

Exactly Solvable Models for Fermionic Symmetry-Enriched Topological Phases and Fermionic 't Hooft Anomaly

Jing-Ren Zhou¹ and Zheng-Cheng Gu^{1,*}

Department of Physics, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong



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The interplay between symmetry and topological properties plays a very important role in modern physics. In the past decade, the concept of symmetry-enriched topological (SET) phases was proposed and their classifications have been systematically studied for bosonic systems. Very recently, the concept of SET phases has been generalized into fermionic systems and their corresponding classification schemes are also proposed. Nevertheless, how to realize all these fermionic SET (fSET) phases in lattice models remains to be a difficult open problem. In this paper, we first construct exactly solvable models for nonanomalous nonchiral $2 + 1$ D fSET phases, namely, the symmetry-enriched fermionic string-net models, which are described by commuting-projector Hamiltonians whose ground states are the fixed-point wave functions of each fSET phase. Mathematically, we provide a partial definition to G -graded super fusion category (when the total symmetry G^f is a direct product of physical symmetry G and fermion-parity symmetry $\mathbb{Z}_2^f$), which is the input data of a symmetry-enriched fermionic string-net model. Next, we construct exactly solvable models for nonchiral $2 + 1$ D fSET phases with 't Hooft anomaly, especially the $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly which is different from the well-known bosonic $H^4[G, U(1)_T]$ anomaly. In our construction, this $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly is characterized by a violation of fermion-parity conservation in some of the surface $\mathcal{F}$ moves (a type of renormalization moves for the ground state wave functions of surface fSET phases), and also by a new fermionic obstruction Θ in the surface pentagon equation. We demonstrate this construction in a concrete example that the surface topological order is a $\mathbb{Z}_4$ gauge theory embedded into a fermion system and the total symmetry $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_4$. We further conjecture that the $H^2(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly is characterized by a possible violation of nonterminating q -type (called σ -type) string in the fusion rules of surface models.

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I. INTRODUCTION

A. Overview

The concept of symmetry has been introduced into modern physics for more than a century. In condensed matter physics, for a long time people believed that the Landau symmetry-breaking theory could describe all possible phases and (continuous) phase transitions. Until the discovery of the fractional quantum Hall effect (FQHE) and high-temperature superconducting cuprates, it was pointed out that systems exhibiting anyon excitations with fractional charge are described by a new type of order beyond the Landau paradigm, namely, the topological orders

characterized by long-range entanglement. Thereafter, the interplay between symmetry and topological properties opened up a new chapter in modern physics. For example, topological insulators and superconductors were proposed and intensively studied in the past two decades. The concept of symmetry protected topological (SPT) order was proposed and their full classifications for interacting systems have also been achieved recently [1–8].

Symmetry-enriched topological (SET) phases are gapped quantum phases with intrinsic topological order and global symmetry, where the symmetry can interact with the topological order in a nontrivial way, thus enriching the classification of these phases. In general, symmetry can also act nontrivially on the anyon excitations (we will use anyon, quasiparticle excitation, and anyon excitation interchangeably) in a SET phase, such that the anyons can carry fractionalized quantum numbers under global symmetry, called symmetry fractionalization [9,10]. For example, the gapped quantum spin liquids in frustrated magnets are SET phases, where the spin-1/2 spinon excitation transforms

*Contact author: zcgu@phy.cuhk.edu.hk

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projectively under the $SO(3)$ spin-rotation symmetry [11]. Aside from symmetry fractionalization, another feature of SET phases is that symmetry may permute the anyon type. For example, considering the toric code topological order with onsite $\mathbb{Z}_2$ symmetry, there are several possible SET phases, in one of which anyons e and m are permuted by the $\mathbb{Z}_2$ symmetry [12]. Anyons and symmetry defects in $2 + 1D$ SET phases are systematically classified in Ref. [13] via G -crossed braided tensor categories for a given symmetry group G . While exactly solvable models for $2 + 1D$ nonchiral SET phases, called symmetry-enriched string-net models, are constructed in Refs. [12,14], which focus on discrete finite group case (the continuous group case is discussed in Ref. [15]).

On the other hand, surface topological orders with 't Hooft anomaly have been studied in many previous works [16–34]. It was shown that $2 + 1D$ bosonic SET phases with 't Hooft anomaly are characterized by an obstruction that is a 4-cocycle in $H^4[G, U(1)_T]$ [35,36], and this obstruction can be canceled by putting it on the surface of a 3D bulk SPT phase classified by a 4-cocycle. If we study the anyon excitations of the surface SET phase, this 4-cocycle $o_4 \in H^4[G, U(1)_T]$ is realized as the obstruction for a G -crossed extension for the unitary modular tensor category that describes the anyons [13]. (G -crossed extension means that we consistently add deconfined symmetry defects into the system, such that the anyons and symmetry defects together form a larger category, called a G -crossed braided tensor category.) Gauging the symmetry is another commonly used tool in studying topological phases with symmetry [37–39]. It transforms a global symmetry into a local gauge symmetry, which brings new topologically ordered states into the system.

Although fruitful results have been achieved for bosonic systems in the past decade, the interplay between symmetry and topological orders is much richer and more complicated in fermionic systems. Here we will focus on $2 + 1D$ fermionic SET (fSET) phases, which are SET phases whose underlying degrees of freedom are fermions (such as electrons). Denoting this physical fermion as f , fermion parity is always conserved in any fermionic system, i.e., there is always a fermion-parity symmetry $\mathbb{Z}_2^f$. Therefore, the total symmetry group is always a central extension of a bosonic symmetry group G by $\mathbb{Z}_2^f$, denoted by G^f . Another feature is that all anyon excitations in an fSET phase always have trivial double braiding with this physical fermion f , whereby f is also called the transparent fermion. The famous fractional quantum Hall (FQH) states are examples of chiral fSET phases [40], where at filling factor $\nu = 1/q$, anyon excitations carry fractional charge e/q under the $U(1)$ charge conservation symmetry. An algebraic framework for the classification of anyons and how they behave under symmetry in fSET phases is studied in Refs. [41,42], and symmetry defects are further included in Ref. [43] as a more complete description. Roughly speaking, G^f -crossed

super modular categories describe anyons and symmetry defects in $2 + 1D$ fSET phases. On the other hand, if we gauge $\mathbb{Z}_2^f$, we obtain G -crossed spin modular categories (a spin modular category is one of the 16 possible $\mathbb{Z}_2^f$ -modular extensions of a super modular category).

Similar to bosonic systems, surface topological orders with 't Hooft anomaly can also be generalized into fermion systems. In the context of quasiparticle excitations, 't Hooft anomalies in $2 + 1D$ fSET phases are distinguished into four layers in Refs. [43,44], characterized by the following obstruction functions: a 1-cocycle $o_1 \in H^1(G, \mathbb{Z}_T)$, a 2-cocycle $o_2 \in H^2(G, \mathbb{Z}_2)$, a 3-cocycle $o_3 \in H^3(G, \mathbb{Z}_2)$, and a 4-cocycle $o_4 \in H^4[G, U(1)_T]$. The four layers of 't Hooft anomalies are in one-to-one correspondence (up to some equivalence relations) to the four layers in classifying $3 + 1D$ fermionic SPT (fSPT) phases: $p + ip$ superconductor layer, Kitaev chain layer, complex fermion layer and bosonic layer [6–8], respectively. This implies the picture that any $2 + 1D$ fSET phase with 't Hooft anomaly is living on the surface of a certain bulk $3 + 1D$ fSPT phase. The $H^4[G, U(1)_T]$ anomaly corresponds to the obstruction to do G -crossed extension or gauging G for the surface bosonic SET phase. While the other three anomalies, we call fermionic 't Hooft anomalies, all correspond to the obstruction to gauging $\mathbb{Z}_2^f$ in different levels.

Although a lot of progress has been made for understanding the intriguing properties of fSET phases in recent years, so far it is still unclear how to systematically realize all these fSET phases in lattice models. The aim of this paper is to construct exactly solvable models for $2 + 1D$ fSET phases, including both nonanomalous phases and phases with fermionic 't Hooft anomaly. In particular, we will propose a generic framework to construct fixed-point wave functions for nonchiral fSET phases based on equivalence class of generalized fermionic symmetric local unitary (gfSLU) transformations. Given a G -graded super fusion category $\mathcal{S}_G$ (described in Subsection II B) as input, we can construct a fermionic symmetry-enriched string-net model that fully describes the fixed-point ground state wave functions and the parent commuting-projector Hamiltonian of a nonanomalous nonchiral $2 + 1D$ fSET phase. We present a simple but illustrative example of the exactly solvable model for the toric code together with a physical fermion with $\mathbb{Z}_4^f \cong \mathbb{Z}_2^f \times_{\omega_2} \mathbb{Z}_2$ symmetry in Subsection II H. We further present two more-involved examples: the fermionic symmetry-enriched string-net models with input $\mathbb{Z}_2$ -graded super fusion category being the super Tambara-Yamagami category and ${}^{\text{SVec}}\text{SU}(2)_6$, both of which are exactly solvable models for nonchiral fSET phases with $\mathbb{Z}_2^f \times \mathbb{Z}_2$ symmetry.

The concept of the equivalence class of fermionic symmetric local unitary transformations can also be generalized into constructing anomalous fSET states by including the bulk $3 + 1D$ fSPT states. In our construction,

TABLE I. Table of notations.

G	Bosonic symmetry group
$\mathbf{1}$	Trivial group element in G
$\mathbb{Z}_2^f$	Fermion-parity symmetry group
ω_2	The 2-cocycle that determines the central extension of G by $\mathbb{Z}_2^f$
G^f	Total fermionic symmetry group
$H^n(G, M)$	n th group cohomology of G valued in M , where M is an Abelian group
$C^n(G, M)$	Group of n -cochains of G valued in M
$B^n(G, M)$	Group of n -coboundaries of G valued in M
f	Physical fermion
$\mathcal{C}$	General fusion or tensor category
$\mathbb{1}$	Tensor unit in $\mathcal{C}$
$\mathcal{Z}_1(\mathcal{C})$	Drinfeld center of $\mathcal{C}$
$\mathcal{Z}_2(\mathcal{C})$	Müger center of $\mathcal{C}$
$\boxtimes$	Deligne tensor product
$\mathbf{Vec}_N$	Category of N -graded vector spaces, where N is a group
$\mathbb{C}^{p q}$	Super vector space with grade-0 dimension p and grade-1 dimension q
$\mathbf{SVec}$	Category of super vector spaces
$\mathbf{Svec}\mathcal{C} \equiv \mathcal{C}/f$	Super fusion category, obtained by the canonical construction of enriched monoidal category, or equivalently, by condensing the fermion f in $\mathcal{Z}_1(\mathcal{C})$
$\mathrm{Hom}(X, Y)$	Hom -space of $X, Y \in \mathcal{C}$, which is the $\mathbb{C}$ -(super) vector space consists of morphisms from X to Y
$\mathrm{End}(X) := \mathrm{Hom}(X, X)$	Endomorphism space of X , $X \in \mathcal{C}$
$\mathcal{S}_G$	G -graded super fusion category
$\mathbf{Svec}\mathcal{C}_G$	G -graded super fusion category constructed by a fermion condensation from a G -graded unitary fusion category $\mathcal{C}_G$
$a_{\mathbf{g}}$	String type, or simple object in $\mathcal{S}_G$
m -type string	$\mathrm{End}(a_{\mathbf{g}}) = \mathbb{C}$
q -type string	$\mathrm{End}(a_{\mathbf{g}}) = \mathbb{C}^{1 1}$
σ -type string	$\mathbb{Z}_2$ -conserved q -type string
α or $ \alpha\rangle$	A fusion state in hom-space $\mathrm{Hom}(a_{\mathbf{g}} \otimes b_{\mathbf{h}}, c_{\mathbf{gh}})$
$\underline{\alpha}$	The vertex supporting fusion state α
$s(\alpha)$	Fermion parity of fusion state α
$s'(\mathbf{g}_0, \alpha)$	Total surface fermion parity of fusion state α with leftmost group element $\mathbf{g}_0$
$S(\mathbf{g})$	Anti-unitarity of $\mathbf{g} \in G$
$U(\mathbf{g})$	Global symmetry transformation of $\mathbf{g} \in G$
$c_{\underline{a}}, c_{\underline{a}}^\dagger$	Fermion annihilation and creation operators on vertex $\underline{a}$ in $2 + 1\mathrm{D}$ system
$C_{ijkl}, C_{ijkl}^\dagger$	Bulk fermion annihilation and creation operators in tetrahedron $\langle ijkl \rangle$ in $3 + 1\mathrm{D}$ system
o_4	Obstruction on G -crossed extension, appearing when there is $H^4[G, U(1)_T]$ 't Hooft anomaly
o_3	Obstruction to extending symmetry fractionalization to its $\mathbb{Z}_2^f$ -modular extended category, appearing when there is $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly
o_2	Obstruction to extending G action to its $\mathbb{Z}_2^f$ -modular extended category, appearing when there is $H^2(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly
Θ	Fermionic obstruction on surface pentagon equation, defined in Eq. (127), appearing when there is $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly

TABLE II. Summary of contents on nonchiral nonanomalous 2 + 1D fermionic topological phases.

	Nonanomalous nonchiral 2 + 1D fermionic topological order	Nonanomalous nonchiral 2 + 1D fSET phase with symmetry G^f
Exactly solvable model	Fermionic string-net model with input super fusion category ${}^{\text{SVec}}\mathcal{C}$	Fermionic symmetry-enriched string-net model with input G -graded super fusion category $\mathcal{S}_G$
Quasiparticles or anyon excitations	$\mathcal{Z}_1({}^{\text{SVec}}\mathcal{C}) \cong {}^{\text{SVec}}\mathcal{Z}_1(\mathcal{C})_0$ (see Eq. (184))	$\mathcal{Z}_1(\mathcal{S}_1)$ with G^f -action (see Conjecture 1) Coupling with G^f -symmetry defects: G^f -crossed super modular category $\mathcal{Z}_1(\mathcal{S}_1)_{G^f}^\times$
Exactly solvable model after gauging $\mathbb{Z}_2^f$ when ω_2 is trivial	String-net model with input unitary fusion category $\mathcal{C}$	Symmetry-enriched string-net model with input G -graded unitary fusion category $\mathcal{C}_G$, where ${}^{\text{SVec}}\mathcal{C}_g = \mathcal{S}_g$, $\forall g \in G$
Anyon excitations after gauging $\mathbb{Z}_2^f$ when ω_2 is trivial	$\mathcal{Z}_1(\mathcal{C})$, which is a spin modular category	$\mathcal{Z}_1(\mathcal{C}_1)$ with G action Coupling with G -symmetry defects: G -crossed spin modular category $\mathcal{Z}_1(\mathcal{C}_1)_G^\times$

we focus on the $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly characterized by a violation of fermion-parity conservation in some of the surface $\mathcal{F}$ moves, and also by a new fermionic obstruction Θ in the surface pentagon equation defined in Eq. (127). We demonstrate this construction in a concrete example that the surface topological order is a $\mathbb{Z}_4$ gauge theory embedded into a fermion system and the total symmetry is $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_4$. We further conjecture that the $H^2(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly can only be supported by surface topological order with Ising σ -type strings (a q -type string [45,46] is called σ -type if it is $\mathbb{Z}_2$ conserved), and this anomaly corresponds to a violation of the $\mathbb{Z}_2$ conservation of σ -type strings in some of the fusion rules, and the obstructed pentagon equation is in a similar structure as the $H^3(G, \mathbb{Z}_2)$ anomaly.

Throughout the paper, we will use the following conventions, and we summarize important notations used in this paper in Table I.

Convention 1. We may use the set of isomorphism classes of simple objects in a fusion category to denote this category. For example, $\mathcal{Z}_1(\mathbf{Vec}_{\mathbb{Z}_2}) = \{1, e, m, \psi\}$ (the notations are introduced in Table I), where $1, e, m, \psi$ are four (representatives of) isomorphism classes of simple objects, which physically correspond to anyon types in the toric code topological order.

Convention 2. We may use the following abbreviation to denote any homogeneous n -cochain $f_n \in C^n(G, M)$ (M is a G module, which is an Abelian group):

$$f_n(012\dots n) := f_n(\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_n), \quad (1)$$

where $\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_n \in G$. In addition, let $f_n \in Z^n(C, M)$ be a n -cocycle. We may abuse the notation $f_n \in H^n(C, M)$ instead of the rigorous notation $[f_n] \in H^n(C, M)$, where $[f_n]$ denotes the cohomology class of f_n .

B. Summary of major results from mathematical perspective

In Table II, we summarize nonchiral nonanomalous 2 + 1D fermionic topological phases, i.e., fermionic topological orders and fSET phases. The exactly solvable model of any nonchiral 2 + 1D fermionic topological order is a fermionic string-net model [46,47], whose input category is a super fusion category. A super fusion category can always be obtained by a fermion condensation (or mathematically canonical construction of an enriched monoidal category, introduced in detail in Appendix A) from a unitary fusion category. Denote a unitary fusion category as $\mathcal{C}$ whose Drinfeld center $\mathcal{Z}_1(\mathcal{C})$ contains a fermion, its fermion condensed category is denoted as $\mathcal{C}/f \equiv {}^{\text{SVec}}\mathcal{C}$, where ${}^{\text{SVec}}\mathcal{C}$ denotes for category $\mathcal{C}$ enriched over $\mathbf{SVec}$, which is a super fusion category. Then we define the Drinfeld center of a super fusion category that describes the anyon excitations of a fermionic string-net model in Sec. IV B as $\mathcal{Z}_1({}^{\text{SVec}}\mathcal{C}) \cong {}^{\text{SVec}}\mathcal{Z}_1(\mathcal{C})_0$. Here $\mathcal{Z}_1(\mathcal{C}) \cong \mathcal{Z}_1(\mathcal{C})_0 \oplus \mathcal{Z}_1(\mathcal{C})_1$ ($0, 1$ are group elements of $\mathbb{Z}_2^f$) is a spin modular category, where its grade-0 sector $\mathcal{Z}_1(\mathcal{C})_0$ is a super modular category, and its grade-1 sector $\mathcal{Z}_1(\mathcal{C})_1$ is the fermion-parity vortex or flux sector. In other words, a spin modular category is a unitary modular category that contains a fermion f , such that f is transparent (up to double braiding) in its grade-0 sector, and ${}^{\text{SVec}}\mathcal{Z}_1(\mathcal{C})_0$ denotes for $\mathcal{Z}_1(\mathcal{C})_0$ enriched over $\mathbf{SVec}$. Gauging the $\mathbb{Z}_2^f$ fermion-parity symmetry for ${}^{\text{SVec}}\mathcal{Z}_1(\mathcal{C})_0$ recovers the bosonic string-net model inputted by unitary fusion category $\mathcal{C}$, whose anyon excitations are described by $\mathcal{Z}_1(\mathcal{C})$. On the other hand, the exactly solvable model of an fSET phase with total symmetry $G^f = \mathbb{Z}_2^f \times_{\omega_2} G$ is a fermionic symmetry-enriched string-net model, whose input category is a G -graded super fusion category $\mathcal{S}_G$, introduced in

TABLE III. Summary of contents on nonchiral 2 + 1D topological phases with 't Hooft anomaly. $[f_n]$ denotes for the cohomology class of n -cocycle f_n . The 't Hooft anomalies are in a structure of layers. That is to say, if the first layer fermionic $H^3(G, \mathbb{Z}_2)$ anomaly vanishes, we then consider whether there is second layer bosonic $H^4[G, U(1)_T]$ anomaly.

	Nonchiral 2 + 1D SET phase with symmetry G with $H^4[G, U(1)_T]$ 't Hooft anomaly	Nonchiral 2 + 1D fSET phase with symmetry G^f (ω_2 is trivial) with $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly
Bulk topological phase	3 + 1D SPT phase with given $[\nu_4] \in H^4[G, U(1)_T]$	3 + 1D fSPT phase with given $[n_3] \in H^3(G, \mathbb{Z}_2)$ and $[\nu_4] \in C^4[G, U(1)_T]/B^4[G, U(1)_T]$ satisfying $d\nu_4 = (-1)^{n_3 \smile 1} n_3$
Surface exactly solvable model	Surface symmetry-enriched string-net model [12], where the pentagon equation is obstructed by ν_4	Fermionic surface symmetry-enriched string-net model, where • $\mathcal{F}$ moves can violate fermion-parity conservation by n_3 ; • The fermionic pentagon equation is obstructed by $\Theta\nu_4$ (see Sec. III, and Θ is a phase factor defined in Eq. (127))
Surface anyon excitations	$\mathcal{Z}_1(\mathcal{C}_1)$ with G -action and with $[o_4] \in H^4[G, U(1)_T]$ obstruction on G -crossed extension (also the obstruction to gauging G), where $[o_4] = [\nu_4]$	$\mathcal{Z}_1(\mathcal{S}_1)$ with G^f action and with $[o_3] \in H^3(G, \mathbb{Z}_2)$ obstruction on extending symmetry fractionalization from $\mathcal{Z}_1(\mathcal{S}_1)$ to $\mathcal{Z}_1(\mathcal{C}_1)$ (also the obstruction to gauging $\mathbb{Z}_2^f$), where $\text{SVec}\mathcal{C}_1 = \mathcal{S}_1$ and $[o_3] = [n_3]$

Sec. II B, and partially defined (when ω_2 is trivial) in Sec. IV A. Its Drinfeld center that describes the anyon excitations of this fSET phase is conjectured to be $\mathcal{Z}_1(\mathcal{S}_1)$ ($\mathbf{1}$ is the trivial group element in G), and the symmetry acts on these anyons through the so-called G^f action (anyon permutation) and symmetry fractionalization [43, 48]. Then performing a G^f -crossed extension [43], i.e., coupling the anyon excitations with G^f -symmetry defects consistently, we obtain a G^f -crossed super modular category $\mathcal{Z}_1(\mathcal{S}_1)_{G^f}^\times$. Then gauging the $\mathbb{Z}_2^f$ symmetry when ω_2 is trivial (we do not know yet how to gauge $\mathbb{Z}_2^f$ when ω_2 is nontrivial), we obtain a bosonic SET phase whose exactly solvable model is a symmetry-enriched string-net model [12, 14] inputted by a G -graded unitary fusion category $\mathcal{C}_G$ such that its center $\mathcal{Z}_1(\mathcal{C}_G)$ is a spin modular category, which is related to $\mathcal{S}_G$ by $\text{SVec}\mathcal{C}_G = \mathcal{S}_G$, i.e., $\mathcal{S}_G$ is obtained by a fermion condensation on $\mathcal{C}_G$. Here $\text{SVec}\mathcal{C}_g = \mathcal{S}_g$, $\forall g \in G$ (see details in Sec. IV A). Briefly, for the trivial group element sector $\text{SVec}\mathcal{C}_1 = \mathcal{S}_1$, $\mathcal{S}_1$ is obtained by a fermion condensation on $\mathcal{C}_1$, as $\mathcal{Z}_1(\mathcal{C}_G)$ is a spin modular category implies that $\mathcal{Z}_1(\mathcal{C}_1)$ is also a spin modular category [see full procedures of fermion condensation in Diagram (186)]; while for other $g \neq \mathbf{1}$ sectors, $\mathcal{S}_g$ is not obtained by fermion condensation, but a so-called canonical construction of enriched category [49] by our constraint that each $\mathcal{C}_g$ sector is closed under SVec -action.

In Table III, we summarize nonchiral 2 + 1D topological phases with 't Hooft anomaly. Since the 't Hooft anomalies are in a structure of layers, given an anomalous fSET phase

for example, we should at first check whether it has fermionic $H^2(G, \mathbb{Z}_2)$ anomaly [the fermionic $H^1(G, \mathbb{Z}_T)$ anomaly corresponds to chiral phases]. If the fermionic $H^2(G, \mathbb{Z}_2)$ anomaly vanishes, we check whether there is fermionic $H^3(G, \mathbb{Z}_2)$ anomaly. Again if the fermionic $H^3(G, \mathbb{Z}_2)$ anomaly vanishes, we check whether there is bosonic $H^4[G, U(1)_T]$ anomaly. Let us consider the simplest case first, i.e., a bosonic SET phases with symmetry G with $H^4[G, U(1)_T]$ anomaly, and we denote its obstruction function as $o_4 \in H^4[G, U(1)_T]$. In the content of surface anyons together with G action of this anomalous bosonic SET phase, o_4 corresponds to the obstruction to perform a G -crossed extension on surface anyons [13, 36]. This $H^4[G, U(1)_T]$ anomaly can be compensated by a bulk 3 + 1D SPT phase characterized by a 4-cocycle $\nu_4 \in H^4[G, U(1)_T]$, where ν_4 is in the same cohomology class as o_4 , i.e., $[o_4] = [\nu_4]$. The surface exactly solvable model of this SET phase with $H^4[G, U(1)_T]$ anomaly is a so-called surface symmetry-enriched string-net model [12], whose pentagon equation is exactly obstructed by ν_4 . On the other hand, for an fSET phase with symmetry G^f (when ω_2 is trivial, as we do not know how to deal with cases with nontrivial ω_2 yet) with $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly, we denote its obstruction function as $o_3 \in H^3(G, \mathbb{Z}_2)$. Since the anomaly is totally due to the anomalous symmetry action, while the surface topological order itself is anomaly-free, the fixed-point ground state wave functions of surface topological order can be described by a super fusion category $\mathcal{S}_1$, whose Drinfeld center that describes the surface anyons is $\mathcal{Z}_1(\mathcal{S}_1)$. In the

content of surface anyons $\mathcal{Z}_1(\mathcal{S}_1)$ together with G^f -action of this anomalous fSET phase, o_3 corresponds on extending symmetry fractionalization from a super modular category $\mathcal{Z}_1(\mathcal{S}_1)$ to its $\mathbb{Z}_2^f$ -modular extension [44], i.e., $\mathcal{Z}_1(\mathcal{C}_1)$ as a spin modular category, where $\text{SVec}\mathcal{C}_1 = \mathcal{S}_1$. This fermionic $H^3(G, \mathbb{Z}_2)$ anomaly can be compensated by a bulk 3 + 1D fSPT phase characterized by a pair (n_3, ν_4) , where n_3 is a 3-cocycle in $H^3(G, \mathbb{Z}_2)$, which is in the same cohomology class as o_3 , i.e., $[o_3] = [n_3]$, and ν_4 is a 4-cochain satisfying $d\nu_4 = (-1)^{n_3 \smile 1^{n_3}}$. The surface exactly solvable model of this fSET phase with fermionic $H^3(G, \mathbb{Z}_2)$ anomaly is a fermionic surface symmetry-enriched string-net model, whose pentagon equation is obstructed by $\Theta\nu_4$, where Θ is a phase factor induced by fermion creation and annihilation operators from the bulk 3 + 1D fSPT phase, defined in Eq. (127). Θ appears as long as there is fermionic 't Hooft anomaly, i.e., $H^3(G, \mathbb{Z}_2)$, $H^2(G, \mathbb{Z}_2)$ or $H^1(G, \mathbb{Z}_T)$ fermionic 't Hooft anomaly.

C. Organization of this paper

The rest of the paper is organized as follows: First, we construct the so-called fermionic symmetry-enriched string-net models in Sec. II, generalizing both symmetry-enriched string-net models and fermionic string-net models [46,47,50], summarized in Sec. II G. They are exactly solvable models for 2 + 1D nonanomalous nonchiral fSET phases. Second, we construct the so-called surface fermionic symmetry-enriched string-net models in Sec. III, summarized in Sec. III D. They are exactly solvable models for 2 + 1D nonchiral surface fSET phases with fermionic 't Hooft anomaly, i.e., fSET phases that live on the surface of bulk 3 + 1D fSPT phases. Since only nonchiral surface states can be realized by our surface model, our approach can only cover the $H^3(G, \mathbb{Z}_2)$ and $H^2(G, \mathbb{Z}_2)$ fermionic 't Hooft anomalies. We conclude that $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly corresponds to a violation of fermion-parity conservation in certain surface $\mathcal{F}$ moves defined in Sec. III A, and also a Θ obstruction in the fermionic pentagon equation. Finally, we discuss the mathematical framework that arises from our lattice model construction for fSET phases in Sec. IV. Further, in Appendix A, we review basic definitions of super fusion category, super modular category and spin modular category, which are broadly used in this paper. In Appendix B, we review the categorical description of the anyons under symmetry action (or G action) in bosonic SET phases [13]. Given a unitary modular tensor category $\mathcal{M}$ that describes anyons, and a symmetry G , there are three pieces of data: G action on $\mathcal{M}$, symmetry fractionalization, and gauging G on $\mathcal{M}$ (i.e., a G -crossed extension followed by a G equivariantization on $\mathcal{M}$). In Appendix C, we review the relation between symmetry-enriched string-net model data and anyon symmetry fractionalization data for bosonic or

fermionic SET phases whose intrinsic topological orders are Abelian gauge theories [12], by which we can show that the o_3 obstruction on anyon level exactly matches with the n_3 obstruction in lattice model level for an example in Remark 3.10. In Appendix D, we review the definitions of obstruction functions $o_2 \in H^2(G, \mathbb{Z}_2)$ and $o_3 \in H^3(G, \mathbb{Z}_2)$ [44,48], which are the obstructions to extending G action and symmetry fractionalization from the super modular category (that describes the anyon excitations) to its $\mathbb{Z}_2^f$ -modular extended category (as a spin modular category), respectively. In Appendix E, we review the anyon data of surface $\mathcal{Z}_1(\text{Vec}_{\mathbb{Z}_4}) \boxtimes \{1, f\}$ topological order with $\mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_4$ symmetry [51], for which we construct its surface fermionic symmetry-enriched string-net model in Sec. III F. In Appendix G, we use another approach called the partition function approach to derive the explicit expressions of bulk $\tilde{\mathcal{F}}$ move and boundary $\mathcal{F}$ move for surface fSET phases with $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly, and we obtain exactly the same expressions as Eq. (114) and Eq. (115). In Appendix I, we list canonical choices of the cochains in 3 + 1D fSPT phases, which will be used when we extend our exactly solvable model construction to $H^2(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly in the future.

II. FERMIONIC SYMMETRY-ENRICHED STRING-NET MODELS

A. Fermionic global symmetry

Denote the fermion-parity symmetry group as $\mathbb{Z}_2^f = \{1, P_f = (-1)^{N_f}\}$, where N_f is the total fermion number operator. The fermionic symmetry group G^f in general can be any central extension of a bosonic symmetry group G by $\mathbb{Z}_2^f$, written in a short exact sequence as

$$1 \rightarrow \mathbb{Z}_2^f \rightarrow G^f \rightarrow G \rightarrow 1, \quad (2)$$

which is specified by a 2-cocycle $\omega_2 \in Z^2(G, \mathbb{Z}_2)$, where G is a discrete finite group. We simply denote G^f as

$$G^f = \mathbb{Z}_2^f \times_{\omega_2} G. \quad (3)$$

Denote a group element in G^f by $(x, \mathbf{g})$, where $x \in \mathbb{Z}_2^f$ and $\mathbf{g} \in G$. Group multiplication in G^f is given by

$$(x, \mathbf{g})(y, \mathbf{h}) = (x + y + \omega_2(\mathbf{g}, \mathbf{h}), \mathbf{gh}). \quad (4)$$

And there might be time-reversal symmetry in G , which is anti-unitary. Therefore, we define a $\mathbb{Z}_2$ grading on G : for any $\mathbf{g} \in G$,

$$S(\mathbf{g}) = \begin{cases} 1, & \text{if } \mathbf{g} \text{ is unitary} \\ *, & \text{if } \mathbf{g} \text{ is anti-unitary} \end{cases}. \quad (5)$$

Denote the global symmetry transformation of $\mathbf{g}$ as $U(\mathbf{g})$. U forms a linear representation of G^f , and therefore it forms a projective representation of G :

$$U(\mathbf{g})U(\mathbf{h}) = P_f^{\omega_2(\mathbf{g},\mathbf{h})}U(\mathbf{gh}). \quad (6)$$

We note that changing $\omega_2 \rightarrow \omega_2 + d\phi_1$, where $\phi_1 \in C^1(G, \mathbb{Z}_2)$ is a 1-cochain and $d\phi_1$ is a 2-coboundary, corresponds to changing the form of symmetry transformation by

$$U(\mathbf{g}) \rightarrow P_f^{\phi_1(\mathbf{g})}U(\mathbf{g}). \quad (7)$$

Therefore, defining the symmetry transformation of G^f needs to specify a particular cocycle ω_2 , but not the cohomology class $[\omega_2]$ [48].

B. G -graded super fusion category

To define a fermionic symmetry-enriched string-net model, we need to input a G -graded super fusion category, denoted as $\mathcal{S}_G$. Mathematically, the general definition of G -graded super fusion category is still unknown. However, we can still say that $\mathcal{S}_G$ must have the following two properties:

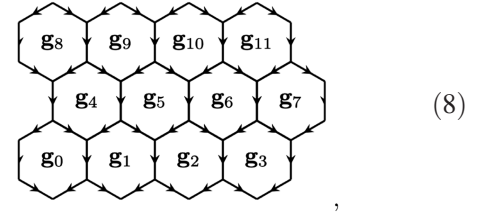
- (i) $\mathcal{S}_G$ is a super fusion category (Definition A.7), i.e., it is a unitary semi-simple rigid monoidal category that is enriched over $\mathbf{SVec}$ (Definition A.1), where enriching over $\mathbf{SVec}$ means that hom-spaces in $\mathcal{S}_G$ are super vector spaces.
- (ii) $\mathcal{S}_G$ has a G -grading structure, i.e., $\mathcal{S}_G = \bigoplus_{\mathbf{g} \in G} \mathcal{S}_{\mathbf{g}}$ and $\forall a_{\mathbf{g}} \in \mathcal{S}_{\mathbf{g}}, b_{\mathbf{h}} \in \mathcal{S}_{\mathbf{h}}, a_{\mathbf{g}} \otimes b_{\mathbf{h}} \in \mathcal{S}_{\mathbf{gh}}$. Here $\mathbf{1}$ is always a super fusion category, where $\mathbf{1}$ is the trivial element in G .

In particular, when ω_2 is trivial (ω_2 is introduced in Sec. II A), $\mathcal{S}_G$ can always be obtained by a fermion condensation (or a canonical construction of enriched monoidal category over $\mathbf{SVec}$) from a G -graded unitary fusion category $\mathcal{C}_G$ [52,53], i.e., $\mathcal{S}_G := {}^{\mathbf{SVec}}\mathcal{C}_G$, explicitly defined in Sec. IV A. But when ω_2 is nontrivial, we do not know how to define the G -graded super fusion category yet. Nonetheless, the two properties listed above are enough for us to define a fermionic symmetry-enriched string-net model. We expect that our physical model brings out the constraints in defining the G -graded super fusion category when ω_2 is nontrivial.

C. Degrees of freedom

Given a finite fermionic global symmetry group $G^f = \mathbb{Z}_2^f \times_{\omega_2} G$ and a G -graded super fusion category $\mathcal{S}_G = \bigoplus_{\mathbf{g} \in G} \mathcal{S}_{\mathbf{g}}$ (though the definition of the G -graded super fusion category is unknown, we only make use of the two properties listed in Sec. II B), a fermionic symmetry-enriched string-net model on a honeycomb lattice consists of the following degrees of freedom:

- (1) $|G|$ -level group element labels $|\mathbf{g}_i\rangle$ in each plaquette, where $\mathbf{g}_i \in G, i \in \mathbb{N}$. Graphically,



where the arrows on links represent the branching structure (arrows are always 2-in-1-out or 1-in-2-out on each vertex). Recall that $U(\mathbf{g}_0)$ denotes for the global symmetry transformation of $\mathbf{g}_0$. We choose the following convention:

$$U(\mathbf{g}_0)|\mathbf{g}_i\rangle = |\mathbf{g}_0\mathbf{g}_i\rangle. \quad (9)$$

The Hilbert space on plaquettes is

$$\mathcal{H}_P = \bigotimes_{p \in P} \mathbb{C}[G], \quad (10)$$

where P is the set of all plaquettes in the lattice, p denotes for each plaquette, and $\mathbb{C}[G]$ is the Hilbert space of linear combinations of group elements in G with coefficient in $\mathbb{C}$.

- (2) G -graded string types $a_{\mathbf{g}}, b_{\mathbf{h}}, \dots \in \mathcal{S}_G$ on each link, where $a_{\mathbf{g}} \in \mathcal{S}_{\mathbf{g}}, b_{\mathbf{h}} \in \mathcal{S}_{\mathbf{h}}, \dots$ (A string type corresponds to a simple object in $\mathcal{S}_G$. So that without causing any ambiguity, we may use the same notation $\mathcal{S}_G$ to denote for the set of isomorphism classes of simple objects in $\mathcal{S}_G$). Moreover, the sector $\mathbf{1}$, graded by the trivial group element $\mathbf{1}$, is always a super fusion category, and $\mathbf{1}, a, b, \dots \in \mathbf{1}$, where $\mathbf{1}$ is the tensor unit in $\mathbf{1}$. Each G -graded string type $a_{\mathbf{g}}$ represents a symmetry defect (or symmetry domain wall) between two plaquettes:

where $\mathbf{g} \equiv \mathbf{g}_0^{-1} \mathbf{g}_1$.

Since hom-spaces in a G -graded super fusion category are in general super vector spaces (see Definition A 1), each string type in $\mathcal{S}_G$ can be either m -type or q -type [45,46]. A string type $a_{\mathbf{g}}$ is called m -type if $\text{End}(a_{\mathbf{g}}) := \text{Hom}(a_{\mathbf{g}}, a_{\mathbf{g}}) = \mathbb{C}$; and is called q -type if $\text{End}(a_{\mathbf{g}}) = \mathbb{C}^{1|1}$. We assign a number $n_{a_{\mathbf{g}}}$ to each $a_{\mathbf{g}} \in \mathcal{S}_G$:

$$n_{a_{\mathbf{g}}} := \dim \text{End}(a_{\mathbf{g}}) = \begin{cases} 1, & \text{if } a_{\mathbf{g}} \text{ is } m\text{-type} \\ 2, & \text{if } a_{\mathbf{g}} \text{ is } q\text{-type} \end{cases}, \quad (12)$$

which is the dimension of the endomorphism space of $a_{\mathbf{g}}$.

