (iii) to case (ii), where the real-space Chern number becomes identically zero for zero applied flux (or $C_k = \Phi$ for general fluxes). Exceptions to this behavior occur at high-symmetry momenta in the Brillouin zone, where real-space zeros can be protected by symmetries and thus remain stable under symmetry-preserving perturbations. However, such symmetry-protected zeros arise only at isolated points (sets of measure zero) in the Brillouin zone.

B. Localized skyrmions generated by perturbed zeros

We now turn to an explicit illustration of the connection between the removal of vortex zeros and a change in real-space Chern number. Effectively, removing a zero of the wave function introduces an additional skyrmion winding of the spinor wave function, which changes the real-space Chern number. To see this intuitively, we consider a two-component spinor. The zero of the wave function is removed by displacing the zeros of the two components relative to each other. At the zero of the bottom layer, the spinor has all its weight in the upper layer, i.e., at the north pole of the Bloch sphere. Conversely, at the zero of the top layer, the spinor has all its weight in the lower layer, i.e., at the south pole. Intuitively, thus, we expect the spinor to cover the entire Bloch sphere in the vicinity of these two points.

We illustrate this further with the (unnormalized) two-component spinor,

$$u_{\epsilon,\mathbf{k}'}(\mathbf{r}) = \left[\mathbf{u}_{0,\mathbf{k}=0}^{\Phi=1}, \epsilon \mathbf{u}_{0,\mathbf{k}'}^{\Phi=1} \right]^T, \quad (35)$$

at applied flux $\Phi = 1$. Here, $\mathbf{u}_{0,\mathbf{k}}^{\Phi=1}$ denotes the lowest-Landau-level wave function, and we assume that most of the weight is in the upper component, $\epsilon \ll 1$. Since the spinor components are equal to Landau-level wave functions at different wave vectors, the positions of their zeros are different. The position of these zeros is known analytically [41],

$$\mathbf{r}_0 = \mathbf{r}_{BA} + \frac{|\mathbf{a}_1|}{|\mathbf{G}_1|} \hat{\mathbf{z}} \times \mathbf{k}, \quad (36)$$

where the basis vectors $\mathbf{a}_1$ and $\mathbf{G}_1$ are defined below Eqs. (2) and (3), respectively. Thus, for $\mathbf{k} = 0$, the Landau-level wave function has a zero at $\mathbf{r}_0 = \mathbf{r}_{BA}$, while for nonzero $\mathbf{k}$ it shifts away from $\mathbf{r}_{BA}$. At $\epsilon = 0$, the spinor $u_{0,\mathbf{k}'}(\mathbf{r})$ points upward everywhere in the unit cell, implying a real-space Chern number $C_{0,\mathbf{k}'} = 0$. For finite ϵ , but $\mathbf{k}' = 0$, the spinor direction $u_{\epsilon,0}(\mathbf{r}) \propto [1, \epsilon]^T$ is still constant everywhere, so that $C_{\epsilon,0} = 0$. In both of these cases, the entire wave function vanishes at $\mathbf{r} = \mathbf{r}_{BA}$, which corresponds to case (iii) of our wave function classification in Sec. II A. The overall zero is removed when $\epsilon \neq 0$ and $\mathbf{k}' \neq 0$, corresponding to case (ii). Since $\Phi = 1$, we thus expect $C_{\epsilon \neq 0, \mathbf{k}' \neq 0} = 1$, suggesting that the two Landau-level-like spinor components produce a skyrmion. For $\epsilon \ll 1$, the wave function is pointing up everywhere except close to $\mathbf{r} = \mathbf{r}_{BA}$, where it can be approximated as $u_{\epsilon,\mathbf{k}'}(\mathbf{r} \sim \mathbf{r}_{BA}) \approx [B_1 z, \epsilon u_{0,\mathbf{k}'}^{\Phi=1}(\mathbf{r}_{BA})]^T$, where B_1 is a constant and $z = (x + iy) - (x_{BA} + iy_{BA})$ is the vortex of the zeroth Landau level. For any finite ϵ , the spinor points down at $\mathbf{r} = \mathbf{r}_{BA}$. Away from $\mathbf{r} = \mathbf{r}_{BA}$, the spinor tilts upward, winding in plane due to the factor of z in the up component. For large $|\mathbf{r} - \mathbf{r}_{BA}|$, the spinor points up, giving rise to a Chern number $C_{\mathbf{k}} = 1$ as expected.

C. Fragility in twisted bilayer TMDs

We proceed to illustrate explicitly how the removal of zeros leads to a sudden change in the real-space Chern number for twisted bilayer TMDs. The adiabatic approximation assumes a wave function of the form of Eq. (20), effectively projecting on the Hilbert space of wave functions with spinor components given by $\chi^+(\mathbf{r})$. Going beyond the adiabatic approximation by allowing the admixture of wave functions with spinors proportional to $\chi^-(\mathbf{r})$ leads to a Bloch wave function of the form

$$u_{\mathbf{k}}(\mathbf{r}) = \chi^+(\mathbf{r})\psi_{\mathbf{k}}(\mathbf{r}) + \beta_{\mathbf{k}}\chi^-(\mathbf{r})\varphi_{\mathbf{k}}(\mathbf{r}). \quad (37)$$

This introduces a second scalar function $\varphi_{\mathbf{k}}(\mathbf{r})$. The second term, with $|\beta_{\mathbf{k}}| \ll 1$, is small close to the adiabatic limit and scales with the twist angle. The scalar wave function $\psi_{\mathbf{k}}(\mathbf{r})$ is a mixture of Landau-level wave functions for nonzero flux, see Eq. (25), and thus exhibits a zero at some position $\mathbf{r}_{\mathbf{k}}$ in the unit cell, which depends on the momentum $\mathbf{k}$. In general, $\varphi_{\mathbf{k}}(\mathbf{r}_{\mathbf{k}}) \neq 0$, meaning that while the Bloch wave function is nearly aligned with the spinor $\chi^+(\mathbf{r})$ across most of the unit cell, it becomes dominated by the oppositely oriented spinor $\chi^-(\mathbf{r})$ in the vicinity of $\mathbf{r}_{\mathbf{k}}$ and thus never vanishes completely. The spinor $\hat{\mathbf{n}}_{\mathbf{k}}(\mathbf{r})$, obtained from Eqs. (13) and (6), closely follows the texture of $\Delta(\mathbf{r})$, such that $\hat{\mathbf{n}}_{\mathbf{k}}(\mathbf{r}) \simeq \hat{\mathbf{n}}_{\Delta}(\mathbf{r})$, except near $\mathbf{r}_{\mathbf{k}}$ where it flips sign, $\hat{\mathbf{n}}_{\mathbf{k}}(\mathbf{r}_{\mathbf{k}}) = -\hat{\mathbf{n}}_{\Delta}(\mathbf{r}_{\mathbf{k}})$.

To make this concrete, in Fig. 2(b) we show a real-space texture $\hat{\mathbf{n}}_{K_M}(\mathbf{r})$ of the full wave function at the K_M point for the topmost band of WSe₂ at $\theta = 1^\circ$. Note that nominally, the adiabatic approximation is expected to be very accurate at these small twist angles [36,37]. And indeed, the texture $\hat{\mathbf{n}}_{K_M}(\mathbf{r})$ largely follows the texture $\hat{\mathbf{n}}_{\Delta}(\mathbf{r})$ defined by the layer-Zeeman field, which is shown in Fig. 2(a). However, exactly at the AB stacking point, where $\psi_{K_M}(\mathbf{r}_{AB}) = 0$, rather than being fully polarized to the top layer like the underlying texture $\hat{\mathbf{n}}_{\Delta}(\mathbf{r})$, it has the opposite layer polarization, yielding a vanishing real-space Chern number $C_{K_M} = 0$, while $C[\hat{\mathbf{n}}_{\Delta}] = -1$.

The same effect occurs also for other momenta $\mathbf{k}$ in the Brillouin zone. We show the real-space Berry curvature $B_{\mathbf{k}}(\mathbf{r})$ at $\mathbf{k} = [-0.2, 0.2]q_1$ in Fig. 2(c), where the admixture of wave functions beyond the adiabatic limit leads to a localized region of positive Berry curvature, which cancels the negative contribution of the rest of the unit cell, leading to a zero real-space Chern number. This positive Berry curvature, which cancels the total Chern number, can be understood as arising from a strongly localized skyrmion which emerges when the position of the zeros in the two layers is split as discussed above in Sec. IV B. This interpretation is confirmed by plotting the layer-resolved densities in Figs. 3(a) and 3(b), showing additional zeros near the center of the unit cell. While these zeros are close to each other, they are nevertheless split by a small amount, generating a localized skyrmion (see Sec. IV B), and ensuring that the total real-space Chern number satisfies the relation $C_{\mathbf{k}} = \Phi$.

For larger twist angles ($\theta = 2.2^\circ$), the adiabatic approximation becomes less accurate, leading to larger admixtures of the $\chi^-(\mathbf{r})$ component. In this regime, the sharp feature is smeared out, as seen in Fig. 2(d), but the real-space Chern number still vanishes.

We have numerically verified that this vanishing of the real-space Chern number is generic for states across the Brillouin zone except for the high-symmetry Γ point. Here, we obtain $C_{\Gamma} = -1$ with its associated zero even away from the adiabatic approximation. We show in Sec. V that the stability of the real-space Chern number at the Γ point is in fact a result of its symmetry properties under the $C_{2y}\mathcal{T}$ and C_{3z} transformations. Unlike for generic points in the Brillouin zone, these leave the Γ point invariant, so that they imply symmetry constraints on the Γ -point wave function. We will see below that the same symmetries allow one to define a robust real-space Chern number for ensembles of wave functions.

D. Finite magnetic flux in TMDs

The adiabatic approximation is also applicable to twisted TMDs in the presence of a finite number of applied flux quanta Φ [64], and is expected to be accurate at small twist angles. The spinor-scalar decomposition in Eq. (20) still holds, while the applied magnetic field adds to the fictitious

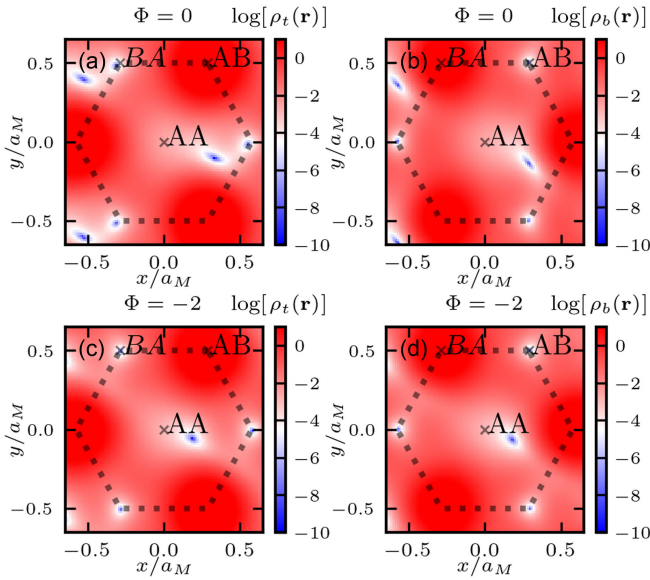


FIG. 3. Wave function layer densities for WSe₂ at $\theta = 1^\circ$. (a) Unit cell map of the log of the top layer density $\rho_t(\mathbf{r})$ at $\Phi = 0$ at a generic momentum $\mathbf{k} = [-0.2, 0.2]q_1$. (b) Same as (a) but for the bottom layer density $\rho_b(\mathbf{r})$. (c),(d) Same as (a),(b), but at flux $\Phi = -2$. The layer zeros are located at the centers of the dark blue regions. Those near the BA and AB sites are inherited from the skyrmion texture of TMDs, which is layer polarized at these positions. At $\Phi = -2$, a second zero per unit cell appears in each layer, as required by the external magnetic flux. Its position changes with the momentum $\mathbf{k}$ and it originates from the zero of the adiabatic scalar function Eq. (25). The (weak) deviation from adiabaticity slightly separates the second zeros in the two layers. A second zero is also present at $\Phi = 0$, but in this case it forms an antivortex, ensuring that the total vorticity across the unit cell vanishes.

field in the Hamiltonian in Eq. (22), which governs the scalar part. For the K valley of twisted WSe₂, for instance, the texture of the spinor part is quantized to a negative winding $C[\hat{n}_\Delta] = -1$. The scalar wave function then experiences an effective flux of $1 + \Phi$ quanta per unit cell. When perturbing away from the adiabatic limit, the zeros in the full Bloch wave function are generically lifted, and $C_k = \Phi$ is expected.

