

Emergent Random Matrix Universality in Quantum Operator Dynamics

Oliver Lunt^{1,2}, Thomas Kriecherbauer³, Kenneth T-R McLaughlin⁴, and Curt von Keyserlingk²

¹*Clarendon Laboratory, Department of Physics, University of Oxford,
Parks Road, Oxford, OX1 3PU, United Kingdom*

²*Department of Physics, King's College London, Strand, London, WC2R 2LS, United Kingdom*

³*Department of Mathematics, Universität Bayreuth, 95440 Bayreuth, Germany*

⁴*School of Science & Engineering, Tulane University, New Orleans, Louisiana 70118-5698, USA*



(Received 15 May 2025; revised 6 October 2025; accepted 23 December 2025; published 23 February 2026)

The high complexity of many-body quantum dynamics means that essentially all analytical or numerical approaches either exploit special structure or are approximate in nature. One such approach—the memory function formalism—involves a carefully chosen split into “fast” and “slow” modes. An approximate model for the fast modes can then be used to solve for Green’s functions $G(z)$ of the slow modes, and the success of this approach depends on the accuracy of the fast space approximation. Using a formulation in operator Krylov space known as the recursion method, we prove the emergence of a universal random matrix description of the fast mode dynamics. This is captured by the “level- n Green’s function” $G_n(z)$, which we show approaches universal scaling forms in the “fast limit” $n \rightarrow \infty$. Notably, this emergent universality can occur in both chaotic and nonchaotic systems, provided their spectral functions are sufficiently smooth. This universality of $G_n(z)$ turns out to be precisely analogous to the universality of eigenvalue correlations in random matrix theory (RMT), even though there is *no explicit randomness* present in the Hamiltonian. Concretely, at finite z we show that $G_n(z)$ approaches the Wigner semicircle law, while if $G(z)$ is the Green’s function of certain hydrodynamical variables, we show that at low frequencies $G_n(z)$ is instead governed by the Bessel universality class from RMT. As an application of this universality, we give a new numerical method, the *spectral bootstrap*, for approximating spectral functions, including hydrodynamic transport data, from a finite number of Lanczos coefficients. Our proof involves a map to a Riemann-Hilbert problem which we solve using a steepest-descent-type method, rigorously controlled in the $n \rightarrow \infty$ limit. Via the steepest-descent procedure, we are led to a related Coulomb gas optimization problem, and we discuss how a recent conjecture—the “Operator Growth Hypothesis”—implies that chaotic operator dynamics can generically be identified with the critical point of a confinement transition in this Coulomb gas. These results elevate the recursion method from a useful numerical technique to a theoretically principled framework with universal content.

DOI: [10.1103/PhysRevX.16.011033](https://doi.org/10.1103/PhysRevX.16.011033)

Subject Areas: Condensed Matter Physics,
Quantum Physics,
Statistical Physics

CONTENTS

I. INTRODUCTION	2	E. Comparison with eigenstate thermalization	9
A. Relation to previous work	4	F. Numerical application: The spectral bootstrap	10
II. SYNOPSIS	5	G. Marginality and quantum chaos	10
A. Emergent universality in the $n \rightarrow \infty$ limit	5	H. Future directions	11
B. Orthogonal polynomials and large- n expansions	6	1. Combining the spectral bootstrap with dissipation-assisted approaches	11
C. Lanczos dynamics and random matrix universality	7	2. Universal dynamics of large operators	11
D. Coulomb gas confinement transition, quantum chaos, and the operator growth hypothesis	8	3. Universality in unitary and open system dynamics	11
		4. Connection to the generalized ETH	12
		5. Krylov complexity and conservation laws	12
		6. New quantum algorithms for quantum dynamics	12
		7. Improved treatment of hydrodynamic tails	12
		I. Reader’s guide	13
		III. BACKGROUND	14
		IV. STATEMENT OF ASSUMPTIONS ON THE SPECTRAL FUNCTION	15
		A. Summary	15

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI.

B. Precise assumptions	16
1. Examples	17
C. Comments on analyticity requirements	17
V. COULOMB GAS, UNIVERSALITY, AND CONFINEMENT	18
A. Coulomb gas definition	18
B. Rescaled equilibrium measure	19
C. Confinement transition	20
VI. UNIVERSALITY OF THE LEVEL- n GREEN'S FUNCTION	21
A. Frequencies in the bulk	22
1. Semicircle law as a fixed point	22
B. Frequencies near $\omega = 0$	23
C. Frequencies near the edges $\omega = \pm\beta_n$	23
VII. HYDRODYNAMIC CONTRIBUTIONS TO THE LANCZOS COEFFICIENTS	23
VIII. HYDRODYNAMICS FROM THE ZERO MODE	25
A. Extracting the low frequency power law	25
B. Extracting hydrodynamic transport coefficients	26
C. Numerical benchmarks	27
IX. THE SPECTRAL BOOTSTRAP: APPROXIMATING THE SPECTRAL FUNCTION AT FINITE FREQUENCIES	28
A. The bulk bootstrap: $\rho = 0$	29
B. Generalization to arbitrary ρ : the Bessel bootstrap	32
X. RANDOM MATRIX UNIVERSALITY	34
A. Background	34
B. Universality in quantum operator dynamics	35
1. Sine universality in the bulk	36
2. Bessel universality at the origin	37
3. Airy universality at the edge	38
C. Scaling of the equilibrium measure	38
XI. THE ROLE OF SMOOTHNESS: FAILURE OF UNIVERSALITY IN A LOCALIZED MODEL	39
ACKNOWLEDGMENTS	39
DATA AVAILABILITY	39

I. INTRODUCTION

Improvements in our understanding of many-body quantum systems can inform the development of new numerical algorithms to simulate them. For example, the realization that gapped 1D ground states have area law entanglement [1] justifies the great success of variational tensor network algorithms [2]. Moving beyond ground states, the study of many-body quantum dynamics presents a long-standing challenge. Due to the rapid growth of entanglement for generic interacting systems [3], the memory requirements for faithful tensor network descriptions typically grow exponentially in time [4]. Given this difficult state of affairs, it is worth asking: are there universal features of many-body quantum dynamics, and if so, how can we utilize this universality to design better algorithms?

Recently an array of algorithms have been proposed that apply new insights from quantum information and quantum chaos to the old problem of hydrodynamics [5–15]. These algorithms work by discarding information about a time-evolved quantum operator that can only be detected using very nonlocal probes, like high-order correlation functions.

If one is primarily interested in physics described by low-order correlation functions, like linear response hydrodynamics, then such approximations can be worthwhile in order to reduce computational requirements. To develop intuition for why this could work, consider the autocorrelation function $C(t) = (O_0|O_0(t))$ of a normalized local operator O_0 evolving in the Heisenberg picture as $O_0(t) = e^{iHt}O_0e^{-iHt}$, and for simplicity let us work at infinite temperature so that $(A|B) = \text{tr}[A^\dagger B]/\text{tr}[\mathbb{1}]$. The correlation function gives the probability amplitude at time t for the time-evolved operator to return to its starting point—that is, it measures the operator “backflow.” Decomposing $O_0(t)$ as a superposition over operator paths, the dominant paths that contribute to $C(t)$ are those that involve only “simple” local operators [13,16]. The intuition is that, at least for ergodic dynamics, once an operator becomes sufficiently non-local, it is very unlikely to shrink down to a small operator again, and will therefore give a negligible contribution to $C(t)$. This suggests that one should be able to neglect such nonlocal operator histories while incurring only a small error in $C(t)$. Where these algorithms differ is in precisely how they choose to discard or approximate these nonlocal operators.

In this paper we give a new approach to systematically approximate the contributions from complicated operator paths. Our approach exploits an emergent random-matrix like universality of quantum operator dynamics. As well as having a practical application in estimating Green’s functions, this therefore also yields a new connection between quantum operator dynamics and random matrix theory. Our results can be phrased in the language of a standard tool from hydrodynamics, the Mori-Zwanzig memory function formalism [17–19]. Given a Hamiltonian H and an initial operator O_0 , consider the Green’s function

$$G(z) = \left(O_0 \left| \frac{1}{z - \mathcal{L}} \right| O_0 \right), \quad (1)$$

where $\mathcal{L}(\cdot) = [H, \cdot]$ is the Liouvillian, and we are using a vectorized notation for operators, $O_0 \leftrightarrow |O_0\rangle$. At the level of linear response, this Green’s function describes the system’s behavior under a perturbation generated by O_0 [19]. In order to link theoretical models with experimental measurements, it is therefore of strong interest to be able to compute objects like $G(z)$. However, doing this analytically is usually intractable for generic interacting systems, and even numerically it remains challenging to compute exactly when the system is large or the frequency z is small, so some approximate treatment is required.

The memory function formalism gives a framework in which to approximately calculate $G(z)$ by separating operators into “fast” and “slow” modes, according to some choice (to be discussed below). The internal dynamics of the fast modes is approximated by making some assumptions about correlation functions between fast operators,

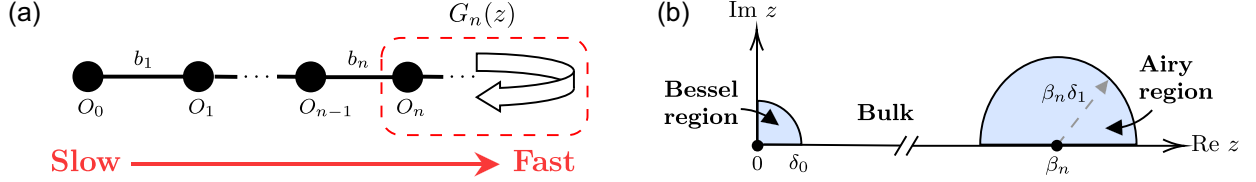


FIG. 1. (a) The semi-infinite 1D operator chain produced by the Lanczos algorithm. The level- n Green's function $G_n(z) = (O_n|(z - \mathcal{L}_n)^{-1}|O_n)$ is the effective Green's function for operator dynamics restricted to operators O_n and above, which we call the fast space (indicated by the dashed box). A good model for the fast space dynamics encoded in $G_n(z)$ can be used to estimate the full Green's function $G(z)$ via the continued fraction in Eq. (3). Intuitively, $G_n(z)$ captures the “backflow” (indicated by the curved arrow) from the fast space to the slow space. (b) We prove that $G_n(z)$ approaches *universal scaling forms* as $n \rightarrow \infty$, with different limits for different regions of the complex z plane. The most prominent example is that $G_n(z)$ approaches the Wigner semicircle law in the bulk of the spectrum, but there can be different behavior near the origin and the spectral edge. This emergent universality is precisely analogous to the universality of eigenvalue correlations of random matrices, even though there is *no explicit randomness* here. For illustration we show only the first quadrant, with the other quadrants obtained by reflection about the axes. Here δ_0 and δ_1 are small $\mathcal{O}(1)$ constants in units of the microscopic couplings, and $\beta_n \approx 2b_n$ determines the bandwidth of the bulk spectrum.

e.g., that they decay exponentially in time. Having reduced this complicated operator dynamics down to the specification of a few free parameters, like exponential decay rates, this fast mode approximation can then be used to close the equations of motion for the slow modes, allowing for an evaluation of their Green's functions. The effectiveness of this approach depends on

- (i) Making a judicious choice of fast and slow modes.
- (ii) The accuracy of the approximation for the fast mode dynamics.

Given the generality of this framework, there are a number of reasonable schemes one could choose to define the fast and slow modes. One approach is “hydrodynamic,” where the slow modes are taken to be those operators that overlap with any conserved quantities related to symmetries of H [19]; they are slow in the sense that their correlation functions typically decay algebraically in time, as opposed to superpolynomial decay for nonconserved operators. This is physically well motivated, but requires a degree of physical insight, particularly if the symmetries are not obvious, such as in integrable systems [20].

We will take a different approach, which in modern language can be thought of in terms of “operator complexity,” although its origins date back to the 1970s [21]. It can be motivated by formally expanding the resolvent $(z - \mathcal{L})^{-1}$ in Eq. (1), from which we see that $G(z)$ will contain contributions from operator paths involving operators of the form $\mathcal{L}^n|O_0\rangle$, $n = 0, 1, \dots$. Generically, the higher the power of n , the more “complicated” and nonlocal these operators become. The rough intuition is that, as these operators become more nonlocal, their internal dynamics becomes faster (this will later be made sharp). One then chooses the slow space to consist of those operators that can be generated by acting only $n < n_{\max}$ times with the Liouvillian $\mathcal{L}$, up to some $n_{\max}$ dictated by computational constraints.

To make this precise, one first forms an orthonormal basis of the operator Krylov space spanned by the operators $\{\mathcal{L}^n|O_0\rangle\}_{n=0}^\infty$. If $\mathcal{L}$ is self-adjoint, this orthogonalization is

called the *Lanczos algorithm*, but it is really just Gram-Schmidt orthogonalization on the Krylov space [22]. This produces two outputs: an orthonormal operator basis $\{O_n\}_{n=0}^\infty$, called the Lanczos basis, and a set of non-negative real numbers $\{b_n\}_{n=1}^\infty$, called the Lanczos coefficients. The Lanczos coefficients have a simple origin: they are just the Gram-Schmidt normalization constants. But they play a privileged role, because it turns out that the matrix representation of the Liouvillian $\mathcal{L}$ restricted to the Lanczos basis is a tridiagonal matrix, where the nonzero matrix elements are given by the Lanczos coefficients:

$$\mathcal{L} = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \quad (2)$$

This matrix can be thought of as a Hamiltonian for an effective 1D hopping model, where the sites of the chain are the basis operators O_n , and the hopping strengths are given by the Lanczos coefficients [see Fig. 1(a)]. The operator dynamics can then be visualized as the evolution of a single-particle wave function $O_0(t)$ on this 1D operator chain. One can associate a notion of operator complexity with this evolution through, say, its average position $\mathcal{K}(t) = \sum_{n=0}^\infty n \times |(O_n|O_0(t))|^2$ along the Lanczos chain. This is known as “operator Krylov complexity,” as introduced in Ref. [11], and has seen a large amount of interest as a tractable measure of operator complexity [23]. Reference [24] proved there is a natural sense in which the Lanczos basis is the “minimal complexity” basis, at least at early times. See Ref. [23] for a review of Krylov complexity, and Ref. [25] for a more general review of quantum dynamics in Krylov space.

Returning to the memory function formalism, having formed this orthonormal operator basis, we take the fast space to be the span of $\{O_m\}_{m \geq n}$ for some large n , with orthogonal fast space projector $\mathcal{Q}_n := \mathbb{1} - \sum_{m=0}^{n-1} |O_m\rangle\langle O_m|$. One useful output of this choice is a continued fraction representation for the original Green's function [26]:

$$G(z) = \frac{1}{z - \frac{b_1^2}{z - \dots - \frac{b_{n-1}^2}{z - b_n^2 G_n(z)}}} \quad (3)$$

The function $G_n(z)$ appearing at level n of the continued fraction is known as the *level- n Green's function*, and is defined as

$$G_n(z) := \left(O_n \left| \frac{1}{z - \mathcal{L}_n} \right| O_n \right), \quad (4)$$

where $\mathcal{L}_n = \mathcal{Q}_n \mathcal{L} \mathcal{Q}_n$ is the Liouvillian restricted to the fast space, corresponding to the lower right block of the tridiagonal matrix in Eq. (2). As such, $G_n(z)$ is a Green's function representing the internal dynamics of the fast space. In terms of the 1D hopping model, it describes dynamics restricted to sites n and above, indicated by the boxed region in Fig. 1(a). The continued fraction representation shows that, if one can accurately describe the fast space dynamics, then one can leverage this to estimate the full many-body Green's function $G(z)$. Choosing such an appropriate “terminator” for $G_n(z)$ forms the basis of a numerical approach called the *recursion method* [26].

A. Relation to previous work

The foundations of the recursion method were developed in the 1970s–1990s [21,26–28]. Making a good choice for $G_n(z)$ is key to the success of the method, and the nature of the continued fraction means this is more nontrivial than, say, simply setting $G_n(z) = 0$ like one might truncate a Taylor series (indeed, this is a very poor approximation [26]). It was quickly recognized that this choice for $G_n(z)$ should be dictated by various “high-level” features of the underlying spectral function

$$\Phi(\omega) := \int_{\mathbb{R}} e^{-i\omega t} (O_0 | O_0(t)) dt, \quad (5)$$

such as its behavior at high and low frequencies, and any gaps in the spectrum. The relevance of $\Phi(\omega)$ is that it turns out to completely determine all the properties of the Lanczos basis (see Secs. II B and III). To some degree, the original practice of the recursion method consisted of finding a toy spectral function with similar high-level features to those expected of $\Phi(\omega)$ on physical grounds, but having analytically computable Lanczos coefficients

and Green's function, which could then be used to terminate the continued fraction [26]. This procedure is not systematic, and depends heavily on the existence of an appropriate solvable toy model, which is not always available. A central goal of our work is to put the recursion method on a more systematic footing, with a particular focus on computing hydrodynamic transport coefficients by estimating $\Phi(\omega)$ at low frequencies.

The recursion method received renewed attention in the late 2010s during an explosion of interest in operator dynamics and scrambling [29], prompted by questions related to the black hole information paradox [30–32] and the dynamics of quantum many-body systems [33,34]. In particular, Ref. [11] stated the “operator growth hypothesis” (OGH). The OGH conjectures that one characteristic of quantum chaotic systems is that operators grow as fast as possible, limited only by the locality of interactions. Locality places an upper limit on operator growth by bounding the possible number of operator trajectories. The OGH can be formulated precisely as the statement that, in a generic quantum chaotic system, the Lanczos coefficients b_n grow *linearly* as $n \rightarrow \infty$ (with a logarithmic correction in one dimensional systems). This is the fastest possible growth rate in lattice systems with a local Hamiltonian [11]. Under mild assumptions on the spectral function $\Phi(\omega)$, linear growth of b_n is implied by $\Phi(\omega)$ decaying exponentially as $|\omega| \rightarrow \infty$ [35]; that is, the OGH is a statement about *high-frequency* behavior. Although this upper bound on b_n does seem to be saturated in many quantum chaotic systems [11], it turns out that some non-chaotic systems also saturate the upper bound [36–38], so satisfying the OGH can be seen only as a necessary but not sufficient requirement for a system to be considered quantum chaotic.

A challenging aspect of the recursion method is that, while high-frequency features are clearly reflected in the gross structure of the Lanczos coefficients, the imprints of low-frequency behavior are reflected in finer structure. Various solvable models have helped decode how different low-frequency phenomena are exhibited in subleading effects, particularly in differences between the Lanczos coefficients for odd and even n , referred to as “staggering.” Reference [26] shows some examples where the spectral function $\Phi(\omega)$ has a power law at low frequencies, $\Phi(\omega \rightarrow 0) \sim |\omega|^\rho$, and how the value of ρ shows up in the coefficient of a subleading staggered term in the Lanczos coefficients. Other authors have studied the signatures of long-lived zero modes [39–42], many-body localization [43], and mass gaps in quantum field theories [44].

Motivated by the OGH, Ref. [11] provides an exactly solvable spectral function with asymptotically linear Lanczos coefficients, and uses this model as a terminator for $G_n(z)$ in order to estimate the energy diffusion constant of the chaotic mixed field Ising model (see also Refs. [45,46] for similar approaches). This toy spectral

function was effectively chosen only to match the high-frequency behavior of $\Phi(\omega)$ through the linear growth of Lanczos coefficients. But the relation between $G(z)$ and $G_n(z)$ in the continued fraction in Eq. (3) holds pointwise in the frequency z , so it is *a priori* unclear why a choice of $G_n(z)$ catered to the high-frequency behavior should have been successful at reproducing a low-frequency property like a diffusion constant. We address this question in Sec. VI.

While these solvable examples are illustrative, there are relatively few general results proving a definitive link between a given low-frequency feature and its signature in Lanczos space. Such results would be valuable for laying solid foundations for new numerical methods. Most existing general results are limited to spectral functions with finite bandwidth [47,48], corresponding to Lanczos coefficients with bounded growth—these results are appropriate for non-interacting systems [26] and finite-size systems. By contrast, interacting systems have spectral functions with infinite support in the thermodynamic limit [26], corresponding to Lanczos coefficients with unbounded growth, and rigorous results [49] on these systems are rare and usually disconnected from the recursion method literature. Within this class, results on systems with asymptotically linear Lanczos coefficients—the generic case according to the OGH—are even more technically challenging, because these systems turn out to be “marginal” [50,51] in a sense we will describe in Secs. II D and II G, and this marginality makes proofs particularly delicate.

Assuming only a few physically motivated conditions on $\Phi(\omega)$, stated in Sec. IV, one of our main results is that, as $n \rightarrow \infty$, $G_n(z)$ approaches *universal scaling forms*, to be discussed further in Secs. II A and VI. This universality can then be exploited to estimate the original Green’s function $G(z)$. As we will explain in Sec. II C, the universality of $G_n(z)$ turns out to be precisely analogous to the universality of eigenvalue correlations of $n \times n$ random matrices. Importantly, our assumptions on the spectral function include the generic case of linearly growing Lanczos coefficients, and much of the technical work in our proof is targeted at this case. One notable difference from previous approaches based on the recursion method [45,46,52] is that we explicitly consider the role of conservation laws in determining the operator backflow encoded in $G_n(z)$. This is particularly relevant for spectral functions $\Phi(\omega)$ that have power-law divergences as $\omega \rightarrow 0$, which we show leads to a new “Bessel universality class” describing the behavior of $G_n(z)$ near $z = 0$. Not accounting for this modified universality can lead to erroneous estimates of the spectral function at low frequencies, as we discuss in Sec. IX. These results thus give a principled theoretical framework for numerical approaches based on the recursion method.

The appearance of random matrix theory (RMT) in this context may seem surprising *a priori*, since we did not explicitly impose any randomness in either the Hamiltonian H or the initial operator O_0 . Of course, RMT is also very

well studied in the context of quantum chaos [53], where it is conjectured that quantum chaotic systems have energy eigenvalue statistics that can be described in terms of random matrix ensembles. A natural question to ask is whether the RMT universality we discuss here is related to the appearance of RMT in quantum chaos, and we delineate the differences in Sec. II E. Connections between the Lanczos algorithm and random matrix theory were also discussed in Ref. [54], but not with a focus on universality akin to that of eigenvalue correlations, as we discuss here; instead they focused on the resulting Lanczos coefficients when H itself is a random matrix.

At a higher level, the recursion method is one of a number of approaches that seek to simplify quantum operator dynamics by somehow approximating the contributions from nonlocal operators. Some approaches, including dissipation-assisted operator evolution (DAOE) [5,13,15,55] and density matrix truncation (DMT) [10], suppress nonlocal operators based on a fairly explicit notion of weight or size. The recursion method is distinct from these approaches in that it seeks to approximate, rather than explicitly discard, the contributions from nonlocal operators $O_{n \gg 1}$. In this sense it has parallels with local information time evolution (LITE) [7,8], but there are also moral similarities with DAOE and DMT since the common assumption is that the contributions from nonlocal operators should be “small” in some sense. One advantage of the Lanczos setting is that it has enough mathematical structure that one can really prove rigorous results about the “operator backflow” that is the subject of these approximations. Furthermore, we view the recursion method not as a direct competitor to those approaches but rather as complementary, and we discuss in Sec. II H how one might utilize a “DAOE-enhanced” Lanczos algorithm to improve the rate of convergence with n for estimates of hydrodynamic transport coefficients.

II. SYNOPSIS

A. Emergent universality in the $n \rightarrow \infty$ limit

When the memory function formalism was first developed it was hoped that, if n is sufficiently large, then all non-universal features of the original dynamical problem would be filtered out into the coefficients $\{b_n\}$, and then the level- n Green’s function $G_n(z)$ could be approximated by assuming it is ultra short-range correlated in time, which amounts to setting $G_n(z)$ to be constant [19,26]. In certain cases this turns out to be well founded. As discussed in Sec. VI, one of our main results is that, as $n \rightarrow \infty$, for z in the “bulk” of the spectrum, $G_n(z)$ approaches the Wigner semicircle law:

$$G_n(z) \approx \frac{2}{\beta_n^2} \left(z - \sqrt{z + \beta_n} \sqrt{z - \beta_n} \right). \quad (6)$$

Here β_n is an effective n -dependent frequency bandwidth approximated by $\beta_n \approx 2b_n$ as $n \rightarrow \infty$, and bulk means

$|z| > \delta_0$ and $|z/\beta_n \pm 1| > \delta_1$ for some small $\mathcal{O}(1)$ constants δ_0, δ_1 in units of the microscopic couplings [see Fig. 1(b)]. That this should be called a semicircle is perhaps clearer from considering the corresponding bulk spectral function,

$$\Phi_n(\omega) = 2 \operatorname{Im} [G_n(\omega - i0^+)] \approx \frac{4}{\beta_n} \sqrt{1 - (\omega/\beta_n)^2}. \quad (7)$$

Note that this semicircle form of $G_n(z)$ is identical to the average global resolvent $(1/n) \operatorname{tr}[1/(z - M)]$ —which encodes dynamics in random matrix theory—for random $n \times n$ matrices M drawn from the Gaussian unitary ensemble [56], rescaled so that the bulk eigenvalue spectrum corresponds to the interval $[-\beta_n, \beta_n]$. Thus we see the emergence of random matrix universality—even though there is no explicit randomness present—in the bulk frequency profile of the “fast space” operator dynamics, i.e., restricted to $\{O_m\}_{m \geq n}$. Since the bandwidth scales like $\beta_n \approx 2b_n$ to leading order in n , if the Lanczos coefficients grow indefinitely, $b_n \rightarrow \infty$, we conclude that $G_n(z) \approx \pm 2i/\beta_n + \mathcal{O}(z/\beta_n^2)$ is approximately constant to leading order in z/β_n , thereby justifying the assumption of short-range time correlations as $n \rightarrow \infty$. This also provides a precise sense in which the large- n dynamics are “fast,” since $G_n(z) = \mathcal{O}(1/\beta_n)$ tends to zero as $n \rightarrow \infty$.

For spectral functions $\Phi(\omega)$ that are complex analytic at $\omega = 0$, corresponding to autocorrelation functions $C(t)$, that decay exponentially in time, we further show that this semicircle behavior persists all the way down to zero frequency. We conjecture that this remains true even if $C(t)$ decays algebraically, provided it is faster than $1/t$ —but that there is a slower decay of the finite- n correction to the semicircle law in this case. This conjecture is based on similar asymptotic analysis in prior work [57], which demonstrated how the assumed degree of differentiability affects the scaling of the error term, not the leading term.

We note that the appearance of the Wigner semicircle law in the level- n Green’s function $G_n(z)$ was recognized by Ref. [58], but only in the context of a *finite*-dimensional system where n is so large that the Lanczos coefficients have plateaued (typically this happens once n is of order the system size), so they can utilize rigorous results from Ref. [59]. Our analysis is instead focused on interacting systems in the thermodynamic limit, where the Lanczos coefficients will generically grow indefinitely. We show that it is the $n \rightarrow \infty$ limit (i.e., not the plateau itself), which results in emergent random matrix universality. Besides the interpretational difference, this is relevant for numerical applications to interacting systems, where one will generically not see a plateau without incurring finite-size effects.

When the autocorrelation function $C(t)$ decays as a power-law slower than $1/t$, this results in power-law behavior of $G_n(z)$ near $z = 0$, so that the semicircle form is no longer appropriate. Such power-law decay in $C(t)$ is typical if the initial operator overlaps with a conserved

quantity with sufficiently slow transport, say diffusive in one dimension [19]. With hydrodynamics in mind, one of our principal goals will be to characterize how signatures of the long-time behavior of the autocorrelation function $C(t)$ imprint themselves on the Lanczos basis, and in turn how this affects the operator backflow encoded in the Green’s functions $G_n(z)$. When the spectral function behaves like a power law at low frequencies, we show that this modifies the behavior of $G_n(z)$ for $z \rightarrow 0$, such that the semicircle form breaks down, and $G_n(z)$ instead has an explicit expression in terms of Bessel functions (see Sec. VI B). This is a reflection of “Bessel universality” that governs the low frequency behavior (see Fig. 1). With these fingerprints of low-frequency behavior in hand, we show how to approximate low-frequency data like hydrodynamic transport coefficients, using only a finite number of Lanczos coefficients as input (see Sec. VIII). We benchmark this approach on a range of physical models, including the mixed field Ising model and the XXZ spin chain, finding results that are competitive with tensor network methods.

Finally, for frequencies near the edge of the spectrum $z = \pm\beta_n$, there is a third “Airy” universality class [see Fig. 1(b)], so-called because $G_n(z)$ approaches a universal scaling form in terms of Airy functions (see Sec. VI C). This is analogous to the modified behavior of eigenvalue correlations near the edge of the spectrum of a random matrix [60]. In Sec. S6 we also show how to utilize this Airy universality to estimate spectral functions at very high frequencies.

B. Orthogonal polynomials and large- n expansions

Our proof makes use of some powerful machinery from complex analysis, developed in the study of orthogonal polynomials [61–67]. This is relevant because the Lanczos algorithm can be naturally phrased in the language of orthogonal polynomials (see Sec. III). Indeed, the n th Lanczos basis operator $|O_n\rangle$ is given by

$$|O_n\rangle = p_n(\mathcal{L})|O_0\rangle, \quad (8)$$

where the polynomials $p_n(\omega)$ are orthonormal with respect to the spectral function, i.e.,

$$\int_{\mathbb{R}} p_m(\omega) p_n(\omega) \frac{\Phi(\omega)}{2\pi} d\omega = \delta_{mn}. \quad (9)$$

One can similarly write an expression for the level- n Green’s function $G_n(z)$ in terms of these orthogonal polynomials [see Eq. (10)]. The complex analytic machinery works by devising a Riemann-Hilbert problem whose solution encodes essential data about these orthogonal polynomials with respect to $\Phi(\omega)$, which in turn gives us information about the Lanczos basis (see Sec. S2 for details [68]). The advantage of this formulation is that it permits approximate solutions, controlled in the limit

$n \rightarrow \infty$, even without knowing the explicit form of the spectral function $\Phi(\omega)$. Instead one only needs to know some “high level” features of $\Phi(\omega)$, like its rate of decay at high frequencies, and whether it has a power-law at low frequencies. Then one can use a technique similar to steepest descent to obtain a $1/n$ expansion of the polynomials $p_n(\omega)$ (Secs. VIII and IX), the recurrence coefficients b_n (Theorem 1), the level- n Green’s function $G_n(z)$ (Sec. VI), and other related quantities. We remark that the $n \rightarrow \infty$ limit is quite natural, in the sense that we expect the operator Krylov space $\mathcal{K} = \text{span}\{\mathcal{L}^n O_0\}_{n \geq 0}$ to be infinite dimensional in the thermodynamic limit, for generic local operators O_0 and generic Hamiltonians. This provides one with the means to perform a large- n expansion of the operator dynamics, even when there is no explicit large parameter in the Hamiltonian.

Since this steepest descent approach involves deformation of contours into the complex plane, our proof technique requires us to make some assumptions about the complex analytic structure of the spectral function $\Phi(\omega)$ near the real frequency axis; this is the main technical caveat to our work, and we discuss it in more detail in Secs. IV and IV C. However, numerical evidence suggests that some of our results may remain valid, at least to leading order in n , assuming only weaker conditions than analyticity, such as differentiability of $\Phi(\omega)$. Indeed, a recent breakthrough result in random matrix theory proved universality assuming only a very mild local continuity condition [69], and it would be interesting to see if these techniques could be adapted for our purposes. Finally, we emphasize that the Riemann-Hilbert formulation is used only for the proof, and readers only interested in practical applications do not need to make use of it.