The quantum dimension of $a_{\mathbf{g}}$ is denoted as $d_{a_{\mathbf{g}}}$, which can be defined through fusion multiplicity around Eq. (21)

or through a certain F move as in Ref. [54]. Then the total quantum dimension for each $\mathcal{S}_g$ sector is defined as

$$D_g^2 = \sum_{a_g \in \mathcal{S}_g} \frac{d_{a_g}^2}{n_{a_g}}. \quad (13)$$

We require that for any $g \in G$, the total quantum dimensions of all g sectors are equal, i.e.,

$$D_1^2 = D_g^2, \quad \forall g \in G. \quad (14)$$

- (3) Fusion states $\alpha, \beta, \dots$ in fusion space $\text{Hom}(a_g \otimes b_h, c_{gh})$ or $\text{Hom}(c_{gh}, a_g \otimes b_h)$ on each vertex, graphically represented as

$$\begin{array}{c} a_g \quad b_h \\ \searrow \quad \swarrow \\ \alpha \\ \downarrow \\ g_0 \\ \downarrow \\ c_{gh} \end{array} \quad \text{or} \quad \begin{array}{c} c_{gh} \\ \downarrow \\ \beta \\ \swarrow \quad \searrow \\ a_g \quad b_h \end{array}. \quad (15)$$

Below let us consider only fusion space $\text{Hom}(a_g \otimes b_h, c_{gh})$, and the property of $\text{Hom}(c_{gh}, a_g \otimes b_h)$ is similar. The hom-space $\text{Hom}(a_g \otimes b_h, c_{gh}) \in \mathbf{SVec}$ is a super vector space, i.e.,

$$\begin{aligned} \text{Hom}(a_g \otimes b_h, c_{gh}) \\ = \text{Hom}(a_g \otimes b_h, c_{gh})_0 \oplus \text{Hom}(a_g \otimes b_h, c_{gh})_1, \end{aligned} \quad (16)$$

where $\text{Hom}(a_g \otimes b_h, c_{gh})_0$ is the grade-0 (parity-even) part, $\text{Hom}(a_g \otimes b_h, c_{gh})_1$ is the grade-1 (parity-odd) part. We define the so-called *fusion multiplicity* by

$$N_{c_{gh}}^{a_g b_h} := \dim \text{Hom}(a_g \otimes b_h, c_{gh}). \quad (17)$$

Different fusion states are labeled by $\alpha, \beta, \dots = 1, \dots, N_{c_{gh}}^{a_g b_h}$. In other words, basis in $\text{Hom}(a_g \otimes b_h, c_{gh})$ are $\{|\alpha\rangle | \alpha = 1, \dots, N_{c_{gh}}^{a_g b_h}\}$. Then we also define bosonic and fermionic fusion multiplicities, respectively:

$$\begin{aligned} B_{c_{gh}}^{a_g b_h} &:= \dim \text{Hom}(a_g \otimes b_h, c_{gh})_0, \\ F_{c_{gh}}^{a_g b_h} &:= \dim \text{Hom}(a_g \otimes b_h, c_{gh})_1, \end{aligned} \quad (18)$$

and obviously,

$$N_{c_{gh}}^{a_g b_h} = B_{c_{gh}}^{a_g b_h} + F_{c_{gh}}^{a_g b_h}. \quad (19)$$

Therefore, the string fusion rules are written as

$$a_g \otimes b_h = \bigoplus_{c_{gh} \in \mathcal{S}_{gh}} \mathbb{C}^{B_{c_{gh}}^{a_g b_h} | F_{c_{gh}}^{a_g b_h}} c_{gh}. \quad (20)$$

We define the *quantum dimension* d_{a_g} of a_g as the largest eigenvalue of matrix $\hat{\eta}_{a_g}$, where

$$\hat{\eta}_{a_g} := \left(\frac{N_{c_{gh}}^{a_g b_h}}{n_{c_{gh}}} : b_h, c_{gh} \in \mathcal{S}_G \right). \quad (21)$$

The Hilbert space of fusion spaces on vertices is

$$\mathcal{H}_V = \bigotimes_{v \in V} \mathcal{H}_v^\pm, \quad (22)$$

where V is the set of all vertices in the lattice, v denotes for each vertex, and

$$\begin{aligned} \mathcal{H}_v^+ &:= \bigoplus_{a_g, b_h, c_{gh} \in \mathcal{S}_G} \text{Hom}(a_g \otimes b_h, c_{gh}), \\ \mathcal{H}_v^- &:= \bigoplus_{a_g, b_h, c_{gh} \in \mathcal{S}_G} \text{Hom}(c_{gh}, a_g \otimes b_h). \end{aligned} \quad (23)$$

- (4) $|G|$ species of physical fermions on each vertex. Labeling the corresponding vertices of fusion states $\alpha, \beta, \dots$ by $\underline{\alpha}, \underline{\beta}, \dots$, the annihilation and creation operators of species- g fermion are denoted as $c_{\underline{\alpha}}^g$ and $c_{\underline{\alpha}}^{g\dagger}$. They satisfy the anticommutation relation:

$$c_{\underline{\alpha}}^g c_{\underline{\beta}}^h = -c_{\underline{\beta}}^h c_{\underline{\alpha}}^g, \quad c_{\underline{\alpha}}^g c_{\underline{\beta}}^{h\dagger} = -c_{\underline{\beta}}^{h\dagger} c_{\underline{\alpha}}^g, \quad (24)$$

for all $\underline{\alpha} \neq \underline{\beta}$, $g, h \in G$. A global symmetry transformation of g_0 acts on $c_{\underline{\alpha}}^g$ as

$$U(g_0) c_{\underline{\alpha}}^g U^\dagger(g_0) = (-1)^{\omega_2(g_0, g)} c_{\underline{\alpha}}^{g_0 g}. \quad (25)$$

We define a *fermion-parity function* $s(\alpha) = 0, 1$ to indicate the parity of a fusion state α is even or odd, i.e.,

$$s(\alpha) = \begin{cases} 0, & \text{if } |\alpha\rangle \in \text{Hom}(a_g \otimes b_h, c_{gh})_0 \\ 1, & \text{if } |\alpha\rangle \in \text{Hom}(a_g \otimes b_h, c_{gh})_1 \end{cases}. \quad (26)$$

Then we decorate fermions on all parity-odd fusion states by a sequence of fermion creation operators. When $s(\alpha) = 1$, we choose the convention that the decorated fermion species depends on the group element in the left-most plaquette, graphically represented as

$$\begin{array}{c} a_g \quad b_h \\ \searrow \quad \swarrow \\ \alpha \\ \downarrow \\ g_0 \\ \downarrow \\ c_{gh} \end{array} \quad (c_{\underline{\alpha}}^{g_0 g})^{s(\alpha)} \quad (27)$$

where the black dot denotes for the physical fermion when $s(\alpha) = 1$.

The fermionic Hilbert space is the following Fock space:

$$\mathcal{H}_f = \bigoplus_{S \subset (V \times G)} \prod_{(v,g) \in S} c_v^{g\dagger} |0\rangle, \quad (28)$$

where $|0\rangle$ is the state of no fermion, and S is any subset of the Cartesian product of set of all vertices V and the group G .

In conclusion, the total Hilbert space is

$$\mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_V \otimes \mathcal{H}_f, \quad (29)$$

where $\mathcal{H}_P$, $\mathcal{H}_V$, $\mathcal{H}_f$ are defined in Eqs. (10), (22), and (28), respectively.

Remark 2.1 A q -type string behaves like a Majorana-Kitaev chain in topological phases [45] (but more generalized in a way that q -type strings may not be $\mathbb{Z}_2$ conserved, while Kitaev chains are $\mathbb{Z}_2$ conserved as the fusion outcomes of two Kitaev chains is vacuum). Therefore, q -type strings have the following two properties: (i) Any q -type string configuration must be closed, or say, q -type string cannot terminate or emerge at any vertex in the lattice model. (ii) Fermions can be created or annihilated by pairs on q -type strings, and a fermion can “slide” along q -type strings freely without energy cost. Accordingly, when there is a q -type string involved in the fusion vertex in graph (15), i.e., at least one of a_g, b_h is q -type, we must have $B_{c_{gh}}^{a_g b_h} = F_{c_{gh}}^{a_g b_h}$. In other words, it is guaranteed that for any bosonic fusion state, there always exists a corresponding fermionic fusion state. Therefore, for a particular fusion state α in $\text{Hom}(a_g \otimes b_h, c_{gh})$, in such cases that there is a q -type string involved, we introduce the notation $\alpha \times f$ (intuitively understood as sliding a fermion f onto α) without any ambiguity to denote for the corresponding fermionic state if α is bosonic, and the corresponding bosonic state if α is fermionic. If we choose the convention that all bosonic fusion states are ordered before fermionic fusion states, i.e., $\alpha = 1, \dots, B_{c_{gh}}^{a_g b_h}, B_{c_{gh}}^{a_g b_h} + 1, \dots, 2B_{c_{gh}}^{a_g b_h}$, then $\alpha \times f = \alpha + B_{c_{gh}}^{a_g b_h}$.

Remark 2.2. The assignment of $|G|$ species of physical fermions, together with Eq. (25), is actually a generalization of the Kramers doublet. For example, let us consider a total symmetry $\mathbb{Z}_4^{Tf} = \mathbb{Z}_2^f \times_{\omega_2} \mathbb{Z}_2^T$, where $\mathbb{Z}_2^T = \{\mathbf{1}, T\}$ is the time-reversal symmetry and $\omega_2(T, T) = 1$. Denote the Kramers doublet as $c^\uparrow$ and $c^\downarrow$, the time-reversal symmetry acts as $U(T)c^\uparrow U(T)^\dagger = c^\downarrow$, $U(T)c^\downarrow U(T)^\dagger = -c^\uparrow$.

Definition 2.3. A q -type string a_g is further called σ -type or Ising type if it satisfies the $\mathbb{Z}_2$ -conservation law as q -type string on fusion:

$$a_g \otimes a_g = \bigoplus_{c_{gg} \in S_{gg}} \mathbb{C}^{n|n} c_{gg}, \quad (30)$$

where all c_{gg} are m -type strings and $n := B_{c_{gh}}^{a_g b_h} = F_{c_{gh}}^{a_g b_h}$.

D. Generalized fermionic symmetric local unitary transformation and fixed-point wave function

We define the following equivalence relation between two gapped fermionic quantum states:

$$|\Phi\rangle \sim |\Phi'\rangle \quad \text{iff} \quad |\Phi'\rangle = \mathcal{T}[e^{i \int d\lambda H_f(\lambda)}]|\Phi\rangle, \quad (31)$$

where the evolution does not close the energy gap, λ is the order parameter, and $\mathcal{T}$ is the path-ordering operator. $H_f(\lambda) = \sum_i \mathcal{O}_i(\lambda)$ is the fermionic Hamiltonian, where i denotes for each local region and $\mathcal{O}_i(\lambda)$ is a product of an even number of local fermionic operators and any number of local bosonic operators. Here $\mathcal{T}[e^{i \int d\lambda H_f(\lambda)}]$ is called a fermionic local unitary (fLU) transformation [47]. We note that a bosonic local unitary transformation is a special case of fLU transformation when there is no local fermionic operator involved.

However, requiring the transformation to be unitary is not general enough. We hereby define the so-called generalized fermionic local unitary (gflU) transformation [46,47]. It generalizes the concept of fLU transformation in a way that the transformation can be projective unitary. Denote a gflU transformation as $\tilde{U}$. $\tilde{U}$ is called *projective unitary* if

$$\tilde{U}^\dagger \tilde{U} = \tilde{P}, \quad \tilde{U} \tilde{U}^\dagger = \tilde{P}', \quad (32)$$

where $\tilde{P}$ and $\tilde{P}'$ are two projectors, i.e., $\tilde{P}^2 = \tilde{P}$ and $\tilde{P}'^2 = \tilde{P}'$.

A generalized fermionic symmetric local unitary (gfSLU) transformation is a gflU transformation that is invariant under any global symmetry action. In classifying 2 + 1D fSET phases, the wave function renormalizations are generated by gfSLU transformations. Then the equivalence classes of wave functions are defined up to gfSLU transformations, which leads to the concept of fixed-point wave function (as a characteristic of each equivalence class).

Recall that by a string-net model [55,56], we mean inputting a set of categorical data (for original string-net model, the input is a unitary fusion category; for the fermionic symmetry-enriched string-net model we define here, the mathematical definition of the input category is only partially known), a quantum lattice model with a full description on fixed-point ground state wave functions and Hamiltonian is outputted. Let $\Psi(-)$ denote for the fixed-point ground state wave function on a certain patch. The topological ground state (defined on a sphere) is a superposition of all possible string-net configurations:

$$|\Psi\rangle = \sum_{\text{all conf.}} \Psi \left(\begin{array}{c} \text{Diagram 1} \end{array} \right) \left| \begin{array}{c} \text{Diagram 2} \end{array} \right\rangle. \quad (33)$$

There exists a gfSLU transformation as a gauge transformation in our model:

$$\left\{ \begin{array}{l} \Psi \left(\begin{array}{c} a_g \quad b_h \\ \searrow \quad \swarrow \\ \alpha \quad \beta \\ \downarrow \\ g_0 \\ \downarrow \\ c_{gh} \end{array} \right) \rightarrow \sum_{\beta} [u_{c_{gh}}^{a_g b_h}]_{\beta}^{\alpha} \Psi \left(\begin{array}{c} a_g \quad b_h \\ \searrow \quad \swarrow \\ \beta \quad \alpha \\ \downarrow \\ g_0 \\ \downarrow \\ c_{gh} \end{array} \right), \text{ if } s(\alpha) = s(\beta) = 0 \\ \Psi \left(\begin{array}{c} a_g \quad b_h \\ \searrow \quad \swarrow \\ \alpha \quad \beta \\ \downarrow \\ g_0 \\ \downarrow \\ c_{gh} \end{array} \right) \rightarrow \sum_{\beta} [v_{c_{gh}}^{a_g b_h}]_{\beta}^{\alpha} \Psi \left(\begin{array}{c} a_g \quad b_h \\ \searrow \quad \swarrow \\ \beta \quad \alpha \\ \downarrow \\ g_0 \\ \downarrow \\ c_{gh} \end{array} \right), \text{ if } s(\alpha) = s(\beta) = 1 \end{array} \right\}, \quad (34)$$

where $u_{c_{gh}}^{a_g b_h}$ and $v_{c_{gh}}^{a_g b_h}$ are two unitary matrices. This corresponds to a unitary transformation in the super vector space $\text{Hom}(a_g \otimes b_h, c_{gh})$. When $s(\alpha) = s(\beta) = 0$, $u_{c_{gh}}^{a_g b_h}$ is a unitary basis transformation in $\text{Hom}(a_g \otimes b_h, c_{gh})_0$. When $s(\alpha) = s(\beta) = 1$, $v_{c_{gh}}^{a_g b_h}$ is a unitary basis transformation in $\text{Hom}(a_g \otimes b_h, c_{gh})_1$.

E. Renormalization moves

The first type of renormalization move, called the $\mathcal{F}$ move, is a gfSLU transformation defined as

$$\Psi \left(\begin{array}{c} a_g \quad b_h \quad c_k \\ \searrow \quad \swarrow \quad \downarrow \\ \alpha \quad \beta \quad \chi \\ \downarrow \\ m_{gh} \\ \downarrow \\ g_0 \\ \downarrow \\ d_{ghk} \end{array} \right) = \sum_{n_{hk} \chi \delta} g_0 [\mathcal{F}_{d_{ghk}}^{a_g b_h c_k}]_{n_{hk}, \chi \delta}^{m_{gh}, \alpha \beta} \Psi \left(\begin{array}{c} a_g \quad b_h \quad c_k \\ \searrow \quad \swarrow \quad \downarrow \\ \chi \quad \delta \quad n_{hk} \\ \downarrow \\ g_0 \\ \downarrow \\ d_{ghk} \end{array} \right), \quad (35)$$

where the $\mathcal{F}$ move can be more simply denoted as $g_0 [\mathcal{F}_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}$. It is a combination of fermion creation and annihilation operators and a bosonic-part F move (mathematically corresponds to the F symbol or associator in the input G -graded super fusion category):

$$g_0 [\mathcal{F}_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} = (c_{\underline{\alpha}}^{g_0 \dagger})^{s(\alpha)} (c_{\underline{\beta}}^{g_0 \dagger})^{s(\beta)} (c_{\underline{\delta}}^{g_0})^{s(\delta)} (c_{\underline{\chi}}^{g_0})^{s(\chi)} \cdot g_0 [F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}, \quad (36)$$

where a gfSLU transformation should preserve fermion parity, i.e.,

$$s(\alpha) + s(\beta) + s(\delta) + s(\chi) = 0 \pmod{2}. \quad (37)$$

Associativity of fusion of strings implies

$$\sum_{e_{gh}} \frac{N_{e_{gh}}^{a_g b_h} N_{d_{ghk}}^{e_{gh} c_k}}{n_{e_{gh}}} = \sum_{f_{hk}} \frac{N_{f_{hk}}^{b_h c_k} N_{d_{ghk}}^{a_g f_{hk}}}{n_{f_{hk}}}. \quad (38)$$

Since the fermion-parity-odd sector and the parity-even sector are independent (due to fermion-parity conservation), this relation can be further split as

$$\begin{aligned} & \sum_{e_{gh}} \frac{B_{e_{gh}}^{a_g b_h} B_{d_{ghk}}^{e_{gh} c_k} + F_{e_{gh}}^{a_g b_h} F_{d_{ghk}}^{e_{gh} c_k}}{n_{e_{gh}}} \\ &= \sum_{f_{hk}} \frac{B_{f_{hk}}^{b_h c_k} B_{d_{ghk}}^{a_g f_{hk}} + F_{f_{hk}}^{b_h c_k} F_{d_{ghk}}^{a_g f_{hk}}}{n_{f_{hk}}}, \end{aligned} \quad (39)$$

$$\begin{aligned} & \sum_{e_{gh}} \frac{B_{e_{gh}}^{a_g b_h} F_{d_{ghk}}^{e_{gh} c_k} + F_{e_{gh}}^{a_g b_h} B_{d_{ghk}}^{e_{gh} c_k}}{n_{e_{gh}}} \\ &= \sum_{f_{hk}} \frac{B_{f_{hk}}^{b_h c_k} F_{d_{ghk}}^{a_g f_{hk}} + F_{f_{hk}}^{b_h c_k} B_{d_{ghk}}^{a_g f_{hk}}}{n_{f_{hk}}}. \end{aligned} \quad (40)$$

Remark 2.4. There are other types of renormalization moves as gfSLU transformations: the $\mathcal{O}$ move and the $\mathcal{Y}$ move that change the number of vertices, and the dual $\mathcal{F}$

move, $\mathcal{H}$ move, and dual $\mathcal{H}$ move that are closely related to the $\mathcal{F}$ move. The definitions of these renormalization moves are listed in Appendix F.

Derived from Eq. (37), the quantum dimensions further satisfy the following equation:

$$d_{a_g} d_{b_h} = \sum_{c_{gh}} \frac{N_{c_{gh}}^{a_g b_h} d_{c_{gh}}}{n_{c_{gh}}} \quad \text{or} \quad \sum_{a_g b_h} \frac{N_{c_{gh}}^{a_g b_h} d_{a_g} d_{b_h}}{n_{a_g} n_{b_h}} = d_{c_{gh}} D^2, \quad (41)$$

where $D^2 = \sum_{\mathbf{g} \in G} D_{\mathbf{g}}^2 = \sum_{a_g \in S_G} ([d_{a_g}^2]/[n_{a_g}])$, and the above two equations are equivalent (i.e., we can derive either one from another). The proof of either of the above two equations is similar to that in Ref. [46]. We note that the condition on $\mathcal{O}$ -move in Eq. (F6) also recovers Eq. (41), which can be shown by substituting the expression in Eq. (F13) into Eq. (F6).

The symmetric condition on $\mathcal{F}$ move requires the following commutative diagram:

$$\begin{array}{ccc} \Psi \left(\begin{array}{c} a_g \quad b_h \quad c_k \\ \alpha \quad \beta \\ e_{gh} \quad \beta \\ 1 \end{array} \right) & \xrightarrow{1[\mathcal{F}_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}} & \Psi \left(\begin{array}{c} a_g \quad b_h \quad c_k \\ \chi \quad \delta \\ 1 \end{array} \right) \\ U(\mathbf{g}_0) \downarrow & & \downarrow U(\mathbf{g}_0) \\ \Psi \left(\begin{array}{c} a_g \quad b_h \quad c_k \\ \alpha \quad \beta \\ e_{gh} \quad \beta \\ \mathbf{g}_0 \end{array} \right) & \xrightarrow{\mathbf{g}_0[\mathcal{F}_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}} & \Psi \left(\begin{array}{c} a_g \quad b_h \quad c_k \\ \chi \quad \delta \\ \mathbf{g}_0 \end{array} \right) \end{array} \quad (42)$$

It implies

$$\begin{aligned} \mathbf{g}_0[\mathcal{F}_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} &= U(\mathbf{g}_0) 1[\mathcal{F}_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} U^\dagger(\mathbf{g}_0) = (-1)^{\omega_2(\mathbf{g}_0, \mathbf{g})s(\chi)} (c_{\underline{\alpha}}^{\mathbf{g}_0 \dagger})^{s(\alpha)} (c_{\underline{\beta}}^{\mathbf{g}_0 \dagger})^{s(\beta)} (c_{\underline{\delta}}^{\mathbf{g}_0})^{s(\delta)} \\ &\quad \times (c_{\underline{\chi}}^{\mathbf{g}_0 \mathbf{g}})^{s(\chi)} (1[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta})^{S(\mathbf{g}_0)}, \end{aligned} \quad (43)$$

where the bosonic-part F move satisfies

$$\mathbf{g}_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} = (-1)^{\omega_2(\mathbf{g}_0, \mathbf{g})s(\chi)} (1[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta})^{S(\mathbf{g}_0)}. \quad (44)$$

The fermionic pentagon equation is written as

$$\sum_{\epsilon} 1[\mathcal{F}_e^{j_{gh} c_k d_1}]_{q, \delta \epsilon}^{m, \beta \chi} 1[\mathcal{F}_e^{a_g b_h q k l}]_{p, \phi \gamma}^{j, \alpha \epsilon} = \sum_{n_{hk} \eta \psi \kappa} 1[\mathcal{F}_m^{a_g b_h c_k}]_{n, \eta \psi}^{j, \alpha \beta} 1[\mathcal{F}_e^{a_g n_{hk} d_1}]_{p, \kappa \gamma}^{m, \psi \chi} [\mathcal{F}_p^{b_h c_k d_1}]_{q, \delta \phi}^{n, \eta \kappa}. \quad (45)$$

Reordering and canceling the fermion creation and annihilation operators gives a sign factor $(-1)^{s(\alpha)s(\delta)}$ as shown in Ref. [47], and by Eq. (44),

$$\sum_{\epsilon} 1[F_e^{j_{gh} c_k d_1}]_{q, \delta \epsilon}^{m, \beta \chi} 1[F_e^{a_g b_h q k l}]_{p, \phi \gamma}^{j, \alpha \epsilon} = (-1)^{(s(\alpha) + \omega_2(\mathbf{g}, \mathbf{h}))s(\delta)} \sum_{n_{hk} \eta \psi \kappa} 1[F_m^{a_g b_h c_k}]_{n, \eta \psi}^{j, \alpha \beta} 1[F_e^{a_g n_{hk} d_1}]_{p, \kappa \gamma}^{m, \psi \chi} (1[F_p^{b_h c_k d_1}]_{q, \delta \phi}^{n, \eta \kappa})^{S(\mathbf{g})}. \quad (46)$$

Remark 2.5. If $s(\alpha) = \omega_2(\mathbf{g}, \mathbf{h})$, Eq. (46) reduces to the bosonic pentagon equation with vanishing sign factor.

Remark 2.6. The most general fermionic pentagon equation is

$$\sum_{\epsilon} \mathbf{g}_0[\mathcal{F}_e^{j_{gh} c_k d_1}]_{q, \delta \epsilon}^{m, \beta \chi} \mathbf{g}_0[\mathcal{F}_e^{a_g b_h q k l}]_{p, \phi \gamma}^{j, \alpha \epsilon} = \sum_{n_{hk} \eta \psi \kappa} \mathbf{g}_0[\mathcal{F}_m^{a_g b_h c_k}]_{n, \eta \psi}^{j, \alpha \beta} \mathbf{g}_0[\mathcal{F}_e^{a_g n_{hk} d_1}]_{p, \kappa \gamma}^{m, \psi \chi} \mathbf{g}_0[\mathcal{F}_p^{b_h c_k d_1}]_{q, \delta \phi}^{n, \eta \kappa}. \quad (47)$$

We can check that it reduces to Eq. (46) after canceling the fermion operators by Eq. (44), where the sign factor is

$$(-1)^{\omega_2(\mathbf{g}_0, \mathbf{g}, \mathbf{h})s(\delta) + \omega_2(\mathbf{g}_0, \mathbf{g})s(\phi) + \omega_2(\mathbf{g}_0, \mathbf{g})s(\eta) + \omega_2(\mathbf{g}_0, \mathbf{g})s(\kappa) + \omega_2(\mathbf{g}_0, \mathbf{g}, \mathbf{h})s(\delta) + s(\alpha)s(\delta)} = (-1)^{[s(\alpha) + \omega_2(\mathbf{g}, \mathbf{h})]s(\delta)}, \quad (48)$$

by $s(\phi) + s(\eta) + s(\kappa) = s(\delta) \pmod{2}$ and $\omega_2(\mathbf{g}_0, \mathbf{g}) + \omega_2(\mathbf{g}_0, \mathbf{g}, \mathbf{h}) + \omega_2(\mathbf{g}_0, \mathbf{g}, \mathbf{h}) = \omega_2(\mathbf{g}, \mathbf{h}) \pmod{2}$.

The $\mathcal{F}$ move as a gfSLU transformation is further required to satisfy the following two projective-unitary conditions [46] [recall Eq. (32)]:

$$\sum_{n_{\mathbf{h}\mathbf{k}}\chi\delta} \mathbf{g}_0 [F_d^{a_{\mathbf{g}}b_{\mathbf{h}}c_{\mathbf{k}}}]_{n,\chi\delta}^{m',\alpha'\beta'} (\mathbf{g}_0 [F_d^{a_{\mathbf{g}}b_{\mathbf{h}}c_{\mathbf{k}}}]_{n,\chi\delta}^{m,\alpha\beta})^* = \begin{cases} \delta_{m_{\mathbf{g}\mathbf{h}}m'_{\mathbf{g}\mathbf{h}}} \delta_{\alpha\alpha'} \delta_{\beta\beta'}, & \text{if } m_{\mathbf{g}\mathbf{h}} \text{ is } m\text{-type} \\ \frac{1}{2} [\delta_{m_{\mathbf{g}\mathbf{h}}m'_{\mathbf{g}\mathbf{h}}} \delta_{\alpha\alpha'} \delta_{\beta\beta'} + \mathbf{g}_0 \Xi_{c_{\mathbf{k}}d}^{a_{\mathbf{g}}b_{\mathbf{h}}m,\alpha\beta} \delta_{m_{\mathbf{g}\mathbf{h}}m'_{\mathbf{g}\mathbf{h}}} \delta_{(\alpha\times f)\alpha'} \delta_{(\beta\times f)\beta'}], & \text{if } m_{\mathbf{g}\mathbf{h}} \text{ is } q\text{-type} \end{cases}; \quad (49)$$

$$\sum_{m_{\mathbf{g}\mathbf{h}}\alpha\beta} (\mathbf{g}_0 [F_d^{a_{\mathbf{g}}b_{\mathbf{h}}c_{\mathbf{k}}}]_{n',\chi'\delta'}^{m,\alpha\beta})^* \mathbf{g}_0 [F_d^{a_{\mathbf{g}}b_{\mathbf{h}}c_{\mathbf{k}}}]_{n,\chi\delta}^{m,\alpha\beta} = \begin{cases} \delta_{n_{\mathbf{h}\mathbf{k}}n'_{\mathbf{h}\mathbf{k}}} \delta_{\chi\chi'} \delta_{\delta\delta'}, & \text{if } n_{\mathbf{h}\mathbf{k}} \text{ is } m\text{-type} \\ \frac{1}{2} [\delta_{n_{\mathbf{h}\mathbf{k}}n'_{\mathbf{h}\mathbf{k}}} \delta_{\chi\chi'} \delta_{\delta\delta'} + \mathbf{g}_0 \Xi_{c_{\mathbf{k}}dn,\chi\delta}^{a_{\mathbf{g}}b_{\mathbf{h}}m,\alpha\beta} \delta_{n_{\mathbf{h}\mathbf{k}}n'_{\mathbf{h}\mathbf{k}}} \delta_{(\chi\times f)\chi'} \delta_{(\delta\times f)\delta'}], & \text{if } n_{\mathbf{h}\mathbf{k}} \text{ is } q\text{-type} \end{cases}, \quad (50)$$

where the notations $\alpha \times f$ and $\beta \times f$ are introduced in Remark 2.1, which is well defined here as $m_{\mathbf{g}\mathbf{h}}$ is q -type for the case in the second line, and similarly for $\chi \times f$ and $\delta \times f$. Here $\mathbf{g}_0 \Xi_{c_{\mathbf{k}}d}^{a_{\mathbf{g}}b_{\mathbf{h}}m,\alpha\beta}$ and $\mathbf{g}_0 \Xi_{c_{\mathbf{k}}dn,\chi\delta}^{a_{\mathbf{g}}b_{\mathbf{h}}m,\alpha\beta}$ are two phase factors satisfying [46]

$$(\mathbf{g}_0 \Xi_{c_{\mathbf{k}}d}^{a_{\mathbf{g}}b_{\mathbf{h}}m,\alpha\beta})^* = \mathbf{g}_0 \Xi_{c_{\mathbf{k}}d}^{a_{\mathbf{g}}b_{\mathbf{h}}m,(\alpha\times f)(\beta\times f)}, \quad (\mathbf{g}_0 \Xi_{c_{\mathbf{k}}dn,\chi\delta}^{a_{\mathbf{g}}b_{\mathbf{h}}m,\alpha\beta})^* = \mathbf{g}_0 \Xi_{c_{\mathbf{k}}dn,(\chi\times f)(\delta\times f)}^{a_{\mathbf{g}}b_{\mathbf{h}}m,\alpha\beta}. \quad (51)$$

The two projective-unitary conditions of dual $\mathcal{H}$ move implies the following two projective-unitary conditions of $\mathcal{F}$ move (see Appendix F2):

$$\begin{aligned} & \sum_{f_{\mathbf{g}\mathbf{h}\mathbf{k}}\chi\delta} \frac{d_{f_{\mathbf{g}\mathbf{h}\mathbf{k}}}}{n_{f_{\mathbf{g}\mathbf{h}\mathbf{k}}}} \mathbf{g}_0 [F_f^{c_{\mathbf{g}}e_{\mathbf{h}}b_{\mathbf{k}}}]_{d,\beta'\chi'}^{a,\alpha'\delta} (\mathbf{g}_0 [F_f^{c_{\mathbf{g}}e_{\mathbf{h}}b_{\mathbf{k}}}]_{d,\beta\chi}^{a,\alpha\delta})^* \\ &= \frac{d_{a_{\mathbf{g}\mathbf{h}}} d_{d_{\mathbf{h}\mathbf{k}}} n_{e_{\mathbf{h}}}}{n_{a_{\mathbf{g}\mathbf{h}}} n_{d_{\mathbf{h}\mathbf{k}}} d_{e_{\mathbf{h}}}} \begin{cases} \delta_{e_{\mathbf{h}}e'_{\mathbf{h}}} \delta_{\alpha\alpha'} \delta_{\beta\beta'}, & \text{if } e_{\mathbf{h}} \text{ is } m\text{-type} \\ \frac{1}{2} (\delta_{e_{\mathbf{h}}e'_{\mathbf{h}}} \delta_{\alpha\alpha'} \delta_{\beta\beta'} + \mathbf{g}_0 \Xi_{tae_{\mathbf{h}},\eta\alpha}^{c_{\mathbf{g}}s} (\mathbf{g}_0 \Xi_{b_{\mathbf{k}}d}^{ste_{\mathbf{h}},\eta\beta})^* \delta_{e_{\mathbf{h}}e'_{\mathbf{h}}} \delta_{(\alpha\times f)\alpha'} \delta_{(\beta\times f)\beta'}), & \text{if } e_{\mathbf{h}} \text{ is } q\text{-type} \end{cases}, \quad (52) \end{aligned}$$

where $s_{\mathbf{g}'}$, $t_{\mathbf{g}'^{-1}\mathbf{h}}$ and the fusion state $\eta: s_{\mathbf{g}'} \otimes t_{\mathbf{g}'^{-1}\mathbf{h}} \rightarrow e_{\mathbf{h}}$ can be arbitrarily chosen; and

$$\begin{aligned} & \sum_{e_{\mathbf{h}}\alpha\beta} \frac{d_{e_{\mathbf{h}}}}{n_{e_{\mathbf{h}}}} (\mathbf{g}_0 [F_{f'}^{c_{\mathbf{g}}e_{\mathbf{h}}b_{\mathbf{k}}}]_{d,\beta\chi'}^{a,\alpha\delta})^* \mathbf{g}_0 [F_f^{c_{\mathbf{g}}e_{\mathbf{h}}b_{\mathbf{k}}}]_{d,\beta\chi}^{a,\alpha\delta} \\ &= \frac{d_{a_{\mathbf{g}\mathbf{h}}} d_{d_{\mathbf{h}\mathbf{k}}} n_{f_{\mathbf{g}\mathbf{h}\mathbf{k}}}}{n_{a_{\mathbf{g}\mathbf{h}}} n_{d_{\mathbf{h}\mathbf{k}}} d_{f_{\mathbf{g}\mathbf{h}\mathbf{k}}}} \begin{cases} \delta_{f_{\mathbf{g}\mathbf{h}\mathbf{k}}f'_{\mathbf{g}\mathbf{h}\mathbf{k}}} \delta_{\alpha\alpha'} \delta_{\beta\beta'}, & \text{if } f_{\mathbf{g}\mathbf{h}\mathbf{k}} \text{ is } m\text{-type} \\ \frac{1}{2} (\delta_{f_{\mathbf{g}\mathbf{h}\mathbf{k}}f'_{\mathbf{g}\mathbf{h}\mathbf{k}}} \delta_{\alpha\alpha'} \delta_{\beta\beta'} + \mathbf{g}_0 \Xi_{dtf,\delta\eta}^{sc_{\mathbf{g}}} (\mathbf{g}_0 \Xi_{b_{\mathbf{k}}tf,\chi\eta}^{sa})^* \delta_{f_{\mathbf{g}\mathbf{h}\mathbf{k}}f'_{\mathbf{g}\mathbf{h}\mathbf{k}}} \delta_{(\alpha\times f)\alpha'} \delta_{(\beta\times f)\beta'}), & \text{if } f_{\mathbf{g}\mathbf{h}\mathbf{k}} \text{ is } q\text{-type} \end{cases}, \quad (53) \end{aligned}$$

where $s_{\mathbf{g}'}$, $t_{\mathbf{g}'\mathbf{g}\mathbf{h}\mathbf{k}}$ and the fusion state $\eta: s_{\mathbf{g}'} \otimes t_{\mathbf{g}'\mathbf{g}\mathbf{h}\mathbf{k}} \rightarrow e_{\mathbf{h}}$ can be arbitrarily chosen. We note that the two projective-unitary conditions of $\mathcal{H}$ move are algebraically equivalent to Eqs. (52) and (53).

F. Parent Hamiltonian

The parent commuting-projector Hamiltonian of the fermionic symmetry-enriched string-net model defined above is

$$H = -\sum_{\nu} C_{\nu} - \sum_{\nu} A_{\nu} - \sum_l D_l - \sum_l Q_l - \sum_p B_p. \quad (54)$$

We introduce all the terms in the following:

- (i) C_{ν} ensures that the fermion number at each vertex ν equals to the grading of the fusion space (or say, the parity of the fusion state) at that vertex for the ground state [57]:

$$C_{\nu} \left| \begin{array}{c} a_{\mathbf{g}} \quad b_{\mathbf{h}} \\ \searrow \quad \swarrow \\ \mathbf{g}_0 \quad \alpha \\ \downarrow \\ c_{\mathbf{g}\mathbf{h}} \end{array} \right\rangle = \delta_{\langle N_f \rangle, s(\alpha)} \left| \begin{array}{c} a_{\mathbf{g}} \quad b_{\mathbf{h}} \\ \searrow \quad \swarrow \\ \mathbf{g}_0 \quad \alpha \\ \downarrow \\ c_{\mathbf{g}\mathbf{h}} \end{array} \right\rangle, \quad (55)$$

where N_f is the fermion number operator, and

$$\langle N_f \rangle := \left\langle \begin{array}{c} a_g \quad b_h \\ \alpha \\ g_0 \downarrow \\ c_{gh} \end{array} \middle| N_f \middle| \begin{array}{c} a_g \quad b_h \\ \alpha \\ g_0 \downarrow \\ c_{gh} \end{array} \right\rangle \in \{0, 1\}. \quad (56)$$

- (ii) A_ν ensures that the string fusion rules are obeyed at each vertex ν for ground state:

$$A_\nu \left| \begin{array}{c} a_g \quad b_h \\ \alpha \\ g_0 \downarrow \\ c_{gh} \end{array} \right\rangle = \begin{cases} \left| \begin{array}{c} a_g \quad b_h \\ \alpha \\ g_0 \downarrow \\ c_{gh} \end{array} \right\rangle, & \text{if } N_{c_{gh}}^{a_g b_h} > 0 \\ 0, & \text{if } N_{c_{gh}}^{a_g b_h} = 0 \end{cases} \quad (57)$$

- (iii) Q_l ensures that the G -grading structure at each link l for ground state:

$$Q_l \left| \begin{array}{c} a_g \\ g_0 \downarrow g_1 \end{array} \right\rangle = \delta_{g, g_0^{-1} g_1} \left| \begin{array}{c} a_g \\ g_0 \downarrow g_1 \end{array} \right\rangle. \quad (58)$$

- (iv) D_l ensures that when link l is decorated by a q -type string, there exist two equivalent fixed-point ground states by “sliding” fermions (recall Remark II.1), the form of which is similar to that in Ref. [46]. D_l is determined by the dimensions of endomorphism space n_{a_g} and the phase factor between all possible equivalent fixed-point states (two fixed-point states are called equivalent if they differ only by a phase factor) on the honeycomb lattice, which can be derived from $\mathcal{Y}$ moves, $\mathcal{H}$ moves, and dual $\mathcal{H}$ moves (these renormalization moves are introduced in Appendix F).
- (v) B_p ensures that the fixed-point ground state is invariant under fusing loops onto the edges of each plaquette p , written as

$$B_p = \frac{1}{\mathcal{D}^2} \sum_{\mathbf{g} \in G} |\mathbf{g}_p \mathbf{g}\rangle \langle \mathbf{g}_p| \sum_{s_g \in S_G} \frac{d_{s_g}}{n_{s_g}} B_p^{s_g}. \quad (59)$$

$B_p^{s_g}$ fuses a loop s_g on the edges ($\mathbf{g} = \mathbf{g}_0^{-1} \mathbf{g}_1$):

$$B_p^{s_g} \left| \begin{array}{c} g_6 \quad g_7 \\ \beta \quad \alpha \\ g_2 \quad g_1 \quad g_5 \\ \gamma \quad \lambda \quad \nu \\ g_3 \quad g_4 \quad \mu \end{array} \right\rangle = \left| \begin{array}{c} g_6 \quad g_7 \\ \beta \quad \alpha \\ g_2 \quad g_1 \quad g_5 \\ \gamma \quad \lambda \quad \nu \\ g_3 \quad g_4 \quad \mu \end{array} \right\rangle = \mathcal{M} \left| \begin{array}{c} g_6 \quad g_7 \\ \beta' \quad \alpha' \\ g_2 \quad g_1 \quad g_5 \\ \gamma' \quad \lambda' \quad \nu' \\ g_3 \quad g_4 \quad \mu' \end{array} \right\rangle, \quad (60)$$

where $\mathcal{M}$ is a sequence of $\mathcal{F}$ moves, $\mathcal{H}$ moves, dual $\mathcal{H}$ moves, and $\mathcal{O}$ moves, similarly to the construction in Ref. [12].

It is easy to check that all above terms are projectors, e.g., $A_\nu^2 = A_\nu$, and commute with each other, e.g., $A_\nu B_p = B_p A_\nu$.

