

Superconductivity via Paramagnon and Magnon Exchange in a 2D Near-Ferromagnetic Full Metal and Ferromagnetic Half-Metal

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We study superconductivity in paramagnetic and ferromagnetically ordered phases in a two-dimensional electron system with parabolic fermionic dispersion and short-range repulsive interaction. In the paramagnetic phase, we find that a weak momentum dependence of a paramagnon propagator parametrically reduces the onset temperature for the pairing compared to that in phenomenological theories, which assume a strong dispersion of a paramagnon, and also changes the topology of the gap function. In the ferromagnetic phase, we show that the order instantly polarizes low-energy fermionic excitations. We derive the fully renormalized pairing interaction between low-energy fermions, mediated by two transverse Goldstone modes, and show that it is attractive in a spatially odd channel. The pairing temperature in the ferromagnetic phase is found to be a fraction of the Fermi energy, significantly larger than in the paramagnetic phase near the transition. Our results are relevant for understanding superconductivity in proximity to itinerant ferromagnetism in multivalley graphene systems, particularly the ones with full valley and spin polarization.

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I. INTRODUCTION

Superconductivity mediated by ferromagnetic fluctuations has been discussed as far back as 1966 when Berk and Schrieffer [1] suggested that, while magnetic fluctuations were detrimental to *s*-wave superconductivity, they do support an unconventional triplet pairing. This idea was further explored in the context of ^3He by Layzer and Fay [2]. Subsequent studies focused on the interplay between superconductivity and non-Fermi-liquid behavior, also caused by ferromagnetic spin fluctuations [3,4], pair-breaking effects by thermal spin fluctuations [5–8], which for spin-triplet pairing act as magnetic impurities [9,10], and the nonanalytic terms in the expansion of the free energy over magnetization [11–13]. An outcome of these studies is that triplet superconductivity does develop and T_c generally increases as the system approaches an instability toward ferromagnetism, but at large magnetic correlation length the transition may become first order.

Less is known about superconductivity coexisting with a magnetic order, although this issue was also discussed by Fay and Appel back in 1980 [14]. It is tempting to assume

that the superconducting region forms a dome above the onset point of ferromagnetism at $T = 0$ [a ferromagnetic quantum-critical point (QCP)], i.e., that superconductivity decreases roughly symmetrically upon deviations from a QCP into the paramagnetic or magnetically ordered phases. A near-symmetric domelike shape of the superconducting region has been experimentally detected in itinerant antiferromagnets, like CePd_2Si_2 [15]. However, in three-dimensional (3D) itinerant ferromagnets, such as UGe_2 [16], URhGe [17], and UCoGe [18], a superconducting region is located largely within a magnetically ordered phase, where superconductivity coexists with ferromagnetism. A similar behavior has been detected in recent experimental studies of two-dimensional (2D) twisted bilayer graphene (see, e.g., Ref. [19] and references therein) and nontwisted graphene-based materials, like Bernal-stacked bilayer graphene (BBG), rhombohedral trilayer and pentagonal layer graphene (RTG and R5G, respectively) in the presence of a displacement field and, in some cases, Ising spin-orbit coupling, induced by placing WSe_2 near graphene sheets [20–30]. In BBG and RTG, superconductivity has been detected within a half-metal state [26,27,29], which was experimentally identified as a ferromagnetic state [29], and in R5G superconductivity has been detected [30] inside a quarter-metal state, which likely has both ferromagnetic and valley order.

On the theory side, several groups [2,6,7,31,32] computed T_c in the paramagnetic phase in 3D due to exchange

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of ferromagnetic spin fluctuations (paramagnons). Fay and Appel [14] computed T_c in 3D both in the paramagnetic phase and in the ferromagnetic phase, due to the exchange of spin-conserving longitudinal fluctuations, and found a near-symmetric superconducting dome around a ferromagnetic QCP. Kirkpatrick *et al.* [33] extended these calculations to include the coupling between longitudinal and transverse (Goldstone) modes and argued that this gives rise to the strong enhancement of T_c in the ferromagnetic phase, compared to the paramagnetic phase. In 2D, T_c in the paramagnetic phase has been computed in, e.g., Refs. [5,8]. Superconductivity in the ferromagnetically ordered state has been recently analyzed theoretically in magic-angle twisted bilayer graphene [19], BBG and RTG with Ising-like spin-orbit coupling [34], and R5G (Ref. [35]). Several groups also analyzed proximity-induced superconductivity in a metal placed next to a magnetically ordered insulator (see Ref. [36] and references therein).

In this communication, we present a comparable analysis of superconductivity on both sides of a ferromagnetic QCP in a 2D metal, using a specific microscopic model of fermions occupying a single valley, with a parabolic dispersion and Hubbard-like interaction at small momentum transfer. We derive the spin-mediated pairing interaction in both the normal and ferromagnetic state within the ladder approximation, by summing up infinite series of ladder diagrams containing particle-hole polarizations. Within this approximation, ferromagnetic order appears as a result of a Stoner instability at $U\nu = 1$, where $\nu = m/(2\pi)$ is the fermionic density of states per spin [37]. In 2D, this transition is strongly first order, going directly to a state with full spin polarization [38], yet the static susceptibility diverges as the transition is approached from the paramagnetic side. Our goal is to understand how this affects superconductivity near the Stoner transition.

Our key results can be briefly summarized as follows. In the paramagnetic phase, we obtain the commonly used Ornstein-Zernike + Landau-damping form of the dynamical spin susceptibility, however with a small prefactor for the static gradient q^2 term. We find that this smallness *reduces* the strength of the attractive pairing interaction in the spin-triplet channel, reduces the magnitude of T_c , and changes its dependence on $U\nu$. It also gives rise to a topologically nontrivial gap function with the sign change on the Matsubara axis. In the ferromagnetically ordered state, we find an attraction from the process involving two gapless magnons. We show that this process gives rise to larger T_c than in the paramagnetic phase even though the corresponding vertex function is reduced in agreement with the Adler principle for an interaction between fermions and a Goldstone boson. We argue that superconductivity around a ferromagnetic QCP in 2D is largely confined to the region where it coexists with full spin polarization.

As we are primarily interested in the comparison of pairing scales in the paramagnetic and ferromagnetic

phases, we restrict our discussion to zero temperature, where the Mermin-Wagner theorem does not hold and long-range order is possible. While finite-temperature effects are certain to be quantitatively important, they should not modify the qualitative comparison of the pairing scales.

The outline of the paper is as follows. In Sec. II, we introduce the model and review the nature of the two-dimensional Stoner transition. In Sec. III, we consider pairing mediated by paramagnons in the paramagnetic phase. In Sec. IV, we consider pairing mediated by Goldstone modes in the ferromagnetic phase. In Sec. V, we summarize our results and discuss their application to 2D materials, including R5G.

II. MODEL AND FERROMAGNETIC ORDER

We consider a 2D model of spinful fermions in two dimensions. Having in mind applications to BBG, RTG, and R5G, we assume that fermionic density is low and approximate fermionic dispersion by a parabola $k^2/(2m)$. We define the Fermi energy as $E_F = k_F v_F/2$ and assume that it is smaller than the bandwidth W . We assume a contact repulsive interaction U between electrons up to a certain momentum transfer $q_c \sim (2mW)^{1/2} > k_F$ and set $U = 0$ at larger momentum transfers. The model is described by the Matsubara action

$$S = \sum_k \bar{\psi}_{k\sigma} \left(-i\omega_n + \frac{k^2}{2m} - \mu \right) \psi_{k\sigma} + \frac{U}{2} \sum_{kk'q} \bar{\psi}_{k+q\sigma} \psi_{k\sigma} \bar{\psi}_{k'\sigma'} \psi_{k'+q\sigma'}, \quad (1)$$

where $k = (\mathbf{k}, i\omega_n)$ combines fermionic momentum and Matsubara frequency and σ denotes spin. We measure the effects of U in terms of the dimensionless coupling

$$c = U\nu, \quad (2)$$

where ν is the density of states [= $m/(2\pi)$ for a parabolic dispersion $\epsilon_k = k^2/(2m)$].

