

How Much Entanglement Is Needed for Topological Codes and Mixed States with Anomalous Symmetry?

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It is known that particles with exotic properties can emerge in systems made of simple constituents such as qubits, due to long-range quantum entanglement. In this paper, we provide quantitative characterizations of entanglement necessary for emergent anyons and fermions by using the geometric entanglement measure (GEM), which quantifies the maximal overlap between a given state and any short-range-entangled states. For systems with emergent anyons, based on the braiding statistics, we show that the GEM scales linearly in the system size regardless of microscopic details. The phenomenon of emergent anyons can also be understood within the framework of quantum error correction (QEC). Specifically, we show that the GEM of any 2D stabilizer codes must be at least quadratic in the code distance. Our proof is based on a generic prescription for constructing string operators, establishing a rigorous and direct connection between emergent anyons and QEC. For systems with emergent fermions, despite that the ground state subspaces could be exponentially huge and their coding properties could be rather poor, we show that the GEM also scales linearly in the system size. Our analysis establishes an intriguing link between quantum anomaly and entanglement: A quantum state respecting anomalous 1-form symmetries must be long-range entangled and have large GEM. Our results also extend to mixed states, such as the ZX-dephased toric code, providing a provably nontrivial class of intrinsically mixed-state phases.

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I. INTRODUCTION

Quantum information processing is intrinsically vulnerable to noises and decoherence that may arise from physical interactions with the environment as well as engineering imperfections in handling qubits. A useful way to fault-tolerantly store and process quantum information is to utilize particles with exotic statistical properties known as anyons [1,2], which provide ideal platforms with naturally arising topological protection from otherwise detrimental errors [3,4]. While anyons may not naturally exist as elementary particles in nature, they do *emerge* as low-energy excitations in strongly correlated many-body quantum systems—most notably in fractional quantum Hall states [5]—demonstrating their physical reality. Theoretical and experimental investigations of anyons remain one of the most active areas of research in modern physics.

The emergent phenomenon of anyons is closely related to long-range entanglement in many-body systems [6–12]. A vivid feature of quantum entanglement is the superposition: Construction of wave functions requires a large number of product basis states. For instance, consider a ground state of the 2D toric code. It can be expressed as a sum of exponentially many computational (Z-basis) states

$$|\Psi_{\text{toric}}\rangle = \frac{1}{\sqrt{2^{n/2}}} \prod_{\text{vertex}} (I + \sum_{\vec{X}} \vec{X}) |0\rangle^{\otimes n} \propto \sum_{\alpha \in \text{closed-loop}} |\alpha\rangle, \quad (1)$$

where each term can be viewed as a closed-loop configuration by drawing a line connecting qubits in $|1\rangle$. This characterization of the toric code is an example of the string-net condensation picture which hosts a wide generalization for nonchiral topological order [13]. Anyonic excitations can be generally created at end points of an open string operator γ applied to the entangled ground state. Nontrivial braiding statistics of anyons implies that the ground state cannot be prepared by a short-depth quantum circuit and, thus, must be long-range entangled [6,14,15].

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The degree of superpositions in a given wave function $|\Psi\rangle$ can be quantitatively measured by the geometric entanglement measure (GEM) of the form [16–21]

$$E(\Psi) = - \max_{|P\rangle: \text{product}} \log_2 |\langle\Psi|P\rangle|^2 \quad (2)$$

which computes the maximum overlap between $|\Psi\rangle$ and product states. Unfortunately, this measure fails to distinguish short-range and long-range entanglement, since short-range-entangled states, such as the cluster state, can score high on the GEM. To circumvent this problem, the depth- t geometric entanglement measure (depth- t GEM) was proposed [22]:

$$E_t(\Psi) = - \max_{U: \text{depth}(U)=t} \log_2 |\langle\Psi|U|0\rangle^{\otimes n}|^2, \quad (3)$$

which considers the maximum overlap between $|\Psi\rangle$ and all short-range-entangled states $U|0\rangle^{\otimes n}$. Here, the maximum is taken over all n -qubit geometrically local (e.g., brick wall) depth- t quantum circuits; a depth-0 circuit is defined as a product of single-qubit gates, and, hence, $E(\Psi) = E_0(\Psi)$.

This generalization of GEM provides a refined characterization of long-range entanglement. According to the conventional definition, a quantum state $|\Psi\rangle$ is said to be long-range entangled if it cannot be (approximately) prepared by a short-depth circuit. In terms of the GEM, this simply amounts to $E_t(\Psi) \neq 0$ [or $E_t(\Psi) > \epsilon$ if approximations are accounted] for small t . However, depending on its underlying entanglement structure, each long-range-entangled state $|\Psi\rangle$ can have strikingly different values of $E_t(\Psi)$. Some long-range-entangled states score very low on the GEM, suggesting that they have a significant overlap with short-range-entangled states despite their high quantum circuit complexity. On the contrary, genuinely long-range-entangled states score very high on the GEM, suggesting that they not only cannot be approximated by short-range-entangled states, but also must be almost orthogonal to any short-range-entangled states. For instance, the Greenberger-Horne-Zeilinger (GHZ) state $(1/\sqrt{2})(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$ is long-range entangled, since it cannot be prepared from a product state by a constant depth circuit. However, the GHZ state scores small for depth- t GEM, namely, $E_t(\text{GHZ}) \sim O(1)$ for any $t \sim O(1)$. This aligns with our intuition concerning the intrinsic instability of entanglement in the GHZ state. On the other hand, as we prove later in this paper, the toric code has $E_t(\text{toric}) \sim O(n)$ for $t \sim O(1)$, reflecting its complex entanglement structure.

In this work, we use the depth- t GEM as a more direct, quantitative characterization of quantum entanglement that is needed for emergent anyons and other exotic physical objects.

A. Emergent anyons

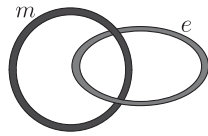
In this paper, we begin by studying two-dimensional spin systems whose ground states support anyonic excitations with nontrivial braiding statistics. While we focus on the toric code for simplicity of presentation, similar results can be generically derived. We use n , k , and d to refer the number of spins, the logarithmic ground state degeneracy, and the code distance, respectively.

Result 1. A ground state $|\Psi\rangle$ of the toric code satisfies

$$E_t(\Psi) = \Omega(n), \quad (4)$$

where the multiplicative coefficient depends only on t . [In this paper, we use standard Bachmann-Landau notation [23] for Ω , Θ , and O symbols. For instance, Eq. (4) implies that there exists $\alpha, n_0 > 0$ such that $E_t(\Psi) > \alpha n$ for all $n > n_0$.]

The proof involves several steps, but the underlying physical picture is simple enough to summarize here. Recall that the toric code supports two types of Abelian anyonic excitations, e and m , that can be created at end points of stringlike operators. Such operators can be deformed while fixing the end points and create the same sets of anyons. The braiding statistics between e and m can be probed as follows:

$$\langle\Psi|\gamma_m^\dagger \gamma_e \gamma_m|\Psi\rangle = \text{diagram} = -1. \quad (5)$$


Here, γ_m represents an open m string, and γ_e represents a closed e loop; in the middle is a spacetime diagram illustrating the braiding. The key observation is that, due to the deformability of γ_m string, the region that supports the pair of m anyons must be decoupled from the rest of the system if $|\Psi\rangle$ was short-range entangled, so that the γ_e loop “does not know” whether there was or was not an m anyon inside. Therefore, $\langle\Psi|\gamma_m^\dagger \gamma_e \gamma_m|\Psi\rangle = 1$. This contradicts the braiding statistics, which proves that the toric code ground states must be long-range entangled. The above beautiful argument is due to Bravyi [24] and was described in Ref. [15].

To prove the claimed linear lower bound on $E_t(\Psi)$, we consider $O(n)$ copies of braiding processes in nonoverlapping subregions where each process contributes $O(1)$ to $E_t(\Psi)$. This argument can be readily extended to ground states of any spin systems that support nontrivial anyonic excitations. Namely, we can show that the nontrivial braiding $\langle\gamma_a^\dagger \gamma_b \gamma_a\rangle \neq 1$ for two anyons a and b is inconsistent with short-range entanglement, and, as such, a similar lower bound on $E_t(\Psi)$ follows.

B. Emergent fermions

Next, we turn our attention to systems that support emergent fermions. To be concrete, we focus on the following variant of the Kitaev honeycomb model [25] (similar results can be derived in more general settings):

$$H_{\text{fermion}} = - \sum_{\square} S_{\square}, \quad S_{\square} = X \begin{array}{c} Y \\ \diagup \quad \diagdown \\ Z \end{array} X. \quad (6)$$

This Hamiltonian naturally emerges when viewing Kitaev's honeycomb model as a subsystem code and then considering the stabilizer group [26,27]. Viewing Eq. (6) as a stabilizer code, a ground state satisfies $S_{\square}|\Psi\rangle = +|\Psi\rangle$, and the code parameters are $k \sim n/2$ and $d = 2$. Despite the exponentially large code subspace 2^k and the small code distance d , a linear lower bound, which is applicable to *arbitrary* ground state, can be proven.

Result 2. A ground state $|\Psi\rangle$ of the honeycomb model satisfies

$$E_t(\Psi) = \Omega(n), \quad (7)$$

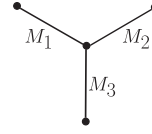
where the multiplicative coefficient depends only on t .

Unlike anyons in the toric code, fermions have trivial braiding statistics (i.e., they are not anyons). Hence, our proof utilizes the fermion exchange statistic crucially. Before proceeding, however, it is worth emphasizing an important difference between how anyons and fermions emerge in the toric code and the honeycomb model, respectively. In the toric code, open string anyon creation operators do not commute with Hamiltonian; hence, anyons are low-energy excitations *outside* the ground state subspace. On the contrary, in the honeycomb model, the fermion creation operators do commute with the Hamiltonian and, hence, fermions emerge *inside* the ground state subspace. More precisely, note the honeycomb model possesses the following two-body logical operators:

$$X-X \quad Y-Y \quad Z-Z \quad (8)$$

that commute with stabilizers $S_{\square}$. Let us pick an arbitrary ground state $|\Psi\rangle$ and regard it as a “vacuum” state with no fermions. Consider an open string operator M which is generated from two-body logical operators. Then, $M|\Psi\rangle$ can be interpreted as a state with two emergent fermions at end points of an open string M . The crucial point is that $M|\Psi\rangle$ still lives in the energy ground space, as M is a logical operator of the code.