Remark II.7. Once the following data in a fermionic symmetry-enriched string-net model is given: the symmetry group G^f , string types a_g , dimensions of endomorphism space n_{a_g} , fusion rules $N_{c_k}^{a_g b_h}, F_{c_k}^{a_g b_h}$ (which imply quantum dimensions d_{a_g}), bosonic-part F moves $g_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}$, in total denoted as

$$\{G^f, a_g, n_{a_g}, d_{a_g}, N_{c_k}^{a_g b_h}, F_{c_k}^{a_g b_h}, g_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}\}, \quad (61)$$

the form of the parent Hamiltonian is directly and uniquely determined [up to gauge transformations in Eq. (34)]:

- (i) Q_l is determined by G through Eq. (58);
- (ii) C_ν and A_ν are determined by fusion rules encoded in $N_{c_k}^{a_g b_h}, F_{c_k}^{a_g b_h}$ through Eqs. (55) and (57) respectively;
- (iii) D_l is determined by n_{a_g} , $\mathcal{Y}$ moves, $\mathcal{H}$ moves, dual $\mathcal{H}$ moves, where by Eqs. (F9), (F13), (F16), and (F19), the bosonic-part Y moves, H moves, dual H moves are all determined by $\{n_{a_g}, d_{a_g}, N_{c_k}^{a_g b_h}, g_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}\}$, while the Grassmann numbers are determined by $F_{c_k}^{a_g b_h}$;
- (iv) B_p is determined by $\mathcal{F}$ moves, $\mathcal{H}$ moves, dual $\mathcal{H}$ moves, $\mathcal{O}$ moves through Eq. (59), thus also determined by $\{n_{a_g}, d_{a_g}, N_{c_k}^{a_g b_h}, F_{c_k}^{a_g b_h}, g_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}\}$.

Therefore, in most of the examples in Sec. IIH, we will only list the set of data $\{G, a_g, n_{a_g}, d_{a_g}, N_{c_k}^{a_g b_h}, F_{c_k}^{a_g b_h}, g_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}\}$, which is complete, and the form of parent Hamiltonian written in renormalization moves can be

directly obtained, thus omitted. However, presenting the Hamiltonian in terms of expressible operators is in general complicated, which will only be shown in a simple example in Remark II.14.

G. Summary

Definition II.8. (Fermionic symmetry-enriched string-net model). Inputting a finite fermionic symmetry group $G^f = \mathbb{Z}_2^f \times_{\omega_2} G$ and a G -graded super fusion category $\mathcal{S}_G$, a fermionic symmetry-enriched string-net model is an exactly solvable model for an fSET phase, constructed on a honeycomb lattice, consists of

- (i) Degrees of freedom introduced in Sec. II C;
 - (ii) Renormalization moves on fixed-point ground states satisfying the symmetric condition, introduced in Sec. II E;
 - (iii) A parent commuting-projector Hamiltonian, introduced in Sec. II F,
- where the full data of fermionic symmetry-enriched string-net model is denoted as $\{G^f, a_g, n_{a_g}, d_{a_g}, N_{c_k}^{a_g b_h}, F_{c_k}^{a_g b_h}, g_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}\}$, satisfying the following consistency equations:

$$N_{c_{gh}}^{a_g b_h} = B_{c_{gh}}^{a_g b_h} + F_{c_{gh}}^{a_g b_h}, \quad (62)$$

$$\sum_{e_{gh}} \frac{N_{e_{gh}}^{a_g b_h} N_{d_{ghk}}^{e_{gh} c_k}}{n_{e_{gh}}} = \sum_{f_{hk}} \frac{N_{f_{hk}}^{b_h c_k} N_{d_{ghk}}^{a_g f_{hk}}}{n_{f_{hk}}}, \quad (63)$$

$$\begin{aligned} & \sum_{e_{gh}} \frac{B_{e_{gh}}^{a_g b_h} B_{d_{ghk}}^{e_{gh} c_k} + F_{e_{gh}}^{a_g b_h} F_{d_{ghk}}^{e_{gh} c_k}}{n_{e_{gh}}} \\ &= \sum_{f_{hk}} \frac{B_{f_{hk}}^{b_h c_k} B_{d_{ghk}}^{a_g f_{hk}} + F_{f_{hk}}^{b_h c_k} F_{d_{ghk}}^{a_g f_{hk}}}{n_{f_{hk}}}, \end{aligned} \quad (64)$$

$$d_{a_g} d_{b_h} = \sum_{c_{gh}} \frac{N_{c_{gh}}^{a_g b_h} d_{c_{gh}}}{n_{c_{gh}}}, \quad (65)$$

$$g_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} = (-1)^{\omega_2(g_0, g)s(\chi)} (1[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta})^{S(g_0)}, \quad (66)$$

$$s(\alpha) + s(\beta) + s(\delta) + s(\chi) = 0 \pmod{2}, \quad (67)$$

$$\begin{aligned} & \sum_e 1[F_e^{j_{gh} c_k d_1}]_{q, \delta \epsilon}^{m, \beta \chi} 1[F_e^{a_g b_h q_{k1}}]_{p, \phi \gamma}^{j, \alpha \epsilon} \\ &= (-1)^{(s(\alpha) + \omega_2(g, h))s(\delta)} \sum_{n_{hk} \eta \psi \kappa} 1[F_m^{a_g b_h c_k}]_{n, \eta \psi}^{j, \alpha \beta} 1[F_e^{a_g n_{hk} d_1}]_{p, \kappa \gamma}^{m, \psi \chi} \\ & \quad \times (1[F_p^{b_h c_k d_1}]_{q, \delta \phi}^{n, \eta \kappa})^{S(g)}, \end{aligned} \quad (68)$$

and the four projective-unitary conditions in Eqs. (49)–(53).

Remark II.9. When there is no nontrivial string type except from symmetry defects, i.e., $\mathcal{S}_1 = \{1\}$, the fermionic symmetry-enriched string-net model defined here reduces to the fixed-point wave function construction of $2 + 1$ D fSPT phases [8].

Remark II.10. Given a bosonic string-net model with input unitary fusion category $\mathcal{C}$, the anyons (or quasiparticle excitations) in the model are described by the Drinfeld center $\mathcal{Z}_1(\mathcal{C})$ of $\mathcal{C}$, which can be shown by calculating the so-called Q algebra or tube algebra [58] defined on the string-net model.

Remark II.11. Given a symmetry-enriched string-net model with input G -graded unitary fusion category $\mathcal{C}_G$, the anyons in the model are described by $\mathcal{Z}_1(\mathcal{C}_1)$ [12, 14], which is a unitary modular tensor or fusion category (UMTC).

Conjecture 1. Given a fermionic symmetry-enriched string-net model with input G -graded super fusion category $\mathcal{S}_G$, the quasiparticle or anyon excitations in the model are described by $\mathcal{Z}_1(\mathcal{S}_1)$, where $1 \in G$ is the trivial element, and $\mathcal{Z}_1(\mathcal{S}_1)$ is a super modular category.

Remark II.12. Physically, a fermionic symmetry-enriched string-net model can be viewed as constructed by the following two steps:

- (1) Constructing a fermionic string-net model [46, 47] with input $\mathcal{S}_G$, where the strings are labeled by simple objects $a_g, b_h, \dots \in \mathcal{S}_G$ ($a, b, \dots \in \mathcal{S}_1$ and $g, h, \dots \in G$). It corresponds to the exactly solvable model for the G -gauged topological phase, i.e., a fermionic topological order without any symmetry;
- (2) Ungauging G for the above string-net model by adding degrees of freedom labeled by G to each plaquette (strings $a_g, b_h, \dots$ become also symmetry defects), as shown in Graph (8), we obtain the fermionic symmetry-enriched string-net model.

Remark II.13. For any G -graded super fusion category, there is no anti-unitary symmetry structure [by anti-unitary symmetry structure, we mean that if $g_0 \in G$ is anti-unitary, the F move is complex conjugated under the symmetry action of g_0 , as shown in Eq. (44)], otherwise it will not be a well-defined fusion category. Nevertheless, though any G -graded super fusion category does not contain anti-unitary symmetry structure, we can add it in the constructed symmetry-enriched string-net model, which is a physical model that cannot be described by any fusion category.

H. Examples

Here we present three examples of fermionic symmetry-enriched string-net models. Recall Eq. (44), if ω_2 is nontrivial or G contains anti-unitary symmetry, the F moves $g_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}$ depend on the leftmost group element g_0 , otherwise they do not.

Convention 3. For the F move $g_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}$, if the fusion outcome of any G -graded string types is unique, it is

simplified as $g_0 F^{a_g b_h c_k}$. If the F move is independent of the group element g_0 in the leftmost plaquette, it is simplified as $[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}$.

1. Toric code topological order stacking with a physical fermion with $\mathbb{Z}_4^f$ symmetry

Let us consider the $\mathcal{Z}_1(\text{Vec}_{\mathbb{Z}_2}) \boxtimes \{1, f\}$ topological order with $\mathbb{Z}_4^f$ symmetry. Here $\mathcal{Z}_1(\text{Vec}_{\mathbb{Z}_2}) \boxtimes \{1, f\} = \{1, e, m, \psi\} \boxtimes \{1, f\}$ describes anyons of the toric code topological order stacking with a physical fermion f , where $\boxtimes$ is the Deligne tensor product, physically understood as stacking of topological phases. Denote group elements in the bosonic symmetry group as $G = \mathbb{Z}_2 = \{0, 1\}$. Then $\omega_2(\mathbf{g}, \mathbf{h}) = \mathbf{gh} \pmod{2}$. Given such a topological order and a fermionic global symmetry, there are several possibilities that how the symmetry interacts with the topological order, i.e., there are several possible fSET phases.

Let us further consider special cases that $s(\alpha) = \omega_2(\mathbf{g}, \mathbf{h}) = \mathbf{gh} \pmod{2}$ in the fermionic symmetry-enriched string-net models as stated in Remark II.5, i.e., Eq. (46) reduces to a bosonic pentagon equation. In classifying such fSET phases up to equivalence relations (i.e., gfsLU transformations), we divide these possible fSET phases into three classes [12]:

- (1) $\mathcal{S}_{\mathbb{Z}_2} = \text{Vec}_{\mathbb{Z}_2 \times \mathbb{Z}_2} = \mathcal{S}_0 \oplus \mathcal{S}_1 = \{1_0, e_0\} \oplus \{1_1, e_1\}$, where $1_1 \otimes 1_1 = e_1 \otimes e_1 = 1_0$. Since ${}^0F^{a_g b_h c_k}$ are the same as the bosonic solution in Ref. [12], we direct conclude that the classification of this subcase is $\mathbb{Z}_2 \times \mathbb{Z}_2$. The first $\mathbb{Z}_2$ classification corresponds to

$${}^0F^{e_0 e_0 e_0} = {}^1F^{e_0 e_0 e_0} = 1, \quad (69)$$

and

$${}^0F^{e_0 e_0 e_0} = {}^1F^{e_0 e_0 e_0} = -1. \quad (70)$$

The second $\mathbb{Z}_2$ classification corresponds to

$$\begin{aligned} ({}^0F^{1_1 1_1 1_1}, {}^0F^{e_1 e_1 e_1}) &= (1, -1), \\ ({}^1F^{1_1 1_1 1_1}, {}^1F^{e_1 e_1 e_1}) &= (-1, 1), \end{aligned} \quad (71)$$

and

$$\begin{aligned} ({}^0F^{1_1 1_1 1_1}, {}^0F^{e_1 e_1 e_1}) &= (-1, -1), \\ ({}^1F^{1_1 1_1 1_1}, {}^1F^{e_1 e_1 e_1}) &= (1, 1). \end{aligned} \quad (72)$$

For the second $\mathbb{Z}_2$ classification, the first solution corresponds to a trivial fSET phase, i.e., the $\mathcal{Z}_1(\text{Vec}_{\mathbb{Z}_2}) \boxtimes \{1, f\}$ topological order stacking with a $\mathbb{Z}_2$ SPT phase, where the fermion f carries half $\mathbb{Z}_2$ charge, i.e.,

$$U_f(1)U_f(1) = -U_f(0), \quad (73)$$

where $U_a(\mathbf{g})$ is the localized symmetry operator of $\mathbf{g}$ near anyon a defined in Appendix B 2 [please do not mix it up with the global symmetry operator $U(\mathbf{g})$]. The second solution corresponds to a fSET phase where anyon e (or m) and f carry half $\mathbb{Z}_2$ charge.

- (2) $\mathcal{S}_{\mathbb{Z}_2} = \text{Vec}_{\mathbb{Z}_4} = \mathcal{S}_0 \oplus \mathcal{S}_1 = \{1_0, e_0\} \oplus \{1_1, e_1\}$, where $1_1 \otimes 1_1 = e_1 \otimes e_1 = e_0$. There are two inequivalent solutions of F moves:

$${}^0F^{abc} = 1, \quad {}^1F^{abc} = (-1)^{abc}, \quad (74)$$

and

$$\begin{aligned} {}^0F^{abc} &= e^{\frac{i\pi}{4}a(b+c-[b+c]_4)}, \\ {}^1F^{abc} &= (-1)^{abc} e^{\frac{i\pi}{4}a(b+c-[b+c]_4)}, \end{aligned} \quad (75)$$

where we relabel $0 := 1_0$, $2 := e_0$, $1 := 1_1$, $3 := e_1$, and $[b+c]_4 := b+c \pmod{4}$. The first solution corresponds to a fSET phase where anyon e (or m) and f carry half $\mathbb{Z}_2$ charge. The second solution corresponds to a fSET phase where all e , m and f carry half $\mathbb{Z}_2$ charge.

- (3) $\mathcal{S}_{\mathbb{Z}_2} = \text{Ising} = \mathcal{S}_0 \oplus \mathcal{S}_1 = \{1, e\} \oplus \{\sigma\}$, where $\sigma \otimes \sigma = 1 \oplus e$. There are two inequivalent solutions of F moves (distinguished by ± 1 in ${}^0[F_{\sigma}^{\sigma\sigma\sigma}]_b^a$):

$$\begin{aligned} {}^0[F_{\sigma}^{\sigma\sigma\sigma}]_{\sigma}^{\sigma} &= {}^0[F_{\sigma}^{\sigma\sigma\sigma}]_{\sigma}^{\sigma} = {}^1[F_{\sigma}^{\sigma\sigma\sigma}]_{\sigma}^{\sigma} = {}^1[F_{\sigma}^{\sigma\sigma\sigma}]_{\sigma}^{\sigma} = -1, \\ {}^0[F_{\sigma}^{\sigma\sigma\sigma}]_b^a &= \frac{\pm 1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \\ {}^1[F_{\sigma}^{\sigma\sigma\sigma}]_b^a &= -{}^0[F_{\sigma}^{\sigma\sigma\sigma}]_b^a, \end{aligned} \quad (76)$$

where $a, b = 1, e$. Both solutions correspond to fSET phases where the $\mathbb{Z}_2$ symmetry permutes e and m , and f carries half $\mathbb{Z}_2$ charge.

Remark II.14. We present the parent Hamiltonian explicitly in expressible operators for the case $\mathcal{S}_{\mathbb{Z}_2} = \text{Vec}_{\mathbb{Z}_4} = \{1_0, e_0\} \oplus \{1_1, e_1\} \equiv \{0, 2\} \oplus \{1, 3\}$ on a honeycomb lattice in terms of Pauli operators and the following defined operators [12]:

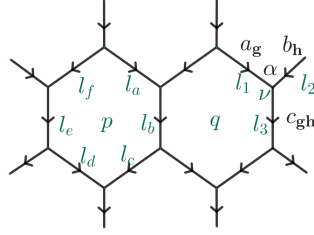
$$U_l|n\rangle_l = i^n|n\rangle_l, \quad V_l|n\rangle_l = |n+1\rangle_l, \quad \text{where } n \in \mathbb{Z}_4, \quad (77)$$

and $|n\rangle_l$ denotes for the state of string type n on link l . Since in this example all strings are m -type, the D_l term vanishes. The Hamiltonian is given by

$$H = -\sum_{\nu} C_{\nu} - \sum_{\nu} A_{\nu} - \sum_l Q_l - \sum_p B_p, \quad (78)$$

where we label the three links around each vertex ν as l_1, l_2, l_3 , and the six links of each plaquette p as $l_a, l_b, l_c, l_d, l_e, l_f$, as shown in the following graph (the lattice vertex, link,

plaquette labels, differ from the state labels, are all colored in green):


(79)

Then all the terms are written explicitly as

$$C_\nu = \frac{1 + P_f(-1)^{\frac{1-U_l^2}{2} \frac{1-U_{l_2}^2}{2}}}{2}, \quad (80)$$

$$A_\nu = U_{l_1}^\dagger U_{l_2}^\dagger U_{l_3} + (U_{l_1}^\dagger U_{l_2}^\dagger U_{l_3})^\dagger, \quad (81)$$

$$Q_l = \frac{1 + U_l^2}{2} \frac{1 + \sigma_p^z \sigma_q^z}{2} + \frac{1 - U_l^2}{2} \frac{1 - \sigma_p^z \sigma_q^z}{2}, \quad (82)$$

$$B_p = \frac{1}{4} \sum_{n=0}^3 \left[\sigma_p^x \left(V_{l_a} V_{l_b} V_{l_c} V_{l_d}^\dagger V_{l_e}^\dagger V_{l_f}^\dagger + (V_{l_a} V_{l_b} V_{l_c} V_{l_d}^\dagger V_{l_e}^\dagger V_{l_f}^\dagger)^\dagger \right) \right]^n, \quad (83)$$

where $P_f = (-1)^{N_f}$ is the fermion-parity operator acting on each vertex ν , the fermion-parity function on each vertex ν is $s(\alpha) = \omega_2(\mathbf{g}, \mathbf{h}) = \mathbf{gh} \pmod{2} = \{[(1 - U_{l_1}^2)/2][(1 - U_{l_2}^2)/2]\}$, and σ_p^z and σ_p^x are Pauli matrices acting on plaquette state $|\mathbf{g}\rangle$ ($\mathbf{g} \in \mathbb{Z}_2$) for each plaquette p . Note that the B_p term can also be written as a sequence of renormalization moves. We can check that this Hamiltonian is invariant under the global $\mathbb{Z}_2$ symmetry operator $\prod_p \sigma_p^x$ and the fermion-parity operator $P_f = (-1)^{N_f}$ acting on all the vertices.

2. Super Tambara-Yamagami category as input with $\mathbb{Z}_2^f \times \mathbb{Z}_2$ symmetry

Let us consider the fermionic symmetry-enriched string-net with input G -graded super fusion category being the super Tambara-Yamagami category (as a $\mathbb{Z}_2$ -graded super fusion category) defined in Example IV.3:

$$\mathcal{S}_{\mathbb{Z}_2} = \mathcal{S}_0 \oplus \mathcal{S}_1 = \{0, \dots, N-1\} \oplus \{\sigma\}, \quad (84)$$

where N is an odd integer, and σ is a q -type string (and further an σ -type string). Let $a, b \in \{0, \dots, N-1\}$. Quantum dimensions are $d_a = 1, d_\sigma = \sqrt{2N}$. Fusion rules are

$$a \otimes b = \begin{cases} \mathbb{C}^{1|0}[a+b]_N, & \text{if } a+b < N \\ \mathbb{C}^{0|1}[a+b]_N, & \text{if } a+b \geq N \end{cases},$$

$$\sigma \otimes a = a \otimes \sigma = \mathbb{C}^{1|1}\sigma, \quad \sigma \otimes \sigma = \bigoplus_a \mathbb{C}^{1|1}a, \quad (85)$$

where $[a+b]_N := a+b \pmod{N}$. Recall Convention 3, the F moves have no difference from those in super Tambara-Yamagami category, which are listed in Ref. [46]. This constructs the exactly solvable model of the following fSET phase: the $\mathcal{Z}_1(\mathcal{S}_0)$ topological order with $\mathbb{Z}_2^f \times \mathbb{Z}_2$ symmetry, where $\mathcal{Z}_1(\mathcal{S}_0)$ is the super modular category that describes the anyon excitations in this model.

3. $\text{SvecSU}(2)_6$ as input with $\mathbb{Z}_2^f \times \mathbb{Z}_2$ symmetry

Let us consider the fermionic symmetry-enriched string-net with input G -graded super fusion category being $\text{SvecSU}(2)_6$ (as a $\mathbb{Z}_2$ -graded super fusion category) defined in Example IV.4:

$$\mathcal{S}_{\mathbb{Z}_2} = \mathcal{S}_0 \oplus \mathcal{S}_1 = \{0, 1\} \oplus \left\{ \frac{1}{2}, \frac{3}{2} \right\}, \quad (86)$$

where $\frac{3}{2}$ is a q -type string (or further a σ -type string) and all other string types are m -type. Quantum dimensions are $d_0 = 1, d_1 = 1 + \sqrt{2}, d_{\frac{1}{2}} = \sqrt{2 + \sqrt{2}}, d_{\frac{3}{2}} = \sqrt{2}\sqrt{2 + \sqrt{2}}$, which satisfy Eq. (14). Fusion rules are

$$\begin{aligned}
1 \otimes 1 &= 0 \oplus \mathbb{C}^{1|1} 1, & \frac{1}{2} \otimes \frac{1}{2} &= 0 \oplus 1, \\
\frac{3}{2} \otimes \frac{3}{2} &= \mathbb{C}^{1|1} 0 \oplus \mathbb{C}^{1|1} 1, & \frac{1}{2} \otimes \frac{3}{2} &= \frac{3}{2} \otimes \frac{1}{2} = \mathbb{C}^{1|1} 1,
\end{aligned} \quad (87)$$

where for simplicity $\mathbb{C}^{1|0}$ is always omitted. Recall Convention 3, the F moves have no difference from those in $\text{SVecSU}(2)_6 \equiv \text{SU}(2)_6/f$, which can be calculated by the fermion condensation algorithm introduced in Ref. [45] or [46], and F moves in $\text{SU}(2)_6$ are listed in Ref. [59]. The calculation is omitted. (The F moves in $\mathcal{S}_0 = \text{SVecSO}(3)_6 \equiv \text{SO}(3)_6/f$ are listed in Ref. [46].) This constructs the exactly solvable model of the following fSET phase: the $\mathcal{Z}_1(\mathcal{S}_0) \cong \mathcal{Z}_1(\text{SVecSO}(3)_6)$ topological order with $\mathbb{Z}_2^f \times \mathbb{Z}_2$ symmetry, where $\mathcal{Z}_1(\text{SVecSO}(3)_6)$ describes the anyon excitations in this model.

Remark II.15. As calculated in Ref. [45] by tube algebra defined on fermionic string-net models, the anyon excitations $\mathcal{Z}_1[\text{SVecSO}(3)_6] = \{m_{00}, m_{02}, m_{20}, m_{22}\}$ (the notations are from Ref. [45]). Performing a $\mathbb{Z}_2^f$ -crossed extension $\mathcal{Z}_1[\text{SVecSO}(3)_6] \cong \text{SVec} \mathcal{Z}_1[\text{SO}(3)_6]_0$ (details

please refer to Sec. IV B and $\mathcal{Z}_1[\text{SO}(3)_6]$ is a spin modular category), we obtain

$$\begin{aligned}
\{\text{SVec} \mathcal{Z}_1[\text{SO}(3)_6]_0\}_{\mathbb{Z}_2^f}^\times &\cong \text{SVec} \mathcal{Z}_1[\text{SO}(3)_6]_0 \\
&\oplus \text{SVec} \mathcal{Z}_1[\text{SO}(3)_6]_1, \quad (88)
\end{aligned}$$

where $\text{SVec} \mathcal{Z}_1[\text{SO}(3)_6]_1 = \{m_{11}, q_{13}, q_{31}, m_{33}\}$ is understood as the sector of fermion-parity vortices.

By Remark IV.7, the ground state degeneracy (GSD) on torus $\mathbb{T}^2$ for four different spin structures are

$$\text{GSD}(\mathbb{T}_{BB}^2) = \text{GSD}(\mathbb{T}_{BN}^2) = \text{GSD}(\mathbb{T}_{NB}^2) = \text{GSD}(\mathbb{T}_{NN}^2) = 4, \quad (89)$$

where B denotes for bounding (anti-periodic) and N for nonbounding (periodic). Denote the ground state Hilbert spaces on torus with the four different spin structures as $A(\mathbb{T}_{BB}^2), A(\mathbb{T}_{BN}^2), A(\mathbb{T}_{NB}^2), A(\mathbb{T}_{NN}^2)$, respectively, which are all four-dimensional. The modular S, T transformations on them have the form

$$\begin{pmatrix} A(\mathbb{T}_{BB}^2) \\ A(\mathbb{T}_{BN}^2) \\ A(\mathbb{T}_{NB}^2) \\ A(\mathbb{T}_{NN}^2) \end{pmatrix} \xrightarrow{S} \begin{pmatrix} S^{BB \rightarrow BB} & 0 & 0 & 0 \\ 0 & 0 & S^{NB \rightarrow BN} & 0 \\ 0 & S^{BN \rightarrow NB} & 0 & 0 \\ 0 & 0 & 0 & S^{NN \rightarrow NN} \end{pmatrix} \begin{pmatrix} A(\mathbb{T}_{BB}^2) \\ A(\mathbb{T}_{BN}^2) \\ A(\mathbb{T}_{NB}^2) \\ A(\mathbb{T}_{NN}^2) \end{pmatrix}, \quad (90)$$

$$\begin{pmatrix} A(\mathbb{T}_{BB}^2) \\ A(\mathbb{T}_{BN}^2) \\ A(\mathbb{T}_{NB}^2) \\ A(\mathbb{T}_{NN}^2) \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 0 & T^{BN \rightarrow BB} & 0 & 0 \\ T^{BB \rightarrow BN} & 0 & 0 & 0 \\ 0 & 0 & T^{NB \rightarrow NB} & 0 \\ 0 & 0 & 0 & T^{NN \rightarrow NN} \end{pmatrix} \begin{pmatrix} A(\mathbb{T}_{BB}^2) \\ A(\mathbb{T}_{BN}^2) \\ A(\mathbb{T}_{NB}^2) \\ A(\mathbb{T}_{NN}^2) \end{pmatrix}, \quad (91)$$

where all the 4×4 matrices $S^{BB \rightarrow BB}, S^{NB \rightarrow BN}, \dots, T^{NN \rightarrow NN}$ are calculated and listed in Ref. [45].

I. fSET phase obtained by partially gauging an fSPT phase

Let $\mathcal{G}$ be a finite group and N be an Abelian normal subgroup of $\mathcal{G}$. Given a $2+1\text{D}$ fSPT phase with $\mathbb{Z}_2^f \times \mathcal{G}$ symmetry (i.e., ω_2 is trivial), we can partially gauge symmetry N and obtain a fermionic N gauge theory with symmetry $\mathbb{Z}_2^f \times G$. This corresponds to the following group extension written in a short exact sequence:

$$1 \rightarrow N \rightarrow \mathcal{G} \rightarrow G \rightarrow 1. \quad (92)$$

Let $a, b, \dots \in N, \mathbf{g}, \mathbf{h}, \dots \in G$ and $a_{\mathbf{g}}, b_{\mathbf{h}}, \dots \in \mathcal{G}$. The group extension (in general may not be central extension) is specified by two pieces of data:

- (i) A group homomorphism $\varphi: G \rightarrow \text{Aut}(N)$, where $\text{Aut}(N)$ is the automorphism group of N .
- (ii) A cohomology class $[\mu] \in H_{\varphi}^2(G, N)$, where the 2-cocycle μ satisfies the following twisted cocycle condition:

$$\varphi_{\mathbf{g}}[\mu(\mathbf{h}, \mathbf{k})]\mu(\mathbf{g}, \mathbf{h}\mathbf{k}) = \mu(\mathbf{g}, \mathbf{h})\mu(\mathbf{g}\mathbf{h}, \mathbf{k}). \quad (93)$$

Then the group multiplication rule in $\mathcal{G}$ is

$$a_{\mathbf{g}} \times b_{\mathbf{h}} = [a\varphi_{\mathbf{g}}(b)\mu(\mathbf{g}, \mathbf{h})]_{\mathbf{g}\mathbf{h}}, \quad (94)$$

Remark II.16. The partially gauging procedure introduced above cannot be directly generalized to cases where N is non-Abelian, as the 2-cocycle $\mu \in Z^2_\varphi(G, N)$ is only well defined when N is Abelian. Therefore, fSET phases obtained by such partially gauged fSPT phases only cover the simple cases that the intrinsic topological order of an fSET phase is described by an Abelian gauge theory. Nevertheless, our fermionic symmetry-enriched string-net model framework describes general nonchiral fSET phases.

Remark II.17. In $2 + 1$ D, the relations among $\mathcal{G} \times \mathbb{Z}_2^f$ -fSPT phase, the fSET phase obtained by partially gauging N for the $\mathcal{G} \times \mathbb{Z}_2^f$ -fSPT phase (which is a N gauge theory with $G \times \mathbb{Z}_2^f$ symmetry), the fermionic topological order (fTO) obtained by fully gauging G for the $\mathcal{G} \times \mathbb{Z}_2^f$ -fSPT phase (which is a fermionic $\mathcal{G}$ gauge theory [50]), and their bosonic counterparts related by fermion condensation (or inversely, gauging $\mathbb{Z}_2^f$) are summarized in the following commutative diagram:

$$\begin{array}{ccc}
 \text{SET phase: } \mathbb{Z}_2 \text{ gauge theory} & \xrightleftharpoons[\text{Gauging } \mathbb{Z}_2^f]{\text{F.c.}} & \mathcal{G} \times \mathbb{Z}_2^f \text{ - fSPT phase} \\
 \text{with } \mathcal{G} \text{ symmetry} & & \\
 \downarrow \text{Partially gauging } N & & \downarrow \text{Partially gauging } N \\
 \text{SET phase: } N \times \mathbb{Z}_2 \text{ gauge theory} & \xrightleftharpoons[\text{Gauging } \mathbb{Z}_2^f]{\text{F.c.}} & \text{fSET phase: } N \text{ gauge theory} \\
 \text{with } G \text{ symmetry} & & \text{with } G \times \mathbb{Z}_2^f \text{ symmetry} \\
 \downarrow \text{Gauging } G & & \downarrow \text{Gauging } G \\
 \text{TO: } \mathcal{G} \times \mathbb{Z}_2 \text{ gauge theory} & \xrightleftharpoons[\text{Gauging } \mathbb{Z}_2^f]{\text{F.c.}} & \text{fTO: fermionic } \mathcal{G} \text{ gauge theory}
 \end{array} \quad (95)$$

where f.c. is short for fermion condensation.

Let us briefly review the classification of $2 + 1$ D fSPT phases (without $p + ip$ superconductor layer) with $\mathbb{Z}_2^f \times \mathcal{G}$ symmetry, and $\mathcal{G}$ does not contain time-reversal symmetry (as we do not know how to gauging time-reversal symmetry yet). The classification is specified by a triplet (n_1, n_2, ν_3) [7,8], where $n_1 \in H^1(\mathcal{G}, \mathbb{Z}_2)$ is the 1-cocycle specifying Kitaev chain decoration, $n_2 \in H^2(\mathcal{G}, \mathbb{Z}_2)$ the 2-cocycle specifying complex fermion decoration, and $\nu_3 \in C^3[\mathcal{G}, U(1)]$ is the 3-cochain describing a $2 + 1$ D bosonic 2-2 move [7], satisfying the following consistency equations:

$$dn_1 = 0 \pmod{2}, \quad (96)$$

$$dn_2 = 0 \pmod{2}, \quad (97)$$

(ii) Fusion rules [see Graph (15)]:

$$a_{\mathbf{g}} \otimes b_{\mathbf{h}} = \begin{cases} \mathbb{C}^{1|0}[a\varphi_{\mathbf{g}}(b)\mu(\mathbf{g}, \mathbf{h})]_{\mathbf{gh}}, & \text{if } a_{\mathbf{g}}, b_{\mathbf{h}} \text{ are all m-type and } n_2(a_{\mathbf{g}}, b_{\mathbf{h}}) = 0 \\ \mathbb{C}^{0|1}[a\varphi_{\mathbf{g}}(b)\mu(\mathbf{g}, \mathbf{h})]_{\mathbf{gh}}, & \text{if } a_{\mathbf{g}}, b_{\mathbf{h}} \text{ are all m-type and } n_2(a_{\mathbf{g}}, b_{\mathbf{h}}) = 1. \\ \mathbb{C}^{1|1}[a\varphi_{\mathbf{g}}(b)\mu(\mathbf{g}, \mathbf{h})]_{\mathbf{gh}}, & \text{if at least one of } a_{\mathbf{g}} \text{ and } b_{\mathbf{h}} \text{ is } \sigma\text{-type} \end{cases} \quad (100)$$

Due to this constrained fusion rule, quantum dimension of an σ -type string is $\sqrt{2}$, and all other string types are m -type with quantum dimension 1.

$$d\nu_3 = (-1)^{n_2 \sim n_2}. \quad (98)$$

Then the full data in the fermionic symmetry-enriched string-net model obtained by partially gauging can be directly read from the data of the original fSPT phase:

- (i) String types: $a_{\mathbf{g}}, b_{\mathbf{h}}, \dots \in \mathcal{S}_G := \mathcal{G}$, which means that the set of isomorphism classes of simple objects in $\mathcal{S}_G$ equals to the set of group elements in $\mathcal{G}$ (recall Convention 1). A Kitaev chain becomes an σ -type string (Definition II.3) after the partial gauging, i.e., whether a string type $a_{\mathbf{g}}$ is a σ type or not is specified by

$$n_{a_{\mathbf{g}}} = n_1(a_{\mathbf{g}}) + 1. \quad (99)$$

- (iii) The bosonic-part F move: (i) If all string types in $\mathcal{G}$ are m -type, or say, n_1 is always trivial in the original $\mathcal{G} \times \mathbb{Z}_2^f$ -fSPT phase, we have

$$1F^{a_g b_h c_k} = \nu_3(a_g, b_h, c_k), \quad (101)$$

where the fusion outcome of any strings is always unique and therefore we omit all indexes related to

multiple fusion outcomes and simplify it as $1F^{a_g b_h c_k}$. (ii) If there exist σ -type strings in $\mathcal{G}$, the F move is obtained from the fSPT phase data $\{n_1, n_2, \nu_3\}$ by first partially gauging $\mathbb{Z}_2^f \times N$ and then performing a fermion condensation (F moves after fermion condensation can be explicitly calculated by the algorithm illustrated in Refs. [45,46]), as shown in the following commutative diagram:

$$\begin{array}{ccc}
 & \mathbb{Z}_2^f \times \mathcal{G} - \text{fSPT phase} \\
 & \text{(with Kitaev-chain layer)} \\
 & \text{described by } \{n_1, n_2, \nu_3\} \\
 & \downarrow \text{Partially gauging } N \\
 \text{Symmetry-enriched string model} & \xleftrightarrow[\text{Gauging } \mathbb{Z}_2^f]{\text{F.c.}} & \text{Fermionic symmetry-enriched string model} \\
 \text{with input } \mathcal{C}_G & & \text{with input } \mathcal{S}_G := \text{SVec} \mathcal{C}_G
 \end{array}, \quad (102)$$

which is part of Diagram (95). In both cases, the fermionic pentagon equation in Eq. (46) reduces exactly to Eq. (98):

$$1F^{(a_g \times b_h) c_k d_l} \cdot 1F^{a_g b_h (c_k \times d_l)} = (-1)^{n_2(a_g, b_h) n_2(c_k, d_l)} \cdot 1F^{a_g b_h c_k} \cdot 1F^{a_g (b_h \times c_k) d_l} \cdot (1F^{b_h c_k d_l})^{S(g)}. \quad (103)$$

Remark II.18. Only a subset of fSET phases can be obtained by partially gauging 2 + 1D fSPT phases. There exist a large class of fSET phases that cannot be realized by partially gauging.

Remark II.19. To help understanding the gauging of 2 + 1D fSPT phases, we give a simple fully gauged example. Let us consider 2 + 1D fSPT phases with $\mathbb{Z}_2^f \times \mathbb{Z}_2$ symmetry, the classification of which is $\mathbb{Z}_8$ [60], denoted by $\nu = 0, \dots, 7$. Gauging the $\mathbb{Z}_2$ symmetry for the $\nu = 2$ phase, we obtain the fermionic toric code topological order, which can be realized by a fermionic string-net model (Example A.10). Gauging the $\mathbb{Z}_2$ symmetry for the $\nu = 1$ phase, we obtain the Majorana toric code topological order. Bosonic-part F moves in the fermionic string-net model of Majorana toric code can be obtained by firstly gauging $\mathbb{Z}_2^f \times \mathbb{Z}_2$ and then performing a fermion condensation, shown in the following commuting diagram:

$$\begin{array}{ccc}
 & \mathbb{Z}_2^f \times \mathbb{Z}_2 - \text{fSPT phase } (\nu = 1) \\
 & \downarrow \text{Gauging } \mathbb{Z}_2 \\
 \text{Ising topological order} & \xleftrightarrow[\text{Gauging } \mathbb{Z}_2^f]{\text{F.c.}} & \text{Majorana toric code topological order}
 \end{array}. \quad (104)$$

We recall that the Ising topological order can be realized by a string-net model inputted by the Ising fusion category $\mathbf{Ising} = \{1, f, \sigma\}$, where $\sigma \otimes \sigma = 1 \oplus f$. The Majorana toric code can be realized by a fermionic string-net model inputted by the super fusion category $\text{SVec} \mathbf{Ising} = \{1, \sigma\}$ (Example A.11), F -moves of which are listed in Ref. [46].

Remark II.20. Consider a 2 + 1D $\mathcal{G} \times \mathbb{Z}_2^f$ -fSPT phase without Kitaev chain layer, i.e., it is described by $\{\nu_3, n_2\}$. It is known that gauging the $\mathbb{Z}_2^f$ symmetry for any of such a $\mathcal{G} \times \mathbb{Z}_2^f$ -fSPT phase gives a SET phase: a toric code with $\mathcal{G}$ symmetry, called a bosonic “shadow” model. This shadow model can also be obtained by gauging the $\mathbb{Z}_2$ symmetry for an auxiliary bosonic SPT phase with symmetry $\tilde{\mathcal{G}}$, where

$\tilde{\mathcal{G}}$ is a central extension of $\mathcal{G}$ over $\mathbb{Z}_2$ determined by the 2-cocycle n_2 [61,62].

III. SURFACE FERMIONIC SYMMETRY-ENRICHED STRING-NET MODELS

It is well known that the boundaries of 2 + 1D SPT phases are either gapless or symmetry broken [63]. However, in 3 + 1D or higher dimensions, aside from the mentioned two possibilities, the boundaries can be gapped and exhibit intrinsic topological order with all symmetry preserved. That is to say, if the bulk $(n + 1)\text{D}$ SPT phase has symmetry G , where $n \geq 3$, its boundary can be an anomalous SET phase also with symmetry G [64].

Here we focus on exactly this case for bulk 3 + 1D fermionic SPT (fSPT) phases with on-site symmetry $G^f = \mathbb{Z}_2^f \times G$ (ω_2 is trivial), i.e., we study the G^f -preserving and gapped boundaries of 3 + 1D fSPT phases, which belong to 2 + 1D anomalous fSET phases [51,65,66]. For such anomalous fSET phases, their symmetry actions cannot be defined on-site purely on the 2 + 1D surface, therefore cannot be gauged within 2 + 1D [67]. Such obstruction to gauging the symmetry is called the 't Hooft anomaly [68]. We further distinguish 't Hooft anomalies into bosonic ones and fermionic ones, corresponding to the obstruction to gauging G and gauging $\mathbb{Z}_2^f$, respectively. While since our exactly solvable model construction only applies to non-chiral phases, our construction limits to nonchiral 2 + 1D fSET phases with 't Hooft anomaly.