C. Lanczos dynamics and random matrix universality

As previously discussed, one of our main results is that, as $n \rightarrow \infty$, the level- n Green’s function $G_n(z)$ approaches universal scaling forms, the most prominent example being the Wigner semicircle law for z in the bulk of the spectrum. In what follows, we will discuss how the universality of $G_n(z)$ and related quantities is analogous, in a precise sense, to the universality of eigenvalue correlations of random matrices. Mathematically, this shared universality can be understood from the fact that both Lanczos operator dynamics and random matrix theory can be formulated in the language of orthogonal polynomials. These polynomials themselves exhibit universal scaling forms in the $n \rightarrow \infty$ limit, which then implies universality in both Lanczos operator dynamics and random matrix theory.

While these formulations in terms of orthogonal polynomials are well known to practitioners of both fields, it is worth spelling out how they work in some detail.

In the Lanczos context, we already saw the connection with orthogonal polynomials in Eqs. (8) and (9): the n th Lanczos operator O_n can be written in terms of the n th orthogonal polynomial $p_n(\omega)$ with respect to the spectral function $\Phi(\omega) = \int_{\mathbb{R}} e^{-i\omega t} C(t) dt$. To get from there to the level- n Green’s function, one can employ the useful identity [59]

$$G_n(z) = \frac{1}{b_n} \frac{C_n(z)}{C_{n-1}(z)},$$

$$\text{where } C_n(z) \equiv \int_{\mathbb{R}} \frac{p_n(\omega) \Phi(\omega)}{z - \omega} \frac{d\omega}{2\pi}. \quad (10)$$

Unfortunately, $G_n(z)$ does not, in general, have an interpretation as a simple low-order correlation function of a random matrix ensemble. In that sense, the result for the bulk of the spectrum is a special case—albeit one that applies almost everywhere in the complex z plane—where $G_n(z)$ approaches the Wigner semicircle law, and can then indeed be identified with a simple RMT correlation function. But nonetheless, even away from the bulk, $G_n(z)$ does still approach universal scaling forms, which can be understood in terms of universality of the orthogonal polynomials $p_n(\omega)$ in the $n \rightarrow \infty$ limit.

In the random matrix theory context, orthogonal polynomials provide a useful tool for computing eigenvalue correlation functions [70]. Consider an ensemble of $n \times n$ random Hermitian matrices M with probability density $P(M) \propto \exp(-\text{tr}[Q(M)])$, for some function Q called the *potential*. The Gaussian unitary ensemble (GUE) corresponds to $Q(x) = x^2$, but it can be useful to consider more general functions Q to introduce correlations between matrix elements. One can instead consider the distribution on the eigensystem of M rather than its matrix elements; for general Q , integrating out the eigenvectors produces a distribution on the eigenvalues given by

$$P_Q(\lambda_1, \dots, \lambda_n) \propto \left(\prod_{i < j} |\lambda_i - \lambda_j|^2 \right) \prod_i e^{-Q(\lambda_i)}, \quad (11)$$

resulting in the familiar Vandermonde determinant $\prod_{i < j} |\lambda_i - \lambda_j|^2$ responsible for eigenvalue repulsion. It is then natural to consider the k -point eigenvalue correlation function $R_k(\lambda_1, \dots, \lambda_k)$ defined by

$$R_k(\lambda_1, \dots, \lambda_k) = \frac{n!}{(n-k)!} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} P_Q(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \dots, \lambda_n) d\lambda_{k+1} \cdots d\lambda_n. \quad (12)$$

The key fact linking RMT and orthogonal polynomials is that R_k can be expressed as a $k \times k$ matrix determinant,

$$R_k(\lambda_1, \dots, \lambda_k) = \det \left[(\hat{K}_n(\lambda_i, \lambda_j))_{1 \leq i, j \leq k} \right], \quad (13)$$

where each matrix element is given by evaluating a 2-point correlation kernel $\hat{K}_n(\lambda_i, \lambda_j)$. The kernel is defined by

$$\hat{K}_n(\lambda_i, \lambda_j) = \sqrt{e^{-Q(\lambda_i)} e^{-Q(\lambda_j)}} \sum_{m=0}^{n-1} p_m(\lambda_i) p_m(\lambda_j), \quad (14)$$

where the p_m are orthogonal polynomials satisfying

$$\int_{\mathbb{R}} p_k(\lambda) p_l(\lambda) e^{-Q(\lambda)} d\lambda = \delta_{kl}. \quad (15)$$

Thus we see that eigenvalue statistics of random matrices can be described in terms of orthogonal polynomials with respect to $e^{-Q(\lambda)}$, the potential defining the random matrix ensemble. One of the deepest facts about random matrices is that their eigenvalue statistics can be *universal*, when probed on a local scale [60]. Here “local scale” means that we consider correlations over $\mathcal{O}(1)$ separations in units of the inverse local eigenvalue density, and “universal” means they are independent of the exact eigenvalue probability distribution. Given the formulation in terms of orthogonal polynomials in Eq. (15), one can derive RMT universality from universality of the orthogonal polynomials themselves.

To relate Lanczos operator dynamics to random matrix theory, let us summarize the discussion so far: the Lanczos basis operators O_n are related to orthogonal polynomials with respect to $\Phi(\omega)/2\pi$ [see Eq. (9)], while RMT eigenvalue statistics are related to orthogonal polynomials with respect to $e^{-Q(\lambda)}$ [see Eq. (15)]. Therefore, under the identification of weight functions:

$$\frac{\Phi(\omega)}{2\pi} \equiv e^{-Q(\omega)}, \quad (16)$$

we see that there is a connection between

n th Lanczos operator $O_n \leftrightarrow n \times n$ random matrices.

Much as one can derive RMT universality from universality of the orthogonal polynomials defining the matrix ensemble, one can derive universality of the level- n Green’s function $G_n(z)$ and related quantities from universality of the orthogonal polynomials with respect to the spectral function $\Phi(\omega)$. Universal asymptotics of orthogonal polynomials is a much older phenomenon than RMT universality, going back to work in the 1920s of Plancherel and Rotach on $n \rightarrow \infty$ asymptotics for the Hermite polynomials [$Q(x) = x^2$] [71]. One of our technical contributions is proving so-called “Plancherel-Rotach asymptotics” of the

orthogonal polynomials for a large class of spectral functions obeying physically-motivated conditions (described in Sec. IV). From there, one is led to universality for the level- n Green’s function.

D. Coulomb gas confinement transition, quantum chaos, and the operator growth hypothesis

Stemming from the foundational work of Dyson [72], a key concept in random matrix theory (RMT) is the Coulomb gas [73]. To give context for our result about a Coulomb gas confinement transition, let us briefly illustrate where this comes from, both in random matrix theory and in Lanczos operator dynamics.

To compute eigenvalue correlation functions in RMT, one needs to consider integrals with respect to the eigenvalue probability distribution $P_Q(\lambda_1, \dots, \lambda_n)$. For reasons that will soon become apparent, it turns out to be important to rescale by an n -dependent factor, $\lambda_i \rightarrow \beta_n x_i$, where the scale factor β_n is determined by Q and the x_i are $\mathcal{O}(1)$; for $Q(\lambda) = \lambda^2$ we have $\beta_n = \sqrt{2n}$. Then, defining the rescaled potential $V_n(x) \equiv Q(\beta_n x)/n$, one can rewrite Eq. (11) as

$$P_Q(\lambda_1, \dots, \lambda_n) \propto e^{-\left[\sum_{i \neq j} \log |x_i - x_j|^{-1} + n \sum_i V_n(x_i) \right]}, \quad (17)$$

and so one expects the leading contributions to come from tuples $\mathbf{x} = (x_1, \dots, x_n)$ for which the exponent is minimal. For any such tuple $\mathbf{x}$, if we introduce the normalized counting measure $d\psi_{\mathbf{x}}(y) = (1/n) \sum_{i=1}^n \delta(y - x_i) dy$, then we are led to consider the Coulomb gas energy functional

$$E_{V_n}[\psi] = \int_{\mathbb{R}} \int_{\mathbb{R}} \log |x - y|^{-1} d\psi(x) d\psi(y) + \int_{\mathbb{R}} V_n(x) d\psi(x). \quad (18)$$

In the $n \rightarrow \infty$ limit, eigenvalue correlation functions will be dominated by configurations ψ which minimize this Coulomb gas energy. The rescaling $\lambda_i \rightarrow \beta_n x_i$ served the purpose of ensuring that the two energy terms are of comparable magnitude as $n \rightarrow \infty$. In general, this minimal configuration gives the *mean eigenvalue density profile* for the random matrix ensemble defined by the potential Q . For $Q(\lambda) = \lambda^2$, the minimizing configuration is the Wigner semicircle law.

Remarkably, the same Coulomb gas ensemble also plays a central role in the $n \rightarrow \infty$ asymptotics of orthogonal polynomials [65]. The easiest way to see this is through the following explicit formula for the orthogonal polynomials $p_n(\omega)$ with respect to the weight function $e^{-Q(\omega)}$:

$$p_n(\omega) \propto \int \cdots \int \left(\prod_{i=1}^n (\omega - \omega_i) \right) \times P_Q(\omega_1, \dots, \omega_n) d\omega_1 \cdots d\omega_n, \quad (19)$$

where the probability density $P_Q(\omega_1, \dots, \omega_n)$ was defined in Eq. (15). In other words, $p_n(\omega)$ can be expressed as an average over all degree- n polynomials, weighted by the *same* probability measure as that which governed the eigenvalue distribution of the random matrix ensemble defined by the potential Q . In this context, the minimal energy configuration of the charge- n Coulomb gas—which we call $\sigma_n(\omega)$ —gives the *limiting density distribution of zeros* of the orthogonal polynomials with respect to $e^{-Q(\omega)}$. In our Riemann-Hilbert steepest descent analysis (Sec. S2 [68]), the support of this minimal energy configuration is analogous to the “oscillatory region” of a WKB approximation, and yields the dominant contribution to the steepest descent problem as $n \rightarrow \infty$.

With the relevance of the Coulomb gas established, we can now discuss its confinement transition and the connection to quantum chaos. For systems with local interactions, one can show that the spectral function must decay at least exponentially as $|\omega| \rightarrow \infty$ [74–76], which translates, via Eq. (16), into the requirement that the potential $Q(\omega)$ grow at least linearly at large ω . This is enough to conclude that the Coulomb gas is always confined, in the sense that, for any finite n , the equilibrium Coulomb gas density $\sigma_n(\omega)$ has finite support. However, within the confined phase, there can be a further phase transition between “weak confinement” and “strong confinement” (see Fig. 2) [77–80]. These phases are distinguished by whether the Coulomb gas density is approximately uniform or not within the bulk of its support.

Importantly, the order parameter for this transition is the equilibrium density at *low frequencies*, $\sigma_n(\omega \approx 0)$. In the weakly confined phase, $\sigma_n(\omega)$ has an algebraic divergence as $\omega \rightarrow 0$, while in the strongly confined phase it is approximately constant there; at the critical point, there is instead a logarithmic divergence as $\omega \rightarrow 0$ (see Fig. 5 for an example). What is interesting is that, despite being diagnosed by the density near $\omega = 0$, the transition itself is primarily driven by the rate of *high frequency* decay of the

spectral function. In particular, the weakly confined phase, the critical point, and the strongly confined phase, correspond to $\Phi(\omega \rightarrow \infty)$ decaying subexponentially, exponentially, and superexponentially respectively (Fig. 2). This critical exponential decay—the slowest possible decay consistent with locality [74–76]—coincides with the behavior that is expected to occur *generically* for chaotic systems with local interactions; this is the content of a recent conjecture in quantum chaos, the operator growth hypothesis [11]. Thus, assuming the OGH, there is a sense in which chaotic systems are generically marginal. Since this confinement transition is related to the density of zeros of the orthogonal polynomials $p_n(\omega)$, which in turn govern the Lanczos operators O_n via Eq. (8), this marginality has implications for the computational efficiency of numerical applications, as we will discuss. One of our contributions (Lemma 2) is showing that this confinement phase structure is robust, in the sense that we prove it occurs for a wide class of spectral functions assuming only a fairly weak specification of how they decay at high frequencies (see Sec. IV).

E. Comparison with eigenstate thermalization

The main technical caveat to our work is that we also impose some regularity and analyticity conditions on $\Phi(\omega)$ in order to prove rigorous statements; we discuss this in detail in Sec. IV. But with these conditions in mind, let us discuss the relationship between the appearance of random matrix theory in our work, and more familiar appearances of RMT in quantum many-body physics, namely through the eigenstate thermalization hypothesis (ETH) [53,81]. The ETH postulates that the matrix elements of a local operator O in the energy eigenbasis of a chaotic local Hamiltonian take the form

$$\langle E_m | O | E_n \rangle = O(\bar{E}) \delta_{mn} + e^{-S(\bar{E})/2} f_O(\bar{E}, \omega) R_{mn}, \quad (20)$$

where $\bar{E} \equiv (E_m + E_n)/2$, $\omega \equiv E_n - E_m$, and $S(\bar{E})$ is the thermodynamic entropy at energy $\bar{E}$. RMT appears through the entropic factor $e^{-S(\bar{E})/2}$ and the random matrix R , which is postulated to be distributed according to a Gaussian random matrix ensemble [53].

An important part of the ETH is that the functions $O(\bar{E})$ and $f_O(\bar{E}, \omega)$ are postulated to be *smooth* functions of their arguments. For our purposes, working in the thermodynamic limit, we do assume some smoothness properties of the spectral function $\Phi(\omega)$, which coincides with the function $|f_O(E_{\text{av}}, \omega)|^2$ at the average energy $E_{\text{av}} = \text{tr}[H]/\text{tr}[1]$ of the infinite temperature state [53]. Therefore, arguably the most natural class of physical systems that one might expect to obey our assumptions about the spectral function are those that obey the ETH. However, the RMT universality we discuss in our work appears to be distinct in nature from the appearance of random matrix theory in the ETH,

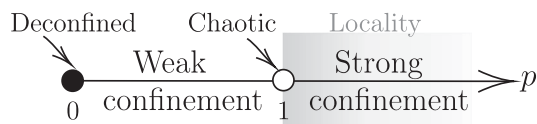


FIG. 2. Phase diagram of Coulomb gas confinement with a single-particle potential $Q(\omega)$, controlled by the large frequency decay of the spectral function $\Phi(\omega) \sim \exp[-Q(\omega)]$. Locality of the Hamiltonian forces the potential to grow polynomially at large ω , $Q(|\omega| \rightarrow \infty) \sim |\omega|^p$, with $p \geq 1$. The operator growth hypothesis [11] posits that chaotic systems generically have $p = 1$. A Coulomb gas with finite charge (i.e., finite n) is confined for all $p > 0$ since the two-particle repulsion is only logarithmic, while the potential is polynomial. However, there is a transition between weak and strong confinement at $p = 1$, precisely at the boundary imposed by locality. This has implications for numerical applications.

since it really owes itself to the *regularity* of spectral functions (and the consequence of that on the Lanczos orthogonal polynomials), which can be achieved even in nonergodic single-particle and integrable systems. Indeed, we also find some evidence of this RMT universality in the integrable XXZ chain, where the presence of integrability should violate the ETH (although a weak version may persist [82]). Validity of the ETH may therefore be a sufficient but not necessary condition for the RMT universality we discuss here, but a fuller investigation of these connections is an important topic for future work.

We also give some evidence in Sec. XI that smoothness of spectral functions is indeed a necessary condition for this fast space universality. We focus on the 1D transverse field Ising model with quenched disorder, which, owing to the disorder, is provably Anderson localized and has point-like spectrum even in the thermodynamic limit [83]. We perform a self-consistency check for the presence of universality in this model—which was passed by the clean version—and find that this check fails in the presence of disorder, suggesting the breakdown of fast space universality.

F. Numerical application: The spectral bootstrap

As described in the Introduction, the recursion method is a numerical technique based around the continued fraction representation for the full Green’s function $G(z)$; traditionally it requires one to make an educated guess for the functional form of the level- n Green’s function $G_n(z)$ [26]. Our approach allows us to rigorously derive the functional behavior of $G_n(z)$, showing how this changes for z in different regions of the complex plane (see Fig. 1). This provides a principled means of terminating the continued fraction and thereby approximating the full Green’s function $G(z)$.

We show how to use this control afforded by the large- n limit to produce an approximation of the spectral function $\Phi(\omega)$ using a finite number of Lanczos coefficients. Our focus is particularly on the low frequency behavior, and we show how to extract diffusion constants, as well as transport coefficients in models with nondiffusive transport, such as the isotropic Heisenberg model which has superdiffusive spin transport [84]. We also show how to generalize these ideas to finite frequencies. The simplest case is when the semicircle form for the level- n Green’s function $G_n(z)$ is valid at the target frequency, because then only a single parameter, the frequency bandwidth β_n , needs to be specified in order to fully fix the semicircle Green’s function—and β_n can be simply approximated from the Lanczos coefficients via $\beta_n = 2b_n[1 + \mathcal{O}(1/n)]$ (see Theorem 1). However, there are also more complicated cases, such as when $\Phi(\omega)$ has a low-frequency power-law, where an additional phase, arising from an integral over the Coulomb gas density, needs to be specified in order to approximate $G_n(z)$. This is the content of our “spectral bootstrap” algorithm. It works by using the appropriate

$n \rightarrow \infty$ asymptotics of the orthogonal polynomials $p_n(\omega)$ to formulate a first-order ordinary differential equation in frequency space involving the spectral function and the desired phase factor. Iteratively solving this ODE gives a finite- n approximation to the spectral function, using only the first n Lanczos coefficients as input.

One interesting aspect of this is that the convergence rate of this approximation at $\omega = 0$ is determined by a combination of the high- and low-frequency behavior of the spectral function $\Phi(\omega)$, owing to the manifestation of the Coulomb gas confinement transition in the Coulomb gas density at low frequencies. We show that the rate of convergence is $1/\text{poly}(n)$ when the spectral function $\Phi(\omega \rightarrow \infty)$ decays superexponentially, regardless of the low-frequency scaling of $\Phi(\omega)$. However, when $\Phi(\omega \rightarrow \infty)$ decays only (quasi-)exponentially—which is the case conjectured to be generic for chaotic quantum systems by the operator growth hypothesis [11]—then the convergence rate is only $1/\text{poly}(\log n)$ if $\Phi(\omega \rightarrow 0) \sim |\omega|^\rho$ with $\rho \neq 0$. We recover $1/\text{poly}(n)$ scaling when $\rho = 0$, but this is likely a consequence of the fact that, for $\rho = 0$, our assumptions on $\Phi(\omega)$ amount to assuming analyticity at $\omega = 0$, i.e. that the correlation function $C(t)$ decays exponentially in time. We conjecture that when $0 < \Phi(0) < \infty$ but $\Phi(\omega)$ is not analytic at $\omega = 0$ due to hydrodynamic tails, then the convergence rate again reduces to $1/\text{poly}(\log n)$. However, despite this worst-case theoretical convergence, in practice we find that the spectral bootstrap can still often give surprisingly accurate estimates using only a modest number of Lanczos coefficients (see, e.g., Fig. 9). Alternatively, since the convergence rate of the spectral bootstrap is better for nonchaotic systems than chaotic ones, one can view it as being complementary to other methods like dissipation-assisted operator evolution [5,13,15,55], whose accuracy is best justified for chaotic systems [13].

G. Marginality and quantum chaos

One curious aspect of this work concerns the connection between “marginality” and chaotic operator dynamics. In our work the spectral function $\Phi(\omega)$ takes center stage, since it completely characterizes the Lanczos basis. In that role it functions mainly as the density of a measure $d\mu(\omega) = [\Phi(\omega)/2\pi]d\omega$ on $\mathbb{R}$, where it defines the particular set of orthogonal polynomials constructed by the Lanczos algorithm. Its origin in quantum operator dynamics is left somewhat implicit, entering only through constraints at high- and low-frequencies coming from considerations of locality and hydrodynamics.

Our numerical algorithm, the spectral bootstrap, gives one solution to the “spectral inverse problem” of recovering the measure $\Phi(\omega)/2\pi$ from its moments (which is equivalent to recovery from the Lanczos coefficients [85]). Relatedly, the “Hamburger moment problem” asks whether there exists a measure μ generating a given set of moments,

and if so whether this measure is unique [86]. If it is unique then we say the moment problem for μ is *determinate*, and if not then it is *indeterminate*. By definition it is then impossible to uniquely recover a measure from its moments if the corresponding moment problem is indeterminate. Crucially, it is well known that the moment problem is determinate for all measures which decay at least *exponentially* at infinity, meaning $\int_{\mathbb{R}} e^{c|x|} d\mu(x) < \infty$ for some $c > 0$ [51]. For example, the “Freud weights” $d\mu(x) = \exp(-|x|^p) dx$ are determinate for $p \geq 1$ and indeterminate for $p < 1$ [51].

What seems curious is that this determinate-indeterminate boundary happens to be precisely where the operator growth hypothesis conjectures generic spectral functions of locally interacting quantum chaotic systems should fall, in the sense that they should generically have exponentially decaying spectral functions, $\Phi(\omega \rightarrow \infty) \sim \exp[-\mathcal{O}(\omega)]$ [11]. It is fitting, then, that in our spectral bootstrap algorithm we find that the error bounds at $\omega = 0$ decay increasingly slowly the slower the high-frequency decay of the spectral function, as discussed in the previous section. These error bounds control how large n must be to approximate $\Phi(\omega = 0)$ to a given precision ϵ , so we see that the inverse problem becomes increasingly difficult as the determinate-indeterminate boundary is approached, before becoming impossible (by definition) in the indeterminate regime.

Thus, according to the OGH, recovery of spectral functions of chaotic systems is *exponentially hard* in n as a function of the inverse precision $1/\epsilon$. From the vantage of the original operator growth problem, it is not so surprising that chaotic systems are hard to simulate; for example, it is known that the operator entanglement entropy generically grows linearly in time for chaotic systems [87], rendering tensor network descriptions challenging. But it is perhaps surprising that, in the context of the Lanczos algorithm, where entanglement entropy plays no obvious role, nonetheless the spectral theory should conspire so as to make this problem hard.

To be clear, in the context of this section, quantum chaos is a *sufficient but not necessary* condition for marginality. More generally the requirement for marginality is (quasi-)linear growth of Lanczos coefficients, and there are examples of nonchaotic systems exhibiting this growth, such as certain operators in many-body localized systems [36].

H. Future directions

1. Combining the spectral bootstrap with dissipation-assisted approaches

In Sec. II F we discussed that, in the Lanczos context, it is actually the high-frequency scaling of the spectral function $\Phi(\omega)$ that determines the computational difficulty of recovering $\Phi(\omega)$ at low frequencies. But it is often the

case that one is principally interested in characterizing the low-frequency behavior of a system, say for computing hydrodynamic transport coefficients, and the accuracy of estimates of high-frequency properties are of lesser concern. That motivates studying potential modifications of the system’s dynamics which cause $\Phi(\omega)$ to decay super-exponentially as $\omega \rightarrow \infty$, thereby improving the error bound scaling, while ideally having minimal impact on $\Phi(\omega)$ near $\omega = 0$. One possibility is to use operator truncation approaches, such as in dissipaton-assisted operator evolution [5] and similar techniques. Provided the modification is such that the effective Liouvillian is still self-adjoint, then the Lanczos approach is valid; anything of the form $\mathcal{L}_{\text{eff}} = \mathcal{D}\mathcal{L}\mathcal{D}$ for some self-adjoint dissipator $\mathcal{D}$ would suffice.

A concrete choice is the DAOE dissipator $\mathcal{D}_{\text{DAOE}}$, which suppresses Pauli strings $\sigma^{\alpha} := \otimes_i \sigma_i^{\alpha_i}$, $\alpha_i \in \{0, 1, 2, 3\}$, according to their Pauli weight $|\alpha| := |\{i : \alpha_i \neq 0\}|$; if $|\alpha|$ is above some chosen cutoff length ℓ_* then the dissipator acts as $\mathcal{D}_{\text{DAOE}}[\sigma^{\alpha}] = e^{-\gamma(|\alpha| - \ell_*)} \sigma^{\alpha}$, where γ is a dissipation strength. Our initial experiments suggest that applying $\mathcal{D}_{\text{DAOE}}$ as part of the Lanczos algorithm generically arrests the growth of the Lanczos coefficients from $b_n \sim \mathcal{O}(n)$ to $b_n \sim \mathcal{O}(1)$, which should exponentially improve the convergence of the spectral bootstrap approach to computing transport coefficients from $\epsilon \sim 1/\text{poly}(\log n)$ to $\epsilon \sim 1/\mathcal{O}(n)$. It also has the benefit of reducing the computational requirements to compute the Lanczos coefficients up to a given n , due to a suppression of operator trajectories. But it is worth investigating to what extent one can smoothly extrapolate the resulting transport coefficients in the limit of vanishing dissipation $\gamma \rightarrow 0$ [5].

2. Universal dynamics of large operators

In this work we have demonstrated that operator dynamics restricted to the large- n “fast space” of Lanczos dynamics exhibits an emergent universality, akin to that of eigenvalue correlations in random matrix theory. The Lanczos setting is useful because it has enough mathematical structure to enable rigorous results, but one could speculate that this universality is more generally a statement about the universal dynamics of large operators restricted to the space of large operators. Practically, one way to investigate this in a tensor network approach would be to use the matrix product operator (MPO) formulation of the DAOE dissipator (see Ref. [5] for details), but change the boundary conditions of the MPO to keep only operators with weight $|\alpha| > \ell_*$, rather than keeping $|\alpha| \leq \ell_*$ as in the standard DAOE approach.

3. Universality in unitary and open system dynamics

All of the results in this paper concern operator dynamics generated by a time-independent and Hermitian Hamiltonian H . An obvious question is whether the notion

of universality discussed in this paper can be generalized to other types of operator dynamics. For Floquet dynamics generated by a unitary U , this question is likely quite tractable, and will involve asymptotics of orthogonal polynomials on the unit circle [88], corresponding to the spectra of unitary operators (for related work see Ref. [89]). Where the situation is both less clear and possibly richer is that of open quantum system dynamics [55,90–93]. With dynamics generated by a Lindbladian $\mathcal{L}$ with nontrivial jump operators, the spectrum of $\mathcal{L}$ now generically lies in the complex plane. It is still possible to get a banded matrix representation of $\mathcal{L}$ —which was necessary to get a simple recursion between $G_n(z)$ and $G_{n+1}(z)$ —by performing the biorthogonal version of the Lanczos algorithm. However, this will now involve polynomials whose orthogonality relation is defined not on a contour but on a subset of the complex plane [22,55,93]. This complex spectrum could lead to new universality classes for open system operator dynamics, but characterizing the nature of this universality is an open question. The technical machinery of polynomials orthogonal on 2D complex domains is much less well developed [94,95], although in certain special cases there have been generalizations of our Riemann-Hilbert approach to this case.

4. Connection to the generalized ETH

One of our main results is the appearance of the Wigner semicircle law in the level- n Green’s function $G_n(z)$ describing the dynamics restricted to the fast space $\{O_m\}_{m \geq n}$. One can heuristically argue for this result using the recursion relation Eq. (62) between $G_n(z)$ and $G_{n+1}(z)$, where the Wigner semicircle appears as a fixed point (see Sec. VI). This recursion bears some resemblance to the R -transform from free probability [96], but “deformed” by the ratio b_{n+1}/b_n . Recently, the language of free probability was used to generalize the eigenstate thermalization hypothesis to higher-order correlation functions [97,98]. It is an interesting topic for future work to explore potential connections between universality in Lanczos dynamics and the generalized ETH.

5. Krylov complexity and conservation laws

Much of the recent interest in Krylov space dynamics has been through the lens of Krylov complexity [23]. Stemming from the original work of Ref. [11], there is by now a reasonably good understanding of how the leading behavior of the Lanczos coefficients affects the early-time behavior of the Krylov complexity $\mathcal{K}(t)$, and in particular how the linear growth $b_n \sim \mathcal{O}(n)$ postulated by the OGH translates into exponential growth $\mathcal{K}(t) \sim \exp[\mathcal{O}(t)]$ of the complexity at early times. There have also been studies of Krylov complexity in finite-sized systems, examining how the finite-size plateau of the Lanczos coefficients arrests the initial complexity growth [99]. However, what is relatively less well understood is the effect of conservation laws

on Krylov complexity, which should be increasingly important for the mid-to-late time complexity of thermodynamically large systems. Our work, and particularly our discussion of Bessel universality, lays the mathematical foundations for a systematic study of these effects. Indeed, if one expands the time-evolved operator $|O_0(t)\rangle = \sum_{n=0}^{\infty} c_n(t)|O_n\rangle$ in the Lanczos basis, then a quick calculation shows that the coefficient $c_n(t)$ is given by $c_n(t) = \int_{\mathbb{R}} e^{i\omega t} p_n(\omega) \Phi(\omega) d\omega / 2\pi$. By exploiting the universal description of the orthogonal polynomials $p_n(\omega)$ we have derived for $n \gg 1$ [e.g., Eq. (99) or expressions in Sec. S4 A1 in [68]], one could probe the large- n contributions to the Krylov complexity.

6. New quantum algorithms for quantum dynamics

Our results focus on the “postprocessing” step of converting Lanczos coefficients into estimates of spectral functions, and as such are agnostic about how these Lanczos coefficients were actually obtained. In classical implementations of the Lanczos algorithm, a lightcone argument suggests that the memory requirement for a brute force computation of the n th Lanczos coefficient scales like $\exp[\mathcal{O}(n)]$, which can become prohibitively large for modest n . (Though see Ref. [52] for a potentially better scaling approach using tensor networks.) While one approach to ameliorating these memory requirements is to use the dissipation-assisted methods discussed above, one could also implement the Lanczos algorithm on a quantum computer [100,101]. One reason to consider this approach over direct real-time evolution is that it does not face the latter’s tradeoff between Trotter error and circuit depth (for a review of related methods see Ref. [102]).

7. Improved treatment of hydrodynamic tails

The main technical caveat to our proof technique is that we need to make some quite strong analyticity assumptions on the spectral function (see Sec. IV A), in order to perform the contour deformation involved in a steepest descent analysis of an associated Riemann-Hilbert problem. It would be desirable to prove our results under weaker assumptions, not just mathematically but also from a physical perspective, since it allows for a better treatment of hydrodynamic tails. In particular, for autocorrelation functions $C(t)$ that decay algebraically, $C(t \rightarrow \infty) \sim 1/t^\alpha$, standard Fourier arguments suggest that the spectral function $\Phi(\omega)$ should possess only a finite number of derivatives at $\omega = 0$. In a sense, our results adequately cover the case where $0 < \alpha < 1$, since then $\Phi(\omega)$ should have a power-law divergence at $\omega = 0$, and we explicitly address this case by decomposing $\Phi(\omega)/2\pi \equiv |\omega|^\rho e^{-\mathcal{Q}(\omega)}$. However, our results do not adequately cover the case where $\alpha > 1$, because in this case $\Phi(\omega)$ is finite at $\omega = 0$ (so $\rho = 0$) but is not infinitely differentiable, whereas we need to assume full analyticity for our proof. We have

conjectured that our results should still be correct to leading order in n , but the lack of full analyticity will result in a more slowly decaying error term. One way to address this rigorously would be to replace our Riemann-Hilbert problem approach with a $\bar{\partial}$ -problem approach [57,103], which is still treatable with nonlinear steepest descent controlled in the $n \rightarrow \infty$ limit.

I. Reader's guide

We outline the contents of the paper below, and have attempted to indicate how readers can focus on those sections most relevant to their interests. There is also a glossary at the end of the Supplemental Material [68].

Section III provides background material, and in particular illustrates the connection between the Lanczos algorithm and orthogonal polynomials generated by the spectral function $\Phi(\omega)$. We recommend all readers to ensure they are familiar with the contents of this section, since it explains the central role that the spectral function plays in our analysis.