The particle exchange statistics can be probed by considering three stringlike operators M_1 , M_2 , and M_3 that share a common end point [28]:



$$\langle \Psi | M_3^\dagger M_2^\dagger M_1^\dagger M_3 M_2 M_1 | \Psi \rangle = e^{i\theta}, \quad (9)$$

where $e^{i\theta} = -1$ for fermions. In the honeycomb model, this relation can be explicitly verified, as three stringlike logical operators anticommute with each other at the overlapping qubit. Using a variant of Bravyi's idea, it can be shown that short-range entanglement and the deformability of stringlike operators would imply $\langle \Psi | M_3^\dagger M_2^\dagger M_1^\dagger M_3 M_2 M_1 | \Psi \rangle = 1$, contradicting with the exchange statistics. Then, with some extra work, the claimed linear lower bound on $E_t(\Psi)$ can be obtained.

C. Anomalous symmetry and mixed state

The aforementioned result on emergent fermions has interesting implications from the perspectives of anomalous symmetries and mixed states.

I. Anomalous symmetries

Recently, there has been considerable progress in understanding the relation between quantum entanglement and anomalous symmetries. To motivate this viewpoint, consider a system with conventional symmetries, such as a $\mathbb{Z}_2$ global symmetry generated by $S = \bigotimes_{j=1}^n X_j$. The system is equipped with well-defined local $\mathbb{Z}_2$ charge generators which are simply X_j 's. The symmetry constraint of the form $S|\Psi\rangle = |\Psi\rangle$ then dictates that the total $\mathbb{Z}_2$ charge in $|\Psi\rangle$, expressed as a product of local $\mathbb{Z}_2$ generators X_j , must be trivial (+1). Furthermore, the system has a natural “vacuum” state, namely, $|+\rangle^{\otimes n}$, for which local generators X_j act trivially, and, thus, is not entangled.

On the contrary, anomalous symmetries do not admit locally definable charge generators. A prototypical example of this phenomenon is a 1D $\mathbb{Z}_2$ anomalous global symmetry, often called the CZX symmetry S_{CZX} [29], which is generated by

$$S_{\text{CZX}} = S_{\text{CZ}} S_X, \quad (10)$$

where $S_{\text{CZ}} = \prod_j \text{CZ}_{j,j+1}$, $S_X = \prod_j X_j$. This operator defines a global $\mathbb{Z}_2$ symmetry as $S_{\text{CZX}}^2 = I$ (for even n) but does not possess local generators. For instance, one might want to consider $\text{CZ}_{j,j+1} X_j$ as a local generator. Unfortunately, a product of $\text{CZ}_{j,j+1} X_j$ does not generate $S_{\text{CZX}} = S_{\text{CZ}} S_X$. Furthermore, $\text{CZ}_{j,j+1} X_j$ does not commute with S_{CZX} , and also they do not commute with each other. As such, consistent meaning as local charge generators cannot be given to them.

This impossibility of decomposing an anomalous symmetry operator into local charge generators is closely

related to quantum entanglement. This viewpoint has been extensively studied from the perspective of symmetry-protected topological (SPT) order, as the CZX symmetry emerges at the boundary of a 2D $\mathbb{Z}_2$ SPT phase [29]. Recently, it has been rigorously shown that any wave functions satisfying $S_{\text{CZX}}|\Psi\rangle = |\Psi\rangle$ must be multipartite entangled [30].

Returning to the honeycomb model, the upshot is that the hexagonal stabilizers $S_\circ$ are another example of anomalous symmetries. More precisely, these are 1-form anomalous symmetries [27,31–33], since $S_\circ$ acts as a codimension-1 operator instead of a global codimension-0 operator as in the CZX model. The anomalous nature of $S_\circ$ symmetries can be seen from the fact that they cannot be cut into two pieces. For example, suppose we cut $S_\circ$ vertically into two parts and write it as a product of the left and right parts, $S_\circ = S_\circ^{(L)} S_\circ^{(R)}$. We immediately notice that $S_\circ^{(L)}$ and $S_\circ^{(R)}$ do not commute with some of other hexagonal stabilizers and, thus, are not well defined as local generators. Instead, one might decompose $S_\circ$ as a product of two-body logical operators from Eq. (8). While this decomposition consists of operators that commute with $S_\circ$ symmetries, two-body operators anticommute with each other and, hence, do not admit interpretations as local generators.

Our result on the honeycomb model can be rephrased as follows.

Result 2' (anomalous symmetry). Any state $|\Psi\rangle$ satisfying anomalous 1-form symmetries $S_\circ|\Psi\rangle = |\Psi\rangle$ in the honeycomb model must be long-range entangled. Furthermore, they satisfies $E_t(\Psi) = \Omega(n)$.

While wave functions with anomalous global symmetries are long-range entangled, in general, they may score very low in terms of the GEM. For instance, the GHZ state $(1/\sqrt{2})(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$ satisfies the CZX symmetry, but it has $E_0(\text{GHZ}) = 1$. This is in contrast with anomalous 1-form symmetries, such as $S_\circ$ symmetries, where the GEM scales linearly with respect to n .

2. Mixed states

Characterization of quantum entanglement for mixed states is a considerably harder question compared to its counterpart for pure states. Quantum circuit complexity of preparing a mixed state has been studied in quantum information theory in the context of topological quantum memory at finite temperature [4,34]. Recently, there has been renewed interest in the possibility of novel quantum phases, that are genuinely intrinsic to mixed state, in the condensed matter community. Generally, a mixed state ρ is said to be short-range entangled if ρ can be expressed as a probabilistic ensemble of short-range-entangled states [34]:

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|, \quad |\psi_j\rangle = U_j |0\rangle^n, \quad (11)$$

where U_j are short-depth circuits. A naturally arising question is whether a mixed state ρ in the ground state space of the honeycomb model is long-range entangled or not. Evidence for long-range entanglement in ρ was reported in Ref. [35], where a subleading contribution in the entanglement negativity [36], similar to topological entanglement entropy, was found. Further connections between anomalous 1-form symmetry and mixed-state topological order were discussed in Refs. [37,38]. Here, as a simple corollary of the aforementioned result, we can prove that ρ in the honeycomb model is indeed long-range entangled.

Result 2'' (mixed state). Any mixed state ρ satisfying the hexagonal constraints $\text{Tr}(\rho S_\circ) = 1$ in the honeycomb model cannot be written as an ensemble of short-range-entangled states.

In fact, we prove a stronger statement by generalizing the depth- t GEM to the mixed-state setting where the maximum fidelity between the target state and all short-range-entangled mixed states is considered. Namely, we prove that a similar linear lower bound on the mixed-state GEM holds for any symmetric ρ .

Lastly, we note that the results also apply to other models of mixed states with anomalous 1-form symmetry, such as the ZX-dephased toric code [27].

D. 2D stabilizer codes

The phenomenon of emergent anyons can be understood within the framework of quantum error correction (QEC) where quantum information is stored in the energy ground state space. This motivates us to ask whether there is a fundamental lower bound on the necessary quantum entanglement for storing quantum information in a bounded volume of space.

In this paper, we reveal a universal lower bound on $E_t(\Psi)$ for certain two-dimensional QEC systems. Specifically, we consider a system of n qubits placed on a 2D lattice and focus on codes for which the code subspace $\mathcal{C}$ is defined by geometrically local stabilizer generators S_j : The code subspace $\mathcal{C}$ can be realized as the energy ground subspace of a local gapped Hamiltonian

$$H = -\sum_j S_j. \quad (12)$$

The QEC capability is usually measured by its code distance. A code is said to have code distance d if logical states are locally indistinguishable on any subset of less than d physical qubits.

Our main result is the following lower bound on the GEM.

Result 3. For a 2D geometrically local stabilizer code, if the code distance $d > d_0$, then any logical state $|\Psi\rangle$ satisfies

$$E_t(\Psi) = \Omega(d^2). \quad (13)$$

Here, both the multiplicative constant and d_0 depend only on t and the size of stabilizer generators.

For the toric code, $d = \Theta(\sqrt{n})$; hence, Result 3 is in accordance with Result 1. There is, however, a crucial difference. The starting point of Result 1 is the emergent anyons, which exist on any geometries including the sphere. The starting point of Result 3, on the other hand, is the QEC property which relies on the ground state degeneracy. Therefore, Result 3 does not apply to the toric code on the sphere *per se*.

Nevertheless, the proof of Result 3 reflects a rigorous and direct connection between QEC and emergent anyons. The main idea is that we can derive the existence of emergent anyons *solely* based on the error correction property. More precisely, as long as the code distance d is sufficiently larger than the size of local stabilizer generators, we can always construct looplike operators γ_b that create only syndrome at end points, as well as stringlike stabilizer operators γ_a , such that $\langle \gamma_a^\dagger \gamma_b \gamma_a \rangle = -1$. Key technical tools in the constructions of loop and stringlike operators are the cleaning lemma [39] and the disentangling lemma [40], which allows us to identify a pair of anticommuting logical operators. Specifically, these logical operators can be designed to be supported on unions of linelike subregions. We then introduce a simple but powerful truncation procedure of stabilizer operators which enables us to obtain the braiding configuration similar to Eq. (5). Such braiding statistics is not compatible with short-range entanglement, which ultimately leads to the desired linear lower bound on the GEM.

The anyon braiding statistics then simply follow by interpreting the syndromes at the end of string operators as anyons. From this point of view, our result implies that anyons always emerge from QEC in 2D stabilizer codes, namely, from the sole assumption of the code distance.

E. Relation to previous work

In a previous work, we derived generic lower bounds on depth- t GEM for QEC codes in several settings [22]. The most relevant to the present paper is the following universal lower bound for a quantum low-density parity-check (qLDPC) code with the code distance d :

$$E_t(\Psi) = \Omega(d), \quad (14)$$

where the multiplicative coefficient depends only on the sparsity of the qLDPC code and the depth t . The linear scaling with d cannot be improved further, as there exist qLDPC codes with the distance $d \sim n$. This lower bound, however, is not tight for quantum codes supported on lattices with geometrically local projectors. For instance, the two-dimensional toric code has $E_0(\Psi) \sim n/2$, but its code distance is $d \sim \sqrt{n}$. Also, the above lower bound is not applicable to the toric “code” without logical qubits (e.g., supported on a sphere), as its derivation crucially

depends on the presence of logical qubits. Finally, the above bound is not particularly useful in studying long-range entanglement in the honeycomb model, as its code distance is $d = 2$.