Our numerical simulations confirm this expectation and show that the same mechanism of zero removal is at play at finite applied flux. As an illustration, in Figs. 3(c) and 3(d), we plot the layer-projected densities at flux $\Phi = 2$. In the adiabatic approximation, the full wave function has a zero in the middle of the unit cell. In the full model, however, the zeros of the two layers are split and generate a localized skyrmion following the mechanism of Sec. IV B. This effect reduces the adiabatic $C[\hat{n}_\Delta] = -1$ to $C_k = -2$ expected generically at $\Phi = -2$.

E. Fragility in realistic TBG

As discussed in Sec. III B, TBG in the chiral limit exhibits two ideal bands, each localized on a distinct

sublattice. They have opposite Chern numbers, both in momentum and real space. The real-space Berry curvature for the A-polarized band is shown in the left-hand inset of Fig. 1(a), and integrated to $C_k = -1$.

Away from the chiral limit, based on our arguments above, the real-space Chern number C_k is expected to vanish due to wave function admixture. The right-hand inset of Fig. 1(a) illustrates this, showing the real-space Berry curvature of the predominantly A-polarized four-component wave function of TBG. There we introduce weak intrasublattice tunneling $w_{AA} = 5$ meV and a sublattice mass $m_s = 5$ meV to split the two bands. The texture remains nearly identical to that in the chiral limit, except for a small region in real space near the zero of the chiral-limit Landau-level wave function [whose position is given in Eq. (36)], where the Berry curvature is large and positive. Although highly localized, this sharp peak nonetheless cancels the total real-space Chern number $C_k = 0$. Increasing w_{AA} and deviating further from the chiral limit smoothens out the sharp singularity.

We note that, alternatively, the $N = 4$ component Bloch wave functions can be reduced to $N = 2$ by projecting onto a given sublattice. This makes the profile of the chiral texture much more robust toward increasing w_{AA} , even though the real-space Chern number collapses to $C_k = 0$ for infinitesimal w_{AA} . Despite the vanishing total real-space Chern number of nonideal bands, the real-space texture inherited from the chiral limit remains clearly visible as illustrated in Fig. 1(a), suggesting that the chiral-limit texture is still relevant even away from the ideal limit. In Sec. VI, we will clarify this observation and show that the chiral-limit texture is in fact recovered when one considers wave function ensembles, rather than individual wave functions.

V. REAL-SPACE TEXTURES OF ENSEMBLES OF WAVE FUNCTIONS

A generalization to wave function ensembles allows one to describe more physically relevant textures. First, it permits the description of the real-space topology of a given band. This ensemble encodes the number of flux quanta per unit cell associated with a fictitious magnetic field affecting the band. Second, we consider the real-space topology of the electronic states at a given energy. This ensemble is relevant for describing tunneling experiments.

In this section, we construct a topological index for such wave function ensembles. In contrast to the case of individual wave functions, its nontriviality does not require fine-tuned zeros in the unit cell.

A. Topology of ensembles

We generalize the unit vector introduced in Eq. (13) for $N = 2$ to describe ensembles of states:

$$\mathbf{A}(\mathbf{r}) = \sum_{\lambda, n, \mathbf{k} \in \text{BZ}} u_{\mathbf{k}, n, \lambda}^\dagger(\mathbf{r}) \boldsymbol{\mu} u_{\mathbf{k}, n, \lambda}(\mathbf{r}) p_{\mathbf{k}, n, \lambda}. \quad (38)$$

The sum is over the Bloch wave functions $u_{\mathbf{k}, n, \lambda}^\dagger(\mathbf{r})$ in the first Brillouin zone, indexed by the band index n and the flavor index λ , which stands for the spin and valley degrees of freedom. The ensemble is specified by the non-negative coefficients $p_{\mathbf{k}, n, \lambda}$ describing the weight with which different wave functions contribute to the ensemble. We define the associated normalized vector,

$$\hat{\mathbf{n}}(\mathbf{r}) = \frac{\mathbf{A}(\mathbf{r})}{|\mathbf{A}(\mathbf{r})|}, \quad (39)$$

which inherits the spatial periodicity of the triangular moiré lattice. Other lattice symmetries will be discussed in Sec. VB below. This normalized vector generalizes Eq. (13) and can thus be used, via Eq. (15), to compute a corresponding real-space Chern number $C[\hat{\mathbf{n}}]$.

The robustness of the topological invariant is conveniently described by introducing the 2×2 Hamiltonian,

$$H(\mathbf{r}) = \mathbf{A}(\mathbf{r}) \cdot \boldsymbol{\mu}, \quad (40)$$

which has two bands as a function of the position $\mathbf{r}$. The Chern number of the upper band of this Hamiltonian is equal to $C[\hat{\mathbf{n}}]$, which is topologically protected by the spectral gap $2|\mathbf{A}(\mathbf{r})|$ between the two bands.

If the sum in Eq. (38) were to reduce to a single momentum, this definition would coincide with the one used for individual wave functions, as given in Eq. (13). A nonzero real-space Chern number at zero flux would be possible only if the wave function vanishes somewhere, i.e., if the real-space gap $2|\mathbf{A}(\mathbf{r})|$ vanishes in the unit cell, implying a lack of robustness. In contrast, when $\hat{\mathbf{n}}(\mathbf{r})$ is constructed from an ensemble of wave functions, the arguments presented in Sec. II do not apply, and a nonzero real-space Chern number $C[\hat{\mathbf{n}}]$ for zero applied flux no longer requires the presence of fine-tuned zeros of the wave functions.

A nonzero value of the texture $\mathbf{A}(\mathbf{r})$ implies a tendency of the wave functions in the ensemble to follow a common preferred spinor direction. A large value of $\mathbf{A}(\mathbf{r})$ implies that, while the underlying system has multiple components, at that point in space, the ensemble wave functions are behaving like those of a single-component system aligned with the direction defined by $\mathbf{A}(\mathbf{r})$. A topological phase transition is possible only when $\mathbf{A}(\mathbf{r})$ vanishes somewhere. At such a point, the wave functions in the ensemble are, on average, not aligned with any particular direction.

This definition can be generalized to $N > 2$ in two ways. First, $u_{\mathbf{k}}$ can be projected on two components, reducing the $N > 2$ problem to an effective two-component problem. In this case, a generalization of Eq. (38) reads

$$\mathbf{A}^{(\alpha)}(\mathbf{r}) = \sum_{\lambda, n, \mathbf{k} \in \text{BZ}} u_{\mathbf{k}, n, \lambda}^\dagger(\mathbf{r}) \boldsymbol{\mu}^{(\alpha)} u_{\mathbf{k}, n, \lambda}(\mathbf{r}) p_{\mathbf{k}, n, \lambda}, \quad (41)$$

where $\boldsymbol{\mu}^{(\alpha)}$ denote the Pauli matrices in the projected subspace. Then, the analysis follows the discussion for two components. Second, one can directly consider the properties of an $N \times N$ real-space effective Hamiltonian $H(\mathbf{r})$ defined from the texture. Depending on the number of spectral gaps, there are at most N real-space Chern numbers whose sum is constrained to be zero. We leave the detailed discussion to Appendix F. In what follows, we focus on the first approach, which leads to a more straightforward classification.

1. Textures at a fixed energy

We first consider textures of wave function ensembles consisting of states at a fixed energy. These are obtained by taking

$$p_{\mathbf{k}, n, \lambda} = \delta(\epsilon_{\mathbf{k}, n, \lambda} - E) \quad (42)$$

and provide energy-resolved real-space textures, which we denote $\mathbf{A}(E, \mathbf{r})$. This corresponds to the density of states typically probed in tunneling experiments [77–82],

The associated real-space topology is obtained from the normalized vector,

$$\hat{\mathbf{n}}_E(\mathbf{r}) = \frac{\mathbf{A}(E, \mathbf{r})}{|\mathbf{A}(E, \mathbf{r})|}, \quad (43)$$

which leads to a real-space Chern number $C[\hat{\mathbf{n}}_E]$. The components of $\hat{\mathbf{n}}_E(\mathbf{r})$ are accessible in tunneling experiments [38, 39].

When the Pauli matrices $\boldsymbol{\mu}^{(\alpha)}$ are in layer space (as for twisted bilayer TMDs and sublattice-projected TBG), only the z component of $\hat{\mathbf{n}}$ is directly accessible, and corresponds to local density of states imbalance between the two layers. The x component would be recoverable from the combination of the z component and the value of the density of states at the midpoint between the two layers. However, we note that the latter is not immediately accessible in STM. On the other hand, when the $\boldsymbol{\mu}^{(\alpha)}$ refer to sublattice space (as we will discuss below for layer-projected twisted bilayer graphene), both the z and x components are accessible in STM experiments [83]. The z component corresponds to sublattice polarization in the density of states, while the x component is recoverable from the value of the z component combined with the value of the local the density of states at the midpoint between the two sublattices. We remark that the y component does not have a direct signature in STM, leading instead to atomic-scale currents.

2. Textures of a band

Next, we consider textures associated with bands. By choosing

$$p_{\mathbf{k},n,\lambda} = \frac{1}{N_{\text{u.c.}}} \delta_{n,b} \delta_{\lambda,\lambda_b}, \quad (44)$$

where $N_{\text{u.c.}}$ is the number of unit cells (number of $\mathbf{k}$ points), we sum over all momenta within the Brillouin zone for a given band b and flavor λ_b , and obtain the intrinsic texture associated with that band. The corresponding real-space Chern number encodes the number of flux quanta of the fictitious magnetic field experienced by the electrons in that band. We postpone a detailed discussion of the physical implications of this field to Sec. VII.

B. Symmetries in real space

Spatial symmetries are generally not very informative for individual wave functions, as such symmetries act not only on the spatial position $\mathbf{r}$ but also on the momentum $\mathbf{k}$. Thus, a symmetry operation maps $u_{\mathbf{k}}$ to a different Bloch function, unless $\mathbf{k}$ is a high-symmetry momentum such as Γ . In contrast, the ensemble average of Eq. (38) sums over different momenta weighted with $p_{\mathbf{k},n,\lambda}$. Thus, provided that $p_{\mathbf{k},n,\lambda}$ is invariant under a symmetry of the system, the real-space Hamiltonian introduced in Eq. (40) inherits that symmetry, transferring it to the space-dependent vector $\hat{\mathbf{n}}(\mathbf{r})$.

Specifically, a (not necessarily spatial) symmetry operation $\mathcal{S}$ acts on general multicomponent wave functions $\Psi_{\lambda}(\mathbf{r})$ as

$$\mathcal{S}\Psi_{\lambda}(\mathbf{r}) = S(\mathbf{r})\Psi_{\mathcal{S}(\lambda)}(\mathcal{R}_{\mathcal{S}}\mathbf{r}), \quad (45)$$

transforming both the real-space position, induced by $\mathcal{R}_{\mathcal{S}}$, the flavor index λ , and the wave function with some possibly space-dependent (anti)unitary matrix $S(\mathbf{r})$. Provided $p_{\mathbf{k},n,\lambda}$ respects the symmetry $\mathcal{S}$, the symmetry properties of the wave functions translate to the real-space Hamiltonian. Explicitly,

$$\mathcal{S}H(\mathbf{r})\mathcal{S}^{-1} = H(\mathbf{r}), \quad (46)$$

which can be rewritten as

$$S(\mathbf{r})H(\mathcal{R}_{\mathcal{S}}\mathbf{r})S^{-1}(\mathbf{r}) = H(\mathbf{r}). \quad (47)$$

Inserting this expression into the definition Eq. (40) of $H(\mathbf{r})$, we obtain

$$\hat{\mathbf{n}}(\mathcal{R}_{\mathcal{S}}\mathbf{r}) = \mathcal{R}_{\mu,\mathcal{S}}\hat{\mathbf{n}}(\mathbf{r}), \quad (48)$$

where $\mathcal{R}_{\mu,\mathcal{S}}\boldsymbol{\mu} = S(\mathbf{r})\boldsymbol{\mu}S^{-1}(\mathbf{r})$ denotes the rotation of the Pauli matrices $\boldsymbol{\mu}$ induced by the matrix $S(\mathbf{r})$.