Section IV is where we state the assumptions on the spectral function under which we can prove our results. We recommend readers follow the high-level summary in Sec. IVA, and skip the precise definitions in Sec. IV B until they are interested in the proof. Section IV C provides some discussion of our analyticity requirements on $\Phi(\omega)$, and can also be skipped on a first reading. For interested readers, we note that various properties of these spectral functions are derived in Sec. S1 B [68], which is not too technical and is independent of the later Riemann-Hilbert analysis.

Section V introduces the Coulomb gas problem associated with the spectral function. Section VA defines the Coulomb gas and introduces some useful notation that will appear in later sections, so we recommend it for all readers. This is enough to understand the later Sec. VI on universality of the level- n Green's function. Section VB introduces some further notation related to a rescaled version of the Coulomb gas; this does appear in the statements of some theorems (e.g., Theorem 3), but is less essential to understand immediately. Section VC is where we discuss the confinement transition in the Coulomb gas. We recommend this subsection to readers interested in the connections between quantum chaos, the operator growth hypothesis, and marginality.

Section VI is where we discuss our results on the universality of the level- n Green's function $G_n(z)$. We expect this section to be of interest to most readers.

In Sec. VII we move focus to signatures of hydrodynamics in Lanczos space, and in Theorem 3 show how a low-frequency power-law $\Phi(\omega \rightarrow 0) \sim |\omega|^\rho$ appears as a subleading “staggered” term in the Lanczos coefficients. However, we do not recommend using this result as a practical method to estimate ρ , instead preferring to focus on the zero mode, to be discussed imminently.

Indeed, in the application-motivated Sec. VIII, we show how a low-frequency power-law affects the *leading* behavior of the zero mode of the Liouvillian restricted to the Krylov space. In Sec. VIII A we focus on the decay of the zero mode amplitudes with n , and how this can be used to extract the value of the exponent ρ . Then in Sec. VIII B we further show how one can estimate hydrodynamic transport coefficients from the zero mode amplitudes. We benchmark this approach on physical models in Sec. VIII C.

The previous Sec. VIII was focused on $\omega = 0$ properties of the spectral function $\Phi(\omega)$, but in Sec. IX we show how to estimate $\Phi(\omega)$ at nonzero frequencies in an algorithm called the *spectral bootstrap*. The examples in the previous section used the $\omega \rightarrow 0$ limit of this algorithm. On a first read of Sec. IX, one could just focus on Sec. IX A, where we illustrate the basic ideas behind the spectral bootstrap.

We expect Sec. X to be of most interest to readers already familiar with the universality of eigenvalue correlations in random matrix theory, although we do lay out the relevant background in Sec. X A. The reason is that we examine quantities whose meaning is most transparent in the random matrix language. For each spectral function $\Phi(\omega)$, we define an associated random matrix ensemble. By using the spectral bootstrap to estimate the equilibrium measure, we are able to test whether the random matrix ensembles associated with various physical models possess the well-known universality of eigenvalue correlations. This can be seen as a consistency check for the universality present in the level- n Green's function $G_n(z)$, since both types of universality have the same origin (universality of orthogonal polynomials). But at present the universality of these eigenvalue correlations does not appear to have a transparent implication for physically relevant quantities.

Finally, in Sec. XI we give evidence that fast space universality indeed requires a degree of smoothness of the spectral function $\Phi(\omega)$ as a function of frequency ω . We demonstrate this by focusing on an Anderson localized model, which provably has a pointlike spectrum, and where we see the failure of a consistency check that was passed by the clean version of this model.

Moving to the Supplemental Material [68], we first derive in Sec. S1 B various implications of our assumptions for the behavior of the spectral function. Then the Riemann-Hilbert analysis proceeds in Sec. S2. By reading only Sec. S2 A, one can appreciate the relevance of the Riemann-Hilbert problem (RHP) to Lanczos dynamics, without going through the technical detail of actually solving the RHP. However, we emphasize that it is *not* necessary for readers to follow the RHP analysis in order to use our proposed spectral bootstrap algorithm.

For readers familiar with RHPs for orthogonal polynomials, we note that much of our analysis in Sec. S2 is quite standard, but we have given explicit details for the benefit of a mixed readership (the same goes for Secs. S3 and S4 [68]). The main novelty is in our local analysis near

the origin, and this requires a detailed study of the equilibrium measure that can be found in Sec. S5. For the sake of completeness, in Sec. S6 we also describe a version of the spectral bootstrap involving “Airy universality” suited for frequencies near the endpoints $\omega = \pm\beta_n$, but since these frequencies often become large for large n , we expect this to be less practically relevant than the versions of the spectral bootstrap described in Sec. IX.

III. BACKGROUND

A time-evolved operator $A(t) := e^{iHt} A e^{-iHt}$ can be expanded as a sum of nested commutators,

$$e^{iHt} A e^{-iHt} = A + it[H, A] + \frac{(it)^2}{2!} [H, [H, A]] + \dots \quad (21)$$

This suggests that if we want to understand the growth of $A(t)$, a natural object to study is the *Krylov space* $\mathcal{K} := \text{span}\{\mathcal{L}^k(A)\}_{k=0}^\infty$, where $\mathcal{L}(\cdot) := [H, \cdot]$ is the Liouvillian superoperator. To probe this space, it is convenient to construct an orthonormal basis for $\mathcal{K}$ using the Gram-Schmidt process. When $\mathcal{L}$ is self-adjoint, this reduces to the *Lanczos algorithm*, where successive basis vectors for $\mathcal{K}$ can be defined by a recurrence relation containing only three terms, rather than all previous basis vectors. We will use a vector notation $|A\rangle$ for operators, and for now we take the inner product to be the Hilbert-Schmidt product $(A|B) := \text{tr}[A^\dagger B]/\text{tr}[1]$, so we are effectively working at infinite temperature. Starting with a self-adjoint operator $|A\rangle$, we initialize the recurrence with $|O_{-1}\rangle := 0$ and $|O_0\rangle := b_0^{-1}|A\rangle$, where $b_0 := \sqrt{(A|A)} \equiv \|A\|$. Then we recursively define

$$|A_n\rangle := \mathcal{L}|O_{n-1}\rangle - b_{n-1}|O_{n-2}\rangle, \quad (22a)$$

$$b_n := \sqrt{(A_n|A_n)}, \quad (22b)$$

$$|O_n\rangle := b_n^{-1}|A_n\rangle. \quad (22c)$$

We will refer to the basis $\{|O_n\rangle\}_{n=0}^\infty$ as the *Lanczos basis*, and throughout will consider the generic case where $\mathcal{K}$ is infinite dimensional in the thermodynamic limit.

As well as the basis vectors themselves, the Lanczos algorithm produces a sequence of numbers $\{b_n\}_{n=1}^\infty$, which we will interchangeably refer to as the *recurrence* or *Lanczos coefficients*. It turns out that these numbers are enough to fully characterize the action of $\mathcal{L}$ within $\mathcal{K}$; indeed, the restriction of $\mathcal{L}$ to $\mathcal{K}$ can be represented in the Lanczos basis by a tridiagonal matrix

$$\mathcal{L} = \begin{pmatrix} 0 & b_1 & 0 & 0 & \dots \\ b_1 & 0 & b_2 & 0 & \dots \\ 0 & b_2 & 0 & b_3 & \dots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \quad (23)$$

That the diagonal elements are all zero is a generic consequence of *Hermiticity*, since $(O|\mathcal{L}|O) = 0$ follows for any self-adjoint O simply by using the definition $\mathcal{L}(\cdot) = [H, \cdot]$ with self-adjoint H , and inductively one can show that $i^n O_n$ is self-adjoint.

The tridiagonal form lends itself to interpreting $\mathcal{L}$ as a tight-binding model on a semi-infinite chain, where the sites are the operators $\{O_n\}_{n=0}^\infty$, and the hopping strengths are the recurrence coefficients $\{b_n\}_{n=1}^\infty$. We can visualize the operator evolution $A(t)$ as a single-particle problem, with a wave function initially localized on site $n = 0$ of the chain, which then spreads out along the chain over time. The autocorrelation function $C(t) := (A|A(t))$ corresponds to the probability amplitude for the operator wave function to be found back at its starting point at time t .

The *spectral function* $\Phi(\omega)$ defined as

$$\Phi(\omega) := \int_{\mathbb{R}} e^{-i\omega t} C(t) dt, \quad (24)$$

plays a significant role in the Lanczos algorithm. In particular, the Lanczos algorithm has a natural formulation in terms of the *orthogonal polynomials* with respect to the weight function $w(\omega) = \Phi(\omega)/2\pi$. To see this, we start from

$$C(t) = (A|e^{i\mathcal{L}t}|A) = \int_{\mathbb{R}} e^{i\omega t} \frac{\Phi(\omega)}{2\pi} d\omega, \quad (25)$$

and evaluate the k th derivative with respect to t at $t = 0$. Assuming $\Phi(\omega)$ decays sufficiently quickly as $|\omega| \rightarrow \infty$ (more on this later), this gives

$$(A|\mathcal{L}^k|A) = \int_{\mathbb{R}} \omega^k \frac{\Phi(\omega)}{2\pi} d\omega. \quad (26)$$

Since $\mathcal{L}$ has real spectrum, $\omega_{ij} = E_i - E_j$, if we then define the inner product between real polynomials $p, q: \mathbb{R} \rightarrow \mathbb{R}$

$$\langle p, q \rangle := (p(\mathcal{L})A|q(\mathcal{L})A), \quad (27)$$

one can use linearity of the inner product and Eq. (26) to write this as

$$\langle p, q \rangle = \int_{\mathbb{R}} p(\omega) q(\omega) \frac{\Phi(\omega)}{2\pi} d\omega, \quad (28)$$

so, as claimed, $\Phi(\omega)/2\pi$ appears as the relevant weight function. A defining property of orthonormal polynomials with respect to an even weight function on $\mathbb{R}$ is that they obey a three-term recursion relation of the form [85,104]

$$b_n p_n(\omega) = \omega p_{n-1}(\omega) - b_{n-1} p_{n-2}(\omega). \quad (29)$$

Comparing this to the Lanczos recursion relation, we deduce that the n th Lanczos vector $|O_n\rangle$ is given by

$$|O_n\rangle = p_n(\mathcal{L})|A\rangle, \quad (30)$$

where $p_n(\omega) = y_n \omega^n + \dots, y_n > 0$, is the n th order orthonormal polynomial with respect to the inner product Eq. (27) [22,105]. Thus, by studying the orthogonal polynomials with respect to the spectral function $\Phi(\omega)/2\pi$, we can understand properties of the Lanczos operators.

Note that this argument is very general, so that we can also easily apply it to operator inner products other than the Hilbert-Schmidt product. In particular, at finite temperature, we can consider the following family of inner products [26]

$$\langle A|B\rangle_\beta^g := \frac{1}{\beta} \int_0^\beta g(\lambda) \langle A^\dagger e^{-\lambda H} B e^{\lambda H} \rangle_\beta d\lambda - \langle A^\dagger \rangle_\beta \langle B \rangle_\beta, \quad (31)$$

where $\langle B \rangle_\beta = \text{tr}[e^{-\beta H} B]/\mathcal{Z}$ is a thermal expectation value, $\mathcal{Z} = \text{tr}[e^{-\beta H}]$ is the partition function, and $g(\lambda)$ is any function defined on the thermal circle $[0, \beta]$ satisfying

$$g(\lambda) \geq 0, \quad g(\beta - \lambda) = g(\lambda), \quad \frac{1}{\beta} \int_0^\beta g(\lambda) d\lambda = 1. \quad (32)$$

One can then perform a version of the Lanczos algorithm where operators are orthogonal with respect to this inner product. Examples include $g(\lambda) = [\delta(\lambda) + \delta(\lambda - \beta)]/2$ for linear response, and the Wightman product $g(\lambda) = \delta(\lambda - \beta/2)$. The autocorrelation function and spectral function are then $C_\beta^g(t) := \langle A|A(t)\rangle_\beta^g$ and $\Phi_\beta^g(\omega) := \int_{-\infty}^\infty e^{-i\omega t} C_\beta^g(t) dt$, and the polynomial inner product defined by $\langle p, q \rangle_\beta^g := (p(L^\dagger)A|q(L)A)_\beta^g$ can be written as

$$\langle p, q \rangle_\beta^g = \int_{-\infty}^\infty p(\omega) q(\omega) \frac{\Phi_\beta^g(\omega)}{2\pi} d\omega, \quad (33)$$

with the generalized spectral function $\Phi_\beta^g(\omega)/2\pi$ again appearing as the weight function. Since all our work is premised on assumptions made about the spectral function (Sec. IV), we expect our conclusions to hold not only at infinite temperature, but also at high enough finite temperatures, given a suitable choice of inner product.

IV. STATEMENT OF ASSUMPTIONS ON THE SPECTRAL FUNCTION

A. Summary

We saw in Sec. III that the spectral function $\Phi(\omega) := \int_{\mathbb{R}} e^{-i\omega t} C(t) dt$ determines all the properties of the Lanczos basis. However, for a given interacting many-body Hamiltonian H , we do not generally have a hope of exactly calculating $\Phi(\omega)$. Instead, the approach we will take is to impose a few physically motivated conditions on $\Phi(\omega)$, and study their implications. Before stating these conditions precisely, we state them at a high level:

- (1) We fix the $|\omega| \rightarrow \infty$ behavior of $\Phi(\omega)$, taking it to decay at least exponentially in $|\omega|$ (but possibly faster). This is provably true at high temperatures for *local* lattice Hamiltonians with a bounded local Hilbert space [74–76], and forms part of the motivation for the operator growth hypothesis [11].
- (2) Since we are interested in signatures of the $|t| \rightarrow \infty$ behavior of $C(t)$, we impose a condition on the $\omega \rightarrow 0$ behavior of its Fourier transform $\Phi(\omega)$. To model $C(t) \sim 1/|t|^{1+\rho}$ as $|t| \rightarrow \infty$, we require $\Phi(\omega) \sim |\omega|^\rho$ as $\omega \rightarrow 0$. Such power-law decay in $C(t)$ is typical if the initial operator overlaps with a conserved quantity with sufficiently slow transport, say diffusive in one dimension [19].
- (3) We take $\Phi(\omega) = \Phi(-\omega)$ to be even, which is satisfied if one uses any of the thermal inner products defined in Eq. (31) satisfying the conditions in Eq. (32). This includes the infinite temperature Hilbert-Schmidt product $\langle A|B \rangle = \text{tr}[A^\dagger B]/\text{tr}[1]$, which we will use in all our numerical tests.
- (4) We require $\Phi(\omega) > 0$ to be strictly positive, which should be generically true for interacting systems in the thermodynamic limit. Furthermore, in a sense we will make precise below, we require $\Phi(\omega)$ to be very smooth. This has both technical and physical implications. At a technical level, it allows us to use a powerful complex analytic proof technique involving a Riemann-Hilbert problem. However, we have evidence that smoothness of spectral functions is not merely an artefact of the proof technique, but rather is a physical requirement to get universality. Indeed, in Sec. XI we give evidence that universality fails in an Anderson localized system, where strong disorder can result in a pure point spectrum [83].

We remark that the analyticity requirements for our proof could likely be weakened to assuming only a few derivatives of $\Phi(\omega)$, by employing a so-called $\bar{\partial}$ -problem approach [57,106,107], rather than using a Riemann-Hilbert problem as we do here. Experience suggests that one usually finds the same leading order asymptotics, but with potentially slower decaying error terms, depending on the degree of differentiability of $\Phi(\omega)$ one assumes.

B. Precise assumptions

In order to state our precise assumptions, we need to make a definition. Since $\Phi(\omega) \geq 0$ and is even, we can always write it in the form $\Phi(\omega) \equiv \exp[-Q(\omega)]$ for some even real-valued function $Q(\omega)$. However, since we are interested in hydrodynamic spectral functions, which can have an algebraic divergence $\Phi(\omega) \sim |\omega|^p$ as $\omega \rightarrow 0$, it will prove helpful to factorize out this divergence, and decompose $\Phi(\omega)$ as

$$\Phi(\omega)/2\pi \equiv |\omega|^p e^{-Q(\omega)}. \quad (34)$$

In principle $\Phi(\omega)$ may also have algebraic behavior near other frequencies, in which case we would also factorize those out [67]. But for simplicity we will focus on the case where there is only an algebraic divergence at $\omega = 0$. The function $Q(\omega)$ so defined is called the *potential* (we will later make precise the sense in which this is a potential).

For technical reasons, it is more convenient to make all our assumptions in terms of $Q(\omega)$ rather than $\Phi(\omega)$ directly. We will translate assumptions about the high-frequency decay of $\Phi(\omega \rightarrow \infty)$ into assumptions about the high-frequency growth of $Q(\omega \rightarrow \infty)$. In particular, we will consider a class of potentials inspired by the “very smooth Freud weights” of Refs. [35,108], which they denote by $\text{VSF}(p)$, with p an exponent governing the degree of the polynomial growth of $Q(\omega) \sim |\omega|^p$ as $|\omega| \rightarrow \infty$. We will consider a subset of $\text{VSF}(p)$, where we add the requirement of analyticity, and also require a specification of the logarithmic corrections to the leading polynomial growth of $Q(\omega)$.

Our Riemann-Hilbert analysis draws heavily from Ref. [62], where they take Q to be a polynomial of even order. However, we are particularly interested in the marginal case where $Q(\omega \rightarrow \infty)$ grows linearly with $|\omega|$, since the operator growth hypothesis [11] conjectures this to be generic for spectral functions in chaotic many-body quantum systems. But it is clearly not possible to simultaneously have (i) $Q(\omega) \sim |\omega|$ as $|\omega| \rightarrow \infty$, and (ii) Q be a polynomial. This was a primary motivation for considering this generalized class $\text{VSLF}(p, q)$ of “polynomial-like” weights.

Definition 1. [$\text{VSLF}(p, q)$: log-Freud potentials of order (p, q)] Let $Q: \mathbb{R} \rightarrow \mathbb{R}$ be real-analytic, even, and satisfy

$$Q'(\omega) > 0, \quad \text{for } \omega \text{ large enough}, \quad (35)$$

$$\lim_{\omega \rightarrow \infty} \left(\frac{\omega Q''(\omega)}{Q'(\omega)} \right) = p - 1, \quad (36)$$

$$\lim_{\omega \rightarrow \infty} \left(\log(\omega) \left[-p + \frac{\omega Q'(\omega)}{Q(\omega)} \right] \right) = q. \quad (37)$$

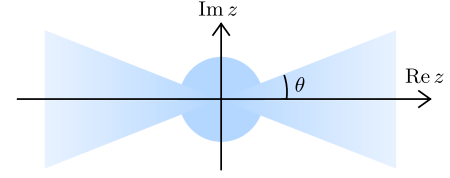


FIG. 3. We require the potential $Q(z)$ defined in Eq. (34) to have an analytic continuation to the shaded region of the complex plane, where $0 < \theta \leq \pi/2$ is any positive angle. This region is the union of the complex cone C_θ [see Eq. (39)] and a disk of constant radius around $z = 0$.

for some $p > 0$ and $q \in \mathbb{R}$. Then we shall call Q a *log-Freud potential* of order (p, q) and write $Q \in \text{VSLF}(p, q)$.

We characterize the behavior of these potentials in Sec. S1 B. In particular, in Eq. (S1.8) we show that these potentials grow as $|\omega| \rightarrow \infty$ like

$$|\omega|^p (\log |\omega|)^{q-\epsilon} \leq Q(\omega) \leq |\omega|^p (\log |\omega|)^{q+\epsilon}, \quad (38)$$

where ϵ can be taken to zero as $|\omega| \rightarrow \infty$; we will informally write this as $Q(\omega) \sim |\omega|^p \log^q |\omega|$. In this sense assumptions Eqs. (36) and (37) are similar to but slightly weaker than assuming $Q \in \Theta(|\omega|^p (\log |\omega|)^q)$. We make the assumption Eq. (35) for technical convenience; it amounts to assuming that the spectral function $\Phi(\omega)$ decays monotonically above some frequency scale [which is $O(1)$ but potentially much larger than any characteristic frequency scale].

In order to apply Riemann-Hilbert techniques, we also need to assume that some of these properties continue to hold in a region of the complex plane near the real axis (illustrated in Fig. 3).

Definition 2. [$\text{CVSLF}(p, q, \theta, \gamma)$: complex log-Freud potentials of order (p, q)] For an angle $0 < \theta \leq \pi/2$, define the “complex cone” C_θ by

$$C_\theta := \{z: |\arg z| < \theta\} \cup \{z: |\arg z| > \pi - \theta\}, \quad (39)$$

using the convention $-\pi < \arg z \leq \pi$. We consider the open cone, so $z = 0$ is not included in C_θ . Now suppose there is some $0 < \theta \leq \pi/2$ and $\gamma > 0$ such that $Q \in \text{VSLF}(p, q)$ can be analytically continued to $C_\theta \cup \{z: |z| < \gamma\}$, the union of C_θ and the disk of radius γ centered at the origin (see Fig. 3). Also assume that Eqs. (36) and (37) generalize to this region, in the sense that for z restricted to C_θ we have

$$\lim_{|z| \rightarrow \infty} \frac{z Q''(z)}{Q'(z)} = p - 1, \quad (40)$$

$$\lim_{|z| \rightarrow \infty} \left(\log(z) \left[-p + \frac{z Q'(z)}{Q(z)} \right] \right) = q. \quad (41)$$

Given these properties, we say that Q is a complex log-Freud potential of order (p, q) , and write $Q \in \text{CVSLF}(p, q, \theta, \gamma)$.

1. Examples

Our proofs will apply for $Q \in \text{CVSLF}(p, q, \theta, \gamma)$ in the cases $p > 1, q \in \mathbb{R}$, and $p = 1, q > -1$, with the latter case conjectured to be generic for quantum chaotic systems, according to the operator growth hypothesis [11]. Note that local interactions guarantee $p \geq 1$ [74–76].

Example 1. All even polynomials with positive leading coefficient, $Q(x) = q_{2m}x^{2m} + \dots$, $q_{2m} > 0$, lie in $\text{CVSLF}(p, q, \theta, \gamma)$ with $p = 2m, q = 0, \theta = \pi/2, \gamma = \infty$.

Example 2. Certain fractional powers of polynomials satisfy our assumptions, e.g. $Q(x) = \sqrt{1+x^2}$ has $p = 1, q = 0, 0 < \theta < \pi/2$, and $0 < \gamma < 1$. A similar example is $Q(x) = \sqrt{1+x^2} \log(1+x^2)$, which has $p = 1, q = 1$.

Example 3. The symmetric Meixner-Pollaczek weights have $w(x) = \exp[-Q(x)] = \Gamma(\lambda + ix)\Gamma(\lambda - ix)$. (Q can then be defined as in Eq. (44) below.) With $\lambda > 0$, we have $p = 1, q = 0, 0 < \theta < \pi/2$, and $0 < \gamma < \lambda$. These weights appear in a rescaled form as the 2-point Wightman spectral function in the SYK model [109,110], and were utilized in Ref. [11] to give an exactly solvable spectral function with exponential decay for use with the recursion method [26].

Example 4. Taking $Q'(z) = \text{erf}(z) = (2/\sqrt{\pi}) \int_0^z e^{-t^2} dt$ gives an example of a $p = 1$ potential which is also entire, unlike the previous $p = 1$ examples. The conditions Eqs. (40) and (41) hold for $0 < \theta < \pi/4$.

C. Comments on analyticity requirements

Regarding the definition of the class CVSLF, the reason we need analyticity in a cone of constant argument rather than, say, a strip of constant width, is that we need this region of analyticity to be invariant under rescaling $z \mapsto z/\beta$ for $\beta > 0$. Note that θ could be very small; we just require that it is a positive constant. We also require $Q(z)$ to be analytic in a disk of constant radius around $z = 0$. Handling the fact that this disk is *not* invariant under rescaling is one of the technical achievements of our work. Note that this analyticity requirement on Q is not in general the same as requiring $\Phi(z)$ to be analytic at $z = 0$, since we are explicitly factorizing out a low-frequency power law, cf. Eq. (34), which is supposed to capture the main nonanalyticity of $\Phi(z)$.

Typically, we are more used to talking about analytic continuations of Green's functions than of spectral functions. Suppose we have a retarded Green's function $G_R(z)$, defined in the lower-half plane by

$$G_R(z) := \left(O_0 \left| \frac{1}{z - \mathcal{L}} \right| O_0 \right) = \int_{\mathbb{R}} \frac{\Phi(\omega) d\omega}{z - \omega} \frac{1}{2\pi}, \quad z \in \mathbb{C}_-. \quad (42)$$

By construction $G_R(z)$ is analytic in the lower-half plane, and $\Phi(\omega) = 2 \text{Im} \lim_{\epsilon \rightarrow 0^+} G_R(\omega - i\epsilon)$. Now suppose that $G_R(z)$ has an analytic continuation across a section of the real line to some subset $\Omega \subseteq \mathbb{C}_+ \cup \mathbb{R}$ of the upper-half plane. We can use this to define an analytic continuation of the spectral function by

$$\Phi(z) := \frac{1}{i} (G_R(z) + G_R(-z)). \quad (43)$$

$\Phi(z)$ is then analytic in $\Omega \cup (-\Omega)$, and satisfies $\Phi(-z) = \Phi(z)$. So, for example, if $G_R(z)$ has a diffusive pole at $z = iDk^2$, then $\Phi(z)$ would have poles at $z = \pm iDk^2$. Having analytically continued $\Phi(z)$ in this way, $Q(z)$ would then be defined such that Eq. (34) is satisfied. In the case $\rho = 0$, this can be achieved by taking

$$Q(z) = -\log \left(\frac{\Phi(0)}{2\pi} \right) - \int_0^z \frac{\Phi'(s)}{\Phi(s)} ds, \quad (44)$$

where the integral is along any path where $\Phi(s) > 0$ and avoiding any poles. Then $\exp[-Q(z)] = \Phi(z)/2\pi$ by construction. A similar construction is possible for $\rho \neq 0$ given an analytic continuation of $|\omega|^\rho$ to the relevant quadrant of the complex plane.

Note that we do *not* show that the violation of these analyticity conditions on Q necessarily leads to a modification of our results. Versions of some of our results have been proven, though not always with explicit constants, only using assumptions about the behavior of the spectral function on the real line, see, e.g., [111,112]. Rather than using Riemann-Hilbert techniques, which rely quite crucially on analyticity, it may also be possible to prove similar statements using a $\bar{\partial}$ steepest descent method [106], which does not require such strong analyticity assumptions on Q , at the expense of getting weaker error bounds.

One potential violation of our analyticity conditions could come from systems with a “diffuson cascade” [113,114], which can lead to an infinite series of poles in k -space retarded Green's functions $G_R(z, k)$ accumulating all the way down to the real z axis, as opposed to the conventional expectation of just a simple pole at $z = iDk^2$ [19]. Handling such singular points would require a modification of our analysis which goes beyond the scope of this work (though see Ref. [50] for some work in this direction). However, despite their analyticity conditions, our calculations give certain predictions that can be tested numerically. We find that they are obeyed well even in some interacting nonintegrable systems that might be expected to have a diffuson cascade. This suggests that, even if we weaken our analyticity conditions, some of our conclusions may continue to hold.

V. COULOMB GAS, UNIVERSALITY, AND CONFINEMENT

To state our later results about hydrodynamics and Lanczos dynamics, we first need to introduce an associated Coulomb gas problem. As outlined in Sec. II D, this Coulomb gas arises naturally both in the context of random matrix theory and orthogonal polynomials, and finding the minimum energy Coulomb gas configuration will give us a handle on the regions which dominantly contribute to the Lanczos coefficients. As we alluded to in Sec. II D, one interesting aspect of this Coulomb gas is that it undergoes a *confinement transition*, and that the operator growth hypothesis implies that quantum chaotic systems are generically at the critical point of this transition. We describe this in more detail in Sec. V C.

A. Coulomb gas definition

Given a potential $Q(\omega)$ determined from the spectral function by Eq. (34), we define a Coulomb gas energy functional E_Q as follows. We take as input a charge density function $\sigma: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ (defined on frequency space), and give it an energy consisting of two terms,

$$E_Q[\sigma] := - \int_{\mathbb{R}} \int_{\mathbb{R}} \log |\omega_1 - \omega_2| \sigma(\omega_1) \sigma(\omega_2) d\omega_1 d\omega_2 + \int_{\mathbb{R}} Q(\omega) \sigma(\omega) d\omega, \quad (45)$$

namely, a logarithmic repulsion between charges, and a single-particle energy determined by the potential Q . For a given Lanczos index n , we imagine distributions with total charge n , so that $\int_{\mathbb{R}} \sigma(\omega) d\omega = n$. We then want to consider the distribution σ_n with charge n that minimizes the energy $E_Q[\sigma]$, so that

$$\sigma_n := \operatorname{argmin}_{\sigma} \left\{ E_Q[\sigma] : \int_{\mathbb{R}} \sigma(\omega) d\omega = n \right\}. \quad (46)$$

We will refer to σ_n as the *equilibrium density*. For large enough n , one can show that σ_n is indeed uniquely defined and continuous under reasonable assumptions on Q [115]. Let us make a few observations about this equilibrium density. Since the potential $Q(x)$ must grow at least linearly for $|x| \rightarrow \infty$ due to constraints from locality, while the two-particle repulsion is only logarithmic, we conclude that for any finite n , the charge will remain at a finite distance from the origin. This means that $\sigma_n(\omega)$ has support within some finite interval we denote by

$$\operatorname{supp} \sigma_n \equiv (-\beta_n, \beta_n), \quad (47)$$

which we refer to as the bulk. This distribution is even, $\sigma_n(-\omega) = \sigma_n(\omega)$, because Q is even. As $n \rightarrow \infty$ the density $\sigma_n(\omega)$ gives the density of zeros of the orthogonal

polynomials with respect to the weight $e^{-Q(\omega)}$ [115], and the dominant contribution to the recurrence coefficients will come from this bulk frequency range $(-\beta_n, \beta_n)$. This interval is analogous to the oscillatory region in a WKB approximation [62,116]. Note that, for technical convenience, we do not include the power law $|\omega|^\rho$ in the Coulomb gas potential, instead handling it by other means (Szegő functions).

The distance β_n is referred to as the *n*th *Mhaskhar-Rakhmanov-Saff* (MRS) number [115,117,118], and is defined for even Q as the positive solution to the integral equation

$$\frac{1}{2\pi} \int_{-1}^1 \frac{\beta_n s Q'(\beta_n s)}{\sqrt{1-s^2}} ds = n. \quad (48)$$

This solution is unique for large enough n . How does β_n scale with n ? Clearly, the slower the growth of the single-particle potential Q , the more the charge will spread out. It turns out that the growth of Q and β_n are related by

$$Q(\omega) \sim |\omega|^p \log^q |\omega| \quad \text{as } |\omega| \rightarrow \infty$$

$$\Updownarrow$$

$$\beta_n \sim \left(\frac{n}{\log^q n} \right)^{1/p} \quad \text{as } n \rightarrow \infty. \quad (49)$$

It is straightforward to verify this scaling when Q is literally a polynomial, e.g., $Q(\omega) = \omega^{2m}$, but it continues to hold when $Q(\omega)$ is sufficiently polynomial-like as $|\omega| \rightarrow \infty$ (see S1). The locality constraint on Q translates into $p \geq 1$, so that β_n grows at most linearly in local systems. It is well known, as we will see in Theorem 1, that the scaling of β_n fixes the leading behavior of the recurrence coefficients b_n , which are given by $b_n \approx \beta_n/2$ to leading order in n [35]. Note that by construction β_n depends only on Q' and so is independent of the hydrodynamic exponent ρ , so the hydrodynamics will only show up at *subleading* orders in the recurrence coefficients.

