

Generalized Bogoliubov-Born-Green-Kirkwood-Yvon Hierarchy for Near-Integrable Dynamics

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We study quantum and classical many-body Hamiltonian systems that combine integrable contact interactions with generic long-range two-body potentials. Starting from an ansatz for the state at time t , which we call the correlated fluid-cell ensemble, we show that the dynamics of local observables at macroscopic times and length scales can be cast into a generalized Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy formulated in terms of the quasiparticle densities of the underlying integrable model and their correlations. We derive this hierarchy and validate these predictions against microscopic molecular-dynamics simulations, finding perfect agreement. At late times, the one-particle distribution relaxes via a Boltzmann-type scattering integral encoding the interplay between integrable contact processes and long-range collisions, whereas higher-point correlations remain strongly nonthermal on thermalization timescales, indicative of a form of incomplete or generalized thermalization. Focusing on long-range dipolar quantum gases, where the relevant matrix elements can be obtained explicitly, we show that our collision integral reduces exactly to the Fermi golden rule result and provide a complete theoretical account of the experimental observations of Tang *et al.* [Phys. Rev. X **8**, 021030 (2018)]. More broadly, our framework extends the BBGKY program to regimes with strong local interactions and applies to a wide class of experimentally relevant systems, from one-dimensional dipolar cold-atom gases to Lennard-Jones molecular fluids.

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I. INTRODUCTION

The study of nonequilibrium many-body physics continues to reveal ever more intricate phenomena arising from complex interactions. Classical theories of chaos, turbulence, and active dynamics remain vibrant research areas, and in recent years, quantum many-body systems have also entered the nonequilibrium arena [1–4]. In low spatial dimensions, integrable models—distinguished by their stable quasiparticles—form a key class of dynamical systems. These quasiparticles govern transport properties, entanglement spreading [5–7], anomalous scrambling [8,9], extended coherence times [10–12], and exotic hydrodynamic behavior [13–26]. Typical examples of interacting integrable systems include bosons, spins, or fermions with two-body contact interactions, such as the Lieb-Liniger gas [27,28] and the Fermi-Hubbard chain [29]. This naturally raises the question: How do quasiparticles and the overall dynamics evolve when the Hamiltonian includes additional, longer-range

interactions? Integrability breaking has attracted considerable interest [30–55], prompted by experiments [56–59] that report anomalously slow thermalization in quasi-one-dimensional settings, a challenge shared across molecular dynamics, plasma physics, and astrophysics [60–76].

Significant progress has been achieved for systems whose unperturbed integrable limit is free—where matrix elements of local operators are typically known, allowing a Boltzmann scattering approach via Fermi’s golden rule. However, for genuinely interacting integrable models, matrix elements depend nontrivially on both the state and the model, and existing methods accurately capture only late-time behavior, leaving short-time dynamics—often of primary experimental interest—largely unexplored.

Despite advances in specific contexts [47,48,77], a general framework is still lacking to compute the dynamics of any integrable model perturbed by weak two-body or higher-order potentials. Developing such a framework would be both theoretically valuable and directly relevant to cold-atom experiments [58], where the precise form of interparticle potentials can be uncertain.

In kinetic-theory texts, the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy is a classic tool [78–83], typically applied to free particles with weak or long-range interactions (as in plasma and astrophysical

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systems), truncating at two- or three-point correlations to describe thermalization and dissipation. Yet, to date, the BBGKY framework has not been implemented in cases where the unperturbed Hamiltonian is an interacting integrable model with well-defined quasiparticles.

Here, we derive a generalization of BBGKY hierarchy for the macroscopic time and scale evolution of systems where the unperturbed (classical or quantum) theory features arbitrarily strong integrable contact interactions. Specifically, we consider a hard-core repulsion $V_a^{\text{int}}(r)$ of radius a supplemented by a long-range field $\varphi(r)$, leading to the two-body potential (see Fig. 1)

$$U(r) = V_a^{\text{int}}(r) + V(r), \quad V(r) = \frac{V_0}{\xi} \varphi(r/\xi), \quad (1)$$

where $V_0 > 0$ (< 0) describes repulsive (attractive) long-range interactions acting over the scale ξ . Crucially, we assume a strict separation of length scales. We require the range of the perturbation ξ to be much larger than the typical microscopic correlation lengths characterizing the state of the unperturbed integrable model.

Notable examples include atomic dipolar systems [23,58,84–93], whose Hamiltonian in terms of the bosonic field $\hat{\psi}(x)$ can be written as

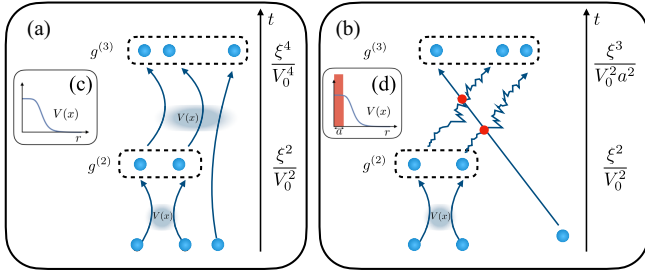


FIG. 1. Schematic evolution of (a) BBGKY and (b) gBBGKY predictions for the two point functions $g^{(2)}$; the two insets show, respectively, (c) the long-range potential for free particles and (d) the combination of long- and short-range interactions in the interacting integrable case. In (a), two particles drawn from the momentum distribution $\rho(\theta)$ scatter via $V(r)$ and become correlated on the timescale ξ^2/V_0^2 . This is the “prethermalization” stage, where correlations build up but no extensive mixing occurs; they then scatter again and generate three-point correlations $g^{(3)}$, which control the “thermalization” stage on timescales $(\xi/V_0)^4$. In (b), with integrable local interactions in homogeneous systems, the two-point correlation is generated by the large-scale interaction $V(r)$, while the three-point function is produced by a different and faster mechanism. Particles propagate and scatter through local interactions, shown as red circles, providing an effective diffusive bath for the two-point functions and leading to the thermalization of one- and two-point functions, namely, generalized thermalization, on timescales $\xi^3/(V_0 a)^2$.

$$H = \int dx \left(\frac{1}{2m} \partial_x \hat{\psi}^\dagger(x) \partial_x \hat{\psi}(x) + a(\hat{\psi}^\dagger(x))^2 (\hat{\psi}(x))^2 - \mu \hat{\psi}^\dagger(x) \hat{\psi}(x) \right) + \int dx dx' V(x-x') \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\psi}^\dagger(x') \hat{\psi}(x'), \quad (2)$$

in which a local contact interaction is again supplemented by a long-range component. Our approach provides a fully predictive theory for a wide class of near-integrable dynamics in both classical and quantum regimes, encompassing bosonic, fermionic, and spin-chain models. In particular, it unifies the description of prethermalization and full thermalization dynamics.

II. BACKGROUND AND SUMMARY OF THE RESULTS

In this work, we focus on the broad class of integrable systems that are characterized by stable quasiparticle excitations. Similar to free systems, these excitations provide a complete description of local properties. In particular, a generic thermodynamic state is specified by the “quasiparticle momentum distribution,” or one-particle (quasi)momenta distribution, denoted by $\rho_\theta(x)$, where we use θ as a generic parametrization of the quasiparticle momenta $k(\theta)$. Under the unperturbed time evolution, the spatial integral of $\rho_\theta(x)$ is conserved.

Such systems arise in both classical and quantum physics [20,94–103], typically in 1 + 1 dimensions, where, in certain cases, scattering processes factorize. Generic perturbations, such as couplings between different one-dimensional channels or the addition of long-range interactions, break integrability. In this situation, quasiparticles become unstable, and most distributions $\rho_\theta(x)$ are no longer stationary. Under weak integrability breaking, typically only thermal distributions remain stationary, as first recognized by Boltzmann.

The evolution from a generic out-of-equilibrium quasiparticle distribution toward a thermal state is the subject of prethermalization and kinetic theory. These frameworks have applications ranging from high-energy collisions and plasma physics to, more recently, quantum many-body dynamics in ultracold-atomic systems [56,58,104–109]. A common approach models this evolution through a collision integral derived from Fermi’s golden rule [30,47,48,110]; the perturbation induces transitions between quasiparticle states with rates determined by the corresponding matrix elements or by classical cross sections. This approximation, however, neglects correlations that build up dynamically, and the computation of the relevant matrix elements is generally difficult. The BBGKY hierarchy provides a more comprehensive framework, as it allows one to track correlations up to arbitrary order. It also enables a systematic derivation of the

collision integral without requiring explicit knowledge of all transition amplitudes, and it goes beyond the late-time approximation inherent in the Boltzmann approach [111,112].

Concretely, we consider a model of particles in the continuum; the extension to lattice systems is immediate, with Hamiltonian, quantum, or classical,

$$\hat{H} = \hat{H}_a^{\text{int}} + \frac{1}{2} \int dx dx' V(x-x') \hat{q}_0(x) \hat{q}_0(x'), \quad (3)$$

where $\hat{q}_0(x)$ is the local particle density operator, and $\hat{H}_a^{\text{int}}$ is an integrable Hamiltonian containing standard kinetic terms together with an integrable interaction $\hat{V}_a^{\text{int}}$. For example, in the Lieb-Liniger model, as in Eq. (2), the integrable interaction is $V^{\text{int}}(x-x') = c\delta(x-x')$, with $c = 0$ and $c \rightarrow \infty$ corresponding to free and hard-core bosons, effectively free fermions, respectively. We denote by a the strength of the integrable interaction. In the limit $a \rightarrow 0$, the integrable potential vanishes and the model reduces to free particle propagation,

$$\lim_{a \rightarrow 0} \hat{H}_a^{\text{int}} = \hat{H}^{\text{free}} = \int dx \frac{[\hat{q}_1(x)]^2}{2m}, \quad (4)$$

with $\hat{q}_1(x)$ the local momentum density. The integrable Hamiltonian $\hat{H}_a^{\text{int}}$ possesses an extensive family of conserved charges $\hat{q}_i(x)$, with $i \in \mathbb{N}$. The generalized BBGKY (gBBGKY) theory describes the time evolution of the expectation values of these densities and of their connected correlations.

To describe the system at finite macroscopic time t , we introduce the following ansatz. The state is given by the “correlated fluid-cell ensemble,” which can be formulated either as a density matrix or as a phase-space density; for a precise definition, see Eq. (14). This state can be viewed as an embedding of a generalized Gibbs ensemble (GGE), $\hat{\rho} \propto \exp(-\sum_i \int dx \beta^i(x) \hat{q}_i(x))$, determined by a set of generalized temperatures $\beta^i(x)$ associated with the local conserved quantities $\hat{q}_i(x)$. The correlated fluid-cell ensemble additionally involves higher-rank Lagrange parameters $\beta_{(n)}^{i_1, \dots, i_n}(x_1, \dots, x_n)$, which encode information about long-range correlations in the system. When these higher-order terms are nonzero, the state is not simply a product of independent local stationary GGE cells. It is instead a collection of correlated fluid cells as illustrated in Fig. 2.

The reason to expect this form is as follows. The hydrodynamic postulates assert that, for macroscopic times $t \gg \tau_{\text{mic}}$, much larger than the microscopic time τ_{mic} , a generic interacting system relaxes locally, giving rise to quasiparticle description. Fast degrees of freedom associated with nonconserved modes decay, and the evolution becomes confined to a reduced manifold of states. Since the slow modes are precisely the conserved densities and their products, we include them here in full generality.

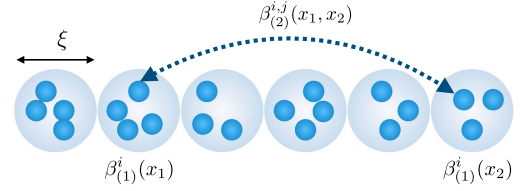


FIG. 2. Schematic representation of the correlated fluid-cell ensemble. Fluid cells of size ξ can be correlated over long distances by the external potential V or by the interplay of contact interactions and ballistic dynamics in the system. The corresponding density matrix is introduced in Eq. (14).

Crucially, the correlated fluid-cell ensemble correctly describes the out-of-equilibrium state of pure integrable systems when observed on macroscopic scales, i.e., much larger than the homogeneous GGE connected correlation functions typical decay length ξ_0 [113].

Our framework fundamentally relies on the assumption that the integrability-breaking potential $V(x-x')$ acts strictly on macroscopic scales $\xi \gg \xi_0$. Because of this scale separation, the integrability-breaking potential does not corrupt the microscopic local structure of the integrable state. Instead, its effect is purely hydrodynamic: It smoothly modifies the local inverse temperatures $\beta^i(x)$ and the higher-rank parameters $\beta_{(n)}^{i_1, \dots, i_n}(x_1, \dots, x_n)$ over time. This clear separation of scales is precisely what justifies our core assumption: Under the condition $\xi \gg \xi_0$, the macroscopic state of the system remains validly described by the correlated fluid-cell ensemble throughout its time evolution. A similar logic can be applied to standard BBGKY hierarchy. The only difference lies in the fact that $\tau_{\text{mic}} \rightarrow 0$ as there is no transient phase needed for local relaxation to GGE and emergence of quasiparticles. Thus, in the usual setting, where one starts from states without correlations, there is little practical need to make the structure of the correlated ensemble explicit. Our formulation therefore generalizes BBGKY hierarchy to situations in which the stationary states of the unperturbed Hamiltonian already displays nontrivial local correlations (in contrast to H^{free} , where local correlations are absent and the hierarchy simplifies substantially). We will thus refer to the conventional hierarchy as the “standard BBGKY” and to the present extension as the “generalized BBGKY” (or gBBGKY). However, it is crucial to emphasize a fundamental conceptual distinction between the two frameworks: While the standard BBGKY hierarchy is an exact microscopic identity valid across all length and timescales prior to any truncation, the gBBGKY framework is inherently a macroscopic effective theory that fundamentally relies on coarse graining and the assumption of local generalized equilibrium at large scales.

In principle, the full hierarchy of multipoint correlations encodes complete macroscopic information about the state, but, to obtain a tractable description, it must be truncated at

some finite order. Truncating the hierarchy at the level of three-point functions yields the explicit evolution equations in Eqs. (32)–(34). Equivalently, these equations can be rewritten in terms of quasiparticle densities $\rho_\theta(x, t)$ and their correlations by the replacement $i \rightarrow \theta$ and by identifying the local densities, or operator or phase-space densities, as $\langle \hat{q}_i(x, t) \rangle \rightarrow \rho_\theta(x, t)$, which also recovers standard textbook results in the limit $a \rightarrow 0$.

In this work, we show that the presence of local interactions strongly affects the hierarchy. The main new feature is the emergence of δ -function jumps in the evolution equations for multipoint correlations, which appear as discontinuities in spatial derivatives at coinciding points. These jumps encode ballistic, nonlinear, namely, interacting, modes, and they persist even when integrability is weakly broken, since ballistic transport remains relevant at intermediate times, thereby driving the evolution of lower-order correlations. This has important physical consequences, see Fig. 1. For $a = 0$, correlations are generated solely by scattering mediated by the perturbing potential $V(x)$. In one dimension, two-body scattering alone cannot thermalize the system, so three-point correlations must be retained, leading to thermalization times scaling as V_0^{-4} , a phenomenon known as “kinetic blocking.” For $a \neq 0$, local integrable interactions allow quasiparticles to scatter via V and then interact again locally with a ballistically propagating particle. These processes generate three-point functions with coincident-point jumps, which, in turn, induce dissipative, diffusive contributions to the dynamics of the two-point correlations and lead to effective thermalization. As a result, thermalization of one- and two-point functions, which we refer to as incomplete or “generalized thermalization,” is reached at times of order $(aV_0)^{-2}$. Higher-point functions, such as three-point functions, remain far from thermal on these timescales, preserving remarkable long-range correlations, as a direct consequence of the interplay between local contact interactions and long-range interactions. Recent preliminary experiments [58] have observed this scaling, and our gBBGKY framework places it on firmer theoretical ground. Moreover, we show that truncation at third order reproduces the dynamics of one- and two-point functions very precisely and at all times, as we demonstrate by comparison with exact numerical simulations. Moreover, we show that truncation at third order reproduces the dynamics of one- and two-point functions throughout the dynamically relevant relaxation process, in excellent agreement with exact numerical simulations.

Before proceeding to the detailed derivation, we explicitly outline the physical boundaries, required inputs, and inherent limitations of the gBBGKY framework:

- (i) Model constraints: The practical application of this framework restricts the class of unperturbed Hamiltonians. The integrable model must admit a well-defined macroscopic description in terms of

generalized hydrodynamics. This requires a known spectrum of stable quasiparticle excitations, established microscopic scattering shifts, and a valid thermodynamic Bethe ansatz to rigorously define the local root densities.

- (ii) Macroscopic effective theory (length scales): The gBBGKY approach is a macroscopic effective theory built upon the correlated fluid-cell ensemble. It relies on a strict scale separation, $\xi \gg \xi_0$, where ξ is the range of the integrability-breaking perturbation and ξ_0 represents the typical microscopic correlation length of the unperturbed model. Consequently, our equations do not resolve multipoint correlations at spatial separations comparable to or smaller than the fluid-cell size.
- (iii) Initial conditions and quenches (timescales): Because the framework fundamentally assumes local equilibrium, it cannot resolve the highly complex, microscopic dephasing transient immediately following a sudden integrable quantum quench, when the system is not in GGE. The macroscopic gBBGKY equations apply only after an initial local relaxation time (τ_{mic}). Furthermore, the framework requires the postquench quasiparticle rapidity distribution to be known and provided as an external input to initialize the time evolution.

After the derivation of the gBBGKY hierarchy, we illustrate its predictive power by studying the nonequilibrium dynamics in the system of hard rods and in dipolar quantum gases. The theoretical predictions will be benchmarked against the numerical simulations of the former and experimental thermalization rates measured in the latter.

A. Organization of the paper

The paper is organized as follows. In Sec. III, we derive the gBBGKY hierarchy, an infinite set of coupled equations for correlation functions of charges and currents. We then show how to truncate this hierarchy consistently, obtaining a closed set of equations for connected charge correlators. This requires neglecting correlations beyond a given order and assuming a separation of scales between local integrable interactions and the long-range integrability-breaking potential. We also show that the generalized hierarchy reduces to the standard one when the integrable interactions vanish. The resulting equations are later solved numerically and validated against numerical simulations.

To clarify the mechanisms behind generalized thermalization, in Sec. IV we derive a kinetic equation, dubbed the generalized Landau equation, from which the thermalization timescale quoted above follows. This derivation assumes a separation of timescales between the fast dynamics of correlations and the slow evolution of the one-body function.

In the next two sections, we apply the gBBGKY theory and generalized Landau equations to two systems. First, in Sec. V, the predictions of the preceding sections are tested

against exact numerical simulations for classical hard rods, hard spheres, with long-range interactions, showing perfect agreement. Second, in Sec. VI, we apply the framework to dipolar cold-atomic gases trapped in an array of quasi-one-dimensional tubes. This requires an extension of the methods to multicomponent systems. It also provides an opportunity to compare the generalized Landau equation, derived here under explicit assumptions, with a heuristic approach where the scattering integral is obtained from Fermi's golden rule. We find exact agreement between the two approaches in the regime where both apply.

In the final section, Sec. VII, we summarize our findings and outline future directions. More technical aspects are relegated to the Appendixes. In the accompanying Supplemental Material [114], we provide derivations of the generalized Landau equation and of the gBBGKY hierarchy for multicomponent systems.

III. THE gBBGKY HIERARCHY

We start by introducing the notation needed to set up the generalized BBGKY hierarchy. We first work with densities of conserved quantities and later move to particles or quasiparticle occupation functions, as is customary in standard formulations of BBGKY hierarchy.

We denote by $\hat{q}_i(x)$ the density of the conserved charge $\hat{Q}_i$, which commutes with the integrable Hamiltonian, $[\hat{H}_a^{\text{int}}, \hat{Q}_i] = 0$, noting that a suitable gauge must be chosen since these densities are defined up to a total derivative. The first standard conserved quantities are particle number $\hat{Q}_0 = \hat{N}$, momentum $\hat{Q}_1 = \hat{P}$, and unperturbed energy

$\hat{Q}_2 = \hat{H}_a^{\text{int}}$, to which we refer as the kinetic energy of quasiparticles. Throughout the derivation, we focus on the quantum formulation, yet all steps can be translated straightforwardly to classical models by replacing the commutator with the Poisson bracket, $i[\cdot, \cdot] \rightarrow \{\cdot, \cdot\}$. We will often refer to the case of free classical particles, where the equations are simplest and essentially all steps are exact, and we also assume Galilean invariance, although violations can be incorporated without difficulty. We set $\hbar = 1$ and start from the Heisenberg equation for the charge densities,

$$\partial_t \hat{q}_i = i[\hat{H}, \hat{q}_i]. \quad (5)$$

We decompose the commutator and introduce currents $\hat{j}_i$ and generalized currents $\hat{j}_{i,k}$, defined by

$$i[\hat{H}_a^{\text{int}}, \hat{q}_i] + \partial_x \hat{j}_i = 0, \quad i[\hat{Q}_i, \hat{q}_k] + \partial_x \hat{j}_{i,k} = 0, \quad (6)$$

where the density does not generate any dynamics, $\hat{j}_{0,i} = 0$. To simplify the Heisenberg equation, we use locality of the charges,

$$i[\hat{q}_i(x), \hat{q}_0(x')] = \hat{j}_{i,0}(x) \delta'(x - x'). \quad (7)$$

Using this relation for commutators between local charges, the quantum generalization of the classical Liouville equation reads [115,116]

$$\partial_t \hat{q}_{i_1}(x_1) + \partial_{x_1} \hat{j}_{i_1}(x_1) = -\frac{1}{2} \int dx_2 V'(x_1 - x_2) \left[\hat{q}_0(x_2) \hat{j}_{i_1,0}(x_1) + \hat{j}_{i_1,0}(x_1) \hat{q}_0(x_2) \right]. \quad (8)$$

Similar equations to Eq. (8) can be written for multipoint operators involving arbitrary products of charge densities. In what follows we drop $\hat{\cdot}$ to lighten the notation. Taking expectation values then generates the hierarchy of dynamical equations

$$\begin{aligned} \partial_t \langle q_{i_1}(x_1) \dots q_{i_n}(x_n) \rangle &= - \sum_{k=1}^n \partial_{x_k} \langle q_{i_1}(x_1) \dots j_{i_k}(x_k) \dots q_{i_n}(x_n) \rangle \\ &\quad - \sum_{k=1}^n \int dy V'(x_k - y) \langle q_{i_1}(x_1) \dots q_0(y) j_{i_k,0}(x_k) \dots q_{i_n}(x_n) \rangle. \end{aligned} \quad (9)$$

To isolate the relevant parts of correlators, we move to connected correlation functions. For $n = 2$, we set $\langle o_1 o_2 \rangle^c := \langle o_1 o_2 \rangle - \langle o_1 \rangle \langle o_2 \rangle$, whereas for $n = 3$ we define

$$\langle o_1 o_2 o_3 \rangle^c := \langle o_1 o_2 o_3 \rangle - \langle o_1 o_2 \rangle^c \langle o_3 \rangle - \langle o_1 o_3 \rangle^c \langle o_2 \rangle - \langle o_2 o_3 \rangle^c \langle o_1 \rangle - \langle o_1 \rangle \langle o_2 \rangle \langle o_3 \rangle, \quad (10)$$

and for $n > 3$ we proceed analogously. The first three layers of the hierarchy can then be written as

$$\partial_t \langle q_{i_1}(x_1) \rangle + \partial_{x_1} \langle j_{i_1}(x_1) \rangle = - \int dx_2 V'(x_1 - x_2) \langle j_{i_1,0}(x_1) \rangle \langle q_0(x_2) \rangle - \int dx_2 V'(x_1 - x_2) \langle j_{i_1,0}(x_1) q_0(x_2) \rangle^c, \quad (11)$$

$$\begin{aligned} & \partial_t \langle q_{i_1}(x_1) q_{i_2}(x_2) \rangle^c + [\partial_{x_1} \langle j_{i_1}(x_1) q_{i_2}(x_2) \rangle^c]_{(1,2)} \\ &= - \left[\int dx_3 V'(x_1 - x_3) (\langle j_{i_1,0}(x_1) \rangle \langle q_{i_2}(x_2) q_0(x_3) \rangle^c + \langle j_{i_1,0}(x_1) q_{i_2}(x_2) \rangle^c \langle q_0(x_3) \rangle + \langle j_{i_1,0}(x_1) q_{i_2}(x_2) q_0(x_3) \rangle^c) \right]_{(1,2)}, \end{aligned} \quad (12)$$

$$\begin{aligned} & \partial_t \langle q_{i_1}(x_1) q_{i_2}(x_2) q_{i_3}(x_3) \rangle^c + [\partial_{x_1} \langle j_{i_1}(x_1) q_{i_2}(x_2) q_{i_3}(x_3) \rangle^c]_{(1,2,3)} \\ &= - \left[\int dx_4 V'(x_1 - x_4) (\langle j_{i_1,0}(x_1) \rangle \langle q_{i_2}(x_2) q_{i_3}(x_3) q_0(x_4) \rangle^c + \langle j_{i_1,0}(x_1) q_{i_2}(x_2) q_{i_3}(x_3) \rangle^c \langle q_0(x_4) \rangle \right. \\ & \quad \left. + \langle j_{i_1,0}(x_1) q_{i_2}(x_2) \rangle^c \langle q_{i_3}(x_3) q_0(x_4) \rangle^c + \langle j_{i_1,0}(x_1) q_{i_3}(x_3) \rangle^c \langle q_{i_2}(x_2) q_0(x_4) \rangle^c + \langle j_{i_1,0}(x_1) q_{i_2}(x_2) q_{i_3}(x_3) q_0(x_4) \rangle^c) \right]_{(1,2,3)}. \end{aligned} \quad (13)$$

Here the symbol $[\dots]_{(i_1, \dots, i_n)}$ denotes the sum of the expression inside the square brackets over all cyclic permutations of the indices $(i_1, \dots, i_n)$.