We recall that given a bosonic topological order whose anyon excitations are described by a unitary modular tensor category (UMTC), and given a symmetry group G , 2 + 1D bosonic G -SET phases are characterized by the following data [13]: (i) How the symmetry permutes anyons, called symmetry action. (ii) The fractional quantum numbers carried by the anyons under the symmetry, called symmetry fractionalization. (iii) The structure of symmetry defects, where introducing symmetry defects to the system consistently mathematically corresponds to a G -crossed extension of the original UMTC. However, some of the symmetry fractionalization classes are not consistent with symmetry defects, which leads to anomalous symmetry fractionalization that can only exist at the surface of some 3 + 1D bulk. The anomaly for a given symmetry fractionalization class is specified by an obstruction function $o_4 \in H^4[G, U(1)_T]$. While 3 + 1D bosonic SPT phases are also classified by the cohomology class $H^4[G, U(1)_T]$. So that intuitively a 2 + 1D SET phase with nontrivial anomaly o_4 should live on the surface of certain bulk 3 + 1D bosonic SPT phase, such that the obstruction is compensated by the bulk SPT phase.

In fermionic cases, things are getting more involved. For nonanomalous fSET phases, it is required that all symmetries can be gauged, including the fermion-parity symmetry $\mathbb{Z}_2^f$. While for anomalous phases, there can be obstruction to extending symmetry fractionalization to the fermion-parity vortices or fluxes when trying gauging the $\mathbb{Z}_2^f$ symmetry, where the anomaly is specified by an obstruction $o_3 \in H^3(G, \mathbb{Z}_2)$ [43]. We recall that 3-cocycles in $H^3(G, \mathbb{Z}_2)$ describe complex fermion decoration in 3 + 1D group super-cohomology model [6] (without Kitaev chain decoration). So that it is natural to put a 2 + 1D fSET phase with $H^3(G, \mathbb{Z}_2)$ anomaly as the surface of some 3 + 1D fSPT phase with complex fermion decoration. Furthermore, there may even be obstruction to

extending symmetry action to fermion-parity vortices when trying gauging $\mathbb{Z}_2^f$, specified by any nontrivial $o_2 \in H^2(G, \mathbb{Z}_2)$. We conjecture that such anomalous state corresponds to the surface of some bulk 3 + 1D fSPT phase with Kitaev chain layer, as the Kitaev chain decoration is exactly described by 2-cocycles in $H^2(G, \mathbb{Z}_2)$ in the general classification of 3 + 1D fSPT phases [7,8]. While the $o_1 \in H^1(G, \mathbb{Z}_T)$ obstruction corresponds to chiral anomalous fSET phases, which are beyond the scope of our surface exactly solvable models for nonchiral phases at present.

Definition III.1. A 2 + 1D fermionic symmetry-enriched topological phase with symmetry $G^f = \mathbb{Z}_2^f \times_{\omega_2} G$ is said to have *fermionic 't Hooft anomaly* if there is an obstruction to gauging $\mathbb{Z}_2^f$.

There is a well-known example of nonchiral surface fSET phase: $\mathcal{Z}_1(\mathbf{Vec}_{\mathbb{Z}_4}) \boxtimes \{1, f\}$ ($\mathbb{Z}_4$ gauge theory stacking with a physical fermion f) with $\mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_4$ symmetry. It is a surface fSET phase with $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly, living on the surface of the root phase of bulk 3 + 1D $\mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_4$ -fSPT phase. The anyon excitations (together with G action and symmetry fractionalization) in this example were studied in Refs. [51,65]. The classification of gapped boundary of the root phase of the 3 + 1D $\mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_4$ -fSPT phase by group extension [such that the group super-cohomology data $n_3 \in H^3(G, \mathbb{Z}_2)$, $n_2 \in H^2(G, \mathbb{Z}_2)$ are trivialized] is studied in Refs. [62,66]. While it lacks an exactly solvable model description, which we will construct as a surface fermionic symmetry-enriched string-net model in the following, summarized in Sec. III D. We will also present the fully-solved data (string types, fusion rules, and surface F moves) for the exactly solvable model of this example in Sec. III F.

A. Group super-cohomology classification of bulk 3 + 1D fSPT phase

First we briefly review the classification of bulk 3 + 1D fSPT phases without $p + ip$ superconductor decoration or Kitaev chain decoration and when ω_2 is trivial (we do not know how to include nontrivial ω_2 in the surface exactly solvable model yet), via the fixed-point wave function approach, or say group super-cohomology approach [6–8]. We construct fixed-point ground state wave functions for 3 + 1D fSPT phases on a triangulated and oriented lattice with the following two layers of degrees of freedom:

- (1) Complex fermion layer: Each complex fermion c_{ijkl} is decorated at the center of each tetrahedron $\langle ijkl \rangle$. Existence of a complex fermion in tetrahedron $\langle ijkl \rangle$ is specified by a 3-cocycle $n_3(\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k, \mathbf{g}_l) \in H^3(G, \mathbb{Z}_2)$, satisfying the symmetric condition (or

say, the transformation rule between homogeneous and inhomogeneous cochains):

$$\begin{aligned} n_3(\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) &= U(\mathbf{g}_0)n_3(\mathbf{1}, \mathbf{g}_0^{-1}\mathbf{g}_1, \mathbf{g}_0^{-1}\mathbf{g}_2, \mathbf{g}_0^{-1}\mathbf{g}_3)U^\dagger(\mathbf{g}_0) \\ &= n_3(\mathbf{g}_0^{-1}\mathbf{g}_1, \mathbf{g}_0^{-1}\mathbf{g}_2, \mathbf{g}_0^{-1}\mathbf{g}_3), \end{aligned} \quad (105)$$

and the 3-cocycle condition (recall Convention 2):

$$\begin{aligned} dn_3(01234) &= n_3(1234) + n_3(0234) + n_3(0134) \\ &\quad + n_3(0124) + n_3(0123) \\ &= 0 \pmod{2}. \end{aligned} \quad (106)$$

- (2) Bosonic SPT layer: $|G|$ level bosonic state $|g_i\rangle$ ($g_i \in G$) on each vertex i . A 3 + 1D 2-3 move (as a retriangulation or renormalization) is specified by a 4-cochain $\nu_4(\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k, \mathbf{g}_l, \mathbf{g}_m) \in C^4[G, U_T(1)]/B^4[G, U_T(1)]$, satisfying the symmetric condition:

$$\begin{aligned} \nu_4(\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4) &= U(\mathbf{g}_0)\nu_4(\mathbf{1}, \mathbf{g}_0^{-1}\mathbf{g}_1, \mathbf{g}_0^{-1}\mathbf{g}_2, \mathbf{g}_0^{-1}\mathbf{g}_3, \mathbf{g}_0^{-1}\mathbf{g}_4)U^\dagger(\mathbf{g}_0) \\ &= \nu_4(\mathbf{g}_0^{-1}\mathbf{g}_1, \mathbf{g}_0^{-1}\mathbf{g}_2, \mathbf{g}_0^{-1}\mathbf{g}_3, \mathbf{g}_0^{-1}\mathbf{g}_4)^{S(\mathbf{g}_0)}, \end{aligned} \quad (107)$$

and the twisted 4-cocycle condition:

$$\begin{aligned} d\nu_4(012345) &= \frac{\nu_4(12345)\nu_4(01345)\nu_4(01235)}{\nu_4(02345)\nu_4(01245)\nu_4(01234)} \\ &= (-1)^{n_3 \smile_1 n_3(012345)}. \end{aligned} \quad (108)$$

In this simple case, 3 + 1D fSPT phases are classified by the doublet (n_3, ν_4) .

B. Surface renormalization moves

Given a global symmetry group $G^f = Z_2^f \times G$ (ω_2 is trivial), a bulk 3 + 1D fSPT phase specified by (n_3, ν_4) , and a surface or anomalous 2 + 1D fSET phase specified by a set of G -graded string types $a_g, b_h, \dots$ with given fusion rules (the G -graded string types no longer form a G -graded super fusion category as the fermionic pentagon equation is obstructed, as we will see below), we define the surface fermionic symmetry-enriched string-net model in this section. gfSLU transformations in this bulk-boundary system are not well-defined purely on the surface, but only well defined if considering the corresponding bulk transformation and the surface transformation as a whole. A renormalization move is a special kind of gfSLU transformation that changes the state configuration or the number of degrees of freedom. Therefore, renormalization moves on surface are also well defined only when their corresponding bulk renormalization moves are put together. Below let Ψ denote for the fixed-point ground state wave function of the whole bulk-boundary system. We define the *surface $\mathcal{F}$ move* and the *bulk $\tilde{\mathcal{F}}$ move* together as a renormalization move, called *bulk-boundary $\mathcal{F}$ move*, which is also a gfSLU transformation:

$$\Psi \left(\begin{array}{c} \text{Diagram 1: A tetrahedron with vertices } \mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 \text{ and a central vertex } \mathbf{g}_*. \text{ Red lines represent surface string types } a_g, b_h, c_k, d_{ghk}. \text{ Green lines represent bulk string types.} \end{array} \right) = \sum_{n_{hk}\chi\delta} \mathbf{g}_0[\mathcal{F}_d^{a_g b_h c_k}]_{n, \chi\delta}^{m, \alpha\beta} \cdot \mathbf{g}_0[\tilde{\mathcal{F}}^{ghk}](\mathbf{g}_*) \Psi \left(\begin{array}{c} \text{Diagram 2: A tetrahedron with vertices } \mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 \text{ and a central vertex } \mathbf{g}_*. \text{ Red lines represent surface string types } a_g, b_h, c_k, d_{ghk}. \text{ Green lines represent bulk string types.} \end{array} \right), \quad (109)$$

where $\mathbf{g}_0[\mathcal{F}_d^{a_g b_h c_k}]_{n, \chi\delta}^{m, \alpha\beta}$ is the surface $\mathcal{F}$ move, it transforms on the surface G -graded string types and fusion states (red), and the bulk $\tilde{\mathcal{F}}$ move $\mathbf{g}_0[\tilde{\mathcal{F}}^{ghk}](\mathbf{g}_*)$ transforms on the fixed-point ground states of the bulk 3 + 1D fSPT phase (green and black). To obtain an explicit expression for the bulk move $\mathbf{g}_0[\tilde{\mathcal{F}}^{ghk}](\mathbf{g}_*)$, we put an auxiliary vertex e with trivial group element $\mathbf{g}_e = \mathbf{1}$ on the other side of the bulk. We note that this side of bulk [blue in (110)] is virtual, and this trivial vertex $\mathbf{g}_e = \mathbf{1}$ is invariant under any symmetry action, i.e., it is always trivial. We find that the above bulk-boundary renormalization move can be divided into the following three steps:

$$\begin{aligned}
& \Psi \left(\begin{array}{c} \text{Diagram 1: Tetrahedron with vertices } \mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 \text{ and } \mathbf{g}_* \text{ at the top. Edges are labeled } a_g, b_h, c_k, d_{ghk}. \text{ Surface fermion operators } c_{\alpha}^{\dagger s(\alpha)}, c_{\beta}^{\dagger s(\beta)}, c_{\delta}^{\dagger s(\delta)}, c_{\chi}^{\dagger s(\chi)} \text{ are shown. } \mathbf{g}_e = 1. \end{array} \right) \\
&= \tilde{\mathcal{F}}(0123*) \Psi \left(\begin{array}{c} \text{Diagram 2: Same as Diagram 1, but with a different configuration of fermion operators. } \mathbf{g}_e = 1. \end{array} \right) \\
&= \sum_{n_{\mathbf{h}\mathbf{k}}\chi\delta} \tilde{\mathcal{F}}(0123*) c_{\underline{\alpha}}^{\dagger s(\alpha)} c_{\underline{\beta}}^{\dagger s(\beta)} c_{\underline{\delta}}^{\dagger s(\delta)} c_{\underline{\chi}}^{\dagger s(\chi)} \mathbf{g}_0 [F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} \Psi \left(\begin{array}{c} \text{Diagram 3: Same as Diagram 1, but with a different configuration of fermion operators. } \mathbf{g}_e = 1. \end{array} \right) \\
&= \sum_{n_{\mathbf{h}\mathbf{k}}\chi\delta} \tilde{\mathcal{F}}(0123*) c_{\underline{\alpha}}^{\dagger s(\alpha)} c_{\underline{\beta}}^{\dagger s(\beta)} c_{\underline{\delta}}^{\dagger s(\delta)} c_{\underline{\chi}}^{\dagger s(\chi)} \mathbf{g}_0 [F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} \tilde{\mathcal{F}}(e0123)^{-1} \Psi \left(\begin{array}{c} \text{Diagram 4: Same as Diagram 1, but with a different configuration of fermion operators. } \mathbf{g}_e = 1. \end{array} \right), \quad (110)
\end{aligned}$$

where $s(\alpha) + s(\beta) + s(\delta) + s(\chi) = 0 \pmod{2}$, $c_{\underline{\alpha}}$ and $c_{\underline{\alpha}}^{\dagger}$ are surface fermion annihilation and creation operators on surface vertex $\underline{\alpha}$, and $\tilde{\mathcal{F}}(0123*)$ is the 2-3 move on the real bulk and $\tilde{\mathcal{F}}(e0123)^{-1}$ is the inverse 2-3 move on the virtual bulk. The 2-3 moves are explicitly expressed as [7]

$$\tilde{\mathcal{F}}(0123*) = \nu_4(0123*) C_{012*}^{\dagger n_3(012*)} C_{023*}^{\dagger n_3(023*)} C_{0123}^{n_3(0123)} C_{013*}^{n_3(013*)} C_{123*}^{n_3(123*)}, \quad (111)$$

$$\tilde{\mathcal{F}}(e0123)^{-1} = \nu_4(e0123)^{-1} C_{0123}^{\dagger n_3(0123)} C_{e023}^{\dagger n_3(e023)} C_{e012}^{\dagger n_3(e012)} C_{e123}^{n_3(e123)} C_{e013}^{n_3(e013)}, \quad (112)$$

where $\nu_4(ijklm) := \nu_4(\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k, \mathbf{g}_l, \mathbf{g}_m)$ (recall Convention 2) is the bosonic-part of the 2-3 move, C_{ijkl} and $C_{ijkl}^{\dagger}$ are bulk fermion annihilation and creation operators in tetrahedron $\langle ijkl \rangle$, and $n_3(ijkl) := n_3(\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k, \mathbf{g}_l) = 0, 1$ specifies the fermion parity in $\langle ijkl \rangle$. Let $\mathbf{g} = \mathbf{g}_0^{-1} \mathbf{g}_1$, $\mathbf{h} = \mathbf{g}_1^{-1} \mathbf{g}_2$, $\mathbf{k} = \mathbf{g}_2^{-1} \mathbf{g}_3$, and $\mathbf{m} = \mathbf{g}_3^{-1} \mathbf{g}_*$. Then eliminating $C_{0123}^{\dagger n_3(0123)}$ with $C_{0123}^{n_3(0123)}$, the bulk-boundary $\mathcal{F}$ move has the following expression:

$$\begin{aligned}
& \mathbf{g}_0 [\mathcal{F}_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} \mathbf{g}_0 [\tilde{\mathcal{F}} \mathbf{g} \mathbf{h} \mathbf{k}] (\mathbf{g}_*) = (-1)^{n_3(0123)(n_3(013*) + n_3(123*))} C_{012*}^{\dagger n_3(012*)} C_{023*}^{\dagger n_3(023*)} C_{013*}^{n_3(013*)} C_{123*}^{n_3(123*)} \\
& \times C_{e023}^{\dagger n_3(e023)} C_{e012}^{\dagger n_3(e012)} C_{e123}^{n_3(e123)} C_{e013}^{n_3(e013)} \frac{\nu_4(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{m})^{S(\mathbf{g}_0)}}{\nu_4(\mathbf{g}_0, \mathbf{g}, \mathbf{h}, \mathbf{k})} \\
& \times c_{\underline{\alpha}}^{\dagger s(\alpha)} c_{\underline{\beta}}^{\dagger s(\beta)} c_{\underline{\delta}}^{\dagger s(\delta)} c_{\underline{\chi}}^{\dagger s(\chi)} \mathbf{g}_0 [F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}, \quad (113)
\end{aligned}$$

where all surface fermion operators commute with bulk fermion operators, e.g., $c_{\underline{\alpha}}^{\dagger} C_{123*} = C_{123*} c_{\underline{\alpha}}^{\dagger}$, and the homogeneous ν_4 are changed to inhomogeneous ones by Eq. (107).

Remark III.2. We can obtain the same expression of bulk-boundary $\mathcal{F}$ move in Eq. (113) by another approach called partition function approach, shown in Appendix G.

1. $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly

Let us carefully examine Eq. (113) that which terms belong to the boundary and which belong to the bulk. Recall Eq. (109), the bulk $\tilde{\mathcal{F}}$ move has the expression:

$$\begin{aligned} g_0[\tilde{\mathcal{F}}^{\mathbf{ghk}}](\mathbf{g}_*) &= (-1)^{n_3(0123)[n_3(013*)+n_3(123*)]} C_{012*}^{\dagger n_3(012*)} \\ &\times C_{023*}^{\dagger n_3(023*)} C_{013*}^{n_3(013*)} C_{123*}^{n_3(123*)} \\ &\times \frac{\nu_4(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{m})^{S(\mathbf{g}_0)}}{\nu_4(\mathbf{g}_0, \mathbf{g}, \mathbf{h}, \mathbf{k})}. \end{aligned} \quad (114)$$

While the surface $\mathcal{F}$ -move $g_0[\mathcal{F}_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}$ should involve the fermion operators from the virtual bulk, i.e., fermion operators on tetrahedrons $\langle e023 \rangle$, $\langle e012 \rangle$, $\langle e123 \rangle$, and $\langle e013 \rangle$, and therefore it generally may not preserve fermion-parity, with the following expression:

$$\begin{aligned} g_0[\mathcal{F}_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} &= c_{\underline{a}}^{\dagger s(\alpha)} c_{\underline{\beta}}^{\dagger s(\beta)} c_{\underline{\delta}}^{s(\delta)} c_{\underline{\chi}}^{s(\chi)} C_{\underline{\beta}}^{\dagger n_3(e023)} \\ &\times C_{\underline{a}}^{\dagger n_3(e012)} C_{\underline{\chi}}^{n_3(e123)} C_{\underline{\delta}}^{n_3(e013)} g_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}, \end{aligned} \quad (115)$$

where $C_{\underline{a}}^{\dagger} := C_{e012}^{\dagger}$, $C_{\underline{\beta}}^{\dagger} := C_{e023}^{\dagger}$, $C_{\underline{\delta}} := C_{e013}$, $C_{\underline{\chi}} := C_{e123}$, which are understood as surface fermions that are pumped from the bulk (therefore they still satisfy the anticommutation relation of bulk fermions, and commute with all surface fermions).

Definition III.3. [Total surface fermion-parity function] The total surface fermion-parity function for each fusion state α in Graph (119) is defined as

$$s'(\mathbf{g}_0, \alpha) = s(\alpha) + n_3(\mathbf{g}_0, \mathbf{g}, \mathbf{h}). \quad (116)$$

Then for the surface $\mathcal{F}$ move above, we have $s'(\mathbf{g}_0, \alpha) = s(\alpha) + n_3(e012)$, $s'(\mathbf{g}_0, \beta) = s(\beta) + n_3(e023)$, $s'(\mathbf{g}_0, \delta) = s(\delta) + n_3(e013)$, $s'(\mathbf{g}_0 \mathbf{g}, \chi) = s(\chi) + n_3(e123)$, and they satisfy

$$\begin{aligned} s'(\mathbf{g}_0, \alpha) + s'(\mathbf{g}_0, \beta) + s'(\mathbf{g}_0, \delta) + s'(\mathbf{g}_0 \mathbf{g}, \chi) \\ (\text{mod } 2) = n_3(0123) \equiv n_3(\mathbf{g}, \mathbf{h}, \mathbf{k}), \end{aligned} \quad (117)$$

i.e., the surface $\mathcal{F}$ move in general violates the fermion-parity conservation by $n_3(\mathbf{g}, \mathbf{h}, \mathbf{k}) \in H^3(G, \mathbb{Z}_2)$. This violation of fermion-parity conservation on surface $\mathcal{F}$ move exactly counts for the $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly.

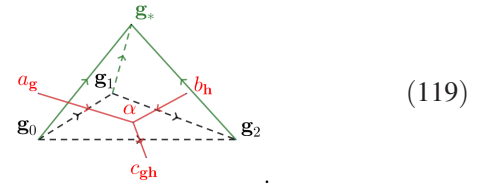
Remark III.4. Since 't Hooft anomalies are anomalies on symmetry actions, in our surface fermionic symmetry-enriched string-net model, string types $a_1, b_1, \dots$ with trivial group element grading still form a super fusion category $\mathcal{S}_1$, i.e., $\mathcal{S}_1 = \{a_1, b_1, \dots\}$.

Remark III.5. In the context of excitations/anyons on surface. Let $o_3 \in H^3(G, \mathbb{Z}_2)$ be the obstruction on extending symmetry fractionalization from $\mathcal{Z}_1(\mathcal{S}_1)$ to $\mathcal{Z}_1(\mathcal{C}_1)$. The n_3 obstruction on fermion-parity conservation for a surface $\mathcal{F}$ move matches with the o_3 anomaly, i.e.,

$$[o_3] = [n_3], \quad (118)$$

where $[]$ denotes for the cohomology class.

A surface string fusion is graphically represented as



The surface string fusion rules now depend on the leftmost group element $\mathbf{g}_0$:

$$g_0[a_g \otimes b_h] = \begin{cases} \bigoplus_{c_{gh}} B_{c_{gh}}^{a_g b_h} |F_{c_{gh}}^{a_g b_h} c_{gh}, & \text{if } n_3(\mathbf{g}_0, \mathbf{g}, \mathbf{h}) = 0 \\ \bigoplus_{c_{gh}} F_{c_{gh}}^{a_g b_h} |B_{c_{gh}}^{a_g b_h} c_{gh}, & \text{if } n_3(\mathbf{g}_0, \mathbf{g}, \mathbf{h}) = 1 \end{cases}, \quad (120)$$

where when $n_3(e012) := n_3(\mathbf{g}_0, \mathbf{g}, \mathbf{h}) = 1$, a bulk fermion is attached to the surface and the surface fermion-parity on vertex α is changed, and thereby the original bosonic fusion states become fermionic, while the original fermionic fusion states become bosonic.

2. 't Hooft anomalous fermionic pentagon equation

The consistency equation for the bulk-boundary $\mathcal{F}$ move in Eq. (109) is a surface pentagon equation together with the corresponding bulk moves:

$$\begin{aligned} \sum_e g_0[\mathcal{F}_e^{j_{gh} c_k d_l}]_{q, \delta \epsilon}^{m, \beta \chi} g_0[\tilde{\mathcal{F}}(\mathbf{ghkl})](\mathbf{g}_*) g_0[\mathcal{F}_e^{a_g b_h c_k d_l}]_{p, \phi \gamma}^{j, \alpha \epsilon} g_0[\tilde{\mathcal{F}}(\mathbf{ghkl})](\mathbf{g}_*) \\ = \sum_{n_{hk} \eta \psi \kappa} g_0[\mathcal{F}_m^{j_{gh} c_k d_l}]_{n, \eta \psi}^{j, \alpha \beta} g_0[\tilde{\mathcal{F}}(\mathbf{ghk})](\mathbf{g}_*) g_0[\mathcal{F}_e^{a_g b_h c_k d_l}]_{p, \kappa \gamma}^{m, \psi \chi} g_0[\tilde{\mathcal{F}}(\mathbf{hkl})](\mathbf{g}_*) g_0[\mathcal{F}_p^{b_h c_k d_l}]_{q, \delta \phi}^{n, \eta \kappa} g_0[\tilde{\mathcal{F}}(\mathbf{hkl})](\mathbf{g}_*). \end{aligned} \quad (121)$$

Using the expression in Eq. (110), eliminating the fermion creation and annihilation operators, we find that the above equation splits into the following two equations:

$$\frac{\nu_4(0234^*) \nu_4(0124^*)}{\nu_4(e0234) \nu_4(e0124)} = \frac{(-1)^{n_3 \sim 1 n_3(01234^*)}}{(-1)^{n_3 \sim 1 n_3(e01234)}} \frac{\nu_4(0123^*) \nu_4(0134^*) \nu_4(1234^*)}{\nu_4(e0123) \nu_4(e0134) \nu_4(e1234)}, \quad (122)$$

$$\sum_{\epsilon} \mathbf{g}_0 [F_e^{j_{gh} c_k d_l}]_{q, \delta \epsilon}^{m, \beta \chi} \mathbf{g}_0 [F_e^{a_g b_h q_{kl}}]_{p, \phi \gamma}^{j, \alpha \epsilon} = (-1)^{s(\alpha) s(\delta)} \sum_{n_{hk} \eta \psi \kappa} \mathbf{g}_0 [F_m^{a_g b_h c_k}]_{n, \eta \psi}^{j, \alpha \beta} \mathbf{g}_0 [F_e^{a_g n_{hk} d_l}]_{p, \kappa \gamma}^{m, \psi \chi} \mathbf{g}_0 [F_p^{b_h c_k d_l}]_{q, \delta \phi}^{n, \eta \kappa}, \quad (123)$$

where we see that Eq. (122) is a combination of two hexagon equations [8] (one for the real bulk, one for the virtual bulk) of 3 + 1D fSPT phases by adding a trivial term $\{\nu_4(01234)/[\nu_4(01234)]\} = 1$, and Eq. (123) is the surface fermionic pentagon equation.

The symmetric condition on the bulk-boundary $\mathcal{F}$ move requires the following commutative diagram:

where by the expression of bulk $\tilde{F}$ move in Eq. (114) and recall that n_3 is invariant under symmetry action as written in Eq. (105), we have $\mathbf{g}_0[\tilde{F}^{ghk}](\mathbf{g}_*) = \nu_4(\mathbf{g}_0, \mathbf{g}, \mathbf{h}, \mathbf{k})^{-1} \cdot 1[\tilde{F}^{ghk}](\mathbf{g}_*)$. It implies

$$\mathbf{g}_0 [F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} = \nu_4(\mathbf{g}_0, \mathbf{g}, \mathbf{h}, \mathbf{k}) C_{e023}^{\dagger n_3(e023)} C_{e012}^{\dagger n_3(e012)} C_{e123}^{n_3(e123)} C_{e013}^{n_3(e013)} 1[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}, \quad (125)$$

where

$$\mathbf{g}_0 [F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} = \mathbf{g}_0 \Theta(a_g, b_h, c_k) \nu_4(\mathbf{g}_0, \mathbf{g}, \mathbf{h}, \mathbf{k}) (1[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta})^{S(\mathbf{g}_0)}. \quad (126)$$

Here the phase factor $\mathbf{g}_0 \Theta(a_g, b_h, c_k)$, called *fermionic obstruction*, is defined as a solution of the following equation:

$$\frac{\mathbf{g}_0 \Theta(j_{gh}, c_k, d_l) \mathbf{g}_0 \Theta(a_g, b_h, q_{kl})}{\mathbf{g}_0 \Theta(a_g, b_h, c_k) \mathbf{g}_0 \Theta(a_g, n_{hk}, d_l) \mathbf{g}_0 \Theta(b_h, c_k, d_l)} = d\nu_4(\mathbf{g}_0, \mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}), \quad (127)$$

together with a required property $\mathbf{g}_0 \Theta(a_g, b_h, c_k) \mathbf{g}_0 \Theta(a_g, b_h, c_k)^{S(\mathbf{g}_0)} = \mathbf{g}_0 \Theta(a_g, b_h, c_k)$. We check in Appendix H that the anomalous symmetry action in Eq. (126) and the surface fermionic pentagon equation in Eq. (123).

Remark III.6. The phase factor $\mathbf{g}_0 \Theta(a_g, b_h, c_k)$ counts for the contribution of the fermion creation and annihilation operators $C_{e023}^{\dagger n_3(e023)} C_{e012}^{\dagger n_3(e012)} C_{e123}^{n_3(e123)} C_{e013}^{n_3(e013)}$ from the bulk 3 + 1D fSPT phase appearing in Eq. (125), by first eliminating all fermion operators in the consistency equation in Eq. (121) and obtaining total phase factor $d\nu_4(e01234) := d\nu_4(\mathbf{g}_0, \mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l})$, and then distributing the total phase factor for each surface F move, which is the

meaning of Eq. (127). We show in Appendix H that only with this phase factor $g_0 \Theta(a_g, b_h, c_k)$, the anomalous symmetry condition in Eq. (126) is consistent with the surface fermionic pentagon equation in Eq. (123).

Replacing Eq. (126) into Eq. (123) and let $g_0 = \mathbf{1}$, we obtained the so-called 't Hooft anomalous fermionic pentagon equation:

$$\sum_{\epsilon} \mathbf{1}[F_e^{j_{gh}c_k d_l}]_{q,\delta\epsilon}^{m,\beta\gamma} \mathbf{1}[F_e^{a_g b_h q_{kl}}]_{p,\phi\gamma}^{j,\alpha\epsilon} = g_0 \Theta(b_h, c_k, d_l) \nu_4(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}) (-1)^{s(\alpha)s(\delta)} \times \sum_{n_{hk}\eta\psi\kappa} \mathbf{1}[F_m^{a_g b_h c_k}]_{n,\eta\psi}^{j,\alpha\beta} \mathbf{1}[F_e^{a_g n_{hk} d_l}]_{p,\kappa\gamma}^{m,\psi\chi} (\mathbf{1}[F_p^{b_h c_k d_l}]_{q,\delta\phi}^{n,\eta\kappa})^{S(\mathbf{g})}, \quad (128)$$

where $g_0 \Theta(b_h, c_k, d_l)$ is the fermionic obstruction, $\nu_4(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l})$ is hereby called the bosonic obstruction.

Remark III.7. Other types of bulk-boundary renormalization moves, such as bulk-boundary $\mathcal{O}$ move, $\mathcal{Y}$ move, $\mathcal{H}$ move and so on can be defined following a similar spirit combining this subsection and Appendix F and omitted here.

Remark III.8. Applying Eq. (126) in the last F move in Eq. (123), full 't Hooft anomalous fermionic pentagon equations for surface F moves with general leftmost group element g_0 are

$$\sum_{\epsilon} g_0 [F_e^{j_{gh}c_k d_l}]_{q,\delta\epsilon}^{m,\beta\gamma} g_0 [F_e^{a_g b_h q_{kl}}]_{p,\phi\gamma}^{j,\alpha\epsilon} = g_0 \Theta(b_h, c_k, d_l)^{S(\mathbf{g})} \frac{\nu_4(g_0 \mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l})}{\nu_4(g_0, \mathbf{h}, \mathbf{k}, \mathbf{l})^{S(\mathbf{g})}} (-1)^{s(\alpha)s(\delta)} \times \sum_{n_{hk}\eta\psi\kappa} g_0 [F_m^{a_g b_h c_k}]_{n,\eta\psi}^{j,\alpha\beta} g_0 [F_e^{a_g n_{hk} d_l}]_{p,\kappa\gamma}^{m,\psi\chi} (g_0 [F_p^{b_h c_k d_l}]_{q,\delta\phi}^{n,\eta\kappa})^{S(\mathbf{g})}, \quad (129)$$

where all these equations with nontrivial g_0 are actually redundant, as Eq. (126) and Eq. (128) completely determine all surface F moves.

C. Surface parent Hamiltonian

The parent Hamiltonian of a surface fermionic symmetry-enriched string-net model with $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly is

$$H' = -\sum_{\nu} C_{\nu}^{\text{surface}} - \sum_{\nu} A_{\nu} - \sum_l D_l - \sum_l Q_l - \sum_p B_p^{\text{surface}}, \quad (130)$$

where the terms A_{ν} , D_l , Q_l are the same as the nonanomalous case in Eq. (54), while C_{ν}^{surface} and B_p^{surface} are modified as

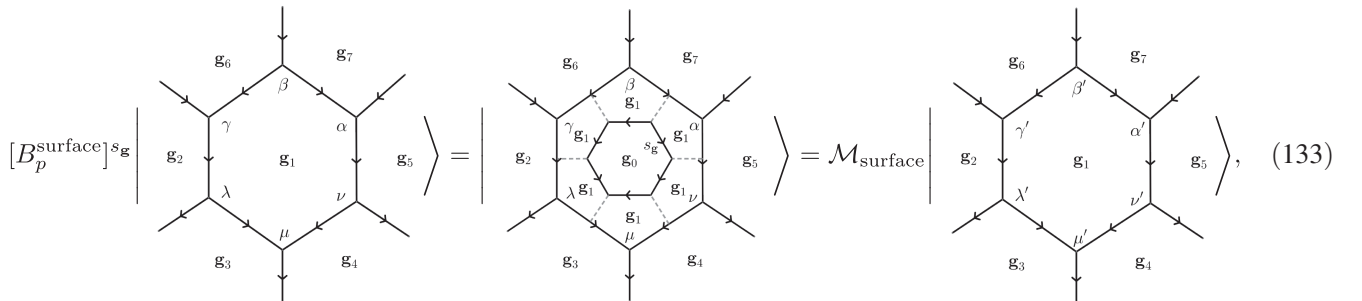
(i) C_{ν}^{surface} equalizes the fermion number $\langle N_f \rangle$ with the total fermion-parity function on surface $s'(\mathbf{g}_0, \alpha)$, written as

$$C_{\nu}^{\text{surface}} \left| \begin{array}{c} a_g \quad b_h \\ \alpha \\ \downarrow \\ g_0 \quad c_{gh} \end{array} \right\rangle = \delta_{\langle N_f \rangle, s'(\mathbf{g}_0, \alpha)} \left| \begin{array}{c} a_g \quad b_h \\ \alpha \\ \downarrow \\ g_0 \quad c_{gh} \end{array} \right\rangle, \quad (131)$$

(ii) B_p^{surface} may change the fermion-parity in general, written as

$$B_p^{\text{surface}} = \frac{1}{\mathcal{D}^2} \sum_{\mathbf{g} \in G} |\mathbf{g}_p \mathbf{g}\rangle \langle \mathbf{g}_p| \sum_{s_g} \frac{d_{s_g}}{n_{s_g}} [B_p^{\text{surface}}]_{s_g}. \quad (132)$$

$[B_p^{\text{surface}}]_{s_g}$ fuses a loop s_g on the edges:



$$[B_p^{\text{surface}}]_{s_g} \left| \begin{array}{c} \beta \\ \gamma \\ \alpha \\ \lambda \\ \mu \\ \nu \end{array} \right\rangle = \left| \begin{array}{c} \beta \\ \gamma \\ \alpha \\ \lambda \\ \mu \\ \nu \end{array} \right\rangle = \mathcal{M}_{\text{surface}} \left| \begin{array}{c} \beta' \\ \gamma' \\ \alpha' \\ \lambda' \\ \mu' \\ \nu' \end{array} \right\rangle, \quad (133)$$

where $\mathcal{M}_{\text{surface}}$ is a sequence of surface $\mathcal{F}$ moves, $\mathcal{H}$ moves, dual $\mathcal{H}$ moves, and $\mathcal{O}$ moves. It in general changes the fermion parity by

$$\begin{aligned} & \sum_{\kappa=\alpha, \beta, \gamma} s'(\mathbf{g}_0, \kappa) - \sum_{\eta=\alpha, \beta, \gamma} s'(\mathbf{g}_1, \eta) \\ &= n_3(\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_6) + n_3(\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_6, \mathbf{g}_7) + n_3(\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_7, \mathbf{g}_5) \\ & \quad + n_3(\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_4, \mathbf{g}_5) + n_3(\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_3, \mathbf{g}_4) + n_3(\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3), \end{aligned} \quad (134)$$

obtained by counting the fermion-parity change in Eq. (117) for all involved surface $\mathcal{F}$ moves, $\mathcal{H}$ moves, and dual $\mathcal{H}$ moves.

This surface parent Hamiltonian H' is still a commuting-projector Hamiltonian.

D. Summary

Definition III.9. (Surface fermionic symmetry-enriched string-net model with $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly). Inputting a finite fermionic symmetry group $G^f = \mathbb{Z}_2^f \times G$, a bulk $3 + 1\text{D}$ G^f -fSPT phase (with up to complex fermion layer) described by $\{n_3, \nu_4\}$, and a set of G -graded string types $\{a_g, b_h, \dots\}$ with given fusion rules such that $\mathcal{S}_1 = \{a_1, b_1, \dots\}$ is a super fusion category, a surface fermionic symmetry-enriched string-net model with $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly on a surface $2 + 1\text{D}$ honeycomb lattice consists of

- (i) Degrees of freedom:
- (1) Group element labels $|\mathbf{g}_i\rangle$ in each plaquette, where $\mathbf{g}_i \in G$.
- (2) G -graded string types $a_g, b_h, \dots$ on each link, where $a_1, b_1, \dots \in \mathcal{S}_1$ and $\mathbf{g}, \mathbf{h}, \dots \in G$. The dimension of endomorphism space n_{a_g} and quantum dimension d_{a_g} are defined similarly as in Sec. II C.