One way to understand the MRS number β_n is as an n -dependent frequency bandwidth. From Eq. (48) it is easy to see that if we introduce a frequency scale ω_0 by mapping $Q(\omega) \mapsto Q(\omega/\omega_0)$, then β_n transforms as $\beta_n \mapsto \omega_0 \beta_n$, so β_n indeed has units of frequency. Furthermore, if we consider the spectral function $\tilde{\Phi}_n(\omega) := \int_{\mathbb{R}} e^{-i\omega t} (O_n | e^{i\mathcal{L}t} | O_n) dt$ for the dynamics of the n th Lanczos operator O_n under the full Liouvillian $\mathcal{L}$, then from Eq. (30) we can see that this is related to the original spectral function by

$$\tilde{\Phi}_n(\omega) = p_n(\omega)^2 \Phi(\omega). \quad (50)$$

[Note $\tilde{\Phi}_n(\omega)$ is not to be confused with the spectral function $\Phi_n(\omega)$ related to $G_n(z)$; they correspond to dynamics generated by $\mathcal{L}$ and $\mathcal{L}_n$, respectively.]

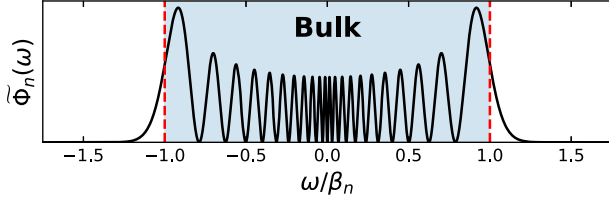


FIG. 4. The n th spectral function $\tilde{\Phi}_n(\omega) = p_n(\omega)^2 \Phi(\omega)$ for the toy spectral function $\Phi(\omega)/2\pi = \text{sech}(\pi\omega)$ and $n = 20$. This governs the dynamics of O_n with respect to the *full* Liouvillian $\mathcal{L}$, not the restricted Liouvillian $\mathcal{L}_n$ (see Fig. 6 for the latter). In this case $\beta_n \approx \sqrt{n(n-1)}$ [120], with the linear scaling $\beta_n \sim \mathcal{O}(n)$ reflecting the exponential decay of $\Phi(\omega \rightarrow \infty)$. One can see that $\tilde{\Phi}_n(\omega)$ is peaked near $|\omega| = \beta_n$, and as $n \rightarrow \infty$ this becomes increasingly sharp. The region $|\omega| \leq \beta_n$ is referred to as the bulk, and is analogous to the oscillatory region in a WKB approximation. For $|\omega| \gg \beta_n$, $\tilde{\Phi}_n(\omega)$ is exponentially small. When $\Phi(\omega \rightarrow 0) \sim |\omega|^p$ has a power-law at $\omega = 0$, the orthogonal polynomials $p_n(\omega)$ behave differently near the origin (see, e.g., Theorem 2).

A remarkable result due to Mhaskar and Saff [119] shows that, under certain conditions on Q , the MRS number β_n provides an asymptotically sharp characterization of where this weighted polynomial is peaked, so $\arg\max_{\omega \geq 0} \tilde{\Phi}_n(\omega) \xrightarrow{n \rightarrow \infty} \beta_n$. One can understand the scaling in Eq. (49) as coming from the tradeoff between the growth of the degree $2n$ polynomial $p_n(\omega)^2$ and the decay of the spectral function $\Phi(\omega \rightarrow \infty) \sim \exp[-\mathcal{O}(\omega^p \log^q \omega)]$. Physically, β_n therefore represents the frequency bandwidth over which the n th spectral function $\tilde{\Phi}_n(\omega)$ is non-negligible. For frequencies $|\omega| \gg \beta_n$, $\tilde{\Phi}_n(\omega)$ will be exponentially small (see Fig. 4 for an example); this region is analogous to the “classically forbidden” region in a WKB analysis, and will give only exponentially small contributions to the Lanczos coefficients.

B. Rescaled equilibrium measure

Given a minimizing density distribution σ_n , it is helpful to define a related distribution

$$\psi_n(x) := \frac{\beta_n}{n} \sigma_n(\beta_n x), \quad (51)$$

which is normalized to 1 and has support $[-1, 1]$. We refer to ψ_n as the rescaled equilibrium density. It is the energy minimizing distribution with charge 1 for the “rescaled potential”

$$V_n(x) := \frac{1}{n} Q(\beta_n x). \quad (52)$$

The rescaling by β_n in the definition of V_n washes out the nonuniversal details of Q when n is large. For example, for

the class of potentials we consider, if $Q(x) \sim x^p$ as $x \rightarrow \infty$, then for any finite x we have

$$\lim_{n \rightarrow \infty} V_n(x) = \kappa_p |x|^p, \quad (53)$$

where $\kappa_p = \Gamma[\frac{1}{2}] \Gamma[p/2] / \Gamma[(p+1)/2]$ [108] (Lemma 3.2). This emergent dominance of the high frequency scaling means that the equilibrium density $\psi_n(x)$ has several displays of *universality* in the $n \rightarrow \infty$ limit, with the same properties as if the potential were the corresponding “Freud potential” $Q^{(p)}(x) \equiv \kappa_p |x|^p$. These potentials have $\beta_n = n^{1/p}$, and an equilibrium measure given by the “Ullman distribution” [115]

$$\psi^{(p)}(x) := \frac{1}{\pi} \int_{|x|}^1 \frac{p u^{p-1}}{\sqrt{u^2 - x^2}} du. \quad (54)$$

One simple manifestation of this is that, quite generally, the density $\psi_n(x)$ vanishes like $\sqrt{1-x^2}$ at the end points $x = \pm 1$; this has a manifestation in the famous Wigner semicircle law of random matrix theory [70], but holds near the edge more generally beyond Gaussian ensembles. Other universal properties of the equilibrium Coulomb gas distribution are more easily stated in terms of a function $h_n(x)$ defined by

$$\psi_n(x) = \frac{1}{2\pi} h_n(x) \sqrt{1-x^2}, \quad (55)$$

for $x \in [-1, 1]$. The values of $h_n(x)$ at $x = 0$ and $x = 1$ appeared in Theorem 1 describing hydrodynamic corrections to the Lanczos coefficients.

For large enough n [but $\mathcal{O}(1)$ in terms of the microscopic couplings], we show that the minimum energy configuration of E_Q is obtained when $h_n(x)$ is given by

$$h_n(x) = \frac{1}{\pi} \int_{-1}^1 \frac{V'_n(s) - V'_n(x)}{s - x} \frac{ds}{\sqrt{1-s^2}}. \quad (56)$$

We then show that $h_n(x)$ displays universal behavior at the origin $x = 0$ and the endpoints $x = \pm 1$, similar to that of the Ullman distribution Eq. (54). This behavior depends only on the $|\omega| \rightarrow \infty$ behavior of Q , provided it is sufficiently smooth near the origin, essentially because of the large- n scaling of $V_n(x)$ given by Eq. (53).

As an example, for our class of spectral functions, at the edge of the bulk we have

Lemma 1. (Informal) If $Q(\omega) \sim |\omega|^p \log^q |\omega|$ as $|\omega| \rightarrow \infty$ for some $p > 0$ and $q \in \mathbb{R}$, then

$$\lim_{n \rightarrow \infty} h_n(1) = 2p. \quad (57)$$

We test this result numerically for the mixed field Ising model in Sec. S6. Note that $h_n(x)$ is even if $Q(x)$ is even, so the same conclusion holds for $h_n(-1)$.

C. Confinement transition

The behavior of $h_n(0)$ is more interesting, since it is sensitive to a *confinement transition* in the Coulomb gas, between strong confinement for $p > 1$ to weak confinement for $p < 1$ (see Fig. 2) [77–80]. Despite this confinement transition being driven primarily by the *high frequency* behavior of the spectral function, it turns out to manifest in a divergent density at *low frequencies*, as codified in Lemma 2. This means it will have an imprint on the signatures of hydrodynamics on the recurrence coefficients b_n , as we saw in Theorem 1. The following result is stated in terms of $h_n(0)$, but can be translated back to a confinement transition in the original equilibrium density $\sigma_n(0)$ using Eqs. (51) and (55).

Lemma 2. (Informal) Suppose $Q(\omega) \sim |\omega|^p \log^q |\omega|$ as $|\omega| \rightarrow \infty$ for some $p \geq 1$ and $q \in \mathbb{R}$.

If $p > 1$, then for all $q \in \mathbb{R}$ we have

$$\lim_{n \rightarrow \infty} h_n(0) = \frac{2p}{p-1}, \quad (58)$$

while in the marginal case $p = 1$, as $n \rightarrow \infty$ we have

$$h_n(0) = (\log n)^{1+o(1)} \quad (59)$$

provided $q > -1$. (Local interactions enforce $q \geq 0$.)

We prove Lemmas 1 and 2 in Sec. S5 [68]. It is interesting to now recall the operator growth hypothesis [11], which posits that operators undergoing chaotic many-body dynamics generically grow as fast as possible under locality constraints. These constraints stipulate that the spectral function must decay at least exponentially with ω , so that $Q(\omega) \sim |\omega|^p$ with $p \geq 1$, and hence b_n grows no faster than $\mathcal{O}(n)$. The OGH therefore amounts to the statement that chaotic many-body systems generically have $p = 1$; here we see that this implies they can be identified with the critical point of a Coulomb gas confinement transition (see Fig. 2). Systems with spectral functions decaying superexponentially ($p > 1$) will instead lie in the strongly confined phase of the Coulomb gas.

The confinement transition can be heuristically understood as follows. We have already seen that the large frequency scaling of the potential $Q(\omega) \sim |\omega|^p \log^q |\omega|$ determines the width β_n of the equilibrium charge density distribution $\sigma_n(\omega)$ via Eq. (49). The confinement transition is related to the relative scaling of the bandwidth β_n and the total charge n . For $p > 1$, β_n grows slower than n , so the average charge density $n/(2\beta_n)$ increases with n ; heuristically, the charge is increasingly “packed in.” A useful

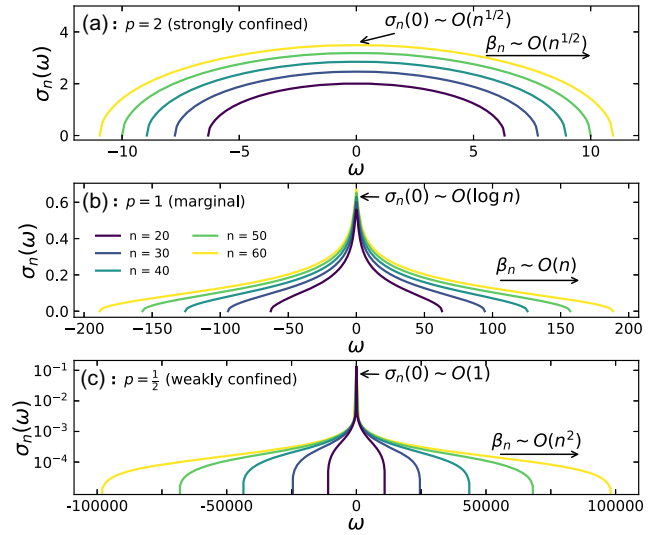


FIG. 5. Illustration of the confinement transition using the potential $Q(\omega) = (1 + \omega^2)^{p/2}$ for different growth exponents p . The values $p = \frac{1}{2}$ and $p = 2$ lie in the weakly and strongly confined phases, respectively, while $p = 1$ is marginal. All systems with local interactions should have $p \geq 1$. The confinement transition can be diagnosed via the equilibrium density $\sigma_n(0)$ at zero frequency. In the strongly confined phase $\sigma_n(0) \sim \mathcal{O}(n/\beta_n)$ grows algebraically with n , reducing to logarithmic growth $\sigma_n(0) \sim \mathcal{O}(\log n)$ at the critical point [or $\sigma_n(0) \sim \mathcal{O}(\log^2 n)$ in one spatial dimension]. In the weakly confined phase, $\sigma_n(0) \sim \mathcal{O}(1)$ does not grow with n .

diagnostic is the charge density at zero frequency, $\sigma_n(\omega = 0)$, which is related to $h_n(0)$ via

$$\sigma_n(0) = \frac{1}{2\pi} \frac{n}{\beta_n} h_n(0). \quad (60)$$

Lemma 2 tells us that $h_n(0)$ is $\mathcal{O}(1)$ for $p > 1$, where the $\mathcal{O}(1)$ refers to scaling with n . Combining this with the scaling of β_n given in Eq. (49), we conclude that $\sigma_n(0) \sim \mathcal{O}(n/\beta_n)$ increases algebraically with n , with the same n scaling as the average charge density. In this sense the charge density is approximately uniform (in ω) in the bulk. An example is shown in Fig. 5(a), where we show the equilibrium density $\sigma_n(\omega)$ for the toy potential $Q(\omega) = (1 + \omega^2)^{p/2}$ with $p = 2$.

By contrast, in the (locality forbidden) weakly confined phase ($p < 1$), the bandwidth β_n grows faster than n , so the average charge density $n/(2\beta_n)$ decreases with n . However, the charge is now distributed differently at low frequencies compared with the rest of the bulk; the density profile is peaked at $\omega = 0$, where it becomes $\sigma_n(0) \sim \mathcal{O}(1)$. An example is shown in Fig. 5(c). One can understand this divergence from the behavior of the derivative $Q'(\omega) \sim \omega^{p-1}$, which for $p < 1$ grows algebraically as ω becomes smaller. For our class of spectral

functions we only assume that this behavior holds at large frequencies, but when n is large this scaling still governs the behavior of the rescaled potential $V'_n(s) = (\beta_n/n)Q'(\beta_n s)$ for $s \sim \mathcal{O}(1)$ [cf. Eq. (53)]. As such, for $x = 0$ the integrand of Eq. (56) diverges algebraically as s approaches zero, until the divergence is eventually cut off by our differentiability assumption on Q . This gives rise to a contribution scaling like $h_n(0) \sim \mathcal{O}(n^{(1-p)/p})$, so that $\sigma_n(0) \sim \mathcal{O}(1)$.

Finally, in the marginal case $p = 1$ relevant for chaotic systems, the bandwidth β_n grows at the same rate as the total charge n (up to logarithmic factors depending on q). By a similar argument to the $p < 1$ case, the derivative $V'_n(s)$ is now approximately constant for $s \sim \mathcal{O}(1)$, so the integrand of Eq. (56) has a logarithmic divergence as $s \rightarrow 0$. This is again cut off at very small $s \sim \mathcal{O}(1/\beta_n)$ by our differentiability assumption on Q , with the end result that $h_n(0)$ diverges like $\mathcal{O}(\log n)$. Any logarithmic corrections to $Q(\omega) \sim |\omega| \log^q |\omega|$ do not affect the logarithmic scaling of $h_n(0)$ provided $q > -1$. The case $q = -1$ corresponds to the transition to weak confinement; this is also where the Hamburger moment problem for $\Phi(\omega)$ becomes indeterminate [121], which follows from Carleman's condition [86]. Note that local interactions enforce $q \geq 0$ if $p = 1$ [74]. The scaling of $\sigma_n(0)$ is then $\sigma_n(0) \sim \mathcal{O}[n h_n(0)/\beta_n]$, giving $\sigma_n(0) \sim \mathcal{O}(\log n)$ for $\beta_n \sim n$ and $\sigma_n(0) \sim \mathcal{O}(\log^2 n)$ for $\beta_n \sim n/\log n$, the latter case relevant in one spatial dimension [11].

Before proceeding, we note that the convergence with n of the equilibrium Coulomb gas density to the asymptotic forms indicated in Lemmas 1 and 2 can be extremely slow, depending on the functional form for Q . When Q is an even-order polynomial, $Q(x) \sim \mathcal{O}(x^{2m})$, then it is known that the finite n corrections to $h_n(0)$ and $h_n(1)$ scale like $\mathcal{O}(n^{-1/2m})$ [62]. When Q is not necessarily an actual polynomial, but merely of polynomial growth as we treat here, then in general the rate of convergence depends on how quickly Q approaches its asymptotic scaling $Q(x) \sim x^p \log^q x$. In toy examples we have also found that convergence of $h_n(0)$ tends to be slower than that of $h_n(1)$, particularly in the quasilinear case $p = 1$, where $h_n(0)$ has a logarithmic divergence. Having said that, this slow convergence will not necessarily be a problem for estimating, e.g., diffusion constants: in Sec. VIII B we show how to eliminate the dependence on these slowly converging Coulomb gas densities. After developing some technical tools, we perform some quantitative checks of the convergence of the equilibrium measure to the Ullman distribution Eq. (54) in Sec. X C. We find, at the values of n typically available for numerical simulations, that the Ullman distribution successfully describes the large-scale qualitative shape of the true equilibrium measure, but at finite n there remain significant fluctuations which must be accounted for in order to get quantitative accuracy in estimates of the spectral function.

VI. UNIVERSALITY OF THE LEVEL- n GREEN'S FUNCTION

In this section we will explore the emergent universality in Green's functions outlined in Sec. II A. Let $\mathcal{L}_n = (\mathcal{L}_{ij})_{i,j \geq n}$ denote the tridiagonal Liouvillian matrix restricted to sites n and above, and let

$$G_n(z) := \left(O_n \left| \frac{1}{z - \mathcal{L}_n} \right| O_n \right), \quad (61)$$

which we refer to as the “level- n Green's function.” In the language of the memory function formalism, one should think of this as the Green's function for dynamics restricted to the “fast space” spanned by $\{O_m\}_{m \geq n}$.

Because of the tridiagonal structure of the Liouvillian, $G_n(z)$ obeys the recursion relation [26]

$$G_n(z) = \frac{1}{z - b_{n+1}^2 G_{n+1}(z)}. \quad (62)$$

Recurring this gives a continued fraction expansion for the original Green's function $G(z) \equiv G_0(z)$ (taking $\|A\| = 1$ for notational simplicity)

$$G(z) = \frac{1}{z - \frac{b_1^2}{z - \frac{b_2^2}{z - \dots - \frac{b_{n-1}^2}{z - b_n^2 G_n(z)}}}} \quad (63)$$

In practice one can only compute some finite number of coefficients $\{b_k\}_{k=1}^n$, so one must somehow terminate the continued fraction expansion at level n . Simply setting $G_n(z) = 0$ is a bad idea because it amounts to terminating the Lanczos chain after site n , which gives unphysical reflections of the operator wave function off the hard boundary, leading to very slow convergence with n . A better approach is to choose a model for $G_n(z)$ which is designed to accurately capture the operator backflow. The “recursion method” [26] is a popular numerical technique where one chooses such a terminator level- n Green's function based on some high level features of the spectral function, like its high-frequency decay and any algebraic singularities. The existence of appropriate terminators is something of a lottery because one needs exact expressions for all three of the Lanczos coefficients, the spectral function, and the Green's function. These solutions are usually expressed in terms of special functions, and there is no guarantee an exactly solvable model will exist with all the appropriate characteristics. With our Riemann-Hilbert approach, we can analyze the $n \rightarrow \infty$ asymptotic behavior of $G_n(z)$ directly, bypassing the need for these exactly solvable models.

We prove that, as $n \rightarrow \infty$, $G_n(z)$ approaches different universal scaling forms in different regions of the complex plane. The simplest case is when z is sufficiently far from

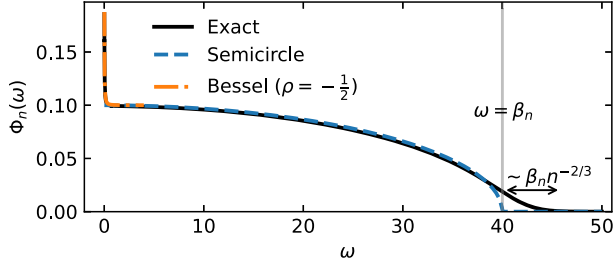


FIG. 6. The spectral function $\Phi_n(\omega) = 2 \text{Im}[G_n(\omega - i0^+)]$ corresponding to the level- n Green's function $G_n(z)$, shown for the toy spectral function $\Phi(\omega) = |\omega|^{-1/2} \text{sech}(\pi\omega)$ and $n = 40$. The Wigner semicircle law Eq. (65) describes the behavior very well in the bulk, but breaks down near $\omega = 0$ due to the low frequency power-law in the spectral function. Instead the behavior near $\omega = 0$ is described by the universal Bessel form corresponding to Eq. (67). Near the endpoint $\omega = \beta_n$ the behavior is described by Airy universality.

the special points $z = 0$ and $z = \pm\beta_n$. In this case we show that $G_n(z)$ approaches the Wigner semicircle law, the same as the average global resolvent $(1/n) \text{tr}[1/(z - M)]$ for random $n \times n$ matrices M drawn from the Gaussian unitary ensemble [56] with a suitably rescaled bandwidth. If the spectral function $\Phi(\omega)$ is complex analytic at $\omega = 0$, this semicircle behavior persists all the way down to $z = 0$. On the other hand, if $\Phi(\omega \rightarrow 0) \sim |\omega|^\rho$ has a low-frequency power law, then the semicircle law breaks down near $\omega = 0$, and $G_n(z)$ instead is described by a form dictated by the Bessel universality class. Finally, near the edge of the spectrum, $z \approx \pm\beta_n$, the behavior of $G_n(z)$ is described by the Airy universality class. We show an example of this in Fig. 6 for the toy spectral function $\Phi(\omega) = |\omega|^{-1/2} \text{sech}(\pi\omega)$, where we focus on the behavior along the real line, as encoded in the corresponding spectral function $\Phi_n(\omega) = 2 \text{Im}[G_n(\omega - i0^+)]$. Since in this case we know the spectral function $\Phi(\omega)$, we can numerically compute $G_n(z)$ using a formula expressing it as a ratio of two Cauchy-Stieltjes transforms (see Eq. S4.42 [68]).

A. Frequencies in the bulk

First let us consider frequencies sufficiently far from the special points $z = 0$ and $z = \pm\beta_n$, where the level- n Green's function $G_n(z)$ takes on a particularly simple form as $n \rightarrow \infty$. In Sec. S4 C [68] we show that, as $n \rightarrow \infty$, $\beta_n G_n(\beta_n z)$ approaches a universal scaling form

$$\beta_n G_n(\beta_n z) \approx 2(z - \sqrt{z+1}\sqrt{z-1}), \quad (64)$$

for all z satisfying $|z| > \delta_0$ and $|z/\beta_n \pm 1| > \delta_1$, where δ_0 and δ_1 are small but $\mathcal{O}(1)$ constants (see Fig. 1), and we use the principal branch of the square root, so the rhs has a branch cut along $[-1, 1]$. Thus, away from the special points $z = 0$ and $z = \pm\beta_n$, the only free parameter determining $G_n(z)$ is the frequency bandwidth β_n , which can be

approximated up to $\mathcal{O}(1/n)$ relative error by $\beta_n \approx 2b_n$ (cf. Theorem 1). When the spectral function $\Phi(\omega)$ is complex analytic at $\omega = 0$, we prove this semicircle behavior extends all the way to $z = 0$. One can then understand the special case Eq. (90) of our formula for recovering $\Phi(\omega = 0)$ from the Lanczos coefficients as being equivalent to substituting $G_n(\pm i0^+) \approx \mp 2i/\beta_n \approx \mp i/b_n$ into the continued fraction expansion for $G(z)$. Although our proof technique in this case requires analyticity at $\omega = 0$, our numerical tests in Sec. VIII B suggest that this scaling $G_n(\pm i0^+) \approx \mp 2i/\beta_n$ may still be accurate to leading order in n , even without such strong analyticity requirements.

Within the bulk frequency range $\delta_0 < |\omega| < (1 - \delta_1)\beta_n$, the spectral function corresponding to Eq. (64) is

$$\begin{aligned} \Phi_n(\omega) &:= \int_{-\infty}^{\infty} e^{-i\omega t} (O_n |e^{iL_n t}| O_n) dt \\ &= i[G_n(\omega + i0^+) - G_n(\omega - i0^+)] \\ &\approx \frac{4}{\beta_n} \sqrt{1 - (\omega/\beta_n)^2}, \end{aligned} \quad (65)$$

namely, the *semicircle law* famous from random matrix theory (RMT) [56]. Thus we see the emergence of RMT universality in the bulk frequency response of the operators O_n restricted to the fast space. To leading order in $\omega/\beta_n \ll 1$, this spectral function is constant, providing some justification for approximating the large- n dynamics by white noise, as was the original intuition for the Mori-Zwanzig memory function formalism [19,26]. In Sec. VI B we will show how this can fail near $\omega = 0$.

1. Semicircle law as a fixed point

Although this is not how we prove it (see Sec. S4 C for the actual proof [68]), a quick way to arrive at Eq. (64) is to *assume* the existence of the limit $\beta_n G_n(\beta_n z) \xrightarrow{n \rightarrow \infty} f(z)$. This is a nontrivial assumption. One way it can fail is that this limit is different for even and odd n —this is what happens near $z = 0$ when $\rho \neq 0$. These concerns aside, if we assume the limit exists, then using the properties $\lim_{n \rightarrow \infty} b_n/\beta_n = 1/2$ (cf. Theorem 1) and $\lim_{n \rightarrow \infty} b_{n+1}/b_n = 1$, one concludes from Eq. (62) that $f(z)$ must solve the fixed point equation

$$f(z) = \frac{1}{z - \frac{1}{4}f(z)}. \quad (66)$$

The rhs of Eq. (64) is then the unique solution of this fixed point, subject to the requirement that $f(z)$ is separately analytic in $\mathbb{C}_\pm$ and approaches zero like $\mathcal{O}(1/z)$ as $z \rightarrow \infty$ [these properties follow from the resolvent representation of $G_n(z)$]. We remark that this fixed point argument is analogous to a method of proving the free central limit theorem in random matrix theory, where the Wigner semicircle law emerges as the fixed point of repeated free

convolutions [96,122]. (Similar to the argument for the standard central limit theorem that the Gaussian distribution is the fixed point of repeated convolutions.)

Without going into the details of the actual proof, let us note that the origin of the semicircle law behavior for $G_n(z)$ seems somewhat different to its origin for Gaussian random matrix ensembles. For the latter, the semicircle law arises because the equilibrium measure of the Gaussian potential $Q(\omega) = -\log[\exp(-\omega^2)] = \omega^2$ is the semicircle law (c.f. Eq. (54) for $p = 2$). In that context, the equilibrium measure governs the average eigenvalue density as $n \rightarrow \infty$. However, here the semicircle law for $G_n(z)$ arises for a much wider class of potentials, even when the corresponding equilibrium measure does *not* approach the semicircle law. The main ingredient to get the semicircle law for $G_n(z)$ seems to be that the equilibrium density $\sigma_n(\omega)$ is nonzero throughout the whole interval $(-\beta_n, \beta_n)$, and that it vanishes like a square root as ω approaches the end points $\pm\beta_n$ (this is called “regular” behavior). We prove this is the case for large enough n for our class of spectral functions (see Lemma S6), subject to the regularity condition that $Q'(\omega) > 0$ for large enough ω . In this sense the universal form Eq. (64) seems quite generic, but it would be possible to violate it locally if, e.g., the spectral function contained spectral gaps.

In the language of the recursion method [26,27], the Green’s function in Eq. (64) is known as the “square root terminator,” and is one of the simplest ways to terminate the infinite continued fraction for the true Green’s function. Indeed, within the approximation $\beta_n \approx 2b_n$, it amounts to setting all $b_{n+k} \mapsto b_n$, $k = 0, 1, 2, \dots$, to be equal, so that $\mathcal{L}_n$ is just a constant tridiagonal matrix with b_n on the off-diagonals. Our results show that, remarkably, this simple square root terminator is sufficient to accurately compute diffusion constants ($\rho = 0$). In fact, we have even more freedom: any terminator satisfying $G_n(\omega \pm i0^+) \approx \mp 2i/\beta_n$ as $\omega \rightarrow 0$ would work just as well. For example, this means that it is *not* necessary to match the high frequency tail of the true spectral function (this is not generally true if $\rho \neq 0$ —see Sec. IX B). In related work, we explore this “stitching freedom” in more detail [123].

We discuss the error term of Eq. (64) in Sec. S4 C [68]. In brief, for $z = \mathcal{O}(1)$ there is a multiplicative error of $\mathcal{O}(1/n)$. For $\rho = 0$ this is true for all z , but if $\rho \neq 0$ the error can become larger as $z \rightarrow 0$, reaching $\mathcal{O}[1/\sigma_n(0)]$ at $z \sim 1/\beta_n$ [corresponding to a physical frequency $\omega = \beta_n z$ of $\mathcal{O}(n^0)$]. This is $\tilde{\mathcal{O}}(1/n^{(p-1)/p})$ for $p > 1$, but only $\mathcal{O}[1/(\log n)^{1+q+o(1)}]$ in the marginal case $p = 1$. (We remind the reader that, for $p = 1$, $q \geq 0$ is guaranteed by locality, with $q \geq 1$ in $d = 1$ spatial dimensions [11,74].)

B. Frequencies near $\omega = 0$

In the limit $\omega \rightarrow 0$, the distinction between even and odd n is important for $\rho \neq 0$; we will discuss the even case. In the order of limits where $\omega \rightarrow 0$ before $n \rightarrow \infty$,

we can show

$$G_{2n}(\omega \pm i0^+) \approx \frac{-2}{\beta_{2n}} \left(\frac{J_{\frac{1}{2}(\rho-1)} \pm iY_{\frac{1}{2}(\rho-1)}}{J_{\frac{1}{2}(\rho+1)} \pm iY_{\frac{1}{2}(\rho+1)}} \right) [\pi\sigma_{2n}(0)\omega], \quad (67)$$

where J_ν and Y_ν are Bessel functions of the first and second kind, respectively, and our shorthand indicates all Bessel functions should be evaluated with the argument $\pi\sigma_{2n}(0)\omega$ [this is a linear approximation of $\pi I_{2n}(\omega) = \pi \int_0^\omega \sigma_{2n}(s)ds \approx \pi\sigma_{2n}(0)\omega$]. See Sec. S4 C for a derivation [68]. The appearance of Bessel functions here is an indication of the Bessel universality governing the behavior near the origin due to the spectral function scaling $\Phi(\omega \rightarrow 0) \sim |\omega|^\rho$. In principle one can use this Bessel universality of $G_n(z)$ to estimate spectral functions near $\omega = 0$. However, in order to eliminate the *a priori* unknown Coulomb gas density $\sigma_{2n}(0)$, we find it more convenient to follow an equivalent approach and work directly with the zero mode in Lanczos space. We will describe this approach in Sec. VIII.

C. Frequencies near the edges $\omega = \pm\beta_n$

Near the edge of the spectrum, meaning $|z/\beta_n \pm 1| < \delta_1$ for a small $\mathcal{O}(1)$ constant δ_1 , the Wigner semicircle law for $G_n(z)$ again breaks down, and instead the behavior is governed by the Airy universality class. This is analogous to the behavior near the turning points of a WKB approximation [116], or the maximal eigenvalue distribution of a random matrix ensemble [124]. We refer the reader to Sec. S4 C [68] for the full expression for $G_n(z)$ in terms of Airy functions, and simply state here the leading behavior precisely at the end point β_n , where we have

$$G_{n,\pm}(\beta_n) \approx \frac{2}{\beta_n} \left(\frac{2\text{Ai}(0)\sqrt{f'_n(1)} + e^{\pm\frac{i\pi}{3}}\sqrt{2}\text{Ai}'(0)(\rho+1)}{2\text{Ai}(0)\sqrt{f'_n(1)} + e^{\pm\frac{i\pi}{3}}\sqrt{2}\text{Ai}'(0)(\rho-1)} \right), \quad (68)$$

with $f'_n(1) = (nh_n(1)/\sqrt{2})^{2/3} \sim \mathcal{O}(n^{2/3})$, and where $\text{Ai}(0) = (3^{2/3}\Gamma[\frac{2}{3}])^{-1}$ and $\text{Ai}'(0) = -(3^{1/3}\Gamma[\frac{1}{3}])^{-1}$. The appearance of the Airy function Ai is a reflection of the Airy universality near the spectral edge. From this expression we see that the corresponding spectral function $\Phi_n(\omega) = \mp 2\text{Im}[G_{n,\pm}(\omega)]$ does not vanish precisely at $\omega = \beta_n$, as the semicircle law would predict, but instead scales to zero like $\mathcal{O}(\beta_n^{-1}n^{-1/3})$. As discussed in Sec. S4 C, this expression for $G_{n,\pm}(\beta_n)$ has a multiplicative error of $\mathcal{O}(1/n)$ [68].