We analyze the model of Eq. (1) within the ladder approximation, i.e., obtain the propagator for ferromagnetic spin fluctuations and the effective four-fermion interaction, mediated by these fluctuations, by summing series of ladder and bubble diagrams. Within this approximation, whose accuracy we discuss below, the system at $T = 0$ exhibits a Stoner quantum phase transition between a paramagnetic metal (PM) and a ferromagnetic metal (FM) at $c = 1$. The self-consistent equation for the ferromagnetic order parameter Δ is

$$\Delta = \frac{U}{2} T \sum_k \text{tr}[\hat{G}_k \hat{\sigma}^z], \quad (3)$$

where $\hat{G}$ is a two-component diagonal matrix Green's function for spin-up and spin-down fermions. This equation must be supplemented by the condition on the total density:

$$n = 2\nu\mu_0 = T \sum_k \text{tr}[\hat{G}_k], \quad (4)$$

where n is the electronic density and $\mu_0 = E_F$ is the chemical potential in the PM state. For $c < 1$, the self-consistent equation has only the trivial solution $\Delta = 0$. For $c \geq 1$, there appears another solution [38]:

$$\Delta = c\mu_0, \quad \mu = (2 - c)\mu_0, \quad (5)$$

in which Δ has a finite value already at $c = 1 + 0^+$. The spin-up and spin-down dispersions become

$$\epsilon_k^\uparrow = \frac{k^2}{2m} - 2\mu_0, \quad \epsilon_k^\downarrow = \frac{k^2}{2m} + 2\mu_0(c - 1). \quad (6)$$

These expressions imply that, immediately upon crossing the transition, the Fermi level in one band jumps upward to accommodate all of the electrons, while the Fermi level of the other band drops to zero. In modern terminology, such a state is a half-metal. We see from Eq. (6) that, as the interaction strength is increased, the chemical potential of the filled band remains fixed, while the chemical potential of the other band sinks into negative values.

It turns out, however, that the first-order nature of the transition shows up only in the ordered state, while, as the transition is approached from the PM side, the static spin susceptibility diverges, as if the transition were continuous. This unconventional behavior can be understood as being due to the fact that, at zero temperature, the dimensionless Landau internal energy is *exactly* quadratic in the spin polarization with no higher-order terms [38]:

$$F(\zeta) = (1 - c)\zeta^2, \quad (7)$$

where $\zeta = (n_\uparrow - n_\downarrow)/n_0$ is the spin polarization, related to Δ and μ by

$$\begin{aligned} \zeta(\Delta, \mu) &= \frac{n_\uparrow(\Delta, \mu) - n_\downarrow(\Delta, \mu)}{n_0} \\ &= \frac{(\mu + \Delta) - (\mu - \Delta)\Theta(\mu - \Delta)}{(\mu + \Delta) + (\mu - \Delta)\Theta(\mu - \Delta)}. \end{aligned} \quad (8)$$

However, once c exceeds one, the slope changes sign, and the equilibrium value of ζ jumps to its largest possible value $\zeta = 1$.

A first-order ferromagnetic transition has been discussed before in connection with nonanalytical corrections to Fermi-liquid theory [11–13]. In 2D, such corrections give rise to a term $-a|\zeta|^3$ with a positive a . This prefactor is nonzero for a parabolic dispersion, though small

numerically. Combining this term with Eq. (7), we find that the transition remains first order into a half-metal with a maximal spin polarization; the only difference is that it occurs at a somewhat smaller $c = 1 - a$. Because a is numerically small, we just neglect the nonanalytic $-a|\zeta|^3$ term in our study.

In the next two sections, we consider an effective pairing interaction between low-energy fermions mediated by fluctuations of the order parameter around its equilibrium value. In the PM phase, we are interested in the p -wave pairing interaction mediated by overdamped paramagnons. We compute this interaction within our microscopic model and show that its p -wave component is weaker than previously thought based on semiphenomenological analysis. On the FM side, where spin-up fermions have a Fermi surface and spin-down fermions are gapped, we first derive the interaction between spin-up and spin-down fermions, mediated by a single Goldstone magnon (a spin fluctuation transverse to the long-range order in spin space) and a direct interaction between low-energy spin-up fermions, mediated by two Goldstone magnons. We then derive the full spatially odd pairing interaction between spin-up fermions by combining two single-magnon scatterings and direct two-magnon scattering. We show that this interaction is attractive and yields a higher pairing scale than in the PM phase.

Before we proceed with the analysis, we comment on the applicability of the ladder approximation that we are using. It has been known for quite some time [39] that this approximation overestimates the strength of fluctuations leading to a Stoner instability. Moreover, numerical studies indicate that, for a system with a quadratic dispersion and short-range U , the true ground state remains a paramagnet even when $c > 1$ (see, e.g., Ref. [40]). Our reasoning to stick with the ladder approximation is fourfold. First, there are clear experimental realizations of itinerant ferromagnetism in both 3D systems, like UGe_2 and ZrZn_2 , and 2D systems, like BBG, RTG, and R5G. Quantum oscillation measurements show [20–22, 25–27] that a ferromagnetic state in 2D examples is a half-metal, and the transition from a full to a half-metal is likely first order. There are no direct measurements of the magnetic susceptibility, but the electronic compressibility $dn/d\mu$ gets enhanced as the system approaches a half-metal state [22, 26, 29]. In the presence of spin-orbit coupling, this may serve as an indirect indication of enhanced spin fluctuations. A first-order transition to a FM together with strong magnetic fluctuations in a PM phase are the outcomes of the analysis of Eq. (1) in the ladder approximation. Second, to get ferromagnetism in BBG, RTG, and R5G, one needs to extend the model to two valleys. However, both analytical [38, 41] and variational Monte Carlo studies [42] show that, for a parabolic dispersion, a two-valley system exhibits a direct first-order transition from a full metal to a quarter-metal with full spin and valley polarizations. From our

perspective, this implies that the system does undergo a first-order transition into a state which can be viewed as a single-valley ferromagnet. Third, valley, and spin orders have also been detected experimentally in ultraclean quantum well AIs [43–45]. This system has two valleys with elliptical fermionic dispersion in each valley. Experiments on AIs revealed two first-order phase transitions upon decreasing fermionic density—the first one into a half-metal state with full valley polarization and the second onto a quarter-metal state with full spin and valley polarizations. Both transitions are accompanied by a rapid increase of the corresponding susceptibility. Near a FM transition, only one valley is relevant (excitations in the other valley are all gapped); hence, the system again can be viewed as a single-valley one undergoing a ferromagnetic transition of the same time as we find in the ladder approximation. Fourth, the analysis of the pairing in the PM state, which we present below, is quite general and is rather similar to that in semiphenomenological theories of pairing by low-energy collective modes [4,46,47]. The only distinction is in that in our microscopic theory the gradient q^2 term in the spin propagator appears with a small coefficient. In the FM phase, the ladder approximation yields the correct quadratic spin-wave spectrum of transverse spin waves, and also the full interaction mediated by spin waves obeys the Adler principle for the interaction between fermions and Goldstone bosons. We believe that in this respect the analysis within the ladder approximation captures the actual features of the pairing in both PM and FM phases.

We discuss some technical aspects of the ladder approximation in PM and FM phases in Appendix D.

III. PARAMAGNETIC PHASE: PAIRING VIA PARAMAGNONS

We begin by considering pairing in the paramagnetic phase, $c < 1$. The first task here is to obtain the form of the effective dynamical four-fermion pairing interaction mediated by a paramagnon. For this, we first recall that, by the Pauli principle, the pairing interaction (also known as pairing vertex) is the fully dressed, irreducible, *antisymmetrized* interaction between fermions $\Gamma(k, p)_{\alpha\beta;\gamma\delta}$ with incoming momenta and spin projections $(\mathbf{k}, \alpha)$ and $(-\mathbf{k}, \beta)$ and outgoing $(\mathbf{p}, \gamma)$ and $(-\mathbf{p}, \delta)$, where both $\mathbf{k}$ and $\mathbf{p}$ are set to be on the Fermi surface (see Fig. 1). To first order in U ,

$$\Gamma_{\alpha\beta;\gamma\delta}(k, p) = U(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}). \quad (9)$$

Using the Fierz identity, this can be reexpressed as

$$\Gamma_{\alpha\beta;\gamma\delta}(k, p) = \frac{U}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \boldsymbol{\sigma}_{\alpha\gamma} \cdot \boldsymbol{\sigma}_{\beta\delta}), \quad (10)$$

where $\sigma^{(i)}$ are Pauli matrices. Expressed this way, the antisymmetrized interaction contains charge and spin components, specified by $\delta_{\dots}$ and $\boldsymbol{\sigma}_{\dots}$ form factors.