Our results in the present work improve the above qLDPC lower bound in various ways by focusing on two-dimensional systems. Result 1 proves that the quadratic scaling $E_t(\text{toric}) = \Omega(d^2)$ [$t \sim O(1)$] for the toric code. Result 3 further improves this characterization for the cases of 2D local stabilizer codes by proving $E_t(\Psi) = \Omega(d^2)$. Result 2 proves that, despite the poor coding properties, all the code word states in the honeycomb model are long-range entangled with $E_t(\Psi) = \Omega(n)$.

F. Plan of the paper

This paper is organized as follows. In Sec. II, we study the toric code as a prototypical example of systems with emergent anyons. In Sec. III, we derive a generic lower bound on the GEM for arbitrary 2D local stabilizer codes. In Sec. IV, we study the honeycomb model and show that wave functions with anomalous 1-form symmetries have $E_t(\Psi) = \Omega(n)$. In Sec. V, we conclude with brief discussions.

II. 2D TORIC CODE

We prove the following linear lower bound on $E_t(\Psi)$ for a toric code ground state.

Theorem 1. There exists an absolute constant $\alpha > 0$ ($\alpha = 10^{-4}$ suffices) such that for any toric code ground state $|\Psi\rangle$, if $t \leq O(\sqrt{n})$, then

$$E_t(\Psi) > \frac{\alpha n}{(t+1)^2}. \quad (15)$$

A. Anyon braiding statistics

Recall that the toric code Hamiltonian is given by

$$H_{\text{toric}} = -\sum_{+} A_{+} - \sum_{\square} B_{\square}, \quad (16)$$

where $A_{+} = \prod_{j \in +} X_j$ and $B_{\square} = \prod_{j \in \square} Z_j$. Qubits reside on edge of an $L \times L$ square lattice, and $+$ ($\square$) denote vertices (plaquettes). A ground state $|\Psi\rangle$ satisfies $A_{+}|\Psi\rangle = B_{\square}|\Psi\rangle = |\Psi\rangle$.

Microscopic details of the construction, however, are not important. A crucial property we use for the proof is the nontrivial braiding statistics of anyons of the toric code. Namely, consider a pair of a closed string operator γ_e and an open string operator γ_m which consist of Pauli Z and X operators, respectively (Fig. 1). Considering a process of braiding e around m , we find

$$\langle \Psi | \gamma_m^\dagger \gamma_e \gamma_m | \Psi \rangle = -1. \quad (17)$$

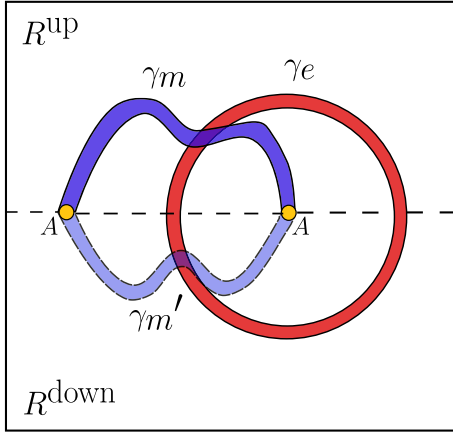


FIG. 1. Two dressed anticommuting string operators for anyon braiding in a deformed toric code. Their width is at most $2(t+1)$. For the purpose of the proof, it suffices to consider a region R with a linear size of $8(t+1)$.

This braiding statistics is stable under deformation by a constant depth circuit U_t . Namely, for a deformed toric code ground state $|\Psi_t\rangle \equiv U_t|\Psi\rangle$, define dressed string operators by $\tilde{\gamma}_m = U_t\gamma_m U_t^\dagger$ and $\tilde{\gamma}_e = U_t\gamma_e U_t^\dagger$. Equation (17) still holds for $|\Psi_t\rangle$ and $\tilde{\gamma}_m, \tilde{\gamma}_e$, but dressed string operators $\tilde{\gamma}_m, \tilde{\gamma}_e$ now have width $O(t)$. One can then consider a quasilocal anyon braiding process as in Fig. 1 by taking a larger region of linear size t .

B. Long-range entanglement

We first prove that a toric code ground state is long-range entangled.

Lemma 1. Given a toric code ground state $|\Psi\rangle$, we have

$$U_t|\Psi\rangle \neq |0\rangle^{\otimes n} \quad (18)$$

for any depth- t unitary U_t with constant t .

Proof. The proof is similar to Bravyi's proof described in Ref. [15]. Suppose that Eq. (18) is false. In terms of a deformed state $|\Psi_t\rangle \equiv U_t|\Psi\rangle$, this is equivalent to $|\Psi_t\rangle = |0\rangle^{\otimes n}$. Since U_t is a finite depth circuit, the braiding process of Eq. (17) can be considered by a pair of dressed string operators $\tilde{\gamma}_m = U_t\gamma_m U_t^\dagger$ and $\tilde{\gamma}_e = U_t\gamma_e U_t^\dagger$. Below, we omit the tilde and denote dressed operators by γ_m, γ_e to simplify the notation.

As in Fig. 1, we use a dressed string operator γ_m to create two m anyons supported on region A . We claim that

$$\gamma_m|\Psi_t\rangle = |\phi_A\rangle \otimes |0_{A^c}\rangle, \quad (19)$$

where $|\phi_A\rangle$ is a state in the Hilbert space $\mathcal{H}_A$ and $|0_{A^c}\rangle$ is the product state of $|0\rangle$ for all qubits outside A . To show this, we divide the system into three subregions: A , R^{up} , and

R^{down} as in Fig. 1. We can find another string operator γ'_m that is supported on $A \cup R^{\text{down}}$ satisfying

$$\gamma'_m|\Psi_t\rangle = \gamma_m|\Psi_t\rangle. \quad (20)$$

Define Π^{up} as an operator that projects $\mathcal{H}_{R^{\text{up}}}$ to $|0_{R^{\text{up}}}\rangle$. We have, by assuming Eq. (18) is false,

$$\begin{aligned} \Pi^{\text{up}}\gamma'_m|\Psi_t\rangle &= \gamma'_m\Pi^{\text{up}}|\Psi_t\rangle = \gamma'_m\Pi^{\text{up}}|0\rangle^{\otimes n} \\ &= \gamma'_m|0\rangle^{\otimes n} = \gamma'_m|\Psi_t\rangle. \end{aligned} \quad (21)$$

Similarly,

$$\Pi^{\text{down}}\gamma_m|\Psi_t\rangle = \gamma_m|\Psi_t\rangle. \quad (22)$$

Combining the above three equations, we have

$$\Pi^{\text{up}}\Pi^{\text{down}}\gamma_m|\Psi_t\rangle = \gamma_m|\Psi_t\rangle, \quad (23)$$

and, hence, Eq. (19) is proved.

Now we apply a loop operator γ_e on $\gamma_m|\Psi\rangle$. Recall that γ_e is a deformed stabilizer, and, hence, $\gamma_e|\Psi_t\rangle = |\Psi_t\rangle$. Assuming Eq. (18) is false, this implies $\gamma_e|0_{A^c}\rangle = |0_{A^c}\rangle$. Therefore, due to Eq. (19), we have

$$\begin{aligned} \gamma_e\gamma_m|\Psi_t\rangle &= |\phi_A\rangle \otimes \gamma_e|0_{A^c}\rangle = |\phi_A\rangle \otimes |0_{A^c}\rangle \\ &= \gamma_m|\Psi_t\rangle. \end{aligned} \quad (24)$$

This, however, contradicts Eq. (17), namely, $\gamma_m\gamma_e\gamma_m|\Psi_t\rangle = -|\Psi_t\rangle$, which completes the proof. ■

It is worth mentioning that the proof based on the Lieb-Robinson bound in Ref. [6] crucially uses the ground state degeneracy, while the above proof (based on Ref. [15]) applies to the toric code with no ground state degeneracy as well.

C. Overlap with short-range-entangled state

The result of the above lemma is equivalent to $E_t(\Psi) > 0$ and is, in general, not sufficient to obtain a lower bound for $E_t(\Psi)$ that would scale with the system size.

We now prove an upper bound on the maximum overlap between ρ_R of a deformed toric code ground state and $|0_R\rangle$ for a patch R of linear size t .

Lemma 2. Let R be any patch of linear size $8(t+1)$. Given a deformed toric code ground state $|\Psi_t\rangle = U_t|\Psi\rangle$ with a depth t circuit U_t , there exists an absolute constant $\epsilon > 0$ such that, in terms of trace norm,

$$\|\rho_R - |0_R\rangle\langle 0_R|\|_1 > \epsilon, \quad (25)$$

where $\rho_R = \text{Tr}_{R^c}(|\Psi_t\rangle\langle\Psi_t|)$, and accordingly the fidelity [here, we use the convention $\mathcal{F}(\rho, \sigma) = (\text{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})^2$] satisfies

$$\mathcal{F}(|0_R\rangle, \rho_R) < 1 - \frac{1}{4}\epsilon^2 = 1 - \epsilon' \quad (26)$$

for an absolute constant $\epsilon' > 0$.

To rephrase, the “local GEM” is not only positive, but must also be bounded away from zero by an absolute gap that is independent of the subregion size.

Proof. Let γ_m and γ_e be dressed string operators. Suppose that Eq. (25) is false. We claim that

$$\gamma_m|0_R\rangle\langle 0_R|\gamma_m^\dagger \approx |0_{A^c}\rangle\langle 0_{A^c}| \otimes \rho'_A, \quad (27)$$

where $\approx$ means that the trace distance is smaller than a constant $O(\epsilon)$. Here, A^c refers to the complement of A inside the patch R ; ρ'_A is positive semidefinite but might be unnormalized, since we allow approximation.