VII. HYDRODYNAMIC CONTRIBUTIONS TO THE LANCZOS COEFFICIENTS

Our next result, Theorem 1, is a statement about the recurrence coefficients $\{b_n\}$ associated with spectral

functions of the form in Eq. (34). It is an asymptotic statement, controlled in the limit $n \rightarrow \infty$ of large Lanczos number. This limit is analogous to the large matrix dimension limit in random matrix theory [65], and similarly will lead to universal behavior.

It is well established that the scaling of $\Phi(\omega)$ as $|\omega| \rightarrow \infty$ determines the *leading* behavior of the b_n as $n \rightarrow \infty$ [35]. What we show in Theorem 1 is that the hydrodynamic power-law imprints itself on the *subleading* behavior of b_n in a universal manner. Some of the quantities appearing in the theorem, namely β_n and $h_n(x)$, are defined in terms of a classical Coulomb gas problem described in Sec. V.

Theorem 1. (Informal) Consider $\Phi(\omega)/2\pi \equiv |\omega|^\rho \exp[-Q(\omega)]$, where $\rho > -1$, and Q is an even function scaling like $Q(\omega) \sim |\omega|^\rho \log^q |\omega|$ as $|\omega| \rightarrow \infty$, for some $p \geq 1$ and $q \in \mathbb{R}$. Then the recurrence coefficients b_n associated with $\Phi(\omega)$ have a large- n expansion given by

$$b_n = \frac{\beta_n}{2} \left[1 + \rho \left(\frac{1}{h_n(1)} - (-1)^n \frac{1}{h_n(0)} \right) \frac{1}{n} + \dots \right], \quad (69)$$

where the scaling of β_n is given in Eq. (49), the scaling of $h_n(1)$ and $h_n(0)$ are given in Lemmas 1 and 2, and the dots indicate terms subleading in n .

The case $p = 1$ of (quasi-)exponential decay—most relevant for generic chaotic systems according to the operator growth hypothesis [11]—turns out to be marginal, with the above expression simplifying to

$$b_n = \frac{\beta_n}{2} \left[1 + \frac{\rho}{2} \left(1 - (-1)^n \frac{1}{(\log n)^{1+o(1)}} \right) \frac{1}{n} + \dots \right], \quad (70)$$

provided $q > -1$. (Local interactions enforce $q \geq 0$.)

If $p > 1$, then this expression instead simplifies to

$$b_n = \frac{\beta_n}{2} \left[1 + \frac{\rho}{2p} (1 - (-1)^n (p-1)) \frac{1}{n} + \dots \right], \quad (71)$$

for all $q \in \mathbb{R}$.

This theorem is proved in Sec. S3, with the scaling of the $\dots$ error term discussed in Sec. S3 B5. It shows that the hydrodynamic power-law, $\Phi(\omega) \sim |\omega|^\rho$ as $\omega \rightarrow 0$, shows up as a *multiplicative correction* to the leading scaling governed by β_n , whose scaling depends only on the $|\omega| \rightarrow \infty$ asymptotics via Eq. (49). The subleading hydrodynamic term displays *staggering* due to the factor $(-1)^n$, but the magnitude of the staggering decays to zero as $n \rightarrow \infty$. Many authors have noted examples where such staggering is sufficient to give singular $\omega \rightarrow 0$ behavior [26,37–40,43,125,126]. Our result proves in a wide class of models that this staggering *necessarily* arises from the low-frequency behavior. Hermiticity is an important ingredient in producing the specific $(-1)^n$ form: it enters in the proof via the identity

$$(-1)^n = \exp \left(2\pi i \int_0^{\beta_n} \sigma_n(\omega) d\omega \right), \quad (72)$$

where $\sigma_n(\omega)$ is the charge density of the Coulomb gas distribution that we defined in Sec. V. This identity follows very generically by symmetry, since the distribution has total charge n by construction, $\int_{-\beta_n}^{\beta_n} \sigma_n(\omega) d\omega = n$, and $\sigma_n(\omega)$ is *even*, the latter coming from the evenness of the spectral function, which is a property of Hermitian systems. Thus we have $\int_0^{\beta_n} \sigma_n(\omega) d\omega = n/2$, giving the staggering factor $(-1)^n$. It is also worth emphasizing that this is a contribution from the $\omega = 0$ behavior of the spectral function; that is why the lower limit of the integral in Eq. (72) is zero. This $(-1)^n$ staggering factor also has the same origins as the sign of the polynomials at zero, $\text{sgn}[p_{2n}(0)] = (-1)^n$ [cf. Eq. (74)], which again is a generic property of even weight functions [104].

Notice that the case $p = 1$ is marginal, in the sense that the staggered multiplicative correction associated with the hydrodynamics scales like $1/n \log n$, rather than $1/n$ for $p > 1$. This is a signature of the Coulomb gas confinement transition discussed in Sec. V. This transition is primarily a consequence of the *high frequency* scaling of the spectral function, but it affects the hydrodynamic signature because of its manifestation in a logarithmically divergent equilibrium charge density at *low frequencies*.

We emphasize that the operator growth hypothesis [11] conjectures that chaotic many-body quantum systems generically have $p = 1$, and so are marginal in this sense. The possibility of a log correction for $p = 1$ was noted in [38]. Note that $p = 1$ scaling is not restricted to chaotic models—certain many-body localized models have been shown to also exhibit this scaling (with a $q = 1$ log correction) [36]. There is numerical evidence that the case $p = 2$, corresponding to $b_n \sim \sqrt{n}$, holds for interacting integrable systems at infinite temperature [11,26,127,128]. We can also consider noninteracting systems as the limit $p \rightarrow \infty$, such that $b_n \sim n^0$ and the spectral function has compact support [see Ref. [47] (Theorem 7.4)].

It is tempting to extract the value of ρ by fitting the Lanczos coefficients to the relevant asymptotic form given in Theorem 1. In practice, we have found that, while this method gives qualitatively correct answers for ρ , it tends to give rather large error bars, since one is attempting to fit the coefficient of a small subleading correction. Instead, we recommend following the procedure outlined in Sec. VIII A for extracting ρ from the leading scaling of the zero mode in the Krylov space, which amplifies the effects of the staggered subleading term coming from the low-frequency power law.

Example 5. As a check of Theorem 1, we can consider the generalized Hermite polynomials, which have weight function $\Phi(\omega)/2\pi = |\omega|^\rho \exp[-\omega^2]$. With $Q(\omega) = \omega^2$, we can determine $\beta_n = \sqrt{2n}$ from Eq. (48),

and $h_n(0) = h_n(1) = 4$ from Eq. (56) (in agreement with Lemmas 1 and 2 with $p = 2$). Substituting into Theorem 1, we get agreement to $\mathcal{O}(1/n)$ with the exact recurrence coefficients, which are known to be $b_n = (1/\sqrt{2})\sqrt{n + \frac{1}{2}[1 - (-1)^n]}\rho$ [129].

VIII. HYDRODYNAMICS FROM THE ZERO MODE

So far we have given a large- n expansion of the Lanczos coefficients b_n , showing how a low-frequency power-law in the spectral function, $\Phi(\omega \rightarrow 0) \sim |\omega|^\rho$, shows up as a subleading correction to the leading scaling of the Lanczos coefficients. In this section we explain how by studying the zero mode $|\omega = 0\rangle_{\mathcal{K}}$ of the Liouvillian $\mathcal{L}$ within the Krylov space, satisfying $\mathcal{L}|\omega = 0\rangle_{\mathcal{K}} = 0$, one can extract the value of the low frequency power-law exponent ρ , as well as hydrodynamic transport coefficients encoded in $\lim_{\omega \rightarrow 0} \Phi(\omega)/|\omega|^\rho$. The advantage of studying the zero mode over directly analyzing the Lanczos coefficients is that the zero mode amplifies the effects of the small subleading terms in the Lanczos coefficients, such that the hydrodynamics is manifest in the *leading* behavior of the zero mode as $n \rightarrow \infty$. This approach to computing transport coefficients can be summarized in three steps:

- (1) By fitting the leading behavior of the Lanczos coefficients $b_n \sim (n/\log^q n)^{1/p}$, determine the growth exponents p and q . Generic models usually have $p = 1$ and $q = 0$ (or $q = 1$ in one dimension).
- (2) By fitting the leading behavior of the zero mode amplitudes $p_{2n}(0)^2$ (defined below) to the scalings given in Corollary 1 and Table I, determine the power-law exponent ρ governing $\Phi(\omega \rightarrow 0) \sim |\omega|^\rho$.
- (3) Having fixed ρ , extract hydrodynamic transport coefficients by computing $\lim_{\omega \rightarrow 0} \Phi(\omega)/|\omega|^\rho$ using Theorem 3.

In the following subsections we will describe the second and third steps in more detail. Throughout this section we will focus on $\omega = 0$ properties of the spectral function. In Sec. IX we will generalize these ideas to $\omega \neq 0$ through an algorithm we call the *spectral bootstrap*; the approach described in this section is the $\omega \rightarrow 0$ limit of the spectral bootstrap.

A. Extracting the low frequency power law

If we expand the zero mode $|\omega = 0\rangle_{\mathcal{K}} = \sum_{n=0}^{\infty} c_n |O_n\rangle$ in the Lanczos basis, then Eq. (30) tells us that

$$\frac{c_n}{c_0} = \frac{p_n(0)}{p_0(0)}, \quad (73)$$

where $p_0(0) = \|A\|^{-1}$ for an initial operator $|A\rangle$. It is a general property of orthogonal polynomials with respect to even weight functions that they have definite parity [104], $p_n(-x) = (-1)^n p_n(x)$, and hence $c_n = 0$

for odd n . Furthermore, by using the recurrence relation in Eq. (29), we find the recursive formula $p_{2n}(0) = -(b_{2n-1}/b_{2n})p_{2n-2}(0)$, and hence

$$\frac{p_{2n}(0)}{p_0(0)} = (-1)^n \frac{b_{2n-1}}{b_{2n}} \frac{b_{2n-3}}{b_{2n-2}} \dots \frac{b_1}{b_2}. \quad (74)$$

Because this quantity involves the ratio of even and odd Lanczos coefficients, it amplifies the effects of the even or odd staggering in the subleading corrections to the Lanczos coefficients indicated in Theorem 1. Previous authors have noted that this is analogous to the origin of the localized zero mode in the Su-Schrieffer-Heeger chain [39,40,130]. Our next result gives us a precise handle on how the low-frequency power-law imprints itself on the scaling of this zero mode, which is a reflection of the Bessel universality class discussed in Sec. VI B.

Theorem 2. (Informal) Given the spectral function $\Phi(\omega)/2\pi \equiv |\omega|^\rho \exp[-Q(\omega)]$, so that $\Phi(\omega \rightarrow 0) \sim |\omega|^\rho$ for some $\rho > -1$, then as $n \rightarrow \infty$ we have

$$p_{2n}(0)^2 = \frac{C_\rho}{e^{-Q(0)}} \frac{[\pi \sigma_{2n}(0)]^\rho}{\beta_{2n}} [1 + o(1)], \quad (75)$$

where σ_{2n} and β_{2n} are, respectively, the density and support of the equilibrium Coulomb gas distribution defined in Sec. V, and the constant C_ρ is given by

$$C_\rho = \frac{2^{1-\rho}}{\Gamma[\frac{1}{2}(1+\rho)]^2}. \quad (76)$$

See Sec. S4 A 1 for a proof; the $o(1)$ error term is given by the rhs of Eq. (S2.126). For the purposes of numerically extracting the value of ρ , it is worth isolating the n dependence of $p_{2n}(0)^2$. The scaling of β_{2n} is given in Eq. (49), and the scaling of $\sigma_{2n}(0) = (2n/\beta_{2n})h_{2n}(0)/2\pi$ can be deduced from Lemma 2 leading to the following.

Corollary 1. With $\Phi(\omega) \equiv |\omega|^\rho \exp[-Q(\omega)]$, for $n \rightarrow \infty$ we have

$$p_{2n}(0)^2 \sim \mathcal{O}\left(\frac{[nh_{2n}(0)]^\rho}{\beta_{2n}^{1+\rho}}\right). \quad (77)$$

With $Q(x) \sim |x|^p \log^q |x|$ as $|x| \rightarrow \infty$, this leads to

$$p_{2n}(0)^2 \sim \begin{cases} n^{-\frac{1}{p} + \rho(1 - \frac{1}{p})} (\log n)^{\frac{q}{p}(1+\rho)}, & \text{for } p > 1, q \in \mathbb{R}, \\ n^{-1} (\log n)^{\rho + q(1+\rho)}, & \text{for } p = 1, q > -1, \end{cases} \quad (78)$$

where the $\sim$ indicates only the overall n dependence, and for $p = 1$ we have dropped the $o(1)$ subleading exponent in $h_n(0) = (\log n)^{1+o(1)}$ from Lemma 2.

TABLE I. Zero mode amplitude $p_{2n}(0)^2$ scaling as a function of the low-frequency power-law exponent $\Phi(\omega \rightarrow 0) \sim |\omega|^\rho$ and the high-frequency behavior $\Phi(\omega \rightarrow \infty) \sim \omega^p \log^q \omega$.

Dynamical class	p	q	$\mathcal{O}[p_{2n}(0)^2]$	$\mathcal{O}(\sum_{m=0}^n p_{2m}(0)^2)$
Chaotic ($d > 1$)	1	0	$(\log n)^\rho/n$	$(\log n)^{1+\rho}$
Chaotic ($d = 1$)	1	1	$(\log n)^{1+2\rho}/n$	$(\log n)^{2(1+\rho)}$
Interacting integrable	2	0	$n^{\frac{1}{2}(-1+\rho)}$	$n^{\frac{1}{2}(1+\rho)}$
Noninteracting	∞	0	n^ρ	$n^{1+\rho}$

In Table I we summarize some of the most physically relevant cases for the scaling of $p_{2n}(0)^2$. The statements about chaotic dynamics are made according to the operator growth hypothesis [11], while the statement about integrable systems is based on numerical evidence only.

Since we must have $1 + \rho > 0$ in order for the spectral function to be Lebesgue integrable across $\omega = 0$, we see that $\sum_n p_{2n}(0)^2$ diverges algebraically for $p > 1$, but polylogarithmically for $p = 1$, so the zero mode is *marginally delocalized* for $p = 1$. Unlike the original Su-Schrieffer-Heeger chain, here the zero mode is not a true localized eigenstate; although its support does decay further into the bulk of the Lanczos chain, it does so too slowly to be localized. The only case in which the zero mode is normalizable is when the spectral function has a delta-function peak at zero frequency. In Fig. 7 we test this

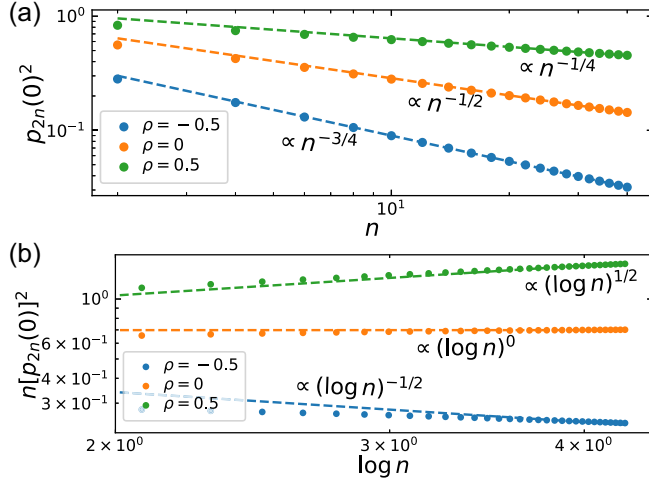


FIG. 7. Scaling of the zero mode amplitudes $p_{2n}(0)^2$ as a function of the power-law exponent ρ governing the low-frequency spectral function $\Phi(\omega \rightarrow 0) \sim |\omega|^\rho$. (a) Toy model $\Phi(\omega)/2\pi = |\omega|^\rho \exp[-Q(\omega)]$ with $Q(\omega) = (1 + \omega^2 + \omega^4)^{1/2}$. This model has $p = 2$, so $p_{2n}(0)^2$ scales algebraically as $p_{2n}(0)^2 \sim n^{\frac{1}{2}(\rho-1)}$. (b) Toy model $\Phi(\omega)/2\pi = |\omega|^\rho \exp[-Q(\omega)]$ with $Q(\omega) = (1 + \omega^2 + \omega^4)^{1/4}$. This model has $p = 1$, so $n p_{2n}(0)^2$ scales polylogarithmically as $n p_{2n}(0)^2 \sim (\log n)^\rho$. [Note the axes are log-scale, so for case (b) the x-axis is linear in $\log \log n$].

scaling numerically for some toy spectral functions, finding good agreement with the predictions of Theorem 2.

Having fixed the values of p and q from the leading behavior of the Lanczos coefficients $b_n \sim (n/\log^q n)^{1/p}$, one can extract the value of the power-law exponent ρ by computing $p_{2n}(0)^2$ and fitting to the relevant asymptotic form indicated in Corollary 1. In practice, we have found this to be more robust than attempting to extract ρ by fitting the subleading terms in Theorem 1, since here one is fitting to the leading behavior of $p_{2n}(0)^2$. Generally this procedure works well for $p > 1$, i.e., sublinear Lanczos coefficients. However, for $p = 1$, where the Lanczos coefficients are (quasi-)linear, we have found that n has to be extremely large before this procedure gives the correct value of ρ , and so in this case it might be best to assume the value of ρ on phenomenological grounds. We attribute this to the potentially very slow convergence with n of the Coulomb gas density $h_n(0)$ to its asymptotic form given in Lemma 2. While this slow convergence seems troubling, our algorithms for extracting hydrodynamic coefficients like diffusion constants, discussed in the next section, do *not* rely on the asymptotics for the Coulomb gas because we ultimately eliminate $h_n(0)$ from the equations that will determine the hydrodynamic coefficients.

B. Extracting hydrodynamic transport coefficients

Now let us apply this to develop a numerical algorithm for computing diffusion constants and other transport coefficients. To warm up, let us consider diffusion. The infinite temperature dc conductivity σ_{dc} , rescaled by the temperature [131], is given by the Kubo formula [132]

$$\sigma_{dc} = \lim_{\tau \rightarrow \infty} \lim_{L \rightarrow \infty} \int_0^\tau \frac{\langle \mathcal{J} | \mathcal{J}(t) \rangle}{L} dt, \quad (79)$$

where $\mathcal{J}$ is the total current for the conserved charge, and L is the system size. Now we set our initial Lanczos operator to be $A := \mathcal{J}/\sqrt{L}$, the zero wave vector component of the Fourier transform of the current density, and take the thermodynamic limit $L \rightarrow \infty$. For a translationally invariant system, the Lanczos algorithm can be run in Fourier space with minimal modification compared with real space [11] (Appendix C). Since $(\mathcal{J} | \mathcal{J}(t)) = (\mathcal{J} | \mathcal{J}(-t))$ at infinite temperature, we have

$$\sigma_{dc} = \frac{1}{2} \int_{-\infty}^{\infty} (A | A(t)) dt = \frac{1}{2} \Phi(0). \quad (80)$$

Given our ansatz $\Phi(\omega)/2\pi = |\omega|^\rho \exp[-Q(\omega)]$, we have

$$\lim_{\omega \rightarrow 0} \frac{\Phi(\omega)}{|\omega|^\rho} = 2\pi e^{-Q(0)}. \quad (81)$$

For regular diffusion we expect the dc conductivity σ_{dc} to be finite, which corresponds to $\rho = 0$. To obtain the

diffusion constant, one can use the Einstein relation $D = \sigma_{\text{dc}}/\chi$, where χ is the static susceptibility.

The first thing to notice is that Eq. (75) contains this factor $\exp[-Q(0)]$ which gives us the value of $\lim_{\omega \rightarrow 0} \Phi(\omega)/|\omega|^\rho$. However, despite being able to exactly compute $p_{2n}(0)$ via Eq. (74), the problem with using Eq. (75) on its own to extract $\exp[-Q(0)]$ is that it also contains the factor $\sigma_n(0)$ related to the density of the equilibrium measure at the origin, and in general computing $\sigma_n(0)$ exactly requires already knowing the spectral function (since one needs to know Q). While Lemma 2 indicates that $\sigma_n(0) = (1/2\pi)(nh_n(0)/\beta_n)$ exhibits some universal scaling as $n \rightarrow \infty$, this asymptotic scaling can take a long time to set in, so given only a modest number of Lanczos coefficients, it would be preferable to rely only on quantities we can compute exactly or approximately using a finite number of Lanczos coefficients. Fortunately, it turns out to be possible to eliminate the leading-order dependence on $\sigma_n(0)$ using the precise asymptotics of the partial sums of the amplitudes $p_{2n}(0)^2$. We define

$$K_n(x, y) := \sum_{m=0}^{n-1} p_m(x) p_m(y), \quad (82)$$

which is known as the ‘‘Christoffel-Darboux kernel’’ in the language of orthogonal polynomials [133]. Under the same assumptions as Theorem 2, we can derive the asymptotic expression

$$K_n(0, 0) = \frac{\hat{c}_\rho}{e^{-Q(0)}} [\pi \sigma_n(0)]^{1+\rho} [1 + o(1)], \quad (83)$$

where the $o(1)$ error term is given in the rhs of Eq. (S2.126), and the constant $\hat{c}_\rho$ is given by

$$\hat{c}_\rho := \frac{1}{2^{1+\rho} \Gamma[\frac{1}{2}(1+\rho)] \Gamma[\frac{1}{2}(3+\rho)]}. \quad (84)$$

Combining Eq. (83) and Theorem 2 gives

$$\pi \sigma_{2n}(0) = 2(1+\rho) \frac{K_{2n}(0, 0)}{\beta_{2n} p_{2n}(0)^2} [1 + o(1)], \quad (85)$$

which allows us to eliminate the equilibrium measure $\sigma_{2n}(0)$, leading to the following result:

Theorem 3. With $\Phi(\omega) \sim |\omega|^\rho$ as $\omega \rightarrow 0$, we have

$$\lim_{\omega \rightarrow 0} \frac{\Phi(\omega)}{|\omega|^\rho} = \lim_{n \rightarrow \infty} \frac{c_\rho [K_{2n}(0, 0)]^\rho}{[\beta_{2n} p_{2n}(0)^2]^{1+\rho}}, \quad (86)$$

where the constant c_ρ is given by

$$c_\rho = \frac{4\pi(1+\rho)^\rho}{\Gamma[\frac{1}{2}(1+\rho)]^2}. \quad (87)$$

For the reader’s convenience, we can use Eqs. (74) and (82) to give the following more explicit expression in terms of Lanczos coefficients:

$$\lim_{\omega \rightarrow 0} \frac{\Phi(\omega)}{|\omega|^\rho} = \lim_{n \rightarrow \infty} c_\rho \|A\|^2 \frac{\left[1 + \sum_{k=1}^{n-1} \left(\frac{b_{2k-1}}{b_{2k}} \dots \frac{b_1}{b_2}\right)^2\right]^\rho}{\left[\beta_{2n} \left(\frac{b_{2n-1}}{b_{2n}} \dots \frac{b_1}{b_2}\right)^2\right]^{1+\rho}}, \quad (88)$$

where A is the initial Lanczos operator, $O_0 = A/\|A\|$.

By Theorem 1 we know that $\beta_{2n} = 2b_{2n}[1 + \mathcal{O}(1/n)]$, so from Eq. (88) we can see that simply approximating $\beta_{2n} \approx 2b_{2n}$ also results in a small $\mathcal{O}(1/n)$ relative error in $\lim_{\omega \rightarrow 0} \Phi(\omega)/|\omega|^\rho$. All of the subtle effects from subleading staggered terms in the Lanczos coefficients are captured in $p_{2n}(0)^2$ and $K_{2n}(0, 0)$, which we can compute exactly using the first $2n$ Lanczos coefficients.

C. Numerical benchmarks

To start our benchmarks, we begin with regular diffusion. In this case $\Phi(0)$ for the current operator is nonzero, corresponding to $\rho = 0$. In this special case, Theorem 3 becomes

$$\Phi(0) = \lim_{n \rightarrow \infty} \frac{4}{\beta_{2n} p_{2n}(0)^2}, \quad (89)$$

$$= \lim_{n \rightarrow \infty} \frac{4\|A\|^2}{\beta_{2n}} \left(\frac{b_{2n}}{b_{2n-1}} \frac{b_{2n-2}}{b_{2n-3}} \dots \frac{b_2}{b_1}\right)^2, \quad (90)$$

where in the second line we used Eq. (74) to replace $p_{2n}(0)^2$. A similar formula appears in Ref. [45], based on considerations motivated by the operator growth hypothesis [11] (see also Ref. [46]). Our derivation shows that this behavior does not necessarily require the system to obey the OGH in order for it to be valid.

We start with the mixed field Ising model,

$$H_{\text{MFIM}} = \sum_{i=1}^L (Z_i Z_{i+1} + g_x X_i + g_z Z_i), \quad (91)$$

which has energy diffusion for generic values of g_x and g_z [3]. The energy current operator is defined by a continuity equation to be $\mathcal{J} = \sum_i g_x (Y_i Z_{i+1} - Z_i Y_{i+1})$, so given the initial Lanczos operator $A := \mathcal{J}/\sqrt{L}$, in the thermodynamic limit we have $\|A\|^2 = \lim_{L \rightarrow \infty} (\mathcal{J}|\mathcal{J})/L = 2g_x^2$, which is needed to compute $p_{2n}(0)$ via Eq. (74). The infinite temperature static energy susceptibility is $\chi = \lim_{L \rightarrow \infty} [\langle H_{\text{MFIM}}^2 \rangle - \langle H_{\text{MFIM}} \rangle^2]/L = 1 + g_x^2 + g_z^2$. Fixing $g_x = 1.4$ and $g_z = 0.9045$, Fig. 8(a) shows the result of using Theorem 3 with $\rho = 0$ to extract the energy diffusion constant D . A linear extrapolation in $1/n$ on the predictions from up to $n = 40$ Lanczos coefficients gives an estimate

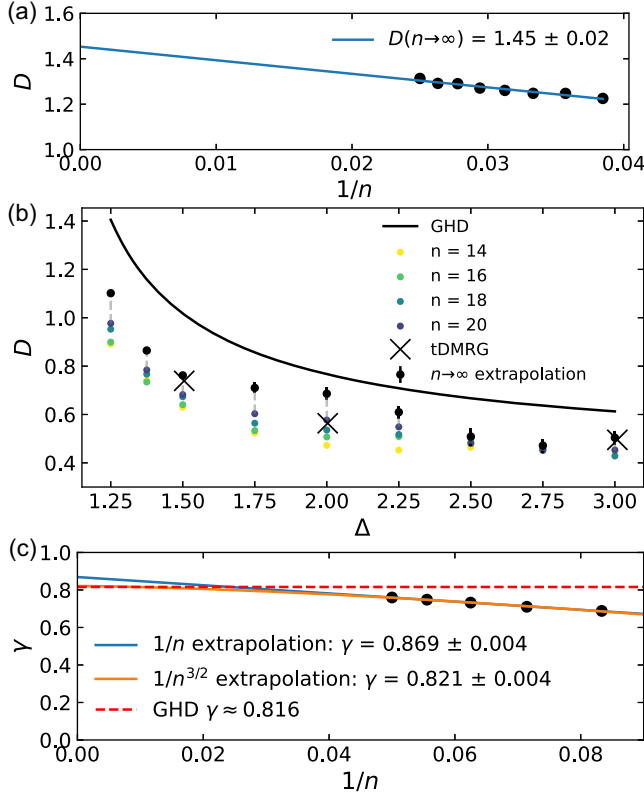


FIG. 8. (a) Energy diffusion constant of the mixed field Ising chain using Theorem 3 with $\rho = 0$ and up to $n = 40$ Lanczos coefficients. (b) Spin diffusion constant of the XXZ spin chain as a function of the anisotropy Δ using Theorem 3 with $\rho = 0$ and up to $n = 20$ Lanczos coefficients, compared with the prediction Eq. (93) from generalized hydrodynamics (GHD) [134,135]. The black dots show the result of a linear extrapolation in $1/n$ to $n \rightarrow \infty$, and the crosses show tDMRG data from Ref. [136]. (c) Coefficient γ of the time-dependent spin diffusion constant $D(t) \sim \gamma t^{1/3}$ for the isotropic Heisenberg chain using Theorem 3 with $\rho = -1/3$ and up to $n = 20$ Lanczos coefficients. Extrapolations in $1/n$ and $1/n^{3/2}$ give slightly different predictions, with the $1/n^{3/2}$ result agreeing better with the GHD prediction $\gamma \approx 0.816$ [137,138].

of $D(n \rightarrow \infty) = 1.45 \pm 0.02$, close to the prediction $D = 1.4$ – 1.45 from tensor network methods [5,10].

For a more challenging example, we study spin diffusion in the XXZ chain,

$$H_{\text{XXZ}} = \frac{1}{4} \sum_i (X_i X_{i+1} + Y_i Y_{i+1} + \Delta Z_i Z_{i+1}), \quad (92)$$

which has diffusive spin transport for $\Delta > 1$ [131]. The spin current operator is defined by a continuity equation to be $\mathcal{J} = \frac{1}{4} \sum_i (X_i Y_{i+1} - Y_i X_{i+1})$, so $\|A\|^2 = 1/8$. The infinite temperature static spin susceptibility is $\chi = \lim_{L \rightarrow \infty} [(\langle S_{\text{tot}}^z \rangle^2) - \langle S_{\text{tot}}^z \rangle^2] / L = 1/4$. Performing a linear extrapolation in $1/n$ on the predictions from up to $n = 20$ Lanczos coefficients, Fig. 8(b) shows that the resulting

diffusion constants come reasonably close to the prediction from generalized hydrodynamics [134,135],

$$D_{\text{XXZ}} = \frac{2 \sinh \eta}{9\pi} \sum_{s=1}^{\infty} (1+s) \left[\frac{s+2}{\sinh \eta s} - \frac{s}{\sinh \eta (s+2)} \right], \quad (93)$$

where $\eta := \text{arcosh} \Delta$. We note that $n = 20$ is still relatively small, but even at these small values the predictions are close to those obtained using tensor network methods like tDMRG [136].