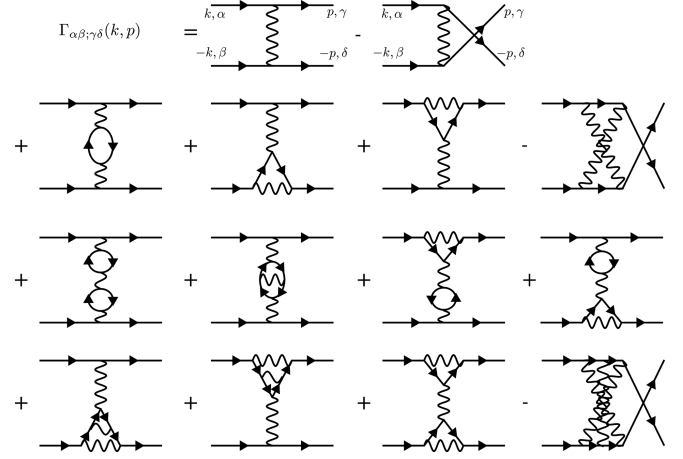


FIG. 1. Ladder and bubble diagrams contributing to the antisymmetrized dressed four-fermion interaction $\Gamma_{\alpha\beta;\gamma\delta}(k, p)$ up to third order in U . Only diagrams that contain $\Pi(k-p)$ are included.

The fully dressed irreducible $\Gamma_{\alpha\beta;\gamma\delta}(k, p)$ within the ladder approximation has been discussed in earlier works [48–50]. We discuss details in Appendix B and here cite the result: The dressed Γ can still be split into charge and spin components, but each component becomes dynamical; i.e., it depends on both the momentum transfer $\mathbf{q} = \mathbf{k} - \mathbf{p}$ and the frequency transfer $\Omega_m = \omega_{m,k} - \omega_{m,p}$. The dependence comes via the dynamical polarization $\Pi(q, \Omega_m)$. In explicit form,

$$\Gamma_{\alpha\beta;\gamma\delta}(q, \Omega_m) = \Gamma^{ch}(q, \Omega_m)\delta_{\alpha\gamma}\delta_{\beta\delta} + \Gamma^{sp}(q, \Omega_m)\boldsymbol{\sigma}_{\alpha\beta} \cdot \boldsymbol{\sigma}_{\gamma\delta}, \quad (11)$$

where

$$\begin{aligned} \Gamma^{ch}(q, \Omega_m) &= \frac{U}{2[1 + U\Pi(q, \Omega_m)]}, \\ \Gamma^{sp}(q, \Omega_m) &= -\frac{U}{2[1 - U\Pi(q, \Omega_m)]}, \end{aligned} \quad (12)$$

and $\Pi(q, \Omega_m)$, subject to $\Pi(0, 0) = \nu$, is the dynamical polarization bubble at momentum transfer q and frequency transfer Ω_m . Equations (11) and (12) show that near a Stoner instability at $U\nu = 1$ the dominant interaction is in the spin channel, mediated by the dynamical propagator of collective ferromagnetic fluctuations.

At $T = 0$, the polarization bubble can be evaluated exactly (see Appendix A):

$$\Pi(q, \Omega_m) = \nu \left(1 - \frac{2k_F}{q} \text{Re} \sqrt{\left(\frac{q}{2k_F} + \frac{i\Omega_m}{v_F q} \right)^2 - 1} \right). \quad (13)$$

For the pairing, we need q between the points at the Fermi surface, $q = |\mathbf{k} - \mathbf{p}| < 2k_F$. We assume and then verify

a posteriori that relevant q are comparable to k_F , while relevant Ω_m are of the order of T_c and are much smaller than $v_F q \sim \mu_0$. In this situation, one can expand Eq. (13) in Ω_m . Substituting the expansion into Eq. (12), we obtain a spin-mediated pairing interaction of the form

$$\Gamma^{sp}(q, \Omega_m) = -\frac{U}{2\left(1 - c + c \frac{|\Omega_m|}{v_F q \sqrt{1 - (q/2k_F)^2}}\right)}. \quad (14)$$

We see that the static interaction is independent of q , as long as $q < 2k_F$. This independence is a known artifact of treating Π as a 2D static polarization bubble of free fermions. The corrections to the bubble, which go beyond the ladder approximation, generate a bq^2 term in the denominator of Eq. (14) (Refs. [51,52]). Inserting this term into Eq. (14), we obtain after a simple rewriting

$$\Gamma^{sp}(q, \Omega_m) = -\frac{\bar{g}}{\xi^{-2} + q^2 + \gamma \frac{|\Omega_m|}{v_F q \sqrt{1 - (q/2k_F)^2}}}, \quad (15)$$

where $\bar{g} = U/(2b)$, $\gamma = \bar{g}k_F/(\pi v_F)$, and $\xi^{-2} = (1 - c)/b$. As b has dimensions of $1/k_F^2$, $\bar{g}$ has dimensions of energy. Comparing with $\Gamma^{sp}(q, \Omega_m)$ in semiphenomenological spin-fermion theories, where the static part of the spin-mediated interaction is assumed to have the Ornstein-Zernike form $1/(\xi^{-2} + q^2)$ and the dynamical part (the Landau-damping term for arbitrary q) is obtained from one-loop bosonic self-energies [4,12,14,33,46,47,53–57], there is one crucial difference. In spin-fermion theories, $\bar{g}$ is treated as a phenomenological spin-fermion vertex, which is assumed to be smaller than the Fermi energy or, at most, comparable to it. In our microscopic theory,

$$\bar{g} = 2\pi\mu_0 \frac{c}{k_F^2 b}. \quad (16)$$

For a $k^2/(2m)$ dispersion, the value of $b k_F^2$ is fully determined by inserting the corrections into the polarization bubble. Such corrections to the order of U^2 have been analyzed in Ref. [52]. The analysis requires care, as there are contributions to $b k_F^2$ from low-energy fermions and from high-energy fermions with momenta of the order of q_c , which, we remind, is the momentum cutoff for Hubbard-like interaction. The low-energy contribution is reduced by Fermi-liquid mass renormalization $\sqrt{1 - c}$ (Ref. [58]), leaving the high-energy contribution to $b k_F^2$ as the dominant one. This last contribution is of the order of one parameterwise but numerically comes out as quite small for realistic $q_c/k_F = O(1)$ [$b k_F^2 \sim 0.01(q_c/k_F)$ for $c \leq 1$; see Eq. (48) and Fig. 5 in Ref. [52]]. We take these results as an input and set $b k_F^2$ to be a small number. Then, the effective interaction $\bar{g} \gg 2\pi\mu_0$, despite that $U\nu \approx 1$; i.e., the actual interaction does not exceed μ_0 .

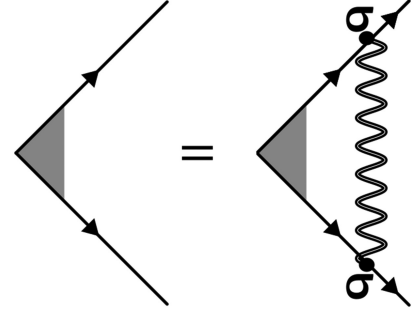


FIG. 2. Linearized equation for the pairing vertex for paramagnon-mediated pairing. Solid directed lines represent fermions, and the double wavy lines represent the paramagnon propagator.