The proof is a more quantitative version of Lemma 1. We have

$$\begin{aligned} & \Pi^{\text{up}}(\gamma_m|0_R\rangle\langle 0_R|\gamma_m^\dagger)\Pi^{\text{up}} \\ & \approx \Pi^{\text{up}}(\gamma_m\rho_R\gamma_m^\dagger)\Pi^{\text{up}} \\ & = \Pi^{\text{up}}(\gamma_{m'}\rho_R\gamma_{m'}^\dagger)\Pi^{\text{up}} = \gamma_{m'}(\Pi^{\text{up}}\rho_R\Pi^{\text{up}})\gamma_{m'}^\dagger \\ & \approx \gamma_{m'}(\Pi^{\text{up}}|0_R\rangle\langle 0_R|\Pi^{\text{up}})\gamma_{m'}^\dagger = \gamma_{m'}(|0_R\rangle\langle 0_R|)\gamma_{m'}^\dagger \\ & \approx \gamma_{m'}\rho_R\gamma_{m'}^\dagger = \gamma_m\rho_R\gamma_m^\dagger \\ & \approx \gamma_m|0_R\rangle\langle 0_R|\gamma_m^\dagger, \end{aligned} \quad (28)$$

where the negation of Eq. (25) is used for each $\approx$. Moreover,

$$\begin{aligned} & \Pi^{\text{down}}(\gamma_m|0_R\rangle\langle 0_R|\gamma_m^\dagger)\Pi^{\text{down}} \\ & = \gamma_m\Pi^{\text{down}}|0_R\rangle\langle 0_R|\Pi^{\text{down}}\gamma_m^\dagger = \gamma_m|0_R\rangle\langle 0_R|\gamma_m^\dagger. \end{aligned} \quad (29)$$

Therefore, Eq. (27) is proved:

$$\begin{aligned} \gamma_m|0_R\rangle\langle 0_R|\gamma_m^\dagger & \approx \Pi^{\text{down}}\Pi^{\text{up}}(\gamma_m|0_R\rangle\langle 0_R|\gamma_m^\dagger)\Pi^{\text{up}}\Pi^{\text{down}} \\ & = |0_{A^c}\rangle\langle 0_{A^c}| \otimes \rho'_A. \end{aligned} \quad (30)$$

Using this approximation, we obtain

$$\begin{aligned} & \text{Tr}(\gamma_e\gamma_m\rho_R\gamma_m^\dagger) \\ & \approx \text{Tr}(\gamma_e\gamma_m|0_R\rangle\langle 0_R|\gamma_m^\dagger) \\ & \approx \text{Tr}[(\gamma_e|0_{A^c}\rangle\langle 0_{A^c}|) \otimes \rho'_A] = \text{Tr}(\gamma_e|0_{A^c}\rangle\langle 0_{A^c}|)\text{Tr}(\rho'_A) \\ & \approx \text{Tr}(\gamma_e|0_{A^c}\rangle\langle 0_{A^c}|) = \text{Tr}(\gamma_e|0_R\rangle\langle 0_R|) \\ & \approx \text{Tr}(\gamma_e\rho_R) = \langle\Psi_t|\gamma_e|\Psi_t\rangle = 1 \end{aligned} \quad (31)$$

which contradicts Eq. (17).

Counting the ϵ 's in the above calculation, we see $\epsilon = \frac{1}{5}$ suffices; hence, $\epsilon' = 0.01$ suffices. [More precisely, Eq. (28) implies that the 1-norm distance between two sides is less than 4ϵ ; the approximation in Eq. (30) is, therefore, accurate up to 4ϵ ; the approximations in Eq. (31) are accurate up to ϵ , 4ϵ , 4ϵ , and ϵ , respectively, leading to a total difference up to 10ϵ . However, the left-hand side in Eq. (31) should equal -1 assuming Eq. (17).] Importantly, ϵ and ϵ' are absolute constants and do not depend on the size of R . ■

We emphasize that the proof applies whenever anyons can be excited and braiding can be performed. It does not depend, for example, on the microscopic details of the stabilizers, the specifics of the lattice, or the translational invariance of the system.

Finally, to complete the proof of Theorem 1, we divide the whole system into multiple regions R_i such that mutual distances are larger than $2(t+1)$. For a (deformed) toric code, we have the following decoupling property:

$$\rho_R = \bigotimes_i \rho_{R_i}, \quad R = \cup_i R_i, \quad (32)$$

since stabilizer generators are geometrically local and each region R_i is correctable [40]. From this and the monotonicity of fidelity, we obtain

$$\begin{aligned} \mathcal{F}(|\Psi_t\rangle, |0^n\rangle) & \leq \mathcal{F}(\rho_R, |0_R\rangle) = \prod_i \mathcal{F}(\rho_{R_i}, |0_{R_i}\rangle) \\ & < (1 - \epsilon')^{n/[100(t+1)^2]}, \end{aligned} \quad (33)$$

where $n/100(t+1)^2$ is a number of disentangled patches. We emphasize that Eq. (32) is essential for Eq. (33); otherwise the first inequality does not hold, with the GHZ state being a counterexample. Equation (33) proves Theorem 1, with the constant $\alpha = -\frac{1}{100}\log_2(1 - \epsilon') \approx 1.4 \times 10^{-4}$.

While we focused on the toric code for concreteness, our proof is applicable to any system featuring emergent anyons with nontrivial braiding statistics. The key ingredients for our argument are the braiding relation Eq. (17) and the deformability of string operators Eq. (20). In more general settings, Eq. (17) takes the value $\mathcal{D}S_{ab}/d_a d_b$, where d_a and d_b are the quantum dimension of the anyons, $\mathcal{D}$ is the total dimension, and S_{ab} is the corresponding element of the modular S matrix [25]. Our proof applies whenever $\mathcal{D}S_{ab}/d_a d_b \neq 1$, namely, whenever the braiding is nontrivial.

III. 2D LOCAL STABILIZER CODES

In the previous section, we derived the GEM lower bound for the toric code based on emergent anyons. The toric code model stands out as a quintessential example of topological quantum error-correcting codes. In this section, we show that a similar bound applies more broadly to all

2D geometrically local stabilizer codes with or without boundaries (the geometric locality here is defined using the standard Euclidean geometry).

Theorem 2. For 2D geometrically local stabilizer codes, denote w as the maximal diameter of stabilizer generators. If $d > \Theta((w + t)^2)$, we have

$$E_t(\Psi) > \alpha \frac{d^2}{w^2(w + t)^2}. \quad (34)$$

for an absolute constant $\alpha > 0$.

Although the starting point here is quantum error correction instead of anyons, our proof strategy is similar to the one for the toric code using nontrivial braiding processes. The key extra ingredient is a generic prescription for constructing string operators. This enables us to derive the existence of anyons with nontrivial braiding statistics based solely on the quantum error correction property.

A. Cleaning lemma

A technical challenge is to construct a pair of open and closed string operators that anticommute with each other without relying on the macroscopic details of stabilizer generators. The crucial tool is a particular strengthening of the cleaning lemma for 2D local stabilizer codes [39].

Consider square regions A_j of linear size $O(d/w)$ which are separated from each other by $O(w)$. Denote the union by $A \equiv \cup_j A_j$ and its complement by $B \equiv A^c$. We call a subsystem B a *mesh* (Fig. 2).

Lemma 3. In a 2D local stabilizer code, all logical operators can be supported on a two-dimensional mesh $B = A^c$.

Proof. The lemma was essentially proven in Ref. [40], and we sketch the key arguments. First, each square is correctable, since the linear size is $O(d/w)$ (Corollary 1 in Ref. [40]). Second, since their mutual separation is larger than the size of stabilizers, the union $A = \cup_j A_j$ is also correctable (Lemma 2 in Ref. [40]). Therefore, due to the

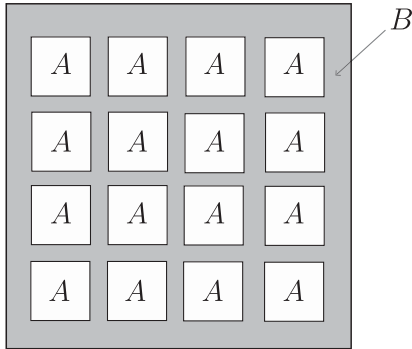


FIG. 2. Each square A_j is correctable and of linear size $O(d/w)$. Squares are separated by $O(w)$, so their union A is also correctable. Hence, all the logical operators can be found in the shaded region, a mesh $B = A^c$.

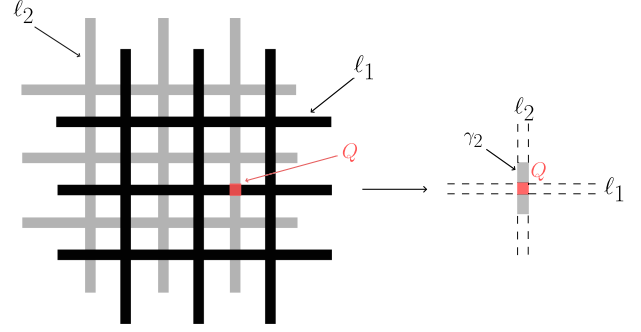


FIG. 3. Left: a pair of logical operators ℓ_1 and ℓ_2 which anticommute with each other at one square Q . Right: a segment of ℓ_2 around Q is called γ_2 .

cleaning lemma (Lemma 1 in Ref. [39]), for any logical operator ℓ , there exists a stabilizer S such that ℓS is supported on the mesh $B = A^c$. ■

We use Lemma 3 to construct a braiding process. Let ℓ_1, ℓ_2 be a pair of anticommuting logical operators. Picking a mesh B_1 as in Lemma 3, ℓ_1 can be supported on B_1 . We now shift B_1 by $O(w)$ in both horizontal and vertical directions and denote the new mesh by B_2 (Fig. 3). Then, ℓ_2 can be supported on B_2 . Observe that the intersection of ℓ_1 and ℓ_2 is a union of squares of linear size $O(w)$. Since ℓ_1 and ℓ_2 anticommute with each other, they must anticommute on at least one square. Denote such a square as Q .