When the integrable model reduces to a free theory, this hierarchy is equivalent to the standard BBGKY hierarchy, as discussed later in Sec. III E. A key difference is that the present formulation is expressed in terms of correlations of conserved charges and currents, whereas the standard BBGKY hierarchy is usually written directly for quasiparticle distributions. Nevertheless, the familiar structures are readily recognized. Truncating the hierarchy at the level of expectation values, namely, setting the two-point functions to zero, yields the Vlasov equation [117–119]. Retaining two-body correlations instead leads, at large times, to either the Landau or the Boltzmann equation, depending on the truncation procedure. Here, we keep three layers of the hierarchy because, due to kinetic blocking, the first two layers are not sufficient to go beyond the mean-field regime [68,69]. Finally, as in the standard BBGKY case [78], the hierarchy can be represented diagrammatically. In Fig. 3, we show the diagrams corresponding to the first three layers written above.

The hierarchy of Eqs. (11)–(13) is not closed. First, as in the standard BBGKY hierarchy, the equation for an n -point function involves the $(n+1)$ -point function. Closing the system therefore requires assuming that connected correlation functions beyond a given order are negligible. Second, the hierarchy contains correlators involving currents, of the form $\langle \dots q_{i_k} j_{i_k} q_{i_{k'}} \dots \rangle^c$ and $\langle \dots q_{i_k} q_0 j_{i_k,0} q_{i_{k'}} \dots \rangle^c$. To obtain a consistent truncation scheme, these correlators must be expressed in terms of correlators involving charge densities only. In a hierarchy built on a noninteracting theory, current operators are linear functionals of the charge densities, so this step is straightforward. In the present setting, it requires identifying the relevant structure of correlations in integrable theories.

To achieve this, two ingredients are needed: the ensemble with respect to which expectation values are computed and the associated fluid-cell averaging procedure. These steps are discussed in full generality below and later specialized to homogeneous systems.

Before proceeding, we summarize the assumptions made throughout the manuscript:

- (i) At large space-time distances, the connected correlators scale as $\langle q_{i_1}(x_1) \dots q_{i_n}(x_n) \rangle^c \sim 1/\xi^{n-1}$, where ξ is the characteristic range of the long-range potential and is much larger than the range a of the integrable interaction. This is discussed in Sec. III A.

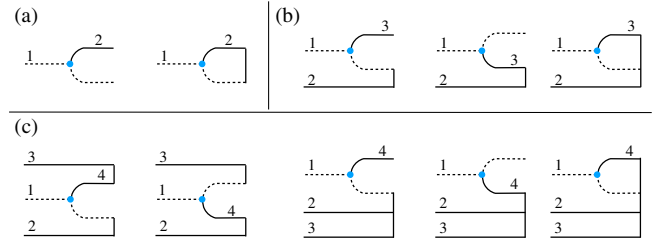


FIG. 3. Diagrams (a)–(c) representing the right-hand sides of the hierarchy equations, Eqs. (11)–(13). Dashed lines represent the current operator $j_{i_1,0}(x_1)$, while solid lines represent charge operators $q_{i_k}(x_k)$, with k indicated by the number above the line. Vertical lines represent connected correlations between operators. Each diagram contains a single vertex associated with the potential $V(x - x')$, which couples a current to an additional operator q_0 . To translate a diagram into an expression, one takes products of connected correlation functions according to the vertical lines, multiplies by $V(x - x_{m+1})$, where $m+1$ labels the extra charge operator, and integrates over x_{m+1} . To draw diagrams at the m th level, one considers one dashed line from which an extra solid line emanates, together with $m-1$ additional solid lines, and then enumerates all possible connections, or partitions, between the $m+1$ lines. The case with no connections is excluded, as it corresponds to the disconnected part of the correlator.

- (ii) The state of the system is described by the correlated fluid-cell ensemble Eq. (14). It encodes the expectation values of local charge densities and the correlations between them, as discussed in Sec. III A. Using this ensemble, one also obtains equations for correlation functions involving currents, see Sec. III B.
- (iii) The resulting partial differential equations, obtained by imposing the previous points in Eqs. (11)–(13), are expanded at large ξ , up to and including order ξ^{-3} , yielding the final hierarchy reported in Sec. III C.

A. The correlated fluid-cell ensemble

To truncate the hierarchy and obtain a closed description of the system at time t , correlations must be retained only up to a chosen maximal order. This mirrors standard hydrodynamics, where a mesoscopic length ℓ separates microscopic fluctuations from macroscopic, observable fields. In the homogeneous setting considered here, the natural choice for this scale is the interaction length ξ of the potential. Although additional length scales may emerge in inhomogeneous systems, we assume that, in the present case, ξ is the only relevant spatial scale.

We postulate that, at macroscopic times, the system relaxes to the state described by the correlated fluid-cell density matrix

$$\begin{aligned} \rho = \exp & \left(- \int dx_1 \beta_{(1)}^i(x_1) q_i(x_1) \right. \\ & - \int_{x_1 \neq x_2} d\vec{x} \beta_{(2)}^{i,j}(x_1, x_2) \delta q_i(x_1) \delta q_j(x_2) \\ & - \int_{x_1 \neq x_2 \neq x_3 \neq x_1} d\vec{x} \beta_{(3)}^{i,j,k}(x_1, x_2, x_3) \delta q_i(x_1) \delta q_j(x_2) \delta q_k(x_3) \\ & \left. + \dots \right). \end{aligned} \quad (14)$$

In the following, the ensemble average $\langle \dots \rangle$ is always taken with respect to density matrix Eq. (14). The first term corresponds to a family of local GGE states, one for each uncorrelated fluid cell at position x . Accordingly, we denote by $\langle \dots \rangle_1$ the average with respect to the zeroth order of the correlated fluid-cell ensemble, namely, the local GGE average given by Eq. (14) with $\beta_{(n>1)} = 0$. In Eq. (14), we also introduce fluctuations around the local GGE, $\delta q_i(x) = q_i(x) - \langle q_i(x) \rangle_1$.

For the states defined by (14), the entropy is maximized independently in each cell, with additional constraints imposed by conservation laws. Setting all higher terms to zero amounts to neglecting nonlocal constraints, which implies that only short-range, thermal-like correlations are retained, decaying on distances much smaller than the scale ξ . As in the standard BBGKY hierarchy, the goal is instead

to keep higher-order correlations in the description of the state. The state Eq. (14) therefore maximizes the entropy under constraints given by not only fixed one-point functions, but also by correlations of arbitrary order. As we will show in this section, the nonlocal generalized temperatures $\beta_{(n)}^{i_1, \dots, i_n}(x_1, \dots, x_n)$ are directly related to the connected n -point functions $\langle q_{i_1}(x_1) \dots q_{i_n}(x_n) \rangle^c$, and, in particular, they display the same scaling with the length scale ξ .

For generic states, computing expectation values with respect to ensemble Eq. (14) is a difficult task, and, in close analogy with (g)BBGKY approach, it calls for a perturbative scheme. Here, a natural truncation arises from the assumed separation of length scales. A key hypothesis for hydrodynamic states is that connected correlation functions satisfy a large-deviation scaling

$$\langle q_{i_1}(x_1) \dots q_{i_n}(x_n) \rangle^c \sim \frac{1}{\xi^{n-1}}, \quad (15)$$

which also implies $\beta_{(n)}^{i_1, \dots, i_n}(x_1, \dots, x_n) \sim \xi^{-n+1}$. Equivalently, as the order n increases, the magnitude of the connected correlator decreases by a factor $1/\xi$ for each additional field. This ensures that higher-order correlations become progressively smaller and can be neglected beyond a finite truncation order. This scaling is central for defining a controlled truncation scheme, both for the gBBGKY hierarchy and for the correlated fluid-cell density matrix. Under the scaling Eq. (15), the density matrix Eq. (14) can be evaluated perturbatively in the parameter ξ^{-1} . As an example, we compute the two-point function using the state Eq. (14), expanded to first order in ξ^{-1} ,

$$\begin{aligned} \langle q_{i_1}(x_1) q_{i_2}(x_2) \rangle^c &= \langle q_{i_1}(x_1) q_{i_2}(x_2) \rangle_1 \\ &+ \int_{x_3 \neq x_4} dx_3 dx_4 \beta_{(2)}^{i_3, i_4}(x_3, x_4) \langle \delta q_{i_3}(x_3) \delta q_{i_4}(x_4) q_{i_1}(x_1) q_{i_2}(x_2) \rangle_1 \\ &+ O(\xi^{-2}). \end{aligned} \quad (16)$$

Using the cumulant expansion for four-point functions and the locality of the GGE, we obtain

$$\begin{aligned} \langle q_{i_1}(x_1) q_{i_2}(x_2) \rangle^c &= \delta(x_1 - x_2) C_{i_1, i_2} \\ &+ 2\beta_{(2)}^{i_3, i_4}(x_1, x_2) C_{i_1, i_3} C_{i_2, i_4} + O(\xi^{-2}), \end{aligned} \quad (17)$$

where we used $\langle q_{i_1}(x_1) q_{i_2}(x_2) \rangle_1^c = \delta(x_1 - x_2) C_{i_1, i_2}$, and where C_{i_1, i_2} denotes the local charge-charge susceptibility matrix. From Eq. (17), we define the long-range charge-charge correlation $g_{i_1, i_2}^{(2)}(x_1, x_2)$ through

$$g_{i_1, i_2}^{(2)}(x_1, x_2) = 2\beta_{(2)}^{i_3, i_4}(x_1, x_2)C_{i_1, i_3}C_{i_2, i_4} + O(\xi^{-2}), \quad (18)$$

so that

$$\langle q_{i_1}(x_1)q_{i_2}(x_2) \rangle^c = \delta(x_1 - x_2)C_{i_1, i_2}(x_1) + g_{i_1, i_2}^{(2)}(x_1, x_2). \quad (19)$$

Similarly, as shown in Appendix A, for the three-point function, we find

$$\begin{aligned} \langle q_{i_1}q_{i_2}q_{i_3} \rangle^c &= \delta(x_1 - x_2)\delta(x_1 - x_3)C_{i_1, i_2, i_3}(x_1) \\ &\quad + [\delta(x_1 - x_2)C_{i_1, i_2, j}C^{j, k}g_{k, i_3}^{(2)}(x_1, x_3)]_{(1,2,3)} \\ &\quad + g_{i_1, i_2, i_3}^{(3)}(x_1, x_2, x_3), \end{aligned} \quad (20)$$

with

$$\begin{aligned} g_{i_1, i_2, i_3}^{(3)}(x_1, x_2, x_3) \\ = 3\beta_{(3)}^{i_1, j_2, j_3}(x_1, x_2, x_3)C_{i_1, j_1}C_{i_2, j_2}C_{i_3, j_3} + O(\xi^{-3}), \end{aligned} \quad (21)$$

and where C_{i_1, i_2, i_3} is the local GGE three-point function, and $C^{i, j}$ is the inverse covariance matrix, $C_{i, j}C^{j, k} = \delta_{i, k}$. We emphasize that the matrices C_{i_1, i_2} and C_{i_1, i_2, i_3} are known functions of the expectation values of the conserved charge densities $\langle q_i \rangle_1$. It is also important to note that the functions $g^{(2)}(x_1, x_2)$, $g^{(3)}(x_1, x_2, x_3)$, and higher orders, as well as the generalized temperatures $\beta_{(n)}$, are smooth everywhere except at coinciding points, $x_i = x_j$, where discontinuities may appear, but no divergences occur. For this reason, to avoid ill-defined expressions, in Eq. (14) the integrals are defined over configurations with noncoinciding points.

B. The expression for the currents in the ensemble

Importantly, once the state (14) is assumed, the averages of all local, or quasilocal, observables can be expressed through cumulant expansions in the charge densities.

In this section, we use the state Eq. (14) to derive the cumulant expansion for the average current

$$\begin{aligned} \langle j_{i_1}(x_1) \rangle &= \langle j_{i_1}(x_1) \rangle_1 \\ &\quad + \int_{x_2 \neq x_3} dx_2 dx_3 \beta_{(2)}^{i_2, i_3}(x_2, x_3) \langle q_{i_2}(x_2) \\ &\quad \times q_{i_3}(x_3) j_{i_1}(x_1) \rangle_1^c + O(\xi^{-2}). \end{aligned} \quad (22)$$

We now observe that, due to the locality of the GGE average, one has

$$\begin{aligned} \langle q_{i_2}(x_2)q_{i_3}(x_3)j_{i_1}(x_1) \rangle_1^c &= 0, \\ \text{for } (|x_2 - x_1| + |x_3 - x_1| > \epsilon), \end{aligned} \quad (23)$$

where ϵ denotes a typical microscopic length scale of the model. Since the generalized temperature $\beta_{(2)}$ is, in general, discontinuous at coinciding points, due to fast microscopic-scale fluctuations encoded in the local GGE, this contribution must be treated with care.

Condition (23) effectively localizes the integral on the rhs of Eq. (22) to a neighborhood of size ϵ around the discontinuity. Within this region, the correlation function decomposes into its local parity-time ($\mathcal{PT}$) symmetric and antisymmetric components, $\beta_{(2)}^{i_1, i_2} = \beta_{(2), \text{sym}}^{i_1, i_2} + \beta_{(2), \text{anti}}^{i_1, i_2}$. Crucially, because the microscopic jump is inherently antisymmetric, the singularity is entirely confined to $\beta_{(2), \text{anti}}^{i_1, i_2}$. This implies that the symmetric component, defined locally as

$$\begin{aligned} \beta_{(2), \text{sym}}^{i_1, i_2}(x + \epsilon, x - \epsilon, t) &= \frac{1}{2} \left[\beta_{(2)}^{i_1, i_2}(x + \epsilon, x - \epsilon, t + \epsilon) \right. \\ &\quad \left. + \beta_{(2)}^{i_1, i_2}(x - \epsilon, x + \epsilon, t - \epsilon) \right], \end{aligned} \quad (24)$$

remains smooth even as $\epsilon \rightarrow 0$. Since charge and current density operators are $\mathcal{PT}$ symmetric [120], they couple exclusively to this smooth symmetric part in Eq. (22). Consequently, the integral projects out the singular antisymmetric term, rendering the limit $\epsilon \rightarrow 0$ regular,

$$\begin{aligned} \langle j_{i_1}(x_1) \rangle &= \langle j_{i_1}(x_1) \rangle_1 \\ &\quad + \beta_{(2), \text{sym}}^{i_2, i_3}(x_1^+, x_1^-) \frac{\delta^2 \langle j_{i_1}(x_1) \rangle_1}{\delta \beta_{(1)}^{i_2} \delta \beta_{(1)}^{i_3}} + O(\xi^{-2}). \end{aligned} \quad (25)$$

This calculation follows analogously for the ensemble average of any $\mathcal{PT}$ -symmetric observable. Also, we used that in a local GGE correlation functions of local densities can be obtained by differentiation with respect to $\beta_{(1)}$. Applying Eq. (18) and the definition of the inverse susceptibility matrix, we obtain

$$\begin{aligned} \langle j_{i_1}(x_1) \rangle &= \langle j_{i_1}(x_1) \rangle_1 \\ &\quad + \frac{1}{2} \frac{\delta^2 \langle j_{i_1}(x_1) \rangle_1}{\delta \langle q_{i_2} \rangle_1 \delta \langle q_{i_3} \rangle_1} g_{i_2, i_3}^{(2), \text{sym}}(x_1^+, x_1^-) + O(\xi^{-2}). \end{aligned} \quad (26)$$

In analogy, we define

$$g_{i_2, i_3}^{(2), \text{sym}}(x_1^+, x_1^-) = \frac{g_{i_2, i_3}^{(2)}(x_1^+, x_1^-) + g_{i_2, i_3}^{(2)}(x_1^-, x_1^+)}{2}, \quad (27)$$

where we dropped the splitting in time, since the correlation functions are always smooth for any t . Finally, introducing the notation

$$H_{i_1}^{i_2, i_3} \equiv \frac{\delta^2 \langle j_{i_1} \rangle_1}{\delta \langle q_{i_2} \rangle_1 \delta \langle q_{i_3} \rangle_1}, \quad (28)$$

we obtain the double-projection formula

$$\begin{aligned} \langle j_{i_1}(x_1) \rangle &= \langle j_{i_1}(x_1) \rangle_1 \\ &+ \frac{1}{2} H_{i_1}^{i_2, i_3} g_{i_2, i_3}^{(2), \text{sym}}(x_1^+, x_1^-) + O(\xi^{-2}). \end{aligned} \quad (29)$$

Similarly, for the current-charge correlator, we find

$$\begin{aligned} \langle j_i(x_1) q_j(x_2) \rangle^c &= A_i^k \langle q_k(x_1) q_j(x_2) \rangle^c \\ &+ \frac{1}{2} H_i^{k, l} g_{k, l, j}^{(3), \text{sym}}(x_1^+, x_1^-, x_2) + O(\xi^{-3}), \end{aligned} \quad (30)$$

where we used the chain rule to introduce the flux Jacobian $A_i^k = \delta \langle j_i \rangle_1 / \delta \langle q_k \rangle_1$, and where $g_{k, l, j}^{(3), \text{sym}}(x_1^+, x_1^-, x_2)$ is defined in analogy with Eq. (27). This simplification is crucial, as it allows one to express the full hierarchy in terms of few-point charge correlators. We also note that the generalized currents $j_{i,0}$ appearing on the right-hand side of Eqs. (11)–(13) are linear in the charge density; therefore, the cumulant expansion truncates after the first term,

$$\langle j_{i_1,0} q_{i_2} \dots \rangle^c = A_{i_1,0}^k \langle q_k q_{i_2} \dots \rangle^c, \quad (31)$$

with $A_{i_1,0}^k = \delta \langle j_{i_1,0} \rangle_1 / \delta \langle q_k \rangle_1$.

C. The gBBGKY hierarchy

In this section, we make use of the assumptions and results introduced in Sec. III A in order to write Eqs. (11)–(13) as a hierarchy of equations for the smooth few-point functions $g_{i_1, \dots, i_n}^{(n)}$. The formulas (19) and (20), a direct consequence of taking expectation values within the ensemble Eq. (14), reveal that the correlators, and thus the hierarchy, feature singular terms. Thus, to understand the dynamics, it is crucial to plug these expressions into the hierarchy along with the expression for the correlation functions containing the currents. To simplify the equations, we restrict to a homogeneous setting, where the one-point functions are space independent, and two-point functions depend only on the relative distances. Equations (11)–(13) reduce then to the following first layers of the gBBGKY hierarchy:

$$\partial_t \langle q_{i_1} \rangle = - \int dx_2 V'(x_1 - x_2) A_{i_1,0}^k g_{k,0}^{(2)}(x_1, x_2), \quad (32)$$

$$\begin{aligned} \partial_t g_{i_1, i_2}^{(2)}(x_1, x_2) &+ \left[\partial_{x_1} A_{i_1}^j g_{j, i_2}^{(2)}(x_1, x_2) + \frac{1}{2} \partial_{x_1} H_{i_1}^{j, k} g_{j, k, i_2}^{(3), \text{sym}}(x_1^+, x_1^-, x_2) \right]_{(1,2)} \\ &= - \left[V'(x_1 - x_2) A_{i_1,0}^j \left(\langle q_j \rangle C_{i_2,0} + g_{j,k}^{(2)}(x_1, x_2) C_{i_2,0}^{k,m} C_{m,i_2,0} \right) + \int dx_3 V'(x_1 - x_3) A_{i_1,0}^j \left(\langle q_j \rangle g_{i_2,0}^{(2)}(x_2, x_3) + g_{j,i_2,0}^{(3)} \right) \right]_{(1,2)} \\ &- \delta(x_1 - x_2) \int dx_3 V'(x_1 - x_3) \left[-C_{i_1, i_2, j} C_{j,k}^{j,k} A_{k,0}^n + \left(A_{i_1,0}^j \delta_{i_2}^k + \delta_{i_1}^j A_{i_2,0}^k \right) C_{j,k,m} C_{m,i_2,0} \right] g_{n,0}^{(2)}(x_1, x_3), \end{aligned} \quad (33)$$

$$\begin{aligned} \partial_t g_{i_1, i_2, i_3}^{(3)}(x_1, x_2, x_3) &+ \left[\partial_{x_1} A_{i_1}^j g_{j, i_2, i_3}^{(3)}(x_1, x_2, x_3) + \frac{1}{2} \partial_{x_1} H_{i_1}^{j, k} g_{j, i_2}^{(2)}(x_1, x_2) g_{k, i_3}^{(2)}(x_1, x_3) \right]_{(1,2,3)} \\ &= \left[-\delta(x_1 - x_2) \left(A_{i_1}^k C_{k, i_2, m} C_{m,i_3}^{m,n} - C_{i_1, i_2, k} C_{k,m}^{k,m} A_m^n + H_{i_1}^{m,n} C_{m,i_2} \right) \partial_{x_1} g_{n, i_3}^{(2)}(x_1, x_3) \right]_{(1,2,3)} + O(V_0^2 \xi^2, V_0 \xi^{-3}), \end{aligned} \quad (34)$$

where, for brevity, in Eq. (34) we truncated higher-order terms in both parameters V_0 and $1/\xi$. Let us comment on the structure of this complex hierarchy of equations. First, in homogeneous systems, the evolution equation for the density, Eq. (32), does not contain the Vlasov term and therefore all its dynamics is given by the evolution of the two-point functions, Eq. (33). Here, on the lhs are the kinetic term describing the ballistic transport of the two-point function and its diffusive corrections. On the rhs, we observe that the

singular contribution to the two-point function creates a contact forcing proportional to V_0/ξ . Finally, the dynamics of the three-point function is written in Eq. (34) up to order $O(V_0/\xi)$. Again, its lhs includes the kinetic terms for the ballistic transport of correlations, while the rhs behaves as singular forcing, which is given by the integrable dynamics, i.e., by the integrable contact interactions. In Appendix E, more details are given in order to derive Eqs. (32)–(34). Finally, we observe that the assumption of large

hydrodynamic length scale ξ is sufficient to have a well-defined truncation scheme for the hierarchy.