- (3) Fusion states $\alpha, \beta, \dots$ in fusion space $\text{Hom}(a_g \otimes b_h, c_{gh})$ on each vertex. Fusion multiplicities $N_{c_k}^{a_g b_h}, B_{c_k}^{a_g b_h}, F_{c_k}^{a_g b_h}$ are defined similarly as in Sec. II C.
- (4) $|G|$ species of physical fermions on each vertex. Still we define

$$s(\alpha) = \begin{cases} 0, & \text{if } |\alpha\rangle \in \text{Hom}(a_g \otimes b_h, c_{gh})_0 \\ 1, & \text{if } |\alpha\rangle \in \text{Hom}(a_g \otimes b_h, c_{gh})_1 \end{cases}. \quad (135)$$

While the total surface parity function of fusion state α with leftmost group element $\mathbf{g}_0$ [as shown in Graph (119)] is

$$s'(\mathbf{g}_0, \alpha) = s(\alpha) + n_3(\mathbf{g}_0, \mathbf{g}, \mathbf{h}). \quad (136)$$

The string fusion rules are therefore $\mathbf{g}_0$ dependent:

$$\mathbf{g}_0[a_g \otimes b_h] = \begin{cases} \bigoplus_{c_{gh}} \mathbb{C}^{B_{c_{gh}}^{a_g b_h} | F_{c_{gh}}^{a_g b_h}} c_{gh}, & \text{if } n_3(\mathbf{g}_0, \mathbf{g}, \mathbf{h}) = 0 \\ \bigoplus_{c_{gh}} \mathbb{C}^{F_{c_{gh}}^{a_g b_h} | B_{c_{gh}}^{a_g b_h}} c_{gh}, & \text{if } n_3(\mathbf{g}_0, \mathbf{g}, \mathbf{h}) = 1 \end{cases}. \quad (137)$$

- (ii) Surface renormalization moves on surface fixed-point ground states satisfying 't Hooft anomalous symmetric condition, introduced in Sec. III B.
- (iii) A surface parent commuting-projector Hamiltonian, introduced in Sec. III C,

where the full data this surface exactly solvable model is denoted as $\{G^f, n_3, \nu_4, a_g, n_{a_g}, d_{a_g}, N_{c_k}^{a_g b_h}, F_{c_k}^{a_g b_h}, \mathbf{g}_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}\}$, satisfying the following consistency equations:

$$N_{c_{gh}}^{a_g b_h} = B_{c_{gh}}^{a_g b_h} + F_{c_{gh}}^{a_g b_h}, \quad (138)$$

$$\sum_{e_{gh}} \frac{N_{e_{gh}}^{a_g b_h} N_{d_{ghk}}^{e_{gh} c_k}}{n_{e_{gh}}} = \sum_{f_{hk}} \frac{N_{f_{hk}}^{b_h c_k} N_{d_{ghk}}^{a_g f_{hk}}}{n_{f_{hk}}}, \quad (139)$$

$$\sum_{e_{gh}} \frac{B_{e_{gh}}^{a_g b_h} B_{d_{ghk}}^{e_{gh} c_k} + F_{e_{gh}}^{a_g b_h} F_{d_{ghk}}^{e_{gh} c_k}}{n_{e_{gh}}} = \sum_{f_{hk}} \frac{B_{f_{hk}}^{b_h c_k} B_{d_{ghk}}^{a_g f_{hk}} + F_{f_{hk}}^{b_h c_k} F_{d_{ghk}}^{a_g f_{hk}}}{n_{f_{hk}}}, \quad (140)$$

$$d_{a_g} d_{b_h} = \sum_{c_{gh}} \frac{N_{c_{gh}}^{a_g b_h} d_{c_{gh}}}{n_{c_{gh}}}, \quad (141)$$

$$\mathbf{g}_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} = \mathbf{g}_0 \Theta(a_g, b_h, c_k) \nu_4(\mathbf{g}_0, \mathbf{g}, \mathbf{h}, \mathbf{k}) (1[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta})^{S(\mathbf{g}_0)}, \quad (142)$$

$$s'(\mathbf{g}_0, \alpha) + s'(\mathbf{g}_0, \beta) + s'(\mathbf{g}_0, \delta) + s'(\mathbf{g}_0, \chi) \pmod{2} = n_3(\mathbf{g}, \mathbf{h}, \mathbf{k}), \quad (143)$$

$$\begin{aligned}
& \sum_{\epsilon} \mathbf{1}[F_e^{j_{gh}c_k d_l}]_{q, \delta \epsilon}^{m, \beta \chi} \mathbf{1}[F_e^{a_g b_h q_{kl}}]_{p, \phi \gamma}^{j, \alpha \epsilon} \\
& = \mathbf{g}\Theta(b_h, c_k, d_l) \nu_4(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}) (-1)^{s(\alpha)s(\delta)} \\
& \quad \times \sum_{n_{hk} \eta \psi \kappa} \mathbf{1}[F_m^{a_g b_h c_k}]_{n, \eta \psi}^{j, \alpha \beta} \mathbf{1}[F_e^{a_g n_{hk} d_l}]_{p, \kappa \gamma}^{m, \psi \chi} (\mathbf{1}[F_p^{b_h c_k d_l}]_{q, \delta \phi}^{n, \eta \kappa})^{S(\mathbf{g})},
\end{aligned} \tag{144}$$

$$\begin{aligned}
& \frac{\mathbf{g}_0 \Theta(j_{gh}, c_k, d_l) \mathbf{g}_0 \Theta(a_g, b_h, q_{kl})}{\mathbf{g}_0 \Theta(a_g, b_h, c_k) \mathbf{g}_0 \Theta(a_g, n_{hk}, d_l) \mathbf{g}_0 \Theta(b_h, c_k, d_l)} \\
& = d\nu_4(\mathbf{g}_0, \mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}),
\end{aligned} \tag{145}$$

where $\mathbf{g}_0 \Theta(a_g, b_h, c_k) \mathbf{g}\Theta(a_g, b_h, c_k)^{S(\mathbf{g}_0)} = \mathbf{g}_0 \mathbf{g}\Theta(a_g, b_h, c_k)$, and the four projective-unitary conditions in Eqs. (49)–(53).

E. Abelian gauge theory surface topological order and surface anyons

Let us consider the simple case that the surface topological order is an Abelian gauge theory (let N be the Abelian gauge group) stacking with a physical fermion f , i.e., the surface anyon types are $\mathcal{Z}_1(\mathbf{S}_1) \cong \mathcal{Z}_1(\mathbf{Vec}_N) \boxtimes \{1, f\}$, where $\mathcal{Z}_1(\mathbf{Vec}_N)$ describes

anyons in the N gauge theory, and given that the corresponding bulk 3 + 1D fSPT phase is classified by (n_3, ν_4) , then the surface fermionic symmetry-enriched string-net model is constructed by the following data

- (i) $\mathcal{S}_1 = \mathbf{Vec}_N = \{a, b, \dots | a, b, \dots = 0, \dots, N-1\}$ is a unitary fusion category.
- (ii) Other G -graded string types $\{a_g, b_h, \dots, \text{for } \mathbf{g}, \mathbf{h}, \dots \neq \mathbf{1}\}$ are constructed by a group extension of G by N , and we denote the extended group as $\mathcal{G}$, written in a short exact sequence as $1 \rightarrow N \rightarrow \mathcal{G} \rightarrow G \rightarrow 1$, i.e., $a_g, b_h, \dots$ are elements in $\mathcal{G}$. Similarly as in Sec. III, the group extension is specified by a group homomorphism $\varphi: G \rightarrow \text{Aut}(N)$ and a cohomology class $[\mu] \in H_\varphi^2(G, N)$ satisfying Eq. (93). Group multiplication rule in $\mathcal{G}$ is

$$a_g \times b_h = [a + \varphi_g(b) + \mu(\mathbf{g}, \mathbf{h})]_{gh}, \tag{146}$$

where since N is Abelian, we denote the group multiplication in N as $+$ here. All the G -graded string types are m -type.

- (iii) The surface string fusion rules when the group element in the leftmost plaquette being $\mathbf{g}_0$ are

$$\mathbf{g}_0[a_g \otimes b_h] = \begin{cases} \mathbb{C}^{1|0}[a + \varphi_g(b) + \mu(\mathbf{g}, \mathbf{h})]_{gh}, & \text{if } n_3(\mathbf{g}_0, \mathbf{g}, \mathbf{h}) = 0 \\ \mathbb{C}^{0|1}[a + \varphi_g(b) + \mu(\mathbf{g}, \mathbf{h})]_{gh}, & \text{if } n_3(\mathbf{g}_0, \mathbf{g}, \mathbf{h}) = 1 \end{cases}. \tag{147}$$

The quantum dimension of any G -graded string type is 1. We note that in this simple case $s(\alpha) = 0$ for any fusion state α , while $s'(\mathbf{g}_0, \alpha) = n_3(\mathbf{g}_0, \mathbf{g}, \mathbf{h})$.

- (iv) The phase factor $\mathbf{g}_0 \Theta(a_g, b_h, c_k)$ is obtained by solving Eq. (127). In this simple case, when $\mathbf{g}_0$ is fixed, $\mathbf{g}_0 \Theta(a_g, b_h, c_k) \in C^3[\mathcal{G}, U(1)]$ is an $U(1)$ -valued 3-cochain, and Eq. (127) is a twisted 3-cocycle condition for each fixed $\mathbf{g}_0$.
- (v) Surface $\mathcal{F}$ moves $\mathbf{1}[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}$ with leftmost group element being $\mathbf{1}$, abbreviated as $\mathbf{1}F^{a_g b_h c_k}$, are obtained by solving the 't Hooft anomalous fermionic pentagon equation in Eq. (128), i.e.,

$$\begin{aligned}
& \mathbf{1}F^{(a_g \times b_h) c_k d_l} \cdot \mathbf{1}F^{a_g b_h (c_k \times d_l)} \\
& = \mathbf{g}\Theta(b_h, c_k, d_l) \nu_4(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}) \mathbf{1}F^{a_g b_h c_k} \\
& \cdot \mathbf{1}F^{a_g (b_h \times c_k) d_l} \cdot \mathbf{1}F^{b_h c_k d_l}.
\end{aligned} \tag{148}$$

And surface $\mathcal{F}$ moves $\mathbf{g}_0 F^{a_g b_h c_k}$ with general leftmost group element $\mathbf{g}_0$ directly obtained by Eq. (126).

In this simple case, we can relate surface fermionic symmetry-enriched string-net model data with the surface

anyon data, as introduced by Ref. [12]. We first write the surface F moves in the following special form:

$$\mathbf{1}F^{a_g b_h c_k} = \chi_c(\mathbf{g}, \mathbf{h}) \alpha_3(\mathbf{g}, \mathbf{h}, \mathbf{k}), \tag{149}$$

where Eq. (148) implies that χ and α_3 should satisfy the following properties:

- (i) χ is a character on N :

$$\chi_a(\mathbf{g}, \mathbf{h}) \chi_b(\mathbf{g}, \mathbf{h}) = \chi_{a+b}(\mathbf{g}, \mathbf{h}). \tag{150}$$

- (ii) Fixing $a, \chi_a \in C^2[G, U(1)]$ is a 2-cochain satisfying

$$\frac{\chi_d(\mathbf{g}, \mathbf{h}, \mathbf{k}) \chi_d(\mathbf{g}, \mathbf{h})}{\chi_d(\mathbf{g}, \mathbf{h}, \mathbf{k}) \chi_d^{S(\mathbf{g})}(\mathbf{h}, \mathbf{k})} = \mathbf{g}\Theta(b_h, c_k, d_l). \tag{151}$$

- (iii) $\alpha_3 \in C^3[G, U(1)]$ is a 3-cochain satisfying

$$\chi_{\mu(\mathbf{k}, \mathbf{l})}(\mathbf{g}, \mathbf{h}) = \nu_4(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}) d\alpha_3(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}). \tag{152}$$

We show derivations of the above properties in Appendix C 2. The surface anyons of Abelian N gauge

theory are described by $\mathcal{Z}_1(\text{Vec}_N) \boxtimes \{1, f\}$. We denote the anyons by

$$\{a = (a_m, a_e) | a_m, a_e \in N\}, \quad (153)$$

where $a_m \in N$ are gauge fluxes, $a_e: N \rightarrow U(1)$ are gauge charges that are one-dimensional irreducible representations of N , which also form a group $\hat{N} \cong N$ called character group, i.e., $a_e \in \hat{N}$. Recall Definition B.7, assuming that the G action (or anyon permutation) $\rho: G \rightarrow \text{Aut}[\mathcal{Z}_1(\text{Vec}_N)]$ is in the following relatively trivial form:

$$\rho_{\mathbf{g}}(a) = [a_m, \rho_{\mathbf{g}}(a_e)], \quad (154)$$

i.e., the G action does not permute any gauge flux, where for nonanomalous Abelian N gauge theory it reduces to $\rho_{\mathbf{g}}(a) = (a_m, a_e^{S(\mathbf{g})})$. Then the anyon valued 2-cocycle characterizing the symmetry fractionalization class $\mathfrak{w} \in H^2_\rho[G, \mathcal{Z}_1(\text{Vec}_N)]$ [see Eq. (B17), and we note that in this trivial symmetry action case, $\mathfrak{w}$ is a 2-cocycle] can be separated into

$$\mathfrak{w} = \mathfrak{w}_e \otimes \mathfrak{w}_m \equiv (\mathfrak{w}_e, \mathfrak{w}_m), \quad (155)$$

where $\mathfrak{w}_e \in H^2_\rho(G, N)$ and $\mathfrak{w}_m \in H^2_\rho(G, \hat{N})$. As shown in Ref. [12], we have the following relation between surface exactly solvable model data and anyon symmetry fractionalization data:

$$\mu(\mathbf{g}, \mathbf{h}) \equiv \mathfrak{w}_e(\mathbf{g}, \mathbf{h}), \quad \chi_{a_m}(\mathbf{g}, \mathbf{h}) \equiv R^{[-, \mathfrak{w}_m(\mathbf{g}, \mathbf{h})]}(a_m, -), \quad (156)$$

where $R^{(a_m, a_e)(b_m, b_e)}$ is the R symbol (half-braiding) between two anyons $a = (a_m, a_e)$ and $b = (b_m, b_e)$.

F. Example: Surface $\mathcal{Z}_1(\text{Vec}_{\mathbb{Z}_4}) \boxtimes \{1, f\}$ topological order with $\mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_4$ symmetry

We briefly review anyon data studies in Refs. [51, 65] in Appendix E. The surface anyon types are $\mathcal{Z}_1(\text{Vec}_{\mathbb{Z}_4}) \boxtimes \{1, f\}$, where $\mathcal{Z}_1(\text{Vec}_{\mathbb{Z}_4})$ is the unitary modular tensor category of anyon types in the $\mathbb{Z}_4$ gauge theory, f is the physical fermion, and $\mathcal{Z}_1(\text{Vec}_{\mathbb{Z}_4}) \boxtimes \{1, f\}$ is a super modular category (see Definition A.14). The corresponding bulk phase is the root phase of $\mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_4$ -fSPT phase, as reviewed below. We construct the surface fermionic symmetry-enriched string-net model for this example according to Sec. III E.

1. Bulk 3+1D $\mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_4$ -fSPT phases

The classification of fSPT phases with $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_4$ symmetry is [69]

$$\mathbb{Z}_2 \times \mathbb{Z}_4, \quad (157)$$

where the classification group structure is a $\mathbb{Z}_2$ complex fermion layer embedded in a $\mathbb{Z}_2 \times \mathbb{Z}_2$ BSPT phase layer. Denote the four phases in the $\mathbb{Z}_4$ classification as $\nu = 0, 1, 2, 3$. Denote group elements in $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ as $\{\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2) | \mathbf{g}_1 = 0, 1, \mathbf{g}_2 = 0, 1, 2, 3\}$, and the group multiplication rule is written as $\mathbf{gh} = (\mathbf{g}_1 + \mathbf{h}_1, \mathbf{g}_2 + \mathbf{h}_2)$.

For the $\nu = 1$ root phase in the $\mathbb{Z}_4$ classification, the (n_3, ν_4) data is

$$n_3(\mathbf{g}, \mathbf{h}, \mathbf{k}) = [\mathbf{g}_2 \mathbf{h}_1 \mathbf{k}_1]_2, \quad \nu_4(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}) = e^{\frac{\pi i}{2} \mathbf{g}_2 \mathbf{h}_1 \mathbf{k}_1 \mathbf{l}_1}, \quad (158)$$

where $[a]_2 := a \pmod{2}$, and they satisfy

$$\begin{aligned} d\nu_4(\mathbf{g}_0, \mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}) &= (-1)^{n_3 \sim_1 n_3(\mathbf{g}_0, \mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l})} \\ &= (-1)^{(\mathbf{g}_0)_2 \mathbf{g}_1 \mathbf{h}_1 \mathbf{k}_1 \mathbf{l}_1}. \end{aligned} \quad (159)$$

[Please do not confuse the notation $\mathbf{g}_0 = [(\mathbf{g}_0)_1, (\mathbf{g}_0)_2] \in \mathbb{Z}_2 \times \mathbb{Z}_4$.] We note that there can be other solutions up to coboundary transformations, e.g., $n_3(\mathbf{g}, \mathbf{h}, \mathbf{k}) = [\mathbf{g}_1 \mathbf{h}_1 \mathbf{k}_2]_2, \nu_4(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}) = e^{\frac{\pi i}{2} \mathbf{g}_1 \mathbf{h}_1 \mathbf{k}_1 \mathbf{l}_2}$ is another solution.

For the $\nu = 2$ phase, which is bosonic, the ν_4 data is

$$\nu_4(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}) = e^{\pi i \mathbf{g}_2 \mathbf{h}_1 \mathbf{k}_1 \mathbf{l}_1}, \quad (160)$$

which is a 4-cocycle.

2. Surface fermionic symmetry-enriched string-net model for the $\nu=1$ bulk fSPT phase

According to Sec. III E, we label the surface string types by

$$\mathcal{G} := \{a_{\mathbf{g}} | a = 0, 1, 2, 3, \mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2) \in \mathbb{Z}_2 \times \mathbb{Z}_4\}. \quad (161)$$

Group multiplication rule in $\mathcal{G}$ is

$$a_{\mathbf{g}} \times b_{\mathbf{h}} = [a + b + \mathbf{g}_1 \mathbf{h}_1]_4, \quad (162)$$

where $[a]_4 = a \pmod{4}$, and $\mu(\mathbf{g}, \mathbf{h}) = \mathbf{g}_1 \mathbf{h}_1$, i.e.,

$$\mathcal{G} \cong \mathbb{Z}_8 \times \mathbb{Z}_4. \quad (163)$$

The surface string fusion rules are

$$a_{\mathbf{g}} [a_{\mathbf{g}} \otimes b_{\mathbf{h}}] = \begin{cases} \mathbb{C}^{1|0}([a + b + \mathbf{g}_1 \mathbf{h}_1]_4)_{\mathbf{gh}}, & \text{if } n_3(\mathbf{g}_0, \mathbf{g}, \mathbf{h}) = 0 \\ \mathbb{C}^{0|1}([a + b + \mathbf{g}_1 \mathbf{h}_1]_4)_{\mathbf{gh}}, & \text{if } n_3(\mathbf{g}_0, \mathbf{g}, \mathbf{h}) = 1 \end{cases}, \quad (164)$$

i.e., $s'(\mathbf{g}_0, \alpha) = n_3(\mathbf{g}_0, \mathbf{g}, \mathbf{h}) = [(\mathbf{g}_0)_2 \mathbf{g}_1 \mathbf{h}_1]_2$ while $s(\alpha) = 0$. Solving Eq. (127), where $d\nu_4(\mathbf{g}_0, \mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}) = (-1)^{(\mathbf{g}_0)_2 \mathbf{g}_1 \mathbf{h}_1 \mathbf{k}_1 \mathbf{l}_1}$, we obtain the fermionic obstruction

$$g_0 \Theta(a_{\mathbf{g}}, b_{\mathbf{h}}, c_{\mathbf{k}}) = (-1)^{(\mathbf{g}_0)_2 \mathbf{g}_1 \mathbf{h}_1 c}. \quad (165)$$

Surface F moves with leftmost group element $\mathbf{1}$ are

$$1F^{a_g b_h c_k} = e^{\frac{\pi i}{2} \mathbf{g}_2 \mathbf{h}_1 c}. \quad (166)$$

They satisfy Eq. (128), or Eq. (148) in surface Abelian gauge theory topological order case, i.e.,

$$\begin{aligned} & 1F^{(a_g \times b_h) c_k d_1} \cdot 1F^{a_g b_h (c_k \times d_1)} \\ &= (-1)^{\mathbf{g}_2 \mathbf{h}_1 \mathbf{k}_1 d} \cdot e^{\frac{\pi i}{2} \mathbf{g}_2 \mathbf{h}_1 \mathbf{k}_1 d} \\ & 1F^{a_g b_h c_k} \cdot 1F^{a_g (b_h \times c_k) d_1} \cdot 1F^{b_h c_k d_1}. \end{aligned} \quad (167)$$

Then by Eq. (126), surface F moves with general leftmost group element $\mathbf{g}_0$ are

$$\mathbf{g}_0 F^{a_g b_h c_k} = (-1)^{(\mathbf{g}_0)_2 \mathbf{g}_1 \mathbf{h}_1 c} e^{\frac{\pi i}{2} (\mathbf{g}_0)_2 \mathbf{g}_1 \mathbf{h}_1 \mathbf{k}_1} e^{\frac{\pi i}{2} \mathbf{g}_2 \mathbf{h}_1 c}. \quad (168)$$

They satisfy Eq. (129), i.e.,

$$\begin{aligned} & \mathbf{g}_0 F^{(a_g \times b_h) c_k d_1} \cdot \mathbf{g}_0 F^{a_g b_h (c_k \times d_1)} \\ &= (-1)^{\mathbf{g}_2 \mathbf{h}_1 \mathbf{k}_1 d} \cdot e^{\frac{\pi i}{2} \mathbf{g}_2 \mathbf{h}_1 \mathbf{k}_1 d} \\ & \mathbf{g}_0 F^{a_g b_h c_k} \cdot \mathbf{g}_0 F^{a_g (b_h \times c_k) d_1} \cdot \mathbf{g}_0 F^{b_h c_k d_1}. \end{aligned} \quad (169)$$

We have checked with *Mathematica* that the surface F moves with general leftmost group element $\mathbf{g}_0$ satisfy the above equation, where Eq. (167) corresponds to the case that $(\mathbf{g}_0)_2 = 0$.

Remark III.10. The anyon data (i.e., anyon types and symmetry fractionalization) obtained from the surface fermionic symmetry-enriched string-net model exactly matches with the anyon data in Refs. [51,65]. Since in this example, the surface topological order is simply an Abelian $\mathbb{Z}_4$ gauge theory, the relation in Eq. (156) that relates the above exactly solvable model data to anyon symmetry fractionalization data are satisfied. Let us rewrite the surface F move by

$$1F^{a_g b_h c_k} = \chi_c(\mathbf{g}, \mathbf{h}) = e^{\frac{\pi i}{2} \mathbf{g}_2 \mathbf{h}_1 c}, \quad (170)$$

where $\alpha_3 = 1$ is trivial here. We can check that the properties in Eq. (150)–(152) are satisfied. Then recall the anyon R symbol in Eq. (E2) and the symmetry fractionalization data in Eq. (E6) for this example, it is direct to see that Eq. (156) is satisfied, i.e.,

$$\mathbf{w}_e(\mathbf{g}, \mathbf{h}) = \mathbf{g}_1 \mathbf{h}_1, \quad \mathbf{w}_m(\mathbf{g}, \mathbf{h}) = \mathbf{g}_2 \mathbf{h}_1. \quad (171)$$

Further, we see that in this example, the anyon-level obstruction function o_3 in Eq. (E7), which is the obstruction to extend the symmetry fractionalization in the super modular category to its minimal modular extended theory (as a spin modular category, see

Definition A.18), is exactly the same as the n_3 data in Eq. (117), i.e., we have

$$o_3(\mathbf{g}, \mathbf{h}, \mathbf{k}) = n_3(\mathbf{g}, \mathbf{h}, \mathbf{k}) = [\mathbf{g}_2 \mathbf{h}_1 \mathbf{k}_1]_2. \quad (172)$$

3. Surface symmetry-enriched string-net model for the $\nu=2$ bulk SPT phase

In this case, the bulk 3 + 1D fSPT phase and surface anomalous SET phase are both purely bosonic. Surface string types are the same as $\nu=1$ case. While the surface string fusion rules do not depend on the leftmost group element $\mathbf{g}_0$ anymore:

$$a_g \otimes b_h = ([a + b + \mathbf{g}_1 \mathbf{h}_1]_4)_{\mathbf{gh}}. \quad (173)$$

Surface F moves are

$$1F^{a_g b_h c_k} = e^{\pi i \mathbf{g}_2 \mathbf{h}_1 c}. \quad (174)$$

They satisfy Eq. (128) in the simple case that the phase factor Θ is trivial:

$$1F^{j_{\mathbf{gh}} c_k d_1} \cdot 1F^{a_g b_h q_{\mathbf{kl}}} = e^{\pi i \mathbf{g}_2 \mathbf{h}_1 \mathbf{k}_1 d} \cdot 1F^{a_g b_h c_k} \cdot 1F^{a_g n_{\mathbf{hk}} d_1} \cdot 1F^{b_h c_k d_1}. \quad (175)$$

G. $H^2(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly

For the surface fermionic symmetry-enriched string-net models defined in Sec. III, we only construct up to the $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly. While exactly solvable models for the $H^2(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly introduced in Appendix D are left for future study. We conjecture that the $H^2(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly is characterized by a violation of $\mathbb{Z}_2$ conservation of σ -type strings (Definition II.3) on each vertex. Formally, let $o_2 \in H^2(G, \mathbb{Z}_2)$ be the obstruction to extend the G action from a super modular category to its $\mathbb{Z}_2^f$ modular extension as a spin modular category (see details in Appendix D), where the super modular category describes the surface anyons. Then in the following fusion rule that involves σ -type string on each vertex in the surface fermionic symmetry-enriched string-net model:

$$a_g \otimes b_h = \bigoplus_{c_{\mathbf{gh}}} \mathbb{C}^{n|n} c_{\mathbf{gh}}. \quad (176)$$

Assume that whether a string type a_g is m -type or q/σ -type is fully determined by the group element label $\mathbf{g}$, we denote the total number of σ -type strings that appear in the above fusion rule as $N_\sigma(\mathbf{g}, \mathbf{h})$. We then conjecture that $N_\sigma(\mathbf{g}, \mathbf{h}) \pmod{2}$ is in the same cohomology class as $o_2(\mathbf{g}, \mathbf{h})$, i.e.,

$$[N_\sigma \pmod{2}] = [o_2] \in H^2(G, \mathbb{Z}_2). \quad (177)$$

Furthermore, there is also a limitation in our surface models that they can only deal with simple cases when ω_2 is trivial. How to construct exactly solvable models for surface fSET phases with 't Hooft anomalies when ω_2 is nontrivial is also left for future study.

A limitation of our constructed (surface) fermionic symmetry-enriched string-net models in this paper is that they only realize nonchiral 2 + 1D fermionic topological phases. However, for example, the $H^1(G, \mathbb{Z}_7)$ fermionic 't Hooft anomaly, which corresponds to the $p + ip$ superconductor layer of bulk 3 + 1D fSPT phase, can only be realized by chiral surface fSET phases. There is an attempt to construct exactly solvable models for chiral topological phases in Ref. [70], combining the ideas of which may generalize our construction to chiral cases.

Remark III.11. By Ref. [71], there is a 3-group structure in a 3 + 1D fSPT phase with $G^f = G \times \mathbb{Z}_2^f$ symmetry characterized by (ν_4, n_3, n_2) (we set ω_2 to be trivial as our constructed exactly solvable models for fermionic 't Hooft anomalies are only for ω_2 trivial cases). After gauging the fermion-parity symmetry $\mathbb{Z}_2^f$, the bosonic shadow has a 3-group structure that consists of:

- (i) A 0-form global symmetry G .
- (ii) A 1-form $\mathbb{Z}_2$ symmetry generated by the Kitaev chain surface (noninvertible by itself, but invertible in the 3-group), whose decoration is specified by $n_2 \in H^2(G, \mathbb{Z}_2)$.
- (iii) A 2-form $\mathbb{Z}_2$ symmetry generated by the Wilson line of complex fermion, whose decoration is specified by $n_3 \in H^3(G, \mathbb{Z}_2)$, which is constrained by $dn_3 = n_2 \cup n_2$.

Then let us consider a surface 2 + 1D fSET phase with H^2 fermionic 't Hooft anomaly on its boundary (before gauging $\mathbb{Z}_2^f$). The boundary H^2 anomaly is compensated by the bulk 1-form symmetry generated by the Kitaev chain. Additionally, when n_2 is a coboundary, the boundary H^3 anomaly is compensated by a bulk 2-form symmetry generated by the complex fermion, where n_3 now satisfies $dn_3 = 0$.

IV. MATHEMATICAL FRAMEWORK

A. G -graded super fusion category when ω_2 is trivial

Here we give a partial definition of the G -graded super fusion category.

Definition IV.1. (G -graded super fusion category when ω_2 is trivial). Let $\mathcal{C}_G = \bigoplus_{\mathbf{g} \in G} \mathcal{C}_{\mathbf{g}}$ be a G -graded (unitary) fusion category such that $\mathcal{Z}_1(\mathcal{C}_G)$ is a spin modular category (Definition A.18) that contains a fermion f (Definition A.15), where $\mathcal{Z}_1(\mathcal{C}_G)$ denotes for the Drinfeld center of $\mathcal{C}_G$. [Or equivalently, there exists a braided monoidal functor $\phi: \mathbf{SVec} \rightarrow \mathcal{Z}_1(\mathcal{C}_G)$]. Then there is a monoidal $\{\mathbb{1}, f\} \equiv \mathbf{SVec}$ -action on $\mathcal{C}_G$. If each sector $\mathcal{C}_{\mathbf{g}}$ is closed under such $\mathbf{SVec}$ -action (each $\mathcal{C}_{\mathbf{g}}$ forms an $\mathbf{SVec}$ -module category), we call

$$\mathbf{SVec}\mathcal{C}_G := \bigoplus_{\mathbf{g} \in G} \mathbf{SVec}\mathcal{C}_{\mathbf{g}} \quad (178)$$

a G -graded (unitary) super fusion category when ω_2 is trivial.

We can translate this definition as the following:

- (i) Given that there exists a braided monoidal functor $\phi: \mathbf{SVec} \rightarrow \mathcal{Z}_1(\mathcal{C}_G)$, $\mathbf{SVec}\mathcal{C}_G$ is a super fusion category obtained by a canonical construction of enriched monoidal category (Proposition A.6 and Definition A.7).
- (ii) For the trivial group element $\mathbf{1} \in G$, $\mathcal{Z}_1(\mathcal{C}_G)$ is a spin modular category implies that $\mathcal{Z}_1(\mathcal{C}_1)$ is also a spin modular category that contains the fermion f . Then $\mathbf{SVec}\mathcal{C}_1$ is a super fusion category obtained by a canonical construction of enriched monoidal category.
- (iii) For all nontrivial $\mathbf{g} \in G$, given each $\mathcal{C}_{\mathbf{g}}$ is a $\mathbf{SVec}$ -module category, each $\mathbf{SVec}\mathcal{C}_{\mathbf{g}}$ is a super category obtained by a canonical construction of enriched category (Definition A.4 and Proposition A.5).

Or say, the G -graded super fusion category $\mathbf{SVec}\mathcal{C}_G$ is obtained by a fermion condensation from a G -graded unitary fusion category $\mathcal{C}_G$ such that the G -grading structure in $\mathcal{C}_G$ is preserved. We illustrate the difficulty on defining the general G -graded super fusion category where ω_2 can be nontrivial in Remark IV.5.

Remark IV.2. When ω_2 is nontrivial, there may not exist a $\mathbb{Z}_2$ -conserved fermion (f is called $\mathbb{Z}_2$ conserved if $f \otimes f = \mathbb{1}$) in the Drinfeld center $\mathcal{Z}_1(\mathcal{C}_G)$ of a G -graded fusion category $\mathcal{C}_G$. For example, $\mathcal{Z}_1(\mathbf{Vec}_{\mathbb{Z}_4})$ and $\mathcal{Z}_1(\mathbf{Vec}_{\mathbb{Z}_8})$ both do not contain the $\mathbb{Z}_2$ -conserved fermion. Therefore, general G -graded super fusion categories cannot be constructed by a fermion condensation from a G -graded fusion category, as the fermion condensation construction only covers fermionic global symmetries with trivial ω_2 .

Example IV.3. [Super Tambara-Yamagami category as a $\mathbb{Z}_2$ -graded super fusion category] The Tambara-Yamagami category $\mathbf{TY}_{\mathbb{Z}_{2N}}$ as a unitary fusion category is a generalization of the Ising fusion category, which can be viewed as a $\mathbb{Z}_2$ -graded unitary fusion category:

$$\mathcal{C}_{\mathbb{Z}_2} := \mathbf{TY}_{\mathbb{Z}_{2N}} = \mathcal{C}_0 \oplus \mathcal{C}_1 = \{0, 1, \dots, 2N-1\} \oplus \{\sigma\}. \quad (179)$$

Let $a, b \in 2N$. Fusion rules are $a \otimes b = [a + b]_{2N}$ ($[a]_{2N}$ means $a \pmod{2N}$) and $\sigma \otimes \sigma = \bigoplus_{a \in \mathbb{Z}_{2N}} a$.

When N is odd, the simple object N (together with the half-braiding) is a fermion in $\mathcal{Z}_1(\mathbf{TY}_{\mathbb{Z}_{2N}})$ [46]. Then condensing N , we obtain the super Tambara-Yamagami category as a $\mathbb{Z}_2$ -graded super fusion category:

$$\mathbf{SVec}\mathcal{C}_{\mathbb{Z}_2} = \mathbf{SVec}\mathcal{C}_0 \oplus \mathbf{SVec}\mathcal{C}_1 = \{0, \dots, N-1\} \oplus \{\sigma\}, \quad (180)$$

where σ is a q -type object. Let $a, b \in \{0, \dots, N-1\}$.

Quantum dimensions and fusion rules are listed in Example II H 2.

Example IV.4. [$\mathbf{SVec} \text{SU}(2)_6$ as a $\mathbb{Z}_2$ -graded super fusion category] $\text{SU}(2)_6$ can be viewed as a $\mathbb{Z}_2$ -graded unitary fusion category. Denote simple objects in $\text{SU}(2)_6$ as $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$.

$$\mathcal{C}_{\mathbb{Z}_2} := \text{SU}(2)_6 = \mathcal{C}_0 \oplus \mathcal{C}_1 = \{0, 1, 2, 3\} \oplus \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\right\}, \quad (181)$$

where $f := 3$ is the fermion, and $\mathcal{C}_0 \cong \text{SO}(3)_6$. F symbols in $\text{SU}(2)_6$ are rather complicated and listed in Ref. [59]. Its Drinfeld center is

$$\begin{aligned} \mathcal{Z}_1[\text{SU}(2)_6] &= \text{SU}(2)_6 \boxtimes \overline{\text{SU}(2)_6} \\ &= \left\{ (j_1, j_2) \mid j_1, j_2 = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3 \right\}, \end{aligned} \quad (182)$$

where $(0, 3)$ and $(3, 0)$ are two fermions. By condensing their bound state $(3, 3)$, or say, trivializing $j = 3$ in $\text{SU}(2)_6$, we obtain a $\mathbb{Z}_2$ -graded super fusion category:

$$\mathbf{SVec} \mathcal{C}_{\mathbb{Z}_2} = \mathbf{SVec} \mathcal{C}_0 \oplus \mathbf{SVec} \mathcal{C}_1 = \{0, 1\} \oplus \left\{\frac{1}{2}, \frac{3}{2}\right\}, \quad (183)$$

where $\frac{3}{2}$ is a q -type object and all other simple objects are m -type. Quantum dimensions and fusion rules are listed in Example II H 3.

Remark IV.5. Here we discuss the difficulty on defining general G -graded super fusion category. Our definition of G -graded super fusion category in Definition IV.1 only applies for the case that ω_2 is trivial. This definition cannot be directly generalized to the general case with nontrivial ω_2 , which is due to that this definition is obtained by performing a fermion condensation for a G -graded unitary fusion category (equivalently, by enriching over $\mathbf{SVec}$ for a G -graded unitary fusion category). However, in the general cases where ω_2 is nontrivial, the structure of fermion condensation is unclear (more specifically, the usual definition of fermion condensation, as illustrated in Remark A.8, does not hold for ω_2 nontrivial case in general). We show that why our definition of the G -graded super fusion category fails in ω_2 nontrivial case by a simple example. Since $\mathbb{Z}_4^f$ is the simplest symmetry group with nontrivial ω_2 , let us consider the $\mathbb{Z}_4^f$ trivial fSPT phase, which can be treated as a special fSET phase where the intrinsic topological order is trivial. Gauging the $\mathbb{Z}_4^f$ symmetry, we obtain a $\mathbb{Z}_4$ gauge theory, described by the category of $\mathbb{Z}_4$ -graded vector spaces $\text{Vec}_{\mathbb{Z}_4}$, which can be treated as a special G -graded unitary fusion category $\mathcal{C}_{\mathbb{Z}_4}$ where each

$\mathcal{C}_g = \mathbb{C}$. According to our way to define the G -graded super fusion category, we should start from the G -graded unitary fusion category $\text{Vec}_{\mathbb{Z}_4}$ and perform a fermion condensation. While its Drinfeld center (physically, anyons of $\mathbb{Z}_4$ gauge theory) $\mathcal{Z}_1(\text{Vec}_{\mathbb{Z}_4}) = \{(a_m, a_e) \mid a_m, a_e \in \mathbb{Z}_4\}$ contains no fermion [recall Definition A.15 and the R symbol $R^{(a_m, a_e)(b_m, b_e)} = e^{2\pi i \frac{a_m b_e}{4}}$, the anyons labeled by $(0, 0), (0, 2), (2, 0), (2, 2)$ are all bosons]. Therefore, fermion condensation cannot be performed on $\text{Vec}_{\mathbb{Z}_4}$. A possible way to solve this problem is to generalize the concept of fermion condensation, e.g., we can condense $\text{Rep}(\mathbb{Z}_4^f)$ [$\mathbb{Z}_4$ super-Tannakian category, or $\text{Rep}(\mathbb{Z}_4, z)$] instead of $\mathbf{SVec} = \{1, f\}$. However, there does not exist a braided monoidal functor from $\text{Rep}(\mathbb{Z}_4^f)$ to $\mathcal{Z}_1(\text{Vec}_{\mathbb{Z}_4})$ (an anyon condensation can be performed if and only if there exists a braided monoidal functor from the category of condensed anyons to the category of all anyons, which is equivalent to the existence of a Lagrangian algebra for the category of all anyons), this naive generalization does not work. Therefore, how to define the G -graded super fusion category in ω_2 nontrivial case remains as an unsolved problem.

B. Drinfeld center of super fusion category

We review the Drinfeld center of a super fusion category [45] in this section. Let $\mathbf{SVec} \mathcal{C}$ be the super fusion category obtained by fermion condensation or the canonical construction in Definition A.7 from a unitary fusion category $\mathcal{C}$. The Drinfeld center of $\mathbf{SVec} \mathcal{C}$ is

$$\mathcal{Z}_1(\mathbf{SVec} \mathcal{C}) \cong \mathbf{SVec} \mathcal{Z}_2[\mathcal{Z}_1(\mathcal{C}), \mathbf{SVec}] \cong \mathbf{SVec} \mathcal{Z}_1(\mathcal{C})_0, \quad (184)$$

where $\mathcal{Z}_2[\mathcal{Z}_1(\mathcal{C}), \mathbf{SVec}]$ is the relative Müger center of $\mathcal{Z}_1(\mathcal{C})$ relative to $\mathbf{SVec}$ (Definition A.13), and

$$\mathcal{Z}_1(\mathcal{C}) \cong \mathcal{Z}_1(\mathcal{C})_0 \oplus \mathcal{Z}_1(\mathcal{C})_1 \quad (185)$$

is a spin modular category (Definition A.13, and 0, 1 are group elements in $\mathbb{Z}_2^f$), and $\mathcal{Z}_1(\mathcal{C})_0$ is a super modular category (Definition A.14). Then $\mathbf{SVec} \mathcal{Z}_1(\mathcal{C})_0$ is defined as $\mathcal{Z}_1(\mathcal{C})_0$ enriched in $\mathbf{SVec}$ by a canonical construction of enriched braided monoidal category (inputted by a symmetric monoidal functor $\mathbf{SVec} \rightarrow \mathcal{Z}_2[\mathcal{Z}_1(\mathcal{C})_0]$) [49]. We clarify the relations among all mentioned categories by the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\text{Fermion condensation}} & \mathbf{SVec} \mathcal{C} \\
\downarrow \text{Drinfeld center} & & \downarrow \text{Drinfeld center} \\
\mathcal{Z}_2^f\text{-equivariantization} & \swarrow & \nwarrow \mathcal{Z}_2^f\text{-crossed extension} \\
\mathcal{Z}_1(\mathcal{C}) & \xrightarrow{\text{Fermion condensation}} & \mathbf{SVec} \mathcal{Z}_1(\mathcal{C})_0 \\
\swarrow \mathcal{Z}_2^f\text{-modular extension} & \xleftarrow{\text{Gauging } \mathbb{Z}_2^f} & \searrow \mathcal{Z}_2^f\text{-equivariantization} \\
& \mathcal{Z}_1(\mathcal{C})_0 &
\end{array}, \quad (186)$$

where the $\mathbb{Z}_2^f$ -modular extension is introduced in Definition A.21, G -equivariantization is introduced in Definition B.14, and the commutativity of the lower-half diagram is introduced in Remark B.18. Further, the $\mathbb{Z}_2^f$ -modular extended category

$$[\mathbf{SVec} \mathcal{Z}_1(\mathcal{C})_0]_{\mathbb{Z}_2^f}^{\times} \cong \mathbf{SVec} \mathcal{Z}_1(\mathcal{C})_0 \oplus \mathbf{SVec} \mathcal{Z}_1(\mathcal{C})_1, \quad (187)$$

where $\mathbf{SVec} \mathcal{Z}_1(\mathcal{C})_0$ is called the *bounding sector* (the sector of anyons), and $\mathbf{SVec} \mathcal{Z}_1(\mathcal{C})_1$ is called the *nonbounding sector* (the sector of fermion-parity vortices) in Ref. [45].