At first glance, a larger effective spin-fermion coupling $\bar{g}$ should boost superconducting T_c . We show, however, that this is not the case, as in this situation the p -wave component of the interaction actually decreases. To see this, we now associate $\Gamma^{sp}(q, \Omega_m)$, given by Eq. (15), with the spin-mediated pairing interaction and analyze the linearized equation for the pairing vertex $\hat{\phi}_k$, where here we set $k = (\mathbf{k}, \omega_n)$. The temperature at which this equation has a nontrivial solution is a superconducting T_c . We assume that T_c is small and use the $T = 0$ form of $\Gamma^{sp}(q, \Omega_m)$. The equation for $\hat{\phi}_k$ is shown graphically in Fig. 2. In analytic form,

$$\hat{\phi}_k = -T \sum_{k' i} G_{k'} G_{-k'} \Gamma^{sp}(k - k') \hat{\sigma}^i \hat{\phi}_{k'} (\hat{\sigma}^i)^T, \quad (17)$$

where G_k is the Matsubara fermion Green's function in the normal state and $\Gamma^{sp}(k - k')$ is given by Eq. (15). Decomposing the pairing vertex into singlet and triplet channels,

$$\hat{\phi}_k = i\hat{\sigma}^2(\phi_k^0 + \mathbf{d}_k \cdot \boldsymbol{\sigma}) \quad (18)$$

and using

$$\sum_i \sigma^i \sigma^y (\sigma^i)^T = -3\sigma^y, \quad \sum_i \sigma^i i\sigma^j \sigma^y (\sigma^i)^T = i\sigma^j \sigma^y, \quad (19)$$

we find that the interaction mediated by ferromagnetic spin fluctuations is repulsive in the singlet channel and attractive in the spin-triplet channel. Without loss of generality, we then choose $\hat{\phi}_k = \phi_n f(\mathbf{k}) \hat{\sigma}^0$, where $f(\mathbf{k})$ is odd in momentum and $\phi_n = \phi_{-n}$. The linearized gap equation [Eq. (17)] then reduces to

$$\hat{\phi}_n f(\mathbf{k}) = -T \sum_{k'} G_{k'} G_{-k'} \Gamma^{sp}(k - k') \phi_{n'} f(\mathbf{k}'). \quad (20)$$

Substituting the forms of the free fermion propagators, keeping the $|\mathbf{k}'|$ dependence only in the fermionic

propagators and setting $|\mathbf{k}| = |\mathbf{k}'| = k_F$ elsewhere, as is usually done in the Eliashberg-like treatment, and integrating over the dispersion of an intermediate fermion, we reexpress Eq. (20) as an integral equation over the angle on the Fermi surface and Matsubara frequency:

$$\phi_n f(\theta) = -\nu \pi T \sum_{n'} \frac{\phi_{n'}}{|\omega_{n'}|} \oint \frac{d\theta'}{2\pi} f(\theta') \times \Gamma^{sp} \left(\omega_n - \omega_{n'}, 2k_F \sin \left| \frac{\theta - \theta'}{2} \right| \right), \quad (21)$$

where $\omega_n = \pi T(2n + 1)$, θ and θ' are the angles between the directions of $\mathbf{k}$ and $\mathbf{k}'$ on the Fermi surface, and $|\mathbf{k} - \mathbf{k}'| = 2k_F \sin(\theta - \theta')/2$. For the electron-phonon problem and many quantum-critical pairing problems, the factorization of the momentum integration can be justified by the condition that a pairing boson is a slow mode compared to an electron [59]. Here, the issue is less clear, but for an order of magnitude estimate of the pairing energy scale using the Eliashberg approximation should not be problematic (see Ref. [60]).

Going forward, we focus on the p -wave ($\ell = 1$) component of $f(\theta)$ [$f(\theta) = \cos \theta$] for which generically the instability is the strongest. Multiplying both sides of the gap equation by $\cos \theta$ and integrating over the Fermi surface, we obtain the effective gap equation as an integral equation in frequency only:

$$\phi_n = -\pi T \sum_{n'} \tilde{\Gamma}_{\ell=1}^{sp}(\omega_n - \omega_{n'}) \frac{\phi_{n'}}{|\omega_{n'}|}, \quad (22)$$

where

$$\tilde{\Gamma}_{\ell=1}^{sp}(\Omega_m) = \nu \int_0^{\pi/2} \frac{d\theta}{\pi} \cos \theta \left[\Gamma^{sp} \left(\Omega_m, 2k_F \sin \frac{\theta}{2} \right) - \Gamma^{sp} \left(\Omega_m, 2k_F \cos \frac{\theta}{2} \right) \right]. \quad (23)$$

Writing the interaction in this way emphasizes that the triplet channel sees the antisymmetrized interaction $\Gamma^{sp}(\mathbf{k} - \mathbf{p}) - \Gamma^{sp}(\mathbf{k} + \mathbf{p})$.

Up to this moment, the analysis was not specific to the value of $\bar{g}/\mu_0$, which, we recall, depends on the magnitude of $b k_F^2$ [Eq. (16)]. In previous studies of Eq. (22), with a semiphenomenological form of $\Gamma^{sp}[\Omega_m, k_F \sin(\theta/2)]$, it was assumed that $b \sim (1/mW)$ such that $b k_F^2 > 1$ and $\bar{g}/(2\pi)\mu_0$ is at most of the order of one. In this situation, relevant momentum transfers $q = 2k_F \sin \theta/2$ are small in $1/(b k_F^2)$, as we see immediately below, and out of the two terms in Eq. (22) only the first one is relevant. Keeping only this term in Eq. (22), approximating $\sin(\theta/2)$ by $\theta/2$ and $\sin \theta$ by θ , integrating in Eq. (22) over θ , and setting $c = 1$, we obtain Eq. (24) in the form

$$\phi_n = 0.039 \left(\frac{\bar{g}^2}{T\mu_0} \right)^{1/3} \sum_{n'} \frac{\phi_{n'}}{|2n' + 1||n - n'|^{1/3}}. \quad (24)$$

From a general point of view, this describes quantum-critical pairing with the exponent $\gamma = 1/3$ (Refs. [4,47,60]). The formally divergent $n = n'$ term in the right-hand side accounts for the pair-breaking effect from thermal fluctuations. As our goal is to determine the characteristic pairing scale at $T = 0$, where thermal fluctuations are not present, we drop this term and treat the temperature T^* , at which Eq. (24) has a solution, as a proxy to the actual superconducting T_c . Previous studies [7,8,32] have found that by keeping the $n = n'$ term one obtains a first-order superconducting transition at a temperature T smaller but comparable to T^* . We expect that the same happens in our case.

It is clear from Eq. (24) that $T^* \sim \mu_0[\bar{g}/(2\pi\mu_0)]^2$. Solving Eq. (24) numerically, we find

$$T^* = 0.065 \mu_0 \left(\frac{\bar{g}}{2\pi\mu_0} \right)^2. \quad (25)$$

A more accurate result for the numerical prefactor (0.022 instead of 0.065) is obtained by including fermionic self-energy $\Sigma(\omega_m)$ into the calculation of T^* (Refs. [4,46,47,60]). However, the functional form of T^* remains intact, because $\bar{g}^2/\mu_0$ is the energy scale at which $\Sigma(\omega_m) \sim \omega_m$. Returning to the integral over θ and using $2\pi T^*$ as a proxy for Ω_m , we estimate typical θ as $O[\bar{g}/(2\pi\mu_0)]$. We see that typical θ are small when $\bar{g}/(2\pi\mu_0)$ is small.