D. The hierarchy for the (quasi)particle occupations and correlations

In this section, we express the gBBGKY hierarchy in terms of quasiparticles of the Hamiltonian $\hat{H}_a^{\text{int}}$. In particular, for integrable systems, the quasiparticles are characterized by Bethe rapidities θ , whose distribution function in the fluid cell (x, t) is denoted by $\rho_\theta(x, t)$ (in the rest of the text, we use Greek indices for rapidities, while Latin ones are used for charge labels). The rapidities are related to the quasiparticles momentum through $k_\theta = k(\theta)$. The density $\rho_\theta(x, t)$ is defined through its relation with the hydrodynamic variables

$$\langle q_i \rangle(x, t) = \int d\theta \rho_\theta(x, t) h_i(\theta), \quad (35)$$

where $h_i(\theta)$ is the single-particle eigenvalue of the charge Q_i . With a complete set of charges, it is possible to invert this expression, having a bijection between the quasiparticle density $\rho_\theta(x, t)$ and the expectation values of the charge density $\langle q_i \rangle$. Hence, it is possible to recast the infinite tower of hydrodynamic equations for the conservation of charges, labeled by the integer i , to the single evolution of $\rho_\theta(x, t)$. We now define the dressing operator acting on a test function $h(\theta)$,

$$h^{\text{dr}} = (1 - Tn)^{-1} \cdot h, \quad (36)$$

where the action of a linear operator is defined as $(A \cdot h)(\theta) = \int d\gamma A(\theta, \gamma) h(\gamma) = A_\theta^\gamma h_\gamma$, assuming the Einstein convention for the integration over repeated indexes. In particular, $T = T_{\theta, \gamma}$ is the scattering shift of the model (which vanishes in the limit of zero contract interactions $a \rightarrow 0$), being independent of the state. The filling function defines the occupation of rapidities with respect to the total density of states $\rho^\dagger = k'/2\pi$ and it is given by $n = \rho/\rho^\dagger$. Moreover, we shall make extensive use of compact matrix notation, for example, by introducing the rotation matrix

$$R_{\theta_1, \theta_2} = [1 - Tn]_{\theta_1, \theta_2} = \delta_{\theta_1, \theta_2} - T_{\theta_1, \theta_2} n_{\theta_2}, \quad (37)$$

which implements the dressing operation as $h^{\text{dr}} = R^{-1} \cdot h$. Crucially, it is also possible to express the density of currents, as

$$\langle j_i(x) \rangle = \int d\theta \rho_\theta(x) v_\theta^{\text{eff}}(x) h_i(\theta) + O(1/\xi), \quad (38)$$

with the effective velocity $v_\theta^{\text{eff}} = (h_2')^{\text{dr}}(\theta)/(h_1')^{\text{dr}}(\theta)$, where for Galilean particles with unit mass $h_2(\theta) = \theta^2/2$ and $h_1(\theta) = \theta$. Similarly, we can define quasiparticle expression for few-point functions as

$$\langle q_{i_1} \dots q_{i_n} \rangle = \int \prod_{k=1}^n \left[d\theta_k h_{i_k}(\theta_k) \right] \langle \rho_{\theta_1} \dots \rho_{\theta_n} \rangle. \quad (39)$$

Another important object is the flux Jacobian, which in quasiparticles reads

$$A_i^j = \int d\theta_1 d\theta_2 h_i(\theta_1) R_{\theta_1, \gamma}^{-t} v_\gamma^{\text{eff}} R_{\gamma, \theta_2}^t h^j(\theta_2), \quad (40)$$

where h^j is defined as the inverse of h_j , such that $h_j(\theta) h^j(\theta') = \delta_{\theta, \theta'}$. For completeness, we also recall $A_\theta^{\theta'} \equiv R_{\theta, \gamma}^{-t} v_\gamma^{\text{eff}} R_{\gamma, \theta'}^t$, with $R^t = 1 - nT$. In a similar manner, we compute the second derivative of the current as

$$H_i^{j,k} = \int d\theta d\alpha d\beta h_i(\theta) H_\theta^{\alpha\beta} h^j(\alpha) h^k(\beta), \quad (41)$$

where we defined (from below onward, we shall use the notation $v_{\mu, \gamma} \equiv v_\mu^{\text{eff}} - v_\gamma^{\text{eff}}$)

$$H_\theta^{\alpha\beta} = \left[R_{\theta, \gamma}^{-t} v_{\mu, \gamma} \frac{T_{\gamma, \mu}^{\text{dr}}}{\rho_\gamma^t} R_{\mu, \alpha}^t R_{\gamma, \beta}^t \right]_{(\alpha, \beta)}. \quad (42)$$

Analogous to what was done in the previous section, we consider the connected few-point correlators and separately recognize the GGE contribution from the regular long-range part. For the two-point function, we have

$$\langle \rho_{\theta_1}(x_1) \rho_{\theta_2}(x_2) \rangle^c = \delta(x_1 - x_2) C_{\theta_1, \theta_2} + g_{\theta_1, \theta_2}^{(2)}(x_1, x_2), \quad (43)$$

where $C_{\theta_1, \theta_2} = R_{\theta_1, \gamma}^{-t} \rho_\gamma f_\gamma R_{\gamma, \theta_2}^{-1}$ is the susceptibility matrix in quasiparticle representation, and $g_{\theta_1, \theta_2}^{(2)}(x_1, x_2)$ is the non-singular contribution. Also, the generic symbol $f_\theta = \{1, 1 - n_\theta, 1 + n_\theta, n_\theta\}$ denotes the statistical factor (classical, fermionic, bosonic, and wave statistics, respectively). Analogously, for the three-point functions, we can write

$$\begin{aligned} \langle \rho_{\theta_1}(x_1) \rho_{\theta_2}(x_2) \rho_{\theta_3}(x_3) \rangle^c &= \delta(x_1 - x_2) \delta(x_1 - x_3) C_{\theta_1, \theta_2, \theta_3} \\ &+ \left[\delta(x_1 - x_2) C_{\theta_1, \theta_2}^\gamma g_{\gamma, \theta_3}^{(2)}(x_1, x_3) \right]_{(1,2,3)} \\ &+ g_{\theta_1, \theta_2, \theta_3}^{(3)}(x_1, x_2, x_3), \end{aligned} \quad (44)$$

where $C_{\theta_1, \theta_2, \theta_3}$ is the local three-point function in a GGE. We also introduce the tensor given by the functional derivative of the susceptibility with respect to the local density

$$C_{\theta_1, \theta_2}^{\theta_3} \equiv \frac{\delta C_{\theta_1, \theta_2}}{\delta \rho_{\theta_3}} = R_{\theta_1, \alpha}^{-t} \left[\delta_{\alpha, \beta} \delta_{\alpha, \theta_3} f_\alpha - \delta_{\alpha, \beta} \varepsilon n_\alpha R_{\alpha, \theta_3}^t + T_{\alpha, \beta}^{\text{dr}} \frac{R_{\alpha, \theta_3}^t}{\rho_\alpha^t} f_\beta \rho_\beta + f_\alpha \rho_\alpha T_{\alpha, \beta}^{\text{dr}} \frac{R_{\beta, \theta_3}^t}{\rho_\beta^t} \right] R_{\beta, \theta_2}^{-1}, \quad (45)$$

with $\varepsilon = \{0, -1, 1\}$, respectively, for classical, Fermi-Dirac, or Bose-Einstein statistics and we also introduce the tensor giving the kinetic term of the three-point correlations,

$$M_{\theta_1, \theta_2}^{\theta_3} = \left[R_{\theta_1, \mu}^{-t} R_{\theta_2, \gamma}^{-t} T_{\mu, \gamma}^{\text{dr}} \frac{\rho_\gamma f_\gamma}{\rho_\mu^t} v_{\mu, \gamma} R_{\mu, \theta_3}^t \right]_{(\theta_1, \theta_2)}, \quad (46)$$

which is proportional to the strength of the local interactions a (see Appendix E 1 for a complete derivation).

With these ingredients, we can write down the first three layers in the quasiparticle representation,

$$\partial_t \rho_{\theta_1} = \int dx_2 d\theta_2 V'(x_1 - x_2) \partial_{\theta_1} g_{\theta_1, \theta_2}^{(2)}(x_1, x_2), \quad (47)$$

$$\begin{aligned} \partial_t g_{\theta_1, \theta_2}^{(2)}(x_1, x_2) &+ \left[\partial_{x_1} (A_{\theta_1}^\gamma g_{\gamma, \theta_2}^{(2)}(x_1, x_2)) + \frac{1}{2} \partial_{x_1} (H_{\theta_1}^{\gamma, \eta} g_{\gamma, \eta, \theta_2}^{(3), \text{sym}}(x_1^+, x_1^-, x_2)) \right]_{(1,2)} \\ &= \left[V'(x_1 - x_2) \int d\theta_3 \partial_{\theta_1} \left(\rho_{\theta_1} C_{\theta_2, \theta_3} + C_{\theta_2, \theta_3}^\gamma g_{\theta_1, \gamma}^{(2)}(x_1, x_2) \right) \right. \\ &\quad \left. + \int dx_3 d\theta_3 V'(x_1 - x_3) \partial_{\theta_1} \left(\rho_{\theta_1} g_{\theta_2, \theta_3}^{(2)}(x_2, x_3) + g_{\theta_1, \theta_2, \theta_3}^{(3)}(x_1, x_2, x_3) \right) \right]_{(1,2)} \\ &\quad + \delta(x_1 - x_2) \int dx_3 d\theta_3 V'(x_1 - x_3) \left(-C_{\theta_1, \theta_2}^\gamma \partial_\gamma + (\partial_{\theta_1} + \partial_{\theta_2}) C_{\theta_1, \theta_2}^\gamma \right) g_{\gamma, \theta_3}^{(2)}(x_1, x_3), \end{aligned} \quad (48)$$

$$\partial_t g_{\theta_1, \theta_2, \theta_3}^{(3)}(x_1, x_2, x_3) + \left[\partial_{x_1} A_{\theta_1}^\gamma g_{\gamma, \theta_2, \theta_3}^{(3)}(x_1, x_2, x_3) \right]_{(1,2,3)} = \left[\delta(x_1 - x_2) M_{\theta_1, \theta_2}^\gamma \partial_{x_1} g_{\gamma, \theta_3}^{(2)}(x_1, x_3) \right]_{(1,2,3)} + O(V_0^2 \xi^2, V_0 \xi^{-3}). \quad (49)$$

Let us comment on the kinetic terms for the two- and three-point functions in Eqs. (48) and (49). The first terms with the matrix $A_\theta^\gamma = [R^{-1} v^{\text{eff}} R]_\theta^\gamma$ represent convective spreading with velocities associated with the different quasiparticles. While in an interacting model this matrix is nondiagonal (due to mode mixing), we shall see in the next section that the latter simply reduces to a diagonal matrix containing the velocities $v^{\text{br}}(\theta)$ as eigenvalues in the limit $a \rightarrow 0$. The second term in Eq. (48) is instead nonzero only in interacting systems. The latter can be seen as an effective diffusion induced by the motion of the three-point functions for the two-point functions. Indeed, its form becomes more familiar in the small-time limit, i.e., for states close to the local GGE. In this limit, we can indeed show the relation (see Supplemental Material [114] for a complete derivation)

$$\lim_{t \rightarrow 0^+} H_{\theta_1}^{\gamma, \eta} g_{\gamma, \eta, \theta_2}^{(3), \text{sym}}(x_1^+, x_1^-, x_2; t) = \mathfrak{D}_{\theta_1}^\gamma \partial_{x_1} g_{\gamma, \theta_2}^{(2)}(x_1, x_2), \quad (50)$$

where $\mathfrak{D}_\theta^\gamma$ is indeed the diffusion matrix [120–122] associated with the local equilibrium state around which we are linearizing. As the system evolves, the validity of this relation

gets worse, due to the buildup of nontrivial long-range three-point correlations; yet, this contribution is fundamental to thermalization of the two-point functions and, consequently, of the one-point functions as we shall see in the next sections. In the limit $a \rightarrow 0$, this contribution vanishes, as there is indeed no internal (convective) diffusion $\mathfrak{D} \approx a^2$, as also confirmed by the analysis in the next section, Sec. III E.

E. Recovering the standard BBGKY hierarchy in the limit $a \rightarrow 0$

The standard BBGKY hierarchy is recovered in the limit of noninteracting unperturbed systems, i.e., $\hat{H}_a^{\text{int}} \rightarrow \hat{H}^{\text{free}}$. It is important here to stress that the gBBGKY formalism is valid both for classical and quantum systems and difference between the two scenarios is fully incorporated into the statistical factor $f = f(n)$. Here, in order to compare with standard textbook expressions [78], we specialize to the classical case and set $f = 1$.

The limit $\hat{H}_a^{\text{int}} \rightarrow \hat{H}^{\text{free}}$ corresponds to zero scattering shift $T \rightarrow 0$. We now carefully analyze the effect of this limit on the objects defined in Sec. III D. First, we observe

that this limit trivializes the dressing operator as $R_{\theta,\theta'} \rightarrow \delta_{\theta,\theta'}$, and hence $h^{\text{dr}} \rightarrow h$. As a consequence, the effective velocity reduces to the particle's velocity $v_{\theta}^{\text{eff}} \rightarrow v_{\theta}^{\text{br}} = \theta$. In a similar way, the flux Jacobian becomes $A'_{\theta} \rightarrow v_{\theta}^{\text{br}} \delta_{\theta,\gamma}$. Since noninteracting particles do not exhibit nontrivial correlations, we also have

$$C_{\theta_1,\theta_2} \rightarrow \delta_{\theta_1,\theta_2} \rho_{\theta_1}, \quad C_{\theta_1,\theta_2}^{\theta_3} \rightarrow \delta_{\theta_1,\theta_2} \delta_{\theta_1,\theta_3}. \quad (51)$$

As a consequence, we observe that the δ contribution appearing in Eq. (48) is vanishing, being proportional to

$$\int d\gamma [\delta_{\theta_1,\theta_2} \delta_{\theta_1,\gamma} \partial_{\gamma} - (\partial_{\theta_1} + \partial_{\theta_2}) \delta_{\theta_1,\theta_2} \delta_{\theta_1,\gamma}] h_{\gamma} = 0. \quad (52)$$

We also observe that, in this limit, we have $H \rightarrow 0$ and $M \rightarrow 0$, both being proportional to T . We also stress that any higher-order hydrodynamic correction in the cumulant expansion is vanishing since noninteracting particles simply spread ballistically. Hence, in the noninteracting limit $T \rightarrow 0$, Eqs. (47)–(49) reduce to

$$\partial_t \rho_{\theta_1} = \int dx d\theta_2 V'(x) \partial_{\theta_1} g_{\theta_1,\theta_2}^{(2)}(x), \quad (53)$$

$$\begin{aligned} & \partial_t g_{\theta_1,\theta_2}^{(2)}(x_1, x_2) + \left[\partial_{x_1} (v_{\theta_1}^{\text{br}} g_{\theta_1,\theta_2}^{(2)}(x_1, x_2)) \right]_{(1,2)} \\ &= \left[V'(x_1 - x_2) \partial_{\theta_1} \left(\rho_{\theta_1} \rho_{\theta_2} + g_{\theta_1,\theta_2}^{(2)} \right) \right. \\ &+ \int dx_3 d\theta_3 V'(x_1 - x_3) \partial_{\theta_1} \rho_{\theta_1} g_{\theta_2,\theta_3}^{(2)}(x_2, x_3) \\ &+ \left. \int dx_3 d\theta_3 V'(x_1 - x_3) \partial_{\theta_1} g_{\theta_1,\theta_2,\theta_3}^{(3)}(x_1, x_2, x_3) \right]_{(1,2)}, \quad (54) \end{aligned}$$

$$\begin{aligned} & \partial_t g_{\theta_1,\theta_2,\theta_3}^{(3)}(x_1, x_2, x_3) \\ &+ \left[\partial_{x_1} (v_{\theta_1}^{\text{br}} g_{\theta_1,\theta_2,\theta_3}^{(3)}(x_1, x_2, x_3)) \right]_{(1,2,3)} + \dots = 0, \quad (55) \end{aligned}$$

which perfectly coincide with the first equations of celebrated BBGKY hierarchy for a homogeneous system. It is also easy to prove that this equivalence is valid for any level of the hierarchy and that can be extended also in inhomogeneous settings. It is important to stress that, through the gBBGKY formalism, we provided a new alternative derivation of BBGKY equations in terms of correlated fluid cells that does not require one to use the full system phase-space distribution function.

F. The $V \rightarrow 0$ limit of the hierarchy

It is natural to ask what happens to the dynamics of correlations when the $V \rightarrow 0$ limit is considered and the

only possible source of interaction is due to the integrable part of the Hamiltonian. First, in the homogenous setting with no initial long-range correlations, i.e., $g^{(n)}(t=0) = 0$ for $n > 1$, there is no dynamics at all in the system, as expected. This is a trivial confirmation that a GGE is a stationary state of our framework when there is no integrability breaking.

This picture changes when the initial state is inhomogeneous. In such case, our hierarchy extends the equations put forward in Ref. [113] and recovers them upon truncation to two layers. These two equations provide the basis for the new understanding of diffusion in integrable models which occurs under the presence of long-range correlations generated by integrable dynamics [123]. Let us mention that gBBGKY formalism is also capable of tracking the dynamics of higher-point functions, even if higher-order hydrodynamic matrices are not explicitly computed in this manuscript. For instance, we expect to observe long-range structures in three-point correlations, generated due to the gradients in two-point function (similar to two-point correlations generated by gradients of one-point functions in Ref. [113]). This can be seen from solution (70) of Eq. (49). There, even in the homogenous case, we see a generation of three-point correlations solely by integrable dynamics, powered by gradients of $g^{(2)}$ (generated in a homogenous setting by the presence of V).

IV. THE GENERALIZED LANDAU EQUATION AND LATE-TIME DYNAMICS

The gBBGKY equations presented above are clearly complicated to analyze and to fully numerically simulate. However, to extract the thermalization dynamics, some approximations can be performed, leading to what is known as the Landau equation, namely, the Boltzmann equation in the limit of small momentum exchange, also known as the grazing collisions limit.

We shall perform a macroscopic rescaling of space $x \rightarrow \xi x$ and time $t \rightarrow \xi t$, and importantly, in what follows, we will assume that the long-range coupling is weak by treating V_0 as a small parameter. Thus, scattering integral descriptions have a narrower range of applicability than the hierarchy, which assumes only large ξ . However, as we show in Sec. V, the scattering integral turns out to be quantitatively predictive also for quite large values of V_0 , provided that $V_0/\xi \ll 1$. Nevertheless, this approach is sufficient to understand the main mechanisms behind the thermalization of the system.

We start the derivation by considering the gBBGKY hierarchy written in terms of quasiparticle occupations and correlations, as introduced in Sec. III D. More precisely, in this section, we explicitly make use of homogeneity by expressing correlations as functions of relative distance only. Hence, for the two-point function, we have $g_{\theta_1,\theta_2}^{(2)}(x_1, x_2) = g_{\theta_1,\theta_2}^{(2)}(z = x_1 - x_2)$ and for the three-point

function $g_{\theta_1, \theta_2, \theta_3}^{(3)}(x_1, x_2, x_3) = g_{\theta_1, \theta_2, \theta_3}^{(3)}(z, z')$, with $z = x_1 - x_2$ and $z' = x_2 - x_3$.

Furthermore, we perform a decomposition of the two-point function $g^{(2)}$ into leading and subleading contributions in the $1/\xi$ expansion, namely,

$$g_{\theta_1, \theta_2}^{(2)}(z) = \frac{1}{\xi} g_{\theta_1, \theta_2}^{(2),1}(z) + \frac{1}{\xi^2} g_{\theta_1, \theta_2}^{(2),2}(z). \quad (56)$$

We also explicitly factorize the scaling parameter of the three-point functions using $g^{(3)} \rightarrow g^{(3)}/\xi^2$. Now, using the definition of the interaction potential (1), the $1/\xi$ dependence is fully explicit. Plugging the decomposition into Eq. (48) for the two-point function and requiring equality both at the order $1/\xi$ and $1/\xi^2$ produces two equations,

$$\begin{aligned} \partial_t g_{\theta_1, \theta_2}^{(2),1}(z) + \partial_z \left(A_{\theta_1}^\gamma g_{\gamma, \theta_2}^{(2),1}(z) - A_{\theta_2}^\gamma g_{\theta_1, \gamma}^{(2),1}(z) \right) \\ = V_0 \varphi'(z) \int d\gamma \left(\partial_{\theta_1} \rho_{\theta_1} C_{\gamma, \theta_2} - \partial_{\theta_2} \rho_{\theta_2} C_{\theta_1, \gamma} \right), \end{aligned} \quad (57)$$

$$\begin{aligned} \partial_t g_{\theta_1, \theta_2}^{(2),2}(z) + \partial_z \left(A_{\theta_1}^\gamma g_{\gamma, \theta_2}^{(2),2}(z) - A_{\theta_2}^\gamma g_{\theta_1, \gamma}^{(2),2}(z) \right) \\ = -\frac{1}{2} \partial_z \left(H_{\theta_1}^{\gamma_1, \gamma_2} g_{\gamma_1, \gamma_2, \theta_2}^{(3), \text{sym}}(0^+, z) - H_{\theta_2}^{\gamma_1, \gamma_2} g_{\theta_1, \gamma_1, \gamma_2}^{(3), \text{sym}}(z, 0^+) \right). \end{aligned} \quad (58)$$

These equations are the effective evolution laws for $g^{(2),1}$ and $g^{(2),2}$. We can already observe that, meanwhile $g^{(2),1}$ is rapidly evolving due to the coupling with the density, the subleading correction $g^{(2),2}$ slowly builds up forced by the three-point function. We start with considering four equations, namely, Eqs. (47), (57), (58), and (49). The goal is to reduce them to a single equation for quasiparticle distribution ρ_θ . We will follow a procedure in spirit similar to classic derivations of the Boltzmann scattering kernel from BBGKY hierarchy [68,69]. The central idea here is the separation of timescales over which different correlations relax. For instance, the one-point function is expected to relax on much longer timescales than the two-point function. Schematically, the derivation will proceed as follows:

- (1) Integrate Eq. (57) in time, expressing $g_{\theta_1, \theta_2}^{(2),1}(z, t)$ as the explicit functional of $\rho_\theta(\tau)$ for all earlier times $\tau < t$.
- (2) Plug the result of the previous step into Eq. (49) and again integrate in time, expressing $g_{\theta_1, \theta_2, \theta_3}^{(3)}(z, z', t)$ as the functional of $\rho_\theta(\tau)$.
- (3) Use this result to compute $g_{\theta_1, \theta_2, \theta_3}^{(3)}(z, z', t)$, plug into Eq. (58), and integrate in time, expressing $g_{\theta_1, \theta_2}^{(2),2}(z, t)$ as the functional of $\rho_\theta(\tau)$.