We now discuss the role of time-reversal symmetry, which enforces $H(\mathbf{r})$ to be real. This in turn implies that the vector $\hat{\mathbf{n}}(\mathbf{r})$ has a vanishing y component and lies entirely in the xz plane. As a result, the real-space Chern number vanishes identically in time-reversal symmetric systems,

$C[\hat{\mathbf{n}}] = 0$. We conclude that nontrivial real-space topology, $C[\hat{\mathbf{n}}] \neq 0$, requires broken time-reversal symmetry. In what follows, we will assume that time-reversal symmetry is broken by spontaneous valley polarization. We note, however, that in the presence of time-reversal symmetry, one may still define a valley-resolved Chern number.

VI. ROBUSTNESS OF REAL-SPACE TOPOLOGY OF WAVE FUNCTION ENSEMBLES

We now apply the concept of the topology of ensembles to twisted TMDs and TBG, demonstrating the robustness of their real-space topology. We will exploit the C_{3z} and $C_{2y}\mathcal{T}$ symmetries and show that they impose the real-space topology to be nontrivial.

A. Robustness in twisted bilayer TMDs

For twisted bilayer TMDs, we now use symmetry arguments to show that the real-space Chern number at a given energy is typically nonzero, $C[\hat{\mathbf{n}}_E] \neq 0$, provided the system valley polarizes due to interactions. The spontaneous valley polarization breaks time-reversal symmetry, but leaves the intravalley C_{3z} and $C_{2y}\mathcal{T}$ symmetries intact. A possible flavor-polarized state is the $\nu = 1$ valley-polarized state, observed in many experiments to date. The C_{3z} and $C_{2y}\mathcal{T}$ symmetries transform wave functions in the K valley as

$$C_{3z}\Psi_K(\mathbf{r}) = \begin{pmatrix} 1 & 0 \\ 0 & e^{ig_5 \cdot \mathbf{r}} \end{pmatrix} \Psi_K(C_{3z}^{-1}\mathbf{r}), \quad (49)$$

$$C_{2y}\mathcal{T}\Psi_K(\mathbf{r}) = e^{iq_1 \cdot \mathbf{r}} \mu_x \Psi_K^*(C_{2y}\mathbf{r}). \quad (50)$$

Assuming that the ensemble of electronic states is invariant under these symmetries, they impose constraints on the effective Hamiltonian of Eq. (40),

$$H(E, C_{3z}\mathbf{r}) = \begin{pmatrix} 1 & 0 \\ 0 & e^{ig_5 \cdot \mathbf{r}} \end{pmatrix} H(E, \mathbf{r}) \begin{pmatrix} 1 & 0 \\ 0 & e^{-ig_5 \cdot \mathbf{r}} \end{pmatrix}, \quad (51)$$

$$H(E, C_{2y}\mathbf{r}) = \mu_x H^*(E, \mathbf{r}) \mu_x, \quad (52)$$

where the second equation uses that $e^{iq_1 \cdot \mathbf{r}}$ commutes with the local Hamiltonian. This in turn imposes conditions on the normalized vector $\hat{\mathbf{n}}_E(\mathbf{r})$ on the Bloch sphere,

$$\hat{\mathbf{n}}_E(C_{3z}\mathbf{r}) = R_z(\mathbf{g}_5 \cdot \mathbf{r}) \hat{\mathbf{n}}_E(\mathbf{r}), \quad (53)$$

$$\hat{\mathbf{n}}_E(C_{2y}\mathbf{r}) = M_z \hat{\mathbf{n}}_E(\mathbf{r}), \quad (54)$$

where M_z denotes the mirror symmetry about the xy plane, and $R_z(\theta)$ a rotation by the angle θ about the z axis. At high-symmetry stacking points, these relations, together with the moiré periodicity, imply that

$$\hat{n}_E(AA)|_z = 0, \quad (55)$$

$$\hat{n}_E(AB) = (0, 0, \pm 1), \quad (56)$$

$$\hat{n}_E(BA) = (0, 0, \mp 1), \quad (57)$$

so that the vector is purely in plane at AA sites and purely out of plane at AB and BA sites. Moreover, the layer polarizations are opposite at the AB and BA sites. Remarkably, these properties imply that

$$C[\hat{n}_E] = \pm 1 \bmod 3 \quad (58)$$

is necessarily nonvanishing. We obtain this result in Appendix G using symmetry indicators of band topology [84].

We now focus separately on energy-resolved textures and textures of entire bands.

1. Energy-resolved textures

As a specific example, we consider homobilayer WSe₂ twisted by an angle $\theta = 2.2^\circ$ far from the adiabatic limit. A numerical evaluation of the real-space Chern number, shown in Fig. 4(a), confirms our prediction. We find $C[\hat{n}_E] = -1$ for all energies within the two top bands. Notably, these bands each possess a momentum-space Chern number of $C = 1$ [64].

We note that these arguments based on $C_{2y}\mathcal{T}$ and C_{3z} can also be applied to the topology of the wave function at Γ , the only point in the BZ that remains invariant under these two symmetries. They explain why, at Γ , the finite real-space Chern number for an individual wave function persists even away from the adiabatic limit.

Our arguments also apply in the valley-unpolarized state, where time reversal precludes a nonzero Chern number. In this case, C_{3z} and C_{2y} symmetries still impose Eqs. (56) and (57), implying opposite layer polarizations at the BA and AB points, consistent with recent experiments [38,39].

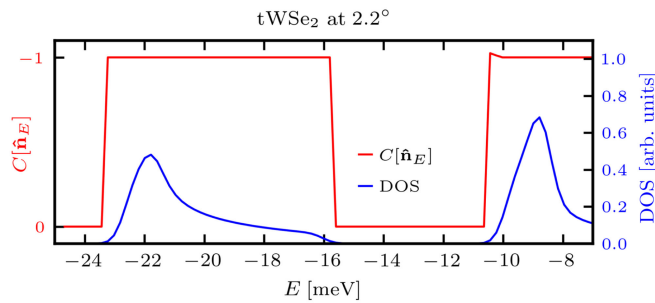


FIG. 4. $C[\hat{n}_E]$ (left-hand axis) and DOS (right-hand axis) for twisted WSe₂ at $\theta = 2.2^\circ$, far away from the adiabatic limit. The real-space Chern number is everywhere nonvanishing in this case. Note that we consider a fully valley-polarized state.

2. Textures of full bands

Turning to real-space textures of individual bands as introduced in Sec. VA 2, the same symmetry arguments imply that, generically, the real-space Chern number of every band is nonzero, $C[\hat{n}] \neq 0$. This implies the generic presence of a net fictitious magnetic field in TMDs. Specifically, we find for the top band of twisted WSe₂ that $C[\hat{n}] = -1$. The same analysis can be applied to other models of twisted TMDs, such as models of MoTe₂, suggested to host states analogous to Landau levels [47,48,59,85–88]. Focusing on a model of twisted MoTe₂ at 2.1° , taking model parameters from Refs. [47,48], we find that the textures of the two bands are consistent with a fictitious magnetic field picture, possessing a real-space Chern number $C[\hat{n}] = -1$. Furthermore, the band follows the texture extremely well, and allows a mapping to Landau levels to be made as we detail in Sec. VII below.

B. Robustness in TBG

We now consider TBG, where μ and σ denote the layer and sublattice Pauli matrices, respectively. The relevant symmetries are C_{3z} , $C_{2y}\mathcal{T}$, and $C_{2z}\mathcal{T}$, which act on wave functions within the K valley as

$$C_{3z}\Psi_K(\mathbf{r}) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\mathbf{g}_5 \cdot \mathbf{r}} \end{pmatrix} \otimes e^{i(2\pi/3)\sigma_z} \Psi_K(C_{3z}^{-1}\mathbf{r}), \quad (59)$$

$$C_{2y}\mathcal{T}\Psi_K(\mathbf{r}) = \mu_x \Psi_K^*(C_{2y}\mathbf{r}), \quad (60)$$

$$C_{2z}\mathcal{T}\Psi_K(\mathbf{r}) = \sigma_x \Psi_K^*(-\mathbf{r}). \quad (61)$$

There are two natural ways to turn $N = 4$ components into a two-component spinor. First, one may project on a given sublattice. Then, the remaining degree of freedom is layer. Alternatively, one projects onto a given layer, and the remaining degree of freedom is sublattice. These two cases determine the available symmetries, which we now analyze in detail.

1. Sublattice-resolved textures

We define the sublattice-projected Pauli matrices for the two sublattices A and B as

$$\mu^{(A)} = \mu \otimes (1 + \sigma_z)/2, \quad (62)$$

$$\mu^{(B)} = \mu \otimes (1 - \sigma_z)/2. \quad (63)$$

Using these, we introduce (for energy-resolved ensembles) $\mathbf{A}^{(A)}(E, \mathbf{r})$ and $\mathbf{A}^{(B)}(E, \mathbf{r})$, along the corresponding renormalized vectors $\hat{n}_E^{(A)}$ and $\hat{n}_E^{(B)}$. They define two real-space Chern numbers $C[\hat{n}_E^{(A)}]$ and $C[\hat{n}_E^{(B)}]$ which are opposite due to $C_{2z}\mathcal{T}$ symmetry. After the sublattice projection, the symmetry analysis previously carried out for twisted TMDs

in Sec. VI A remains applicable, as the two key symmetries, $C_{2y}\mathcal{T}$ and C_{3z} , are diagonal in sublattice space. This allows us to recover the same symmetry-based arguments and formulas derived for twisted TMDs, with in-plane layer polarization at AA sites and out-of-plane polarization at AB and BA sites. The only difference with twisted TMDs lies in the sublattice-dependent phase in Eq. (59), which disappears when analyzing how H and $\hat{n}_E$ transform under C_{3z} . By employing symmetry indicators, we find—just as in twisted TMDs—that the real-space Chern number is always nonzero and follows Eq. (58).

This remarkable result is illustrated for TBG in Fig. 5(a), where we compute the real-space Chern numbers of the two sublattices at the magic angle, $\theta = 1.09^\circ$, using a realistic $w_{AA} = 80$ meV. We find that the real-space Chern numbers for the two sublattices are consistently nonzero and opposite in sign. This demonstrates that the real-space topology of the chiral limit extends to the original

Bistritzer-Macdonald model [1] with a realistic corrugation. We expect this topology to manifest in scanning tunneling experiments when focusing on a single sublattice.

Considering the sublattice-projected textures of the entire TBG flat bands, we find a robust fictitious magnetic field, opposite on each sublattice. This allows the mapping to an effective two-band Landau level problem, as detailed in Sec. VII below.