Remark IV.6. A fermion condensation on a spin modular category (Definition A.18) $\mathcal{Z}_1(\mathcal{C})$ consists of two steps:

- (1) Anyons with nontrivial braiding with the fermion will be confined, which is the inverse step of the $\mathbb{Z}_2^f$ -modular extension.
- (2) Two anyons differ by the fermion will be identified, which is the inverse step of the $\mathbb{Z}_2^f$ equivariantization.

Remark IV.7. According to Ref. [45], there is a way to obtain the ground state degeneracy (GSD) on torus $\mathbb{T}^2$. Denote the four spin structures on torus as BB, NB, BN, NN , where B stands for bounding (anti-periodic) and NB for nonbounding (periodic). We have

$$\begin{aligned}
\text{GSD}(\mathbb{T}_{BB}^2) &= \text{GSD}(\mathbb{T}_{BN}^2) \\
&= \text{Number of simple objects in } \mathbf{SVec} \mathcal{Z}_1(\mathcal{C})_0,
\end{aligned} \quad (188)$$

or say, the number of anyon types in the bounding sector, and

$$\begin{aligned}
\text{GSD}(\mathbb{T}_{NB}^2) &= \text{GSD}(\mathbb{T}_{NN}^2) \\
&= \text{Number of simple objects in } \mathbf{SVec} \mathcal{Z}_1(\mathcal{C})_1,
\end{aligned} \quad (189)$$

or say, the number of fermion-parity vortex types in the nonbounding sector.

Example IV.8. (Anyons of Majorana toric code) Recall the notations in Example A.11. Denote (isomorphism classes of) simple objects in Ising fusion category as $\mathbf{Ising} = \{1, f, \sigma\}$, and simple objects in its Drinfeld center as $\mathcal{Z}_1(\mathbf{Ising}) \cong \mathbf{Ising} \boxtimes \overline{\mathbf{Ising}} = \{1, f, \sigma\} \boxtimes \{1, \bar{f}, \bar{\sigma}\}$, which describes anyons in the Ising string-net model. Then anyons in the Majorana toric code topological order is described by the Drinfeld center of a super fusion category $\mathcal{Z}_1(\mathbf{SVec} \mathbf{Ising}) \cong \mathbf{SVec} \mathcal{Z}_1(\mathbf{Ising})_0 = \{m_1, m_f, m_\sigma\}$ (full data please refer to Ref. [45]), where $d_{m_1} = d_{m_f} = 1, d_{m_\sigma} = \sqrt{2}$.

Then we have two different ways of gauging $\mathbb{Z}_2^f$:

- (i) Performing a $\mathbb{Z}_2^f$ equivariantization on $\mathbf{SVec} \mathcal{Z}_1(\mathbf{Ising})_0$, we obtain $\mathcal{Z}_1(\mathbf{Ising})_0 = \{1, f, \bar{f}, f\bar{f}, \bar{\sigma}, f\bar{\sigma}\}$ (and $\mathcal{Z}_1(\mathbf{Ising})_1 = \{\sigma, \bar{f}\sigma, \sigma\bar{\sigma}\}$). Then performing a $\mathbb{Z}_2^f$ -modular extension on $\mathcal{Z}_1(\mathbf{Ising})_0$, we obtain $\mathcal{Z}_1(\mathbf{Ising})$.
- (ii) Performing a $\mathbb{Z}_2^f$ -crossed extension on $\mathbf{SVec} \mathcal{Z}_1(\mathbf{Ising})_0$, we obtain $[\mathbf{SVec} \mathcal{Z}_1(\mathbf{Ising})_0]_{\mathbb{Z}_2^f}^{\times} = \{m_1, m_f, m_\sigma\} \oplus \{q_1, q_f, q_\sigma\}$ [45], where $d_{q_1} = d_{q_f} = \sqrt{2}, d_{q_\sigma} = 2$. Then performing a $\mathbb{Z}_2^f$ equivariantization, we obtain $\mathcal{Z}_1(\mathbf{Ising})$.

We conclude these relations in the following commutative diagram:

$$\begin{array}{ccc}
\mathbf{Ising} & \xrightarrow{\text{Fermion condensation}} & \text{MTC} := \mathbf{SVec} \mathbf{Ising} \\
\downarrow \text{Drinfeld center} & & \downarrow \text{Drinfeld center} \\
\mathcal{Z}_2^f\text{-equivariantization} & \swarrow & \nwarrow \mathcal{Z}_2^f\text{-crossed extension} \\
\mathcal{Z}_1(\mathbf{Ising}) & \xrightarrow{\text{Fermion condensation}} & \mathbf{SVec} \mathcal{Z}_1(\mathbf{Ising})_0 \\
\swarrow \mathcal{Z}_2^f\text{-modular extension} & \xleftarrow{\text{Gauging } \mathbb{Z}_2^f} & \searrow \mathcal{Z}_2^f\text{-equivariantization} \\
& \mathcal{Z}_1(\mathbf{Ising})_0 &
\end{array}. \quad (190)$$

Example IV.9. (Drinfeld center of $\text{SVec } \frac{1}{2}\text{E}_6$) $\frac{1}{2}\text{E}_6 = \{1, x, y\}$ is a unitary fusion category, where the quantum dimensions are $d_1 = d_y = 1, d_x = 1 + \sqrt{3}$, and the fusion rules are

$$\begin{aligned} x \otimes x &= 1 \oplus 2x \oplus y, & x \otimes y &= y \otimes x = x, \\ y \otimes y &= 1. \end{aligned} \quad (191)$$

F -symbols (or associators) are listed in Ref. [72]. Its Drinfeld center $\mathcal{Z}_1(\frac{1}{2}\text{E}_6) = \{1, Y, X_1, X_2, X_3, X_4, X_5, U, V, W\}$ has 10 simple objects as introduced in Ref. [72], where Y is a fermion (Definition A.15). Since y in $\frac{1}{2}\text{E}_6$ (together with half-braiding) is promoted to be the fermion Y in its Drinfeld center, a fermion condensation can be performed on $\frac{1}{2}\text{E}_6$, and the resultant super fusion category is denoted as $\frac{1}{2}\text{E}_6/y \equiv \text{SVec } \frac{1}{2}\text{E}_6$. Fusion rules in $\text{SVec } \frac{1}{2}\text{E}_6$ are

$$1 \otimes x = x \otimes 1 = \mathbb{C}^{1|1}x, \quad x \otimes x = \mathbb{C}^{1|1}1 \oplus \mathbb{C}^{2|2}x. \quad (192)$$

F symbols are listed in Ref. [46]. Its Drinfeld center $\mathcal{Z}_1(\text{SVec } \frac{1}{2}\text{E}_6) = \text{SVec } \mathcal{Z}_1(\frac{1}{2}\text{E}_6)_0 = \{m_1, m_2, m_3\}$ as introduced in Ref. [45], where $d_{m_1} = 1, d_{m_2} = 2 + \sqrt{3}, d_{m_3} = 1 + \sqrt{3}$. Then we have two different ways of gauging $\mathbb{Z}_2^f$:

- (i) Performing a $\mathbb{Z}_2^f$ equivariantization on $\text{SVec } \mathcal{Z}_1(\frac{1}{2}\text{E}_6)_0$, we obtain $\mathcal{Z}_1(\frac{1}{2}\text{E}_6)_0 = \{1, Y, U, V, X_4, X_5\}$ (and $\mathcal{Z}_1(\frac{1}{2}\text{E}_6)_1 = \{W, X_1, X_2, X_3\}$). Then performing a $\mathbb{Z}_2^f$ -modular extension on $\mathcal{Z}_1(\frac{1}{2}\text{E}_6)_0$, we obtain $\mathcal{Z}_1(\frac{1}{2}\text{E}_6)$.
- (ii) Performing a $\mathbb{Z}_2^f$ -crossed extension on $\text{SVec } \mathcal{Z}_1(\frac{1}{2}\text{E}_6)_0$, we obtain $[\text{SVec } \mathcal{Z}_1(\frac{1}{2}\text{E}_6)_0]_{\mathbb{Z}_2^f}^\times = \{m_1, m_2, m_3\} \oplus \{q_1, q_2, m_4\}$ [45], where $d_{q_1} = 3 + \sqrt{3}, d_{q_2} = d_{m_4} = 1 + \sqrt{3}$. Then performing a $\mathbb{Z}_2^f$ equivariantization, we obtain $\mathcal{Z}_1(\frac{1}{2}\text{E}_6)$.

We conclude these relations in the following commutative diagram:

$$\begin{array}{ccc} \frac{1}{2}\text{E}_6 & \xrightarrow{\text{Fermion condensation}} & \frac{1}{2}\text{E}_6/y \equiv \text{SVec } \frac{1}{2}\text{E}_6 \\ \downarrow \text{Drinfeld center} & & \downarrow \text{Drinfeld center} \\ \mathcal{Z}_1(\frac{1}{2}\text{E}_6) & \xleftarrow{\mathbb{Z}_2^f\text{-equivariantization}} [\text{SVec } \mathcal{Z}_1(\frac{1}{2}\text{E}_6)_0]_{\mathbb{Z}_2^f}^\times \xleftarrow{\mathbb{Z}_2^f\text{-crossed extension}} \text{SVec } \mathcal{Z}_1(\frac{1}{2}\text{E}_6)_0 \\ \mathcal{Z}_1(\frac{1}{2}\text{E}_6) & \xrightleftharpoons[\text{Gauging } \mathbb{Z}_2^f]{\text{Fermion condensation}} & \text{SVec } \mathcal{Z}_1(\frac{1}{2}\text{E}_6)_0 \\ \mathcal{Z}_1(\frac{1}{2}\text{E}_6)_0 & \xleftarrow{\mathbb{Z}_2^f\text{-modular extension}} \mathcal{Z}_1(\frac{1}{2}\text{E}_6) \xleftarrow{\mathbb{Z}_2^f\text{-equivariantization}} & \end{array} \quad (193)$$

Example IV.10. (Drinfeld center of $\text{SVec } \text{SU}(2)_6$). The Drinfeld center $\mathcal{Z}_1[\text{SVec } \text{SU}(2)_6] \cong \text{SVec } [\text{SO}(3)_6 \boxtimes \overline{\text{SO}(3)}_6]$. Its $\mathbb{Z}_2^f$ -crossed extension is denoted as $\{\text{SVec } \mathcal{Z}_1[\text{SU}(2)_6]_0\}_{\mathbb{Z}_2^f}^\times \cong \text{SVec } [\text{SU}(2)_6 \boxtimes \overline{\text{SU}(2)}_6]$ [45,73]. We conclude these relations in the following commutative diagram:

$$\begin{array}{ccc} \text{SU}(2)_6 & \xrightarrow{\text{Fermion condensation}} & \text{SVec } \text{SU}(2)_6 \\ \downarrow \text{Drinfeld center} & & \downarrow \text{Drinfeld center} \\ \text{SU}(2)_6 \boxtimes \overline{\text{SU}(2)}_6 & \xleftarrow{\mathbb{Z}_2^f\text{-equivariantization}} \text{SVec } (\text{SU}(2)_6 \boxtimes \overline{\text{SU}(2)}_6) \xleftarrow{\mathbb{Z}_2^f\text{-crossed extension}} \text{SVec } (\text{SO}(3)_6 \boxtimes \overline{\text{SO}(3)}_6) \\ \text{SU}(2)_6 \boxtimes \overline{\text{SU}(2)}_6 & \xrightleftharpoons[\text{Gauging } \mathbb{Z}_2^f]{\text{Fermion condensation}} & \text{SVec } (\text{SO}(3)_6 \boxtimes \overline{\text{SO}(3)}_6) \\ \text{SO}(3)_6 \boxtimes \overline{\text{SO}(3)}_6 & \xleftarrow{\mathbb{Z}_2^f\text{-modular extension}} \text{SU}(2)_6 \boxtimes \overline{\text{SU}(2)}_6 \xleftarrow{\mathbb{Z}_2^f\text{-equivariantization}} & \end{array} \quad (194)$$

Recall Definition IV.1, when ω_2 is trivial, a G -graded super fusion category $\mathcal{S}_G := \text{SVec } \mathcal{C}_G$, where $\mathcal{S}_1 := \text{SVec } \mathcal{C}_1$ is always a super fusion category. Therefore, $\mathcal{S}_1 := \text{SVec } \mathcal{C}_1$ always satisfies the commutative diagram (186). Then recall Conjecture 1,

the Drinfeld center $\mathcal{Z}_1(\mathcal{S}_1)$ of the trivial grading sector $\mathcal{S}_1$ describes the anyon or quasiparticle excitations of a fermionic symmetry-enriched string-net model inputted by $\mathcal{S}_G$.

C. Drinfeld center of G -graded super fusion category when ω_2 is trivial

Conjecture 2. The Drinfeld center of a G -graded super fusion category ${}^{\text{SVec}}\mathcal{C}_G$ (ω_2 is trivial) is

$$\mathcal{Z}_1({}^{\text{SVec}}\mathcal{C}_G) \cong {}^{\text{SVec}}\mathcal{Z}_2[\mathcal{Z}_1(\mathcal{C}_G), \text{SVec}] \cong {}^{\text{SVec}}\mathcal{Z}_1(\mathcal{C}_G)_0, \quad (195)$$

where $\mathcal{Z}_2[\mathcal{Z}_1(\mathcal{C}_G), \text{SVec}]$ is the relative Müger center of spin modular category $\mathcal{Z}_1(\mathcal{C}_G)$ relative to SVec , and ${}^{\text{SVec}}\mathcal{Z}_1(\mathcal{C}_G)_0$ is defined as $\mathcal{Z}_1(\mathcal{C}_G)_0$ enriched in SVec by canonical construction [inputted by a symmetric monoidal functor $\text{SVec} \rightarrow \mathcal{Z}_2[\mathcal{Z}_1(\mathcal{C}_G)_0]$] [49].

Conjecture 3. There exists a G^f action (i.e., anyon permutation by G^f , see Sec. B 1) on $\mathcal{Z}_1(\mathcal{S}_1)$. We conjecture that after gauging G , i.e., performing a G -crossed extension and then a G equivariantization, we obtain $\mathcal{Z}_1({}^{\text{SVec}}\mathcal{C}_G) \cong {}^{\text{SVec}}\mathcal{Z}_1(\mathcal{C}_G)_0$, and the following diagram commutes:

$$\begin{array}{ccccc}
 & & \text{Gauging } G & & \\
 & & \curvearrowright & & \\
 \mathcal{Z}_1({}^{\text{SVec}}\mathcal{C}_1) \cong {}^{\text{SVec}}\mathcal{Z}_1(\mathcal{C}_1)_0 & \xrightarrow{\text{G-crossed extension}} & G\text{-crossed super-modular} & \xrightarrow{G\text{-equivariantization}} & (\mathcal{Z}_1({}^{\text{SVec}}\mathcal{C}_1)_G^\times)^G \cong \mathcal{Z}_1({}^{\text{SVec}}\mathcal{C}_G) \\
 \text{with } G^f\text{-action} & & \text{category } \mathcal{Z}_1({}^{\text{SVec}}\mathcal{C}_1)_G^\times & & \cong {}^{\text{SVec}}\mathcal{Z}_1(\mathcal{C}_G)_0 \\
 \downarrow \text{Gauging } \mathbb{Z}_2^f & & \searrow \text{Gauging } G^f & & \downarrow \text{Gauging } \mathbb{Z}_2^f \\
 \mathcal{Z}_1(\mathcal{C}_1) \text{ with } G\text{-action} & \xrightarrow{\text{G-crossed extension}} & G\text{-crossed spin-modular} & \xrightarrow{G\text{-equivariantization}} & (\mathcal{Z}_1(\mathcal{C}_1)_G^\times)^G \cong \mathcal{Z}_1(\mathcal{C}_G) \\
 & & \text{category } \mathcal{Z}_1(\mathcal{C}_1)_G^\times & & \\
 & & \text{Gauging } G & &
 \end{array} \quad (196)$$

where we note that in Ref. [43], a G^f -crossed extension is performed on the super modular category $\mathcal{Z}_1({}^{\text{SVec}}\mathcal{C}_1)$, while here we perform a G -crossed extension on it in order that the diagram looks “symmetric”.

Combining Diagrams (186) and (196), we have the following commutative diagram that reveals the spirit of the fermionic symmetry-enriched string-net model construction of 2 + 1D fSET phases when ω_2 is trivial, i.e., inputting a G -graded super fusion category ${}^{\text{SVec}}\mathcal{C}_G$, taking its Drinfeld center and then ungauging G (the inverse process of gauging G), we obtain anyon excitations in this exactly solvable model:

$$\begin{array}{ccccc}
 {}^{\text{SVec}}\mathcal{C}_G & \xrightarrow{\text{Drinfeld center}} & \mathcal{Z}_1({}^{\text{SVec}}\mathcal{C}_G) & \xrightarrow{\text{Ungaunging } G} & \mathcal{Z}_1({}^{\text{SVec}}\mathcal{C}_1) \text{ with } G^f\text{-action} \\
 \uparrow \text{F.c.} & & \uparrow \text{F.c.} & & \uparrow \text{F.c.} \\
 \mathcal{C}_G & \xrightarrow{\text{Drinfeld center}} & \mathcal{Z}_1(\mathcal{C}_G) & \xrightarrow{\text{Ungaunging } G} & \mathcal{Z}_1(\mathcal{C}_1) \text{ with } G\text{-action}
 \end{array} \quad (197)$$

where F.c. is short for fermion condensation, and the bosonic case in the second row is stated in Refs. [12,14].

Example IV.11. (Fermionic $\mathbb{Z}_2$ -enriched string-net model inputted by ${}^{\text{SVec}}\text{SU}(2)_6$). Let us consider the fermionic $\mathbb{Z}_2$ -enriched string-net model inputted by ${}^{\text{SVec}}\text{SU}(2)_6$ as a $\mathbb{Z}_2$ -graded super fusion category. The Drinfeld center $\mathcal{Z}_1({}^{\text{SVec}}\text{SU}(2)_6)$ describes the anyon excitations in this exactly solvable model after gauging G . Then by ungauging G , we obtain anyons with $\mathbb{Z}_2$ symmetry action described by $\mathcal{Z}_1({}^{\text{SVec}}\mathcal{C}_1) \cong \mathcal{Z}_1[{}^{\text{SVec}}\text{SO}(3)_6]$. These relations are concluded in the following commutative diagram:

$$\begin{array}{ccccc}
 {}^{\text{SVec}}\text{SU}(2)_6 & \xrightarrow{\text{Drinfeld center}} & {}^{\text{SVec}}(\text{SO}(3)_6 \boxtimes \overline{\text{SO}(3)_6}) & \xrightarrow[\text{G}=\mathbb{Z}_2]{\text{Ungaunging}} & \mathcal{Z}_1({}^{\text{SVec}}\text{SO}(3)_6) \\
 \uparrow \text{F.c.} & & \uparrow \text{F.c.} & & \uparrow \text{F.c.} \\
 \text{SU}(2)_6 & \xrightarrow{\text{Drinfeld center}} & \text{SU}(2)_6 \boxtimes \overline{\text{SU}(2)_6} & \xrightarrow[\text{G}=\mathbb{Z}_2]{\text{Ungaunging}} & \text{SO}(3)_6 \boxtimes \overline{\text{SO}(3)_6}
 \end{array} \quad (198)$$

V. SUMMARY AND DISCUSSION

In this paper, we first construct exactly solvable models for $2 + 1$ D nonchiral nonanomalous fSET phases. Recall that the total symmetry is $G^f = \mathbb{Z}_2^f \times_{\omega_2} G$. A new feature of our constructed fermionic symmetry-enriched string-net models is that ω_2 appears as a phase factor in the symmetry action Eq. (44), and hence also involved in the fermionic pentagon equation in Eq. (46). For the fermionic symmetry-enriched string-net models defined in Sec. II, the input data is a G -graded super fusion category, which is only partially defined in Sec. IV A when ω_2 is trivial. Therefore, a complete definition of G -graded super fusion category is desired. Next, we construct exactly solvable models for nonchiral surface fSET phases with $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly when ω_2 is trivial. Each such anomalous phase is the surface of a bulk $3 + 1$ D fSPT phase described by a pair (n_3, ν_4) , where $n_3 \in H^3(G, \mathbb{Z}_2)$ and ν_4 is a 4-cochain that satisfies $d\nu_4 = (-1)^{n_3 \smile 1_{n_3}}$. A generalization to ω_2 nontrivial cases is expected in our future work. There are two key features in our constructed surface fermionic symmetry-enriched string-net models:

- (i) The fermion-parity conservation of a surface $\mathcal{F}$ move is broken by $n_3 \in H^3(G, \mathbb{Z}_2)$ in Eq. (117), which is due to the fact that bulk fermions can run onto the surface, and we have $[n_3] = [o_3]$ (Remark III.5). While the fermion-parity is conserved if we perform the renormalization move for both bulk and boundary together, i.e., the bulk-boundary $\mathcal{F}$ move defined in Eq. (109).
- (ii) The symmetry action is anomalous (this is exactly the characteristics of a 't Hooft anomaly) in a way that a symmetry action involves a nontrivial phase factor $\Theta\nu_4$ as shown in Eq. (126), where the phase factor Θ appears as a contribution of fermion creation and annihilation operators from the bulk fSPT phase, as defined in Eq. (127). Hence the surface fermionic pentagon equation in Eq. (128) is also obstructed by the phase factor $\Theta\nu_4$.

One open question is how to compute the full anyon data in general fermionic symmetry-enriched string-net models. The *tube algebra for fermionic string-net* models has been developed to calculate the anyon types, fusion rules, ground state degeneracy and modular S, T transformations on torus [45]. However, the full anyon data also includes the G -action (anyon permutation) and symmetry fractionalization data, the computation of which remains unknown. Up to now, we only know how to calculate the anyon data in the simple case where the intrinsic topological order of a (fermionic) SET phase is an Abelian gauge theory and the G action does not permute any gauge flux, with or without 't Hooft anomaly (illustrated in Sec. III E). Therefore, a complete construction of *tube algebra for (fermionic) symmetry-enriched string-net models* is

desired, where the G -action and symmetry fractionalization data can be extracted.

Bulk-boundary gauging is also a very interesting future direction. Fully gauging the symmetry for a bulk $3 + 1$ D symmetry-protected topological (SPT) phase together with an anomalous SET phase on its surface simultaneously, we obtain a bulk $3 + 1$ D topological order together with its gapped boundary (a surface anomalous $2 + 1$ D topological order). This has been studied for a particular example of bulk $3 + 1$ D $\mathbb{Z}_2 \times \mathbb{Z}_2$ -SPT phase with boundary projective semions [74]. A complete framework of such bulk-boundary gauging process defined on lattice models is desired, which can also be generalized to fermionic systems and related to fermion condensation. For the example we studied in Sec. III F, i.e., a bulk $3 + 1$ D $\mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_4$ -SPT phase with boundary $\mathbb{Z}_4$ gauge theory anyons stacking with the physical fermion, we expect that after gauging the $\mathbb{Z}_2 \times \mathbb{Z}_4$ symmetry for both boundary and bulk, we will obtain a bulk fermionic $3 + 1$ D $\mathbb{Z}_2 \times \mathbb{Z}_4$ twisted gauge theory, whose gapped boundary is an anomalous fermionic $\mathbb{Z}_8 \times \mathbb{Z}_4$ gauge theory. With the fact that almost all $3 + 1$ D bosonic or fermionic topological orders can be obtained by gauging $3 + 1$ D bosonic or fermionic SPT phases [75], we conjecture that the categorical symmetry correspondence [76–80] between $3 + 1$ D and $2 + 1$ D is a simple case of such bulk-boundary gauging when the bulk SPT phase is trivial. A systematic study on such a topic also provides lattice model descriptions for the $2 + 1$ D gapped boundaries of a given $3 + 1$ D symmetry topological field theory (SymTFT) [81,82] as a by-product.

Finally, constructing exactly solvable models for the $H^2(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly is another future direction. Surface fermionic symmetry-enriched string-net models for fSET phases with $H^2(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly are expected to be technically involved. However, once the bulk-boundary gauging defined on lattice models illustrated above is fully understood, bulk-boundary ungauging certain $3 + 1$ D topological order together with one of its gapped boundaries may provide a systematic way to construct a lattice model for an fSET phases with the $H^2(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: SUPER FUSION CATEGORY AND SUPER MODULAR CATEGORY

Assuming basic knowledge on unitary (braided) fusion categories, this section is a brief review of some definitions in Refs. [41,42,44,48,57,83–85].

1. Super fusion category

Definition A.1. [The category of super vector spaces with grade-even linear maps] The category of super vector spaces over $\mathbb{C}$ with only grade-even linear maps, denoted as $\mathbf{SVec}$, is a symmetric fusion category that consists of

- (i) Objects: $\mathbb{Z}_2$ -graded vector spaces $V = V_0 \oplus V_1$. We denote a super vector space as $\mathbb{C}^{p|q}$, where $p = \dim(V_0)$, $q = \dim(V_1)$. A nonzero element $v \in V_k$, $k = 0, 1$ is called homogeneous and we define the parity function $|v| = k$.
- (ii) Morphisms: Grade-even linear maps that preserve the gradings $\mathbf{SVec}(V, W) = (W_0 \otimes V_0^*) \oplus (W_1 \otimes V_1^*)$.
- (iii) Tensor product: $V \otimes W = (V_0 \otimes W_0 \oplus V_1 \otimes W_1) \oplus (V_0 \otimes W_1 \oplus V_1 \otimes W_0)$.
- (iv) Braidings:

$$c_{V,W}: V \otimes W \rightarrow W \otimes V$$

$$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v, \quad (\text{A1})$$

where $v \in V$ and $w \in W$ are homogeneous.

Remark A.2. In some mathematical contexts, $\mathbf{SVec}$ denotes for the category of $\mathbb{Z}_2$ -graded vector spaces with both grade-even and grade-odd linear maps. We use this $\mathbf{SVec}$ notation by a different meaning from these contexts.

Remark A.3. $\mathbf{SVec}$ and $\mathbf{Vec}_{\mathbb{Z}_2}$ are equivalent as fusion categories, where $\mathbf{Vec}_{\mathbb{Z}_2}$ is the category of $\mathbb{Z}_2$ -graded vector spaces, while $\mathbf{Vec}_{\mathbb{Z}_2}$ does not emphasize on the braiding structure.

Definition A.4. (Super category). A super category is an $\mathbf{SVec}$ -enriched category, denoted as ${}^{\mathbf{SVec}}\mathcal{C}$. (Please see the definition of enriched category in, e.g., Ref. [49].)

Roughly speaking, important data in an $\mathbf{SVec}$ -enriched category ${}^{\mathbf{SVec}}\mathcal{C}$ are

- (i) Objects: $X \in {}^{\mathbf{SVec}}\mathcal{C}$.
- (ii) Hom-objects: ${}^{\mathbf{SVec}}\mathcal{C}(X, Y) \in \mathbf{SVec}$ for every pair of $X, Y \in \mathcal{S}$.

$\mathbf{SVec}$ is called the background category. $\mathcal{C}$ is called the underlying category, defined by

- (i) Objects: $\text{Ob}(\mathcal{C}) = \text{Ob}({}^{\mathbf{SVec}}\mathcal{C})$.
- (ii) Morphisms: $\mathcal{C}(X, Y) := \mathbf{SVec}[\mathbb{C}, {}^{\mathbf{SVec}}\mathcal{C}(X, Y)]$ for every $X, Y \in \mathcal{S}$.
- (iii) Identity morphism and composition of morphisms: Omitted and please refer to Ref. [49].

Intuitively, the underlying category $\mathcal{C}$ takes the grade-0 hom-object of ${}^{\mathbf{SVec}}\mathcal{C}$ as its hom-space.

Proposition A.5. Let $\mathcal{A}$ be a monoidal category. Given a left $\mathcal{A}$ -module category $\mathcal{C}$, we obtain an $\mathcal{A}$ -enriched category ${}^{\mathcal{A}}\mathcal{C}$. This is called a *canonical construction of enriched category*.

Proof. See Ref. [49].

Proposition A.6. Let $\mathcal{A}$ be a braided monoidal category. Given a monoidal left $\mathcal{A}$ -module category $\mathcal{C}$, we obtain an $\mathcal{A}$ -enriched monoidal category ${}^{\mathcal{A}}\mathcal{C}$. (Definition of enriched monoidal category please also refer to Ref. [49].) This is called a *canonical construction of enriched monoidal category*.

Proof. See Ref. [49].

Definition A.7. (Super fusion category). Let $\mathcal{C}$ be a (unitary) fusion category and $\phi: \mathbf{SVec} \rightarrow \mathcal{Z}_1(\mathcal{C})$ be a braided monoidal functor, where $\mathcal{Z}_1(\mathcal{C})$ is the Drinfeld center of $\mathcal{C}$. Then $\mathcal{C}$ equipped with the following $\mathbf{SVec}$ -module action

$$\mathbf{SVec} \times \mathcal{C} \xrightarrow{\phi \times \text{id}} \mathcal{Z}_1(\mathcal{C}) \times \mathcal{C} \xrightarrow{I \times \text{id}} \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} \quad (\text{A2})$$

is a monoidal left $\mathbf{SVec}$ module, where I is the forgetful functor. Then according to Proposition A.6, the $\mathbf{SVec}$ -enriched monoidal category ${}^{\mathbf{SVec}}\mathcal{C}$ obtained by canonical construction is called a (unitary) super fusion category.

A simple object $X \in {}^{\mathbf{SVec}}\mathcal{C}$ is called m -type if ${}^{\mathbf{SVec}}\mathcal{C}(X, X) \cong \mathbb{C}^{1|0}$, and q -type if ${}^{\mathbf{SVec}}\mathcal{C}(X, X) \cong \mathbb{C}^{1|1}$.

Remark A.8. Let $\mathcal{C}$ be a unitary fusion category. The existence of the braided monoidal functor $\phi: \mathbf{SVec} \cong \{1, f\} \rightarrow \mathcal{Z}_1(\mathcal{C})$ implies that $\mathcal{Z}_1(\mathcal{C})$ contains a fermion f (Definition A.15), i.e., $\mathcal{Z}_1(\mathcal{C})$ is a spin modular category (Definition A.18).

The above definition of super fusion category by canonical construction physically corresponds to a *fermion condensation* [45,86] in the string-net model with input $\mathcal{C}$, and the resultant fermionic string-net model is inputted by a super fusion category, denoted as $\mathcal{C}/f \equiv {}^{\mathbf{SVec}}\mathcal{C}$ in Ref. [45].

Remark A.9. F symbols (or associators) in a super fusion category satisfy the fermionic or super pentagon equation [47,84].

Example A.10. (Fermion toric code). A fermionic toric code model is a fermionic string-net model. Input of a fermionic toric code is a (unitary) super fusion category $\mathbf{FTC} := {}^{\mathbf{SVec}}\mathbf{Vec}_{\mathbb{Z}_4} = \{1, s\}$ with only m -type objects, where $\mathbf{Vec}_{\mathbb{Z}_4} = \{1, s, f, sf\}$ is a unitary fusion category. Fusion rules and F symbols please refer to Ref. [50].

Example A.11. (Majorana toric code). A Majorana toric code model is a fermionic string-net model. Input of a Majorana toric code is a super fusion category $\mathbf{MTC} := {}^{\mathbf{SVec}}\mathbf{Ising} = \{1, \sigma\}$ with q -type object σ , where $\mathbf{Ising} = \{1, f, \sigma\}$ is the Ising unitary fusion category. Fusion rules and F symbols please refer to Ref. [46].

2. Super modular category and spin modular category

Definition A.12. (premodular category). A premodular category is a spherical braided fusion category.

Definition A.13. ((Relative) Müger center). Let $\mathcal{D}$ be a premodular category. The Müger center $\mathcal{Z}_2(\mathcal{D})$ of $\mathcal{D}$ is the full subcategory

$$\mathcal{Z}_2(\mathcal{D}) = \{X \in \mathcal{D} | c_{X,Y}c_{Y,X} = \text{id}_{Y \otimes X}, \forall Y \in \mathcal{D}\}, \quad (\text{A3})$$

where $c_{X,Y}: X \otimes Y \rightarrow Y \otimes X$ is the braiding natural isomorphism in $\mathcal{D}$.

Let $\mathcal{B}$ be a subcategory of $\mathcal{D}$. The relative Müger center $\mathcal{Z}_2(\mathcal{D}, \mathcal{B})$ of $\mathcal{D}$ relative to $\mathcal{B}$ is the full subcategory

$$\mathcal{Z}_2(\mathcal{D}, \mathcal{B}) = \{X \in \mathcal{D} | c_{X,Y}c_{Y,X} = \text{id}_{Y \otimes X}, \forall Y \in \mathcal{B}\}. \quad (\text{A4})$$

Definition A.14. (Symmetric, modular, and super modular category). Let $\mathcal{D}$ be a premodular category and $\mathbf{Vec}$ be the category of vector spaces. $\mathcal{D}$ is called symmetric if $\mathcal{Z}_2(\mathcal{D}) \cong \mathcal{D}$. $\mathcal{D}$ is called modular if $\mathcal{Z}_2(\mathcal{D}) \cong \mathbf{Vec}$. $\mathcal{D}$ is called super modular if $\mathcal{D}$ is unitary and $\mathcal{Z}_2(\mathcal{D}) \cong \mathbf{SVec}$.

Definition A.15. (Fermion). Let $\mathcal{D}$ be a pre-modular category. A *fermion* f , if it exists, is a distinguished object in $\mathcal{D}$ satisfying

$$f \otimes f \cong \mathbb{1}, \quad \theta_f = -1,$$

where $\mathbb{1}$ is the tensor unit in $\mathcal{D}$, and θ_a is the topological spin or twist of object a .

Remark 16. In other words, a premodular category $\mathcal{D}$ is a super modular if its Müger center is generated by a fermion f , where $\{\mathbb{1}, f\} \cong \mathbf{SVec}$.

Definition A.17. (Split). A super modular category $\mathcal{D}$ is called *split* if there exists a modular category $\mathcal{B}$ such that $\mathcal{D} \cong \mathcal{B} \boxtimes \mathbf{SVec}$, where $\boxtimes$ is the Deligne tensor product.

Definition A.18. (Spin modular category). A spin modular category $(\check{\mathcal{D}}, f)$ is a modular category $\check{\mathcal{D}}$ together with a distinguished fermion f .

Remark A.19. There is a canonical $\mathbb{Z}_2$ grading on a spin modular category

$$\check{\mathcal{D}} \cong \check{\mathcal{D}}_0 \oplus \check{\mathcal{D}}_1, \quad (\text{A5})$$

where $M_{a,f} = 1$ if $a \in \check{\mathcal{D}}_0$ and $M_{a,f} = -1$ if $a \in \check{\mathcal{D}}_1$ ($M_{a,b}$ is the monodromy operator of objects a and b). The grade-0 sector $\check{\mathcal{D}}_0$ is a super modular category. Physically, anyons in the grade-1 sector $\check{\mathcal{D}}_1$ are fermion-parity vortices (i.e., symmetry defects of $\mathbb{Z}_2^f$).

Example A.20. The toric code anyons $\mathbf{K}^{(0)} = \{\mathbb{1}, e, m, \psi\}$ form a spin modular category, where ψ is the fermion. We relabel

$$\mathbb{1} \rightarrow I_0, \quad \psi \rightarrow \psi_0, \quad e \rightarrow I_1, \quad m \rightarrow \psi_1,$$

i.e., $\mathbf{K}^{(0)} \cong \mathbf{K}_0^{(0)} \oplus \mathbf{K}_1^{(0)}$, and $\mathbf{K}_0^{(0)} = \{\mathbb{1}, \psi\} \cong \mathbf{SVec}$, $\mathbf{K}_1^{(0)} = \{e, m\}$.

Definition A.21. (Minimal modular extension). Let $\mathcal{D}$ be a premodular category. A minimal modular extension of $\mathcal{D}$ is a modular category $\mathcal{E}$ such that $\mathcal{D} \subset \mathcal{E}$ and $\text{FPdim}(\mathcal{E}) = \text{FPdim}(\mathcal{D})\text{FPdim}[\mathcal{Z}_2(\mathcal{D})]$, where $\text{FPdim}(\mathcal{D})$ is the Frobenius-Perron dimension of $\mathcal{D}$.

If $\mathcal{Z}_2(\mathcal{D}) \cong \mathbf{SVec}$, this is also called a $\mathbb{Z}_2^f$ -modular extension.

Remark A.22. For any unitary (fusion) category $\mathcal{C}$, the Frobenius-Perron dimension coincides with categorical dimension, i.e., $\text{FPdim}(X) = d_X := \tilde{S}_{1,X}$, $\forall X \in \mathcal{C}$ ($\tilde{S}$ is the unnormalized S matrix in $\mathcal{C}$), and $\text{FPdim}(\mathcal{C}) = \dim(\mathcal{C}) := \sum_{X \in \mathcal{C}} d_X^2$.

Remark A.23. If $\mathcal{D}$ is super modular, a minimal modular extension of $\mathcal{D}$ is a spin modular category $(\check{\mathcal{D}}, f)$, where $\check{\mathcal{D}}_0 \cong \mathcal{D}$ and $\dim(\check{\mathcal{D}}) = 2 \dim(\mathcal{D})$. Physically, let the super modular category $\mathcal{D}$ describe the anyon types (including the fermion f) of a fermionic topological phase, a $\mathbb{Z}_2^f$ -equivariantization together with a $\mathbb{Z}_2^f$ -modular extension of $\mathcal{D}$ corresponds to gauging the $\mathbb{Z}_2^f$ fermion-parity symmetry [see Diagram (186), and gauging $\mathbb{Z}_2^f$ is the inverse process of a fermion condensation].

Theorem A.24. Every super modular category admits minimal modular extensions, and there are precisely 16 inequivalent such extensions [87–89].

Example A.25. There are 16 minimal modular extensions for the simplest super modular category $\mathbf{SVec}$, also known as the Kitaev's 16-fold way [54]. The 16 minimal modular extensions are denoted as $\mathbf{K}^{(\nu)}$, where $\nu \in \mathbb{Z}_{16}$, and the $\nu = 0$ case is exactly the toric code anyon in Example A.20. The fermion-parity vortex sector always has $\dim(\mathbf{K}_1^{(\nu)}) = 2$, corresponding to either Abelian vortices $\mathbf{K}_1^{(\nu \text{ is even})} = \{I_1, \psi_1\}$, or non-Abelian Ising vortices $\mathbf{K}_1^{(\nu \text{ is odd})} = \{\sigma_1\}$ ($d_{\sigma_1} = \sqrt{2}$). Topological spin of any vortex a_1 is $\theta_{a_1} = e^{i\frac{\pi}{8}\nu}$, which is related to the chiral central charge $c_- = (\nu/2) \pmod{8}$.

Remark A.26. Anyons and symmetry defects of 2 + 1D fSET phases (including anomalous ones) are described by G^f -crossed super modular categories. Anyons and defects of nonanomalous 2 + 1D fSET phases are described by G -crossed spin modular categories [43]. 2 + 1D fSET phases with fermionic 't Hooft anomalies correspond to those G^f -crossed super modular categories where there are obstructions on modular extension or gauging the $\mathbb{Z}_2^f$ symmetry.

APPENDIX B: REVIEW OF CLASSIFICATION OF ANYONS AND SYMMETRY DEFECTS IN 2 + 1D BOSONIC SET PHASES

This section is a brief review on Refs. [13,90]. Let $\mathcal{M}$ be an unitary modular tensor category (UMTC) describing

anyons of a $2 + 1$ D bosonic topological order, and let G be a group describing the symmetry.