- (4) Plug this result into Eq. (47) and perform approximations related to separation of timescales, which finally yield the generalized Landau kinetic equation.

We elaborate on the derivation in what follows. We assume that at $t = 0$ the state is a GGE state with $g^{(n)} = 0$ for $n \geq 2$. This assumption is not necessary as the system will eventually forget about the initial correlations, but simplifies the calculation which anyway leads to universal, late-time physics.

The first step consists of considering Eq. (57) and viewing it as a linear equation (we assume here that A_θ^γ is a constant operator) for $g_{\theta_1, \theta_2}^{(2),1}(z)$ with a ρ -dependent source term on the rhs given by

$$\mathcal{S}_{\theta_1, \theta_2}^{(2),1}[\rho](z) = V_0 \varphi'(z) \int d\gamma \left(\partial_{\theta_1} \rho_{\theta_1} C_{\gamma, \theta_2} - \partial_{\theta_2} \rho_{\theta_2} C_{\theta_1, \gamma} \right). \quad (59)$$

Thus, Eq. (57) can formally be solved using the Green's function method (for details, see Appendix E 2), which gives

$$\begin{aligned} g_{\theta_1, \theta_2}^{(2),1}[\rho](z, t) = R_{\theta_1, \xi_1}^{-t} R_{\theta_2, \xi_2}^{-t} \int_0^t d\tau \int \frac{dk}{2\pi} \int dz' \\ \times e^{ik(z+z'-v_{\xi_1, \xi_2} \tau)} R_{\xi_1, \eta_1}^t R_{\xi_2, \eta_2}^t \mathcal{S}_{\eta_1, \eta_2}^{(2),1}[\rho](z', t-\tau). \end{aligned} \quad (60)$$

In this way, we have expressed $g_{\theta_1, \theta_2}^{(2),1}(z, t)$ with ρ_θ at all earlier times. The structure of a linear equation with ρ -dependent source term turns out to be typical in this derivation and we encounter it also in the second step, which amounts to plugging Eq. (60) into Eq. (49). The linear equation features a source term on the rhs which reads

$$\begin{aligned} \mathcal{S}_{\theta_1, \theta_2, \theta_3}^{(3)}[\rho](z, z') = \delta(z) M_{\theta_1, \theta_2}^\gamma \partial_z g_{\gamma, \theta_3}^{(2),1}(z+z') \\ + \delta(z') M_{\theta_2, \theta_3}^\gamma \partial_z g_{\theta_1, \gamma}^{(2),1}(z) \\ + \delta(z') M_{\theta_2, \theta_3}^\gamma \partial_{z'} g_{\theta_1, \gamma}^{(2),1}(z+z'). \end{aligned} \quad (61)$$

Using a similar technique as used to get Eq. (60), we explicitly find the functional $g_{\theta_1, \theta_2, \theta_3}^{(3)}[\rho](z, z', t)$. In the third step, we compute $g_{\theta_1, \theta_2, \theta_3}^{(3), \text{sym}}[\rho]$ and plug it to Eq. (58). Once again, this equation is integrated in time yielding $g_{\theta_1, \theta_2}^{(2),1}[\rho](z, t)$. In the last step, we insert the expressions for $g_{\theta_1, \theta_2}^{(2),1}[\rho](z, t)$ and $g_{\theta_1, \theta_2}^{(2),2}[\rho](z, t)$ into the equation for ρ_θ , which now becomes closed as all the higher correlations have been eliminated through time integration. We find that the equation has two contributions,

$$\partial_t \rho_\theta = \frac{1}{\xi} \mathcal{I}^{(2),1}[\rho](\theta) + \frac{1}{\xi^2} \mathcal{I}^{(2),2}[\rho](\theta). \quad (62)$$

Restoring the microscopic time variable, the two terms are, respectively, proportional to V_0^2/ξ^2 and $a^2 V_0^2/\xi^3$. In Secs. IV A and IV B, we derive them and show that $\mathcal{I}^{(2),1}$ is vanishing, proving that the leading thermalization rate is $a^2 V_0^2/\xi^3$.

A. Kinetic blocking at times ξ^2/V_0^2

We start by considering the first term, which reads

$$\mathcal{I}^{(2),1}[\rho](\theta_1) = \int dz d\theta_2 V_0 \varphi'(z) \partial_{\theta_1} g_{\theta_1, \theta_2}^{(2,1)}[\rho](z), \quad (63)$$

with $g_{\theta_1, \theta_2}^{(2,1)}[\rho](z)$ given by Eq. (60). We perform an approximation that amounts to replacing $\rho_\theta(z, t - \tau)$ with $\rho_\theta(z, t)$ under the integral and extending the integration range in time to infinity. Next, we use an identity $\int_0^\infty d\tau e^{i\tau x} = \pi \delta(x) + i\mathcal{P}(1/x)$, where $\mathcal{P}$ denotes principal value. The term with principal value vanishes due to the symmetry of the integrand, and the term with the δ function yields (we also use $\int d\theta R_{\theta, \gamma}^{-t} = 2\pi \rho_\gamma^t$)

$$\begin{aligned} \mathcal{I}^{(2),1}[\rho](\theta_1) &= \int \frac{dk}{2\pi} |k| V_0^2 \hat{\varphi}^2(k) \partial_{\theta_1} R_{\theta_1, \gamma_1}^{-t} \rho_{\gamma_2}^t \delta(v_{\gamma_1, \gamma_2}) \\ &\quad \times R_{\gamma_1, \eta_1}^t R_{\gamma_2, \eta_2}^t \int d\gamma_3 (\partial_{\eta_1} \rho_{\eta_1} C_{\gamma_3, \eta_2} \\ &\quad - \partial_{\eta_2} \rho_{\eta_2} C_{\eta_1, \gamma_3}), \end{aligned} \quad (64)$$

where $\hat{\varphi}(k) = \int dx e^{ikx} \varphi(x)$. Crucially, the Dirac δ condition $v_{\gamma_1, \gamma_2} = 0$ implies $\gamma_1 = \gamma_2$ and, hence, the integral vanishes identically,

$$\mathcal{I}^{(2),1}[\rho](\theta_1) = 0. \quad (65)$$

This is a feature of one-dimensional (1D) systems known as kinetic blocking. It is strictly related to the nature of two-

particle scattering in 1D, which can only induce momentum exchanges between particles. The kinetic blocking can be avoided in systems with nonmonotonous effective velocity, for instance, in lattice. Then, so-called umklapp scatterings allow the system to thermalize due to two-body collisions [38,39]. For the present discussion, however, we focus on systems with monotonous effective velocity. Another scenario involves systems of coupled 1D gases, as described in Sec. VI. If 1D gases have different densities or different strengths of integrable interactions, this opens a channel for scattering that redistributes the momenta, albeit not very effectively, leading only to prethermal stationary states [49,124]. Hence, here, in order to capture the thermalization phenomena, we have to include higher-order terms, which describe three-particle scatterings. These processes are encoded in $\mathcal{I}^{(2),2}[\rho](\theta)$, which we now analyze.

B. (Generalized) thermalization at times $\xi^3/(V_0 a)^2$

The derivation of $\mathcal{I}^{(2),2}$ proceeds as sketched above and the approximations made along the way are of the same form as in derivation of $\mathcal{I}^{(2),1}$. Namely, every time we time integrate the equation for the correlation function, we replace $\rho_{\theta_1}(z, t - \tau)$ with $\rho_{\theta_1}(z, t)$ under the integral and extend the integration range in time to infinity. In order to obtain the collision term, three such integrations have to be performed, leading to a much more complex collision term as compared to Eq. (64). The details of this quite long calculation are exposed in Supplemental Material [114]. Here, we present the final result. We simplify the notation $\mathcal{I} \equiv \mathcal{I}^{(2),2}$ and find two type of terms,

$$\mathcal{I} = \mathcal{I}^{\text{cross}} + \mathcal{I}^{\text{self}}. \quad (66)$$

Both terms are given by $\mathcal{I}^\alpha = \partial_{\theta_1} R_{\theta_1, \gamma}^{-t} \mathcal{I}_0^\alpha(\gamma)$, where $\alpha \in \{\text{cross}, \text{self}\}$ and using the notation $v_{\theta_2, \theta_1} = v_{\theta_2}^{\text{eff}} - v_{\theta_1}^{\text{eff}}$, we have

$$\mathcal{I}_0^{\text{cross}}[\rho](\theta_1) = 2\pi V_0^2 \int dk k^2 \hat{\varphi}^2(k) \int d\theta_2 d\theta_3 \left[\frac{(\rho_{\theta_3}^t)^2}{\rho_{\theta_1}^t} (T_{\theta_2, \theta_1}^{\text{dr}})^2 \frac{|v_{\theta_2, \theta_1}|}{v_{\theta_3, \theta_2} v_{\theta_1, \theta_3}^2} - \rho_{\theta_1}^t (T_{\theta_3, \theta_2}^{\text{dr}})^2 \frac{|v_{\theta_3, \theta_2}|}{v_{\theta_1, \theta_3} v_{\theta_2, \theta_1}^2} \right] \mathcal{B}_{\theta_1, \theta_2, \theta_3}, \quad (67)$$

$$\begin{aligned} \mathcal{I}_0^{\text{self}}[\rho](\theta_1) &= 2\pi V_0^2 \int d\theta_2 d\theta_3 \int dk k^2 \hat{\varphi}(k) \rho_{\theta_3}^t T_{\theta_1, \theta_2}^{\text{dr}} \frac{v_{\theta_2, \theta_1}}{v_{\theta_1, \theta_3}} \left[\hat{\varphi} \left(\frac{v_{\theta_2, \theta_1}}{v_{\theta_1, \theta_3}} k \right) \frac{\rho_{\theta_2}^t}{\rho_{\theta_1}^t} \frac{T_{\theta_3, \theta_1}^{\text{dr}}}{|v_{\theta_1, \theta_3}| |v_{\theta_3, \theta_2}|} \right. \\ &\quad \left. - 2\hat{\varphi} \left(\frac{v_{\theta_2, \theta_1}}{v_{\theta_3, \theta_2}} k \right) \frac{T_{\theta_2, \theta_3}^{\text{dr}}}{v_{\theta_1, \theta_3} |v_{\theta_2, \theta_3}|} \right] \mathcal{B}_{\theta_1, \theta_2, \theta_3}, \end{aligned} \quad (68)$$

with the $\mathcal{B}$ tensor given by

$$\mathcal{B}_{\theta_1, \theta_2, \theta_3} = \left[\frac{v_{\theta_3, \theta_2}}{\rho_{\theta_1}^t} R_{\theta_1, \eta}^t \partial_{\eta} \rho_{\eta} \rho_{\theta_3} f_{\theta_3} \rho_{\theta_2} f_{\theta_2} \right]_{(\theta_1, \theta_2, \theta_3)}. \quad (69)$$

It can be shown that the derived collision integral has all the expected features: Thermal states are stationary states, it conserves particle number, momentum, and energy, and satisfies the H theorem with positive entropy increase.

Restoring the microscopic times in the kinetic equation, Eq. (62), and using that $T^{\text{dr}} \sim a$, we can finally observe that the scattering integral given by Eqs. (67) and (68) predict a thermalization rate proportional to $(V_0/\xi)^2 a^2/\xi$. The scaling can be understood as follows: Two factors of V_0/ξ are necessary to form correlated two-point functions on large scales, then, two local scatterings with other particles carry a^2 , and the spatial inhomogeneity of the two-point functions, inducing its convective diffusion in space, carries another factor $1/\xi$. See also in Fig. 1: Particles correlate due to the long-range potential and then diffuse in the convective flow carried by the sea of ballistic particles. This picture may look counterintuitive, as, due to integrability breaking, one would naively expect the ballistic channels to decay at large times. However, this is not the case, as higher-order correlations preserve the ballistic structure to much longer times than ξ^3 , giving an effective bath of convective particles inducing density waves, which is the main and only mechanism for particle diffusion in interacting integrable models, see, for example, Refs. [113,121,125–129].

The reason for distinguishing the two terms is motivated by two different dynamical mechanisms; see Fig. 4 related to convective diffusion. The first term, the cross term, originates from the diffusion of the two-point correlations where the two correlated particles never interact via contact interaction. This can be the case when, for example, two quasi-one-dimensional systems interact with a long-range potential that acts only between the two different tubes. The second term, the self term, originates from the self-induced diffusion of the two particles that can also interact via contact interaction; namely, one of the two particles acts as a convective density wave for the other and vice versa. As shown in Supplemental Material [114], this scattering term is absent when the long-range interaction acts only between different tubes, but is present when it acts also on the same

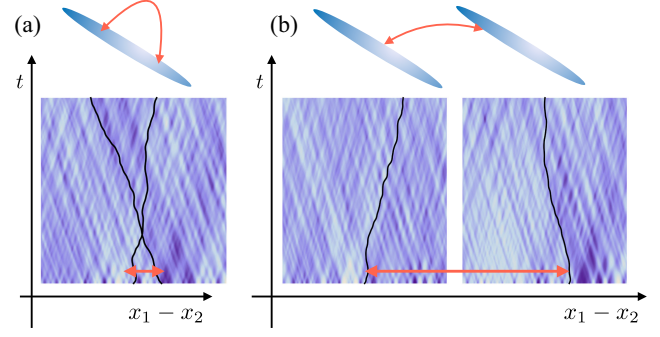


FIG. 4. Schematic representation of the two thermalization mechanisms leading to the two terms in Eq. (66). (a) The dynamics giving the self + cross term: Two correlated particles at a distance ξ experience fluctuations due to convective waves carried by the other quasiparticles, giving cross diffusion, and they also scatter among each other, giving a self-diffusion to the two-point functions. (b) The dynamics giving the cross term: Two correlated particles never interact with contact interactions; therefore, they only experience the convective density waves through the system, giving standard diffusion to the two-point functions.

tube. In the situation where the long-range potential acts across different tubes and on the same one, the scattering integral from Eq. (66) should include also the cross terms due to other tubes. We will discuss such scenarios in Sec. VI when we apply this formalism to experimental setups with dipolar quantum gases.

It is important to stress that, even if one-point functions thermalize on timescales of order $\xi^3/(V_0 a)^2$, higher-point functions remain far from thermal, which is why we have dubbed this process generalized thermalization, in contrast with standard thermalization. In particular, three-point functions exhibit long-range correlations that do not decay on these timescales and arise entirely from local integrable interactions, mirroring the behavior of two-point functions in fully integrable dynamics [113]. In order to derive the collision integral, in Supplemental Material [114] we derive an explicit expression for the dynamics of the three-point functions given by Eq. (49); here we report its large-time limit, namely, in the regime where time is of the order $\xi^3(V_0 a)^{-2}$,

$$\begin{aligned} g_{\theta_1, \theta_2, \theta_3}^{(3)}(z, z', t \gg \xi^3(V_0 a)^{-2}) &= \left[\frac{1}{2} R_{\theta_1, \gamma_1}^{-t} R_{\theta_2, \gamma_2}^{-t} R_{\theta_3, \gamma_3}^{-t} \frac{1}{v_{\gamma_1, \gamma_2}} [\text{sgn}(z) + \text{sgn}(v_{\gamma_1, \gamma_2})] R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t R_{\gamma_3, \mu_3}^t M_{\mu_1, \mu_2}^\beta \partial_1 g_{\beta, \mu_3}^{(2), 1} \left(\frac{v_{\gamma_2, \gamma_3}}{v_{\gamma_1, \gamma_2}} z - z' \right) \right] \\ &+ \left[\frac{1}{2} R_{\theta_1, \gamma_1}^{-t} R_{\theta_2, \gamma_2}^{-t} R_{\theta_3, \gamma_3}^{-t} \frac{1}{v_{\gamma_1, \gamma_3}} [\text{sgn}(z + z') + \text{sgn}(v_{\gamma_1, \gamma_3})] R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t R_{\gamma_3, \mu_3}^t M_{\mu_1, \mu_3}^\beta \partial_1 g_{\beta, \mu_2}^{(2), 1} \left(\frac{v_{\gamma_3, \gamma_2}}{v_{\gamma_1, \gamma_3}} z + \frac{v_{\gamma_1, \gamma_2}}{v_{\gamma_1, \gamma_3}} z' \right) \right] \\ &+ \left[\frac{1}{2} R_{\theta_1, \gamma_1}^{-t} R_{\theta_2, \gamma_2}^{-t} R_{\theta_3, \gamma_3}^{-t} \frac{1}{v_{\gamma_2, \gamma_3}} [\text{sgn}(z') + \text{sgn}(v_{\gamma_2, \gamma_3})] R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t R_{\gamma_3, \mu_3}^t M_{\mu_2, \mu_3}^\beta \partial_1 g_{\beta, \mu_1}^{(2), 1} \left(z + \frac{v_{\gamma_2, \gamma_1}}{v_{\gamma_2, \gamma_3}} z' \right) \right], \end{aligned} \quad (70)$$

where the symbol ∂_1 means that the derivative is performed on the spatial argument of the function, the symbol $M_{\theta_1, \theta_2}^{\theta_3} \sim a$ is defined in Eq. (46), and the two-point functions $g_{\theta_1, \theta_2}^{(2),1}(z)$ are given by their asymptotic value in time. Given that the latter decays rapidly in z , the expression above shows that there exist distinct directions in the (z, z') plane, e.g., along lines $z' \simeq (v_{\gamma_2, \gamma_3}/v_{\gamma_1, \gamma_2})z$ from the first term, along which long-range three-point correlations persist well beyond the thermalization timescales. Therefore, at times of order $\xi^3/(V_0 a)^2$, thermalization is incomplete (generalized), and the deviation from a standard thermal state can be diagnosed by higher-point functions.

C. Thermalization dynamics in the limit $a \rightarrow 0$

In the limit $a \rightarrow 0$, the generalized Landau equation, i.e., the contributions of Eqs. (67) and (68), vanishes, as it is proportional to a^2 . As noted above, the thermalization mechanism encoded by the generalized Landau equation is present only for perturbed interacting integrable models. In the case $a = 0$, thermalization is driven solely by the long-range interaction. A kinetic equation for weak long-range potentials in this regime was derived in Refs. [68,69]. Using the same approximations introduced in Sec. IV, the standard BBGKY equations governing thermalization, analogous to Eqs. (57) and (58), read

$$\partial_t g_{\theta_1, \theta_2}^{(2),1}(x_1, x_2) + [v^{\text{br}}(\theta_1) \partial_{x_1} g_{\theta_1, \theta_2}^{(2),1}(x_1, x_2)]_{(1,2)} = [V_0 \varphi'(x_1 - x_2) \partial_{\theta_1} \rho_{\theta_1} \rho_{\theta_2}]_{(1,2)}, \quad (71)$$

$$\begin{aligned} \partial_t g_{\theta_1, \theta_2}^{(2),2}(x_1, x_2) + [v^{\text{br}}(\theta_1) \partial_{x_1} g_{\theta_1, \theta_2}^{(2),2}(x_1, x_2)]_{(1,2)} \\ = \left[\int d\theta_3 dx_3 V_0 \varphi'(x_1 - x_3) \partial_{\theta_1} g_{\theta_1, \theta_2, \theta_3}^{(3)}(x_1, x_2, x_3) \right]_{(1,2)}. \end{aligned} \quad (72)$$

Similarly, the leading equation for the evolution of $g^{(3)}$ contains a forcing term proportional to the potential. As a consequence, the four nested equations for ρ , $g^{(2),1}$, $g^{(2),2}$, and $g^{(3)}$, which must be integrated to derive the scattering integral, have the interaction potential on the right-hand side, yielding a thermalization rate proportional to $(V_0)^4/\xi^4$, as also demonstrated numerically in the next section. In this case, the picture is simpler (see Fig. 4), the thermalization rate is set first by the formation of two-point functions, namely, $(V_0/\xi)^2$, and then by the formation of three-point functions, necessary for thermalization, giving this way another factor of $(V_0/\xi)^2$ and contributing to the final thermalization rate $(V_0/\xi)^4$.

V. HARD SPHERES WITH LONG-RANGE INTERACTIONS

In order to numerically verify the gBBGKY hierarchy of equations, we apply it to a gas of classical impenetrable hard spheres (Tonks gas) with diameter a [130–133], interacting with the long-range potential of Eq. (1). In particular, we consider the potential

$$V(z) = \frac{V_0}{\xi} \frac{1}{1 + |z/\xi|^3}, \quad (73)$$

which mimics the dipolar interactions emerging in one-dimensional cold atoms, as we shall see in the next section, as well as standard Lennard-Jones interacting potentials [60–64] (especially in the case of $V_0 < 0$). Hard sphere (or rods) gas in one dimension is integrable in the absence of other interactions, namely, in the limit $V_0 = 0$, with a well-known hydrodynamic description [133–136], which shares the same structure as quantum integrable systems, modulus changing the statistical factor [137], related to quasiparticles' statistics $f = 1$, and constant scattering shift $T_{\theta, \theta'} = -a/(2\pi)$. We focus on a homogeneous system with constant density of particles, and we compare gBBGKY prediction with exact microscopic simulations.

We initialize the time-evolution from a finite-temperature state in which half of the particles are boosted with velocity θ_0 and half with velocity $-\theta_0$, $\rho_{\beta; \theta_0} \equiv \sum_{\sigma=\pm} e^{-\beta(\theta - \sigma\theta_0)^2}/Z$, and with $g_{\theta_1, \theta_2}^{(2)}(t=0) = 0$. The state is stationary under the

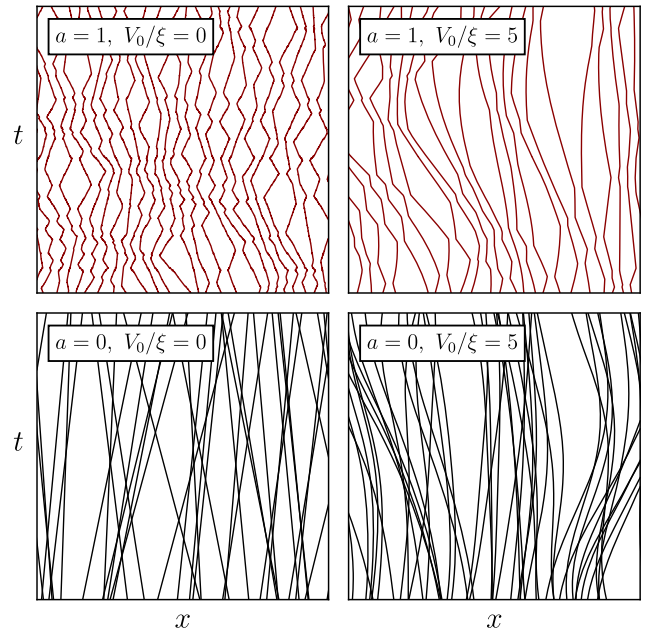


FIG. 5. Examples of microscopic trajectories of hard spheres with long-range interaction given by the potential Eq. (73), both with $a = 0$ (free particles) and finite $a = 1$. In the latter case, we can notice in the trajectories with finite V_0 the interplay of local scatterings and long-range interaction.