Finally, to test the method for $\rho \neq 0$, we study spin transport in the isotropic Heisenberg chain (i.e., $\Delta = 1$ for the XXZ spin chain). This has recently been discovered to have superdiffusive spin transport at infinite temperature [84], with a current-current correlator decaying like $(\mathcal{J}|\mathcal{J}(t)) \sim (\chi\gamma/3)t^{-2/3}$ as $t \rightarrow \infty$, leading to a time-dependent diffusion constant growing like $D(t) \sim \gamma t^{1/3}$. The value of γ can be extracted from the Lanczos coefficients of the spin current operator using Theorem 3 with $\rho = -1/3$ via the relation

$$\gamma = \frac{\chi\sqrt{3}}{\Gamma[\frac{1}{3}]} \lim_{\omega \rightarrow 0} \frac{\Phi(\omega)}{|\omega|^{-1/3}}, \quad (94)$$

where we used the fact that $\int_{\mathbb{R}} e^{-i\omega t} |t|^{-2/3} dt = \sqrt{3}\Gamma[1/3]|\omega|^{-1/3}$. Generalized hydrodynamics gives the prediction $\gamma = (2/3)(10\pi/27)^{4/3} \approx 0.816$ [137,138]. Using only $n = 20$ Lanczos coefficients, we obtain predictions close to this result (Fig. 8(c)). Interestingly, we find extrapolating in $n^{-3/2}$ gives better agreement with the GHD prediction than extrapolating in n^{-1} . Finally, we emphasize that, since this example has $\rho \neq 0$, not accounting for the Bessel universality governing the $\omega \approx 0$ behavior can lead to erroneous results, as we will illustrate in more detail in Sec. IX B and Fig. 10. This is where our approach differs the most from other recent approaches based on the recursion method [45,46,52].

IX. THE SPECTRAL BOOTSTRAP: APPROXIMATING THE SPECTRAL FUNCTION AT FINITE FREQUENCIES

In this section we explain how to extend ideas from the previous section to extract the spectral function at finite frequencies. The high-level summary is that, using the appropriate $n \rightarrow \infty$ asymptotics of the orthogonal polynomials, we formulate a first-order differential equation involving the spectral function and the equilibrium density of the Coulomb gas defined in Sec. V. Together with an initial condition at $\omega = 0$ provided to us by Hermiticity [i.e., $\Phi(-\omega) = \Phi(\omega)$], this differential equation can be iteratively solved to give a finite n approximation to the spectral function at finite frequencies. The convergence of this approximation is determined by the convergence of the polynomial asymptotics. We refer to this general procedure

of using orthogonal polynomial asymptotics to give a differential equation for the spectral function as the *spectral bootstrap*. We emphasize that the approach to hydrodynamic transport coefficients in Sec. VIII B used the $\omega \rightarrow 0$ limit of these asymptotics, and so can be viewed as the initial $\omega = 0$ step of the spectral bootstrap.

In order for this approach to work well, it is important to derive asymptotics taking into account all the relevant “high-level” features of the spectral function, such as algebraic singularities, spectral gaps, etc. We will focus on a particular instance of this idea, where the spectral function $\Phi(\omega)/2\pi \equiv |\omega|^\rho \exp[-Q(\omega)]$ has a power law at $\omega = 0$ but is otherwise smooth. To warm up we will start by taking $\rho = 0$, so that the spectral function is also smooth at $\omega = 0$. The procedure for $\rho \neq 0$ is conceptually similar, but the equations involved are a little more complicated. For $\rho = 0$, the whole frequency range $\omega \in (-\beta_n, \beta_n)$ is referred to as the bulk (see Fig. 4), and the following procedure is controlled for all frequencies inside the bulk and away from the edges $\omega \approx \pm\beta_n$. Since β_n increases with n , for computationally accessible values of n the size of the bulk is large enough to capture the frequency range where the spectral function is non-negligible.

We remark that if one is only interested in recovering the spectral function $\Phi(\omega)$ in the bulk, there is a much simpler procedure than what we are about to describe:

- (1) Approximate the level- n Green’s function by the universal “semicircle form”

$$G_n(z) \approx \frac{2}{\beta_n^2} (z - \sqrt{z + \beta_n} \sqrt{z - \beta_n}),$$

from Eq. (64), where we approximate $\beta_n \approx 2b_n$.

- (2) Substitute $G_n(z)$ into the continued fraction in Eq. (63) to obtain the full Green’s function $G(z)$.
- (3) Compute $\Phi(\omega) = 2 \operatorname{Im} G(\omega - i0^+)$.

For $\rho = 0$, this simple procedure gives numerically identical results to the spectral bootstrap algorithm we will describe in the next section. However, for $\rho \neq 0$, the behavior of $\Phi(\omega)$ near $\omega = 0$ is strongly influenced by the behavior of the Coulomb gas density $\sigma_n(\omega)$, which can have strong fluctuations as a function of ω . Unfortunately, even for finite n , computing $\sigma_n(\omega)$ exactly requires already knowing the full spectral function $\Phi(\omega)$, which would defeat the point. What we show is how to iteratively compute an approximation to $\sigma_n(\omega)$, controlled in the $n \rightarrow \infty$ limit, and in turn how this can be used to obtain a finite n approximation of the spectral function $\Phi(\omega)$. As a by-product, having access to $\sigma_n(\omega)$ is also useful in checking the emergence of random matrix universality in

this quantum operator growth problem—we will discuss this in more detail in Sec. X.

As well as the bulk and Bessel versions of the spectral bootstrap we will describe imminently, one can also formulate a version suited for frequencies near the spectral edge $|\omega| \approx \beta_n$. In this region there is Airy universality, as in Sec. VIC, and one must formulate polynomial asymptotics which are relevant for this universality class. We defer the exposition of this Airy bootstrap to Sec. S6.

A. The bulk bootstrap: $\rho = 0$

In terms of the Coulomb gas density $\sigma_n(x)$ discussed in Sec. V, we define

$$\theta_n(\omega) := -\pi \int_{\omega}^{\beta_n} \sigma_n(\omega') d\omega' - \frac{\pi}{4}. \quad (95)$$

We will see shortly that $\theta_n(\omega)$ acts as a phase factor in a WKB-like asymptotic for the orthogonal polynomials. However, let us first briefly discuss some consequences of Hermiticity. For an arbitrary spectral function, all one knows about σ_n is that it integrates to n over some interval whose right end point is β_n . But for an *even* spectral function, which for us is guaranteed by Hermiticity, we know that $\sigma_n(\omega)$ is even and supported on $[-\beta_n, \beta_n]$, so that $\int_0^{\beta_n} \sigma_n(x) dx = n/2$ (the same integral gave the staggering factor $(-1)^n$ in the recurrence coefficients, cf. Theorem 1). Thus we can rewrite $\theta_n(\omega)$ as

$$\theta_n(\omega) = \pi I_n(\omega) - \frac{n\pi}{2} - \frac{\pi}{4}, \quad (96)$$

where $I_n(\omega)$ is defined as

$$I_n(\omega) := \int_0^{\omega} \sigma_n(\omega') d\omega'. \quad (97)$$

Note that $I_n(0) = 0$, independent of the density distribution σ_n . This is important because it sets the initial condition for the first-order differential equation we will derive below. Although this is manifest from its definition, this discussion shows that it is really Hermiticity that gives us an initial condition at $\omega = 0$. For a noneven weight function, one would only have an initial condition for $\theta_n(\omega)$ at $\omega = \beta_n$.

The Riemann-Hilbert analysis gives us access to detailed asymptotic formulas for the orthogonal polynomials, controlled as $n \rightarrow \infty$. In the orthogonal polynomials literature, these are known as Plancherel-Rotach asymptotics [71].

We have the following generalization of Theorem 2 for $\rho = 0$ to finite frequencies (see Sec. S4 [68] for a proof)

$$p_{n-1}(\omega) \approx \frac{-1}{\sqrt{\Phi(\omega)}} \sqrt{\frac{2}{\beta_n}} \left[\left(\frac{\beta_n - \omega}{\beta_n + \omega} \right)^{1/4} \cos \theta_n(\omega) + \left(\frac{\beta_n + \omega}{\beta_n - \omega} \right)^{1/4} \sin \theta_n(\omega) \right], \quad (98)$$

$$p_n(\omega) \approx \frac{1}{\sqrt{\Phi(\omega)}} \sqrt{\frac{2}{\beta_n}} \left[\left(\frac{\beta_n - \omega}{\beta_n + \omega} \right)^{1/4} \cos \theta_n(\omega) - \left(\frac{\beta_n + \omega}{\beta_n - \omega} \right)^{1/4} \sin \theta_n(\omega) \right], \quad (99)$$

where at this level of approximation we are setting $\beta_n \approx 2b_n$ (see Theorem 1 for $\rho = 0$). The first thing to notice is that $p_{n-1}(\omega)$ and $p_n(\omega)$ are both pointwise proportional to $1/\sqrt{\Phi(\omega)}$; this important feature is quite generic for orthogonal polynomials, and we will see that it continues to hold in other frequency regimes. We can combine these equations and use Eq. (96) to get the first of our bootstrap equations:

$$\Phi(\omega) \approx \frac{4}{\beta_n} \frac{1}{p_{n-1}(\omega)^2 + p_n(\omega)^2} \frac{\beta_n - (-1)^n \omega \sin[2\pi I_n(\omega)]}{\sqrt{\beta_n^2 - \omega^2}}. \quad (100)$$

Note that we used *two* polynomials, p_{n-1} and p_n , in order to avoid dividing by zero: it is a general property of orthogonal polynomials that the zeros of p_{n-1} and p_n interlace [139], which guarantees $p_{n-1}(\omega)^2 + p_n(\omega)^2 > 0$.

Given n Lanczos coefficients $\{b_k\}_{k=1}^n$, the orthogonal polynomials $p_k(\omega)$ can be computed exactly up to order n using the recursion relation Eq. (29), so if we knew the phase function $I_n(\omega)$, then we would be able to recover the spectral function $\Phi(\omega)$ from Eq. (100). However, $I_n(\omega) = \int_0^\omega \sigma_n(\omega') d\omega'$ is defined in terms of the equilibrium measure σ_n , and computing that *exactly* via Eqs. (51), (55) and (56) requires prior knowledge of Φ , precisely the function we are trying to estimate in the first place.

Our solution is to derive a large- n approximation for $\sigma_n(\omega)$ using the polynomial asymptotics Eqs. (98) and (99). If we differentiate either of these equations with respect to ω , we will get terms involving the derivative $I'_n(\omega) = \sigma_n(\omega)$, which is what we want. However, we will also get terms involving the derivative $\Phi'(\omega)$, which is a new unknown. The trick is to consider a certain “determinantal” combination of derivatives. In particular, the Christoffel-Darboux formula [104] shows that the diagonal Christoffel-Darboux kernel $K_n(\omega, \omega) = \sum_{k=0}^{n-1} p_k(\omega)^2$ can be expressed as

$$K_n(\omega, \omega) = b_n \det \begin{pmatrix} p_{n-1}(\omega) & p'_{n-1}(\omega) \\ p_n(\omega) & p'_n(\omega) \end{pmatrix}. \quad (101)$$

This determinantal structure ensures exact cancellation of the terms involving the unwanted derivative $\Phi'(\omega)$. More explicitly, if we write the asymptotics in Eqs. (98) and (99) as $p_{n-1}(\omega) \equiv \Phi(\omega)^{-1/2} \alpha_{n-1}(\omega)$ and $p_n(\omega) \equiv \Phi(\omega)^{-1/2} \alpha_n(\omega)$, then we have (suppressing ω arguments for ease of notation)

$$\begin{aligned} K_n &= b_n \det \begin{pmatrix} \Phi^{-1/2} \alpha_{n-1} & (\Phi^{-1/2} \alpha_{n-1})' \\ \Phi^{-1/2} \alpha_n & (\Phi^{-1/2} \alpha_n)' \end{pmatrix}, \\ &= b_n \Phi^{-1} \det \begin{pmatrix} \alpha_{n-1} & \alpha'_{n-1} \\ \alpha_n & \alpha'_n \end{pmatrix} \\ &\quad - \frac{b_n}{2} \Phi' \Phi^{-2} \det \begin{pmatrix} \alpha_{n-1} & \alpha_{n-1} \\ \alpha_n & \alpha_n \end{pmatrix}, \\ &= b_n \Phi^{-1} [\alpha_{n-1} \alpha'_n - \alpha'_n \alpha_n]. \end{aligned}$$

Thus the term proportional to $\Phi'(\omega)$ vanishes as claimed due to rank deficiency. This argument does not rely on the detailed form of the polynomial asymptotics, other than the fact that $p_{n-1}(\omega)$ and $p_n(\omega)$ are both pointwise proportional to $\Phi(\omega)^{-1/2}$, and so will generalize to other frequency regimes discussed in subsequent sections.

Now, carrying out the relevant derivatives of Eqs. (98) and (99) and rearranging, we get the second of our bootstrap equations:

$$\sigma_n(\omega) \approx \frac{\Phi(\omega)}{2\pi} K_n(\omega, \omega) + (-1)^n \frac{\beta_n \cos[2\pi I_n(\omega)]}{2\pi(\beta_n^2 - \omega^2)}. \quad (102)$$

The kernel $K_n(\omega, \omega)$ can be calculated in terms of the orthogonal polynomials using Eq. (101), which we remind the reader can be computed exactly in terms of the Lanczos coefficients $\{b_k\}_{k=1}^n$ using the three-term recurrence Eq. (29). Equation (102) was proven by different means for a similar class of spectral functions in [111] (Theorem 9.5), but only with the first term on the rhs, which dominates the second term by a factor of $\mathcal{O}(\beta_n)$ when ω is $\mathcal{O}(1)$.

Since $\partial_\omega I_n(\omega) = \sigma_n(\omega)$, Eqs. (100) and (102) constitute a first-order ordinary differential equation for $I_n(\omega)$ which is closed at leading order in n , and whose solution yields $\Phi(\omega)$ as a byproduct. Crucially, we also have the initial condition $I_n(0) = 0$, allowing us to iteratively solve this differential equation and thereby obtain a finite- n approximation to the spectral function. Given a choice of frequency spacing $\delta\omega \ll 1$ and a maximum frequency $\omega_{\max}$ satisfying $0 < \omega_{\max} \ll \beta_n$, the algorithm works as follows:

- (0) Set $\omega = 0$ and $I_n(0) = 0$.
- (1) Compute $\Phi(\omega)$ using Eq. (100).
- (2) Compute $\sigma_n(\omega)$ using Eq. (102).
- (3) Set $I_n(\omega + \delta\omega) = I_n(\omega) + \sigma_n(\omega) \times \delta\omega$.
- (4) Increment $\omega \mapsto \omega + \delta\omega$.
- (5) Repeat steps 1–4 until $\omega = \omega_{\max}$, then terminate.

As an example we will again consider the mixed field Ising model (MFIM). We will compare the results to those obtained through the “simple procedure” described at the

start of the section where we replace the level- n Green's function $G_n(z)$ by the universal “semicircle” form in Eq. (64), as well as to the spectral function resulting from Fourier transforming the real-time autocorrelation function $(\mathcal{J}|\mathcal{J}(t))/\sqrt{L}$ obtained using time-evolving block decimation (TEBD). For the spectral bootstrap we compute $n = 40$ Lanczos coefficients, going up to $\omega_{\max} = 0.99\beta_n$, with $\beta_n \approx 2b_n \approx 46.1$ for $n = 40$ in the MFIM. For the real-time evolution, we go up to $t_{\max} = 10$ using a time step of $dt = 0.01$, with a maximum bond dimension of $\chi_{\max} = 512$ and a system size $L = 201$ large enough that finite-size effects are negligible over these timescales. We can see from the main panel of Fig. 9(a) that the spectral bootstrap and semicircle approximation give numerically identical results throughout this bulk frequency range, providing an important self-consistency check. The spectral bootstrap also generally agrees well with the Fourier transform data at

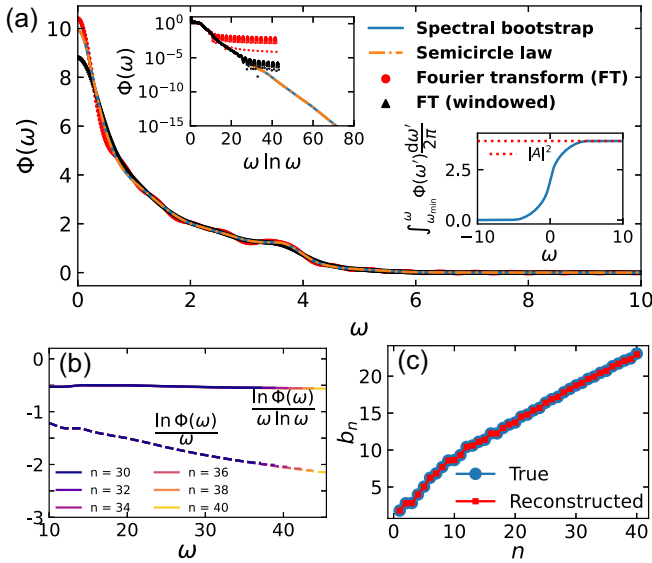


FIG. 9. (a) The $n = 40$ approximation to the spectral function of the energy current operator of the mixed field Ising model, with a comparison between the spectral bootstrap and approximating the level- n Green's function by the semicircle form in Eq. (64), and the Fourier transform (FT) of the real-time correlator $C(t) = (\mathcal{J}|\mathcal{J}(t))/L$ up to $t_{\max} = 10$. We also show the Fourier transform of $C(t)$ multiplied by the cosine window $\frac{1}{2}[1 + \cos(\pi t/t_{\max})]$ in order to reduce spectral leakage due to the finite $t_{\max}$. The left inset demonstrates the high-frequency scaling $\Phi(\omega) \sim \exp[-\mathcal{O}(\omega \log \omega)]$ expected in one spatial dimension [11]. The right inset verifies the sum rule for the approximate spectral function obtained using the spectral bootstrap. (b) The spectral function $\Phi(\omega)$ obtained using the spectral bootstrap, rescaled in two ways, as evidence that $\Phi(\omega \rightarrow \infty) \sim \exp[-\mathcal{O}(\omega \log \omega)]$ rather than $\Phi(\omega \rightarrow \infty) \sim \exp[-\mathcal{O}(\omega)]$. (c) Reconstructed Lanczos coefficients b_n , calculated using the estimate of the spectral function obtained using the spectral bootstrap, and compared with the exact Lanczos coefficients. The close agreement is strong evidence that the spectral bootstrap is producing a good approximation to the true spectral function.

low to moderate frequencies. There is some deviation around $\omega = 0$, but it is hard to do a fair comparison because it is not clear how to systematically convert between n and $t_{\max}$. As a rough indication of computational effort, our Julia implementation of the Lanczos algorithm took ~ 30 min and ~ 60 GiB RAM on a single CPU core to compute $n = 40$ Lanczos coefficients for the MFIM, while our TEBD simulation performed using TeNPy [140] took ~ 30 h on 8 cores using ~ 2 GiB RAM. However, the TEBD runtime is highly parameter dependent, and could be reduced by using a larger dt or a smaller $\chi_{\max}$. At any rate, the initial $\omega = 0$ step of the spectral bootstrap corresponds to the $\rho = 0$ case of the procedure for extracting diffusion constants described in Sec. VIII B, and in Fig. 8(a) we already demonstrated quantitative accuracy in the $n \rightarrow \infty$ limit.

More noticeable differences between the spectral bootstrap and the Fourier transformed correlator can be found at high frequencies, as shown in the left inset of Fig. 9(a). What is striking about the spectral bootstrap data is that we are able to resolve the log-correction due to geometric constraints on operator growth in 1D [11]: Lanczos coefficients growing like $b_n \sim n/\log n$ correspond to a spectral function decaying like $\Phi(\omega \rightarrow \infty) \sim \exp[-\mathcal{O}(\omega \log \omega)]$ [cf. Eq. (49)], rather than the generic exponential bound $\Phi(\omega \rightarrow \infty) \sim \exp[-\mathcal{O}(\omega)]$ expected in higher dimensions. Further evidence for this is shown in Fig. 9(b), where we find that $\log[\Phi(\omega)]/\omega \log \omega$ plateaus at large frequencies, while without the log correction $\log[\Phi(\omega)]/\omega$ continues to decay. The estimates appear to be well converged with n provided we restrict to $|\omega| < \beta_n$ (which is why the smaller n curves stop at smaller ω). Convincingly resolving this log correction had previously required going to much larger n using a Monte Carlo algorithm that exploits the lack of a sign problem for operator growth in the MFIM [36,141].

Getting the correct high-frequency tail from the Fourier transformed real-time correlation function can be challenging due to spectral leakage arising from cutting off the Fourier transform integral at a finite $t_{\max}$. Indeed, we can see in the left inset to Fig. 9(a) that the spectral function from the Fourier transform does not have the expected quasi-exponential high-frequency decay. The standard fix to this problem of spectral leakage is to multiply the real-time correlator by a windowing function before taking the Fourier transform [142]. We also show in Fig. 9(a) the results from multiplying by a simple finite cosine window $\frac{1}{2}[1 + \cos(\pi t/t_{\max})]$. This extends the maximum frequency at which the Fourier transform data agrees with the high frequency tail of the spectral bootstrap, but still eventually leads to an unphysical plateau at very high frequencies. This high frequency tail could likely be improved further by both reducing dt and increasing $t_{\max}$, at the cost of increased computational resources. Figure 9(b) shows that the high-frequency tail of the spectral bootstrap is

converged with n at least up to $\omega \approx 40$, so this Lanczos approach appears to be particularly well suited to reliably extracting high-frequency information.

To further verify the accuracy of the spectral function approximation $\Phi_{\text{est}}(\omega)$ obtained using the spectral bootstrap, we attempt to reconstruct the exact sequence of Lanczos coefficients by carrying out the three-term recurrence relation Eq. (29) for the orthogonal polynomials, but computing the norm

$$b_n^2 = \int_{\mathbb{R}} [\omega p_{n-1}(\omega) - b_{n-1} p_{n-2}(\omega)]^2 \frac{\Phi(\omega)}{2\pi} d\omega \quad (103)$$

by numerically evaluating the integral with $\Phi(\omega)$ approximated by $\Phi_{\text{est}}(\omega)$. We can only compute this integral up to the maximum frequency $\omega_{\text{max}} = 0.99\beta_n$ for which we performed the spectral bootstrap, but this is sufficient to capture almost all of the norm contributing to b_n , for reasons discussed in Sec. V. The result is shown in Fig. 9(c), where we find very close agreement between the exact and the reconstructed Lanczos sequence, thus providing strong evidence that the spectral bootstrap yields a good approximation to the true spectral function [see Fig. 11(c) for more examples]. As a final nontrivial check, the right inset to Fig. 9(a) shows that the sum rule $\int_{\mathbb{R}} \Phi(\omega) d\omega / 2\pi = (A|A)$ is verified very accurately. Using the spectral function extracted in the range $\omega \in [-0.99\beta_n, 0.99\beta_n]$, we find that the sum rule is satisfied up to a relative error of $\sim 10^{-6}$.

Before proceeding, we remark that it is also possible to start this bulk bootstrap at some nonzero frequency ω_0 as follows. Use the semicircle approximation [Eq. (64)] to the level- n Green's function $G_n(\omega_0 \pm i0^+)$ to approximate the spectral function $\Phi(\omega_0)$, as described at the start of Sec. IX. Then numerically solve Eq. (100) to determine

$I_n(\omega_0) \bmod 1$ and use Eq. (102) to compute $\sigma_n(\omega_0)$. Then one can proceed with the main routine of the bulk spectral bootstrap. This ability to start away from the special points $\omega = 0, \pm\beta_n$ is a particular feature of the bulk bootstrap, owing to the fact that the bulk Green's function in Eq. (64) contains only a single free parameter, β_n , which can be directly estimated from the Lanczos coefficients.

B. Generalization to arbitrary ρ : the Bessel bootstrap

The algorithm outlined in the previous section made use of the $n \rightarrow \infty$ asymptotics of the orthogonal polynomials. When $\rho \neq 0$ these asymptotics are modified, particularly near $\omega = 0$, the location of the power law. Once we have the correct asymptotics, however, then the differential equation can be solved iteratively as before. We will aim to capture the “envelope function” $\Phi(\omega)/|\omega|^\rho \equiv 2\pi \exp[-Q(\omega)]$, since this is what is needed to determine the values of hydrodynamic transport coefficients.

For the initial step at $\omega = 0$, we use the result of Theorem 3, which we restate here, together with a result relating this to the equilibrium density at the origin:

$$e^{-Q(0)} \approx \frac{2(1+\rho)^\rho}{\Gamma[\frac{1}{2}(1+\rho)]^2} \frac{[K_n(0,0)]^\rho}{[\beta_n(p_{n-1}(0)^2 + p_n(0)^2)]^{1+\rho}}, \quad (104)$$

$$\sigma_n(0) \approx \frac{2}{\pi} \left(\Gamma\left[\frac{1}{2}(1+\rho)\right] \Gamma\left[\frac{1}{2}(3+\rho)\right] e^{-Q(0)} K_n(0,0) \right)^{\frac{1}{1+\rho}}. \quad (105)$$

These are the $\omega \rightarrow 0^+$ limits of the following equations which we will use for $\omega > 0$. For the envelope function we have

$$e^{-Q(\omega)} \approx \frac{1}{p_{n-1}(\omega)^2 + p_n(\omega)^2} \frac{1}{\sqrt{\beta_n^2 - \omega^2}} \frac{\pi I_n(\omega)}{\omega^\rho} \left[\left(J_{\frac{1}{2}(\rho-1)}^2 + J_{\frac{1}{2}(\rho+1)}^2 \right) (\pi I_n(\omega)) - (-1)^n \frac{\omega}{\beta_n} \left(2J_{\frac{1}{2}(\rho-1)} J_{\frac{1}{2}(\rho+1)} \right) (\pi I_n(\omega)) \cos \left\{ \rho \arcsin \left(\frac{\omega}{\beta_n} \right) \right\} \right], \quad (106)$$

where $I_n(\omega)$ is defined as before in Eq. (97), J_α is a Bessel function of the first kind, and we are using the shorthand $(J_\alpha^2 + J_\beta^2)(x) \equiv J_\alpha^2(x) + J_\beta^2(x)$, $(J_\alpha J_\beta)(x) \equiv J_\alpha(x) J_\beta(x)$, etc. For the equilibrium density we have

$$\sigma_n(\omega) \approx \frac{4 K_n(\omega, \omega) \omega^\rho e^{-Q(\omega)}}{\pi I_n(\omega)} \frac{1}{[J_{\frac{1}{2}(\rho-1)}^2 + J_{\frac{1}{2}(\rho+1)}^2 - J_{\frac{1}{2}(\rho-3)} J_{\frac{1}{2}(\rho+1)} - J_{\frac{1}{2}(\rho-1)} J_{\frac{1}{2}(\rho+3)}] (\pi I_n(\omega))}. \quad (107)$$

We derive these expressions in Sec. S4 B. One can check that they reduce to Eqs. (100) and (102) upon sending $\rho \rightarrow 0$. Unlike for $\rho = 0$, where the asymptotics were valid out to $\omega = \mathcal{O}(\beta_n)$, these asymptotics are, in principle, valid

only for ω within some $\mathcal{O}(1)$ window around $\omega = 0$, essentially the “hydrodynamic” frequency regime where we expect the power law to dominate the behavior of the spectral function. Note that, taking $\omega = \mathcal{O}(1)$, Eqs. (106)

and (107) have been slightly simplified by dropping some terms which are subleading as $n \rightarrow \infty$; see Eqs. (S4.22) and (S4.23) for the full expressions. In practice, we have found that these simplified expressions give numerical results very close to those of the full expressions, provided n is large enough that $\omega/\beta_n \ll 1$.

The appearance of the Bessel functions in these equations is a signature of Bessel universality, akin to that found in random matrix ensembles whose probability measures behave as a power-law near the origin [67,143]. For larger ω , one could then switch to the asymptotics in the bulk, which should then be valid up until $|\omega| \lesssim \beta_n$ like those stated for $\rho = 0$, but in practice we have found that just using the Bessel asymptotics gives good results for the frequency regime where the spectral function is non-negligible.

With these asymptotics in hand, we now have another first-order differential equation which we can iteratively solve as follows to obtain a finite n approximation to the spectral function. We initialize by setting $\omega = 0$ and $I_n(0) = 0$, computing $e^{-Q(0)}$ via Eq. (104) and $\sigma_n(0)$ via Eq. (105), then setting $I_n(\delta\omega) = I_n(0) + \sigma_n(0) \times \delta\omega$ and incrementing $\omega \mapsto \omega + \delta\omega$. Then the main routine is conceptually identical to the procedure outlined for $\rho = 0$:

- (1) Compute $e^{-Q(\omega)}$ using Eq. (106).
- (2) Compute $\sigma_n(\omega)$ using Eq. (107).
- (3) Set $I_n(\omega + \delta\omega) = I_n(\omega) + \sigma_n(\omega) \times \delta\omega$.
- (4) Increment $\omega \mapsto \omega + \delta\omega$.
- (5) Repeat steps 1–4 until $\omega = \omega_{\max}$, then terminate.

To illustrate this generalized spectral bootstrap (SB), we try to reconstruct the envelope functions of some toy spectral functions of the form $\Phi(\omega)/2\pi = |\omega|^\rho \exp[-Q(\omega)]$ for $\rho = -\frac{1}{2}$, again comparing with the recursion method (RM) [26]. Figure 10(a) shows the comparison for $Q(x) = \frac{1}{2}(x^6 - 5x^4 + 20x^2 + 1)^{1/3}$. This function has no special significance other than the fact that $Q(x) \sim \mathcal{O}(x^2)$, so we have $b_n \sim \mathcal{O}(\sqrt{n})$. Here there is an exactly solvable model available for the recursion method which has the relevant features of Gaussian decay and a zero frequency power law [26]. The model spectral function is $\tilde{\Phi}(\omega) = \{[2\pi/\omega_0]/\Gamma[\frac{1}{2}(1+\rho)]\} |\omega/\omega_0|^\rho \exp[-(\omega/\omega_0)^2]$, where ω_0 and ρ are tunable parameters. Crucially, its Lanczos coefficients are known to be $b_{2k-1} = \omega_0 \sqrt{(2k-1+\rho)/2}$ and $b_{2k} = \omega_0 \sqrt{2k/2}$, and its Green's function is $\tilde{G}(z) = (iz/\omega_0^2) \exp[-(z/\omega_0)^2] E_{\frac{1}{2}(1+\rho)}[-(z/\omega_0)^2]$, where $E_a(x)$ is the generalized exponential integral. We recover the envelope function from the RM using $e^{-Q(\omega)} = |\omega|^{-\rho} \Phi(\omega)/2\pi$ with $\Phi(\omega) = 2 \text{Im} G(\omega - i\epsilon)$, $\epsilon = 10^{-14}$. In this case, the SB and the RM with $\rho = -1/2$ give essentially identical results. We also show the result of $e^{-Q(\omega)} = |\omega|^{1/2} \Phi(\omega)/2\pi$ where $\Phi(\omega)$ is computed using the SB with $\rho = 0$, showing that one can get large errors near $\omega = 0$ if one does not take into account the power law, although at larger ω the agreement does become better.