In our case, $b k_F^2$ is a small number, and $\bar{g}/(2\pi E_F)$ is a large number. In this situation, we cannot assume that θ is small and then have to keep both terms in $\tilde{\Gamma}^{sp}$ in Eq. (22). Substituting the full form of $\tilde{\Gamma}^{sp}$ into the gap equation, we obtain

$$\phi_n = \frac{1}{8} \frac{\bar{g}}{2\pi\mu_0} \pi T \sum_{n'} \Psi(|\omega_n - \omega_{n'}|) \frac{\phi_{n'}}{|\omega_{n'}|}, \quad (26)$$

where $\Psi(\Omega_m)$, proportional to $\tilde{\Gamma}^{sp}(\Omega_m)$, is

$$\Psi(\Omega_m) \equiv \int_0^{\pi/2} \frac{d\theta}{\pi} \frac{\sin^2 \theta \cos^2 \theta}{(\sin \theta \sin^2 \frac{\theta}{2} + u)(\sin \theta \cos^2 \frac{\theta}{2} + u)}, \quad (27)$$

where $u = (\pi \bar{g} |\Omega_m| / [4(2\pi\mu_0)^2])$. We assume and justify *post hoc* that, for large $\bar{g}/(2\pi\mu_0)$, T^* is such that, for $\Omega_m \sim T^*$, u is a large number. Then, $\Psi(\omega_m) = (2\pi\mu_0)^4 / (\pi^2 \bar{g}^2 \Omega_m^2)$. Substituting into Eq. (26), we reexpress the gap equation as

$$\phi_n = \frac{1}{32\pi^4} \frac{(2\pi\mu_0)^3}{\bar{g} T^2} \sum_{n'} \frac{\phi_{n'}}{|2n' + 1||n - n'|^2}. \quad (28)$$

From a general perspective, this corresponds to quantum-critical pairing with the exponent $\gamma = 2$ (Refs. [4,47]). From dimensional analysis, $T^* \sim (2\pi\mu_0)^{3/2}/\bar{g}^{1/2}$. Solving Eq. (28) numerically, again dropping the $n = n'$ term, we obtain

$$T^* = 0.13\mu_0 \left(\frac{2\pi\mu_0}{\bar{g}} \right)^{1/2}. \quad (29)$$

Substituting $2\pi T^*$ as a proxy for Ω_m into $u = \pi\bar{g}\Omega_m/[4(2\pi\mu_0)^2]$, we obtain $u = 0.1[\bar{g}/(2\pi\mu_0)]^{1/2}$. We see that u is indeed large when $\bar{g}/(2\pi\mu_0)$ is a sufficiently large number.

We see from Eq. (29) that, at large spin-fermion coupling $\bar{g}$, the characteristic pairing scale decreases as $1/\sqrt{\bar{g}}$, contrary to a naive expectation that a larger interaction should give rise to a larger pairing scale. Digging into the gap equation, we see that the reason is that, in a situation when the Landau-damping term in Γ^{sp} becomes the dominant one, the p -wave component of the pairing interaction drops as the Landau-damping term is invariant under $\theta \rightarrow \pi - \theta$, which changes the sign of the p -wave form factor.

The two expressions for T^* , Eqs. (25) and (29), can be combined to the scaling formula

$$T^* = 0.13\mu_0 (bk_F^2)^{1/2} \Phi(bk_F^2), \quad (30)$$

where $\Phi(0) = 1$ and $\Phi(x \gg 1) = 1/(2x^{5/2})$. We recall that Eq. (29) is the result of previous, semiphenomenological studies, which assumed that the bq^2 term in the static Γ^{sp} is large enough such that $bk_F^2 > 1$, while Eq. (30) is the result of our analysis of the microscopic model with a parabolic dispersion, for which bk_F^2 is small.

In Fig. 3, we present the results of the numerical solution of the gap equation for different bk_F^2 both at the FM QCP ($c = 1$) and away from it ($c < 1$). For numerical calculation, we defined $u_n = (\phi_n/\sqrt{|\omega_n|})$ in order to make the kernel of the gap equation [Eq. (22)] symmetric and truncated the sum over Matsubara frequencies at a maximum $n_{\max} = 50$:

$$u_n = - \sum_{n'=-n_{\max}}^{n_{\max}} \frac{\pi T \tilde{\Gamma}_{\ell=1}^{sp}(n-n')}{\sqrt{|\omega_n \omega_{n'}|}} u_{n'} \quad (n' \neq n). \quad (31)$$

We identify T^* with the temperature at which the largest eigenvalue of the $(2n_{\max} + 1) \times (2n_{\max} + 1)$ kernel matrix

$$K_{nn'} = \begin{cases} -\frac{\pi T \tilde{\Gamma}_{\ell=1}^{sp}(n-n')}{\sqrt{|\omega_n \omega_{n'}|}} & n' \neq n, \\ 0 & n' = n \end{cases} \quad (32)$$

is equal to 1.

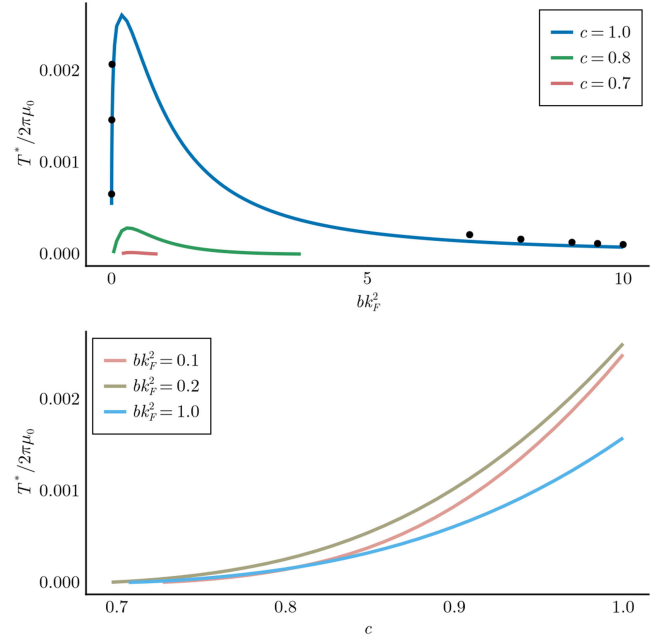


FIG. 3. The pairing scale T^* for paramagnon-mediated triplet pairing near a FM QCP, as a function of the prefactor b for the q^2 term in the bosonic propagator and deviations from a FM QCP, measured by $1 - c$. Dots on the upper plots represent the analytical results [Eqs. (29) and (25)] for the asymptotic behavior of T^* at small and large bk_F^2 , respectively.

In the top panel in Fig. 3, we plot $T^*/(2\pi\mu_0)$ as a function of $bk_F^2 = (2\pi\mu_0)/\bar{g}$ for different c . We clearly see that, at small bk_F^2 , T^* scales as $(bk_F^2)^{1/2}$. The square-root dependence Eq. (29) is overlaid on the numerical results in the top panel in Fig. 3 for small bk_F^2 and shows good qualitative agreement. At larger bk_F^2 , T^* passes through a maximum and then decreases as $1/(bk_F^2)^2$. We emphasize that this last behavior, which, we remind, was found in previous studies assuming that relevant momentum transfers are small, holds only for numerically large bk_F^2 . For realistic $bk_F^2 = O(1)$, the system is in the crossover region between the two regimes. In the lower panel, we plot $T^*/(2\pi\mu_0)$ as a function of c for different b . We clearly see that the magnitude of T^* drops from $c = 1$.

Finally, we note that there is a qualitative change between superconductivity at small and large bk_F^2 . We see from the bottom panel in Fig. 4 that, at small bk_F^2 , the gap function ϕ_n , changes sign twice on the Matsubara axis. On a more careful look, we found that this behavior originates from the fact that, at $b = 0$, the dynamical pairing interaction vanishes in the p -wave channel, if we use Eq. (14), but turns out to be repulsive if we do not expand in frequency (see Appendix C for details). This repulsion competes with the attraction induced by the bq^2 term in the spin propagator in Eq. (15) and for small bk_F^2 accounts for the sign changes of ϕ_n .