An UMTC $\mathcal{M}$ is characterized by a set of gauge-invariant data $\{d_a, \theta_a, S_{ab} | a, b \in \mathcal{M}\}$, where d_a is the quantum dimension of anyon a , θ_a is the topological spin of a , and S_{ab} is the topological S matrix of anyons a and b .

Remark B.1. A spin modular category (Definition A.18) is a special UMTC that contains a fermion f .

Remark B.2. If we can assign a phase factor $e^{i\phi_a}$ for each anyon $a \in \mathcal{M}$ such that

$$e^{i\phi_a} e^{i\phi_b} = e^{i\phi_c}, \quad \text{whenever } N_c^{ab} \neq 0 \quad (\text{B1})$$

($N_c^{ab} = \sum_{x \in \mathcal{M}} [(S_{ax} S_{bx} S_{cx}^*) / S_{1x}]$ is the fusion multiplicity, and 1 is the tensor unit in $\mathcal{M}$), then we have

$$e^{i\phi_a} = M_{ae}^*, \quad \forall a \in \mathcal{M}, \quad (\text{B2})$$

for some Abelian anyon $e \in \mathcal{A} \subseteq \mathcal{M}$ ($\mathcal{A}$ denotes for the subcategory of Abelian anyons in $\mathcal{M}$), where $M_{ab} = [(S_{ab}^* S_{11}) / (S_{1a} S_{1b})]$ is the monodromy operator between anyons a and b . Such phase factors form an Abelian group, denoted as $K(\mathcal{M})$, i.e. $K(\mathcal{M}) := \{e^{i\phi_a} | e^{i\phi_a} e^{i\phi_b} = e^{i\phi_c} \text{ whenever } N_c^{ab} \neq 0, a, b, c \in \mathcal{M}\}$.

1. G action on $\mathcal{M}$

Definition B.3. (Braided auto-equivalence). A braided auto-equivalence in $\mathcal{M}$ is a braided monoidal functor $\varphi: \mathcal{M} \rightarrow \mathcal{M}$.

Given two braided auto-equivalences φ and φ' , a natural isomorphism $\gamma: \varphi \rightarrow \varphi'$ consists of a collection of isomorphisms $\{\gamma_a: \varphi(a) \rightarrow \varphi'(a) | a \in \mathcal{M}\}$ such that they satisfy the naturality condition and preserve both the monoidal and braiding structures.

Remark B.4. We define the following map Υ as a “natural isomorphism” on a fusion vector:

$$\Upsilon|a, b; c, \mu\rangle = \frac{\gamma_a \gamma_b}{\gamma_c} |a, b; c, \mu\rangle, \quad (\text{B3})$$

where $a, b, c \in \mathcal{M}$, $|a, b; c, \mu\rangle \in \text{Hom}(a \otimes b, c)$ is a fusion vector and μ denotes for different orthogonal fusion vectors. Here $\gamma_a, \gamma_b, \gamma_c$ are phase factors, which are also morphisms for a, b, c of natural isomorphism γ (we abuse notations to denote a morphism and the phase factor produced by this morphism).

Remark B.5. Equivalence classes of braided auto-equivalences in $\mathcal{M}$ (up to natural isomorphisms) form a group, denoted as $\text{Aut}(\mathcal{M})$.

Definition B.6. [The category of braided auto-equivalences in $\mathcal{M}$] The category of braided auto-equivalences in $\mathcal{M}$, denoted as $\underline{\text{Aut}}(\mathcal{M})$, is a monoidal category consists of

- (i) Object: A braided auto-equivalence $\varphi: \mathcal{M} \rightarrow \mathcal{M}$;
- (ii) Morphism: A monoidal natural isomorphism $\gamma: \varphi \rightarrow \varphi'$;

(iii) Tensor product: Composition of functors.

Definition B.7. (Symmetry or G action on $\mathcal{M}$). Let $\underline{G}$ be the monoidal category whose objects are group elements in G and tensor product is the group multiplication. A group action of G on $\mathcal{M}$ is a monoidal functor $\underline{\rho}: \underline{G} \rightarrow \underline{\text{Aut}}(\mathcal{M})$.

Spelling it out, the monoidal functor $\underline{\rho}$ consists of

- (i) Braided monoidal functors $\underline{\rho}_{\mathbf{g}}: \mathcal{M} \rightarrow \mathcal{M}, \forall \mathbf{g} \in G$;
- (ii) Monoidal natural isomorphisms $\underline{\kappa}_{\mathbf{g}, \mathbf{h}}: \underline{\rho}_{\mathbf{g}} \rightarrow \underline{\rho}_{\mathbf{h}} \rightarrow \underline{\rho}_{\mathbf{g} \cdot \mathbf{h}}$

such that $\forall \mathbf{g}, \mathbf{h}, \mathbf{k} \in G, \forall a \in \mathcal{M}$, the following diagram commutes:

$$\begin{array}{ccc} \underline{\rho}_{\mathbf{g}\mathbf{h}\mathbf{k}}(a) & \xrightarrow{(\underline{\kappa}_{\mathbf{g}\mathbf{h}, \mathbf{k}})_a} & \underline{\rho}_{\mathbf{g}\mathbf{h}} \circ \underline{\rho}_{\mathbf{k}}(a) \\ \downarrow (\underline{\kappa}_{\mathbf{g}, \mathbf{h}\mathbf{k}})_a & & \downarrow (\underline{\kappa}_{\mathbf{g}, \mathbf{h}})_{\underline{\rho}_{\mathbf{k}}(a)} \\ \underline{\rho}_{\mathbf{g}} \circ \underline{\rho}_{\mathbf{h}\mathbf{k}}(a) & \xrightarrow{\underline{\rho}_{\mathbf{g}} \circ (\underline{\kappa}_{\mathbf{h}, \mathbf{k}})_a} & \underline{\rho}_{\mathbf{g}} \circ \underline{\rho}_{\mathbf{h}} \circ \underline{\rho}_{\mathbf{k}}(a) \end{array} \quad (\text{B4})$$

Remark B.8. $\underline{\kappa}_{\mathbf{g}, \mathbf{h}}$ is the natural transformation that preserves monoidal structure in the definition of monoidal functor, and Diagram (B4) is exactly the coherence relation of a monoidal functor. $\{(\underline{\kappa}_{\mathbf{g}, \mathbf{h}})_a: \underline{\rho}_{\mathbf{g}\mathbf{h}}(a) \rightarrow \underline{\rho}_{\mathbf{g}} \circ \underline{\rho}_{\mathbf{h}}(a) | a \in \mathcal{M}\}$ is the collection of morphisms in the definition of a natural transformation.

Definition B.9. [Obstruction to define a G action on $\mathcal{M}$] Let $\rho: G \rightarrow \text{Aut}(\mathcal{M})$ be a group homomorphism equipped with

- (i) Braided monoidal functors $\rho_{\mathbf{g}}: \mathcal{M} \rightarrow \mathcal{M}, \forall \mathbf{g} \in G$;
- (ii) Monoidal natural isomorphisms $\kappa_{\mathbf{g}, \mathbf{h}}: \rho_{\mathbf{g}} \rightarrow \rho_{\mathbf{h}} \rightarrow \rho_{\mathbf{g} \cdot \mathbf{h}}, \forall \mathbf{g}, \mathbf{h} \in G$,

where $\kappa_{\mathbf{g}, \mathbf{h}}|a, b; c, \mu\rangle = \{[\beta_a(\mathbf{g}, \mathbf{h})\beta_b(\mathbf{g}, \mathbf{h})] / [\beta_c(\mathbf{g}, \mathbf{h})]\} |a, b; c, \mu\rangle$ (recall Theorem. B.4), i.e., $(\kappa_{\mathbf{g}, \mathbf{h}})_a = \beta_a(\mathbf{g}, \mathbf{h})$ (we abuse notations to denote the morphism and the phase factor). Then the obstruction to define a G action on $\mathcal{M}$ is the obstruction to lift the group homomorphism $\rho: G \rightarrow \text{Aut}(\mathcal{M})$ to a monoidal functor $\underline{\rho}: \underline{G} \rightarrow \underline{\text{Aut}}(\mathcal{M})$, which is defined by the following diagram:

$$\begin{array}{ccc} & \rho_{\mathbf{g}\mathbf{h}\mathbf{k}}(a) & \\ \swarrow (\kappa_{\mathbf{g}, \mathbf{h}\mathbf{k}})_a = \beta_a(\mathbf{g}, \mathbf{h}\mathbf{k}) & & \searrow \beta_a(\mathbf{g}\mathbf{h}, \mathbf{k}) \\ \rho_{\mathbf{g}} \circ \rho_{\mathbf{h}\mathbf{k}}(a) & & \rho_{\mathbf{g}\mathbf{h}} \circ \rho_{\mathbf{k}}(a) \\ \downarrow \beta_{\rho_{\mathbf{g}}(a)}(\mathbf{h}, \mathbf{k}) & & \downarrow \beta_a(\mathbf{g}, \mathbf{h}) \\ \rho_{\mathbf{g}} \circ \rho_{\mathbf{h}} \circ \rho_{\mathbf{k}}(a) & \xrightarrow{\Omega_a(\mathbf{g}, \mathbf{h}, \mathbf{k})} & \rho_{\mathbf{g}} \circ \rho_{\mathbf{h}} \circ \rho_{\mathbf{k}}(a) \end{array} \quad (\text{B5})$$

i.e., the obstruction is the phase factor

$$\Omega_a(\mathbf{g}, \mathbf{h}, \mathbf{k}) = \beta_{\rho_g(a)}(\mathbf{h}, \mathbf{k}) \beta_a^{-1}(\mathbf{g}\mathbf{h}, \mathbf{k}) \beta_a(\mathbf{g}, \mathbf{h}\mathbf{k}) \beta_a^{-1}(\mathbf{g}, \mathbf{h}). \quad (\text{B6})$$

Remark B.10. $\{\beta_a(\mathbf{g}, \mathbf{h}), \forall \mathbf{g}, \mathbf{h} \in G, a \in \mathcal{M}\}$ is not a set of cochains, as $\beta_a(\mathbf{g}, \mathbf{h})$ has a dependence on anyon a . However, fixing a , $\{\beta_a(\mathbf{g}, \mathbf{h}), \forall \mathbf{g}, \mathbf{h} \in G\}$ is a set of cochains. Thereby, the $\Omega_a(\mathbf{g}, \mathbf{h}, \mathbf{k})$ defined above is in general not a cochain or further coboundary.

Proposition B.11. The obstruction phase factor $\Omega_a(\mathbf{g}, \mathbf{h}, \mathbf{k})$ satisfies

$$\Omega_a(\mathbf{g}, \mathbf{h}, \mathbf{k}) \Omega_b(\mathbf{g}, \mathbf{h}, \mathbf{k}) = \Omega_c(\mathbf{g}, \mathbf{h}, \mathbf{k}), \quad (\text{B7})$$

whenever $N_{ab}^c \neq 0$ [this can be checked directly by Eq. (B6)]. Then recalling Remark B.2,

$$\Omega_a(\mathbf{g}, \mathbf{h}, \mathbf{k}) = M_{a\mathfrak{D}(\mathbf{g}, \mathbf{h}, \mathbf{k})}^*, \quad (\text{B8})$$

where $\mathfrak{D}(\mathbf{g}, \mathbf{h}, \mathbf{k}) \in C^3(G, \mathcal{A})$ is an anyon-valued 3-cochain.

Remark B.12. Proposition B.11 transforms a set of anyon-dependent phases $\{\Omega_a(\mathbf{g}, \mathbf{h}, \mathbf{k}), \forall \mathbf{g}, \mathbf{h}, \mathbf{k} \in G, a \in \mathcal{M}\}$ to a set of cochains $\{\mathfrak{D}(\mathbf{g}, \mathbf{h}, \mathbf{k}), \forall \mathbf{g}, \mathbf{h}, \mathbf{k} \in G\}$.

Remark B.13. We can check that $\mathfrak{D}(\mathbf{g}, \mathbf{h}, \mathbf{k})$ satisfies the twisted 3-cocycle condition (depending on ρ), i.e., $\mathfrak{D}(\mathbf{g}, \mathbf{h}, \mathbf{k}) \in Z_\rho^3(G, \mathcal{A})$, and we can define the coboundary transformation on it (omitted). Denoting the equivalence class of group homomorphism ρ as $[\rho]$ (up to natural isomorphism), and equivalence class of $\mathfrak{D}$ as $[\mathfrak{D}]$ (up to coboundary transformation), we have

$$[\mathfrak{D}] \in H_{[\rho]}^3(G, \mathcal{A}), \quad (\text{B9})$$

where $H_{[\rho]}^3(G, \mathcal{A})$ is the anyon-valued 3rd-cohomology group that depends on $[\rho]$.

2. Symmetry fractionalization

Let $U(\mathbf{g})$ be the global symmetry operator of $\mathbf{g} \in G$, which forms linear representation of G , i.e.,

$$U(\mathbf{g})U(\mathbf{h}) = U(\mathbf{g}\mathbf{h}). \quad (\text{B10})$$

Let $|\Psi_{a_1, \dots, a_n}\rangle$ be a state with n separated anyons $a_1, \dots, a_n$ such that all the anyons can fuse to vacuum, i.e.,

$$a_1 \otimes \dots \otimes a_n = \mathbb{1} \oplus \dots \quad (\text{B11})$$

Then we assume that any symmetry operator $U(\mathbf{g})$ can be expressed in the following way:

$$U(\mathbf{g})|\Psi_{a_1, \dots, a_n}\rangle = \prod_{j=1}^n U_{a_j}(\mathbf{g}) \rho_{\mathbf{g}} |\Psi_{a_1, \dots, a_n}\rangle, \quad (\text{B12})$$

where $U_{a_j}(\mathbf{g})$ is a local unitary operator of $\mathbf{g}$ action localized near anyon a_j , and $\rho_{\mathbf{g}}$ is the anyon permutation by $\mathbf{g} \in G$ introduced in Definition B.9, which is called *symmetry localization*.

Further, combining Eq. (B10) and Eq. (B12), the local unitary operators form a “twisted” projective representation of the symmetry group:

$$\begin{aligned} U_{a_j}(\mathbf{g}) \rho_{\mathbf{g}} U_{a_j}(\mathbf{h}) \rho_{\mathbf{g}}^{-1} |\Psi_{a_1, \dots, a_n}\rangle \\ = \eta_{a_j}(\mathbf{g}, \mathbf{h}) U_{a_j}(\mathbf{g}\mathbf{h}) |\Psi_{a_1, \dots, a_n}\rangle, \end{aligned} \quad (\text{B13})$$

which is called *symmetry fractionalization*, and $\eta_{a_j}(\mathbf{g}, \mathbf{h})$ is a phase factor satisfying (recall Definition B.9)

$$\prod_{j=1}^n \eta_{a_j}(\mathbf{g}, \mathbf{h}) = \kappa_{\mathbf{g}, \mathbf{h}} = \prod_{j=1}^n \beta_{a_j}(\mathbf{g}, \mathbf{h}). \quad (\text{B14})$$

Here $\eta_a(\mathbf{g}, \mathbf{h})$ in general differs from $\beta_a(\mathbf{g}, \mathbf{h})$ by another phase factor

$$\omega_a(\mathbf{g}, \mathbf{h}) = \frac{\beta_a(\mathbf{g}, \mathbf{h})}{\eta_a(\mathbf{g}, \mathbf{h})}, \quad (\text{B15})$$

which satisfies

$$\prod_{j=1}^n \omega_{a_j}(\mathbf{g}, \mathbf{h}) = 1. \quad (\text{B16})$$

Then recalling Theorem B.2 and Eq. (B11), $\omega_a(\mathbf{g}, \mathbf{h})$ can be written as the braiding phase with some Abelian anyon:

$$\omega_a(\mathbf{g}, \mathbf{h}) = M_{a\mathfrak{w}(\mathbf{g}, \mathbf{h})}^*, \quad (\text{B17})$$

where $\mathfrak{w}(\mathbf{g}, \mathbf{h}) \in \mathcal{A} \subset \mathcal{M}$ is an Abelian anyon valued 2-cochain.

Further, requiring associativity of the global symmetry operator, i.e., $(R_{\mathbf{g}} R_{\mathbf{h}}) R_{\mathbf{k}} = R_{\mathbf{g}} (R_{\mathbf{h}} R_{\mathbf{k}})$, leads to the following condition on $\eta_a(\mathbf{g}, \mathbf{h})$:

$$\frac{\eta_{\rho_{\mathbf{g}}(a)}(\mathbf{h}, \mathbf{k}) \eta_a(\mathbf{g}, \mathbf{h}\mathbf{k})}{\eta_a(\mathbf{g}\mathbf{h}, \mathbf{k}) \eta_a(\mathbf{g}, \mathbf{h})} = 1. \quad (\text{B18})$$

Combining Eq. (B6), Eq. (B15), and Eq. (B18), we have

$$\Omega_a(\mathbf{g}, \mathbf{h}, \mathbf{k}) = \omega_{\rho_{\mathbf{g}}(a)}(\mathbf{h}, \mathbf{k}) \omega_a^{-1}(\mathbf{g}\mathbf{h}, \mathbf{k}) \omega_a(\mathbf{g}, \mathbf{h}\mathbf{k}) \omega_a^{-1}(\mathbf{g}, \mathbf{h}). \quad (\text{B19})$$

Recalling Eq. (B8) and Eq. (B17),

$$\begin{aligned}\mathfrak{D}(\mathbf{g}, \mathbf{h}, \mathbf{k}) &= \rho_{\mathbf{g}}(\mathfrak{w}(\mathbf{h}, \mathbf{k})) \overline{\mathfrak{w}(\mathbf{g}\mathbf{h}, \mathbf{k})} \mathfrak{w}(\mathbf{g}, \mathbf{h}\mathbf{k}) \overline{\mathfrak{w}(\mathbf{g}, \mathbf{h})} \\ &= d\mathfrak{w}(\mathbf{g}, \mathbf{h}, \mathbf{k}),\end{aligned}\quad (\text{B20})$$

where $\bar{a}$ denotes for the anti-particle of anyon $a \in \mathcal{M}$. This means that a well-defined symmetry fractionalization requires $\mathfrak{D}(\mathbf{g}, \mathbf{h}, \mathbf{k})$ to be a 3-coboundary. In other words, any nontrivial $[\mathfrak{D}] \in H^3_{[\rho]}(G, \mathcal{A})$ is an obstruction to define symmetry fractionalization.

Then when $\mathfrak{D}$ is a 3-coboundary, except from $\mathfrak{w}$,

$$\mathfrak{w}'(\mathbf{g}, \mathbf{h}) = \mathfrak{w}(\mathbf{g}, \mathbf{h}) \otimes \mathbf{t}(\mathbf{g}, \mathbf{h}) \quad (\text{B21})$$

is always another solution to Eq. (B20), where $\mathbf{t}(\mathbf{g}, \mathbf{h}) \in H^2_{\rho}(G, \mathcal{A})$ torsorially classify different symmetry fractionalization classes.

3. G -crossed extension and G equivariantization

Omitted. Crucial points of a G -crossed extension are the definition of G -crossed braiding and the three consistency equations in Figs. 9 and 10 in Ref. [13]. The G -crossed extended category of $\mathcal{M}$ is denoted as $\mathcal{M}_G^\times$. Physically, a G -crossed extension of $\mathcal{M}$ corresponds to adding G -symmetry defects to the system such that they couple with the anyons in a consistent way.

By a G equivariantization in $\mathcal{M}_G^\times$, the symmetry defects become anyons, and the resultant theory is a UMTc, denoted by $(\mathcal{M}_G^\times)^G$.

Definition B.14. [G equivariantization] A G -equivariant object in a fusion category $\mathcal{C}$ is a pair $(X, \{u_{\mathbf{g}}\}_{\mathbf{g} \in G})$ consisting an $X \in \mathcal{C}$ and a collection of isomorphisms $u_{\mathbf{g}}: \rho_{\mathbf{g}}(X) \cong X$ ($\rho_{\mathbf{g}}$ is a G action on $\mathcal{C}$ defined in Definition B.7) such that the following diagram commutes:

$$\begin{array}{ccc} \rho_{\mathbf{g}} \circ \rho_{\mathbf{h}}(X) & \xrightarrow{\rho_{\mathbf{g}\mathbf{h}} \circ u_{\mathbf{g}}} & \rho_{\mathbf{g}}(X) \\ \uparrow \kappa_{\mathbf{g}, \mathbf{h}} & & \downarrow u_{\mathbf{g}} \\ \rho_{\mathbf{g}\mathbf{h}}(X) & \xrightarrow{u_{\mathbf{g}\mathbf{h}}} & X \end{array}, \quad (\text{B22})$$

$$\begin{array}{ccc} \mathcal{Z}_1(\mathcal{C}) \text{ with } G\text{-action} & \xrightarrow{G\text{-crossed extension}} & G\text{-crossed MTC } \mathcal{Z}_1(\mathcal{C})_G^\times \\ \downarrow G\text{-equivariantization} & \searrow \text{Gauging } G & \downarrow G\text{-equivariantization} \\ \mathcal{Z}_2(\mathcal{Z}_1(\mathcal{C}_G), \text{Rep}(G)) & \xrightarrow{G\text{-modular extension}} & (\mathcal{Z}_1(\mathcal{C})_G^\times)^G \cong \mathcal{Z}_1(\mathcal{C}_G) \end{array}, \quad (\text{B27})$$

where $\mathcal{C}_G$ is a G -graded unitary fusion category such that $\mathcal{C}_1 \cong \mathcal{C}$ ($\mathbf{1}$ is trivial element in G). We note that our definition of gauging G [39,91] is a G -crossed extension followed by a G -equivariantization, which is slightly different from that in Ref. [13]. This

where $\kappa_{\mathbf{g}, \mathbf{h}}: \rho_{\mathbf{g}\mathbf{h}} \rightarrow \rho_{\mathbf{g}} \circ \rho_{\mathbf{h}}$ is the monoidal natural isomorphism.

Equivariant objects in $\mathcal{C}$ form a fusion category, called the equivariantization, denoted by $\mathcal{C}^G$. Further, we have $\text{FPdim}(\mathcal{C}^G) = |G| \text{FPdim}(\mathcal{C})$ [53].

Definition B.15. (Orbit under G). The orbit of $a \in \mathcal{M}_G^\times$ under G is defined to be the following set:

$$[a] = \{\rho_{\mathbf{g}}(a), \forall \mathbf{g} \in G\}. \quad (\text{B23})$$

Definition B.16. [Stablizer subgroup] Let a be a representative in orbit $[a]$. The stablizer subgroup of a is defined to be

$$G_a = \{\mathbf{g} \in G | \rho_{\mathbf{g}}(a) = a\}. \quad (\text{B24})$$

Remark B.17. Simple objects in $(\mathcal{M}_G^\times)^G$ are pairs

$$([a], \pi_a), \quad (\text{B25})$$

where π_a is an irreducible projective representation of G_a , i.e.,

$$\pi_a(\mathbf{g})\pi_a(\mathbf{h}) = \eta_a(\mathbf{g}, \mathbf{h})\pi_a(\mathbf{g}\mathbf{h}), \quad \mathbf{g}, \mathbf{h} \in G_a, \quad (\text{B26})$$

where $\eta_a(\mathbf{g}, \mathbf{h})$ is exactly the projective phase in symmetry fractionalization in the G -crossed theory in Eq. (B13).

Remark B.18. If $\mathcal{M}$ describes anyons of a nonchiral topological order, i.e., $\mathcal{M} \cong \mathcal{Z}_1(\mathcal{C})$ for some unitary fusion category $\mathcal{C}$ (this means that this nonchiral topological order can be constructed as a string-net model with input $\mathcal{C}$), and given that there is a G -action on $\mathcal{Z}_1(\mathcal{C})$, the following diagram commutes:

commutative diagram implies that this nonchiral topological order $\mathcal{Z}_1(\mathcal{C})$ with symmetry G , i.e., a nonchiral SET phase, can be constructed as a symmetry-enriched string-net model with input G -graded fusion category $\mathcal{C}_G$.

APPENDIX C: RELATING SYMMETRY-ENRICHED STRING-NET MODEL DATA WITH ANYON SYMMETRY FRACTIONALIZATION DATA FOR ABELIAN GAUGE THEORY

1. Nonanomalous cases

Partially gauging a $2 + 1$ D bosonic SPT phase to obtain an SET phase is discussed in Ref. [12]. Here we briefly review an relation between symmetry-enriched string-net model data and symmetry fractionalization data for anyons in Abelian gauge theory.

Using the same notation as in Sec. II I (the short exact sequence is $1 \rightarrow N \rightarrow \mathcal{G} \rightarrow G \rightarrow 1$, where N is an Abelian gauge group), given a $2 + 1$ D SPT phase classied by a 3-cocycle $\nu_3 \in H^3(\mathcal{G}, U_T(1))$, and given a group homomorphism $\varphi: G \rightarrow \text{Aut}(N)$ and a cohomology class $[\mu] \in H_\varphi^2(G, N)$ that specify the partial gauging, the F move or F symbol in the symmetry-enriched string-net model exactly equals to the 3-cocycle of the original SPT phase:

$$1F^{a_g b_h c_k} = \nu_3(a_g, b_h, c_k), \quad (C1)$$

where $a, b, \dots \in N$, $\mathbf{g}, \mathbf{h}, \dots \in G$ and $a_g, b_h, \dots \in \mathcal{G}$. In the following, we consider 3-cocycles that take the following form:

$$\nu_3(a_g, b_h, c_k) = \chi_c(\mathbf{g}, \mathbf{h})\alpha_3(\mathbf{g}, \mathbf{h}, \mathbf{k}), \quad (C2)$$

where the 3-cocycle condition $d\nu_3 = 1$ implies that χ and α_3 should satisfy the following properties:

(i) χ is a character on N :

$$\chi_a(\mathbf{g}, \mathbf{h})\chi_b(\mathbf{g}, \mathbf{h}) = \chi_{a+b}(\mathbf{g}, \mathbf{h}), \quad (C3)$$

where $+$ denotes for the group multiplication in N .

- (ii) Fixing a , $\chi_a \in Z^2[G, U(1)]$ is a 2-cocycle satisfying $\chi_a(\mathbf{g}, \mathbf{h})\chi_a(\mathbf{g}, \mathbf{h}, \mathbf{k}) = \chi_a^{S(\mathbf{g})}(\mathbf{h}, \mathbf{k})\chi_a(\mathbf{g}, \mathbf{h}, \mathbf{k})$.
- (iii) $\alpha_3 \in C^3[G, U(1)]$ is a 3-cochain satisfying

$$d\alpha_3(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}) = \chi_{\mu(\mathbf{k}, \mathbf{l})}(\mathbf{g}, \mathbf{h}). \quad (C4)$$

The input of this symmetry-enriched string-net model is the G -graded unitary fusion category $\mathcal{C}_G := (\text{Vec}_N)_G$.

2. Cases with $H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly

Recall Eq. (128) for surface topological order being Abelian gauge theory:

$$\frac{1F^{(a_g \otimes b_h) c_k d_1} 1F^{a_g b_h (c_k \otimes d_1)}}{1F^{a_g b_h c_k} 1F^{a_g (b_h \otimes c_k) d_1} 1F^{b_h c_k d_1}} = {}^{\mathfrak{g}}\Theta(b_h, c_k, d_1)\nu_4(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}). \quad (C5)$$

Plug in the special form in Eq. (149):

$$\begin{aligned} & \frac{\chi_d(\mathbf{g}, \mathbf{h}, \mathbf{k})\chi_{c+d+\mu(\mathbf{k}, \mathbf{l})}(\mathbf{g}, \mathbf{h})}{\chi_c(\mathbf{g}, \mathbf{h})\chi_d(\mathbf{g}, \mathbf{h}, \mathbf{k})\chi_d^{S(\mathbf{g})}(\mathbf{h}, \mathbf{k})} \\ &= {}^{\mathfrak{g}}\Theta(b_h, c_k, d_1)\nu_4(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}) \\ & \times \frac{\alpha_3^{S(\mathbf{g})}(\mathbf{h}, \mathbf{k}, \mathbf{l})\alpha_3(\mathbf{g}, \mathbf{h}, \mathbf{k})\alpha_3(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l})}{\alpha_3(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l})\alpha_3(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l})}. \end{aligned} \quad (C6)$$

First set $\mathbf{l} = \mathbf{1}$. Under the normalization $\alpha_3(\mathbf{1}, -, -) = \alpha_3(-, \mathbf{1}, -) = \alpha_3(-, -, \mathbf{1}) = 1$ and $\nu_4(-, -, -, \mathbf{1}) = 1$, the right-hand side is only left with ${}^{\mathfrak{g}}\Theta(b_h, c_k, d_1)$, and the equation becomes

$$\begin{aligned} & \chi_d(\mathbf{g}, \mathbf{h}, \mathbf{k})\chi_{c+d}(\mathbf{g}, \mathbf{h}) \\ &= {}^{\mathfrak{g}}\Theta(b_h, c_k, d_1)\chi_c(\mathbf{g}, \mathbf{h})\chi_d(\mathbf{g}, \mathbf{h}, \mathbf{k})\chi_d^{S(\mathbf{g})}(\mathbf{h}, \mathbf{k}). \end{aligned} \quad (C7)$$

If we further set $c = \mathbf{1}$ and use the normalization $\chi_1 = 1$, we find χ_d satisfies the following condition:

$$\frac{\chi_d(\mathbf{g}, \mathbf{h}, \mathbf{k})\chi_d(\mathbf{g}, \mathbf{h})}{\chi_d(\mathbf{g}, \mathbf{h}, \mathbf{k})\chi_d^{S(\mathbf{g})}(\mathbf{h}, \mathbf{k})} = {}^{\mathfrak{g}}\Theta(b_h, c_k, d_1). \quad (C8)$$

Equation (C7) then implies $\chi_{c+d}(\mathbf{g}, \mathbf{h}) = \chi_c(\mathbf{g}, \mathbf{h})\chi_d(\mathbf{g}, \mathbf{h})$. Finally, Eq. (C6) is reduced to

$$\chi_{\mu(\mathbf{k}, \mathbf{l})}(\mathbf{g}, \mathbf{h}) = \nu_4(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l})d\alpha_3(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}). \quad (C9)$$

APPENDIX D: $H^2(G, \mathbb{Z}_2)$ AND $H^3(G, \mathbb{Z}_2)$ FERMIONIC 'T HOOFT ANOMALIES IN THE CONTEXT OF ANYONS

This Section is a brief review on Ref. [44]. Let $\mathcal{D}$ be a super modular category that describe the anyons of a $2 + 1$ D fSET phase, and $(\check{\mathcal{D}}, f)$ be a spin modular category that is the minimal modular extension of $\mathcal{D}$, i.e., $\check{\mathcal{D}} = \check{\mathcal{D}}_0 \oplus \check{\mathcal{D}}_1$ and $\check{\mathcal{D}}_0 \cong \mathcal{D}$.

Given a G action on $\mathcal{D}$, i.e., a group homomorphism $\rho: G \rightarrow \text{Aut}(\mathcal{D})$, there might be obstruction to extend the G action to $\check{\mathcal{D}}$. Let us define a map $\check{\rho}: G \rightarrow \text{Aut}(\check{\mathcal{D}})$, and a restriction map

$$\begin{aligned} r: \text{Aut}(\check{\mathcal{D}}) &\rightarrow \text{Aut}(\mathcal{D}) \\ [\check{\rho}_g] &\mapsto [\rho_g], \end{aligned} \quad (D1)$$

where $[\]$ is the equivalence class up to natural isomorphisms (recall Sec. B 1), and $\ker r = \mathbb{Z}_2 \cong \{1, f\}$ (we mean the group formed by the fusion rules in $\{1, f\}$ is isomorphic to $\mathbb{Z}_2$) as the vacuum in $\mathcal{D}$ is $\mathcal{Z}_2(\mathcal{D}) = \{1, f\}$ (braids trivially with all other anyons). If $\check{\rho}$ fails to be a group homomorphism, i.e., the group multiplication rule of G is not preserved in $\text{Aut}(\check{\mathcal{D}})$, we say there is an *obstruction to extend the G action from $\mathcal{D}$ to $\check{\mathcal{D}}$* , measured by

$$o_2(\mathbf{g}, \mathbf{h}) := \check{\rho}_{\mathbf{g}\mathbf{h}} \check{\rho}_{\mathbf{h}}^{-1} \check{\rho}_{\mathbf{g}}^{-1}, \quad (\text{D2})$$

where o_2 restricts to a trivial map in $\text{Aut}(\mathcal{D})$, i.e., $o_2 \in \ker r = \mathbb{Z}_2$. So that o_2 is a 2-cochain in $C^2(G, \mathbb{Z}_2)$. Requiring associativity on $\check{\rho}_{\mathbf{g}}$ gives the 2-cocycle condition of o_2 , and after defining the 2-coboundary transformation properly, we conclude that $o_2 \in H^2(G, \mathbb{Z}_2)$.

If the above $H^2(G, \mathbb{Z}_2)$ obstruction vanishes, i.e., o_2 is a coboundary, there might be obstruction to extend the symmetry fractionalization from $\mathcal{D}$ to $\check{\mathcal{D}}$. Given a well-defined symmetry fractionalization on $\mathcal{D}$, i.e., the obstruction function Ω_a (for all $a \in \mathcal{D}$) in Eq. (B19) is trivial, or say, the Abelian anyon valued cochain $\mathfrak{D}$ defined in Eq. (B8) is a coboundary. Denote the obstruction function to defined symmetry fractionalization in $\check{\mathcal{D}}$ as $\check{\Omega}_a$, and the corresponding Abelian anyon valued 3-cochain $\check{\mathfrak{D}}$ is defined by

$$\check{\Omega}_a(\mathbf{g}, \mathbf{h}, \mathbf{k}) = M_{a\check{\mathfrak{D}}(\mathbf{g}, \mathbf{h}, \mathbf{k})}^*, \quad (\text{D3})$$

for all $a \in \check{\mathcal{D}}$. Recall the phase factor in Eq. (B15) and its corresponding Abelian anyon valued cochain in Eq. (B17) defined in $\mathcal{D}$. We extend them to $\check{\mathcal{D}}$, denoted as $\check{\omega}_a$ and $\check{\mathfrak{w}}$, where

$$\check{\omega}_a(\mathbf{g}, \mathbf{h}) = M_{a\check{\mathfrak{w}}(\mathbf{g}, \mathbf{h})}^*, \quad (\text{D4})$$

for all $a \in \check{\mathcal{D}}$, such that $r(\check{\omega}_a) = \omega_a$. We say there is an *obstruction to extend the symmetry fractionalization from $\mathcal{D}$ to $\check{\mathcal{D}}$* if $\check{\mathfrak{D}}$ fails to be a 3-coboundary, measured by the following mismatch

$$O_a(\mathbf{g}, \mathbf{h}, \mathbf{k}) = \check{\Omega}_a(\mathbf{g}, \mathbf{h}, \mathbf{k})^{-1} d\check{\omega}_a(\mathbf{g}, \mathbf{h}, \mathbf{k}), \quad (\text{D5})$$

for all $a \in \check{\mathcal{D}}$. We note that $O_a = 1$ for all $a \in \mathcal{D}$. Let us define

$$O_a(\mathbf{g}, \mathbf{h}, \mathbf{k}) = M_{aO_3(\mathbf{g}, \mathbf{h}, \mathbf{k})}^*, \quad (\text{D6})$$

for all $a \in \check{\mathcal{D}}$. Then the mismatch between $\check{\Omega}_a$ and $d\check{\omega}_a$ in Eq. (D5), such that $\check{\mathfrak{w}}$ fails to be a 3-coboundary, is transformed into the following anyon valued 3-cochain:

$$o_3(\mathbf{g}, \mathbf{h}, \mathbf{k}) = \overline{\check{\mathfrak{D}}(\mathbf{g}, \mathbf{h}, \mathbf{k})} \otimes d\check{\mathfrak{w}}(\mathbf{g}, \mathbf{h}, \mathbf{k}) \in \{\mathbb{1}, f\}. \quad (\text{D7})$$

The obstruction o_3 is valued in $\mathcal{Z}_2(\mathcal{D}) = \{\mathbb{1}, f\}$ because changing $\check{\mathfrak{w}} \rightarrow \check{\mathfrak{w}} \otimes f$ does not affect the restriction map $r(\check{\omega}_a) = \omega_a$, or say, o_3 should braid trivially with anyons in $\mathcal{D}$ in order to preserve $r(\check{\omega}_a) = \omega_a$. We see $dO_a = 1$, which implies the 3-cocycle condition of o_3 , and after defining 3-coboundary transformation on o_3 properly, we conclude that $o_3 \in H^3(G, \mathbb{Z}_2)$ ($\{\mathbb{1}, f\} \cong \mathbb{Z}_2$ as groups).

Remark D.1. If restricting the definition of o_3 in $a \in \mathcal{D}$ in Eq. (D6), and recall that $O_a = 1$ for $a \in \mathcal{D}$, we have

$$\begin{aligned} o_3(\mathbf{g}, \mathbf{h}, \mathbf{k}) &= \overline{\mathfrak{D}(\mathbf{g}, \mathbf{h}, \mathbf{k})} \otimes d\mathfrak{w}(\mathbf{g}, \mathbf{h}, \mathbf{k}) \\ &= d\mathfrak{w}(\mathbf{g}, \mathbf{h}, \mathbf{k}) \in \{\mathbb{1}, f\}, \end{aligned} \quad (\text{D8})$$

where $\mathfrak{D}$ is trivial as symmetry fractionalization is well defined in $\mathcal{D}$.