2. Layer-projected textures

Another natural option is to project onto a single layer, say the top layer, by defining

$$\mu^{(t)} = [(1 + \mu_z)/2] \otimes \sigma. \quad (64)$$

Using this, we can define $A^{(t)}(E, \mathbf{r})$. However, now the symmetries available within a single valley reduce to $C_{2z}\mathcal{T}$ and C_{3z} . These constrain the vector $A^{(t)}(E, \mathbf{r})$ through

$$\begin{aligned} A^{(t)}(E, \mathbf{r}) &= R_z(2\pi/3)A^{(t)}(E, \mathbf{r}), \\ A^{(t)}(E, -\mathbf{r}) &= M_z A^{(t)}(E, \mathbf{r}), \end{aligned} \quad (65)$$

respectively, and similarly for the normalized vector $\hat{n}_E^{(t)}(\mathbf{r})$. Crucially, C_{3z} requires that, at the high-symmetry AA, AB, and BA sites, the normalized vector is fully sublattice polarized, $\hat{n}_E^{(t)}(\mathbf{r}) = (0, 0, \pm 1)$. Furthermore, $C_{2z}\mathcal{T}$ constrains

$$\hat{n}_E^{(t)}(AB) = -\hat{n}_E^{(t)}(BA), \quad (66)$$

such that the sublattice polarizations are opposite at the AB and BA sites. At AA sites, the two symmetry constraints of Eq. (65) are only compatible with a vanishing vector, $A^{(t)}(E, AA) = 0$, corresponding to a gap closing for the effective Hamiltonian Eq. (40) and an ill-defined real-space Chern number.

A spectral gap of $H(E, \mathbf{r})$ is opened by applying a sublattice mass which breaks $C_{2z}\mathcal{T}$ but preserves the threefold rotation symmetry C_{3z} and the complete sublattice polarization at AB and BA points. Therefore, the constraint of Eq. (66) still applies by continuity despite the breaking of $C_{2z}\mathcal{T}$. Using symmetry indicators as in the previous section, we therefore obtain the real-space Chern number:

$$C[\hat{n}_E^{(t)}] = \pm 1 \bmod 3. \quad (67)$$

This prediction is confirmed by a numerical evaluation of the real-space Chern number presented in Fig. 5(d) for the two central bands of TBG with a realistic value for the corrugation. A sublattice potential $m_s = 5$ meV has been added to break $C_{2z}\mathcal{T}$. Note that the jump by three of the Chern number is consistent with Eq. (67), which only

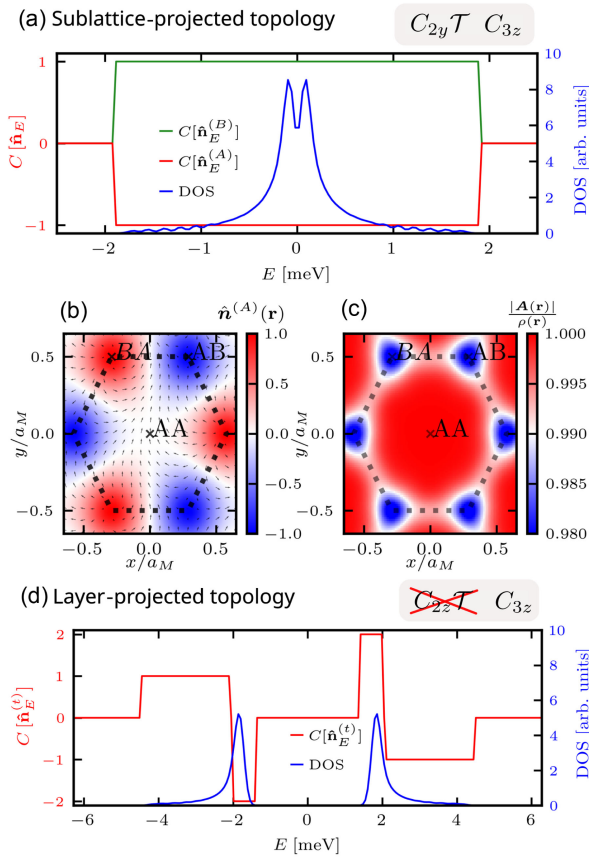


FIG. 5. TBG. (a) Sublattice-projected topology. In a realistic TBG model ($m_s = 0$, $w_{AA} = 80$ meV), the sublattice-projected Chern number $C[\hat{n}_E^{(A/B/t)}]$ is necessarily nonzero as a result of C_{3z} and $C_{2y}\mathcal{T}$ symmetries. (b) A-sublattice projected texture of the TBG flat bands. (c) Value of $|A(\mathbf{r})|/\rho(\mathbf{r})$ in the unit cell for the sublattice-projected texture of the TBG flat bands. (d) Layer-projected topology. A finite Chern number arises when $C_{2z}\mathcal{T}$ is broken by a sublattice mass $m_s = 5$ meV. We keep a realistic corrugation $w_{AA} = 80$ meV.

constrains it mod 3. It corresponds to a gap closing of the fictitious Hamiltonian $H(E, \mathbf{r})$ near $E = \pm 2$ meV at the three C_{3z} -related AB-BA midpoints in the unit cell. We expect this topology to become manifest in scanning tunneling experiments. Intriguingly, all three components of $\hat{\mathbf{n}}_E^{(i)}$ are observable, suggesting that the real-space Chern number can be directly probed. When we consider the texture of the two central bands, we find $C = 1$ for the lower band and $C = -1$ for the upper band.

VII. BAND-ENSEMBLE-INDUCED MAGNETIC FIELD AND LANDAU-LEVEL-LIKE WAVE FUNCTIONS

The fractional Chern insulator phase typically occurs in a Chern band when the interaction is larger than the bandwidth but small compared to the band gaps [30]. Under this regime, all the wave functions in the band are correlated by the interaction, leading to entangled many-body states. Looking for additional insights into the relevant band, in this section, we show how real-space band ensemble textures provide a useful framework for interpreting FCI states observed in recent moiré experiments. We begin by quantitatively characterizing the polarization of the band to an optimal layer texture (or, more generally, other orbital labels such as sublattice). To this end, we introduce a polarization metric that quantifies the fraction of the band aligned with the texture defined in Sec. VA 2. We then project the full Hamiltonian onto this texture, obtaining an effective single-component Hamiltonian. We show that this Hamiltonian experiences an emergent magnetic field given by the Berry curvature of the texture and that it reproduces the band dispersion beyond the adiabatic approximation. Finally, we analyze the properties of the wave function along the texture direction and identify a natural mapping to (generalized) Landau levels. In this picture, the interacting problem within the band reduces to an effective single-component model, paving the way for the construction of many-body FCI wave functions. In the following, we develop the theory for a two-component system relevant to twisted TMDs, and in the last subsection, we extend the analysis to TBG.

A. Quantifying the band alignment—Are the bands single-component-like?

We first quantify how well a band is aligned with a texture. We focus on a two-component (or layer) system, but the approach is easily generalizable to a larger number of components. For the ensemble of a band b in valley λ_b , its texture is [concretizing Eq. (38)] given by the three-dimensional vector,

$$\mathbf{A}(\mathbf{r}) = \frac{1}{N_{\text{u.c.}}} \sum_{\mathbf{k} \in \text{BZ}} u_{\mathbf{k},b,\lambda_b}^\dagger(\mathbf{r}) \boldsymbol{\mu} u_{\mathbf{k},b,\lambda_b}(\mathbf{r}), \quad (68)$$

which allows a definition of the corresponding normalized vector $\hat{\mathbf{n}}(\mathbf{r}) = \mathbf{A}(\mathbf{r})/|\mathbf{A}(\mathbf{r})|$. The length of the texture vector is upper bounded by the local density,

$$|\mathbf{A}(\mathbf{r})| \leq \rho(\mathbf{r}) = \frac{1}{N_{\text{u.c.}}} \sum_{\mathbf{k} \in \text{BZ}} u_{\mathbf{k},b,\lambda_b}^\dagger(\mathbf{r}) u_{\mathbf{k},b,\lambda_b}(\mathbf{r}), \quad (69)$$

with equality reached when all the Bloch waves in the ensemble are aligned in the same direction. To quantify the average alignment across the unit cell, we construct the projection operator to the direction of the texture $P_{\hat{\mathbf{n}}}(\mathbf{r}) = \frac{1}{2}[1 + \boldsymbol{\mu} \cdot \hat{\mathbf{n}}(\mathbf{r})]$, which acts in the layer space at each position $\mathbf{r}$. We then define the polarization index,

$$\kappa = \int_{\text{u.c.}} d\mathbf{r} \frac{1}{N_{\text{u.c.}}} \sum_{\mathbf{k} \in \text{BZ}} u_{\mathbf{k},b,\lambda_b}^\dagger(\mathbf{r}) P_{\hat{\mathbf{n}}}(\mathbf{r}) u_{\mathbf{k},b,\lambda_b}(\mathbf{r}) \quad (70)$$

$$= \int_{\text{u.c.}} d\mathbf{r} \frac{\rho(\mathbf{r}) + |\mathbf{A}(\mathbf{r})|}{2} \in [0, 1], \quad (71)$$

to characterize the band alignment. We show in Appendix D that, for a fixed band, the band-defined texture [defined in Eq. (68)] maximizes κ over all possible textures, and κ then measures how closely the band follows this optimal texture.

The Bloch wave functions can be decomposed as

$$u_{\mathbf{k},b,\lambda_b}(\mathbf{r}) = \alpha_{\mathbf{k}} \chi_{\hat{\mathbf{n}}}(\mathbf{r}) \psi_{\mathbf{k}}(\mathbf{r}) + \beta_{\mathbf{k}} \chi_{\hat{\mathbf{n}}}^\perp(\mathbf{r}) \varphi_{\mathbf{k}}(\mathbf{r}), \quad (72)$$

with $|\alpha_{\mathbf{k}}|^2 + |\beta_{\mathbf{k}}|^2 = 1$, where $\psi_{\mathbf{k}}$ and $\varphi_{\mathbf{k}}$ are normalized, and the spinor $\chi_{\hat{\mathbf{n}}}(\mathbf{r})$ satisfies $P_{\hat{\mathbf{n}}}(\mathbf{r}) \chi_{\hat{\mathbf{n}}}(\mathbf{r}) = \chi_{\hat{\mathbf{n}}}(\mathbf{r})$. In this form, $\kappa = 1 - (1/N_{\text{u.c.}}) \sum_{\mathbf{k}} |\beta_{\mathbf{k}}|^2$, so that large κ corresponds to small $\beta_{\mathbf{k}}$, i.e., wave functions dominated by the spinor $\chi_{\hat{\mathbf{n}}}$. This decomposition generalizes Eq. (11), with a $\mathbf{k}$ -independent spinor, beyond the ideal cases discussed in Sec. III. Accordingly, κ quantifies the proximity to Eq. (11), indicating the fraction of the band following the texture $\hat{\mathbf{n}}(\mathbf{r})$.

We now provide the calculated values of the alignment strength κ for twisted bilayer TMDs, which turn out to be remarkably large. For WSe₂, we find that the value of κ decreases with twist angle. However, even at $\theta = 2.2^\circ$, it is still very large, having $\kappa = 0.96$. For the realistic model of MoTe₂ at $\theta = 2.1^\circ$, suggested to host Landau-level analogs, we find good alignment of $\kappa = 0.98$ for the topmost band and $\kappa = 0.92$ for the second topmost band. The band alignment to the optimal texture is thus excellent for twisted bilayer TMDs. Below, we present two approaches that exploit this observation to project onto a single-component Hamiltonian or wave function and reveal the Landau-level structure of these models.

B. Effective single-component Hamiltonian and energy dispersion

For κ sufficiently close to one, the wave function $\psi_{\mathbf{k}}(\mathbf{r})$ dominates the decomposition Eq. (72) and is approximately a solution of a single-component Hamiltonian,

$$H_{\text{eff}} = \chi_{\hat{n}}^{\dagger}(\mathbf{r}) H \chi_{\hat{n}}(\mathbf{r}), \quad (73)$$

which is obtained after projecting the original Hamiltonian onto the optimal texture $P_{\hat{n}} H P_{\hat{n}}$. We derive this result in Appendix E. Even though we are not in the adiabatic limit, the derivation follows closely the same logic as in Sec. III A. Specializing to twisted TMDs, we recover for H_{eff} the Hamiltonian given in Eq. (22). The main difference is that the effective vector potential

$$\tilde{A}(\mathbf{r}) = -i\chi_{\hat{n}}^{\dagger}(\mathbf{r}) \nabla_{\mathbf{r}} \chi_{\hat{n}}(\mathbf{r}) \quad (74)$$

as well as the kinetic potential $D(\mathbf{r})$ are determined by the average band texture $\hat{n}(\mathbf{r})$ and not by the interlayer coupling $\Delta(\mathbf{r})$ as in the adiabatic limit of twisted TMDs. Thus, the effective magnetic field in this Hamiltonian is that of the optimal band texture.