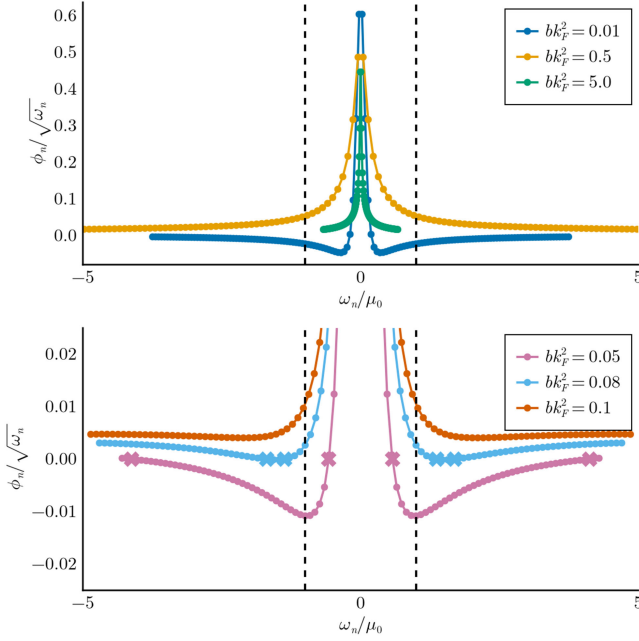


FIG. 4. Top: eigenfunction of the kernel matrix $K_{nn'}$ with maximal eigenvalue for different bk_F^2 and c . For small or large bk_F^2 , the eigenfunctions ϕ_n are mostly confined to low frequencies, justifying the asymptotic forms of the polarization bubble $\Pi(q, \Omega_m)$ in Eq. (22). For intermediate bk_F^2 , the eigenfunctions have significant contributions from high frequencies $\omega_n \sim \mu_0$, where the expansion in ω_n does not hold and the calculation of T^* has to be done numerically. Bottom: details of the frequency dependence of the eigenfunction for the maximal eigenvalue. At small bk_F^2 , the eigenfunction changes sign twice on the Matsubara axis. The first sign change is indicated by the cross marks. As bk_F^2 increases, the position of two sign changes move closer together until they collide and move away from the Matsubara axis (see Appendix C).

A zero of ϕ_n on the Matsubara axis is the origin of a dynamical vortex (the phase of the gap function changes by 2π under anticlockwise rotation around this point) [61,62]. A dynamical vortex in the upper half-plane of frequency affects the behavior of the complex gap function along the physical real frequency axis, $\phi(\omega) = |\phi(\omega)|e^{i\psi(\omega)}$ —it gives rise to a 2π phase slip between $\omega = -\infty$ and $+\infty$ (or, equivalently, by π between $\omega = 0$ and $\omega = \infty$). The two vortices in the upper half-plane give rise to 4π phase slip on the real axis. As bk_F^2 increases, these two vortices come closer to each other, merge, and then disperse in opposite directions away from the Matsubara axis, bending toward the real axis. They eventually cross the real axis at $b = b_c$ and continue dispersing into the lower half-plane (Ref. [62]). Once the vortices leave the upper half-plane, they no longer cause a 4π phase slip. From this perspective, b_c can be thought of as a point of a topological transition from a topologically nontrivial pairing state with dynamical vortices in the upper half-plane of frequency causing 4π phase slip on the real

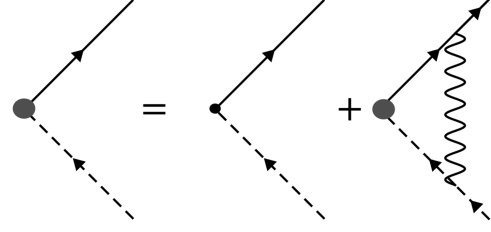


FIG. 5. Bethe-Salpeter equation for the renormalized transverse spin vertex $\hat{\sigma}_+$. Solid lines are for fermions in the occupied (spin-up) band, and dashed lines are for fermions in the unoccupied (spin-down) band. The wavy line is the bare Hubbard interaction U . The equation for $\hat{\sigma}_-$ is obtained by reversing the fermion lines.

frequency axis to a topologically trivial pairing state with no vortices.

IV. FERROMAGNETIC PHASE: PAIRING VIA GOLDSTONE MODES

We now derive the pairing interaction in the ferromagnetically ordered state. We begin by deriving the magnon propagator and the fermion-magnon vertex. For this, we note that, for order along the z direction, Goldstone magnons are the poles of transverse susceptibilities χ_{xx} and χ_{yy} . Both the $\hat{\sigma}_x$ and $\hat{\sigma}_y$ vertices contain combinations $\bar{\psi}_{k+q, \omega_n + \Omega_m}^\uparrow \psi_{k, \omega_n}^\downarrow$ and $\bar{\psi}_{k+q, \omega_n + \Omega_m}^\downarrow \psi_{k, \omega_n}^\uparrow$. To get magnons, we then introduce two trial vertices $\hat{\sigma}_x$ and $\hat{\sigma}_y$ (or, equivalently, $\hat{\sigma}_+$ and $\hat{\sigma}_-$), as shown in Fig. 5, and convert each vertex into a fully dressed one by including an infinite series of renormalizations set by U . To be consistent with the analysis in Sec. II, we keep only ladder renormalizations. The ladder summation is straightforward and yields the susceptibility $\chi_{\uparrow\downarrow}(q, \Omega_m)$ —the ratio of the fully dressed and the bare vertices—in the form

$$\chi_{\uparrow\downarrow}(q, \Omega_m) = \frac{1}{1 - U\Pi_{\uparrow\downarrow}(q, \Omega_m)}, \quad (33)$$

where

$$\begin{aligned} \Pi_{\uparrow\downarrow}(q, \Omega_m) &= - \int \frac{d^2k}{4\pi^2} \int \frac{d\omega_n}{2\pi} G^\uparrow(k+q, \omega_n + \Omega_m) G^\downarrow(k, \omega_n). \end{aligned} \quad (34)$$

The convention of notations is such that the outgoing fermion has spin up, and Ω_m and q are incoming frequency and momentum, respectively. Similarly,

$$\chi_{\downarrow\uparrow}(q, \Omega_m) = \frac{1}{1 - U\Pi_{\downarrow\uparrow}(q, \Omega_m)}, \quad (35)$$

where again Ω_m and q are incoming momenta. Evaluating Π and expanding in Ω_m and q , we obtain

$$\begin{aligned}\chi_{\uparrow\downarrow}(q, \Omega_m) &= \frac{2\mu_0 c}{i\Omega_m + \frac{q^2}{2m} \frac{c-1}{c}}, \\ \chi_{\downarrow\uparrow}(q, \Omega_m) &= \frac{2\mu_0 c}{-i\Omega_m + \frac{q^2}{2m} \frac{c-1}{c}}.\end{aligned}\quad (36)$$

We see that $\chi_{\downarrow\uparrow}(q, -\Omega_m) = \chi_{\uparrow\downarrow}(q, \Omega_m)$. This is relevant to our analysis below.

The physical χ_{xx} and χ_{yy} are linear combinations of $\chi_{\uparrow\downarrow}(q, \Omega_m)$ and $\chi_{\downarrow\uparrow}(q, \Omega_m)$. Each contains two poles corresponding to two magnon modes running in opposite directions. Both poles are located in the lower half-plane of complex frequency, infinitesimally close to the real axis.

Our next goal is to obtain an effective four-fermion interaction mediated by Goldstone bosons. For this, we notice that (i) a fermion-magnon vertex is between a spin-up and spin-down fermion (see Fig. 6) and (ii) a magnon-mediated pairing interaction must contain one incoming and one outgoing spin-up fermion and one incoming and one outgoing spin-down fermion (see Fig. 7). There is no magnon-mediated pairing interaction with two incoming spin-up fermions and two outgoing spin-down fermions and vice versa.

In Fig. 8, we show diagrammatically how to obtain a magnon-mediated four-fermion interaction starting from the Hubbard U between the densities of spin-up and spin-down fermions and collecting ladder series of diagrams with $\Pi_{\uparrow\downarrow}(q, \Omega_m)$ in each cross section (for technical details, see Appendix D). The ladder summation is straightforward and yields the magnon-mediated interaction in the form

$$\Gamma_1(q, \Omega_m) = U \chi_{\uparrow\downarrow}(q, \Omega_m) = U \frac{2\mu_0 c}{i\Omega_m + \frac{q^2}{2m} \frac{c-1}{c}}. \quad (37)$$

This specific form is for the fully spin-polarized state, where at the same momentum the energy of a spin-up fermion is $\epsilon_k^\uparrow = k^2/(2m) - 2\mu_0$, while for a spin-down

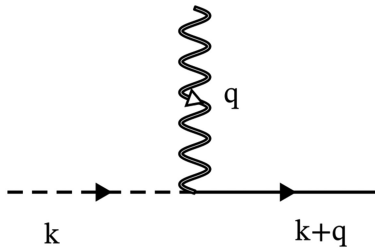


FIG. 6. Fermion-magnon vertex. The solid and dashed lines denote fermions from the occupied (spin-up) band and the unoccupied (spin-down) band, respectively, and the double wavy line is the magnon propagator. Note the vertex involves one spin-up and one spin-down fermion.