B. Braiding process

The goal is to find a braiding process of the form $\gamma_2 \gamma_1 \gamma_2 |\Psi\rangle = -|\Psi\rangle$, where γ_1 is a looplike stabilizer and γ_2 is a deformable open string operator which anticommutes with γ_1 . To construct γ_1 , we repeat the above argument for a slightly deformed mesh B'_1 and construct ℓ'_1 , a logical Pauli operator that is equivalent to ℓ_1 and is supported on B'_1 . This is possible as long as the complement of B'_1 consists of squares whose perimeter remains $O(d/w)$. Then, $\ell_1 \ell'_1$ is a stabilizer operator supported on $B_1 \cup B'_1$ (Fig. 4).

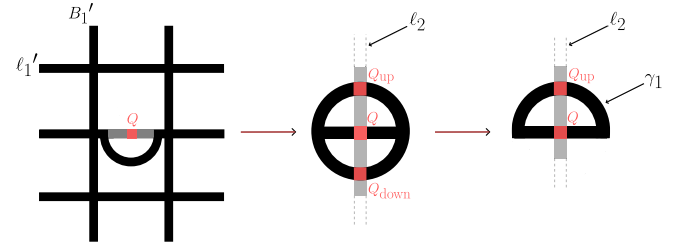


FIG. 4. Left: an equivalent logical operator ℓ'_1 supported on a deformed mesh B'_1 . Middle: starting from a stabilizer operator $S = \ell_1 \ell'_1$ supported on $B_1 \cup B'_1$, we truncate S to obtain a stabilizer operator γ_1 supported around Q . This stabilizer intersects with ℓ_2 at three squares. Right: we further truncate the stabilizer to obtain γ_1 which intersects with ℓ_2 at two squares.

We construct a looplike stabilizer γ_1 from $\ell_1 \ell'_1$ by “truncating” its support away from the intersection Q . For a given region R , let R^+ denote the set of qubits j such that $\text{disc}(j, R) \leq w$. The following lemma allows us to truncate a large stabilizer S into a smaller stabilizer S' supported on R^+ with possible changes only at the boundary while keeping the operator content on R .

Lemma 4. For any stabilizer operator S and any region R , there exists a stabilizer operator S' , such that (i) $\text{supp}(S') \subseteq R^+$ and (ii) $\text{supp}(SS') \cap R = \emptyset$ (equivalently, $S|_R = S'|_R$).

Proof. Since S is a stabilizer operator, it can be written as a product of local stabilizer generators $S = \prod_a S_a$. Specifically, we can decompose S as follows:

$$S = \prod_{\text{supp}(S_a) \cap R = \emptyset} S_a \prod_{\text{supp}(S_b) \cap R \neq \emptyset} S_b. \quad (35)$$

Define $S' = \prod_{\text{supp}(S_b) \cap R \neq \emptyset} S_b$. Since the diameter of each S_a is less than w , we know $\text{supp}(S') \subseteq R^+$. Moreover, $\text{supp}(SS') \cap R = \emptyset$ since $SS' = \prod_{\text{supp}(S_a) \cap R = \emptyset} S_a$. ■

The construction of γ_1 proceeds in two steps. First, let us apply the above lemma to truncate $\ell_1 \ell'_1$ into a circular region containing the square Q . The resulting stabilizer operator is supported on a region consisting of a circle and a horizontal line; see the middle in Fig. 4 for illustration. Second, we observe that this truncated stabilizer intersects with ℓ_2 at Q_{up} , Q , and Q_{down} . Since this operator anticommutes with ℓ_2 at Q , it must anticommute with ℓ_2 at either Q_{up} or Q_{down} . Without loss of generality, we assume that it is Q_{up} . We can then further shrink the stabilizer operator to a semicircular region, as shown on the right in Fig. 4, by removing the stabilizer generators below the horizontal line.

This is the desired looplike stabilizer which we shall use as γ_1 . Note that γ_1 intersects and anticommutes with ℓ_2 at both Q and Q_{up} .

The construction of γ_2 proceeds in an analogous manner. We start by finding ℓ'_2 in a deformed mesh B'_2 and then truncate $\ell_2 \ell'_2$ so that it will be supported only near Q_{up} .

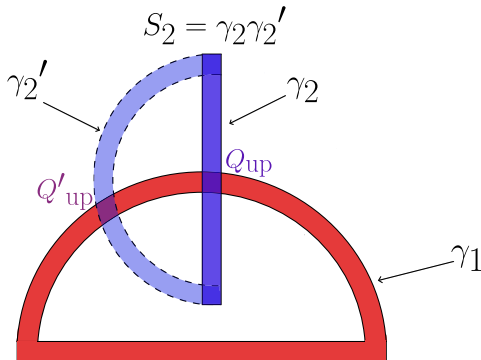


FIG. 5. Braiding process using truncated stabilizer operators.

We can then construct a looplike stabilizer operator S_2 which intersects and anticommutes with γ_1 at Q_{up} and Q'_{up} ; see Fig. 5 for an illustration. Finally, we decompose S_2 into two segments, $S_2 = \gamma_2 \gamma'_2$, such that γ_2 and γ'_2 intersect with γ_1 at Q_{up} and Q'_{up} , respectively. We then obtain the desired braiding processes

$$\gamma_2 \gamma_1 \gamma_2 |\Psi\rangle = -|\Psi\rangle, \quad \gamma'_2 \gamma_1 \gamma'_2 |\Psi\rangle = -|\Psi\rangle. \quad (36)$$

C. Overlap with short-range-entangled states

The above arguments assume $t = 0$. Applying a finite-depth circuit, the correctable squares now have width $\Theta(d/w) - \Theta(t) = \Theta(d/w)$, and the separation is now $\Theta(w + t)$. With the braiding processes, the same argument as in Lemma 1 implies $|\Psi\rangle \neq U_t |0\rangle^{\otimes n}$ if $(d/w) > \Theta(w + t)$. We further lower bound $E_t(\Psi)$ using a similar strategy. Starting from a mesh B of width $\Theta(w + t)$, consider a family of meshes, each obtained by shifting the previous meshes along the diagonal by $\Theta(w + t)$. For each pair of meshes, we can repeat the above argument and find an anticommuting intersection and a local upper bound as in Lemma 2. Moreover, each anticommuting intersection must be separated from each other by $\Theta(w + t)$ and is, hence, decoupled from each other. The number of such meshes is at least $\Theta\{d/[w(w + t)]\}$; hence, the number of anticommuting intersections is at least $\Theta\{d^2/[w^2(w + t)^2]\}$. This completes the proof of Theorem 2.

IV. HONEYCOMB MODEL

The original honeycomb model, introduced by Kitaev, is given by [25]

$$H_{\text{Kitaev}} = -J_X \sum_{X\text{-links}} X-X - J_Y \sum_{Y\text{-links}} Y-Y - J_Z \sum_{Z\text{-links}} Z-Z \quad (37)$$

where qubits reside on vertices of a honeycomb lattice. The model is exactly solvable via mapping to Majorana fermions. For $J_Z \gg J_X, J_Y > 0$, its low-energy physics can be approximated by the toric code at the leading order in perturbation theory. Hence, from our argument in Sec. II, we can deduce that the ground state of H_{Kitaev} in the “toric code phase” satisfies $E_t(\Psi) > O(n)$ for constant t .

Here, we focus on another honeycomb Hamiltonian, denoted as H_{fermion} , that reflects the symmetry properties of the original H_{Kitaev} . The starting point is an observation that the original model H_{Kitaev} possesses peculiar looplike symmetry operators

$$[H_{\text{Kitaev}}, S_{\square}] = 0, \quad S_{\square} = X \begin{array}{c} Y \quad Z \\ \diagdown \quad \diagup \\ X \end{array} \quad (38)$$

where $S_\square$ can be generated by multiplying all the two-body terms on each hexagon. The ground state $|\Psi_{\text{toric}}\rangle$ of H_{Kitaev} in the toric code phase ($J_Z \gg J_X, J_Y > 0$) satisfies $S_\square |\Psi_{\text{toric}}\rangle = |\Psi_{\text{toric}}\rangle$ (i.e., it is vortex-free [25]).

The honeycomb model with emergent fermions, which we address in this paper, can be formulated as a 2D stabilizer Hamiltonian

$$H_{\text{fermion}} = - \sum_{\square} S_{\square}, \quad S_{\square} = X \begin{array}{c} Y \quad Z \\ \diagdown \quad \diagup \\ \square \\ \diagup \quad \diagdown \\ Z \quad Y \end{array} X \quad (39)$$

where qubits reside on vertices of a hexagonal lattice and $\square$ are hexagonal plaquettes. A ground state of H_{fermion} satisfies $S_{\square} |\Psi\rangle = |\Psi\rangle$. We are interested in quantifying the entanglement of wave functions that live in the ground space of H_{fermion} .

What are the key differences between H_{Kitaev} and H_{fermion} ? First of all, note that the toric code ground state $|\Psi_{\text{toric}}\rangle$ of H_{Kitaev} is contained in the ground space of H_{fermion} . In fact, the ground space of H_{fermion} is huge. Viewed as a stabilizer code, there are $n/2$ stabilizer generators $S_{\square}$ with one redundancy relation $\prod_{\square} S_{\square} = I$. Hence, the number of logical qubits is $k = 1 + (n/2)$, which suggests an exponentially large ground state space. Second, as a quantum error-correcting code, H_{fermion} is not fault tolerant. Indeed, logical operators of the code, which must commute with all $S_{\square}$, are generated by two-body Pauli operators:

$$X \text{---} X \quad Y \text{---} Y \quad Z \text{---} Z \quad (40)$$

and, hence, the code distance is $d = 2$.

Because of an exponential ground state degeneracy and a small code distance, one might think that the honeycomb model has a ground state $|\Psi\rangle$ which would have a large overlap with a short-range-entangled state. Contrary to this naive expectation, we obtain a linear lower bound on $E_t(\Psi)$ for any state $|\Psi\rangle$ in the exponentially large ground space of the honeycomb model.