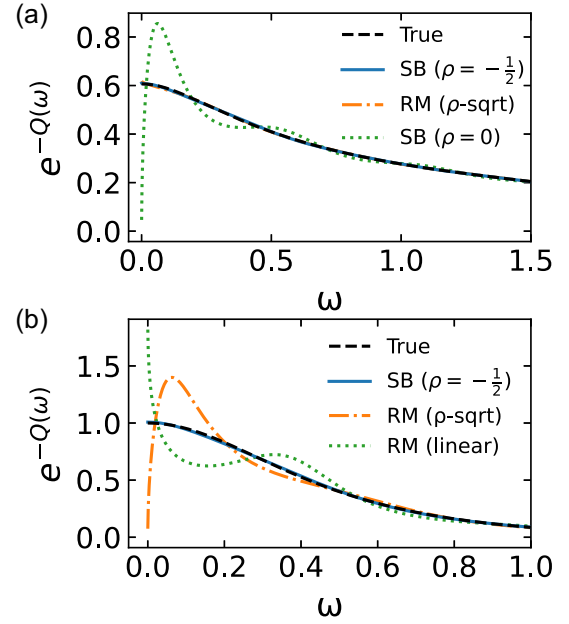


FIG. 10. Reconstructing the spectral envelope function $\exp[-Q(\omega)]$ for a toy spectral function $\Phi(\omega)/2\pi = |\omega|^{-1/2} \exp[-Q(\omega)]$, using the spectral bootstrap (SB) and the recursion method (RM). (a) $Q(\omega) = \frac{1}{2}(\omega^6 - 5\omega^4 + 20\omega^2 + 1)^{1/3}$, which has $b_n \sim \mathcal{O}(\sqrt{n})$. Here there is an appropriate exact “stitching function” for the RM (see main text), so the RM and the $\rho = -1/2$ SB work equally well. The $\rho = 0$ SB curve illustrates the errors one can make by not accounting for the power law. (b) Same as (a), except now $Q(\omega) = -\log \text{sech} \pi \omega$, which has $b_n \sim \mathcal{O}(n)$. Only the $\rho = -1/2$ SB accurately recovers the spectral function. Now there is no available stitching function with the appropriate power-law behavior, so the RM performs poorly. For the RM one has exact solutions with the right low frequency power law but the wrong high frequency tail ($\rho = \text{sqr t}$), or no low frequency power law but the right high frequency tail (linear), neither of which gives the correct result at low frequencies.

This first example showed that, when we have available an exactly solvable spectral function that captures the relevant features of the high-frequency decay and the low-frequency power law, then the recursion method can work very well. One problem with the recursion method, though, is that the existence of these special solutions is not guaranteed for all problems of interest (see Ref. [26] for a review). In order for the recursion method to work well, one needs to have an exact model solution for all three of the spectral functions, the Green's function, and the Lanczos coefficients. Unfortunately, while there is a power-law solution available with Gaussian high-frequency decay, to our knowledge there is no such solution available which has both a low-frequency power law and *exponential* decay. This is potentially a problem because the operator growth hypothesis posits that such exponential decay is generic for chaotic quantum systems [11]. Figure 10(b) shows that when $\rho \neq 0$ it is important to get both of these ingredients

correct to accurately capture the $\omega \rightarrow 0$ behavior of the spectral function. We consider the toy spectral function $\Phi(\omega)/2\pi = |\omega|^{-1/2} \text{sech}(\pi\omega)$, which decays exponentially as $\omega \rightarrow \infty$. The $\rho = -\frac{1}{2}$ spectral bootstrap captures the true spectral function very well. However, since there is now no available exact solution, the RM performs poorly. One could try using the Meixner-Pollaczek solution from Ref. [11] to capture the high-frequency tail while ignoring the low frequency power-law, which is labeled “RM (linear).” Or one could try using the exact solution from the previous example to get the low frequency power-law correct, at the cost of getting the high-frequency tail wrong, which is labeled “RM (ρ -sqrt).” However, both of these approaches result in large errors at low frequencies, and so only the $\rho = -\frac{1}{2}$ spectral bootstrap correctly captures the true low frequency behavior of the spectral function. This highlights one of the advantages of the spectral bootstrap over the recursion method: it does not require any exact solutions, and so can be equally employed for a wide range of growth behaviors of the Lanczos coefficients.

From the discussion in Sec. VI, we know that for $\rho = 0$ there is a great deal of freedom in choosing the stitching function for the recursion method, and in particular it is *not* necessary for the stitching function to have a high frequency decay matching that of the true spectral function; one only needs that the level- n Green’s function satisfies $G_n(\omega \pm i0^+) \approx \mp i/b_n$ as $\omega \rightarrow 0$ and $n \rightarrow \infty$, which is much more generic. One then might wonder why for $\rho \neq 0$ we no longer have such freedom, since it appears to be necessary to also match the high frequency behavior (the relevance of matching the low frequency behavior is perhaps more intuitive). This is because the nontrivial low-frequency behavior of $\Phi(\omega \rightarrow 0) \sim |\omega|^\rho$ couples to the equilibrium density $\sigma_n(0)$ when $\rho \neq 0$ [cf. Theorem 1 and Eq. (75)], and $\sigma_n(0)$ depends nontrivially on the high frequency decay of the spectral function (cf. Lemma 2), being sensitive to a Coulomb gas confinement transition. This coupling is seen most explicitly in Eq. (67), where we show that the level $2n$ Green’s function $G_{2n}(\omega \pm i0^+)$ is primarily a function of $\pi\sigma_{2n}(0)\omega$ as $\omega \rightarrow 0$ and $n \rightarrow \infty$.

X. RANDOM MATRIX UNIVERSALITY

While we developed the spectral bootstrap with the practical aim of recovering the spectral function $\Phi(\omega)$ from its Lanczos coefficients, as a by-product we also get an

estimate of the equilibrium measure $\sigma_n(\omega)$. We can use this to perform various self-consistency checks. In particular, while we cannot usually verify that a given many-body spectral function obeys all the assumptions under which we derive our results, we can at least check that our estimates of $\Phi(\omega)$ and $\sigma_n(\omega)$ are consistent with the behavior expected given those assumptions. One notable example is the emergence of “universality” akin to that governing eigenvalue correlations in random matrix theory [60]. Just as universality appears in the $n \rightarrow \infty$ of $n \times n$ random matrices, here it should appear in the limit $n \rightarrow \infty$ of large Lanczos index.

From a physical point of view, this RMT-like universality is interesting because it seems to be “superuniversal,” in the sense that it can appear not only for chaotic models, which might be expected from studies of eigenstate thermalization [53], but also in integrable and non-interacting models, provided they have sufficiently smooth spectral functions in the thermodynamic limit. What changes between these classes of operator dynamics is the n dependence of the frequency scale on which this RMT universality appears, due to the different scaling of b_n .

We expect this section to be of most interest to readers already familiar with universality of eigenvalue correlations in random matrix theory. Nonetheless, we first provide the necessary background before illustrating our results.

A. Background

Given a spectral function $\Phi(\omega)$, we can define a unitarily-invariant ensemble of $n \times n$ Hermitian random matrices whose eigenvalues are distributed according to

$$P(\lambda) d^n \lambda = \frac{1}{Z_n} \left(\prod_{i < j} |\lambda_i - \lambda_j|^2 \right) \prod_i \frac{\Phi(\lambda_i)}{2\pi} d\lambda_i, \quad (108)$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$ with the ordering $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$, and Z_n is a normalization constant. The bracketed prefactor is the Vandermonde determinant which gives rise to level repulsion. We consider this random matrix ensemble because its eigenvalue distribution will be governed by the same Coulomb gas as that controlling the Lanczos operators via the orthogonal polynomials $p_n(\omega)$.

For any $1 \leq m \leq n-1$, let

$$R_m(\lambda_1, \dots, \lambda_m) = \frac{n!}{(n-m)!} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} P(\lambda_1, \dots, \lambda_m, \lambda_{m+1}, \dots, \lambda_n) d\lambda_{m+1} \dots d\lambda_n \quad (109)$$

denote the m -point eigenvalue correlation function. A remarkable property of these random matrix ensembles is that all of the correlation functions can be written in terms of a determinant of a 2-point correlation kernel [70],

$$R_m(\lambda_1, \dots, \lambda_m) = \det \left[(\hat{K}_n(\lambda_i, \lambda_j))_{1 \leq i, j \leq m} \right]. \quad (110)$$

The correlation kernel $\hat{K}_n$ is a weighted version of the Christoffel-Darboux kernel K_n defined in Eq. (82),

$$\hat{K}_n(\omega_a, \omega_b) = \sqrt{w(\omega_a)w(\omega_b)} K_n(\omega_a, \omega_b), \quad (111)$$

where $w(\omega) = \Phi(\omega)/2\pi$ is the weight function defining the random matrix ensemble. In the Lanczos language, from Eq. (30) we can see that $\hat{K}_n(\omega_a, \omega_b)$ is the integral kernel (in frequency space) of the projection superoperator $\mathcal{P}_n$ on to the first n Lanczos operators [65]

$$\mathcal{P}_n = \sum_{m=0}^{n-1} |O_m\rangle\langle O_m|, \quad (112)$$

which is the complement of the projector $\mathcal{Q}_n = \mathbb{1} - \mathcal{P}_n$ used to construct the level- n Green's function $G_n(z)$ (Sec. VI). By studying $\hat{K}_n(\omega_a, \omega_b)$, we can characterize this operator projection in frequency space.

A further remarkable property of random matrix ensembles is that for large- n their eigenvalue correlations often display *universality* [60]. This manifests in $\hat{K}_n(\omega_a, \omega_b)$ approaching a universal form in different sections of the spectrum when probed on the appropriate scale, independent of the precise form of the weight w . For example, for ω in the bulk of the spectrum, $|\omega| \ll \beta_n$, we get

$$\frac{1}{\sigma_n(\omega)} \hat{K}_n\left(\omega + \frac{u}{\sigma_n(\omega)}, \omega + \frac{v}{\sigma_n(\omega)}\right) \xrightarrow{n \rightarrow \infty} \mathbb{S}(u, v), \quad (113)$$

for $u, v \in \mathbb{R}$, so for large- n the local eigenvalue correlations are described by the *sine kernel*

$$\mathbb{S}(u, v) = \frac{\sin[\pi(u - v)]}{\pi(u - v)}. \quad (114)$$

Note that the rhs of Eq. (113) is translation invariant, provided ω is within the bulk of the spectrum.

If the weight function has a power law at the origin, $w(x) = |x|^\rho e^{-\mathcal{Q}(x)}$, then this affects the local eigenvalue correlations near the origin. Rather than the sine kernel, at $\omega = 0$ we instead have *Bessel universality*:

$$\frac{1}{\sigma_n(0)} \hat{K}_n\left(\frac{u}{\sigma_n(0)}, \frac{v}{\sigma_n(0)}\right) \xrightarrow{n \rightarrow \infty} e^{-\frac{i\rho}{2}(\arg u + \arg v)} \mathbb{J}_{\rho/2}(u, v), \quad (115)$$

where $\mathbb{J}_{\rho/2}$ is the Bessel kernel given by

$$\mathbb{J}_{\rho/2}(u, v) = \pi \sqrt{u} \sqrt{v} \frac{J_{\frac{\rho+1}{2}}(\pi u) J_{\frac{\rho-1}{2}}(\pi v) - J_{\frac{\rho-1}{2}}(\pi u) J_{\frac{\rho+1}{2}}(\pi v)}{2(u - v)} \quad (116)$$

with J_α a Bessel function of the first kind.

Finally, for eigenvalues near the edge of the spectrum, $\omega \approx \beta_n$, the bulk sine universality gives way to *Airy universality*, such that

$$\frac{\beta_n}{c_n n^{2/3}} \hat{K}_n\left(\beta_n + \frac{\beta_n}{c_n n^{2/3}} u, \beta_n + \frac{\beta_n}{c_n n^{2/3}} v\right) \xrightarrow{n \rightarrow \infty} \mathbb{A}(u, v), \quad (117)$$

where $c_n := (h_n(1)/\sqrt{2})^{2/3}$ is $\mathcal{O}(1)$ [note $c_n n^{2/3} = f'_n(1)$ by Eq. (S6.2)], and the Airy kernel $\mathbb{A}$ is given by

$$\mathbb{A}(u, v) = \frac{\text{Ai}(u)\text{Ai}'(v) - \text{Ai}(v)\text{Ai}'(u)}{u - v}, \quad (118)$$

where Ai is the Airy function of the first kind.

Theorem 4. For our class of spectral functions (see Sec. IV for a definition), the corresponding correlation kernels $\hat{K}_n$ satisfy the universal scalings Eqs. (113), (115) and (117) in the relevant sections of the spectrum.

This theorem follows as a direct consequence of the $n \rightarrow \infty$ asymptotics we established for the orthogonal polynomials, such as Eqs. (98) and (99) in the bulk of the spectrum. See Sec. S4 for the full asymptotic expressions [68].

B. Universality in quantum operator dynamics

While we can prove the emergence of universality for spectral functions obeying our assumptions, it would be gratifying to see this also emerge in more familiar many-body quantum models, even though we cannot explicitly verify that our assumptions hold. Indeed, universality is a very generic phenomenon [69,144,145], particularly in the bulk, and it may well hold even if some of our assumptions are violated (such as our analyticity requirements). Using the spectral bootstrap algorithm described in Sec. IX, we can estimate both the weight function $w(\omega) = \Phi(\omega)/2\pi$ and the equilibrium measure $\sigma_n(\omega)$ for a given many-body model, using only a finite number of its Lanczos coefficients. We can then perform a self-consistent check that these approximations reproduce the universal scalings Eqs. (113), (115) and (117).

The evidence we will present for universality in many-body models comes with a qualification. To test for universality requires explicitly knowing the spectral function $\Phi(\omega)$ and the equilibrium measure $\sigma_n(\omega)$, but for most interacting many-body models this is not analytically tractable. For models where we do not know $\Phi(\omega)$, we use our spectral bootstrap algorithm to first obtain a finite- n approximation to $\Phi(\omega)$ and $\sigma_n(\omega)$. The qualification is that this algorithm *assumes* the validity of Plancherel-Rotach asymptotics for the orthogonal polynomials [e.g., Eqs. (99) and (98)], from which universality can be derived as a consequence [62]. (But we do prove these asymptotics hold under reasonable assumptions on the spectral function.) To avoid circularity, we perform independent cross-checks of

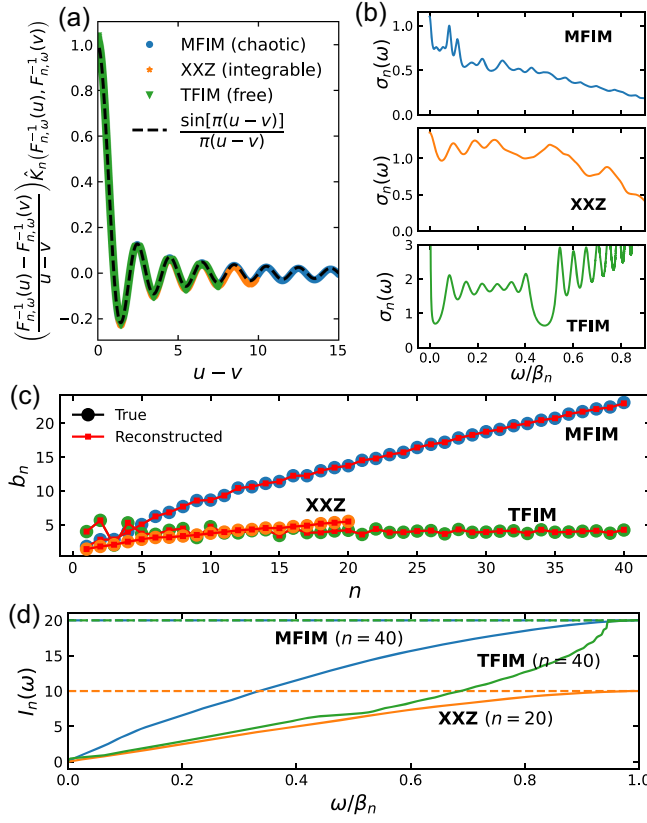


FIG. 11. (a) Sine kernel universality [cf. Eqs. (120) and (114)] in the bulk spectra of the chaotic MFIM, the interacting integrable XXZ chain, and the noninteracting TFIM. There are no free fitting parameters; everything is extracted from the Lanczos coefficients using the spectral bootstrap (Sec. IX A). (b) Equilibrium densities $\sigma_n(\omega)$ estimated using the spectral bootstrap algorithm. Note the significant variation of $\sigma_n(\omega)$ with ω , which necessitates the unfolding procedure. (c) Reconstructed Lanczos coefficients b_n , calculated using the estimate of the spectral function obtained using the spectral bootstrap, and compared with the exact Lanczos coefficients. (d) Cumulative integrals $I_n(\omega) = \int_0^\omega \sigma_n(\omega') d\omega'$ of the equilibrium measures from panel (b), showing that they integrate to $I_n(\omega = \beta_n) = n/2$, as expected from Eqs. (46) and (47).

these estimates of $\Phi(\omega)$ and $\sigma_n(\omega)$, such as the ability to reconstruct the numerically exact Lanczos coefficients [Fig. 11(c)]. If these checks are passed convincingly, which turns out to be the case for the physical models we consider, then this gives strong evidence that we have obtained accurate estimates of the true spectral function and equilibrium measure. With those in hand, it is then reasonable to test for universality.

Before proceeding, we make a practical consideration. The statements of universality in Eqs. (113) and (115)–(117) are made under the approximation that the equilibrium density $\sigma_n(\omega)$ is locally constant. This approximation becomes increasingly well justified as $n \rightarrow \infty$, but for the finite n we have available to us numerically, this approximation is not necessarily so good, particularly near

the origin for exponentially decaying spectral functions which are at the Coulomb gas confinement transition [see Fig. 11(b) for some physical examples]. We will adopt a practice that is standard in the random matrix theory literature, namely, “unfolding” the spectrum [146], which involves transforming to new coordinates that account for variation of the local density. Fixing a frequency ω at which we want to test for universality, we define

$$F_{n,\omega}(x) := \int_\omega^x \sigma_n(s) ds, \quad (119)$$

$$= I_n(x) - I_n(\omega),$$

where $I_n(\omega) = \int_0^\omega \sigma_n(s) ds$ was defined in Eq. (97). Now note that, if we approximate $\sigma_n(s) \approx \sigma_n(\omega)$ as constant in Eq. (119), then each of the arguments of the kernel $\hat{K}_n$ in Eq. (113) can be expressed in terms of the inverse function $F_{n,\omega}^{-1}(u) \approx \omega + u/\sigma_n(\omega)$. Then the statement Eq. (113) of bulk universality at frequency ω can be rewritten as

$$\left(\frac{F_{n,\omega}^{-1}(u) - F_{n,\omega}^{-1}(v)}{u - v} \right) \hat{K}_n(F_{n,\omega}^{-1}(u), F_{n,\omega}^{-1}(v)) \xrightarrow{n \rightarrow \infty} \mathbb{S}(u, v). \quad (120)$$

The point is that, even in the situation where the equilibrium density $\sigma_n(\omega)$ is not constant and $F_{n,\omega}$ must be inverted numerically, this formulation of bulk universality will account for that variation, and so one can still expect to observe the sine kernel. [The inverse $F_{n,\omega}^{-1}$ is well defined because $F_{n,\omega}$ is the integral of a positive function $\sigma_n(\omega) > 0$, so is increasing.] The statement Eq. (115) of Bessel universality can be expressed in the same way by setting $\omega = 0$, with only the rhs changing to the Bessel kernel instead of the sine kernel. The statement Eq. (117) of Airy universality around $\omega = \beta_n$ can also be made with the same lhs, but with $F_{n,\beta_n}(x)$ redefined as

$$(\text{Airy}) \quad F_{n,\beta_n}(x) := f_n(x/\beta_n), \quad (121)$$

where f_n is defined in Eq. (S6.1).

1. Sine universality in the bulk

First we check for sine universality in the bulk of the spectrum, taking as examples the chaotic mixed field Ising model, the interacting integrable XXZ chain, and the noninteracting transverse field Ising model (TFIM). We take the initial Lanczos operator to be the total energy current for the MFIM, the total spin current for the $\Delta = 2$ XXZ model, and $(1/\sqrt{L}) \sum_x Y_x Y_{x+1}$ for the TFIM (with XX interactions), and compute $n = 40, 20, 40$ Lanczos coefficients respectively. Since in each case we expect the autocorrelation function $C(t)$ to decay faster than $1/t$, such that $0 < |\Phi(\omega = 0)| < \infty$, we set $\rho = 0$ and start testing

for bulk universality at $\omega = 0$. Note that, for $\rho = 0$, our assumptions on the spectral function (Sec. IV) required for our Riemann-Hilbert analysis amount to assuming analyticity of $\Phi(\omega)$ at $\omega = 0$, which does not explicitly account for possible effects of power-law decay of $C(t)$ faster than $1/t$ on the analytic structure of $\Phi(\omega)$. The test of bulk universality we are about to perform thereby gives us a probe to see whether this non-analyticity affects the universality class near $\omega = 0$.

We estimate the weight function $w(\omega) = \Phi(\omega)/2\pi$ and the equilibrium density $\sigma_n(\omega)$ throughout the bulk frequency spectrum by carrying out the “bulk bootstrap” algorithm, described in Sec. IX, in the frequency range $\omega \in [0, 0.99\beta_n]$, with $\beta_{n=40} \approx 46.1$ for MFIM, $\beta_{n=20} \approx 11.2$ for XXZ, and $\beta_{n=40} \approx 8.5$ for TFIM. As a check of the accuracy of our estimate of $\Phi(\omega)$, like in Fig. 9(c) we again use it to reconstruct the Lanczos coefficients for these models by evaluating the three-term recurrence Eq. (29), with the results shown in Fig. 11(c), where we find very good agreement. For the MFIM energy current, in Fig. 9(a) we also found good agreement with the spectral function obtained by Fourier transforming the real-time autocorrelation function obtained by TEBD, providing another independent accuracy test. To check the accuracy of the equilibrium measure $\sigma_n(\omega)$, as shown in Fig. 11(b), we compute the cumulative integral $I_n(\omega) = \int_0^\omega \sigma_n(\omega') d\omega'$. By the definitions Eqs. (46) and (47) and the even symmetry $\sigma_n(-\omega) = \sigma_n(\omega)$, we should have the sum rule $I_n(\omega = \beta_n) = n/2$. As shown in Fig. 11(d), we find this is well satisfied in all three models. This sum rule is not enforced by the spectral bootstrap, so this is a nontrivial check of the accuracy of our estimates of $\sigma_n(\omega)$.

With this assurance, we then use these estimates of $\Phi(\omega)$ and $\sigma_n(\omega)$ to numerically compute the inverse function $F_{n,\omega}^{-1}$, and then evaluate the lhs of Eq. (120), with the kernel $K_n(x, y) = \sum_{m=0}^{n-1} p_m(x)p_m(y)$ computed from the exact recurrence coefficients $\{b_n\}$ using the three-term recurrence Eq. (29). The results are shown in Fig. 11(a), where we see that all three models collapse on to the sine kernel $\mathbb{S}(u, v) = \sin[\pi(u - v)]/\pi(u - v)$, indicating the emergence of sine universality in the bulk. We emphasize that there are no free fitting parameters—everything is computed from the Lanczos coefficients—so getting such close agreement is nontrivial. Note that the curve continues for larger separations $u - v$ for the MFIM than the XXZ model and then the TFIM simply because the frequency bandwidth β_n is larger, allowing us to probe larger separations before hitting the edge of the spectrum. This verification of bulk universality can be seen as an *a posteriori* explanation of the effectiveness of our spectral bootstrap algorithm to estimate the spectral functions of these physical models.

2. Bessel universality at the origin

Next we check for Bessel universality near $\omega = 0$, which is expected when the spectral function has a power-law at

the origin, $\Phi(\omega \rightarrow 0) \sim |\omega|^\rho$. We will consider the energy density $h_0 = \frac{1}{2}(Z_{-1}Z_0 + Z_0Z_1) + g_x X_0 + g_z Z_0$ at site zero of the mixed field Ising model, which should have $\Phi(\omega \rightarrow 0) \sim |\omega|^{-1/2}$ (i.e., $\rho = -\frac{1}{2}$) due to energy diffusion [3]. We compute $n = 40$ Lanczos coefficients for this model, and then use the Bessel bootstrap described in Sec. IX B to extract the spectral function and the equilibrium measure. After checking again that these estimates successfully reproduce the true Lanczos coefficients (not shown), we then evaluate the lhs of Eq. (120) for $v = -u$ and using $F_{n,\omega=0}$ [note the rhs of Eq. (120) should now be replaced by $e^{-i\pi\rho/2} \mathbb{J}_{\rho/2}(u, -u)$]. For clarity, we opt to plot as a function of the physical frequency ω , with $u = F_{n,0}(\omega)$. Note that, due to the even symmetry of the spectral function, we have $v = -u = F_{n,0}(-\omega)$. To account for the $\omega \rightarrow 0$ divergent power-law of both the spectral function and the $\rho = -\frac{1}{2}$ Bessel kernel, we rescale both sides by $|u|^{-\rho} = |F_{n,0}(\omega)|^{-\rho}$ when plotting. The results are shown in Fig. 12(a). We find good agreement with the Bessel kernel for $\mathcal{O}(1)$ frequencies, providing evidence for Bessel universality at $\omega = 0$ for this model. There is gradually increasing disagreement for larger frequencies, but this is not surprising since Bessel universality is only expected to hold in an $\mathcal{O}(1)$ -sized region around the origin [cf. Fig. 1(b) and Sec. IX B].

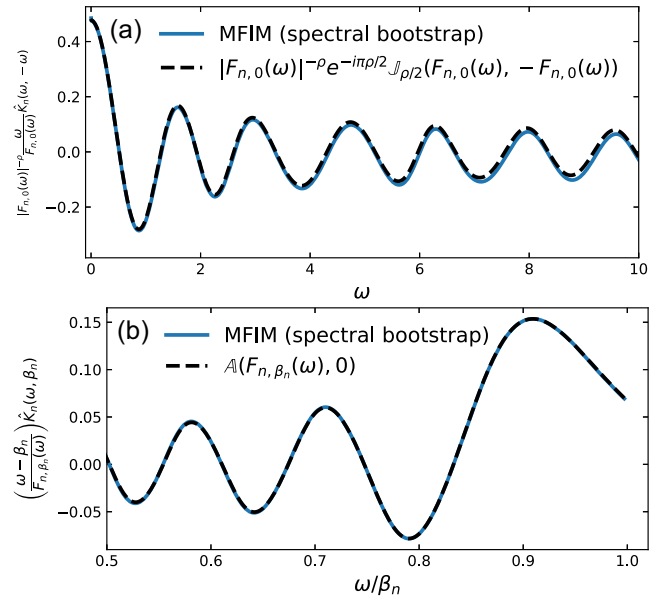


FIG. 12. (a) Bessel universality near $\omega = 0$ in the mixed field Ising model. The initial Lanczos operator is the local energy density; since energy transport is diffusive in this model, the spectral function should scale like $\Phi(\omega \rightarrow 0) \sim |\omega|^\rho$ with $\rho = -\frac{1}{2}$. There is good agreement with the corresponding Bessel kernel $\mathbb{J}_{\rho/2}$ [Eq. (116)] in an $\mathcal{O}(1)$ region around the origin. (b) Airy universality near the spectral edge $\omega = \beta_n$ in the MFIM. We compare the results from the spectral bootstrap to the Airy kernel $\mathbb{A}$ [Eq. (118)], finding very close agreement.

3. Airy universality at the edge

Finally we check for Airy universality at the edge of the spectrum, $\omega \approx \beta_n$. We consider again the mixed field Ising model, and take the energy current as our initial Lanczos operator, computing $n = 40$ Lanczos coefficients. We use the Airy bootstrap of Sec. S6 to approximate the spectral function $\Phi(\omega)$ and the equilibrium measure $\sigma_n(\omega)$ for frequencies in a range $\omega \in [\omega_{\min}, \beta_n]$ close to the edge $\omega = \beta_n$. With these estimates, we evaluate the lhs of Eq. (120), with the rhs now replaced by the Airy kernel $\mathbb{A}(u, v)$ [Eq. (118)]. Note that, for Airy universality, the unfolding map F_n must now be defined according to Eq. (121). Since our spectral bootstrap algorithm estimates the weight only up to the spectral edge $\omega = \beta_n$, we will fix $v = 0$, where $F_{n, \omega=\beta_n}^{-1}(v=0) = \beta_n$, and take the other argument u to be negative. Again for clarity we plot in terms of the physical frequency ω , so negative $u = F_{n, \beta_n}(\omega)$ corresponds to frequencies below $\omega = \beta_n$. The results are shown in Fig. 12(b). We find very good agreement with the Airy kernel, indicating the emergence of Airy universality near the edge of the spectrum in the mixed field Ising model.

C. Scaling of the equilibrium measure

Since our class of spectral functions behaves like the Freud weights $w^{(p)}(x) = \exp(-\kappa_p |x|^p)$ at large frequency scales [cf. Eq. (53)], it is also interesting to compare our estimates of the rescaled equilibrium measure $\psi_n(x) = (\beta_n/n)\sigma_n(\beta_n x)$ with the corresponding function $\psi^{(p)}(x)$ for the Freud weights, which is given by the Ullman distribution defined in Eq. (54) [115]. For our physical test systems, we will use the same models as in Sec. X B 1—the mixed field Ising model, the XXZ chain, and the transverse field Ising model—take the same initial Lanczos operators, and extract the equilibrium measure in the same way. Since these models have Lanczos coefficients growing like $b_n \sim \mathcal{O}(n/\log n)$, $b_n \sim \mathcal{O}(n^{1/2})$, and $b_n \sim \mathcal{O}(1)$ respectively, we will compare to the following cases of the Ullman distribution:

$$\psi^{(p=1)}(x) = \frac{1}{\pi} \operatorname{artanh}(\sqrt{1-x^2}), \quad (122)$$

$$\psi^{(p=2)}(x) = \frac{2}{\pi} \sqrt{1-x^2}, \quad (123)$$

$$\psi^{(p \rightarrow \infty)}(x) = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}. \quad (124)$$

Notice that $\psi^{(p=2)}(0)$ and $\psi^{(p \rightarrow \infty)}(0)$ are finite, while $\psi^{(p=1)}(x)$ has a logarithmic divergence as $x \rightarrow 0$. This is an instance of the Coulomb gas confinement transition discussed in Sec. V, which occurs at $p = 1$. The divergence of $\psi^{(p \rightarrow \infty)}(x)$ as $|x| \rightarrow 1$ is consistent with the scaling $h_n(1) \rightarrow^{n \rightarrow \infty} 2p$ in Lemma 1. For $p \rightarrow \infty$, the rescaled

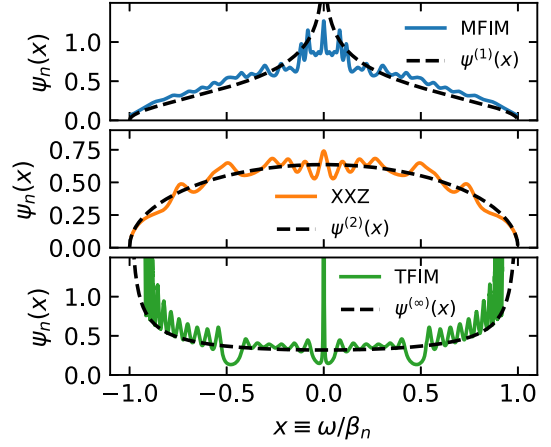


FIG. 13. Rescaled equilibrium measures $\psi_n(x) = (\beta_n/n)\sigma_n(\beta_n x)$ for the mixed field Ising model, the XXZ chain, and the transverse field Ising model, as extracted using the spectral bootstrap. Given that they, respectively, have $b_n \sim \mathcal{O}(n/\log n)$, $b_n \sim \mathcal{O}(\sqrt{n})$, and $b_n \sim \mathcal{O}(1)$, we compare with the equilibrium measures $\psi^{(p)}(x)$ for the Freud weights with equivalent growth rates, as indicated in Eqs. (122) to (124). In each case, $\psi^{(p)}$ captures the qualitative shape of the equilibrium measure for the physical model, but there are significant fluctuations, which have implications for quantitative extraction of the spectral function.

potential $V_n(x) = Q(\beta_n x)/n$ approaches a box potential which is zero for $|x| \leq 1$ and infinite outside, and the logarithmic repulsion between charges causes a build up of charge at the boundaries $x = \pm 1$ of the box.