Another consequence of using the optimal texture to construct the effective Hamiltonian is that the validity of this construction does not impose constraints on the rate of variation of the underlying Hamiltonian, unlike the adiabatic approximation [35–37]. As a result, it is significantly more broadly applicable, extending the effective Hamiltonian approach to larger twist angles and to higher-harmonic models, where the kinetic energy dominates over interlayer coupling. In particular, the only requirement is that κ is close to one for the band of interest.

To demonstrate the power of this approach, we consider the effective Hamiltonian for a realistic model of MoTe₂ at $\theta = 2.1^\circ$, which includes higher-harmonic terms. In this regime, the adiabatic approximation breaks down. The texture defined by the Hamiltonian $\hat{n}_{\Delta}(\mathbf{r})$, shown in Fig. 6(a), exhibits sharp domain walls arising from lattice relaxation [47,48]. These rapidly varying features make the adiabatic approximation inapplicable. In sharp contrast, the texture defined by the topmost band [Fig. 6(b)] is much smoother, making it a more suitable starting point for constructing a single-component Hamiltonian.

Consistent with this picture, the numerically computed energies of the adiabatic effective Hamiltonian show strong disagreement with the full spectrum [Fig. 6(c)]. Remarkably, using instead the average band texture to construct the effective Hamiltonian of Eq. (73), we reproduce the energies of the topmost band with good accuracy [Fig. 6(d)]. The agreement is still acceptable even for the second topmost band.

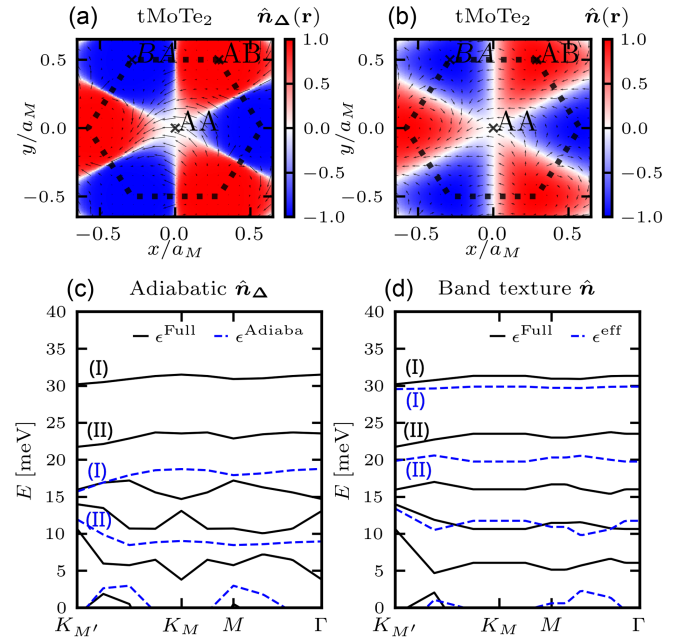


FIG. 6. (a) Texture defined from the Hamiltonian [using Eq. (19)] of twisted MoTe₂ at $\theta = 2.1^\circ$. Because of lattice relaxation, the Hamiltonian contains sharp domains between upper and lower layer polarization. The adiabatic approximation is inapplicable for such a Hamiltonian. (b) Ensemble texture of the upper band [computed using Eq. (68)] of twisted MoTe₂ at $\theta = 2.1^\circ$. The variations in the band texture are much smoother than in (a), and this texture can be used to define a single-component Hamiltonian describing the upper band. (c),(d) Comparison of the full spectrum (black lines, top two bands labeled with roman numerals) of MoTe₂ at $\theta = 2.1^\circ$ with the spectrum of the effective Hamiltonian (blue dashed lines, top two bands labeled with roman numerals) obtained by projecting onto (c) the texture defined from the Hamiltonian (adiabatic approximation), and (d) the texture obtained from the band. The adiabatic texture is a poor approximation for the energies, while the band-defined texture-defined effective Hamiltonian approximates the upper part of the spectrum well.

C. Single-component wave function and Landau-level content

In the flat-band regime relevant for FCIs, the physics is largely determined by the wave functions and their overlaps. We have just discussed how the energies and eigenstates of a high- κ band map onto an effective single-component Hamiltonian with a texture-induced magnetic field. Alternatively, it is possible to directly project the Bloch wave function, given in Eq. (72), onto the dominant spinor texture $\chi_{\hat{n}}$. For the topological real-space textures introduced and discussed in Secs. V and VI, the resulting single-component function can be expanded as $\alpha_{\mathbf{k}} \psi_{\mathbf{k}}(\mathbf{r}) = \sum_{n=0}^{+\infty} c_n^{\mathbf{k}} \mathbf{u}_{n,\mathbf{k}}^{\Phi=1}(\mathbf{r})$. Here, $\mathbf{u}_{n,\mathbf{k}}^{\Phi=1}$ denotes the n th Landau-level wave function with momentum $\mathbf{k}$ and one flux quantum per unit cell. The coefficients $c_n^{\mathbf{k}}$ quantify the overlap with each Landau level. Importantly, their value is

bounded by the texture polarization via

$$\frac{1}{N_{\text{u.c.}}} \sum_{\mathbf{k}} \sum_{n=0}^{+\infty} |c_n^{\mathbf{k}}|^2 = \sum_{n=0}^{+\infty} w_n = \kappa, \quad (75)$$

and can be further separated into the weights w_n of each Landau level. The approach can also be further refined by considering generalized Landau levels which possess ideal but nonuniform quantum geometry encoded in a Kähler potential [42,44,53,59,89]. In all cases, the projection of the single-component wave function onto Landau levels is key for building trial wave functions describing different FCI phases. Crucially, the value of κ of our optimal texture gives an upper bound for such a projection.

For large κ , the interacting physics of the band is essentially that of a single-component band $\psi_{\mathbf{k}}$. In particular, the spinor structure only affects the interacting terms to order d/a_M compared to a monolayer problem, where d is the interlayer distance and a_M is the moiré wavelength. Thus, trial states can be constructed just like in a single-component problem.

In the simplest cases, considering generalized Landau levels is not necessary, and standard Landau levels with a uniform quantum geometry are sufficient to capture the physics in the bands. To illustrate the validity of this approach, we consider a realistic model of MoTe₂ at $\theta = 2.1^\circ$, which has been suggested to host Landau-level analogs [47,48]. We perform a spinor projection followed by the Landau-level decomposition of Eq. (75). We find excellent alignment with the average spinor texture for the topmost and second topmost bands, with $\kappa = 0.98$ and $\kappa = 0.92$, respectively. Furthermore, the topmost band has most of its weight in the zeroth Landau level, $w_0 = 0.971$, while the second band predominantly overlaps with the first Landau level, $w_1 = 0.88$. These results indicate that the top two bands of MoTe₂ at $\theta = 2.1^\circ$ closely correspond to the lowest two Landau levels at an effective flux $\Phi_2 = 1$. This is consistent with previous Hofstadter calculations showing that these bands develop a parabolic dispersion under an external magnetic field $\Phi = -1$ [64]. This correspondence naturally suggests the formation of Laughlin-like states in the topmost band and non-Abelian states in the second band.

D. Application to TBG bands

We now apply this analysis to sublattice-projected textures in TBG. In this system, we find a robust fictitious magnetic field, opposite on each sublattice. In particular, all the wave functions are essentially aligned, with $\kappa \simeq 0.998$ even for realistic corrugation, see Eq. (70), and follow the texture shown in Fig. 5(b). This suggests a natural decomposition of the TBG flat-band eigenfunctions as

$$u_{\mathbf{k}}^{\text{TBG}}(\mathbf{r}) = \chi^A(\mathbf{r})\tilde{\psi}_{\mathbf{k}}^1(\mathbf{r}) + \chi^B(\mathbf{r})\tilde{\psi}_{\mathbf{k}}^{-1}(\mathbf{r}) + \mathcal{O}[1 - \kappa], \quad (76)$$

where the spinors $\chi^{A/B}$ are fully A/B -sublattice polarized, the unnormalized scalar wave functions $\tilde{\psi}_{\mathbf{k}}^{\Phi_2}(\mathbf{r})$ experience a net magnetic field of $\Phi_2 = \pm 1$ flux quanta per unit cell, and the neglected terms are suppressed by a factor of $1 - \kappa \ll 1$ for TBG. The fictitious magnetic field implied by the textures can be made explicit by projecting the TBG Hamiltonian onto the spinors $\chi^A(\mathbf{r})$ and $\chi^B(\mathbf{r})$, turning a four-component problem into a two-component problem.

To illustrate this better, Fig. 5(c) plots the value of $|\mathbf{A}(\mathbf{r})|/\rho(\mathbf{r})$ in the unit cell, showing that the tracking is extremely good throughout the unit cell. However, a crucial difference (as will be discussed below in Sec. VIII) from the chiral limit is that the charge density is strongly peaked at the AA point of the unit cell, so that a large proportion of the unit cell is effectively not sampled.

VIII. HEAVY FERMION MODEL FOR TBG

Although realistic TBG is relatively far from the analytically accessible chiral limit [41], our findings show that the real-space topology carries over from the chiral limit, even quantitatively. To better understand this good agreement, we discuss real-space topology in the context of the topological heavy fermion model [49–51,90–94]. The THF is a minimal model involving six fermion species, which captures the band structure of the two central bands of TBG along with their topological properties. Two of these fermion variables, denoted f_1 and f_2 , describe localized Wannier states centered at the AA points of the moiré lattice. These orbitals form a triangular lattice with very weak overlap between neighboring Wannier states, resulting in negligible kinetic energy (dispersion). The overlap between these Wannier states and the two central bands is excellent (96%) throughout most of the Brillouin zone. However, the overlap vanishes at the Γ point, reflecting a topological obstruction [95–101], which requires the introduction of four light fermion species, denoted c_j ($j = 1, \dots, 4$). Although their overlap with the central bands is small, these fermions carry the entire topology of the central bands (with Berry curvature concentrated near the Γ point) and are essentially delocalized in the sense that they are not Wannierizable. Details of the model are reviewed in Appendix H. The THF model is effective in describing correlation effects in TBG, in particular the coexistence between localized moments and conducting states [14,102–105].

As we now elaborate, regions with strong topological features are concentrated near the AB and BA sites, away from the AA-centered Wannier functions. We illustrate the spatial segregation of orbitals in Fig. 7(a). The Wannier orbitals f_1, f_2 are well localized around the AA points, exhibiting very small densities at the edges of the unit cell. In contrast, the orbitals c_1, c_2 mainly occupy the regions around the BA and AB points, forming a hexagonal lattice. The remaining conduction electrons, c_3 and c_4 , exhibit

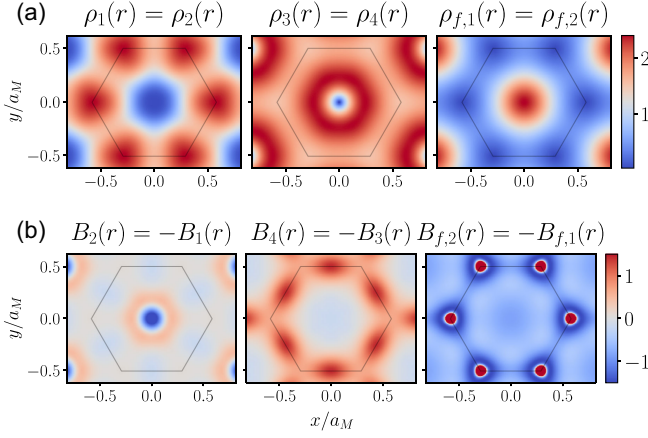


FIG. 7. (a) Electronic densities, summed over the sublattice and layer indices, of the six orbitals ($c_1, c_2, c_3, c_4, f_1, f_2$), two by two equal. The unit cell, centered at the AA site, is depicted as a light hexagon, with its edges corresponding to the BA and AB sites. (b) Real-space Berry curvatures for the six orbitals of the topological heavy fermion model with the corrugation parameter $w_{AA}/w_{AB} = 0.7$.

vanishing density at the AA points and display a circular ring structure, which remains significant at the edges of the unit cell, particularly at the BA and AB points.