Theorem 3. There exists an absolute constant $\alpha > 0$ such that for any ground state $|\Psi\rangle$ of the honeycomb model H_{fermion} , if $t \leq O(\sqrt{n})$, then

$$E_t(\Psi) > \frac{\alpha n}{(t+1)^2}. \quad (41)$$

A. Fermion exchange statistics

A key property of the honeycomb model is that its ground space supports emergent fermions. Consider an open string operator M which is generated by multiplying two-body logical operators [Eq. (40)]. Let us pick an

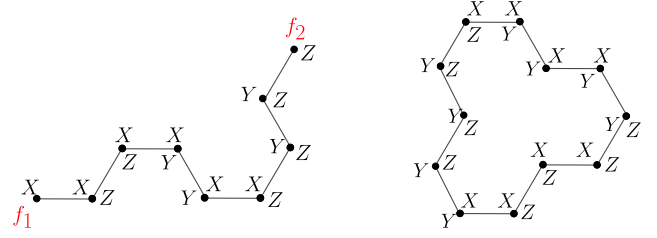


FIG. 6. Left: an open string operator. Right: a symmetry operator in the honeycomb model.

arbitrary ground state $|\Psi\rangle$ and regard it as a “vacuum” state with no fermions. Then, $M|\Psi\rangle$ can be interpreted as a state with two emergent fermions at end points of an open string M . A stabilizer generator $S_{\square}$ implements a process of creating and annihilating a pair of fermions, hence leaving the vacuum state unchanged $S_{\square} |\Psi\rangle = |\Psi\rangle$. These interpretations can be verified by a standard mapping of two-body generators to Majorana fermion operators as in Ref. [25].

It is worth emphasizing that a state $M|\Psi\rangle$ with a pair of fermions, as well as the vacuum state $|\Psi\rangle$, lives in the ground space of the honeycomb model. This is in contrast with the toric code, where anyons emerge as excitations that depart from the ground space of the Hamiltonian.

An immediate challenge is that fermions have trivial braiding statistics (i.e., they are not anyons), and, thus, the argument from Sec. II is not applicable. In the honeycomb model, this is reflected in that an open string operator M always commutes with a closed string operator S , since S is generated by stabilizer generators $S_{\square}$ (see Fig. 6).

Instead, we rely on the fermion exchange statistics to obtain a linear lower bound on $E_t(|\Psi\rangle)$. The topological spin of two fermions can be extracted by the hopping operator algebra due to Levin and Wen [28], which captures the effect of exchange via creation and annihilation processes:

$$\langle \Psi | M_3^\dagger M_2^\dagger M_1^\dagger M_3 M_2 M_1 | \Psi \rangle = e^{i\theta}, \quad (42)$$

where M_i are open string operators as in Fig. 7 and $e^{i\theta} = -1$ for fermions. For the honeycomb model,

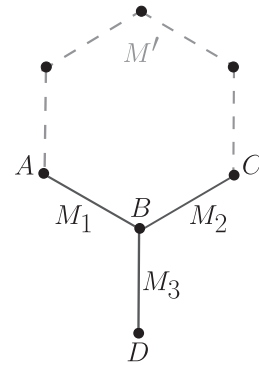


FIG. 7. Exchange statistics of two fermions.

Eq. (42) can be readily verified as three open string operators M_1 , M_2 , and M_3 with a common end point anticommute with each other. The exchange statistics is stable under deformation by a constant depth circuit U_t by considering dressed string operators and quasilocal fermions.

B. Long-range entanglement

We first prove that a ground state of the honeycomb model is long-range entangled.

Lemma 5. For any ground state $|\Psi\rangle$ of the honeycomb model, we have

$$U_t|\Psi\rangle \neq |0\rangle^{\otimes n} \quad (43)$$

for any depth- t unitary U_t with constant t .

Proof. Suppose that Eq. (43) is false, namely, $|\Psi_t\rangle \equiv U_t|\Psi\rangle = |0\rangle^{\otimes n}$. Let M_1 , M_2 , and M_3 be dressed open string operators satisfying Eq. (42). Using the deformability of string operators, one obtains

$$M_3(M_2M_1)|\Psi_t\rangle = |\psi_{BD}\rangle|\psi_{AC}\rangle|0\rangle^{\otimes(ABCD)^c}, \quad (44)$$

where $|\psi_{BD}\rangle$ and $|\psi_{AC}\rangle$ are some wave functions supported on BD and AC .

A crucial observation is that both $|\psi_{BD}\rangle$ and $|\psi_{AC}\rangle$ are factorized:

$$|\psi_{BD}\rangle|\psi_{AC}\rangle = |\psi_B\rangle|\psi_D\rangle|\psi_A\rangle|\psi_C\rangle. \quad (45)$$

To show this, let us construct a deformed string operator M' which connects A and C as in Fig. 7, such that the loop operator $M'M_2M_1$ is a product of some stabilizer generators $S_\square$ and, thus, commutes with all string operators. [If string operators can be implemented via short-depth circuits (as in the honeycomb model), this statement can be immediately proven. Namely, since the neighborhoods of regions B and D are initially unentangled, B and D cannot be entangled via a short-depth circuit that implements M_3 .] We then have

$$\begin{aligned} M_3(M_2M_1)|\Psi\rangle &= M'^{\dagger}M'M_3(M_2M_1)|\Psi\rangle = M'^{\dagger}M_3(M'M_2M_1)|\Psi\rangle \\ &= M'^{\dagger}(M'M_2M_1)M_3|\Psi\rangle = (M_2M_1)M_3|\Psi\rangle. \end{aligned} \quad (46)$$

We can, however, write the right-hand side as $M_2(M_1M_3)|\Psi_t\rangle$ and decompose it as $|\psi_{AD}\rangle|\psi_{BC}\rangle|0\rangle^{\otimes(ABCD)^c}$ similar to Eq. (44), implying

$$|\psi_{BD}\rangle|\psi_{AC}\rangle = |\psi_{AD}\rangle|\psi_{BC}\rangle. \quad (47)$$

The right-hand side implies B and D are unentangled, and so are A and C . This shows that $|\psi_{BD}\rangle$ and $|\psi_{AC}\rangle$ must be factorized as in Eq. (45).

By properly choosing the phases of $|\psi_B\rangle$, $|\psi_C\rangle$, and $|\psi_D\rangle$ in order, we can assume

$$\begin{aligned} M_1|\Psi_t\rangle &= |\psi_A\rangle|\psi_B\rangle|0\rangle^{\otimes(AB)^c}, \\ M_2M_1|\Psi_t\rangle &= |\psi_A\rangle|\psi_C\rangle|0\rangle^{\otimes(AC)^c}, \\ M_3|\Psi_t\rangle &= |\psi_B\rangle|\psi_D\rangle|0\rangle^{\otimes(BD)^c}. \end{aligned} \quad (48)$$

Using the first two equations, we obtain $M_2|\psi_B\rangle|0\rangle^{\otimes(AB)^c} = |\psi_C\rangle|0\rangle^{\otimes(AC)^c}$. Therefore,

$$\begin{aligned} M_1M_2M_3|\Psi_t\rangle &= M_1M_2|\psi_B\rangle|\psi_D\rangle|0\rangle^{\otimes(BD)^c} = M_1|\psi_C\rangle|\psi_D\rangle|0\rangle^{\otimes(CD)^c} \\ &= |\psi_A\rangle|\psi_B\rangle|\psi_C\rangle|\psi_D\rangle|0\rangle^{\otimes(ABCD)^c} \\ &= M_3(M_2M_1)|\Psi_t\rangle, \end{aligned} \quad (49)$$

which contradicts with Eq. (42) with $e^{i\theta} = -1$. ■

C. Overlap with short-range-entangled state

We can prove the following upper bound on the maximum overlap between ρ_R and $|0_R\rangle$. Similar to Lemma 2, it shows a “local GEM” that remains bounded away from zero by an absolute gap that is independent of the subregion size.

Lemma 6. Let R be any patch of linear size $\geq 8(t+1)$. Given any deformed honeycomb model ground state $|\Psi_t\rangle = U_t|\Psi\rangle$ with a depth- t circuit U_t , there exists an absolute constant $\epsilon > 0$ such that, in terms of the trace norm,

$$\|\rho_R - |0_R\rangle\langle 0_R|\|_1 > \epsilon, \quad (50)$$

where $\rho_R = \text{Tr}_{R^c}(|\Psi_t\rangle\langle\Psi_t|)$ and, accordingly,

$$\mathcal{F}(|0_R\rangle, \rho_R) < 1 - \frac{1}{4}\epsilon^2 = 1 - \epsilon' \quad (51)$$

for an absolute constant $\epsilon' > 0$.

The proof can be readily obtained by making the proof of Lemma 5 more quantitative with each equation replaced with $\approx$ in terms of the trace distance (see the Appendix for the full proof). We emphasize that, similar to Lemma 2, all we need to assume on ρ_R is the deformability of string operators (resulting from 1-form symmetry) and the fermionic exchange statistics (resulting from the anomalous nature of symmetry). In particular, the same result, with the same constant ϵ' , would apply to any state that is invariant under $S_\square$ in the region of interest.

Finally, we prove a linear lower bound on $E_t(\Psi)$ for a honeycomb model ground state by adding up the “local GEM.” Let us divide the whole system into multiple regions R_j with $R = \cup_{j=1}^m R_j$. Unlike the toric code, the honeycomb model does not satisfy the decoupling property as in Eq. (32), since the code distance is small. Instead,

we proceed by considering postprojection states sequentially and bound the total projection amplitude.

Let Π_j be an operator that projects $\mathcal{H}_{R_j}$ to $|0_{R_j}\rangle$. Recall that

$$\mathcal{F}(|\Psi_t\rangle, |0\rangle^{\otimes n}) \leq \text{Tr}(\Pi_R \rho_R). \quad (52)$$

We can express the right-hand side as (define $\Pi_0 = I$)

$$\text{Tr}(\Pi_R \rho_R) = \prod_{j=1}^m \mathcal{F}_j, \quad \text{where } \mathcal{F}_j \equiv \frac{\text{Tr}(\Pi_j \cdots \Pi_1 \rho_R)}{\text{Tr}(\Pi_{j-1} \cdots \Pi_1 \rho_R)}. \quad (53)$$

We claim that $\mathcal{F}_j < 1 - \epsilon'$ for some $\epsilon' > 0$ and $\forall 1 \leq j \leq m$. To prove this, we define a normalized postprojection state by

$$\rho^{(j)} \equiv \frac{\Pi_j \cdots \Pi_1 \rho_R \Pi_1 \cdots \Pi_j}{\text{Tr}(\Pi_j \cdots \Pi_1 \rho_R)} \quad (0 \leq j \leq m-1), \quad (54)$$

where $\mathcal{H}_{R_1}, \dots, \mathcal{H}_{R_j}$ have been projected to $|0\rangle$'s.