APPENDIX E: ANYON DATA OF SURFACE $\mathcal{Z}_1(\text{Vec}_{\mathbb{Z}_4}) \boxtimes \{\mathbb{1}, f\}$ TOPOLOGICAL ORDER WITH $\mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_4$ SYMMETRY

This section is a brief review on Ref. [51]. The surface fSET phase of the $\nu = 1/3 + 1\text{D}$ bulk $\mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_4$ fSPT phase (recall Sec. III F 1) have quasiparticle/anyon types $\mathcal{Z}_1(\text{Vec}_{\mathbb{Z}_4}) \boxtimes \{\mathbb{1}, f\}$, which is a super modular category, where $\mathcal{Z}_1(\text{Vec}_{\mathbb{Z}_4})$ describes anyons in the $\mathbb{Z}_4$ gauge theory. We choose the trivial solution that all F symbols are 1, whose K matrix is $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$. Here $\{\mathbb{1}, f\}$ is the trivial fermionic theory, whose K matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We label the anyon types by

$$\{a = (a_m, a_e, a_f) | a_m, a_e = 0, 1, 2, 3, a_f = 0, 1\}, \quad (\text{E1})$$

where a_m and a_e label gauge fluxes and gauge charges of the $\mathbb{Z}_4$ gauge theory, respectively, and $a_f = 0, 1$ labels whether the excitation is bosonic or fermionic. The R symbols are

$$R^{a,b} = e^{2\pi i (\frac{a_m b_e}{4} + \frac{a_f b_f}{2})}. \quad (\text{E2})$$

Recall Sec. B 1, the symmetry action on anyons is characterized by a group homomorphism [13]:

$$\rho: G \rightarrow \text{Aut}(\mathcal{Z}_1(\text{Vec}_{\mathbb{Z}_4}) \boxtimes \{\mathbb{1}, f\}). \quad (\text{E3})$$

Denote the group elements as $\mathbf{g} = \{(\mathbf{g}_1, \mathbf{g}_2) | \mathbf{g}_1 = 0, 1 \in \mathbb{Z}_2, \mathbf{g}_2 = 0, 1, 2, 3 \in \mathbb{Z}_4\}$. For each group element,

$$\begin{aligned} \rho_{\mathbf{g}}: \mathcal{Z}_1(\text{Vec}_{\mathbb{Z}_4}) \boxtimes \{\mathbb{1}, f\} &\rightarrow \mathcal{Z}_1(\text{Vec}_{\mathbb{Z}_4}) \boxtimes \{\mathbb{1}, f\} \\ a \mapsto \rho_{\mathbf{g}}(a) &= (a_m, a_e + 2\mathbf{g}_2 a_m, a_f + \mathbf{g}_2 a_m). \end{aligned} \quad (\text{E4})$$

The group multiplication law of $\rho_{\mathbf{g}}$ is satisfied up to a phase factor $\kappa_{\mathbf{g}, \mathbf{h}}$:

$$\kappa_{\mathbf{g}, \mathbf{h}} \rho_{\mathbf{g}} \rho_{\mathbf{h}} = \rho_{\mathbf{g}\mathbf{h}}. \quad (\text{E5})$$

In this example, we can simply take $\kappa_{\mathbf{g}, \mathbf{h}} = 1$. The anyon valued 2-cochain characterizing the symmetry fractionalization class defined in Eq. (B17) is

$$\mathfrak{w}(\mathbf{g}, \mathbf{h}) = (\mathbf{g}_1 \mathbf{h}_1, \mathbf{g}_2 \mathbf{h}_1, 0). \quad (\text{E6})$$

We can read all symmetry fractionalization phase factors η_a from Eq. (B15) and Eq. (B17), e.g., we see that under the $\mathbb{Z}_2$ symmetry, $e := (0, 1, 0)$ carries $\frac{1}{4}$ charge and $m := (1, 0, 0)$ carries no fractional charge. Recall Remark D.1 for the expression of o_3 when restricting to the super modular category, we have

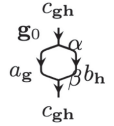
$$\begin{aligned} o_3(\mathbf{g}, \mathbf{h}, \mathbf{k}) &= d\mathfrak{w}(\mathbf{g}, \mathbf{h}, \mathbf{k}) \\ &= \rho_{\mathbf{g}}(\mathfrak{w}(\mathbf{h}, \mathbf{k})) \times \overline{\mathfrak{w}(\mathbf{g}\mathbf{h}, \mathbf{k})} \\ &\quad \times \mathfrak{w}(\mathbf{g}, \mathbf{h}\mathbf{k}) \times \overline{\mathfrak{w}(\mathbf{g}, \mathbf{h})} \\ &= (0, 0, [\mathbf{g}_2 \mathbf{h}_1 \mathbf{k}_1]_2), \end{aligned} \quad (\text{E7})$$

which is a nontrivial 3-cocycle in $H^3(G, \mathbb{Z}_2)$, where $[n]_2 = n \pmod{2}$. This $o_3 \in H^3(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly can be compensated by the complex fermion layer data $n_3 \in H^3(G, \mathbb{Z}_2)$ of a certain bulk 3 + 1D fSPT phase with the same symmetry group [44], where n_3 is in the same cohomology class as o_3 , making the whole bulk-boundary system consistent.

APPENDIX F: OTHER RENORMALIZATION MOVES IN FERMIONIC SYMMETRY-ENRICHED STRING-NET MODELS

1. $\mathcal{O}$ move and $\mathcal{Y}$ move

Let us consider the following local patch of a fermionic symmetry-enriched string-net model:



$$(\text{F1})$$

We define renormalization moves on such local patch as

- (i) When $c_{\mathbf{gh}}$ is m -type, the $\mathcal{O}$ move as a gfSLU transformation is defined as

$$\Psi \left(\begin{array}{c} c_{\mathbf{gh}} \\ \mathbf{g}_0 \downarrow \uparrow \\ a_{\mathbf{g}} \quad b_{\mathbf{h}} \\ \uparrow \downarrow \\ c_{\mathbf{gh}} \end{array} \right) = \mathbf{g}_0 [\mathcal{O}_{c_{\mathbf{gh}}}^{a_{\mathbf{g}} b_{\mathbf{h}}, \alpha}] \Psi \left(\begin{array}{c} c_{\mathbf{gh}} \\ \mathbf{g}_0 \downarrow \end{array} \right), \quad (\text{F2})$$

where we line up the basis choice in $\text{Hom}(a_{\mathbf{g}} \otimes b_{\mathbf{h}}, c_{\mathbf{gh}})$ on vertex $\underline{\beta}$ and $\text{Hom}(c_{\mathbf{gh}}, a_{\mathbf{g}} \otimes b_{\mathbf{h}}) = \text{Hom}(a_{\mathbf{g}} \otimes b_{\mathbf{h}}, c_{\mathbf{gh}})^*$ on vertex $\underline{\alpha}$ (fusion states in the dual space are also denoted as $\alpha, \beta, \dots$) by the gauge transformation in Eq. (34), and

$$\mathbf{g}_0 [\mathcal{O}_{c_{\mathbf{gh}}}^{a_{\mathbf{g}} b_{\mathbf{h}}, \alpha}] = (c_{\underline{\alpha}}^{\mathbf{g}_0 \dagger})^{s(\alpha)} (c_{\underline{\beta}}^{\mathbf{g}_0 \dagger})^{s(\alpha)} \mathbf{g}_0 [\mathcal{O}_{c_{\mathbf{gh}}}^{a_{\mathbf{g}} b_{\mathbf{h}}, \alpha}]. \quad (\text{F3})$$

The $\mathcal{O}$ moves satisfy the unitary condition

$$\sum_{a_{\mathbf{g}} b_{\mathbf{h}} \alpha} \mathbf{g}_0 [\mathcal{O}_{c_{\mathbf{gh}}}^{a_{\mathbf{g}} b_{\mathbf{h}}, \alpha}] (\mathbf{g}_0 [\mathcal{O}_{c_{\mathbf{gh}}}^{a_{\mathbf{g}} b_{\mathbf{h}}, \alpha}])^* = 1. \quad (\text{F4})$$

- (ii) When $c_{\mathbf{gh}}$ is q -type, the three-vertex $\tilde{\mathcal{O}}$ move as a gfSLU transformation is defined similarly as in Ref. [46] and omitted here. While we can still define two-vertex $\mathcal{O}$ moves as in Eq. (F3) when $c_{\mathbf{gh}}$ is q -type, the value of which are related to three-vertex $\tilde{\mathcal{O}}$ moves. The unitary condition of $\tilde{\mathcal{O}}$ move implies the following condition on two-vertex $\mathcal{O}$ move:

$$2 \sum_{a_{\mathbf{g}} b_{\mathbf{h}} \alpha} \mathbf{g}_0 [\mathcal{O}_{c_{\mathbf{gh}}}^{a_{\mathbf{g}} b_{\mathbf{h}}, \alpha}] (\mathbf{g}_0 [\mathcal{O}_{c_{\mathbf{gh}}}^{a_{\mathbf{g}} b_{\mathbf{h}}, \alpha}])^* = 1. \quad (\text{F5})$$

Combining Eqs. (F4) and (F5), we obtain

$$n_{c_{\mathbf{gh}}} \sum_{a_{\mathbf{g}} b_{\mathbf{h}} \alpha} \mathbf{g}_0 [\mathcal{O}_{c_{\mathbf{gh}}}^{a_{\mathbf{g}} b_{\mathbf{h}}, \alpha}] (\mathbf{g}_0 [\mathcal{O}_{c_{\mathbf{gh}}}^{a_{\mathbf{g}} b_{\mathbf{h}}, \alpha}])^* = 1. \quad (\text{F6})$$

The $\mathcal{Y}$ -move as a gfSLU transformation is defined as

$$\Psi \left(\begin{array}{c} a_{\mathbf{g}} \quad b_{\mathbf{h}} \\ \mathbf{g}_0 \downarrow \downarrow \end{array} \right) = \sum_{c_{\mathbf{gh}} \alpha} \mathbf{g}_0 [\mathcal{Y}_{c_{\mathbf{gh}}, \alpha}^{a_{\mathbf{g}} b_{\mathbf{h}}}] \Psi \left(\begin{array}{c} a_{\mathbf{g}} \quad b_{\mathbf{h}} \\ \mathbf{g}_0 \downarrow \uparrow \\ c_{\mathbf{gh}} \end{array} \right), \quad (\text{F7})$$

where

$$\mathbf{g}_0 [\mathcal{Y}_{c_{\mathbf{gh}}, \alpha}^{a_{\mathbf{g}} b_{\mathbf{h}}}] = (c_{\underline{\beta}}^{\mathbf{g}_0})^{s(\alpha)} (c_{\underline{\alpha}}^{\mathbf{g}_0})^{s(\alpha)} \mathbf{g}_0 [\mathcal{Y}_{c_{\mathbf{gh}}, \alpha}^{a_{\mathbf{g}} b_{\mathbf{h}}}]. \quad (\text{F8})$$

We can show that the $\mathcal{Y}$ move is related to the $\mathcal{O}$ move by [46]

$$\mathbf{g}_0 [\mathcal{Y}_{c_{\mathbf{gh}}, \alpha}^{a_{\mathbf{g}} b_{\mathbf{h}}}] = \frac{1}{n_{c_{\mathbf{gh}}} \mathbf{g}_0 [\mathcal{O}_{c_{\mathbf{gh}}}^{a_{\mathbf{g}} b_{\mathbf{h}}, \alpha}]}. \quad (\text{F9})$$

2. Dual $\mathcal{F}$ move, $\mathcal{H}$ move, and dual $\mathcal{H}$ move

The dual $\mathcal{F}$ move as a gfSLU transformation is defined as

$$\Psi \left(\begin{array}{c} d_{\mathbf{ghk}} \\ \mathbf{g}_0 \downarrow \uparrow \\ a_{\mathbf{g}} \quad b_{\mathbf{h}} \quad c_{\mathbf{k}} \end{array} \right) = \sum_{m_{\mathbf{gh}} \alpha \beta} \mathbf{g}_0 [\tilde{\mathcal{F}}_d^{a_{\mathbf{g}} b_{\mathbf{h}} c_{\mathbf{k}}}]^{m, \alpha \beta} \Psi \left(\begin{array}{c} d_{\mathbf{ghk}} \\ \mathbf{g}_0 \downarrow \downarrow \\ m_{\mathbf{gh}} \quad \alpha \quad \beta \\ a_{\mathbf{g}} \quad b_{\mathbf{h}} \quad c_{\mathbf{k}} \end{array} \right), \quad (\text{F10})$$

where

$$\mathbf{g}_0[\tilde{\mathcal{F}}_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} = (c_{\underline{\delta}}^{\mathbf{g}_0 \dagger})^{s(\delta)} (c_{\underline{\chi}}^{\mathbf{g}_0 \dagger})^{s(\chi)} (c_{\underline{\alpha}}^{\mathbf{g}_0})^{s(\alpha)} (c_{\underline{\beta}}^{\mathbf{g}_0})^{s(\beta)} \mathbf{g}_0[\tilde{F}_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}. \quad (\text{F11})$$

The dual $\mathcal{F}$ move is related to the $\mathcal{F}$ move by [46]

$$\mathbf{g}_0[\tilde{F}_d^{a_g b_h c_k}]_{n_{\text{hk}}, \chi \delta}^{m_{\text{gh}}, \alpha \beta} = \frac{n_{\text{hk}}}{n_{m_{\text{gh}}}} \mathbf{g}_0[F_d^{a_g b_h c_k}]_{n_{\text{hk}}, \chi \delta}^{m_{\text{gh}}, \alpha \beta} \mathbf{g}_0[O_{n_{\text{hk}} d}^{b_h c_k \chi}] \mathbf{g}_0[O_{d_{\text{gh}} k}^{a_g n_{\text{hk}}, \delta}] \mathbf{g}_0[O_{m_{\text{gh}} h}^{a_g b_h, \alpha}]^{-1} \mathbf{g}_0[O_{d_{\text{gh}} k}^{m_{\text{gh}} c_k, \beta}]^{-1}. \quad (\text{F12})$$

Then the two projective-unitary conditions of dual $\mathcal{F}$ move should reduce to the two projective-unitary conditions of $\mathcal{F}$ move in Eqs. (49) and (50), which can be satisfied by the following ansatz:

$$\mathbf{g}_0[O_{c_{\text{gh}}}^{a_g b_h, \alpha}] = 1 \quad [O_{c_{\text{gh}}}^{a_g b_h, \alpha}] = \Phi_{c_k}^{a_g b_h, \alpha} \frac{1}{D} \sqrt{\frac{d_{a_g} d_{b_h}}{n_{a_g} n_{b_h} n_{c_{\text{gh}}} d_{c_{\text{gh}}}}} \delta_{c_{\text{gh}}}^{a_g b_h}, \quad (\text{F13})$$

where $\Phi_{c_k}^{a_g b_h, \alpha}$ is a phase factor [46], $\delta_{c_{\text{gh}}}^{a_g b_h} = 1$ for $N_{c_{\text{gh}}}^{a_g b_h} > 0$ and $\delta_{c_{\text{gh}}}^{a_g b_h} = 0$ for $N_{c_{\text{gh}}}^{a_g b_h} = 0$, and $D = \sqrt{\sum_{a_g \in \mathcal{S}_G} d_{a_g}^2}$.

The $\mathcal{H}$ move as a gfSLU transformation is defined as

$$\Psi \left(\begin{array}{c} a_{\text{gh}} b_k \\ \alpha \downarrow \uparrow \beta \\ \mathbf{g}_0 \swarrow \searrow \\ e_h \\ c_g d_{\text{hk}} \end{array} \right) = \sum_{f_{\text{ghk}} \chi \delta} \mathbf{g}_0[\mathcal{H}_{b_k d}^{c_g a}]_{f, \chi \delta}^{e_h, \alpha \beta} \Psi \left(\begin{array}{c} a_{\text{gh}} b_k \\ \chi \downarrow \uparrow \\ \mathbf{g}_0 \swarrow \searrow \\ f_{\text{ghk}} \\ \delta \\ c_g d_{\text{hk}} \end{array} \right), \quad (\text{F14})$$

where

$$\begin{aligned} \mathbf{g}_0[\mathcal{H}_{b_k d}^{c_g a}]_{f, \chi \delta}^{e_h, \alpha \beta} &= (c_{\underline{\alpha}}^{\mathbf{g}_0 \dagger})^{s(\alpha)} (c_{\underline{\beta}}^{\mathbf{g}_0 \dagger})^{s(\beta)} (c_{\underline{\delta}}^{\mathbf{g}_0})^{s(\delta)} (c_{\underline{\chi}}^{\mathbf{g}_0})^{s(\chi)} \\ &\times \mathbf{g}_0[H_{b_k d}^{c_g a}]_{f, \chi \delta}^{e_h, \alpha \beta}. \end{aligned} \quad (\text{F15})$$

The $\mathcal{H}$ move is related to the $\mathcal{F}$ move by [46]

$$\mathbf{g}_0[H_{b_k d}^{c_g a}]_{f, \chi \delta}^{e_h, \alpha \beta} = n_{a_{\text{gh}}} \mathbf{g}_0 Y_{f, \delta}^{c_g d} \cdot (\mathbf{g}_0[F_f^{c_g e_h b_k}]_{d, \beta \delta}^{a, \alpha \chi})^* \cdot \mathbf{g}_0 O_a^{c_g e_h, \alpha}. \quad (\text{F16})$$

The dual $\mathcal{H}$ move as a gfSLU transformation is defined as

$$\Psi \left(\begin{array}{c} a_{\text{gh}} b_k \\ \mathbf{g}_0 \downarrow \uparrow \beta \\ \alpha \swarrow \searrow \\ e_h \\ c_g d_{\text{hk}} \end{array} \right) = \sum_{f_{\text{ghk}} \chi \delta} \mathbf{g}_0[\tilde{\mathcal{H}}_{b_k d}^{c_g a}]_{f, \chi \delta}^{e_h, \alpha \beta} \Psi \left(\begin{array}{c} a_{\text{gh}} b_k \\ \chi \downarrow \uparrow \\ \mathbf{g}_0 \swarrow \searrow \\ f_{\text{ghk}} \\ \delta \\ c_g d_{\text{hk}} \end{array} \right), \quad (\text{F17})$$

where

$$\mathbf{g}_0[\tilde{\mathcal{H}}_{b_k d}^{c_g a}]_{f, \chi \delta}^{e_h, \alpha \beta} = (c_{\underline{\beta}}^{\mathbf{g}_0 \dagger})^{s(\beta)} (c_{\underline{\alpha}}^{\mathbf{g}_0 \dagger})^{s(\alpha)} (c_{\underline{\delta}}^{\mathbf{g}_0})^{s(\delta)} (c_{\underline{\chi}}^{\mathbf{g}_0})^{s(\chi)} \mathbf{g}_0[\tilde{H}_{b_k d}^{c_g a}]_{f, \chi \delta}^{e_h, \alpha \beta}. \quad (\text{F18})$$

The dual $\mathcal{H}$ -move is related to the $\mathcal{F}$ move by [46]

$$\mathbf{g}_0[\tilde{H}_{b_k d}^{c_g a}]_{f, \chi \delta}^{e_h, \alpha \beta} = n_{b_k} \mathbf{g}_0 Y_{f, \delta}^{c_g d} \cdot \mathbf{g}_0[F_f^{a e_h d}]_{b_k, \beta \chi}^{c_g, \alpha \delta} \cdot \mathbf{g}_0 O_{b_k}^{e_h d, \beta}. \quad (\text{F19})$$

Then the two projective-unitary conditions of dual $\mathcal{H}$ -move implies the two projective-unitary conditions of $\mathcal{F}$ move in Eqs. (52) and (53), proof of which is similar to that in Ref. [46]. We note that the two projective-unitary conditions of $\mathcal{H}$ move also implies two projective-unitary conditions of $\mathcal{F}$ move, which are equivalent to (i.e., can be derived from) Eqs. (52) and (53).

APPENDIX G: PARTITION FUNCTION APPROACH ON THE $H^3(G, \mathbb{Z}_2)$ FERMIONIC 'T HOOFT ANOMALY

Here we study the gapped symmetry-preserving boundary theory of 3 + 1D fSPT phases via the partition function approach. Since we do not know how to construct the fermionic partition function with Kitaev chain decoration in the bulk yet. Our method here can only deal with the simple case that the bulk fSPT phases can contain at most complex fermion decoration. The 3 + 1D bulk fermionic partition function in this simple case is given by [6]

$$\mathcal{Z} = \sum_{\{\mathbf{g}_i\}} \int \prod_{\text{interior}\langle ijkl \rangle} d\theta_{ijkl}^{n_3(ijkl)} d\bar{\theta}_{ijkl}^{n_3(ijkl)} \prod_{\text{interior}\langle ijk \rangle} (-1)^{m_2(ijk)} \prod_{\langle ijklm \rangle} \mathcal{V}_4^{\sigma(ijklm)}(ijklm), \quad (\text{G1})$$

where $\sigma(ijklm) = \pm$ is the orientation of the 4-simplex $\langle ijklm \rangle$, and m_2 is a $\mathbb{Z}_2$ -valued function satisfying $m_2(ijk) + m_2(ikl) + m_2(ijl) + m_2(jkl) = dm_2(ijkl) = n_3(ijkl) \pmod{2}$, which contributes as the spin structure term. [Recall Convention 2, $m_2(ijk) := m_2(\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k)$ and so on, where $\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k \in G$.] Further we define

$$\mathcal{V}_4^+(01234) = (-1)^{m_2(013)+m_2(134)+m_2(123)} \theta_{1234}^{n_3(1234)} \theta_{0134}^{n_3(0134)} \theta_{0123}^{n_3(0123)} \bar{\theta}_{0234}^{n_3(0234)} \bar{\theta}_{0124}^{n_3(0124)} \nu_4(01234), \quad (\text{G2})$$

$$\mathcal{V}_4^-(01234) = (-1)^{m_2(024)} \theta_{0124}^{n_3(0124)} \theta_{0234}^{n_3(0234)} \bar{\theta}_{0123}^{n_3(0123)} \bar{\theta}_{0134}^{n_3(0134)} \bar{\theta}_{1234}^{n_3(1234)} \nu_4(01234)^{-1}. \quad (\text{G3})$$

We relate the bulk 3 + 1D fSPT states with the 2 + 1D surface by adding a single vertex to the bulk and connecting the bulk vertex to all surface vertices, and we denote the single vertex in the bulk as $\mathbf{g}_*$. Each surface $\mathcal{F}$ move is associated with a bulk $\tilde{\mathcal{F}}$ move. In order to construct the wave function of the bulk move, we add a trivial vertex $\mathbf{g}_e = \mathbf{1}$ in one higher dimension and write down the partition function. Let $\mathbf{g} = \mathbf{g}_0^{-1} \mathbf{g}_1$, $\mathbf{h} = \mathbf{g}_1^{-1} \mathbf{g}_2$ and $\mathbf{k} = \mathbf{g}_2^{-1} \mathbf{g}_3$. A surface $\mathcal{F}$ move $\mathbf{g}_0[\mathcal{F}_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}$ and its corresponding bulk move $\mathbf{g}_0[\tilde{\mathcal{F}}^{\mathbf{g} \mathbf{h} \mathbf{k}}](\mathbf{g}_*)$ together have expression

$$\begin{aligned} & \mathbf{g}_0[\mathcal{F}_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} \mathbf{g}_0[\tilde{\mathcal{F}}^{\mathbf{g} \mathbf{h} \mathbf{k}}](\mathbf{g}_*) \\ &= c_{\underline{\alpha}}^{\dagger s(\alpha)} c_{\underline{\beta}}^{\dagger s(\beta)} c_{\underline{\delta}}^{s(\delta)} c_{\underline{\chi}}^{s(\chi)} \mathbf{g}_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} \\ & \times \int d\theta_{e01*}^{n_3(e01*)} d\bar{\theta}_{e01*}^{n_3(e01*)} d\theta_{e02*}^{n_3(e02*)} d\bar{\theta}_{e02*}^{n_3(e02*)} d\theta_{e03*}^{n_3(e03*)} d\bar{\theta}_{e03*}^{n_3(e03*)} d\theta_{e12*}^{n_3(e12*)} d\bar{\theta}_{e12*}^{n_3(e12*)} d\theta_{e13*}^{n_3(e13*)} d\bar{\theta}_{e13*}^{n_3(e13*)} d\theta_{e23*}^{n_3(e23*)} d\bar{\theta}_{e23*}^{n_3(e23*)} \\ & \times (-1)^{m_2(e0*)+m_2(e1*)+m_2(e2*)+m_2(e3*)} \mathcal{V}_4^-(e123*) \mathcal{V}_4^-(e013*) \mathcal{V}_4^+(e023*) \mathcal{V}_4^+(e012*) \\ &= c_{\underline{\alpha}}^{\dagger s(\alpha)} c_{\underline{\beta}}^{\dagger s(\beta)} c_{\underline{\delta}}^{s(\delta)} c_{\underline{\chi}}^{s(\chi)} \mathbf{g}_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} \int d\theta_{0123}^{n_3(0123)} d\bar{\theta}_{0123}^{n_3(0123)} \mathcal{V}_4^-(e0123) \mathcal{V}_4^+(0123*) \\ &= (-1)^{n_3(0123)(n_3(123*)+n_3(013*))} c_{\underline{\alpha}}^{\dagger s(\alpha)} c_{\underline{\beta}}^{\dagger s(\beta)} c_{\underline{\delta}}^{s(\delta)} c_{\underline{\chi}}^{s(\chi)} \mathbf{g}_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} \\ & \times \theta_{e013}^{n_3(e013)} \theta_{e123}^{n_3(e123)} \bar{\theta}_{e012}^{n_3(e012)} \bar{\theta}_{e023}^{n_3(e023)} \theta_{123*}^{n_3(123*)} \theta_{013*}^{n_3(013*)} \bar{\theta}_{023*}^{n_3(023*)} \bar{\theta}_{012*}^{n_3(012*)} \frac{\nu_4(0123*)}{\nu_4(e0123)}. \end{aligned} \quad (\text{G4})$$

We should take care of which part belongs to the surface and which belongs to the bulk. Actually the complex fermion operators in tetrahedrons $\langle e012 \rangle$, $\langle e023 \rangle$, $\langle e013 \rangle$, and $\langle e123 \rangle$ all belong to the surface, and they change the fermion parities on each vertex on the surface. Therefore, for the anomalous surface $\mathcal{F}$ move, we have

$$\mathbf{g}_0[\mathcal{F}_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta} = c_{\underline{\alpha}}^{\dagger s(\alpha)} c_{\underline{\beta}}^{\dagger s(\beta)} c_{\underline{\delta}}^{s(\delta)} c_{\underline{\chi}}^{s(\chi)} \theta_{e013}^{n_3(e013)} \theta_{e123}^{n_3(e123)} \bar{\theta}_{e012}^{n_3(e012)} \bar{\theta}_{e023}^{n_3(e023)} \cdot \mathbf{g}_0[F_d^{a_g b_h c_k}]_{n, \chi \delta}^{m, \alpha \beta}, \quad (\text{G5})$$

which may violate fermion-parity conservation by $n_3(0123) = n_3(\mathbf{g}, \mathbf{h}, \mathbf{k}) \pmod{2}$ in general. While for the bulk move, we have

$$\mathbf{g}_0[\tilde{\mathcal{F}}^{\mathbf{g} \mathbf{h} \mathbf{k}}](\mathbf{g}_*) = (-1)^{n_3(0123)(n_3(123*)+n_3(013*))} \theta_{123*}^{n_3(123*)} \theta_{013*}^{n_3(013*)} \bar{\theta}_{023*}^{n_3(023*)} \bar{\theta}_{012*}^{n_3(012*)} \frac{\nu_4(0123*)}{\nu_4(e0123)}. \quad (\text{G6})$$

The surface anomalous $\mathcal{F}$ move and the bulk $\tilde{\mathcal{F}}$ move together constitute a gfSLU transformation.

APPENDIX H: CONSISTENCY BETWEEN THE ANOMALOUS SYMMETRIC CONDITION AND THE SURFACE FERMIONIC PENTAGON EQUATION

In this Appendix, we check that the anomalous symmetry condition in Eq. (126) is consistent with the surface fermionic pentagon equation in Eq. (123). By Eq. (126) and recall the property ${}^{\mathbf{g}_0}\Theta(a_{\mathbf{g}}, b_{\mathbf{h}}, c_{\mathbf{k}}){}^{\mathbf{g}_0}\Theta(a_{\mathbf{g}}, b_{\mathbf{h}}, c_{\mathbf{k}})^{S(\mathbf{g}_0)} = {}^{\mathbf{g}_0}\Theta(a_{\mathbf{g}}, b_{\mathbf{h}}, c_{\mathbf{k}})$,

$$\begin{aligned} {}^{\mathbf{g}_0}\mathbf{g}[F_p^{b_{\mathbf{h}}c_{\mathbf{k}}d_1}]_{q,\delta\phi}^{n,\eta\kappa} &= {}^{\mathbf{g}_0}\Theta(b_{\mathbf{h}}, c_{\mathbf{k}}, d_1)\nu_4(\mathbf{g}_0\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l})(1[F_p^{b_{\mathbf{h}}c_{\mathbf{k}}d_1}]_{q,\delta\phi}^{n,\eta\kappa})^{S(\mathbf{g}_0\mathbf{g})} \\ &= {}^{\mathbf{g}_0}\Theta(b_{\mathbf{h}}, c_{\mathbf{k}}, d_1)\frac{\nu_4(\mathbf{g}_0\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l})}{\nu_4(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l})^{S(\mathbf{g}_0)}}(\mathbf{g}[F_p^{b_{\mathbf{h}}c_{\mathbf{k}}d_1}]_{q,\delta\phi}^{n,\eta\kappa})^{S(\mathbf{g}_0)}, \end{aligned} \quad (\text{H1})$$

where $S(\mathbf{g}_0)S(\mathbf{g}) = S(\mathbf{g}_0\mathbf{g})$. Replacing the above equation into Eq. (123),

$$\begin{aligned} &\sum_{\epsilon} {}^{\mathbf{g}_0}\Theta(j_{\mathbf{gh}}, c_{\mathbf{k}}, d_1)\nu_4(\mathbf{g}_0, \mathbf{gh}, \mathbf{k}, \mathbf{l})(1[F_e^{j_{\mathbf{gh}}c_{\mathbf{k}}d_1}]_{q,\delta\epsilon}^{m,\beta\chi})^{S(\mathbf{g}_0)} \cdot {}^{\mathbf{g}_0}\Theta(a_{\mathbf{g}}, b_{\mathbf{h}}, q_{\mathbf{kl}})\nu_4(\mathbf{g}_0, \mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l})(1[F_e^{a_{\mathbf{g}}b_{\mathbf{h}}q_{\mathbf{kl}}}]_{p,\phi\gamma}^{j,\alpha\epsilon})^{S(\mathbf{g}_0)} \\ &= (-1)^{s(\alpha)s(\beta)} \sum_{n_{\mathbf{hk}}\eta\psi\kappa} {}^{\mathbf{g}_0}\Theta(a_{\mathbf{g}}, b_{\mathbf{h}}, c_{\mathbf{k}})\nu_4(\mathbf{g}_0, \mathbf{g}, \mathbf{h}, \mathbf{k})(1[F_m^{a_{\mathbf{g}}b_{\mathbf{h}}c_{\mathbf{k}}}]_{n,\eta\psi}^{j,\alpha\beta})^{S(\mathbf{g}_0)} \cdot {}^{\mathbf{g}_0}\Theta(a_{\mathbf{g}}, n_{\mathbf{hk}}, d_1)\nu_4(\mathbf{g}_0, \mathbf{g}, \mathbf{hk}, \mathbf{l})(1[F_e^{a_{\mathbf{g}}n_{\mathbf{hk}}d_1}]_{p,\kappa\gamma}^{m,\psi\chi})^{S(\mathbf{g}_0)} \\ &\quad \times {}^{\mathbf{g}_0}\Theta(b_{\mathbf{h}}, c_{\mathbf{k}}, d_1)\frac{\nu_4(\mathbf{g}_0\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l})}{\nu_4(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l})^{S(\mathbf{g}_0)}}(\mathbf{g}[F_p^{b_{\mathbf{h}}c_{\mathbf{k}}d_1}]_{q,\delta\phi}^{n,\eta\kappa})^{S(\mathbf{g}_0)}. \end{aligned} \quad (\text{H2})$$

Then by Eq. (127), and taking an $S(\mathbf{g}_0)$ conjugate for the whole equation, we obtain

$$\sum_{\epsilon} 1[F_e^{j_{\mathbf{gh}}c_{\mathbf{k}}d_1}]_{q,\delta\epsilon}^{m,\beta\chi} 1[F_e^{a_{\mathbf{g}}b_{\mathbf{h}}q_{\mathbf{kl}}}]_{p,\phi\gamma}^{j,\alpha\epsilon} = (-1)^{s(\alpha)s(\delta)} \sum_{n_{\mathbf{hk}}\eta\psi\kappa} 1[F_m^{a_{\mathbf{g}}b_{\mathbf{h}}c_{\mathbf{k}}}]_{n,\eta\psi}^{j,\alpha\beta} 1[F_e^{a_{\mathbf{g}}n_{\mathbf{hk}}d_1}]_{p,\kappa\gamma}^{m,\psi\chi} \mathbf{g}[F_p^{b_{\mathbf{h}}c_{\mathbf{k}}d_1}]_{q,\delta\phi}^{n,\eta\kappa}. \quad (\text{H3})$$

APPENDIX I: CANONICAL CHOICE OF COCHAINS

This appendix will be useful if we further consider the exactly solvable model for 2 + 1D surface SET phases with the $H^2(G, \mathbb{Z}_2)$ fermionic 't Hooft anomaly. It has been proved in Ref. [92] that we can choose a gauge for any n -cocycle ω_n such that

$$\omega_n(\underbrace{\mathbf{1}, \dots, \mathbf{1}}_{m \text{ terms}}, \mathbf{g}_1, \dots, \mathbf{g}_{n+1-m}) = 1, \quad (\text{I1})$$

where m is the number of repeating index 1 ($\mathbf{1} \in G$ is the trivial element) with $2 \leq m \leq n+1$ and $\mathbf{g}_1 \neq \mathbf{1}$.

Here we show that we can also do such a canonical choice for the data in describing general fSPT phases [8], which are cochains ν_4, n_3, n_2 and cocycle n_1 , in the case that $\beta_4(01234) := s_1(01)n_2(134)n_2(123) = 0 \pmod{2}$. Recall that for any n -cochain $f_n(\mathbf{1}, \mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \dots)$, we write it in abbreviation as $f_n(e0123\dots)$. They satisfy

$$dn_1 = 0, \quad (\text{I2})$$

$$dn_2 = \omega_2 \smile n_1 + s_1 \smile n_1 \smile n_1 \pmod{2}, \quad (\text{I3})$$

$$dn_3 = \omega_2 \smile n_2 + n_2 \smile n_2 + s_1 \smile (n_2 \smile n_2) \pmod{2}, \quad (\text{I4})$$

$$d\nu_4 = \mathcal{O}_5[n_3], \quad (\text{I5})$$

where

$$\begin{aligned} \mathcal{O}_5[n_3](012345)|_{\beta_4=0} &= (-1)^{(\omega_2 \smile n_3 + n_3 \smile n_3 \smile n_3 \smile n_3 \smile n_3 \smile n_3)(012345) + \omega_2(013)dn_3(12345)} \\ &\quad \times (-1)^{dn_3(02345)dn_3(01235) + \omega_2(023)[dn_3(01245) + dn_3(01235) + dn_3(01234)]} \\ &\quad \times i^{dn_3(01245)dn_3(01234)} \pmod{2} \times (-i)^{[dn_3(12345) + dn_3(02345) + dn_3(01345)]dn_3(01235)} \pmod{2} \\ &= (-1)^{(\omega_2 \smile n_3 + n_3 \smile n_3 \smile n_3 \smile n_3 \smile n_3 \smile n_3 + dn_3 \smile n_1 n_2 + \zeta_5(n_2 + \omega_2, n_2))(012345)} \times i^{[n_2 + \omega_2]_2 \smile (n_2 \smile n_2)}, \end{aligned} \quad (\text{I6})$$

where the above simplification is from Ref. [71], and $\zeta_5(a_2, b_2)(012345) = a_2(023)a_2(012)b_2(345)b_2(235)$ for any $a_2, b_2 \in Z^2(G, \mathbb{Z}_2)$, and $s_1(\mathbf{g}_0, \mathbf{g}_1) \in \{0, 1\}$ indicates $\mathbf{g}_0^{-1}\mathbf{g}_1$ is anti-unitary or not.

Since ω_2 and s_1 are cocycles, we can at first choose

$$\omega_2(\mathbf{1}, \mathbf{1}, \mathbf{g}_1) = \omega_2(\mathbf{1}, \mathbf{1}, \mathbf{1}) = 0 \pmod{2}, \quad (\text{I7})$$

$$s_1(\mathbf{1}, \mathbf{1}) = 0 \pmod{2}. \quad (\text{I8})$$

1. $\mathbb{Z}_2$ -valued cochain n_2

First we show we can choose

$$n_2(\mathbf{1}, \mathbf{1}, \mathbf{1}) = 0 \pmod{2}. \quad (\text{I9})$$

Following the same argument in Ref. [92], we introduce a coboundary $b_2(\mathbf{1}, \mathbf{1}, \mathbf{1}) = d\mu_1(\mathbf{1}, \mathbf{1}, \mathbf{1}) = \mu_1(\mathbf{1}, \mathbf{1})$, where μ_1 is a $\mathbb{Z}_2$ -valued 1-cochain. If we require that $\mu_1(\mathbf{1}, \mathbf{1}) = n'_2(\mathbf{1}, \mathbf{1}, \mathbf{1})$, then after the gauge transformation $n_2 = n'_2 + b_2 \pmod{2}$, we have $n_2(\mathbf{1}, \mathbf{1}, \mathbf{1}) = 0 \pmod{2}$.

Second we show

$$n_2(\mathbf{1}, \mathbf{1}, \mathbf{g}_1) = 0 \pmod{2}. \quad (\text{I10})$$

We have

$$dn_2(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1) = n_2(\mathbf{1}, \mathbf{1}, \mathbf{g}_1). \quad (\text{I11})$$

And we check $dn_2(eee1) = \omega_2 \smile n_1(eee1) + s_1 \smile n_1 \smile n_1(eee1) = 0 \pmod{2}$.

2. $\mathbb{Z}_2$ -valued cochain n_3

First we show

$$n_3(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}) = 0 \pmod{2}. \quad (\text{I12})$$

We have

$$dn_3(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}) = n_3(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}). \quad (\text{I13})$$

And we check $dn_3(eeeee) = \omega_2 \smile n_2(eeeee) + n_2 \smile n_2(eeeee) + s_1 \smile (n_2 \smile n_2)(eeee) = 0 \pmod{2}$.

Second we show we can choose

$$n_3(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1) = 0 \pmod{2}, \quad (\text{I14})$$

This is also done by the gauge transformation $n_3 = n'_3 + b_3 \pmod{2}$, where $b_3(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1) = d\mu_2(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1) = \mu_2(\mathbf{1}, \mathbf{1}, \mathbf{g}_1) + \mu_2(\mathbf{1}, \mathbf{1}, \mathbf{1})$. By requiring $n'_3(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1) = \mu_2(\mathbf{1}, \mathbf{1}, \mathbf{g}_1) + \mu_2(\mathbf{1}, \mathbf{1}, \mathbf{1})$, we fall into this gauge.

Third we show in this gauge,

$$n_3(\mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_2) = 0 \pmod{2}. \quad (\text{I15})$$

We have

$$\begin{aligned} dn_3(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_2) &= n_3(\mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_2) + n_3(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_2) + n_3(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1) \\ &= n_3(\mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_2). \end{aligned} \quad (\text{I16})$$

And we check $dn_3(eee12) = \omega_2 \smile n_2(eee12) + n_2 \smile n_2(eee12) + s_1 \smile (n_2 \smile n_2)(eee12) = 0 \pmod{2}$.

3. $U(1)$ -valued cochain ν_4

First we show we can choose

$$\nu_4(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}) = 1. \quad (\text{I17})$$

We introduce a coboundary $b_4(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}) = d\mu_3(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}) = \mu_3(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$, where μ_3 is a $U(1)$ -valued 3-cochain. If we require that $\mu_3(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}) = \nu'_4(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$, then after the gauge transformation $\nu_4 = \nu'_4 b_4^{-1}$, we have $\nu_4(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}) = 1$.

And we can also choose

$$\nu_4(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_2) = 1. \quad (\text{I18})$$

We still consider gauge transformation $\nu_4 = \nu'_4 b_4^{-1}$, where $b_4(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_2) = d\mu_3(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_2) = \mu_3(\mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_2) \times \mu_3^{-1}(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_2) \mu_3(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1)$. By requiring $\nu'_4(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_2) = \mu_3(\mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_2) \mu_3^{-1}(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_2) \mu_3(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1)$, we fall into this gauge.

Then we show in this gauge,

$$\nu_4(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1) = 1, \quad (\text{I19})$$

$$\nu_4(\mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) = 1. \quad (\text{I20})$$

We have

$$d\nu_4(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1) = \nu_4(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1), \quad (\text{I21})$$

$$\begin{aligned} d\nu_4(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) &= \frac{\nu_4(\mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) \nu_4(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_3)}{\nu_4(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_2, \mathbf{g}_3) \nu_4(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_2)} \\ &= \nu_4(\mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3). \end{aligned} \quad (\text{I22})$$

And we check from Eq. (I6) that $d\nu_4(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1) = d\nu_4(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) = 1$.

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