We contrast these densities with the real-space Berry curvatures of the various orbitals displayed in Fig. 7(b). The topological features of the Wannier orbitals f_1, f_2 emerge in the BA and AB regions, where their densities are negligibly small. Similarly, the light fermions c_1, c_2 display topological characteristics in low-density regions and overlap only weakly with the central bands. In contrast, the light fermions c_3 and c_4 carry most of the real-space topology, despite having only a small overlap with the central bands overall. c_3 (c_4) carries a real-space Chern number of -1 ($+1$). We have verified numerically, by computing the average sublattice polarizations of the different orbitals, that deviations from the chiral limit affect the Wannier states more significantly than the light fermions. Physically, increasing the interlayer tunneling w_{AA} between A atoms when moving away from the chiral limit is expected to impact AA stacking regions more strongly than AB or BA regions. Consequently, we find that the real-space topology of TBG is primarily inherited from the c_3 and c_4 light conduction electrons and remains largely insensitive to the Wannier orbitals as well as more robust to deviations from the chiral limit.

IX. DISCUSSION

To conclude, we demonstrated the existence of a robust real-space Chern number characterizing the fictitious magnetic field experienced by electrons in moiré systems. This Chern number universally applies to twisted bilayer graphene and twisted bilayer transition metal dichalcogenides without assuming delicate parameter regimes. It extends

previous insights obtained for the special limits of chiral twisted bilayer graphene and the adiabatic approximation to transition metal dichalcogenides, which turn out to apply only to fine-tuned cases where the wave functions exhibit zeros in the unit cell.

We showed that, remarkably, the real-space topology of wave function ensembles in twisted bilayer TMDs is always nontrivial. We identified in particular the two spatial symmetries, C_{3z} and $C_{2y}\mathcal{T}$, as the essential ingredients enforcing a nontrivial real-space Chern number. Extending this framework to twisted bilayer graphene, we showed that the same symmetries protect a nontrivial real-space topology when wave functions are projected onto a single sublattice. Intriguingly, as shown in Fig. 8, the real-space Berry curvature profiles in the chiral limit and for individual and ensembles of wave functions at realistic corrugation are quantitatively very similar. In fact, far from the singularity of Fig. 8(b), the comparison between the three textures is quantitatively excellent as illustrated in Fig. 8(d).

Signatures of nontrivial real-space textures have been recently observed in twisted bilayer TMDs [38,39]. Not only does our theory provide a natural explanation for these experimental observations, it also shows that the same C_{3z} and $C_{2y}\mathcal{T}$ -protected real-space topology should in fact also manifest in twisted bilayer graphene, suggesting the detailed study of real-space textures in twisted bilayer graphene as a promising experimental possibility.

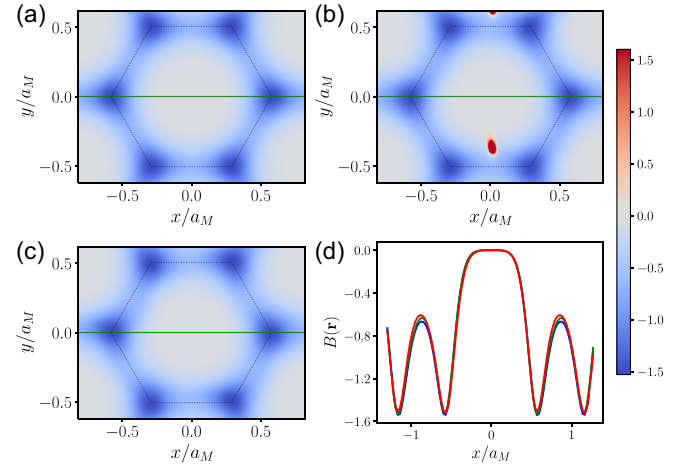


FIG. 8. Real-space topology of twisted bilayer graphene for individual and ensemble of Bloch wave functions, projected onto the A sublattice. The Berry curvature $B(\mathbf{r})$, characterizing the spatial change in orientation of the Bloch spinor wave functions, is represented over the unit cell (dotted hexagon) centered at AA sites. (a) In the chiral limit. (b) Individual and (c) ensemble of wave functions for realistic corrugation, $w_{AA}/w_{BA} = 0.7$. The real-space Berry curvatures integrate to -1 in (a) and (c), and to 0 in (b) due to the strong positive contribution (dark red region). (d) Linecuts of the three color maps (a)–(c) along the green line. Panel (b) has been calculated with $\mathbf{k} = (\mathbf{g}_3 - \mathbf{g}_1)/4$ and (c) with $E = 2.8$ meV, but the results are largely insensitive to the value of E .

A particularly exciting experimental direction would be the study of layer-projected topology, where all three components of the vector $\hat{n}_E^{(i)}$ are observable [83]. Based on our results, we expect this vector to define a topologically nontrivial texture provided the $C_{2z}\mathcal{T}$ symmetry is broken.

The existence of a robust fictitious magnetic field in twisted bilayer graphene raises several interesting theoretical questions. For instance, the decomposition in Eq. (76) suggests that an adiabatic approach, akin to the one applied successfully to twisted bilayer TMDs [35–37], should extend to realistic models of TBG. However, because of the additional sublattice degree of freedom, the resulting adiabatic Hamiltonian will have two components, which experience opposite fictitious magnetic fields. Such an approach could provide additional insights into the analytical structure of the form factors, which is crucial in determining the correlation physics of the system. [49,105,106].

Our framework of band textures developed in Sec. VII can also be directly applied to other moiré systems to study whether a fictitious magnetic field is present, as well as to discover relevant approximation schemes. Indeed, our method may serve as a simple and general zeroth-order framework for characterizing moiré systems. Beginning with the single-particle bands, one computes their real-space textures and the associated topology. These textures can then be used to construct an effective single-component Hamiltonian that features an effective magnetic field, implying ansatz wave functions in the form of a spinor factor multiplied by a Landau-level-like orbital. Having pointed out the importance of the C_{3z} and $C_{2y}\mathcal{T}$ symmetries for the nontrivial quantum anomalous Hall physics in TBG and twisted bilayer TMDs, we expect emerging moiré systems possessing these symmetries [107] to display similar phenomenology. Other important systems, such as hBN-aligned rhombohedral multilayer graphene, or helical trilayer graphene, lack these symmetries. The former features an interaction-induced Chern insulator [19–21,108–110], suggesting the tantalizing possibility of an interaction-induced fictitious magnetic field.

An interesting direction would be to extend our framework to treat intervalley-coherent orders in TBG, and look for the real-space topological properties of the different valley-coherent states [111,112] as well as the incommensurate Kekulé spiral order [113]. With intervalley coherence and preserved spin S_z conservation, twisted bilayer graphene wave functions have $N = 8$ components, requiring either a projection onto a layer and sublattice or the analysis of the topological properties of an $N = 8$ -dimensional real-space Hamiltonian, as detailed in Appendix F.

Finally, the definition of the energy-resolved texture can be extended to many-body states beyond the mean-field approximation by using a layer-resolved local spectral function [79]. In this case, our symmetry-based arguments should still hold, which raises the possibility that the finite

real-space Chern number of twisted bilayer TMDs is relevant even in the fractional Chern insulator phase [22–27]. Moreover, it has recently been predicted [114] that an FCI could emerge from an isolated zero-Chern band with interactions that do not couple to other bands. If confirmed, this suggests that nontrivial momentum-space topology may play only a secondary role in the formation of FCIs [115], calling for future investigations of the precise roles of real-space topology and fictitious magnetic fields in the formation of FCIs as well as their connection to quantum geometry.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: BLOCH STATES IN A MAGNETIC FIELD

We describe the magnetic field using a general linear vector potential:

$$\mathbf{A} = \beta \mathbf{r}, \quad \beta = \begin{pmatrix} \beta_{xx} & \beta_{xy} \\ \beta_{yx} & \beta_{yy} \end{pmatrix}. \quad (\text{A1})$$

Demanding that $\nabla \times \mathbf{A} = B\hat{\mathbf{z}}$ gives the condition

$$\beta_{yx} - \beta_{xy} = B. \quad (\text{A2})$$

We define the guiding-center operator,

$$\mathbf{R} = \mathbf{r} - \frac{1}{eB} \hat{\mathbf{z}} \times \boldsymbol{\pi}, \quad (\text{A3})$$

with the kinetic momentum $\boldsymbol{\pi} = \mathbf{p} - e\mathbf{A}$. This can be used to express the magnetic translation operators:

$$T_{\mathbf{a}} = e^{ieB(\hat{\mathbf{z}} \times \mathbf{R}) \cdot \mathbf{a}}. \quad (\text{A4})$$

In particular, we define the translation operators,

$$T_j = e^{ieB(\hat{\mathbf{z}} \times \mathbf{R}) \cdot \mathbf{a}_j}, \quad (\text{A5})$$

associated with the two lattice vectors $\mathbf{a}_j$ with $j = 1, 2$. (In explicit calculations, we choose without loss of generality that $\mathbf{a}_1 \parallel \hat{\mathbf{x}}$, while the direction of $\mathbf{a}_2$ is arbitrary). For our choice of gauge, the translation operators take the explicit form

$$\begin{aligned} T_1 &= e^{ieB(\mathbf{a}_1 \times \hat{\mathbf{z}}) \cdot \mathbf{r} - ie\mathbf{A} \cdot \mathbf{a}_1 + i\mathbf{p} \cdot \mathbf{a}_1}, \\ T_2 &= e^{ieB(\mathbf{a}_2 \times \hat{\mathbf{z}}) \cdot \mathbf{r} - ie\mathbf{A} \cdot \mathbf{a}_2 + i\mathbf{p} \cdot \mathbf{a}_2}. \end{aligned} \quad (\text{A6})$$

We now choose a gauge such that this simplifies in a manner similar to the Landau gauge.

We first demand that $T_1 = e^{i\mathbf{p} \cdot \mathbf{a}_1}$, i.e., that

$$\begin{aligned} 0 &= eB(\mathbf{a}_1 \times \hat{\mathbf{z}}) \cdot \mathbf{r} - e\mathbf{a}_1^\top \cdot \boldsymbol{\beta} \cdot \mathbf{r} \\ &= [eB(\mathbf{a}_1 \times \hat{\mathbf{z}})^\top - e\mathbf{a}_1^\top \cdot \boldsymbol{\beta}] \cdot \mathbf{r}. \end{aligned} \quad (\text{A7})$$

Thus, the expression in the square brackets must vanish,

$$\begin{aligned} 0 &= eB(\hat{\mathbf{a}}_1 \times \hat{\mathbf{z}})^\top - e\hat{\mathbf{a}}_1^\top \cdot \boldsymbol{\beta} \\ &= -eB\hat{\mathbf{y}}^\top - e\hat{\mathbf{x}}^\top \cdot \boldsymbol{\beta}, \end{aligned} \quad (\text{A8})$$

which implies (recall we choose our coordinate system to satisfy $\mathbf{a}_1 \parallel \hat{\mathbf{x}}$)

$$\beta_{xx} = 0, \quad \beta_{xy} = -B. \quad (\text{A9})$$

Combining with Eq. (A2) gives

$$\beta_{yx} = 0. \quad (\text{A10})$$

We still have β_{yy} left at our disposal.