Crucially, we observe that, since $\cup_{i=1}^j R_j$ does not intersect R_{j+1} , the normalized postprojection state $\rho^{(j)}$ still satisfies the deformability of the string operators and the fermion exchange statistics inside the patch R_{j+1} , namely,

$$\text{Tr}(M_3^\dagger M_2^\dagger M_1^\dagger M_3 M_2 M_1 \rho^{(j)}) = -1 \quad (55)$$

whenever the M 's are supported in R_{j+1} . Therefore, we can apply Lemma 6 to $\rho^{(j)}$ and region R_{j+1} to obtain

$$\mathcal{F}_{j+1} = \mathcal{F}(\rho^{(j)}, \Pi_{j+1}) < 1 - \epsilon', \quad (56)$$

where ϵ' is j independent. Multiplying everything together, we arrive at

$$\mathcal{F}(|\Psi_t\rangle, |0\rangle^{\otimes n}) < (1 - \epsilon')^{cn/(t+1)^2}, \quad (57)$$

which proves Theorem 3.

D. Mixed state

Quantum circuit complexity of preparing a mixed state has been studied in quantum information theory in the context of topological quantum memory at finite temperature. Recently, there has been renewed interest in classifications of mixed-state phases in the condensed matter community. Here, we discuss the implication of our results from the perspective of mixed-state phases.

Following Hastings [34], we say that a mixed state ρ is short-range entangled if ρ can be expressed as a probabilistic ensemble of short-range-entangled states:

$$\rho = \sum_j p_j |\psi_j\rangle \langle \psi_j|, \quad |\psi_j\rangle = U_j |0\rangle^n. \quad (58)$$

For a technical reason, it is conventional to assume that the depth of U_j is at most $t \sim O(\text{polylog}(n))$ in defining short-range entanglement in ρ . Previous works showed that two-dimensional commuting projector Hamiltonians and three-dimensional fracton models have short-range-entangled Gibbs states according to this definition [34,41].

We are interested in the quantum circuit complexity of the maximally mixed state ρ that satisfies $\text{Tr}(\rho S_\square) = 1$. For any decomposition $\rho = \sum_j p_j |\psi_j\rangle \langle \psi_j|$, $\text{Tr}(\rho S_\square) = 1$ and $\|S_\square\| = 1$ would imply

$$S_\square |\psi_j\rangle = 1 \quad (59)$$

for all j . It follows from Theorem 3 that each $|\psi_j\rangle$ is long-range entangled. Hence, we arrive at the following result.

Corollary 1. Any mixed state ρ in the honeycomb model H_{fermion} ground state subspace is long-range entangled.

Finally, it will be useful to generalize the notion of depth- t GEM to mixed states. A natural generalization is to consider the maximum fidelity between the state of interest and an ensemble of short-range-entangled states

$$E_t(\rho) = - \max_{\sigma \in \text{SRE}(t)} \log_2 \mathcal{F}(\rho, \sigma). \quad (60)$$

Here, $\text{SRE}(t)$ denotes the set of mixed states that are representable as ensembles of depth- t short-range-entangled pure states.

Corollary 2. For a symmetric mixed state ρ in the honeycomb model, if $t \leq O(\sqrt{n})$, then

$$E_t(\rho) > \frac{\alpha n}{(t+1)^2}, \quad (61)$$

where $\alpha > 0$ is an absolute constant.

Proof. Given mixed state ρ , consider a quantum channel $\mathcal{N}$ that simultaneously measures stabilizer generators S and records the outcomes. Denote the outcome probability distributions as $P_\rho(s_1, \dots, s_{n/2})$, where $s_j = \pm 1$ are syndrome values of $S_\square$. Define P_σ similarly. Because of the monotonicity of the fidelity under quantum channels, we have

$$\mathcal{F}(\rho, \sigma) \leq \mathcal{F}[\mathcal{N}(\rho), \mathcal{N}(\sigma)] = \mathcal{F}(P_\rho, P_\sigma). \quad (62)$$

Assuming that ρ is a symmetric state: $\text{Tr}(S_\square \rho) = 1$, we have $P_\rho(+1, \dots, +1) = 1$ and $P_\rho(s_1, \dots, s_{n/2}) = 0$ for any other $(s_1, \dots, s_{n/2})$. Assuming $\sigma \in \text{SRE}(t)$, we can write it as $\sigma = \sum_j p_j |\psi_j\rangle \langle \psi_j|$, where $|\psi_j\rangle$ are short-range entangled. Then, we have

$$\begin{aligned} \mathcal{F}(P_\rho, P_\sigma) &= P_\sigma(+1, \dots, +1) = \text{Tr}(\Pi_S \sigma) \\ &= \sum_j p_j \text{Tr}(\Pi_S |\psi_j\rangle \langle \psi_j|), \end{aligned} \quad (63)$$

where Π_S is a projection to the symmetric subspace. (One can get the same inequality by considering the projection measurement $\{\Pi_S, 1 - \Pi_S\}$.)

Finally, we claim

$$\text{Tr}(\Pi_S |\psi_j\rangle\langle\psi_j|) < e^{-\Theta(n)}. \quad (64)$$

To prove this, consider a normalized symmetric pure state

$$|\Psi_j\rangle = \frac{\Pi_S |\psi_j\rangle}{\sqrt{\langle\psi_j|\Pi_S|\psi_j\rangle}}. \quad (65)$$

It follows from Theorem 3 that $|\langle\Psi_j|\psi_j\rangle|^2 < e^{-\Theta(n)}$, which leads to Eq. (64). The desired inequality

$$\mathcal{F}(\rho, \sigma) < e^{-\Theta(n)} \quad (66)$$

then follows from Eq. (63). ■

V. DISCUSSIONS

We comment on a few open problems.

- (i) A similar quantitative characterization of entanglement may be obtained for symmetry-protected topological order by considering an overlap with trivial wave functions that are prepared by short-depth symmetric circuits (i.e., symmetry-protected GEM).
- (ii) The GEM can quantitatively distinguish genuine long-range entanglement from “trivial” long-range entanglement. For instance, we have $E_t(\text{toric}) \sim \Theta(n)$ and $E_t(\text{GHZ}) \sim O(1)$. One potential manifestation of different GEM scaling is (in)stability of long-range entanglement under local decoherence. It will be interesting to make this intuition concrete by establishing a connection between fault tolerance and GEM.
- (iii) Studies of depth- t GEM for critical systems (those described by conformal field theories) may also be an interesting research avenue.
- (iv) Ground states of a topologically ordered Hamiltonian can be generically prepared by $O(L)$ -depth quantum circuits and, as such, have $E_t(\Psi) \approx 0$ for $t \sim \Omega(L)$. In a system of n qubits, however, there exist quantum states whose preparation requires $e^{O(n)}$ -depth quantum circuits. An example of a quantum state with exponential complexity is a Haar random state. It will be interesting to study the behavior of $E_t(\Psi)$ for states prepared from local random unitary circuits.
- (v) Recently, there has been significant progress in understanding noninvertible symmetries. It is interesting to study which classes of noninvertible symmetries require long-range entanglement. In fact, we expect that some aspects of noninvertible symmetries and anomalous symmetries are fundamentally akin

to each other and can be treated on an equal footing. For example, one can implement some anomalous symmetries locally by quantum channels, in a spirit similar to Ref. [42].

Note added. Recently, version 2 of Ref. [35] appeared on arXiv. This updated version establishes that a mixed state with anomalous 1-form symmetry exhibits long-range entanglement. Version 1 of Ref. [35] does not contain this result.

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APPENDIX: PROOF OF LEMMA 6

Here, we prove Lemma 6, copied below for convenience.

Lemma 6. Let R be any patch of linear size $\geq 8(t+1)$. Given any deformed honeycomb model ground state $|\Psi_t\rangle = U_t|\Psi\rangle$ with a depth- t circuit U_t , there exists an absolute constant $\epsilon > 0$ such that, in terms of the trace norm,

$$\|\rho_R - |0_R\rangle\langle 0_R|\|_1 > \epsilon, \quad (\text{A1})$$

where $\rho_R = \text{Tr}_{R^c}(|\Psi_t\rangle\langle\Psi_t|)$ and, accordingly,

$$\mathcal{F}(|0_R\rangle, \rho_R) < 1 - \frac{1}{4}\epsilon^2 = 1 - \epsilon' \quad (\text{A2})$$

for an absolute constant $\epsilon' > 0$.

Proof. Similar to the proof of Lemma 2, we use $\approx$ to denote closeness in the trace distance

$$\rho \approx \sigma \Leftrightarrow \|\rho - \sigma\|_1 < O(\epsilon). \quad (\text{A3})$$

For pure states, we use two different notions of closeness:

$$\begin{aligned} \psi_1 \approx \psi_2 &\Leftrightarrow \|\psi_1\rangle\langle\psi_1| - |\psi_2\rangle\langle\psi_2|\|_1 < O(\epsilon), \\ |\psi_1\rangle \sim |\psi_2\rangle &\Leftrightarrow \| |\psi_1\rangle - |\psi_2\rangle \| < O(\epsilon). \end{aligned} \quad (\text{A4})$$

Here, the norm in the second line is the vector norm in Hilbert space. While $\approx$ ignores the phase information,

the approximation $\sim$ cares about the phases. It is clear that $\psi_1 \approx \psi_2$ if and only if we can choose phases so that $\psi_1 \sim \psi_2$.

Now, assuming $\rho_R \approx |0_R\rangle\langle 0_R|$ in a region R , let us derive a contradiction. In the following, M_i are dressed operators, and we omit the subscript t .