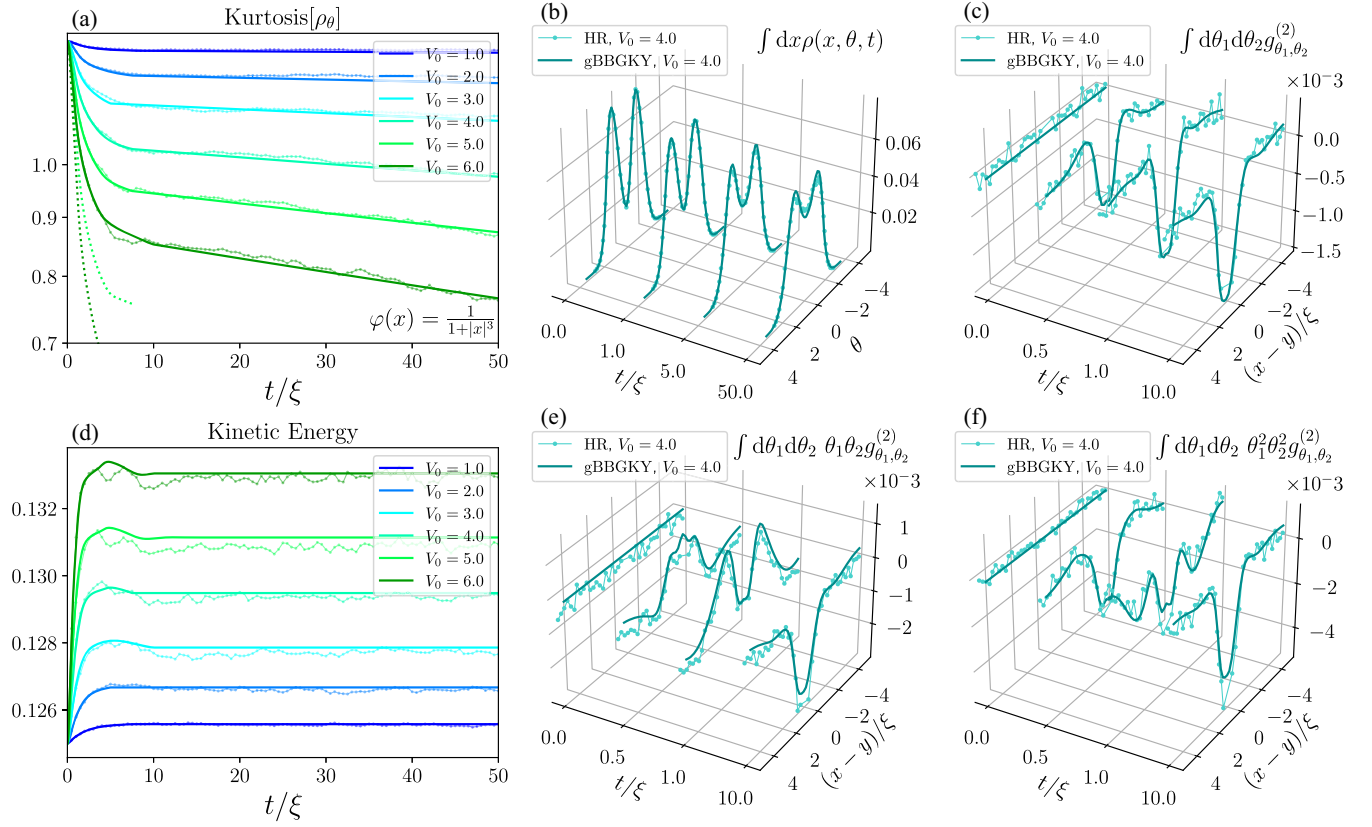


FIG. 6. Numerical results for long-range interacting hard spheres (HR) with $a = 1$, $\xi = 20$, and $\varphi = 1/(1 + |x|^3)$, starting from the two boosted thermal states $\rho_{\beta; \theta_0}$ with $\beta = 2.5$, $\theta_0 = 1$, $\bar{\rho} = 0.2$. The system is homogeneous on a ring. In all the figures, the dots represent the molecular dynamics of the system, while the solid line represent the gBBGKY prediction. (a) The dynamics of kurtosis of rapidity distribution, for $\xi = 20$ and with $V_0 = \{1, 2, \dots, 6\}$, as function of the rescaled time t/ξ . The numerical solution of gBBGKY hierarchy is detailed in Appendix C. In particular, for $t < t_L$ the solid line represents the solution of the two coupled equations, Eqs. (C1) and (C2), while for $t > t_L$ it shows the solution of the generalized Landau equation, Eq. (66). We observe a convincing agreement between molecular dynamics and theoretical predictions for all the values of integrability-breaking potential V_0 . The additional dashed lines represent solutions of the standard BBGKY hierarchy, i.e., gBBGKY with $a = 0$, truncated to two first levels. As expected, this purely perturbative theory does not predict the correct system's evolution. Similarly, (d) shows the dynamics of kinetic energy $\int d\theta \rho_\theta^2/2$. (b) The evolutions of distribution of rapidity ρ_θ for different times. (c),(e),(f) The evolution two-point correlation function $\int d\theta_1 d\theta_2 \theta_1^\alpha \theta_2^\alpha g_{\theta_1, \theta_2}^{(2)}$, respectively, with $\alpha = \{0, 1, 2\}$. As expected, the gBBGKY theory is able to predict the evolution of both one- and two-point functions.

unperturbed Hamiltonian $\hat{H}_a^{\text{int}}$, but it evolves nontrivially as long as V_0 is finite. As a probe of thermalization dynamics, we compute the kurtosis of the (normalized) quasiparticle distribution and the two-point correlations. The kurtosis is expected to be zero in the thermal state, where quasiparticle distribution is Gaussian.

Solving the set of gBBGKY equations, Eqs. (47)–(49), is numerically too expensive in terms of computational time and memory usage. In fact, it requires one to save and perform operations on the three-point function $g_{\theta_1, \theta_2, \theta_3}^{(3)}(x_1 - x_2, x_2 - x_3)$, being, even in a homogeneous system, a five-order tensor. In order to reduce the complexity of the numerical computation, we separately solve the dynamics at small and large times. For small times $t < t_L$, we solve the gBBGKY hierarchy Eqs. (47) and (48) by replacing the contribution proportional to H in Eq. (48)

with the known Kubo diffusion term (explicitly known for integrable systems), as given by Eq. (50), and neglecting the remaining contribution from $g^{(3)}$. For large times $t \geq t_L$, we evolve the system by solving the Landau equation, Eq. (66), using as initial state the gBBGKY solution obtained at time $t = t_L$. See Appendix C for additional information about the numerical solution of the gBBGKY hierarchy.

In Fig. 6, we show how gBBGKY theory reproduces the full molecular dynamics, predicting the evolution of kurtosis as well as the evolution of the two-point function. We can observe two regimes in the quasiparticle evolution: a fast evolution at short times $t < t_L$, driven by the change in the interaction energy due to the buildup of the density-density correlations in the system, and the kinetic regime $t > t_L$, where the system slowly evolves toward the thermal

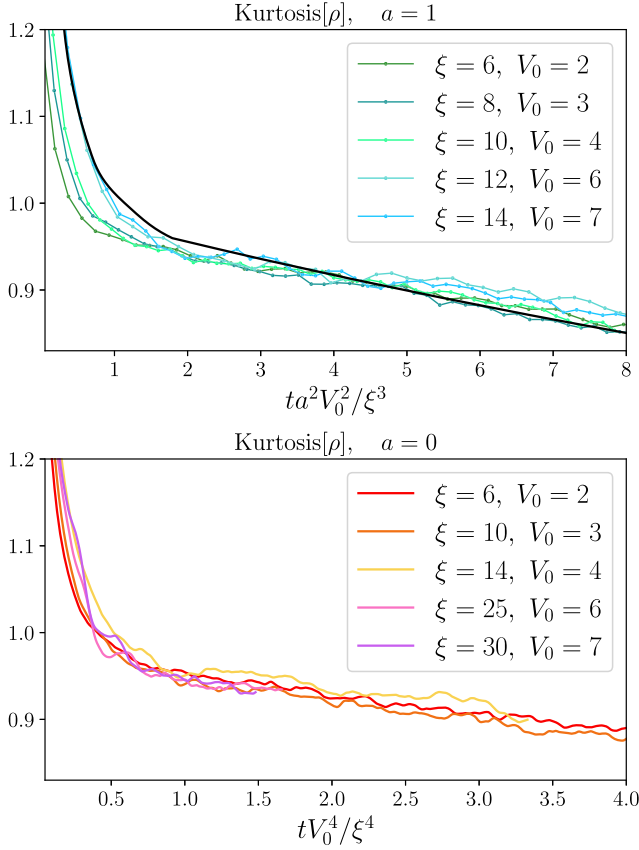


FIG. 7. Numerical confirmation of the predicted thermalization rate $\propto a^2 V_0^2 / \xi^3$, when $a \neq 0$ and $\propto V_0^4 / \xi^4$, when $a = 0$. The two plots show the kurtosis of ρ_θ for different values of V_0, ξ , in the case $a = 1$ (top) and $a = 0$ (bottom), respectively, as function of the rescaled times $\bar{t} = ta^2 V_0^2 / \xi^3$ and $\bar{t} = tV_0^4 / \xi^4$. The dynamics first undergoes a fast small-time evolution and then, for $\bar{t} > 1$, the late-time thermalization dynamics. We show that, under this appropriate time scaling, at late time the dynamics induced by different potential parameters perfectly collapse, confirming that the thermalization rate is $\propto a^2 V_0^2 / \xi^3$ for $a \neq 0$ and $\propto V_0^4 / \xi^4$ for $a = 0$. More precisely, in this figure we introduce a constant offset parameter that takes into account the differences in the small-time behavior, in order to renormalize the starting point for the late-time dynamics. The black curve represents the solution of gBBGKY equations for $\xi = 14, V_0 = 7$, being in good agreement with the numerical simulation.

state. In this regime, the kinetic energy is mostly constant and the dynamics is well described by the Landau equation, which well describes the (generalized) thermalization with rate $(aV_0)^2 / \xi^2$, see Fig. 7, as predicted in the previous section.

We can observe the striking difference between the dynamics of point particles $a = 0$ and the ones with finite a already from a single trajectory snapshot, see Fig. 5: The dynamics with $a = 0$ and V_0 finite is given by smoothly long-range interacting particles, while the one with finite a is given both by long-range interactions and local scatterings, even at long times, where the thermalization mechanisms start to kick in. The difference between these two

cases is also highlighted in Fig. 6(a), where we show how the solution of the gBBGKY equations with $a = 0$ (dashed lines), i.e., BBGKY equations, is far from the hard-sphere molecular dynamics from any arbitrary macroscopic small time. Interestingly, the short-time dynamics with $a = 0$ appears to be faster, compared to the interacting case with $a = 1$, namely, in the presence of contact interaction, the system appears to be jammed. However, at long times, the scaling of the thermalization rates in the two cases shows the opposite behavior: V_0^4 / ξ^4 in the free case and $(aV_0)^2 / \xi^2$, resulting in a much faster (generalized) thermalization in the interacting case $a > 0$.

VI. APPLICATION TO DIPOLAR ONE-DIMENSIONAL QUANTUM GASES

Integrability breaking due to additional long-range interactions arises naturally in quantum gases of atoms or molecules with large magnetic moments [84]. In these systems, the basic Hamiltonian is given by Eq. (2). When confined to one dimension, the van der Waals interaction can be modeled effectively by a contact potential [138], and the gas is described by the integrable Lieb-Liniger (LL) Hamiltonian [27,28],

$$H_{LL} = -\frac{\hbar^2}{2m} \sum_i \partial_{x_i}^2 + c \sum_{i < j} \delta(x_i - x_j), \quad (74)$$

where m is the mass of the bosons and c is the interaction strength.

If the atoms carry a nonzero magnetic moment, they also interact via a dipolar potential, which decays more slowly in space and must be treated separately. In the geometry considered by us (see Fig. 8), dipolar interactions couple both the atoms inside a tube (intratube interactions) as well as atoms in neighboring tubes (intertube interactions). We discuss first the intratube potential. Upon integrating out the two “frozen” transverse degrees of freedom, the effective one-dimensional dipolar coupling becomes [139,140]

$$V_{DDI}^{1D}(z) = V(\theta_{DDI}) \left[v_{DDI}^{1D}(u) - \frac{8}{3} \delta(u) \right], \quad u = \frac{z}{l_\perp}, \quad (75)$$

where $l_\perp = \sqrt{\hbar / (m\omega_\perp)}$ is the transverse oscillator length. The prefactor is

$$V(\theta_{DDI}) = \frac{\mu_0 \mu^2}{4\pi} \frac{1 - 3 \cos^2 \theta_{DDI}}{4l_\perp^3}, \quad (76)$$

and the dimensionless shape function is

$$v_{DDI}^{1D}(u) = -2|u| + \sqrt{2\pi}(1 + u^2)e^{u^2/2} \text{erfc}(|u|/\sqrt{2}). \quad (77)$$

Here, μ is the dipole moment, and θ_{DDI} is the angle between the dipoles (aligned by an external field) and the tube axis; by

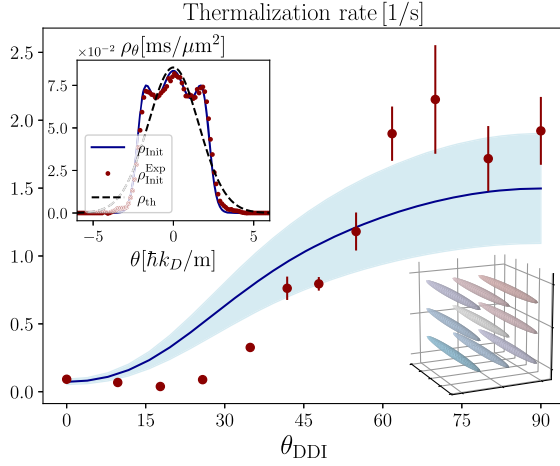


FIG. 8. Thermalization rate (blue solid line) as defined in Eq. (84) for the experimental parameters of Ref. [58], obtained from gBBGKY, together with shaded area indicating uncertainty resulting from 13% density variation in the experiment. The bottom right inset shows the geometry of the quasi-one-dimensional tubes loaded with dipolar atoms. The top left inset shows the initial state ρ_{init} (solid line) compared with the experimentally measured initial state (dots) [determined from the data presented in Figs. 2(c)–2(e) of Ref. [58]] together with the final, thermal state ρ_{th} (dashed line).

tuning θ_{DDI} , one effectively controls the strength of the dipolar interaction.

Following Ref. [58], we consider ^{162}Dy atoms ($\mu = 9.93\mu_B$) loaded into a two-dimensional lattice of one-dimensional tubes (lattice spacing $a_l = 370$ nm), each with average density $\bar{\rho} = 0.8 \mu\text{m}^{-1}$ (see sketch in Fig. 8). Transversal confinement is provided by harmonic trap with oscillator length $l_{\perp} = 57$ nm. In addition to intratube dipolar interactions, atoms in different tubes (i, j) interact with ones in the tube (0,0) via

$$V_{\text{inter}}^{i,j}(x) = \frac{\mu_0 \mu^2}{4\pi} \frac{1 - 3(\hat{r} \cdot \hat{B})^2}{r^3}, \quad (78)$$

where $\vec{r} = (x, ia_l, ja_l)$ and $\hat{B} = (\cos \theta_{\text{DDI}}, 0, \sin \theta_{\text{DDI}})$ denotes the magnetic field direction.

At $t = 0$, a Bragg pulse splits the atoms into two momentum components $\pm 2\hbar k_D$, with $k_D = 2\pi/\lambda$ and $\lambda = 741$ nm. The gas then prethermalizes to a state described by a reference density $\rho_{\text{init}}(\theta)$ (Fig. 8), which closely matches the experimental observations at the end of the first stage of the dynamics.

Although the dipolar potential decays as $V_{\text{DDI}}^{\text{ID}}(x) \sim x^{-3}$ at large distances, it also contains a short-range component. To separate the integrable hard-core part from the genuine long-range contribution, we note that the LL coupling c has two parts: the van der Waals term

$$c_{\text{vdW}} = -\frac{2\hbar^2}{ma_{\text{1D}}}, \quad a_{\text{1D}} = -435 \text{ nm}, \quad (79)$$

and a dipolar term

$$c_{\text{DDI}}(\theta_{\text{DDI}}) = \frac{4V(\theta_{\text{DDI}})}{3}. \quad (80)$$

Hence, $c(\theta_{\text{DDI}}) = c_{\text{vdW}} + c_{\text{DDI}}(\theta_{\text{DDI}})$.

To estimate the effect of the long-range potential, we invoke the characteristic distance ξ . In the LL model, the scattering shift defines an effective hard-sphere diameter for the quasiparticle interaction [19,141]

$$\alpha(\theta - \theta') = \frac{2(c/m)}{(c/\hbar)^2 + (\theta - \theta')^2}. \quad (81)$$

Estimating $\theta - \theta' \approx \hbar k_D/m$ (reflecting the initial momentum spread) and comparing with the dipolar length scales yields

$$\xi_{\text{intra}} = \frac{l_{\perp}}{a} = 0.81, \quad \xi_{\text{inter}} = \frac{a_l}{a} = 5.28. \quad (82)$$

Thus, intratube dipolar interactions can be absorbed into the contact coupling, while intertube interactions act as a truly long-range perturbation. This observation is consistent with the modeling of recent experimental results in Refs. [23,93] where the authors show that intratube dipolar effects mostly renormalize the coupling c at equilibrium [23] and during dynamics, slightly in contrast with the original interpretation in Ref. [58]. Moreover, we observe that summing the contributions to the Landau collision integral from tubes with indices $-2 \leq i, j \leq 2$ (we assume that each tube is in the same state) suffices to capture the relaxation rate, since more distant tubes have negligible effect.

To model the two-stage dynamics observed under the Bragg-split protocol (cf. the hard-rod gas), we adapt our theory to the experimental parameters. The key observable in Ref. [58] is the thermalization rate as a function of θ_{DDI} . We consider a measure of distance to (generalized) thermalization $\text{DT}(t)$ very similar to the one studied in Ref. [58]. We fit a Gaussian distribution $\tilde{\rho}_{\theta}(t)$ to the state $\rho_{\theta}(t)$ at each time t , then $\text{DT}(t)$ is computed as

$$\text{DT}(t) = \int d\theta [\rho_{\theta}(t) - \tilde{\rho}_{\theta}(t)]^2, \quad (83)$$

measuring the distance of the state at time t from a Gaussian distribution. After the first phase of dynamics, an exponential decay is observed, with

$$\log \text{DT}(t) \approx -t/t_{\text{therm}} + \dots \quad (84)$$

We compute t_{therm} from the expression of the Landau collision integral, giving a very good agreement with the experimental rates, see Fig. 8, providing this way a first direct application of gBBGKY framework to an

experimental setting. We should stress that, in our approach, we track the Gaussianity of the rapidity distribution ρ_θ , whereas in Ref. [58] the momentum distribution is measured. However, these two are expected to be close due to high temperature of the state. This is especially clear in the final state ρ_{th} , see inset of Fig. 8, which is very well approximated by a Gaussian, just like final momentum distributions measured in Ref. [58].

What is more, our result can also be used to deduct important aspects of the thermalization rates. Indeed, a more conventional theoretical approach would linearize the collision integral around the thermal distribution ρ_{th} and study spectrum of the linearized collision operator. Equivalently, one can track perturbations in the conserved charges δq_n ,

$$\partial_t \langle \delta q_i \rangle = -\Gamma_i^j \langle \delta q_j \rangle, \quad (85)$$

where matrix Γ is symmetric, positive semidefinite, and has three zero modes (particle number, momentum, and energy). See Supplemental Material [114] for the explicit expression and the derivation of matrix Γ . The first nonzero eigenvalue γ_{gap} sets $t_{\text{therm}} = \gamma_{\text{gap}}^{-1}$. However, this approach predicts thermalization times of order 10^3 s—far longer than experimentally accessible—as it captures the slowest decaying mode, whereas the current state-of-the-art experiments can probe only the fastest relaxation channels.

A. Fermi's golden rule approach and check of the generalized Landau equation

The advantage of restricting to this case of perturbed Lieb-Liniger gas also offers us the possibility of giving a nontrivial check of the generalized Landau equation, Eq. (66), and therefore of our gBBGKY approach via a different approach. Indeed, the dynamics of a quantum nearly integrable system can be also addressed with the Fermi's golden rule (FGR), which can be used to write the Boltzmann equation [47,48,142]. In this section, we give a short review of these results and compare them to the generalized Landau equation obtained from our hierarchy.

The FGR approach relies heavily on the knowledge of form factors, which are the matrix elements of the perturbing operator computed for integrable theory. In the case of integrability-breaking perturbation in the form of additional long-range interactions, the relevant form factors are one- and two-point functions of the density operator. In general, these objects are not known, however, in the limit of small excitations between the states, there are exact results in the LL model [143–147]. As we will see, the small excitation limit will be automatically guaranteed by the long-range limit of the integrability-breaking potential, rendering the FGR Boltzmann kernel computable in the regime of interest of the present work.

Once again we assume that the Hamiltonian is $\hat{H} = \hat{H}_{\text{LL}} + \hat{V}$ where $\hat{V} = \int dx \hat{v}(x)$ is the perturbation,

$$\hat{v}(x) = \frac{1}{2} \int dy V(x-y) \hat{q}_0(x) \hat{q}_0(y). \quad (86)$$

The LL model has fermionic statistics and we define density of holes $\rho^h = \rho^t - \rho$. In the homogeneous system, the dynamics is given by the Boltzmann-like equation [47,48]

$$\begin{aligned} \partial_t \rho_\theta &= \mathcal{I}[\rho](\theta) \\ &= \int d\mathbf{p} d\mathbf{h} \mathbb{F}_{\theta, \mathbf{p}} B(\mathbf{p} \rightarrow \mathbf{h}) [\rho_{\mathbf{p}}^h \rho_{\mathbf{h}} - \rho_{\mathbf{h}}^h \rho_{\mathbf{p}}], \end{aligned} \quad (87)$$

with $\rho^h(\mathbf{p}) = \prod_{p \in \mathbf{p}} \rho_p^h$, $\rho(\mathbf{h}) = \prod_{h \in \mathbf{h}} \rho_h$, and the measure involving different particle-hole scattering processes reads

$$d\mathbf{p} d\mathbf{h} = \sum_{N=0}^{\infty} \frac{1}{(N!)^2} \prod_{i=1}^N dp_i dh_i. \quad (88)$$

Apart from the density factors, there are two ingredients in Eq. (87). First, function $B(\mathbf{p} \rightarrow \mathbf{h})$ is related to particle-hole form factors of the perturbing operator $\langle \rho | v | \rho; \mathbf{p}, \mathbf{h} \rangle$ [148],

$$B(\mathbf{p} \rightarrow \mathbf{h}) = (2\pi)^2 \delta(k) \delta(\varepsilon) |\langle \rho | v | \rho; \mathbf{p}, \mathbf{h} \rangle|^2. \quad (89)$$

Assuming particle-hole symmetry, we have $B(\mathbf{p} \rightarrow \mathbf{h}) = B(\mathbf{h} \rightarrow \mathbf{p})$. The Dressed energy and momentum are

$$\begin{aligned} k_\theta &= \theta - \int d\mu n_\mu F_{\mu, \theta}, \\ \varepsilon_\theta &= \frac{\theta^2}{2} - \int d\mu \mu n_\mu F_{\mu, \theta}, \end{aligned} \quad (90)$$

where $F_{\mu, \theta}$ is the backflow function [149] of the underlying integrable model. Explicitly, the terms inside Dirac δ functions are then $k = \sum_{p \in \mathbf{p}} k_p - \sum_{h \in \mathbf{h}} k_h$ and $\varepsilon = \sum_{p \in \mathbf{p}} \varepsilon_p - \sum_{h \in \mathbf{h}} \varepsilon_h$. For later use, we note that $k'_\lambda = 2\pi \rho'_\lambda$ and $v_\lambda = \varepsilon'_\lambda / k'_\lambda$. The second ingredient in Eq. (87) is the backflow effect implemented by the term $\mathbb{F}_{\theta, \mathbf{p}}$, which reads

$$\mathbb{F}_{\theta, \mathbf{p}} = \sum_{p \in \mathbf{p}} \mathbb{F}_{\theta, p}, \quad \mathbb{F}_{\theta, p} = \delta_{\theta, p} + \partial_\theta (n_\theta F_{\theta, p}). \quad (91)$$

The main practical difficulty for evaluating the collision term is the resulting infinite sum over different numbers of particle-hole excitations. Indeed, writing the sum in integration measure explicitly, we have

$$\mathcal{I}[\rho](\theta) = \mathbb{F}_{\theta, \nu} \sum_{m=1}^{\infty} \mathcal{I}_0^{(m)}[\rho](\nu), \quad (92)$$

where we have factored out the dressing operator $\mathbb{F}$ defining bare collision integrals $\mathcal{I}_0^{(m)}$. Because of this structure, an additional small parameter is needed to effectively truncate the sum. Such situation will happen in our case of long-range interactions, where the potential allows only for small momentum transfer, which is equivalent to considering ξ as a large parameter. With these ingredients allows us to truncate the sum to the leading term in ξ , which corresponds to three-particle-hole ($m = 3$) excitations. The $m = 2$ term is zero, due to the kinetic blocking effect described in Sec. IV A.