We now consider T_2 . We demand that T_2 separates into a product of two commuting exponential factors involving $\mathbf{p}$ and $\mathbf{r}$, respectively. The term $eB(\mathbf{a}_2 \times \hat{\mathbf{z}}) \cdot \mathbf{r}$ in the exponent already contains only the coordinate perpendicular to the $\mathbf{a}_2$ axis, so that it commutes with $\mathbf{p} \cdot \mathbf{a}_2$. We still need to ensure that

$$\begin{aligned} e\mathbf{A} \cdot \mathbf{a}_2 &= e\mathbf{a}_2^\top \cdot \boldsymbol{\beta} \cdot \mathbf{r} \\ &= (a_{2x}, a_{2y}) \begin{pmatrix} 0 & -B \\ 0 & \beta_{yy} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (-a_{2x}B + a_{2y}\beta_{yy})y \end{aligned} \quad (\text{A11})$$

depends only on the coordinate perpendicular to $\mathbf{a}_2$. This is the case only if we chose the prefactor of y to vanish, which fixes

$$\beta_{yy} = B \frac{a_{2x}}{a_{2y}}. \quad (\text{A12})$$

We have now fully determined the Landau-type vector potential,

$$\mathbf{A} = B \frac{\hat{\mathbf{z}} \cdot (\mathbf{a}_1 \times \mathbf{r})}{\hat{\mathbf{z}} \cdot (\mathbf{a}_1 \times \mathbf{a}_2)} \hat{\mathbf{z}} \times \mathbf{a}_2, \quad (\text{A13})$$

as well as the associated translation operators along the lattice vectors,

$$T_1 = e^{i\mathbf{p} \cdot \mathbf{a}_1}, \quad (\text{A14})$$

$$T_2 = e^{ieB(\mathbf{a}_2 \times \hat{\mathbf{z}}) \cdot \mathbf{r} + i\mathbf{p} \cdot \mathbf{a}_2}. \quad (\text{A15})$$

We can classify eigenstates according to the eigenvalues of T_1 and T_2 provided that these two operators commute. We find

$$T_1 T_2 = e^{ieB(\mathbf{a}_2 \times \hat{\mathbf{z}}) \cdot \mathbf{a}_1} T_2 T_1, \quad (\text{A16})$$

and read off that T_1 and T_2 commute if the unit cell is threaded by an integer number Φ of flux quanta,

$$B\hat{\mathbf{z}} \cdot (\mathbf{a}_1 \times \mathbf{a}_2) = \Phi \frac{2\pi\hbar}{e}. \quad (\text{A17})$$

We will now assume that this is the case.

We define Bloch states as simultaneous eigenstates of T_1 and T_2 :

$$T_j e^{i\mathbf{k} \cdot \mathbf{r}} u_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{a}_j} e^{i\mathbf{k} \cdot \mathbf{r}} u_{\mathbf{k}}(\mathbf{r}). \quad (\text{A18})$$

Using the explicit expressions for the T_j , this gives

$$\begin{aligned} u_{\mathbf{k}}(\mathbf{r} + \mathbf{a}_1) &= u_{\mathbf{k}}(\mathbf{r}), \\ u_{\mathbf{k}}(\mathbf{r} + \mathbf{a}_2) &= e^{-ieB(\mathbf{a}_2 \times \hat{\mathbf{z}}) \cdot \mathbf{r}} u_{\mathbf{k}}(\mathbf{r}). \end{aligned} \quad (\text{A19})$$

Finally, expressing the magnetic field through the integer number of flux quanta, we find

$$\mathbf{A} = \frac{2\pi\hbar\Phi}{e} \frac{\hat{\mathbf{z}} \cdot (\mathbf{a}_1 \times \mathbf{r})}{[\hat{\mathbf{z}} \cdot (\mathbf{a}_1 \times \mathbf{a}_2)]^2} \hat{\mathbf{z}} \times \mathbf{a}_2 \quad (\text{A20})$$

as well as

$$\begin{aligned} u_{\mathbf{k}}(\mathbf{r} + \mathbf{a}_1) &= u_{\mathbf{k}}(\mathbf{r}), \\ u_{\mathbf{k}}(\mathbf{r} + \mathbf{a}_2) &= e^{-i2\pi\Phi[\hat{\mathbf{z}} \cdot (\mathbf{r} \times \mathbf{a}_2)/\hat{\mathbf{z}} \cdot (\mathbf{a}_1 \times \mathbf{a}_2)]} u_{\mathbf{k}}(\mathbf{r}). \end{aligned} \quad (\text{A21})$$

We can also define the reciprocal lattice vectors,

$$\mathbf{G}_1 = 2\pi \frac{\mathbf{a}_2 \times \hat{\mathbf{z}}}{\hat{\mathbf{z}} \cdot (\mathbf{a}_1 \times \mathbf{a}_2)}, \quad \mathbf{G}_2 = 2\pi \frac{\hat{\mathbf{z}} \times \mathbf{a}_1}{\hat{\mathbf{z}} \cdot (\mathbf{a}_1 \times \mathbf{a}_2)}, \quad (\text{A22})$$

to simplify the notation

$$\mathbf{A} = -\frac{\Phi}{2\pi e} (\mathbf{G}_2 \cdot \mathbf{r}) \mathbf{G}_1 \quad (\text{A23})$$

and

$$\begin{aligned} u_{\mathbf{k}}(\mathbf{r} + \mathbf{a}_1) &= u_{\mathbf{k}}(\mathbf{r}), \\ u_{\mathbf{k}}(\mathbf{r} + \mathbf{a}_2) &= e^{-i\Phi \mathbf{G}_1 \cdot \mathbf{r}} u_{\mathbf{k}}(\mathbf{r}). \end{aligned} \quad (\text{A24})$$

APPENDIX B: DETAILS ON THE TMD MODEL

The Hamiltonian of a twisted bilayer TMD reads

$$H_{\text{TMD}}^K = -\frac{\hbar^2(\mathbf{k} - \frac{\mu_z \mathbf{q}_1}{2})^2}{2m^*} \mu_0 + \mathbf{\Delta}(\mathbf{r}) \cdot \boldsymbol{\mu} + \Delta_0(\mathbf{r}) \mu_0, \quad (\text{B1})$$

where m^* is the effective mass of the holes, and where $\mu_{0,x,y,z}$ are the Pauli matrices in layer space, and where $\mathbf{q}_1 = [0, 4\pi\theta/(3a_0)]$ is the momentum offset between the valence band maxima of the two layers, where a_0 is the lattice constant and θ is the twist angle. $\mathbf{\Delta}(\mathbf{r})$ and $\Delta_0(\mathbf{r})$ are both periodic functions in the moiré unit cell, given in terms of layer potentials and interlayer tunneling terms as

$$\Delta_0 = \frac{1}{2}[V_t(\mathbf{r}) + V_b(\mathbf{r})], \quad (\text{B2})$$

$$\Delta_x = \text{Re}[w(1 + e^{i\mathbf{g}_5 \cdot \mathbf{r}} + e^{i\mathbf{g}_6 \cdot \mathbf{r}})], \quad (\text{B3})$$

$$\Delta_x = -\text{Im}[w(1 + e^{i\mathbf{g}_5 \cdot \mathbf{r}} + e^{i\mathbf{g}_6 \cdot \mathbf{r}})], \quad (\text{B4})$$

$$\Delta_z = \frac{1}{2}[V_b(\mathbf{r}) - V_t(\mathbf{r})], \quad (\text{B5})$$

where w is the tunneling strength and where the layer potentials in the first harmonic approximation are parametrized by an amplitude V and a phase ψ :

$$V^b(\mathbf{r}) = \left(V e^{i\psi} \sum_{j=1,3,5} e^{i\mathbf{r} \cdot \mathbf{g}_j} \right) + \text{H.c.}, \quad (\text{B6})$$

$$V^t(\mathbf{r}) = \left(V e^{-i\psi} \sum_{j=1,3,5} e^{i\mathbf{r} \cdot \mathbf{g}_j} \right) + \text{H.c.}, \quad (\text{B7})$$

where H.c. denotes the complex conjugate. We use the parameters $(a_0, m^*, V, \psi, w) = (0.332 \text{ nm}, 0.43 m_e, 9 \text{ meV}, -128^\circ, 18 \text{ meV})$, taken from Ref. [58].

APPENDIX C: NONVANISHING SPINOR WAVE FUNCTION

The Hamiltonian of graphene moiré systems can be expressed, in the chiral limit, as

$$H = \begin{pmatrix} 0 & \mathcal{D}^\dagger \\ \mathcal{D} & 0 \end{pmatrix}, \quad (\text{C1})$$

when written in the sublattice basis. The A-polarized ideal bands then correspond to the zero modes of the differential operator,

$$\mathcal{D} = -2iv_F \bar{\partial} + A(\mathbf{r}), \quad (\text{C2})$$

where $\bar{\partial} = (\partial_x + i\partial_y)/2$, and $A(\mathbf{r})$ is a spatially varying $N \times N$ matrix, with N denoting the number of layers. Similarly, the B-polarized ideal bands correspond to the zero modes of $\mathcal{D}^\dagger$.

Let us assume that the spinor function $\mathcal{B}(\mathbf{r})$ appearing in the decomposition of Eq. (26) vanishes at some position, which we take to be the origin without loss of generality. The Landau-level component of the wave function, $u_{0,\mathbf{k}}^{\Phi=1}(\mathbf{r})$, also exhibits a zero that depends on the choice of momentum $\mathbf{k}$. Choosing $\mathbf{k} = \mathbf{k}_0$ such that this zero coincides with the origin, the Bloch part of the ideal wave function, $u_{\mathbf{k}_0}^{\text{Ideal}}(\mathbf{r})$, develops a double zero, and can be expanded as

$$u_{\mathbf{k}_0}^{\text{Ideal}}(\mathbf{r}) \simeq C_1 \bar{z}z + \dots, \quad (\text{C3})$$

for $z \rightarrow 0$, with $C_1 \neq 0$. The Landau-level factor contributes a holomorphic term $\sim z$, while the spinor component contributes a $\bar{z}$ dependence corresponding to an antivortex centered at the origin. Substituting the expansion of Eq. (C3) into the zero-mode equation, $\mathcal{D}u_{\mathbf{k}_0}^{\text{Ideal}}(\mathbf{r}) = 0$, we obtain, for small z ,

$$-2iv_F C_1 z + \mathcal{O}(z^2, |z|^2, \bar{z}^2) = 0, \quad (\text{C4})$$

which implies $C_1 = 0$ and thus a contradiction. This completes the proof that the spinor $\mathcal{B}(\mathbf{r})$ is nowhere vanishing. In contrast, we note that for a double-Dirac differential operator $\mathcal{D} = \bar{\partial}^2 + A(\mathbf{r})$, the expansion in Eq. (C3) is compatible [40] with $C_1 \neq 0$, and the spinor $\mathcal{B}(\mathbf{r})$ may host an antivortex, as predicted in periodically strained graphene [70,71].