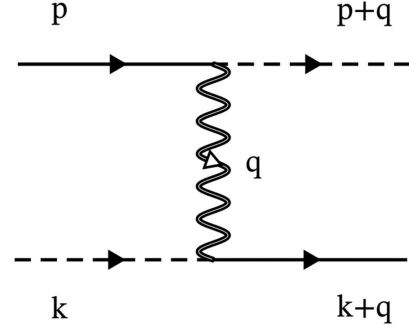


FIG. 7. Effective four-fermion interaction mediated by a single magnon. The solid fermion lines are in the occupied (spin-up) band, and the dashed ones are in the unoccupied (spin-down) band. The double wavy line is the magnon propagator. The effective interaction has one outgoing and one incoming spin-up fermion and one outgoing and one incoming spin-down fermion.

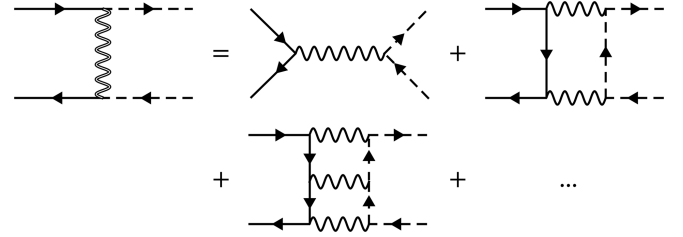


FIG. 8. Diagrammatic representation of the magnon-mediated four-fermion interaction in the ferromagnetic state. The solid fermion lines represent gapless (spin-up) fermions, and the dashed fermion lines represent the gapped (spin-down) fermions. The double wavy lines represent the magnon propagator, and the undirected wavy line is the Hubbard interaction U .

fermion it is $\epsilon_k^\downarrow = k^2/(2m) + 2\mu_0(c-1)$. Notice that the prefactor in Eq. (37) does not vanish at $q = \Omega_m = 0$, for the magnon pole structure is q independent. A similar result has been obtained recently in Ref. [34]. According to Ref. [34], the interaction in Fig. 8 should be viewed as a nondiagonal term in the 2×2 basis. We note in passing that this result is specific to an ordered ferromagnet, in which spin-up and spin-down excitations are split. For an ordered antiferromagnet, the situation is different, because excitations in the ordered state remain spin degenerate, and there is a direct magnon-mediated interaction between low-energy fermions. The side vertex for such interaction contains an overall factor of q , in agreement with the Adler principle for Goldstone bosons interacting with low-energy fermions [63–67].

To obtain the effective interaction with only spin-up fermions with momenta on the Fermi surface, we need to keep the magnon-mediated interaction to second order. The corresponding diagram is shown in Fig. 9. It involves the convolution of the two magnon propagators. Because we assumed that the magnon momentum q is small, fermionic

$\mathbf{k}_F$ and $\mathbf{p}_F$ have to be close; i.e., $\delta\mathbf{k} = \mathbf{p}_F - \mathbf{k}_F$ has to be small. We label this interaction as $\Gamma_{2a}(\mathbf{k}_F - \mathbf{p}_F) = \Gamma_{2a}(\delta\mathbf{k})$. In explicit form,

$$\Gamma_{2a}(\delta\mathbf{k}) = -U^2 \int \frac{d^2q}{4\pi^2} \frac{d\Omega_m}{2\pi} G^\downarrow(\mathbf{k}_F - \mathbf{q}, -\Omega_m) G^\downarrow(\mathbf{k}_F + \mathbf{q} + \delta\mathbf{k}, -\Omega_m) \chi_{\uparrow\downarrow}(q, \Omega_m) \chi_{\downarrow\uparrow}(\mathbf{q} + \delta\mathbf{k}, -\Omega_m) \quad (38)$$

$$= -U^2 \int \frac{d^2q}{4\pi^2} \frac{d\Omega_m}{2\pi} G^\downarrow(\mathbf{k}_F - \mathbf{q}, -\Omega_m) G^\downarrow(\mathbf{k}_F + \mathbf{q} + \delta\mathbf{k}, -\Omega_m) \chi_{\uparrow\downarrow}(q, \Omega_m) \chi_{\uparrow\downarrow}(\mathbf{q} + \delta\mathbf{k}, \Omega_m). \quad (39)$$

We emphasize that both propagators have the same sign of $i\Omega_m$ in their pole structure. Notice the overall minus sign in the right-hand side of Eq. (39).

Next, a simple experimentation shows that there also exists a direct coupling between two low-energy spin-up excitations and *two* magnons. We show the corresponding vertex $\gamma_{2\text{ mag}}(\mathbf{q}, \Omega, \delta\mathbf{k})$ in Fig. 10 (for technical details, see Appendix D). In explicit form,

$$\begin{aligned} \gamma_{2\text{ mag}}(\mathbf{q}, \Omega, \delta\mathbf{k}) &= -U \int \frac{d^2l}{4\pi^2} \frac{d\omega_n}{2\pi} G^\uparrow(\mathbf{l}, \omega_n) \\ &\quad \times G^\downarrow(\mathbf{l} - \mathbf{q}, \omega_n - \Omega_m) G^\uparrow(\mathbf{l} + \delta\mathbf{k}, \omega_n). \end{aligned} \quad (40)$$

Combining the two vertices $\gamma_{2\text{ mag}}$ and two magnon propagators, we obtain the effective interaction between low-energy spin-up fermions, shown in Fig. 11, which we label $\Gamma_{2b}(\delta\mathbf{k})$:

$$\Gamma_{2b}(\delta\mathbf{k}) = -U^2 \int \frac{d^2q}{4\pi^2} \frac{d\Omega_m}{2\pi} \gamma_{2\text{ mag}}^2(\mathbf{q}, \Omega, \delta\mathbf{k}) \quad (41)$$

$$\times \chi_{\uparrow\downarrow}(q, \Omega_m) \chi_{\uparrow\downarrow}(\mathbf{q} + \delta\mathbf{k}, \Omega_m). \quad (42)$$

The overall sign in the right-hand side of Eq. (42) is again minus. And, finally, there exist two cross terms involving one two-magnon vertex and two single-magnon vertices. The corresponding diagrams are shown in Fig. 12. We label this contribution $\Gamma_{2c}(\delta\mathbf{k})$:

$$\begin{aligned} \Gamma_{2c}(\delta\mathbf{k}) &= U^2 \int \frac{d^2q}{4\pi^2} \frac{d\Omega_m}{2\pi} \gamma_{2\text{ mag}}(\mathbf{q}, \Omega, \delta\mathbf{k}) \\ &\quad \times \left(G^\downarrow(\mathbf{k}_F - \mathbf{q}, -\Omega_m) + G^\downarrow(\mathbf{k}_F + \mathbf{q} + \delta\mathbf{k}, -\Omega_m) \right) \\ &\quad \times \chi_{\uparrow\downarrow}(q, \Omega_m) \chi_{\uparrow\downarrow}(\mathbf{q} + \delta\mathbf{k}, \Omega_m). \end{aligned} \quad (43)$$

Notice the overall plus sign in the right-hand side of Eq. (43). Combining all three contributions, we obtain for the full four-fermion interaction mediated by two magnon propagators

$$\begin{aligned} \Gamma_{2,\text{tot}}(\delta\mathbf{k}) &= -U^2 \int \frac{d^2q}{4\pi^2} \frac{d\Omega_m}{2\pi} S[\mathbf{q}, \Omega_m, \delta\mathbf{k}] \\ &\quad \times \chi_{\uparrow\downarrow}(q, \Omega_m) \chi_{\uparrow\downarrow}(\mathbf{q} + \delta\mathbf{k}, \Omega_m), \end{aligned} \quad (44)$$

where

$$\begin{aligned} S[\mathbf{q}, \Omega_m, \delta\mathbf{k}] &= \left(\gamma_{2\text{ mag}}(\mathbf{q}, \Omega_m, \delta\mathbf{k}) + \frac{1}{2} \left(G^\downarrow(\mathbf{k}_F - \mathbf{q}, -\Omega_m) + G^\downarrow(\mathbf{k}_F + \mathbf{q} + \delta\mathbf{k}, -\Omega_m) \right) \right)^2 \\ &\quad - \frac{1}{4} \left(G^\downarrow(\mathbf{k}_F - \mathbf{q}, -\Omega_m) - G^\downarrow(\mathbf{k}_F + \mathbf{q} + \delta\mathbf{k}, -\Omega_m) \right)^2 \end{aligned} \quad (45)$$