From now on, we will assume that all transferred momenta $p_i - h_i$ are small (this approximation is discussed in Supplemental Material [114]). We therefore introduce new variables $\lambda_i = (p_i + h_i)/2$ and $q_i = k(p_i) - k(h_i)$ and obtain the $m = 3$ collision integral in the form

$$\mathcal{I}_0^{(3)}[\rho](\theta) = \frac{1}{3!} \frac{1}{(2\pi)^3} \partial_\theta \int d\lambda d\mathbf{q} B(\lambda, \mathbf{q}) \times \left(\sum_{i=1}^3 \frac{q_i}{2k'_{\lambda_i}} \delta(\theta - \lambda_i) \right) \left(\sum_{i=1}^3 q_i \frac{\varepsilon'_{\lambda_i}}{k'_{\lambda_i}} \right), \quad (93)$$

with integration measure $d\lambda d\mathbf{q} = \prod_{i=1}^3 d\lambda_i dq_i \rho_{\lambda_i} f_{\lambda_i}$ where for fermionic statistics $f = 1 - n$. Moreover, in that limit, the backflow operator becomes $\mathbb{F}_{\theta,p} = [q/(2k'_\lambda)] \partial_\theta R_{\theta,\lambda}^{-1}$ and implements the usual dressing procedure. To move forward, we have to analyze now $m = 3$ form factors of the density-density operator. Consider form-factor of bilocal operator $\langle \rho | \hat{q}_0(x) \hat{q}_0(0) | \rho; \mathbf{p}, \mathbf{h} \rangle$. We plug the resolution of the identity between the two operators to obtain

$$\begin{aligned} & \langle \rho | \hat{q}_0(x) \hat{q}_0(0) | \rho; \mathbf{p}, \mathbf{h} \rangle \\ &= \sum_{\alpha} \langle \rho | \hat{q}_0(x) | \rho; \alpha \rangle \langle \rho; \alpha | \hat{q}_0(0) | \rho; \mathbf{p}, \mathbf{h} \rangle. \end{aligned} \quad (94)$$

The leading contribution to the composite form factor comes then from bipartite partitions of the three-particle-hole excitation $(\mathbf{p}, \mathbf{h})$. More explicitly, for $(\mathbf{p}, \mathbf{h}) = (p_1, p_2, p_3, h_1, h_2, h_3)$, there are six relevant partitions. Three of them are one-particle-hole excitations: $\alpha = (p_1, h_1)$, $\alpha = (p_2, h_2)$, and $\alpha = (p_3, h_3)$. The other three are two-particle-hole excitations $\alpha = (p_1, p_2, h_1, h_2)$, $\alpha = (p_2, p_3, h_2, h_3)$, and $\alpha = (p_1, p_3, h_1, h_3)$. Using reparametrization invariance of the thermodynamic form factors [150], a property that an excitation common to the bra and ket can be absorbed into the thermodynamic state ρ , the contributions from one- and two-particle-hole excitations are pairwise identical up to the x -dependent term. The expressions for the form factors (assuming small momentum transfer) are [143,145]

$$\langle \rho | \hat{q}_0(0) | \rho; p_1, h_1 \rangle = k'_{\lambda_1}, \quad (95)$$

$$\langle \rho | \hat{q}_0(0) | \rho; p_1, p_2, h_1, h_2 \rangle = 2\pi T_{\lambda_1, \lambda_2}^{\text{dr}} \frac{(q_1 + q_2)^2}{q_1 q_2}, \quad (96)$$

and lead to six relevant contributions that can be written as follows:

$$\langle \rho | \hat{v} | \rho; \mathbf{p}, \mathbf{h} \rangle = (2\pi)^2 \left(\hat{V}(q_3) T_{\lambda_1, \lambda_2}^{\text{dr}} k'_{\lambda_3} \frac{(q_1 + q_2)^2}{q_1 q_2} \right)_{(1,2,3)}. \quad (97)$$

The factor B appearing in the collision integral is then

$$B(\lambda, \mathbf{q}) = (2\pi)^4 \delta(q) \delta(vq) \left[\left(T_{\lambda_1, \lambda_2}^{\text{dr}} k'_{\lambda_3} \hat{V}(q_3) \frac{q_3^2}{q_1 q_2} \right)_{(1,2,3)} \right]^2, \quad (98)$$

where we defined $\delta(q) = \delta(q_1 + q_2 + q_3)$, $\delta(vq) = \delta(v_1 q_1 + v_2 q_2 + v_3 q_3)$, and used that $\hat{V}(k) = \hat{V}(-k)$. Crucially, expanding the square we find two types of contributions, which correspond one-to-one to the two contributions in Eq. (66). The cross ones are proportional to $(\hat{V}(q_i))^2$, whereas the self terms involve $\hat{V}(q_i) \hat{V}(q_{j \neq i})$ factors. The collision integral has consequently two contributions, which turn out to be exactly equal to the two contributions of Eqs. (67) and (68), therefore giving an independent, and mathematically highly nontrivial, verification of the gBBGKY approach.

VII. CONCLUSIONS

In this work, we have extended the celebrated BBGKY hierarchy to systems with integrable contact interactions. We derived the hierarchy from a correlated fluid-cell ensemble, which enables accurate characterization of multipoint correlations both in the standard BBGKY framework (where the unperturbed Hamiltonian is free) and in more general settings.

For classical hard spheres with long-range interactions, we tested our theory against molecular-dynamics simulations and observed excellent agreement, even at relatively strong coupling. For quantum gases described by the Lieb-Liniger model with long-range interactions, we showed that the generalized Landau equation aligns with the Fermi golden rule approach, providing a rigorous validation of both our theory and the thermalization hypothesis. Finally, we established a theoretical foundation for the experimental results of Ref. [58], demonstrating that dynamics in these quasi-one-dimensional systems are encompassed by

gBBGKY hierarchy. The latter, in the presence of strong contact interactions, accurately captures short-time dynamics during the prethermal regime, driven by the rapid buildup of correlations that our formalism tracks precisely. Conventional kinetic equations fail to describe this transient phase, which is often the only accessible regime in experiments and simulations. Thus, the gBBGKY provides both a conceptual framework and quantitative predictions for prethermal behavior.

At late times, gBBGKY approach overcomes the well-known kinetic blocking in one dimension. For perturbed free systems, thermalization is driven by three-body scatterings generated by the external potential V , yielding a rate that scales as V_0^4 . By contrast, with contact interactions, the combined effect of long-range interactions and convective diffusion among particles produces a thermalization rate of order $(Va)^2$. Importantly, such thermalization is not complete (generalized thermalization) as only one- and two-point functions reach their canonical thermal values. Higher-order correlations remain far from thermal over the same window of times and retain their integrable structure, leading to finite long-range correlations that would be absent in a thermal Gibbs ensemble but which are instead correctly captured by the correlated fluid-cell ensemble.

Our method is broadly applicable: Its central ingredient is a thermodynamic description of the unperturbed integrable model, yielding a universal framework for diverse classical and quantum systems. Unlike approaches based on Fermi's golden rule and explicit matrix elements, the gBBGKY hierarchy enables direct access to the Boltzmann scattering integrals and the time evolution of multipoint correlations in a wide range of perturbed integrable models. It can be applied to lattice systems—such as long-range spin chains and fermionic models [89,151–154]—inspired by recent experiments. Additionally, it opens a path to connect kinetic theories of interacting waves (via wave turbulence) [59,155] with general nonlinear wave systems [156–158]. We expect that the gBBGKY hierarchy will be an important tool in further studies of thermalization and prethermalization across a wide range of physical settings, from low-dimensional cold-atom experiments to complex condensed-matter and optical systems with tunable interactions.

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DATA AVAILABILITY

The data are available upon reasonable request from the authors.

APPENDIX A: CORRELATION FUNCTIONS IN THE CORRELATED FLUID-CELL ENSEMBLE

In this appendix, we will derive the formulas for correlation functions in the ensemble Eq. (14). We will also connect the various ingredients of these correlators to Lagrange multipliers of the ensemble. The main idea behind this calculation consists of expanding the measure (14) in $1/\xi$ [according to Eq. (15)] in order to compute the corrections to the local GGE prediction. First, we consider the equal-time two-point function

$$\begin{aligned} \langle q_i(x) q_j(x') \rangle^c &= \delta(x - x') C_{i,j} \\ &+ \int dy dy' \beta_{(2)}^{k,l}(y, y') \langle \delta q_k(y) \delta q_l(y') q_i(x) q_j(x') \rangle_1. \end{aligned} \quad (\text{A1})$$

In order to write the correlator in a cumulant expansion, we now use the relation

$$\begin{aligned} \langle \delta a \delta b \delta c \delta d \rangle &= \langle abcd \rangle^c + \langle ac \rangle^c \langle bd \rangle^c \\ &+ \langle ad \rangle^c \langle bc \rangle^c + \langle acd \rangle^c \langle \delta b \rangle + \langle bcd \rangle^c \langle \delta a \rangle, \end{aligned} \quad (\text{A2})$$

where we observe that $\langle \delta q_i \rangle = 0$ and that $\langle abcd \rangle^c$ is higher order in ξ^{-1} . Hence, Eq. (A1) reduces to

$$\begin{aligned} &= \delta(x - x') C_{i,j} \\ &+ \int dy dy' \beta_{(2)}^{k,l}(y, y') \langle q_k(y) q_i(x) \rangle_1^c \langle q_l(y') q_j(x') \rangle_1^c \\ &= \delta(x - x') C_{i,j} + \beta_{(2)}^{k,l}(x, x') C_{i,k}(x) C_{j,l}(x') \\ &= \delta(x - x') C_{i,j} + g_{i,j}^{(2)}(x, x'), \end{aligned} \quad (\text{A3})$$

where in the last line we identified $g_{i,j}^{(2)}(x, x') \equiv 2\beta_{(2)}^{k,l}(x, x') C_{i,k}(x) C_{j,l}(x') + O(\xi^{-2})$. Also, in the last two lines, we explicitly computed the correlation functions on the local GGE.

We now compute the three-point function using the same technique,

$$\begin{aligned}
\langle q_i(x)q_j(x')q_k(x'') \rangle^c &= \delta(x-x')\delta(x-x'')C_{i,j,k}(x) + \int dy dy' \beta_{(2)}^{m,n}(y,y') \langle \delta q_m(y) \delta q_n(y') q_i(x) q_j(x') q_k(x'') \rangle_1 \\
&\quad + \int dy dy' dy'' \beta_{(3)}^{m,n,p}(y,y',y'') \langle \delta q_m(y) \delta q_n(y') \delta q_p(y'') q_i(x) q_j(x') q_k(x'') \rangle_1 \\
&= \delta(x-x')\delta(x-x'')C_{i,j,k}(x) + 2[\delta(x'-x'')\beta_{(2)}^{m,n}(x,x')C_{m,i}(x)C_{n,j,k}(x')]_{(i,j,k)} \\
&\quad + 3\beta_{(3)}^{m,n,p}(x,x',x'')C_{i,m}(x)C_{j,n}(x')C_{k,p}(x'') \\
&= \delta(x-x')\delta(x-x'')C_{i,j,k}(x) + [\delta(x'-x'')g_{i,q}^{(2)}(x,x')C_{q,n}(x')C_{n,j,k}(x')]_{(i,j,k)} + g_{i,j,k}^{(3)}(x,x',x''). \quad (\text{A4})
\end{aligned}$$

In particular, in the last line we defined $g_{i,j,k}^{(3)}(x,x',x'') = 3\beta_{(3)}^{m,n,p}(x,x',x'')C_{i,m}(x)C_{j,n}(x')C_{k,p}(x'')$, and we used the formula for $g^{(2)}$ derived in Eq. (A3). Finally, we observe that the last line coincides with Eq. (20).

APPENDIX B: gBBGKY HIERARCHY AND GENERALIZED LANDAU EQUATION FOR LONG-RANGE INTERACTING HARD SPHERES

In this appendix, we explicitly write the main hydrodynamic functions for the hard-sphere model that permits us to explicitly write the gBBGKY hierarchy and the generalized Landau equation in this simpler case. The hard-sphere gas is a classical system of billiard balls with radius a and unitary mass. The rapidities of the particles in the system are defined as the set of particle velocities. The scattering shift is simply $T_{\theta_1,\theta_2} = -a/2\pi$. Hence, the rotation matrix is given by $R_{\theta_1,\theta_2} = \delta_{\theta_1,\theta_2} + a/2\pi 1_{\theta_1} n_{\theta_2}$, giving $\rho^i = (1 + a\bar{\rho})/2\pi$ and $n_\theta = 2\pi\rho_\theta/(1 - a\bar{\rho})$, being $\bar{\rho} = \int d\theta \rho_\theta$. Since in this model the rotation matrix can be explicitly inverted, the dressing operator can be computed explicitly $R_{\theta_1,\theta_2}^{-1} = \delta_{\theta_1,\theta_2} - a 1_{\theta_1} \rho_{\theta_2}$, as well as the effective velocity

$$v_{\theta_1}^{\text{eff}} = \frac{\theta_1 - a \int d\theta_2 \theta_2 \rho_{\theta_2}}{1 - a\bar{\rho}}. \quad (\text{B1})$$

The susceptibility matrix for this system is given by

$$C_{\theta_1,\theta_2} = \delta_{\theta_1,\theta_2} \rho_{\theta_1} + a\bar{\rho}(a\bar{\rho} - 2)\rho_{\theta_1}\rho_{\theta_2}. \quad (\text{B2})$$

Hence, we can also compute its functional derivative with respect to the density

$$\begin{aligned}
C_{\theta_1,\theta_2}^{\theta_3} &= \delta_{\theta_1,\theta_3} \delta_{\theta_2,\theta_3} + a^2 \rho_{\theta_1} \rho_{\theta_2} 1_{\theta_3} \\
&\quad + a(a\bar{\rho} - 2)\rho_{\theta_2} \delta_{\theta_1,\theta_3} + a(a\bar{\rho} - 2)\rho_{\theta_1} \delta_{\theta_2,\theta_3}. \quad (\text{B3})
\end{aligned}$$

This tensor is entering in the gBBGKY hierarchy as

$$\begin{aligned}
&\int d\theta_3 C_{\theta_2,\theta_3}^{\theta_1} g_{\theta_1,\theta_3}^{(2)}(x_1, x_2) \\
&= (1 - a\bar{\rho})^2 g_{\theta_1,\theta_2}^{(2)}(x_1, x_2) \\
&\quad - 2a(1 - a\bar{\rho})\rho_{\theta_1} \int d\gamma g_{\theta_3,\theta_2}^{(2)}(x_1, x_2). \quad (\text{B4})
\end{aligned}$$

The flux Jacobian $A_{\theta_1}^{\theta_2}$ can be easily computed using the explicit relations for the dressing operator, the rotation matrix, and the effective velocity. We also define $\mathfrak{D}$ as the linear response diffusion matrix of the system, that for hard spheres reads

$$\mathfrak{D}_{\theta_1}^{\theta_2} = a^2(1 - a\bar{\rho}) \left[\delta_{\theta_1,\theta_2} \int d\theta_3 \rho_{\theta_3} |v_{\theta_1,\theta_3}| - \rho_{\theta_1} |v_{\theta_1,\theta_2}| \right]. \quad (\text{B5})$$

Finally, we can explicitly write the generalized Landau equation for the hard-sphere model. For simplicity, we write it considering systems with vanishing total momentum $\int d\theta_2 \theta_2 \rho_{\theta_2} = 0$,

$$\begin{aligned}
\mathcal{I}_{\theta_1}^{\text{cross}} &= \frac{1}{2\pi} a^2 (1 - a\bar{\rho})^3 \int dk k^2 [\hat{V}(k)]^2 \\
&\quad \times \int d\theta_2 d\theta_3 \left(\frac{|v_{\theta_3,\theta_1}^{\text{br}}|}{v_{\theta_2,\theta_3}^{\text{br}} (v_{\theta_1,\theta_2}^{\text{br}})^2} - \frac{|v_{\theta_2,\theta_3}^{\text{br}}|}{v_{\theta_1,\theta_2}^{\text{br}} (v_{\theta_3,\theta_1}^{\text{br}})^2} \right) \\
&\quad \times \mathcal{B}_{\theta_1,\theta_2,\theta_3}^{\text{br}}, \quad (\text{B6})
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{\theta_1}^{\text{self}} &= \frac{1}{2\pi} a^2 (1 - a\bar{\rho})^3 \int d\theta_3 d\theta_2 \int dk k^2 \hat{V}(v_{\theta_1,\theta_3}^{\text{br}} k) \\
&\quad \times \left(\hat{V}(v_{\theta_1,\theta_2}^{\text{br}} k) \frac{v_{\theta_3,\theta_1}^{\text{br}} v_{\theta_1,\theta_2}^{\text{br}}}{v_{\theta_3,\theta_2}^{\text{br}}} - 2\hat{V}(v_{\theta_2,\theta_3}^{\text{br}} k) \frac{(v_{\theta_2,\theta_3}^{\text{br}})^2 v_{\theta_3,\theta_1}}{(v_{\theta_1,\theta_2}^{\text{br}})^2} \right) \\
&\quad \times \mathcal{B}_{\theta_1,\theta_2,\theta_3}^{\text{br}}, \quad (\text{B7})
\end{aligned}$$

where we defined the bare velocity difference as $v_{\theta_1,\theta_2}^{\text{br}} = \theta_1 - \theta_2$ and where

$$\begin{aligned} \mathcal{B}_{\theta_1, \theta_2, \theta_3}^{\text{br}} &\equiv v_{\theta_3, \theta_2}^{\text{br}} \partial_{\theta_1} \rho_{\theta_1} \rho_{\theta_3} \rho_{\theta_2} \\ &+ v_{\theta_2, \theta_1}^{\text{br}} \rho_{\theta_1} \partial_{\theta_3} \rho_{\theta_3} \rho_{\theta_2} + v_{\theta_1, \theta_3}^{\text{br}} \rho_{\theta_1} \rho_{\theta_3} \partial_{\theta_2} \rho_{\theta_2}. \end{aligned} \quad (\text{B8})$$

APPENDIX C: ADDITIONAL DETAILS ON THE NUMERICAL SOLUTION OF gBBGKY HIERARCHY

In this appendix, we detail the numerical methods used to produce the results shown in Fig. 6. Let us start from the microscopic simulation of the hard-sphere system with long-range interactions. The molecular dynamics is solved using the algorithm introduced in Ref. [159], combining symplectic integration of equation of motion and the explicit solution of hard-core collisions. More precisely, we proceeded setting a time interval Δt_S for the symplectic integrator and computing the minimal time before the first hard-core interaction in the system $\Delta t_{\text{Int}} \approx \min_i \{(x_{i+1} - x_i) / (\theta_{i+1} - \theta_i)\}$. At any time, if $\Delta t_S > \Delta t_{\text{Int}}$, we perform the

symplectic integration of equation of motion using an explicit Runge-Kutta method of order 8 implemented by the DOP853 method in `scipy.integrate.solve_ivp` function with time step Δt_S . If, instead, $\Delta t_S < \Delta t_{\text{Int}}$, we perform the symplectic integration with time step Δt_{Int} , and then we explicitly compute the hard-core collision between the two colliding rods, i.e., exchanging the colliding rods' velocities. In particular, we choose $\Delta t_S = 5 \times 10^{-3}$. This value guarantees the correct convergence of the numerical method for the set of parameters considered in the simulations. The results are averaged over $N_{\text{sym}} = 1000$ initial configurations. The initial state is a GGE state; see Ref. [141] for a detailed explanation.

We now discuss the solution of gBBGKY equations. In the results shown in Fig. 6, we split the dynamics in two stages, solving the system separately for $t < t_L$ and $t > t_L$. For the small-time dynamics $t < t_L$, we solved the following two gBBGKY equations for a hard-sphere system (see Appendix B for the explicit functions),

$$\partial_t \rho_{\theta_1} = \frac{1}{\xi} \int dx_2 d\theta_2 V'(x_1 - x_2) \partial_{\theta_1} g_{\theta_1, \theta_2}^{(2)}(x_1, x_2), \quad (\text{C1})$$

$$\begin{aligned} &\partial_t g_{\theta_1, \theta_2}^{(2)}(x_1, x_2) + \left[\partial_{x_1} A_{\theta_1}^\gamma g_{\gamma, \theta_2}^{(2)}(x_1, x_2) - \frac{1}{2} \partial_{x_1} \mathfrak{D}_{\theta_1}^\gamma g_{\gamma, \theta_2}^{(2)} \right]_{(1,2)} \\ &= \int d\theta_3 \partial_{\theta_1} \left[\int dx_3 V'(x_1 - x_3) \rho_{\theta_1} g_{\theta_2, \theta_3}^{(2)}(x_2, x_3) + V'(x_1 - x_2) (\rho_{\theta_1} C_{\theta_2, \theta_3} + C_{\theta_2, \theta_3}^\gamma g_{\theta_1, \gamma}^{(2)}(x_1, x_2)) \right]_{(1,2)} \\ &+ \delta(x_1 - x_2) \int dx_3 d\theta_3 V'(x_1 - x_3) \left(-C_{\theta_1, \theta_2}^\gamma \partial_\gamma + (\partial_{\theta_1} + \partial_{\theta_2}) C_{\theta_1, \theta_2}^\gamma \right) g_{\gamma, \theta_3}^{(2)}(x_1, x_3), \end{aligned} \quad (\text{C2})$$

using a fourth-order Runge-Kutta method with time interval equal to 0.02. In particular, we used a grid of 100 points in space and rapidity in the intervals, with $-10\xi < x < 10\xi$ and $-7 < \theta < 7$. The space interval has been chosen large enough to have vanishing correlations at the boundaries at any time, in order to avoid spurious reflective effects in the dynamics of $g^{(2)}$. For $t > t_L$, we instead solved the generalized Landau equation. The precise value of t_L has been considered as a “fitting” parameter, hence fixed in the interval $\xi < t_L < \xi^2$, in order to accurately reproduce the molecular-dynamics results.