APPENDIX D: TEXTURES DEFINED FROM A BAND MAXIMIZE κ

In the main text, we claimed that the texture defined from the band Eq. (68) maximizes κ [Eq. (70)] across all possible projectors on a single component, thus defining the optimal single-component direction for that band. To prove this, consider the value of $\kappa_{\hat{m}}$ for some arbitrary

spinor direction $\hat{\mathbf{m}}(\mathbf{r})$, defined using the projector onto $\hat{\mathbf{m}}(\mathbf{r})$, defined as $P_{\hat{\mathbf{m}}}(\mathbf{r}) = [1 + \boldsymbol{\mu} \cdot \hat{\mathbf{m}}(\mathbf{r})/2]$. We have

$$\kappa_{\hat{\mathbf{m}}} = \int_{\text{u.c.}} d\mathbf{r} \sum_{\mathbf{k} \in \text{BZ}} u_{\mathbf{k},m}^\dagger(\mathbf{r}) P_{\hat{\mathbf{m}}}(\mathbf{r}) u_{\mathbf{k},m}(\mathbf{r}) \quad (\text{D1})$$

$$= \int_{\text{u.c.}} d\mathbf{r} \sum_{\mathbf{k} \in \text{BZ}} u_{\mathbf{k},m}^\dagger(\mathbf{r}) \frac{1 + \boldsymbol{\mu} \cdot \hat{\mathbf{m}}(\mathbf{r})}{2} u_{\mathbf{k},m}(\mathbf{r}) \quad (\text{D2})$$

$$= \int_{\text{u.c.}} d\mathbf{r} \frac{\rho(\mathbf{r}) + |\mathbf{A}(\mathbf{r})| [\hat{\mathbf{m}}(\mathbf{r}) \cdot \hat{\mathbf{n}}(\mathbf{r})]}{2} \quad (\text{D3})$$

$$\leq \int_{\text{u.c.}} d\mathbf{r} \frac{\rho(\mathbf{r}) + |\mathbf{A}(\mathbf{r})|}{2} = \kappa. \quad (\text{D4})$$

APPENDIX E: EFFECTIVE HAMILTONIAN FOR κ NEAR 1

We consider the Bloch Hamiltonian $H_{\mathbf{k}} = e^{-i\mathbf{k} \cdot \mathbf{r}} H e^{i\mathbf{k} \cdot \mathbf{r}}$ for a two-component system. H is the Hamiltonian, given for instance by Eq. (18) for a single valley in the case of TMDs. The periodic Bloch function solves

$$H_{\mathbf{k}} u_{\mathbf{k},b,\lambda_b}(\mathbf{r}) = \epsilon_{\mathbf{k},b,\lambda_b} u_{\mathbf{k},b,\lambda_b}(\mathbf{r}) \quad (\text{E1})$$

with the energy eigenvalue $\epsilon_{\mathbf{k},b,\lambda_b}$. We insert the decomposition Eq. (72) onto the two spinors $\chi_{\hat{\mathbf{n}}}$ and $\chi_{\hat{\mathbf{n}}}^\perp$ in Eq. (E1). Using their orthogonality, $\chi_{\hat{\mathbf{n}}}^\dagger(\mathbf{r}) \chi_{\hat{\mathbf{n}}}^\perp(\mathbf{r}) = 0$, we multiply on the left Eq. (E1) by $\chi_{\hat{\mathbf{n}}}(\mathbf{r})$ and $\chi_{\hat{\mathbf{n}}}^\perp(\mathbf{r})$ to arrive at a system of two coupled equations:

$$\begin{pmatrix} \chi_{\hat{\mathbf{n}}}^\dagger(\mathbf{r}) H_{\mathbf{k}} \chi_{\hat{\mathbf{n}}}(\mathbf{r}) & \chi_{\hat{\mathbf{n}}}^\dagger(\mathbf{r}) H_{\mathbf{k}} \chi_{\hat{\mathbf{n}}}^\perp(\mathbf{r}) \\ [\chi_{\hat{\mathbf{n}}}^\perp(\mathbf{r})]^\dagger H_{\mathbf{k}} \chi_{\hat{\mathbf{n}}}(\mathbf{r}) & [\chi_{\hat{\mathbf{n}}}^\perp(\mathbf{r})]^\dagger H_{\mathbf{k}} \chi_{\hat{\mathbf{n}}}^\perp(\mathbf{r}) \end{pmatrix} \begin{pmatrix} \psi_{\mathbf{k}}(\mathbf{r}) \\ \frac{\beta_{\mathbf{k}}}{\alpha_{\mathbf{k}}} \varphi_{\mathbf{k}}(\mathbf{r}) \end{pmatrix} = \epsilon_{\mathbf{k},b,\lambda_b} \begin{pmatrix} \psi_{\mathbf{k}}(\mathbf{r}) \\ \frac{\beta_{\mathbf{k}}}{\alpha_{\mathbf{k}}} \varphi_{\mathbf{k}}(\mathbf{r}) \end{pmatrix}. \quad (\text{E2})$$

The band is nearly aligned for κ close to one. In that case, to leading order in $|\beta_{\mathbf{k}}|/|\alpha_{\mathbf{k}}| \ll 1$, the second term is negligible and we obtain an approximate equation for $\psi_{\mathbf{k}}$,

$$H_{\text{eff}} \psi_{\mathbf{k}}(\mathbf{r}) \approx \epsilon_{\mathbf{k},b,\lambda_b} \psi_{\mathbf{k}}(\mathbf{r}), \quad (\text{E3})$$

corresponding to a *single-component* effective Hamiltonian $H_{\text{eff}} = \chi_{\hat{\mathbf{n}}}^\dagger(\mathbf{r}) H_{\mathbf{k}} \chi_{\hat{\mathbf{n}}}(\mathbf{r})$, as given in Eq. (73) of the main text.

These manipulations clarify the emergence of the fictitious magnetic field encoded in the texture of Eq. (14), as the spatial derivative $-i\nabla_{\mathbf{r}}$ in the Bloch Hamiltonian $H_{\mathbf{k}}$ generates the Berry connection of the texture magnetic field; see Eq. (74).

APPENDIX F: ENSEMBLE-AVERAGED TOPOLOGICAL INDEX

We detail here the introduction of a robust topological index accounting for the texture of moiré bands. We start with a Bloch spinor wave function $u_{\mathbf{k},n,\lambda}(\mathbf{r})$ with two components ($N = 2$) and define the 2×2 real-space Hamiltonian:

$$H'(\mathbf{r}) = \sum_{\lambda,n,\mathbf{k} \in \text{BZ}} u_{\mathbf{k},n,\lambda}(\mathbf{r}) u_{\mathbf{k},n,\lambda}^\dagger(\mathbf{r}) p_{\mathbf{k},n,\lambda}. \quad (\text{F1})$$

By construction, $H'(\mathbf{r})$ is Hermitian and can be expanded in terms of the three Pauli matrices and the identity matrix μ_0 . This yields

$$H'(\mathbf{r}) = K(\mathbf{r}) \sigma_0 + \frac{1}{2} H(\mathbf{r}), \quad (\text{F2})$$

with $K(\mathbf{r}) = \frac{1}{2} \sum_{\lambda,n,\mathbf{k} \in \text{BZ}} u_{\mathbf{k},n,\lambda}^\dagger(\mathbf{r}) u_{\mathbf{k},n,\lambda}(\mathbf{r}) p_{\mathbf{k},n,\lambda}$ as a global shift and the matrix $H(\mathbf{r})$ being equal to the Hamiltonian in Eq. (38). The two matrices H and H' share the same eigenvectors. We denote the eigenvector corresponding to the highest eigenvalue $\lambda_{\text{max}}(\mathbf{r})$ as $\psi_E(\mathbf{r})$, which we take to be normalized, $\psi_E^\dagger(\mathbf{r}) \psi_E(\mathbf{r}) = 1$.

In the extreme limit that the sum in Eq. (F1) includes only a single momentum $\mathbf{k}_0$, the maximal eigenvalue satisfies $\lambda_{\text{max}}(\mathbf{r}) = 1$ for all $\mathbf{r}$, and the corresponding eigenvector is simply $\psi_E(\mathbf{r}) \propto u_{\mathbf{k}_0}(\mathbf{r})$. Moreover, the second eigenvalue of H' is identically zero. In the general case, $\lambda_{\text{max}}(\mathbf{r})$ deviates from one and the second eigenvalue from zero. Both eigenvalues form bandlike functions of $\mathbf{r}$, which are periodic over the moiré lattice. Assuming the gap remains open, recover $C[\hat{\mathbf{n}}]$ given in Sec. VA as the Chern number of the top band of this Hamiltonian.

The definition of the moiré-periodic Hamiltonian Eq. (F1) remains the same with N components, in which case $H'(\mathbf{r})$ is an $N \times N$ matrix. When the sum is restricted to a single momentum $\mathbf{k}_0$, the spectrum of H' is simple: $u_{\mathbf{k}_0}(\mathbf{r})$ is an eigenstate with eigenvalue 1, the $N - 1$ remaining eigenvalues are all vanishing in the subspace orthogonal to $u_{\mathbf{k}_0}(\mathbf{r})$. With the ensemble average beyond an individual state, the spectrum is modified and the bands become dispersive. Depending on the number of spectral gaps, there are at most N real-space Chern numbers whose sum is constrained to be zero.

Analyzing the properties of this Hamiltonian permits the isolation of a relevant low-energy subspace. For instance, in TBG, where there are $N = 4$ components, such an analysis would reveal the presence of a relevant two-dimensional subspace. Each dimension would correspond to the chiral limit textures on the A and B sublattices. Crucially, even without *a priori* knowledge of the chiral limit, such an analysis would allow the presence of fictitious magnetic fields in TBG to be revealed.

APPENDIX G: SYMMETRY INDICATORS

We now use symmetry indicators of band topology [84] to show that the properties in Eqs. (55)–(57) imply that $C[\hat{n}_E]$ is necessarily nonzero. This approach can be applied to the real-space Hamiltonian in Eq. (40). The Chern number of its upper band is equal to $C[\hat{n}_E]$. By symmetry indicators, this Chern number is given in terms of the C_{3z} eigenvalues of the positive-energy eigenstates at the high-symmetry stacking points AA , AB , and BA as

$$e^{i2\pi C[\hat{n}_E]/3} = \theta(AA)\theta(AB)\theta(BA), \quad (G1)$$

where $\theta(\mathbf{r})$ is the C_{3z} eigenvalue at point $\mathbf{r}$. For the AA stacking point, we have the trivial eigenvalue $\theta(AA) = 1$. For the AB and BA points, the eigenvalues depend on the orientation, i.e., the choice of the $\pm$ sign in Eq. (56). Using that $e^{i\mathbf{g}_5 \cdot \mathbf{r}_{BA}} = e^{i4\pi/3}$, we find for $\hat{n}_E(AB) = (0, 0, 1)$ that $\theta(AB) = 1, \theta(BA) = e^{i4\pi/3}$. On the other hand, for $\hat{n}_E(AB) = (0, 0, -1)$, we use that $e^{i\mathbf{g}_5 \cdot \mathbf{r}_{AB}} = e^{i2\pi/3}$, obtaining $\theta(AB) = e^{i2\pi/3}, \theta(BA) = 1$. In both cases, we thus obtain that the real-space Chern number

$$C[\hat{n}_E] = \pm 1 \pmod{3}. \quad (G2)$$

APPENDIX H: TOPOLOGICAL HEAVY FERMION MODEL

The Bloch Hamiltonian reproducing the two central bands is given by the 6×6 matrix, in the basis of the six fermion species $(c_1, c_2, c_3, c_4, f_1, f_2)$,

$$H_{\mathbf{p}} = \begin{pmatrix} 0 & v_1(p_x + i\sigma_z p_y) & f(\mathbf{p}) \\ v_1(p_x - i\sigma_z p_y) & M\sigma_z & 0 \\ f(\mathbf{p}) & 0 & 0 \end{pmatrix}, \quad (H1)$$

where $\mathbf{p}$ is relative to the Γ point. Far from Γ , the coupling matrix,

$$f(\mathbf{p}) = [\gamma + v_2(p_x \sigma_x + p_y \sigma_y)]e^{-(p\lambda)^2/2}, \quad (H2)$$

between the Wannier orbitals f_1, f_2 and the conduction electrons c_1, c_2 becomes exponentially small. The central bands thus match the Wannier representation and exhibit very weak energy dispersion. As one approaches the Γ point, the orbital content of the two central bands gradually shifts toward c_3, c_4 , with the reduced Hamiltonian at Γ ,

$$\begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}, \quad (H3)$$

having eigenenergies $\pm M$. The overlap between the central bands and the conduction electrons c_1, c_2 peaks at most at 5%.

To better clarify the structure of the six-band Hamiltonian Eq. (H1), we focus on the conduction electrons and write the projected Bloch Hamiltonian:

$$H_{\mathbf{p}}^c = \begin{pmatrix} 0 & v_1(p_x + i\sigma_z p_y) \\ v_1(p_x - i\sigma_z p_y) & M\sigma_z \end{pmatrix}. \quad (H4)$$

It is precisely the same as the K -valley effective Hamiltonian of (untwisted) Bernal-stacked bilayer graphene in the absence of trigonal warping. The model exhibits quadratic band touching at the Γ point, with a Berry phase winding of 2π around Γ . This nontrivial phase winding originates from the K -valley Dirac cones of the two individual monolayers. The two Dirac cones merge to form a quadratic band touching, and they are responsible for the topological obstruction [49].

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