The comparisons between the rescaled equilibrium measures $\psi_n(x)$ for the physical models and the corresponding Ullman distributions are shown in Fig. 13. Again we note there are no free fitting parameters. In each case the Ullman distribution $\psi^{(p)}(x)$ gives a good qualitative description of the large-scale profile of the equilibrium measure, but the physical equilibrium measures have significant fluctuations on smaller scales. These are a reflection of the fact that the nonuniversal structure of these spectral functions has not yet been washed out by the rescaling by β_n . Given these noticeable fluctuations, if one is aiming for quantitative accuracy in recovering a spectral function from Lanczos coefficients, it is not usually enough to simply approximate the equilibrium measure $\sigma_n(\omega)$ by a rescaled Ullman distribution $(n/\beta_n)\psi^{(p)}(\omega/\beta_n)$. Rather, one has to account for these fluctuations by using something like the spectral bootstrap to more accurately extract the equilibrium measure.

We emphasize that we considered this class of “Freud-like” weights for our proofs because they are sufficiently regular to be amenable to rigorous proofs, but we expect the expressions for the polynomial asymptotics to be more universal, at least to leading order in $n \rightarrow \infty$. Thus we do not think there is a sense in which we have inadvertently assumed Freud-like behavior, so any agreement should be

interpreted as a statement about the physical behavior of these models.

XI. THE ROLE OF SMOOTHNESS: FAILURE OF UNIVERSALITY IN A LOCALIZED MODEL

In order to prove our results about emergent universality in the $n \rightarrow \infty$ limit, we needed to assume that the spectral function $\Phi(\omega)$ was very *smooth* as a function of ω (see assumptions in Sec. IV A). In this section we give some evidence that this is not merely a requirement for our proof technique, but rather has physical relevance. In particular, we will demonstrate how a nonsmooth spectral function can result in a failure of large- n universality. To be concrete, we consider the 1D transverse-field Ising model with quenched disorder, defined by

$$H = \sum_{x \in \mathbb{Z}} (X_x X_{x+1} + h_x Z_x), \quad (125)$$

where the transverse fields $\{h_x\}$ are now independent random variables sampled from a Gaussian distribution $\mathcal{N}(0, \sigma^2)$. In one dimension, for any nonzero variance $\sigma^2 \neq 0$, this model is provably Anderson localized [83]. As such, the spectrum of H is pure pointlike even in the thermodynamic limit [147], as opposed to the continuous spectrum that one usually obtains in the absence of disorder. We fix $\sigma = 4$ for this discussion, and consider a single disorder realization rather than averaging over disorder. We have checked that the following results do not depend significantly on the particular disorder realization.

We check for the effects of a pointlike spectrum by performing the same consistency check that we performed in Figs. 9(c) and 11(c). That is, we *assume* the validity of the universal scaling forms underlying the spectral bootstrap, and use that to produce an estimate $\Phi_{\text{est}}(\omega)$ of the spectral function using the first $n = 50$ Lanczos coefficients. With this estimate, we can then attempt to

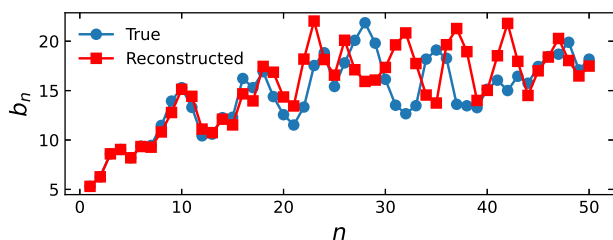


FIG. 14. Reconstructed Lanczos coefficients b_n for the disordered transverse-field Ising model, which is Anderson localized. The reconstructed Lanczos coefficients are calculated using the estimate of the spectral function obtained using the spectral bootstrap, and compared with the exact Lanczos coefficients. The large discrepancies between the exact Lanczos coefficients and the reconstructed coefficients suggests the failure of fast space universality in this disordered model, owing to the lack of smoothness of the spectral function.

reconstruct the Lanczos coefficients solely by performing numerical integration weighted by Φ_{est} , as described in the text surrounding Eq. (103). If universality is valid in this model, then the reconstructed Lanczos coefficients should match the true ones used to estimate Φ_{est} . This reconstruction worked well for the clean version of the TFIM [Fig. 11(c)]. However, as shown in Fig. 14, for the disordered version the reconstruction fails badly, suggesting the breakdown of fast space universality in this model, owing to the lack of a smooth spectrum.

ACKNOWLEDGMENTS

We thank Alexander Avdoshkin, Xiangyu Cao, Pieter Claeys, Fabian Essler, Michael Flynn, Jorge Kurchan, Jean-Bernard Lasserre, Doron Lubinsky, Sheehan Olver, Daniel Parker, and Gabriele Pinna for helpful discussions. Some of the numerical computations were performed using King's College London's CREATE cluster [148]. The data that support the findings of this article are openly available [149]. O. L. was supported by EPSRC through Grant No. EP/Y005058/2. K. M. gratefully acknowledges the support of a Royal Society Wolfson Fellowship (Grant No. RSWVF/R2/212003). C. v K. was supported by a UKRI FLF through MR/T040947/1, MR/T040947/2, and MR/Z000297/1.

DATA AVAILABILITY

The data that support the findings of this article are openly available [149].

- [1] M. B. Hastings, *An area law for one-dimensional quantum systems*, *J. Stat. Mech.* (2007) P08024.
- [2] S. R. White, *Density matrix formulation for quantum renormalization groups*, *Phys. Rev. Lett.* **69**, 2863 (1992).
- [3] H. Kim and D. A. Huse, *Ballistic spreading of entanglement in a diffusive nonintegrable system*, *Phys. Rev. Lett.* **111**, 127205 (2013).
- [4] N. Schuch, M. M. Wolf, F. Verstraete, and J. I. Cirac, *Entropy scaling and simulability by matrix product states*, *Phys. Rev. Lett.* **100**, 030504 (2008).
- [5] T. Rakovszky, C. W. von Keyserlingk, and F. Pollmann, *Dissipation-assisted operator evolution method for capturing hydrodynamic transport*, *Phys. Rev. B* **105**, 075131 (2022).
- [6] C. D. White, M. Zaletel, R. S. K. Mong, and G. Refael, *Quantum dynamics of thermalizing systems*, *Phys. Rev. B* **97**, 035127 (2018).
- [7] T. Klein Kvarning, L. Herviou, and J. H. Bardarson, *Time-evolution of local information: Thermalization dynamics of local observables*, *SciPost Phys.* **13**, 080 (2022).
- [8] C. Artiago, C. Fleckenstein, D. Aceituno Chávez, T. K. Kvarning, and J. H. Bardarson, *Efficient large-scale many-body quantum dynamics via local-information time evolution*, *PRX Quantum* **5**, 020352 (2024).

- [9] C. D. White, *Effective dissipation rate in a Liouvillian-graph picture of high-temperature quantum hydrodynamics*, *Phys. Rev. B* **107**, 094311 (2023).
- [10] S. Yi-Thomas, B. Ware, J. D. Sau, and C. D. White, *Comparing numerical methods for hydrodynamics in a one-dimensional lattice spin model*, *Phys. Rev. B* **110**, 134308 (2024).
- [11] D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, and E. Altman, *A universal operator growth hypothesis*, *Phys. Rev. X* **9**, 041017 (2019).
- [12] T. Prosen and M. Žnidarič, *Matrix product simulations of non-equilibrium steady states of quantum spin chains*, *J. Stat. Mech.* (2009) P02035.
- [13] C. von Keyserlingk, F. Pollmann, and T. Rakovszky, *Operator backflow and the classical simulation of quantum transport*, *Phys. Rev. B* **105**, 245101 (2022).
- [14] N. S. Srivatsa, O. Lunt, T. Rakovszky, and C. von Keyserlingk, *Probing hydrodynamic crossovers with dissipation-assisted operator evolution*, *arXiv:2408.08249*.
- [15] J. Lloyd, T. Rakovszky, F. Pollmann, and C. von Keyserlingk, *Ballistic to diffusive crossover in a weakly interacting Fermi gas*, *Phys. Rev. B* **109**, 205108 (2024).
- [16] A. Nahum, S. Roy, S. Vijay, and T. Zhou, *Real-time correlators in chaotic quantum many-body systems*, *Phys. Rev. B* **106**, 224310 (2022).
- [17] R. Zwanzig, *Memory effects in irreversible thermodynamics*, *Phys. Rev.* **124**, 983 (1961).
- [18] H. Mori, *A continued-fraction representation of the time-correlation functions*, *Prog. Theor. Phys.* **34**, 399 (1965).
- [19] D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (CRC Press, Boca Raton, 2019).
- [20] E. Ilievski, M. Medenjak, and T. Prosen, *Quasilocally conserved operators in the isotropic Heisenberg spin-1/2 chain*, *Phys. Rev. Lett.* **115**, 120601 (2015).
- [21] R. Haydock, V. Heine, and M. J. Kelly, *Electronic structure based on the local atomic environment for tight-binding bands*, *J. Phys. C* **5**, 2845 (1972).
- [22] Y. Saad, *Numerical Methods for Large Eigenvalue Problems* (Manchester University Press, John Wiley and Sons, Halsted Press, New York, 1992).
- [23] E. Rabinovici, A. Sánchez-Garrido, R. Shir, and J. Sonner, *Krylov complexity*, *arXiv:2507.06286*.
- [24] V. Balasubramanian, P. Caputa, J. M. Magan, and Q. Wu, *Quantum chaos and the complexity of spread of states*, *Phys. Rev. D* **106**, 046007 (2022).
- [25] P. Nandy, A. S. Matsoukas-Roubeas, P. Martínez-Azcona, A. Dymarsky, and A. del Campo, *Quantum dynamics in Krylov space: Methods and applications*, *Phys. Rep.* **1125–1128**, 1 (2025).
- [26] V. S. Viswanath and G. Müller, *The Recursion Method: Application to Many-Body Dynamics* (Springer, Berlin Heidelberg, 2013).
- [27] R. Haydock, *The recursive solution of the Schrodinger equation*, in *Solid State Physics*, edited by H. Ehrenreich, F. Seitz, and D. Turnbull (Academic Press, New York, 1980), Vol. 35, pp. 215–294.
- [28] *The Recursion Method and Its Applications: Proceedings of the Conference, Imperial College, London, England 1984*, edited by D. G. Pettifor, D. L. Weaire, M. Cardona, P. Fulde, K. Von Klitzing, and H.-J. Queisser, Springer Series in Solid-State Sciences Vol. 58 (Springer, Berlin, Heidelberg, 1987).
- [29] B. Swingle, *Unscrambling the physics of out-of-time-order correlators*, *Nat. Phys.* **14**, 988 (2018).
- [30] P. Hayden and J. Preskill, *Black holes as mirrors: Quantum information in random subsystems*, *J. High Energy Phys.* **09** (2007) 120.
- [31] Y. Sekino and L. Susskind, *Fast scramblers*, *J. High Energy Phys.* **10** (2008) 065.
- [32] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, *The entropy of Hawking radiation*, *Rev. Mod. Phys.* **93**, 035002 (2021).
- [33] C. W. von Keyserlingk, T. Rakovszky, F. Pollmann, and S. L. Sondhi, *Operator hydrodynamics, OTOCs, and entanglement growth in systems without conservation laws*, *Phys. Rev. X* **8**, 021013 (2018).
- [34] A. Nahum, S. Vijay, and J. Haah, *Operator spreading in random unitary circuits*, *Phys. Rev. X* **8**, 021014 (2018).
- [35] D. S. Lubinsky, H. N. Mhaskar, and E. B. Saff, *A proof of Freud's conjecture for exponential weights*, *Constr. Approx.* **4**, 65 (1988).
- [36] X. Cao, *A statistical mechanism for operator growth*, *J. Phys. A* **54**, 144001 (2021).
- [37] A. Dymarsky and M. Smolkin, *Krylov complexity in conformal field theory*, *Phys. Rev. D* **104**, L081702 (2021).
- [38] B. Bhattacharjee, X. Cao, P. Nandy, and T. Pathak, *Krylov complexity in saddle-dominated scrambling*, *J. High Energy Phys.* **05** (2022) 174.
- [39] D. J. Yates, A. G. Abanov, and A. Mitra, *Dynamics of almost strong edge modes in spin chains away from integrability*, *Phys. Rev. B* **102**, 195419 (2020).
- [40] D. J. Yates, A. G. Abanov, and A. Mitra, *Lifetime of almost strong edge-mode operators in one-dimensional, interacting, symmetry protected topological phases*, *Phys. Rev. Lett.* **124**, 206803 (2020).
- [41] D. J. Yates, A. G. Abanov, and A. Mitra, *Long-lived period-doubled edge modes of interacting and disorder-free Floquet spin chains*, *Commun. Phys.* **5**, 1 (2022).
- [42] N. Tausendpfund, A. Mitra, and M. Rizzi, *Almost strong zero modes at finite temperature*, *Phys. Rev. Res.* **7**, 023245 (2025).
- [43] F. Ballar Trigueros and C.-J. Lin, *Krylov complexity of many-body localization: Operator localization in Krylov basis*, *SciPost Phys.* **13**, 037 (2022).
- [44] A. Avdoshkin, A. Dymarsky, and M. Smolkin, *Krylov complexity in quantum field theory, and beyond*, *J. High Energy Phys.* **06** (2024) 066.
- [45] J. Wang, M. H. Lamann, R. Steinigeweg, and J. Gemmer, *Diffusion constants from the recursion method*, *Phys. Rev. B* **110**, 104413 (2024).
- [46] M. Füllgraf, J. Wang, R. Steinigeweg, and J. Gemmer, *Lanczos-Pascal approach to correlation functions in chaotic quantum systems*, *Phys. Rev. Lett.* **135**, 250401 (2025).
- [47] P. G. Nevai, *Orthogonal Polynomials* (American Mathematical Society, Providence, 1979).
- [48] M. Vanlessen, *Strong asymptotics of the recurrence coefficients of orthogonal polynomials associated to the generalized Jacobi weight*, *J. Approx. Theory* **125**, 198 (2003).

- [49] M. Vanlessen, *Strong asymptotics of Laguerre-type orthogonal polynomials and applications in random matrix theory*, *Constr. Approx.* **25**, 125 (2007).
- [50] T. Kriecherbauer and K. T.-R. McLaughlin, *Strong asymptotics of polynomials orthogonal with respect to Freud weights*, *Int. Math. Res. Not.* **1999**, 299 (1999).
- [51] E. Levin and D. S. Lubinsky, *Orthogonal polynomials for weights close to indeterminacy*, *J. Approx. Theory* **147**, 129 (2007).
- [52] D. E. Parker, X. Cao, and M. P. Zaletel, *Local matrix product operators: Canonical form, compression, & control theory*, *Phys. Rev. B* **102**, 035147 (2020).
- [53] L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, *From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics*, *Adv. Phys.* **65**, 239 (2016).
- [54] V. Balasubramanian, J. M. Magan, and Q. Wu, *Tridiagonalizing random matrices*, *Phys. Rev. D* **107**, 126001 (2023).
- [55] N. S. Srivatsa and C. von Keyserlingk, *Operator growth hypothesis in open quantum systems*, *Phys. Rev. B* **109**, 125149 (2024).
- [56] T. Tao, *Topics in Random Matrix Theory* (American Mathematical Society, Providence, 2012).
- [57] K. T.-R. McLaughlin and P. D. Miller, *The δ steepest descent method and the asymptotic behavior of polynomials orthogonal on the unit circle with fixed and exponentially varying nonanalytic weights*, *Int. Math. Res. Pap.* **2006**, 48673 (2006).
- [58] A. Kar, L. Lamprou, M. Rozali, and J. Sully, *Random matrix theory for complexity growth and black hole interiors*, *J. High Energy Phys.* **01** (2022) 016.
- [59] W. Van Assche, *Orthogonal polynomials, associated polynomials and functions of the second kind*, *J. Comput. Appl. Math.* **37**, 237 (1991).
- [60] A. B. J. Kuijlaars, *Universality*, in *The Oxford Handbook on Random Matrix Theory* (Oxford University Press, New York, 2011).
- [61] P. Deift and X. Zhou, *A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation*, *Ann. Math.* **137**, 295 (1993).
- [62] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou, *Strong asymptotics of orthogonal polynomials with respect to exponential weights*, *Commun. Pure Appl. Math.* **52**, 1491 (1999).
- [63] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou, *A Riemann–Hilbert approach to asymptotic questions for orthogonal polynomials*, *J. Comput. Appl. Math.* **133**, 47 (2001).
- [64] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou, *Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory*, *Commun. Pure Appl. Math.* **52**, 1335 (1999).
- [65] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach*, Courant Lecture Notes, Vol. 3 (American Mathematical Society, Providence, 2000).
- [66] A. B. Kuijlaars, *Riemann–Hilbert analysis for orthogonal polynomials*, in *Orthogonal Polynomials and Special Functions: Leuven 2002, Lecture Notes in Mathematics*, edited by E. Koelink and W. Van Assche (Springer, Berlin, Heidelberg, 2003), pp. 167–210.
- [67] A. Kuijlaars and M. Vanlessen, *Universality for eigenvalue correlations at the origin of the spectrum*, *Commun. Math. Phys.* **243**, 163 (2003).
- [68] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/r9v1-nxj1> for details of the proof, an analysis of the properties of the spectral functions considered, an exposition of the Airy bootstrap, and a glossary of symbols.
- [69] B. Eichinger, M. Lukić, and H. Woracek, *Necessary and sufficient conditions for universality limits*, *arXiv:2409.18045*.
- [70] M. L. Mehta, *Random Matrices* (Elsevier, New York, 2004).
- [71] M. Plancherel and W. Rotach, *Sur les valeurs asymptotiques des polynômes d'Hermite*, *Commentarii mathematici Helvetici* **1**, 227 (1929).
- [72] F. J. Dyson, *Statistical theory of the energy levels of complex systems. I*, *J. Math. Phys. (N.Y.)* **3**, 140 (1962).
- [73] P. Forrester, *Log-Gases and Random Matrices (LMS-34)* (Princeton University Press, Princeton, NJ, 2010).
- [74] D. A. Abanin, W. De Roeck, and F. Huveneers, *Exponentially slow heating in periodically driven many-body systems*, *Phys. Rev. Lett.* **115**, 256803 (2015).
- [75] I. Arad, T. Kuwahara, and Z. Landau, *Connecting global and local energy distributions in quantum spin models on a lattice*, *J. Stat. Mech.* (2016) 033301.
- [76] J.-B. Bru and W. de Siqueira Pedra, *Lieb–Robinson Bounds for Multi-Commutators and Applications to Response Theory*, SpringerBriefs in Mathematical Physics, Vol. 13 (Springer International Publishing, Cham, 2017).
- [77] C. M. Canali, M. Wallin, and V. E. Kravtsov, *Nonuniversality in random-matrix ensembles with soft level confinement*, *Phys. Rev. B* **51**, 2831 (1995).
- [78] V. Freilikher, E. Kanzieper, and I. Yurkevich, *Unitary random-matrix ensemble with governable level confinement*, *Phys. Rev. E* **53**, 2200 (1996).
- [79] V. Freilikher, E. Kanzieper, and I. Yurkevich, *Theory of random matrices with strong level confinement: Orthogonal polynomial approach*, *Phys. Rev. E* **54**, 210 (1996).
- [80] T. Claeys, I. Krasovsky, and O. Minakov, *Weak and strong confinement in the Freud random matrix ensemble and gap probabilities*, *Commun. Math. Phys.* **402**, 833 (2023).
- [81] M. Srednicki, *Chaos and quantum thermalization*, *Phys. Rev. E* **50**, 888 (1994).
- [82] V. Alba, *Eigenstate thermalization hypothesis and integrability in quantum spin chains*, *Phys. Rev. B* **91**, 155123 (2015).
- [83] M. Aizenman and S. Warzel, *Random Operators* (American Mathematical Society, Providence, 2015).
- [84] S. Gopalakrishnan and R. Vasseur, *Superdiffusion from non-Abelian symmetries in nearly integrable systems*, *Annu. Rev. Condens. Matter Phys.* **15**, 159 (2024).
- [85] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*, Numerical Mathematics and Scientific Computation (Oxford University Press, Oxford, New York, 2004).
- [86] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis* (SIAM, 1965).

- [87] C. Jonay, D. A. Huse, and A. Nahum, *Coarse-grained dynamics of operator and state entanglement*, [arXiv:1803.00089](#).
- [88] B. Simon, *Orthogonal Polynomials on the Unit Circle: Part I: Classical Theory* (American Mathematical Society, Providence, 2009).
- [89] P. Suchsland, R. Moessner, and P. W. Claeys, *Krylov complexity and Trotter transitions in unitary circuit dynamics*, *Phys. Rev. B* **111**, 014309 (2025).
- [90] A. Bhattacharya, P. Nandy, P. P. Nath, and H. Sahu, *Operator growth and Krylov construction in dissipative open quantum systems*, *J. High Energy Phys.* **12** (2022) 081.
- [91] B. Bhattacharjee, X. Cao, P. Nandy, and T. Pathak, *Operator growth in open quantum systems: Lessons from the dissipative SYK*, *J. High Energy Phys.* **03** (2023) 054.
- [92] B. Bhattacharjee, P. Nandy, and T. Pathak, *Operator dynamics in Lindbladian SYK: A Krylov complexity perspective*, *J. High Energy Phys.* **01** (2024) 094.
- [93] A. Bhattacharya, P. Nandy, P. P. Nath, and H. Sahu, *On Krylov complexity in open systems: An approach via bi-Lanczos algorithm*, *J. High Energy Phys.* **12** (2023) 066.
- [94] F. Balogh, M. Bertola, S.-Y. Lee, and K. D. T.-R. McLaughlin, *Strong asymptotics of the orthogonal polynomials with respect to a measure supported on the plane*, *Commun. Pure Appl. Math.* **68**, 112 (2015).
- [95] M. Kieburg, A. B. J. Kuijlaars, and S. Lahiry, *Orthogonal polynomials in the normal matrix model with two insertions*, [arXiv:2408.12952](#).
- [96] J. A. Mingo and R. Speicher, *Free Probability and Random Matrices*, Fields Institute Monographs Vol. 35 (Springer, New York, 2017).
- [97] S. Pappalardi, L. Foini, and J. Kurchan, *Eigenstate thermalization hypothesis and free probability*, *Phys. Rev. Lett.* **129**, 170603 (2022).
- [98] S. Pappalardi, F. Fritzsche, and T. Prosen, *Full eigenstate thermalization via free cumulants in quantum lattice systems*, *Phys. Rev. Lett.* **134**, 140404 (2025).
- [99] J. Barbón, E. Rabinovici, R. Shir, and R. Sinha, *On the evolution of operator complexity beyond scrambling*, *J. High Energy Phys.* **10** (2019) 264.
- [100] T. E. Baker, *Lanczos recursion on a quantum computer for the Green's function and ground state*, *Phys. Rev. A* **103**, 032404 (2021).
- [101] W. Kirby, M. Motta, and A. Mezzacapo, *Exact and efficient Lanczos method on a quantum computer*, *Quantum* **7**, 1018 (2023).
- [102] M. Motta, W. Kirby, I. Liepuoniute, K. J. Sung, J. Cohn, A. Mezzacapo, K. Klymko, N. Nguyen, N. Yoshioka, and J. E. Rice, *Subspace methods for electronic structure simulations on quantum computers*, *Electron. Struct.* **6**, 013001 (2024).
- [103] K. T.-R. McLaughlin and P. D. Miller, *The dbar steepest descent method for orthogonal polynomials on the real line with varying weights*, [arXiv:0805.1980](#).
- [104] G. Szegő, *Orthogonal Polynomials*, Colloquium Publications (American Mathematical Society, Providence, 1939), Vol. 23.
- [105] W. Mück and Y. Yang, *Krylov complexity and orthogonal polynomials*, *Nucl. Phys.* **B984**, 115948 (2022).
- [106] K. T.-R. McLaughlin and P. D. Miller, *The steepest descent method for orthogonal polynomials on the real line with varying weights*, *Int. Math. Res. Not.* **2008**, rnn075 (2008).
- [107] M. Dieng, K. D. T.-R. McLaughlin, and P. D. Miller, *Dispersive asymptotics for linear and integrable equations by the $\bar{d}$ -steepest descent method*, in *Nonlinear Dispersive Partial Differential Equations and Inverse Scattering*, edited by P. D. Miller, P. A. Perry, J.-C. Saut, and C. Sulem (Springer, New York, 2019), pp. 253–291.
- [108] D. S. Lubinsky and E. B. Saff, *Uniform and mean approximation by certain weighted polynomials, with applications*, *Constr. Approx.* **4**, 21 (1988).
- [109] J. Maldacena and D. Stanford, *Remarks on the Sachdev-Ye-Kitaev model*, *Phys. Rev. D* **94**, 106002 (2016).
- [110] M. Dodelson, C. Iossa, R. Karlsson, and A. Zhiboedov, *A thermal product formula*, *J. High Energy Phys.* **01** (2024) 036.
- [111] E. Levin and D. S. Lubinsky, *Orthogonal Polynomials for Exponential Weights* (Springer Science & Business Media, New York, 2012).
- [112] T. Kasuga and R. Sakai, *Orthonormal polynomials with generalized Freud-type weights*, *J. Approx. Theory* **121**, 13 (2003).
- [113] L. V. Delacrétaz, *Heavy operators and hydrodynamic tails*, *SciPost Phys.* **9**, 034 (2020).
- [114] A. Raj, P. Glorioso, S. Gopalakrishnan, and V. Oganessian, *Diffusion cascade in a model of interacting random walkers*, [arXiv:2412.05222](#).
- [115] E. Saff and V. Totik, *Logarithmic Potentials with External Fields* (Springer, Berlin, Heidelberg, 1997).
- [116] B. C. Hall, *Quantum Theory for Mathematicians*, Graduate Texts in Mathematics (Springer, New York, 2013), Vol. 267.
- [117] H. N. Mhaskar and E. B. Saff, *Extremal problems for polynomials with exponential weights*, *Trans. Am. Math. Soc.* **285**, 203 (1984).
- [118] E. A. Rakhmanov, *On asymptotic properties of polynomials orthogonal on the real axis*, *Math. USSR-Sb.* **47**, 155 (1984).
- [119] H. N. Mhaskar and E. B. Saff, *Where does the sup norm of a weighted polynomial live?*, *Constr. Approx.* **1**, 71 (1985).
- [120] Y. Chen and M. E. H. Ismail, *Asymptotics of extreme zeros of the Meixner-Pollaczek polynomials*, *J. Comput. Appl. Math.* **82**, 59 (1997).
- [121] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments*, Mathematical Surveys No. 1 (American Mathematical Society, Providence, 1943).
- [122] L. Erdős, *The matrix Dyson equation and its applications for random matrices*, [arXiv:1903.10060](#).
- [123] G. Pinna, O. Lunt, and C. von Keyserlingk, *Approximation theory for Green's functions via the Lanczos algorithm*, *Phys. Rev. B* **112**, 054435 (2025).
- [124] C. A. Tracy and H. Widom, *Level-spacing distributions and the Airy kernel*, *Commun. Math. Phys.* **159**, 151 (1994).
- [125] E. Rabinovici, A. Sánchez-Garrido, R. Shir, and J. Sonner, *Krylov localization and suppression of complexity*, *J. High Energy Phys.* **03** (2022) 211.
- [126] E. Rabinovici, A. Sánchez-Garrido, R. Shir, and J. Sonner, *Krylov complexity from integrability to chaos*, *J. High Energy Phys.* **07** (2022) 151.

- [127] M. H. Lee, *Ergodic theory, infinite products, and long time behavior in hermitian models*, *Phys. Rev. Lett.* **87**, 250601 (2001).
- [128] R. Heveling, J. Wang, and J. Gemmer, *Numerically probing the universal operator growth hypothesis*, *Phys. Rev. E* **106**, 014152 (2022).
- [129] G. Mastroianni and G. V. Milovanović, *Interpolation Processes—Basic Theory and Applications*, Springer Monographs in Mathematics (Springer, Berlin, Heidelberg, 2008).
- [130] W. P. Su, J. R. Schrieffer, and A. J. Heeger, *Solitons in polyacetylene*, *Phys. Rev. Lett.* **42**, 1698 (1979).
- [131] B. Bertini, F. Heidrich-Meisner, C. Karrasch, T. Prosen, R. Steinigeweg, and M. Žnidarič, *Finite-temperature transport in one-dimensional quantum lattice models*, *Rev. Mod. Phys.* **93**, 025003 (2021).
- [132] R. Kubo, *Statistical-mechanical theory of irreversible processes. I. General theory and simple applications to magnetic and conduction problems*, *J. Phys. Soc. Jpn.* **12**, 570 (1957).
- [133] B. Simon, *The Christoffel-Darboux Kernel*, [arXiv:0806.1528](https://arxiv.org/abs/0806.1528).
- [134] S. Gopalakrishnan and R. Vasseur, *Kinetic theory of spin diffusion and superdiffusion in XXZ spin chains*, *Phys. Rev. Lett.* **122**, 127202 (2019).
- [135] J. De Nardis, D. Bernard, and B. Doyon, *Diffusion in generalized hydrodynamics and quasiparticle scattering*, *SciPost Phys.* **6**, 049 (2019).
- [136] C. Karrasch, J. E. Moore, and F. Heidrich-Meisner, *Real-time and real-space spin and energy dynamics in one-dimensional spin- $\frac{1}{2}$ systems induced by local quantum quenches at finite temperatures*, *Phys. Rev. B* **89**, 075139 (2014).
- [137] J. De Nardis, S. Gopalakrishnan, E. Ilievski, and R. Vasseur, *Superdiffusion from emergent classical solitons in quantum spin chains*, *Phys. Rev. Lett.* **125**, 070601 (2020).
- [138] J. De Nardis, S. Gopalakrishnan, R. Vasseur, and B. Ware, *Stability of superdiffusion in nearly integrable spin chains*, *Phys. Rev. Lett.* **127**, 057201 (2021).
- [139] G. Freud, *Orthogonal Polynomials* (Pergamon Press, New York, 1971).
- [140] J. Hauschild and F. Pollmann, *Efficient numerical simulations with Tensor Networks: Tensor Network Python (TeNPy)*, *SciPost Phys. Lecture Notes* **5** (2018), [10.21468/SciPostPhysLectNotes.5](https://doi.org/10.21468/SciPostPhysLectNotes.5).
- [141] A. De, U. Borla, X. Cao, and S. Gazit, *Stochastic sampling of operator growth dynamics*, *Phys. Rev. B* **110**, 155135 (2024).
- [142] S. R. White and I. Affleck, *Spectral function for the $S = 1$ Heisenberg antiferromagnetic chain*, *Phys. Rev. B* **77**, 134437 (2008).
- [143] G. Akemann, P. H. Damgaard, U. Magnea, and S. Nishigaki, *Universality of random matrices in the microscopic limit and the Dirac operator spectrum*, *Nucl. Phys.* **B487**, 721 (1997).
- [144] L. Erdős, *Universality of Wigner random matrices: A survey of recent results*, *Russ. Math. Surv.* **66**, 507 (2011).
- [145] D. S. Lubinsky, *Bulk universality holds in measure for compactly supported measures*, *J. d'Anal. Math.* **116**, 219 (2012).
- [146] T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, *Random-matrix physics: Spectrum and strength fluctuations*, *Rev. Mod. Phys.* **53**, 385 (1981).
- [147] I. Ya. Gol'dshtein, S. A. Molchanov, and L. A. Pastur, *A pure point spectrum of the stochastic one-dimensional Schrödinger operator*, *Funct. Anal. Appl.* **11**, 1 (1977).
- [148] *King's Computational Research, Engineering and Technology Environment (CREATE)* (2024).
- [149] O. Lunt, *Data for Emergent random matrix universality in quantum operator dynamics*, [10.5281/zenodo.15387574](https://doi.org/10.5281/zenodo.15387574) (2025).