For the data presented in Fig. 8, we use grid of 150 points in rapidity space with $-4.4\hbar k_d/m \leq \theta \leq 4.4\hbar k_d/m$. The initial state presented in the inset of Fig. 8 is then evolved for a short time using the expression for the intertube collision integral provided in Supplemental Material [114].

From that dynamics, we extract the rate of decay of $\log DT$.

APPENDIX D: gBBGKY HIERARCHY FOR INHOMOGENEOUS SYSTEMS

In this appendix, we write the first equations of gBBGKY hierarchy for inhomogeneous systems. The derivation follows exactly the same procedure introduced in Sec. III. For brevity reasons, we show here only the first two equations of the hierarchy. Higher-order equations can be derived using the same technique. We first consider the set of generic equations, Eqs. (11) and (12). Hence, the next steps are to use the cumulant expansion for charge-current correlators (see Sec. III B) and to factorize the singular component of few-point functions (see Sec. III A). After these steps, the equations read

$$\partial_t \langle q_{i_1}(x_1) \rangle + \partial_{x_1} \left(A_{i_1}^k \langle q_k(x_1) \rangle + \frac{1}{2} H_{i_1}^{j,k} g_{j,k}^{(2),\text{sym}}(x_1^+, x_1^-) \right) = - \int dx_2 V'(x_1 - x_2) A_{i_1,0}^k (\langle q_k(x_1) \rangle \langle q_0(x_2) \rangle + g_{k,0}^{(2)}(x_1, x_2)), \quad (\text{D1})$$

$$\begin{aligned} & \partial_t g_{i_1, i_2}^{(2)}(x_1, x_2) + \left[\partial_{x_1} A_{i_1}^j g_{j, i_2}^{(2)}(x_1, x_2) + \frac{1}{2} \partial_{x_1} H_{i_1}^{j,k} g_{j,k, i_2}^{(3),\text{sym}}(x_1^+, x_1^-, x_2) \right]_{(1,2)} \\ &= -\delta(x_1 - x_2) \langle q_{i_1} q_{i_2} j_k^- \rangle_1^c C^{kl} \partial_{x_1} \langle q_l(x_1) \rangle - \left[V'(x_1 - x_2) A_{i_1,0}^j \left(\langle q_j(x_1) \rangle C_{i_2,0} + g_{j,k}^{(2)}(x_1, x_2) C^{k,m} C_{m, i_2,0} \right) \right. \\ &+ \left. \int dx_3 V'(x_1 - x_3) A_{i_1,0}^j \left(\langle q_j(x_1) \rangle g_{i_2,0}^{(2)}(x_2, x_3) + \langle q_0(x_3) \rangle g_{j, i_2}^{(2)}(x_1, x_2) + g_{j, i_2,0}^{(3)} \right) \right]_{(1,2)} \\ &- \delta(x_1 - x_2) \int dx_3 V'(x_1 - x_3) \left[-C_{i_1, i_2, j} C^{j,k} A_{k,0}^n + \left(A_{i_1,0}^j \delta_{i_2}^k + \delta_{i_1}^j A_{i_2,0}^k \right) C_{j,k,m} C^{m,n} \right] g_{n,0}^{(2)}(x_1, x_3), \quad (\text{D2}) \end{aligned}$$

with $j_i^- = j_i - A_i^j q_j$. In the last equation, we can observe few important differences with respect to the homogeneous case. First, the dynamical equation for density, Eq. (D1), gets more involved, since now its nontrivial flow of particles is also allowed. It is important to observe that the first term on the rhs is a generalization to the celebrated Vlasov forcing term. We also observe that Eq. (D1) is coupled to two-point functions also through the second term in the current cumulant expansion, in the exact way as for the purely integrable evolution; see the recent results in Ref. [113]. The equation for two-point functions, Eq. (D2), has an additional nontrivial contribution proportional to a Dirac δ function, originating from the complex propagation of initial local fluctuations, also already introduced in Ref. [113].

APPENDIX E: EXTRA DETAILS ON THE DERIVATION OF THE gBBGKY HIERARCHY

In this appendix, we detail the main steps needed in order to derive Eqs. (32)–(34) from the hierarchy Eqs. (11)–(13) written in terms of complete correlation functions. First, we specialize these equations to homogeneous systems, hence Eq. (32) gets particularly simple for q_i being independent on the space coordinate. In addition, the disconnected forcing term is vanishing because of the symmetry of the potential. Similarly, the singular contribution to two-point functions is not entering in the equation since $\varphi'(0) = 0$. Equation (33) describes the evolution of regular connected two-point functions. This equation is derived from Eq. (12) by performing three main steps: expanding currents through hydrodynamic projections, subtracting disconnected components, and finally separating singular and regular parts of the correlators. The most involved part of the calculation is related to the calculation of singular terms on the rhs, i.e., proportional to the Dirac δ . Hence, in this appendix, we describe the main steps of their derivation. First, we compute the contributions coming from the δ part on the lhs of Eq. (12),

$$\begin{aligned} & \delta(x_1 - x_2) \partial_t C_{i_1, i_2} + \left[\partial_{x_1} \delta(x_1 - x_2) A_{i_1}^j C_{j, i_2} \right] \\ &= \delta(x_1 - x_2) \int dx_3 V'(x_1 - x_3) \\ &\quad \times C_{i_1, i_2, j} C^{j,k} A_{k,0}^n g_{n,0}^{(2)}(x_1, x_3), \quad (\text{E1}) \end{aligned}$$

where we used $\delta C_{i_1, i_2} / \langle \delta q_k \rangle_1 = C_{i_1, i_2, j} C^{j,k}$ and where we used Eq. (32). The additional singular contribution to the rhs of Eq. (12) comes from the singular part of the three-point function $\langle j_{i,0}(x) q_j(y) q_0 \rangle^c$ for coinciding x, y . All the remaining δ contributions from the rhs are integrated, i.e., appear as regular terms. We now focus on Eq. (34). The procedure to derive this equation follows the same logic as Eq. (33), but here we simply stop at the leading-order terms in $O(V_0, 1/\xi)$. First, we observe that, in the case of the three-point function, the hydrodynamic projection formula is more involved already for Euler scales as, for any local observable $\hat{o}$, we have [123]

$$\begin{aligned} \langle o(x) q_j(y) q_k(z) \rangle^c &= \frac{\delta \langle o(x) \rangle_1}{\delta \langle q_i(x) \rangle_1} \langle q_i(x) q_j(y) q_k(z) \rangle^c \\ &+ \frac{\delta^2 \langle o(x) \rangle_1}{\delta \langle q_i(x) \rangle_1 \delta \langle q_m(z) \rangle_1} \\ &\times \langle q_m(x) q_k(z) \rangle^c \langle q_i(x) q_j(y) \rangle^c. \quad (\text{E2}) \end{aligned}$$

More precisely, in this equation, the regular part of the second term on the rhs appears as higher order in V_0 with respect to the regular part of the first one, hence we did not explicitly include it in Eq. (34). Then, we proceed to separate the singular part from it. In doing that, we used the following simplifications coming from homogeneity: $\partial_t C_{i,j,k} \sim O(V_0^2)$ and $\partial_x C_{i,j,k} = 0$.

1. Derivation of the equation for the three-point functions

In this section, we derive the rhs of Eq. (49). As a starting point, we consider the rhs of Eq. (34). In particular, it turns out to be writable in a more convenient way as

$$\begin{aligned}
 & A_i^m C_{m,j,p} C^{p,q} - C_{i,j,m} C^{m,p} A_p^q + H_i^{m,q} C_{m,j} \\
 &= \frac{\delta \langle j_i \rangle_1}{\delta \langle q_m \rangle_1} \frac{\delta}{\delta \langle q_q \rangle_1} \frac{\delta \langle q_m \rangle_1}{\delta \beta^j} - \frac{\delta}{\delta \langle q_p \rangle_1} \left[\frac{\delta \langle q_i \rangle_1}{\delta \beta^j} \right] \frac{\delta \langle j_p \rangle_1}{\delta \langle q_q \rangle_1} \\
 &+ \frac{\delta^2 \langle j_i \rangle_1}{\delta \langle q_m \rangle_1 \delta \langle q_q \rangle_1} \frac{\delta \langle q_m \rangle_1}{\delta \beta^j} \\
 &= \frac{\delta^2 \langle j_i \rangle_1}{\delta \beta^j \delta \beta^p} \frac{\delta \beta^p}{\delta \langle q_q \rangle_1} - \frac{\delta}{\delta \langle q_p \rangle_1} \left[\frac{\delta \langle q_i \rangle_1}{\delta \beta^j} \right] \frac{\delta \langle j_p \rangle_1}{\delta \langle q_q \rangle_1} \\
 &= \langle q_i q_j j_p \rangle_1^c C^{p,q} - \langle q_i q_j q_m \rangle_1^c C^{m,p} A_p^q = \langle q_i q_j j_p \rangle_1^c C^{p,q},
 \end{aligned} \tag{E3}$$

where we used that $AC = CA^t$. Here, we defined $j_i^- = j_i - A_i^j q_j$. Hence, as shown in Ref. [141], in the quasiparticle picture it turns out to be equal to

$$\langle q_i q_j j_p \rangle_1^c C^{p,q} = \int d\theta_1 d\theta_2 h_i(\theta_1) h_j(\theta_2) M_{\theta_1, \theta_2}^{\theta_3} h^q(\theta_3), \tag{E4}$$

with the M tensor defined in Eq. (46).

2. Solutions with Green's function method

The aim of this appendix is to justify the solution Eq. (60), which is crucial for derivation of the generalized Landau kinetic equation. Our starting point is the following equation (we suppress upper indices for lighter notation):

$$\partial_t g_{\theta_1, \theta_2}(z) + \partial_z (A_{\theta_1}^\gamma g_{\gamma, \theta_2}(z) - A_{\theta_2}^\gamma g_{\theta_1, \gamma}(z)) = \mathcal{S}_{\theta_1, \theta_2}[\rho](z). \tag{E5}$$

We start with moving to normal modes in order to diagonalize A operators $g_{\theta_1, \theta_2}(z) = R_{\theta_1, \xi_1}^{-t} R_{\theta_2, \xi_2}^{-t} \bar{g}_{\xi_1, \xi_2}(z)$ and similarly for $\mathcal{S}_{\theta_1, \theta_2}[\rho](z)$. We get an equation

$$\partial_t \bar{g}_{\theta_1, \theta_2}(z) + v_{\theta_1, \theta_2} \partial_z \bar{g}_{\theta_1, \theta_2}(z) = \bar{\mathcal{S}}_{\theta_1, \theta_2}[\rho](z), \tag{E6}$$

for which the Green's function can be found with Fourier transform and reads

$$G_{\theta_1, \theta_2}(z, z'; t, t') = \delta(z + z' - v_{\theta_1, \theta_2} t) \theta_H(t - t'), \tag{E7}$$

where θ_H is the Heaviside step function. Hence, we find

$$\begin{aligned}
 \bar{g}_{\theta_1, \theta_2}(z) &= \int dz' \int_0^t d\tau \int \frac{dk}{2\pi} e^{ik(z+z'-v_{\theta_1, \theta_2} \tau)} \\
 &\times \bar{\mathcal{S}}_{\theta_1, \theta_2}[\rho](z', t - \tau).
 \end{aligned} \tag{E8}$$

Finally, we rotate back to $g_{\theta_1, \theta_2}(z)$ to obtain the final result Eq. (60).

APPENDIX F: COMPUTATIONAL COMPLEXITY OF gBBGKY AND GENERALIZATION TO MULTISPECIES MODELS

In practical applications, evaluating the gBBGKY hierarchy requires numerically solving a set of coupled integro-differential equations. To assess the computational complexity of this framework, let us consider the numerical cost of solving the hierarchy truncated at the second layer, meaning we retain the coupled evolution of the one-point root densities and the two-point correlation structures, Eqs. (47) and (48), while higher-order connected correlations are set to zero. For an integrable model with a single quasiparticle species, such as the hard-rod gas or the repulsive Lieb-Liniger gas, the state is parametrized by a single rapidity variable θ . Numerically, the rapidity space is discretized into N_θ grid points. The computational bottleneck in solving the hierarchy lies in the standard generalized hydrodynamics dressing operations, which require matrix inversions that scale polynomially with N_θ . For more generic integrable models that feature bound states, such as the Heisenberg XXX spin chain or the attractive Lieb-Liniger model, the excitation spectrum is significantly richer. In the thermodynamic limit, these bound states are classified as “strings” of length n , where $n \in \mathbb{N}$. Consequently, the thermodynamic state is no longer described by a single root density, but rather by an infinite set of root densities $\rho_n(\theta)$. The algebraic structure of the gBBGKY hierarchy remains identical, but the integration over the rapidity space is formally upgraded to include a sum over all string species: $\int d\theta \rightarrow \sum_n \int d\theta$. In any numerical implementation, this sum must be truncated to a maximum string length $n_{\max}$. This truncation dictates the scaling of the computational cost. For the first layer, Eq. (47), instead of a single density $\rho(\theta)$, the solver must track and update $n_{\max}$ coupled root densities $\rho_n(\theta)$. For the second layer, Eq. (48), the two-point correlation structures now involve cross-correlations between different string species n and m . Therefore, the solver must track an $n_{\max} \times n_{\max}$ matrix of coupled two-point functions. Effectively, the dimension of the local rapidity grid expands from N_θ to $n_{\max} N_\theta$. Hence, with this truncation, the total number of dynamic variables at each spatial grid point scales as $O(n_{\max}^2)$. Correspondingly, the matrix operations required for the dressing transformations and the convective transport will scale polynomially with this expanded grid size. While this $O(n_{\max}^2)$ scaling makes the numerical integration

for generic multispecies models computationally heavier than for single-species models, it is crucial to emphasize that the scaling remains strictly polynomial, i.e., the framework does not introduce any exponential bottlenecks.

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- [1] P. Calabrese, F. H. Essler, and G. Mussardo, *Introduction to “quantum integrability in out of equilibrium systems”*, *J. Stat. Mech.* (2016) 064001.
 - [2] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, *Colloquium: Nonequilibrium dynamics of closed interacting quantum systems*, *Rev. Mod. Phys.* **83**, 863 (2011).
 - [3] M. Cheneau, P. Barmettler, D. Poletti, M. Endres, P. Schauss, T. Fukuhara, C. Gross, I. Bloch, C. Kollath, and S. Kuhr, *Light-cone-like spreading of correlations in a quantum many-body system*, *Nature (London)* **481**, 484 (2012).
 - [4] D. A. Huse, R. Nandkishore, V. Oganesyan, A. Pal, and S. L. Sondhi, *Many-body localization and thermalization in quantum statistical mechanics*, *Phys. Rev. B* **88**, 014206 (2013).
 - [5] M. Fagotti and P. Calabrese, *Evolution of entanglement entropy following a quantum quench: Analytic results for the XY chain in a transverse magnetic field*, *Phys. Rev. A* **78**, 010306(R) (2008).
 - [6] V. Alba and P. Calabrese, *Entanglement dynamics after quantum quenches in generic integrable systems*, *SciPost Phys.* **4**, 017 (2018).
 - [7] V. Alba, B. Bertini, and M. Fagotti, *Entanglement evolution and generalized hydrodynamics: Interacting integrable systems*, *SciPost Phys.* **7**, 005 (2019).
 - [8] V. Alba and P. Calabrese, *Quantum information scrambling after a quantum quench*, *Phys. Rev. B* **100**, 115150 (2019).
 - [9] J. Lopez-Piqueres, B. Ware, S. Gopalakrishnan, and R. Vasseur, *Operator front broadening in chaotic and integrable quantum chains*, *Phys. Rev. B* **104**, 104307 (2021).
 - [10] P. Fendley, *Strong zero modes and eigenstate phase transitions in the XYZ/interacting Majorana chain*, *J. Phys. A* **49**, 30LT01 (2016).
 - [11] P. R. Pasnoori, Y. Tang, J. Lee, J. H. Pixley, N. Andrei, and P. Azaria, *Spin fractionalization and zero modes in the spin- $\frac{1}{2}$ XXZ chain with boundary fields*, *Phys. Rev. B* **112**, 075121 (2025).
 - [12] A. Burshtein and M. Goldstein, *Inelastic decay from integrability*, *PRX Quantum* **5**, 020323 (2024).
 - [13] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, *Emergent hydrodynamics in integrable quantum systems out of equilibrium*, *Phys. Rev. X* **6**, 041065 (2016).
 - [14] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, *Transport in out-of-equilibrium XXZ chains: Exact profiles of charges and currents*, *Phys. Rev. Lett.* **117**, 207201 (2016).
 - [15] S. Dubovsky, L. Hui, A. Nicolis, and D. T. Son, *Effective field theory for hydrodynamics: Thermodynamics, and the derivative expansion*, *Phys. Rev. D* **85**, 085029 (2012).
 - [16] P. Ruggiero, P. Calabrese, B. Doyon, and J. Dubail, *Quantum generalized hydrodynamics*, *Phys. Rev. Lett.* **124**, 140603 (2020).
 - [17] A. Lucas and K. C. Fong, *Hydrodynamics of electrons in graphene*, *J. Phys. Condens. Matter* **30**, 053001 (2018).
 - [18] N. Malvania, Y. Zhang, Y. Le, J. Dubail, M. Rigol, and D. S. Weiss, *Generalized hydrodynamics in strongly interacting 1d Bose gases*, *Science* **373**, 1129 (2021).
 - [19] M. Schemmer, I. Bouchoule, B. Doyon, and J. Dubail, *Generalized hydrodynamics on an atom chip*, *Phys. Rev. Lett.* **122**, 090601 (2019).
 - [20] A. Scheie, N. E. Sherman, M. Dupont, S. E. Nagler, M. B. Stone, G. E. Granroth, J. E. Moore, and D. A. Tennant, *Detection of Kardar–Parisi–Zhang hydrodynamics in a quantum Heisenberg spin-1/2 chain*, *Nat. Phys.* **17**, 726 (2021).
 - [21] F. Møller, C. Li, I. Mazets, H.-P. Stimming, T. Zhou, Z. Zhu, X. Chen, and J. Schmiedmayer, *Extension of the generalized hydrodynamics to the dimensional crossover regime*, *Phys. Rev. Lett.* **126**, 090602 (2021).
 - [22] F. Cataldini, F. Møller, M. Tajik, J. Sabino, S.-C. Ji, I. Mazets, T. Schweigler, B. Rauer, and J. Schmiedmayer, *Emergent Pauli blocking in a weakly interacting Bose gas*, *Phys. Rev. X* **12**, 041032 (2022).
 - [23] K.-Y. Li, Y. Zhang, K. Yang, K.-Y. Lin, S. Gopalakrishnan, M. Rigol, and B. L. Lev, *Rapidity and momentum distributions of one-dimensional dipolar quantum gases*, *Phys. Rev. A* **107**, L061302 (2023).
 - [24] V. B. Bulchandani, R. Vasseur, C. Karrasch, and J. E. Moore, *Bethe-Boltzmann hydrodynamics and spin transport in the XXZ chain*, *Phys. Rev. B* **97**, 045407 (2018).
 - [25] V. B. Bulchandani, C. Karrasch, and J. E. Moore, *Superdiffusive transport of energy in one-dimensional metals*, *Proc. Natl. Acad. Sci. U.S.A.* **117**, 12713 (2020).
 - [26] L. Piroli, J. De Nardis, M. Collura, B. Bertini, and M. Fagotti, *Transport in out-of-equilibrium XXZ chains: Non-ballistic behavior and correlation functions*, *Phys. Rev. B* **96**, 115124 (2017).
 - [27] E. H. Lieb and W. Liniger, *Exact analysis of an interacting Bose gas. I. The general solution and the ground state*, *Phys. Rev.* **130**, 1605 (1963).
 - [28] E. H. Lieb, *Exact analysis of an interacting Bose gas. II. The excitation spectrum*, *Phys. Rev.* **130**, 1616 (1963).
 - [29] F. H. L. Essler, H. Frahm, F. Göhmann, A. Klümper, and V. E. Korepin, *The One-Dimensional Hubbard Model* (Cambridge University Press, Cambridge, England, 2005).
 - [30] K. Mallayya, M. Rigol, and W. De Roeck, *Prethermalization and thermalization in isolated quantum systems*, *Phys. Rev. X* **9**, 021027 (2019).
 - [31] J. Lopez-Piqueres and R. Vasseur, *Integrability breaking from backscattering*, *Phys. Rev. Lett.* **130**, 247101 (2023).
 - [32] T. Goldfriend and J. Kurchan, *Quasi-integrable systems are slow to thermalize but may be good scramblers*, *Phys. Rev. E* **102**, 022201 (2020).
 - [33] A. J. Friedman, S. Gopalakrishnan, and R. Vasseur, *Diffusive hydrodynamics from integrability breaking*, *Phys. Rev. B* **101**, 180302(R) (2020).
 - [34] A. Bastianello, A. De Luca, and R. Vasseur, *Hydrodynamics of weak integrability breaking*, *J. Stat. Mech.* (2021) 114003.