First, the same as Eq. (27) and its proof, a string operator creates two excitations at end points which are approximately decoupled from the rest of the system:

$$M_3|0_R\rangle \approx |\psi\rangle_{BD}|0\rangle^{\otimes(BD)^c}. \quad (\text{A5})$$

With a similar equation for M_2M_1 , we get the approximate version of Eq. (44):

$$\begin{aligned} M_3(M_2M_1)|0_R\rangle &\approx M_3|\psi\rangle_{AC}|0\rangle^{\otimes(AC)^c} \\ &\approx |\psi\rangle_{AC}|\psi\rangle_{BD}|0\rangle^{\otimes(ABCD)^c}. \end{aligned} \quad (\text{A6})$$

Here, in the second approximation, we use $M_3(|0\rangle\langle 0|)^{\otimes(AC)^c} M_3^\dagger = \text{Tr}_{AC}(M_3|0_R\rangle\langle 0_R|M_3^\dagger)$ and Eq. (A5). Also, similarly, we have

$$M_2(M_1M_3)|0_R\rangle \approx |\psi\rangle_{BC}|\psi\rangle_{AD}|0\rangle^{\otimes(ABCD)^c}. \quad (\text{A7})$$

Next, we show that both $|\phi\rangle_{AC}$ and $|\phi\rangle_{BD}$ approximately factorize. From $\rho_R \approx |0_R\rangle\langle 0_R|$ and $M_3M_2M_1\rho_R = M_2M_1M_3\rho_R$, we get $M_3M_2M_1|0_R\rangle \approx M_2M_1M_3|0_R\rangle$. Now take Tr_{BC} on Eqs. (A6) and (A7) and use the monotonicity of the trace distance under partial trace, and we get

$$\phi_A \otimes \phi_D \approx |\psi\rangle\langle\psi|_{AD}, \quad (\text{A8})$$

where $\phi_A = \text{Tr}_C(|\psi\rangle\langle\psi|_{AC})$ and $\phi_D = \text{Tr}_B(|\psi\rangle\langle\psi|_{BD})$. It follows that ϕ_A and ϕ_D are approximately pure states as measured by the purity [for any two states σ_1 and σ_2 , we have $\text{Tr}(\sigma_1^2 - \sigma_2^2) \leq 2\|\sigma_1 - \sigma_2\|_\infty \leq 2\|\sigma_1 - \sigma_2\|_1$]:

$$\text{Tr}\phi_A^2 \cdot \text{Tr}\phi_D^2 = \text{Tr}((\phi_A \otimes \phi_D)^2) = 1 - O(\epsilon), \quad (\text{A9})$$

which further implies that they are close to pure states in the trace norm. [For any σ , denote the eigenvalues as λ_i , and then $\lambda_{\max} = \sum_i \lambda_{\max} \lambda_i \geq \sum_i \lambda_i^2 = \text{Tr}(\sigma^2)$. Hence, $\|\sigma - P_{\max}\|_1 = 2(1 - \lambda_{\max}) \leq 2[1 - \text{Tr}(\sigma^2)]$, where $P_{\max}$ is the projector to the eigenspace corresponding to the largest eigenvalue.] Therefore, we arrive at the approximate version of Eq. (45):

$$\begin{aligned} M_2M_1M_3|0_R\rangle &\approx M_3M_2M_1|0_R\rangle \\ &\approx |\psi\rangle_A|\psi\rangle_C|\psi\rangle_B|\psi\rangle_D|0\rangle^{\otimes(ABCD)^c}. \end{aligned} \quad (\text{A10})$$

Finally, we can derive a contradiction by considering the phase. The proof is the same as that of Lemma 5. We simply replace $=$ by $\sim$, which keeps the phase information. ■

- [1] J. Leinaas and J. Myrheim, *On the theory of identical particles*, *Il Nuovo Cimento* **37**, 132 (1977).
- [2] F. Wilczek, *Magnetic flux, angular momentum, and statistics*, *Phys. Rev. Lett.* **48**, 1144 (1982).
- [3] A. Y. Kitaev, *Fault-tolerant quantum computation by anyons*, *Ann. Phys. (Amsterdam)* **303**, 2 (2003).
- [4] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, *Topological quantum memory*, *J. Math. Phys. (N.Y.)* **43**, 4452 (2002).
- [5] R. B. Laughlin, *Anomalous quantum Hall effect: An incompressible quantum fluid with fractionally charged excitations*, *Phys. Rev. Lett.* **50**, 1395 (1983).
- [6] S. Bravyi, M. B. Hastings, and F. Verstraete, *Lieb-Robinson bounds and the generation of correlations and topological quantum order*, *Phys. Rev. Lett.* **97**, 050401 (2006).
- [7] A. Kitaev and J. Preskill, *Topological entanglement entropy*, *Phys. Rev. Lett.* **96**, 110404 (2006).
- [8] M. Levin and X.-G. Wen, *Detecting topological order in a ground state wave function*, *Phys. Rev. Lett.* **96**, 110405 (2006).
- [9] Y. Zhang, T. Grover, A. Turner, M. Oshikawa, and A. Vishwanath, *Quasiparticle statistics and braiding from ground-state entanglement*, *Phys. Rev. B* **85**, 235151 (2012).
- [10] I. H. Kim and B. J. Brown, *Ground-state entanglement constrains low-energy excitations*, *Phys. Rev. B* **92**, 115139 (2015).
- [11] B. Shi, *Verlinde formula from entanglement*, *Phys. Rev. Res.* **2**, 023132 (2020).
- [12] B. Shi, K. Kato, and I. H. Kim, *Fusion rules from entanglement*, *Ann. Phys. (Amsterdam)* **418**, 168164 (2020).
- [13] M. A. Levin and X.-G. Wen, *String-net condensation: A physical mechanism for topological phases*, *Phys. Rev. B* **71**, 045110 (2005).
- [14] J. Haah, *An invariant of topologically ordered states under local unitary transformations*, *Commun. Math. Phys.* **342**, 771 (2016).
- [15] D. Aharonov and Y. Touati, *Quantum circuit depth lower bounds for homological codes*, [arXiv:1810.03912](https://arxiv.org/abs/1810.03912).
- [16] T.-C. Wei and P. M. Goldbart, *Geometric measure of entanglement and applications to bipartite and multipartite quantum states*, *Phys. Rev. A* **68**, 042307 (2003).
- [17] A. Botero and B. Reznik, *Scaling and universality of multipartite entanglement at criticality*, [arXiv:0708.3391](https://arxiv.org/abs/0708.3391).
- [18] R. Orús, *Universal geometric entanglement close to quantum phase transitions*, *Phys. Rev. Lett.* **100**, 130502 (2008).
- [19] R. Orús, *Geometric entanglement in a one-dimensional valence-bond solid state*, *Phys. Rev. A* **78**, 062332 (2008).
- [20] T.-C. Wei, *Entanglement under the renormalization-group transformations on quantum states and in quantum phase transitions*, *Phys. Rev. A* **81**, 062313 (2010).
- [21] R. Orús, T.-C. Wei, O. Buerschaper, and M. Van den Nest, *Geometric entanglement in topologically ordered states*, *New J. Phys.* **16**, 013015 (2014).
- [22] S. Bravyi, D. Lee, Z. Li, and B. Yoshida, *How much entanglement is needed for quantum error correction?*, *Phys. Rev. Lett.* **134**, 210602 (2025).
- [23] D. E. Knuth, *Big omicron and big omega and big theta*, *ACM SIGACT News* **8**, 18 (1976).

- [24] S. Bravyi (private communication).
- [25] A. Kitaev, *Anyons in an exactly solved model and beyond*, *Ann. Phys. (Amsterdam)* **321**, 2 (2006).
- [26] M. Suchara, S. Bravyi, and B. Terhal, *Constructions and noise threshold of topological subsystem codes*, *J. Phys. A* **44**, 155301 (2011).
- [27] T. D. Ellison, Y.-A. Chen, A. Dua, W. Shirley, N. Tantivasadakarn, and D. J. Williamson, *Pauli topological subsystem codes from Abelian anyon theories*, *Quantum* **7**, 1137 (2023).
- [28] M. Levin and X.-G. Wen, *Fermions, strings, and gauge fields in lattice spin models*, *Phys. Rev. B* **67**, 245316 (2003).
- [29] X. Chen, Z.-X. Liu, and X.-G. Wen, *Two-dimensional symmetry-protected topological orders and their protected gapless edge excitations*, *Phys. Rev. B* **84**, 235141 (2011).
- [30] L. A. Lessa, M. Cheng, and C. Wang, *Mixed-state quantum anomaly and multipartite entanglement*, *Phys. Rev. X* **15**, 011069 (2025).
- [31] R. Verresen and A. Vishwanath, *Unifying Kitaev magnets, kagomé dimer models, and ruby Rydberg spin liquids*, *Phys. Rev. X* **12**, 041029 (2022).
- [32] K. Inamura and K. Ohmori, *Fusion surface models: 2+1d lattice models from fusion 2-categories*, *SciPost Phys.* **16**, 143 (2024).
- [33] R. Liu, H. T. Lam, H. Ma, and L. Zou, *Symmetries and anomalies of Kitaev spin-s models: Identifying symmetry-enforced exotic quantum matter*, *SciPost Phys.* **16**, 100 (2024).
- [34] M. B. Hastings, *Topological order at nonzero temperature*, *Phys. Rev. Lett.* **107**, 210501 (2011).
- [35] Z. Wang, Z. Wu, and Z. Wang, *Intrinsic mixed-state topological order*, *PRX Quantum* **6**, 010314 (2025).
- [36] T.-C. Lu, T. H. Hsieh, and T. Grover, *Detecting topological order at finite temperature using entanglement negativity*, *Phys. Rev. Lett.* **125**, 116801 (2020).
- [37] R. Sohal and A. Prem, *Noisy approach to intrinsically mixed-state topological order*, *PRX Quantum* **6**, 010313 (2025).
- [38] T. D. Ellison and M. Cheng, *Toward a classification of mixed-state topological orders in two dimensions*, *PRX Quantum* **6**, 010315 (2025).
- [39] S. Bravyi and B. Terhal, *A no-go theorem for a two-dimensional self-correcting quantum memory based on stabilizer codes*, *New J. Phys.* **11**, 043029 (2009).
- [40] S. Bravyi, D. Poulin, and B. Terhal, *Tradeoffs for reliable quantum information storage in 2d systems*, *Phys. Rev. Lett.* **104**, 050503 (2010).
- [41] K. Siva and B. Yoshida, *Topological order and memory time in marginally-self-correcting quantum memory*, *Phys. Rev. A* **95**, 032324 (2017).
- [42] M. Okada and Y. Tachikawa, *Noninvertible symmetries act locally by quantum operations*, *Phys. Rev. Lett.* **133**, 191602 (2024).