- [35] P. Orlov, A. Tiutiakina, R. Sharipov, E. Petrova, V. Gritsev, and D. V. Kurlov, *Adiabatic eigenstate deformations and weak integrability breaking of Heisenberg chain*, *Phys. Rev. B* **107**, 184312 (2023).
- [36] D. Szász-Schagrin, B. Pozsgay, and G. Takács, *Weak integrability breaking and level spacing distribution*, *SciPost Phys.* **11**, 037 (2021).
- [37] B. Pozsgay, R. Sharipov, A. Tiutiakina, and I. Vona, *Adiabatic gauge potential and integrability breaking with free fermions*, *SciPost Phys.* **17**, 075 (2024).
- [38] B. Bertini, F. H. L. Essler, S. Groha, and N. J. Robinson, *Prethermalization and thermalization in models with weak integrability breaking*, *Phys. Rev. Lett.* **115**, 180601 (2015).
- [39] B. Bertini, F. H. L. Essler, S. Groha, and N. J. Robinson, *Thermalization and light cones in a model with weak integrability breaking*, *Phys. Rev. B* **94**, 245117 (2016).
- [40] F. H. L. Essler, S. Kehrein, S. R. Manmana, and N. J. Robinson, *Quench dynamics in a model with tuneable integrability breaking*, *Phys. Rev. B* **89**, 165104 (2014).
- [41] J. Lopez-Piqueres, B. Ware, S. Gopalakrishnan, and R. Vasseur, *Hydrodynamics of nonintegrable systems from a relaxation-time approximation*, *Phys. Rev. B* **103**, L060302 (2021).
- [42] F. M. Surace and O. Motrunich, *Weak integrability breaking perturbations of integrable models*, *Phys. Rev. Res.* **5**, 043019 (2023).
- [43] M. Łebek, M. Panfil, and R. M. Konik, *Prethermalization in coupled one-dimensional quantum gases*, *SciPost Phys.* **17**, 007 (2024).
- [44] J. A. Scaramazza, P. Smacchia, and E. A. Yuzbashyan, *Consequences of integrability breaking in quench dynamics of pairing Hamiltonians*, *Phys. Rev. B* **99**, 054520 (2019).
- [45] A. Bastianello, *Lack of thermalization for integrability-breaking impurities*, *Europhys. Lett.* **125**, 20001 (2019).
- [46] G. del Vecchio del Vecchio, A. D. Luca, and A. Bastianello, *Transport through interacting defects and lack of thermalisation*, *SciPost Phys.* **12**, 060 (2022).
- [47] J. Durnin, M. J. Bhaseen, and B. Doyon, *Nonequilibrium dynamics and weakly broken integrability*, *Phys. Rev. Lett.* **127**, 130601 (2021).
- [48] M. Panfil, S. Gopalakrishnan, and R. M. Konik, *Thermalization of interacting quasi-one-dimensional systems*, *Phys. Rev. Lett.* **130**, 030401 (2023).
- [49] M. Łebek and M. Panfil, *Chapman-Enskog theory and crossover between diffusion and superdiffusion for nearly integrable quantum gases*, *SciPost Phys.* **18**, 186 (2025).
- [50] S. Lepri, R. Livi, and A. Politi, *Too close to integrable: Crossover from normal to anomalous heat diffusion*, *Phys. Rev. Lett.* **125**, 040604 (2020).
- [51] L. Delacrétaz, A. L. Fitzpatrick, E. Katz, and M. Walters, *Thermalization and hydrodynamics of two-dimensional quantum field theories*, *SciPost Phys.* **12**, 119 (2022).
- [52] A. Miron, J. Cividini, A. Kundu, and D. Mukamel, *Derivation of fluctuating hydrodynamics and crossover from diffusive to anomalous transport in a hard-particle gas*, *Phys. Rev. E* **99**, 012124 (2019).
- [53] S. Chen, J. Wang, G. Casati, and G. Benenti, *Nonintegrability and the Fourier heat conduction law*, *Phys. Rev. E* **90**, 032134 (2014).
- [54] H. Zhao and W. g. Wang, *Fourier heat conduction as a strong kinetic effect in one-dimensional hard-core gases*, *Phys. Rev. E* **97**, 010103(R) (2018).
- [55] S. Lepri, R. Livi, and A. Politi, *Too close to integrable: Crossover from normal to anomalous heat diffusion*, *Phys. Rev. Lett.* **125**, 040604 (2020).
- [56] M. Gring, M. Kuhnert, T. Langen, T. Kitagawa, B. Rauer, M. Schreitl, I. Mazets, D. A. Smith, E. Demler, and J. Schmiedmayer, *Relaxation and prethermalization in an isolated quantum system*, *Science* **337**, 1318 (2012).
- [57] F. Möller, C. Li, I. Mazets, H.-P. Stimming, T. Zhou, Z. Zhu, X. Chen, and J. Schmiedmayer, *Extension of the generalized hydrodynamics to the dimensional crossover regime*, *Phys. Rev. Lett.* **126**, 090602 (2021).
- [58] Y. Tang, W. Kao, K.-Y. Li, S. Seo, K. Mallayya, M. Rigol, S. Gopalakrishnan, and B. L. Lev, *Thermalization near integrability in a dipolar quantum Newton's cradle*, *Phys. Rev. X* **8**, 021030 (2018).
- [59] C. Colléaux, J. Skipp, J. Laurie, and S. Nazarenko, *Semi-local one-dimensional optical wave turbulence*, *arXiv*: 2412.14153.
- [60] S. Lepri, P. Sandri, and A. Politi, *The one-dimensional Lennard-Jones system: Collective fluctuations and breakdown of hydrodynamics*, *Eur. Phys. J. B* **47**, 549 (2005).
- [61] M. Mareschal and A. Amellal, *Thermal conductivity in a one-dimensional Lennard-Jones chain by molecular dynamics*, *Phys. Rev. A* **37**, 2189 (1988).
- [62] M. Eltoha, X. Wang, C. M. Griffin, and F. Robicheaux, *Simulations of classical three-body thermalization in one dimension*, *Phys. Rev. E* **110**, 014114 (2024).
- [63] A. Parameshwaran and B. S. Gupta, *Kinetics of vapor-liquid transition of active matter system in confined geometry*, *Phys. Rev. E* **111**, 025405 (2025).
- [64] G. Benettin, G. Orsatti, and A. Ponno, *On the role of the integrable Toda model in one-dimensional molecular dynamics*, *J. Stat. Phys.* **190**, 131 (2023).
- [65] O. C. Eldridge and M. Feix, *Numerical experiments with a plasma model*, *Phys. Fluids* **6**, 398 (1963).
- [66] J. M. Dawson, *Thermal relaxation in a one-species, one-dimensional plasma*, *Phys. Fluids* **7**, 419 (1964).
- [67] J. L. Rouet and M. R. Feix, *Relaxation for one-dimensional plasma: Test particles versus global distribution behavior*, *Phys. Fluids B* **3**, 1830 (1991).
- [68] J.-B. Fouvry, B. Bar-Or, and P.-H. Chavanis, *Kinetic theory of one-dimensional homogeneous long-range interacting systems sourced by $1/N^2$ effects*, *Phys. Rev. E* **100**, 052142 (2019).
- [69] J.-B. Fouvry, P.-H. Chavanis, and C. Pichon, *Kinetic theory of one-dimensional homogeneous long-range interacting systems with an arbitrary potential of interaction*, *Phys. Rev. E* **102**, 052110 (2020).
- [70] J. Berges, G. S. Denicol, M. P. Heller, and T. Preis, *Far from equilibrium hydrodynamics of nonthermal fixed points*, *arXiv*:2504.18754.
- [71] J.-B. Fouvry and M. Roule, *Kinetic blockings in long-range interacting inhomogeneous systems*, *Phys. Rev. E* **108**, 054108 (2023).

- [72] G. B. Rybicki, *Exact statistical mechanics of a one-dimensional self-gravitating system*, *Astrophys. Space Sci.* **14**, 56 (1971).
- [73] J. Reidl, Charles J., and B. N. Miller, *Gravity in one dimension—quasi-equilibrium*, *Astrophys. J.* **332**, 619 (1988).
- [74] L. Milanović, H. Posch, and W. Thirring, *Statistical mechanics and computer simulation of systems with attractive positive power-law potentials*, *Phys. Rev. E* **57**, 2763 (1998).
- [75] T. Tsuchiya, N. Gouda, and T. Konishi, *Relaxation processes in one-dimensional self-gravitating many-body systems*, *Phys. Rev. E* **53**, 2210 (1996).
- [76] B. N. Miller, K. R. Yawn, and V. P. Youngkins, *Recent progress in studies of one-dimensional gravitating systems*, in *Dynamical Systems, Plasmas and Gravitation* (Springer, Berlin, Heidelberg, 1996), pp. 164–175.
- [77] M. Łebek and M. Panfil, *Navier-Stokes equations for nearly integrable quantum gases*, *Phys. Rev. Lett.* **134**, 010405 (2025).
- [78] R. Balescu, *Equilibrium and Non-Equilibrium Statistical Mechanics* (Wiley-Interscience, New York, 1975).
- [79] J. G. Kirkwood, *The statistical mechanical theory of transport processes II. Transport in gases*, *J. Chem. Phys.* **15**, 72 (1947).
- [80] J. G. Kirkwood, *The statistical mechanical theory of transport processes I. General theory*, *J. Chem. Phys.* **14**, 180 (1946).
- [81] J. Yvon, *La théorie statistique des fluides et l'équation d'état*, *Actual. Sci. & Indust. No. 203* (Hermann, Paris, 1935).
- [82] N. N. Bogoliubov and K. P. Gurov, *Kinetic equations in quantum mechanics*, *J. Exp. Theor. Phys.* **17**, 614 (1947).
- [83] V. I. Gerasimenko, *Kinetic equations and hierarchies of evolution equations of quantum systems*, *arXiv: 2107.10872*.
- [84] L. Chomaz, I. Ferrier-Barbut, F. Ferlaino, B. Laburthe-Tolra, B. L. Lev, and T. Pfau, *Dipolar physics: A review of experiments with magnetic quantum gases*, *Rep. Prog. Phys.* **86**, 026401 (2023).
- [85] C. Chen, G. Bornet, M. Bintz, G. Emperauger, L. Leclerc, V. S. Liu, P. Scholl, D. Barredo, J. Hauschild, S. Chatterjee, M. Schuler, A. M. Läuchli, M. P. Zaletel, T. Lahaye, N. Y. Yao, and A. Browaeys, *Continuous symmetry breaking in a two-dimensional Rydberg array*, *Nature (London)* **616**, 691 (2023).
- [86] M. Płodzień, G. Lochead, J. de Hond, N. J. van Druten, and S. Kokkelmans, *Rydberg dressing of a one-dimensional Bose-Einstein condensate*, *Phys. Rev. A* **95**, 043606 (2017).
- [87] J. Rincón, M. Ganahl, and G. Vidal, *Lieb-Liniger model with exponentially decaying interactions: A continuous matrix product state study*, *Phys. Rev. B* **92**, 115107 (2015).
- [88] E. Orignac, S. De Palo, L. Salasnich, and R. Citro, *Breathing mode of a quantum droplet in a quasi-one-dimensional dipolar Bose gas*, *Phys. Rev. A* **109**, 043316 (2024).
- [89] I. Frérot, P. Naldesi, and T. Roscilde, *Entanglement and fluctuations in the XXZ model with power-law interactions*, *Phys. Rev. B* **95**, 245111 (2017).
- [90] L. Chomaz, S. Baier, D. Petter, M. J. Mark, F. Wächtler, L. Santos, and F. Ferlaino, *Quantum-fluctuation-driven crossover from a dilute Bose-Einstein condensate to a macrodroplet in a dipolar quantum fluid*, *Phys. Rev. X* **6**, 041039 (2016).
- [91] A. Recati and S. Stringari, *Supersolidity in ultracold dipolar gases*, *Nat. Rev. Phys.* **5**, 735 (2023).
- [92] G. Emperauger, M. Qiao, C. Chen, F. Caleca, S. Bocini, M. Bintz, G. Bornet, R. Martin, B. Gély, L. Klein, D. Barredo, S. Chatterjee, N. Yao, F. Mezzacapo, T. Lahaye, T. Roscilde, and A. Browaeys, *Tomonaga-Luttinger liquid behavior in a Rydberg-encoded spin chain*, *Phys. Rev. X* **15**, 031021 (2025).
- [93] K. Yang, Y. Zhang, K.-Y. Li, K.-Y. Lin, S. Gopalakrishnan, M. Rigol, and B. L. Lev, *Phantom energy in the nonlinear response of a quantum many-body scar state*, *Science* **385**, 1063 (2024).
- [94] D. Bernard and B. Doyon, *Conformal field theory out of equilibrium: A review*, *J. Stat. Mech.* (2016) 064005.
- [95] B. Doyon, S. Gopalakrishnan, F. Møller, J. Schmiedmayer, and R. Vasseur, *Generalized hydrodynamics: A perspective*, *Phys. Rev. X* **15**, 010501 (2025).
- [96] P. Schüttelkopf, M. Tajik, N. Bazhan, F. Cataldini, S.-C. Ji, J. Schmiedmayer, and F. Møller, *Characterising transport in a quantum gas by measuring Drude weights*, *Science* **391**, 290 (2025).
- [97] R. Vasseur and J. E. Moore, *Nonequilibrium quantum dynamics and transport: From integrability to many-body localization*, *J. Stat. Mech.* (2016) 064010.
- [98] M. L. Kerr and K. V. Kheruntsyan, *The theory of generalised hydrodynamics for the one-dimensional Bose gas*, *AAPPS Bull.* **33**, 25 (2023).
- [99] V. Alba, B. Bertini, M. Fagotti, L. Piroli, and P. Ruggiero, *Generalized-hydrodynamic approach to inhomogeneous quenches: Correlations, entanglement and quantum effects*, *J. Stat. Mech.* (2021) 114004.
- [100] F. H. Essler, *A short introduction to generalized hydrodynamics*, *Physica (Amsterdam)* **631A**, 127572 (2023), lecture Notes of the 15th International Summer School of Fundamental Problems in Statistical Physics.
- [101] S. Scopa, P. Ruggiero, P. Calabrese, and J. Dubail, *One-particle density matrix and momentum distribution of the out-of-equilibrium one-dimensional Tonks-Girardeau gas: Analytical results at large n* , *Phys. Rev. A* **108**, 013324 (2023).
- [102] P. Suret, S. Randoux, A. Gelash, D. Agafontsev, B. Doyon, and G. El, *Soliton gas: Theory, numerics, and experiments*, *Phys. Rev. E* **109**, 061001 (2024).
- [103] A. Bastianello, A. Tikan, F. Copie, S. Randoux, and P. Suret, *Observation of a generalized Gibbs ensemble in photonics*, *Phys. Rev. A* **113**, 013514 (2026).
- [104] T. Kinoshita, T. Wenger, and D. S. Weiss, *A quantum Newton's cradle*, *Nature (London)* **440**, 900 (2006).
- [105] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, *Observation of many-body localization of interacting fermions in a quasirandom optical lattice*, *Science* **349**, 842 (2015).
- [106] W. Kao, K.-Y. Li, K.-Y. Lin, S. Gopalakrishnan, and B. L. Lev, *Topological pumping of a 1d dipolar gas into strongly correlated prethermal states*, *Science* **371**, 296 (2021).

- [107] W. Morong, F. Liu, P. Becker, K. S. Collins, L. Feng, A. Kyprianidis, G. Pagano, T. You, A. V. Gorshkov, and C. Monroe, *Publisher correction: Observation of stark many-body localization without disorder*, *Nature (London)* **601**, E13 (2022).
- [108] S. Birkammer, A. Bastianello, and M. Knap, *Prethermalization in one-dimensional quantum many-body systems with confinement*, *Nat. Commun.* **13**, 7663 (2022).
- [109] Y. Le, Y. Zhang, S. Gopalakrishnan, M. Rigol, and D. S. Weiss, *Observation of hydrodynamization and local prethermalization in 1d Bose gases*, *Nature (London)* **618**, 494 (2023).
- [110] T. Young, J. LLoyd, and C. von Keyserlingk, *Breakdown of Fermi's golden rule in 1d systems at non-zero temperature*, *arXiv:2508.00254*.
- [111] P. A. E. M. Janssen, *Nonlinear four-wave interactions and freak waves*, *J. Phys. Oceanogr.* **33**, 863 (2003).
- [112] J. Garnier, M. Lisak, and A. Picozzi, *Toward a wave turbulence formulation of statistical nonlinear optics*, *J. Opt. Soc. Am. B* **29**, 2229 (2012).
- [113] F. Hübner, L. Biagetti, J. De Nardis, and B. Doyon, *Diffusive hydrodynamics from long-range correlations*, *Phys. Rev. Lett.* **134**, 187101 (2025).
- [114] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/sx1x-py11> for additional computations supporting the results presented in the main text.
- [115] B. Doyon and T. Yoshimura, *A note on generalized hydrodynamics: Inhomogeneous fields and other concepts*, *SciPost Phys.* **2**, 014 (2017).
- [116] J. Durin, A. D. Luca, J. De Nardis, and B. Doyon, *Diffusive hydrodynamics of inhomogeneous Hamiltonians*, *J. Phys. A* **54**, 494001 (2021).
- [117] A. A. Vlasov, *The vibrational properties of an electron gas*, *Sov. Phys. Usp.* **10**, 721 (1968).
- [118] W. Braun and K. Hepp, *The Vlasov dynamics and its fluctuations in the $1/n$ limit of interacting classical particles*, *Commun. Math. Phys.* **56**, 101 (1977).
- [119] P. H. Chavanis, J. Sommeria, and R. Robert, *Statistical mechanics of two-dimensional vortices and collisionless stellar systems*, *Astrophys. J.* **471**, 385 (1996).
- [120] J. De Nardis, D. Bernard, and B. Doyon, *Diffusion in generalized hydrodynamics and quasiparticle scattering*, *SciPost Phys.* **6**, 049 (2019).
- [121] M. Medenjak, J. De Nardis, and T. Yoshimura, *Diffusion from convection*, *SciPost Phys.* **9**, 075 (2020).
- [122] S. Gopalakrishnan, D. A. Huse, V. Khemani, and R. Vasseur, *Hydrodynamics of operator spreading and quasiparticle diffusion in interacting integrable systems*, *Phys. Rev. B* **98**, 220303(R) (2018).
- [123] B. Doyon, G. Perfetto, T. Sasamoto, and T. Yoshimura, *Ballistic macroscopic fluctuation theory*, *SciPost Phys.* **15**, 136 (2023).
- [124] P. Lisiak, M. Łebek, and M. Panfil, *Burgers equation from non-thermal stationary states in nearly-integrable gases*, *arXiv:2512.09726*.
- [125] Z. Krajnik, J. Schmidt, V. Pasquier, E. Ilievski, and T. Prosen, *Exact anomalous current fluctuations in a deterministic interacting model*, *Phys. Rev. Lett.* **128**, 160601 (2022).
- [126] S. Gopalakrishnan, E. McCulloch, and R. Vasseur, *Non-Gaussian diffusive fluctuations in Dirac fluids*, *Proc. Natl. Acad. Sci. U.S.A.* **121**, e2403327121 (2024).
- [127] E. McCulloch, R. Vasseur, and S. Gopalakrishnan, *Balistic modes as a source of anomalous charge noise*, *Phys. Rev. E* **111**, 015410 (2025).
- [128] T. Yoshimura and Žiga Krajnik, *Anomalous current fluctuations from Euler hydrodynamics*, *Phys. Rev. E* **111**, 024141 (2025).
- [129] B. Doyon, *Hydrodynamic noise in one dimension: Projected kubo formula and its vanishing in integrable models*, *SciPost Phys.* **20**, 111 (2026).
- [130] H. Spohn, *Large Scale Dynamics of Interacting Particles* (Springer-Verlag, Berlin, 1991).
- [131] M. Flicker, *Pair distribution function of a one-dimensional hard rod gas*, *J. Math. Phys. (N.Y.)* **9**, 171 (1968).
- [132] A. Robledo and J. Rowlinson, *The distribution of hard rods on a line of finite length*, *Mol. Phys.* **58**, 711 (1986).
- [133] B. Doyon and H. Spohn, *Dynamics of hard rods with initial domain wall state*, *J. Stat. Mech.* (2017) 073210.
- [134] H. Spohn, *Large Scale Dynamics of Interacting Particles* (Springer, Berlin, 1991).
- [135] C. Boldrighini, R. L. Dobrushin, and Y. M. Sukhov, *One-dimensional hard rod caricature of hydrodynamics*, *J. Stat. Phys.* **31**, 577 (1983).
- [136] C. Boldrighini and Y. M. Suhov, *One-dimensional hard-rod caricature of hydrodynamics: "Navier-Stokes correction" for local equilibrium initial states*, *Commun. Math. Phys.* **189**, 577 (1997).
- [137] B. Doyon, *Lecture notes on generalised hydrodynamics*, *SciPost Physics Lecture Notes* **18**, (2020), [10.21468/scipostphyslecturenotes.18](https://scipostphyslecturenotes.18).
- [138] M. Olshanii, *Atomic scattering in the presence of an external confinement and a gas of impenetrable bosons*, *Phys. Rev. Lett.* **81**, 938 (1998).
- [139] F. Deuretzbacher, J. C. Cremon, and S. M. Reimann, *Ground-state properties of few dipolar bosons in a quasi-one-dimensional harmonic trap*, *Phys. Rev. A* **81**, 063616 (2010).
- [140] F. Deuretzbacher, J. C. Cremon, and S. M. Reimann, *Erratum: Ground-state properties of few dipolar bosons in a quasi-one-dimensional harmonic trap* [*Phys. Rev. A* **81**, 063616 (2010)]; **87**, 039903(E) (2013).
- [141] A. Urilyon, L. Biagetti, J. Kethapalli, and J. De Nardis, *Simulating generalized fluids via interacting wave packet evolution*, *Phys. Rev. B* **113**, 014314 (2026).
- [142] K. Mallayya and M. Rigol, *Prethermalization, thermalization, and Fermi's golden rule in quantum many-body systems*, *Phys. Rev. B* **104**, 184302 (2021).
- [143] J. De Nardis and M. Panfil, *Density form factors of the 1d Bose gas for finite entropy states*, *J. Stat. Mech.* (2015) P02019.
- [144] J. De Nardis and M. Panfil, *Exact correlations in the Lieb-Liniger model and detailed balance out-of-equilibrium*, *SciPost Phys.* **1**, 015 (2016).
- [145] J. De Nardis and M. Panfil, *Particle-hole pairs and density-density correlations in the Lieb-Liniger model*, *J. Stat. Mech.* (2018) 033102.

- [146] A. C. Cubero and M. Panfil, *Thermodynamic bootstrap program for integrable QFT's: Form factors and correlation functions at finite energy density*, *J. High Energy Phys.* **01** (2019) 104.
- [147] M. Panfil, *The two particle-hole pairs contribution to the dynamic correlation functions of quantum integrable models*, *J. Stat. Mech.* (2021) 013108.
- [148] Note the additional missing factor of 2π with respect to Ref. [47].
- [149] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, England, 1993).
- [150] M. Panfil and R. M. Konik, *Extending the thermodynamic form factor bootstrap program: Multiple particle-hole excitations, crossing symmetry, and reparameterization invariance*, *J. High Energy Phys.* **10** (2023) 072.
- [151] C. Monroe, W. C. Campbell, L.-M. Duan, Z.-X. Gong, A. V. Gorshkov, P. W. Hess, R. Islam, K. Kim, N. M. Linke, G. Pagano, P. Richerme, C. Senko, and N. Y. Yao, *Programmable quantum simulations of spin systems with trapped ions*, *Rev. Mod. Phys.* **93**, 025001 (2021).
- [152] P. Jurcevic, B. P. Lanyon, P. Hauke, C. Hempel, P. Zoller, R. Blatt, and C. F. Roos, *Quasiparticle engineering and entanglement propagation in a quantum many-body system*, *Nature (London)* **511**, 202 (2014).
- [153] B. Kloss and Y. Bar Lev, *Spin transport in a long-range-interacting spin chain*, *Phys. Rev. A* **99**, 032114 (2019).
- [154] J. T. Schneider, S. J. Thomson, and L. Sanchez-Palencia, *Entanglement spectrum and quantum phase diagram of the long-range XXZ chain*, *Phys. Rev. B* **106**, 014306 (2022).
- [155] S. Nazarenko, *Wave Turbulence*, Lecture Notes in Physics Vol. 825 (Springer, Berlin, Heidelberg, 2011).
- [156] L. Fache, H. Damart, F. Copie, T. Bonnemain, T. Congy, G. Roberti, P. Suret, G. El, and S. Randoux, *Dissipation-driven emergence of a soliton condensate in a nonlinear electrical transmission line*, *Phys. Rev. Lett.* **134**, 147201 (2025).
- [157] A. Mucci, P. Suret, F. Copie, S. Randoux, R. Mullyadzhano, and A. Gelash, *Manipulation of strongly interacting solitons in optical fiber experiments*, *Phys. Rev. Lett.* **134**, 193804 (2025).
- [158] L. Fache, F. Copie, P. Suret, and S. Randoux, *Perturbed nonlinear evolution of optical soliton gases: Growth and decay in integrable turbulence*, *Phys. Rev. Lett.* **135**, 157201 (2025).
- [159] L. Biagetti, G. Cecile, and J. De Nardis, *Three-stage thermalization of a quasi-integrable system*, *Phys. Rev. Res.* **6**, 023083 (2024).