A comment is in order here. As written, all terms in Eq. (45), including $\gamma_{2\text{ mag}}(\mathbf{q}, \Omega_m, \delta\mathbf{k})$, have poles at negative $i\Omega_m$, i.e., in the same half-plane of Ω_m where $\chi_{\uparrow\downarrow}(q, \Omega_m)$ and $\chi_{\uparrow\downarrow}(\mathbf{q} + \delta\mathbf{k}, \Omega_m)$ have poles. As the consequence, the integral over Ω_m in $\Gamma_{2,\text{tot}}(\delta\mathbf{k})$ in Eq. (44) vanishes. Because magnon poles are at small $i\Omega_m = -q^2(c-1)/(2mc)$ and the poles in $G^\downarrow(\mathbf{k}_F - \mathbf{q}, -\Omega_m)$ are at much larger $i\Omega_m = -2\mu_0 c$, the vanishing of the frequency integral can be viewed as a cancellation between the contributions from low and high energies. Such a cancellation is, however, a peculiarity of the model with a single band at all energies. In a real

situation, there are other bands (or core levels), located at negative energies. These additional bands at finite negative energies do not affect the excitations near the Fermi level, which produce gapless magnons, but they induce an additional poles to the propagator of a spin-down fermion $G^\downarrow(\mathbf{k}_F - \mathbf{q}, -\Omega_m)$ at positive $i\Omega_m$ comparable to (finite and negative) energies of these additional bands. As a result, there is no cancellation between the contributions to $\Gamma_{2,\text{tot}}$ from high and low energies. Below, we consider only the low-energy contributions. For this, we set the upper cutoff in momentum integration at $q = q_0 < k_F$ and $|\Omega| = \Omega_0 < \mu_0$.

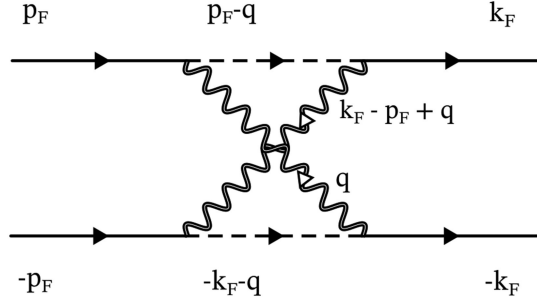


FIG. 9. Effective magnon-mediated interaction between two spin-up fermions. The solid lines are fermionic propagators in the occupied (spin-up) band, and the dashed lines are fermionic propagators in the unoccupied (spin-down) band. The double wavy line is the magnon propagator.

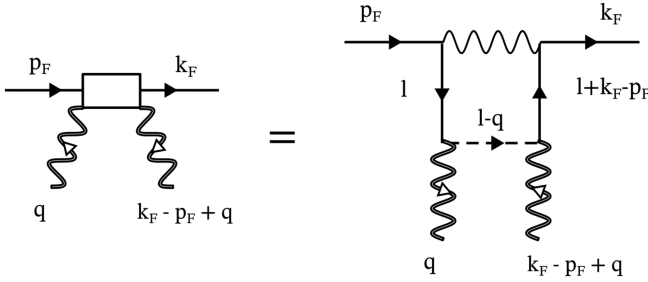


FIG. 10. Vertex for the interaction between two spin-up fermions and two magnons.

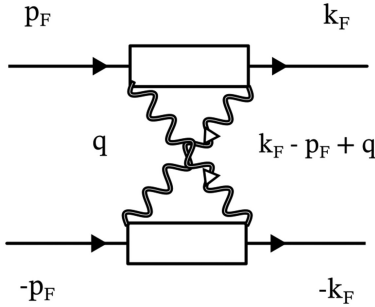


FIG. 11. Contribution $\Gamma_{2b}(\delta\mathbf{k})$ to the effective interaction between two spin-up fermions from two two-magnon vertices Γ_{lm} .

We discuss the form of $S[\mathbf{q}, \Omega_m, \delta\mathbf{k}]$ at low energies below, but first we analyze the structure of the equation for the pairing vertex $\phi(\mathbf{k}_F)$. The linearized equation (the equation for the pairing scale T^* defined as in the previous section) is

$$\phi(\mathbf{k}) = - \int \frac{d^2\mathbf{p}}{(2\pi)^2} \phi(\mathbf{p}) \Gamma_{2,\text{tot}}(\mathbf{k} - \mathbf{p}). \quad (46)$$

Restricting to momenta near k_F and performing a conventional frequency summation and integration over fermionic

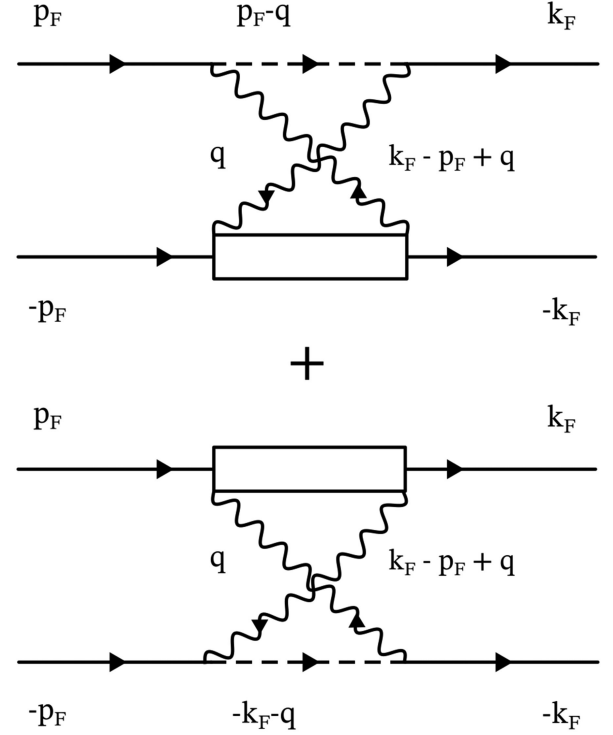


FIG. 12. Contribution $\Gamma_{2c}(\delta\mathbf{k})$ to the effective interaction between two spin-up fermions from the process involving one two-magnon vertex $\gamma_{2\text{ mag}}$ and two single-magnon vertices.

dispersion, we reexpress Eq. (46) as the integral over the Fermi surface:

$$\phi(\mathbf{k}_F) = -L\nu \int \frac{d\mathbf{p}_F}{2\pi k_F} \phi(\mathbf{p}_F) \Gamma_{2,\text{tot}}(\mathbf{k}_F - \mathbf{p}_F), \quad (47)$$

where the integration is along the Fermi surface and $L = \log(\mu_0/T^*)$. As usual, fermion statistics demand that the pairing vertex be antisymmetric under fermion exchange. In our case, the pairing involves only spin-up fermions; hence, the pairing vertex must be odd parity $\phi(\mathbf{k}_F) = -\phi(-\mathbf{k}_F)$. In this situation, the momentum-independent part of $\Gamma_{2,\text{tot}}$ cancels out in the gap equation. With this in mind, we subtract the constant piece from $\Gamma_{2,\text{tot}}$ and assume that the rest of $\Gamma_{2,\text{tot}}(\mathbf{k}_F - \mathbf{p}_F)$, which we label as $\bar{\Gamma}_{2,\text{tot}}(\mathbf{k}_F - \mathbf{p}_F)$, is sizable at momentum transfers smaller than k_F and is small at larger momentum transfers. Under this assumption, we can neglect the term $\bar{\Gamma}_{2,\text{tot}}(\mathbf{k}_F + \mathbf{p}_F)$ and simplify the gap equation to

$$\phi(\mathbf{k}_F) = -L\nu \int \frac{d\delta\mathbf{k}}{2\pi k_F} \phi(\mathbf{k}_F + \delta\mathbf{k}) \bar{\Gamma}_{2,\text{tot}}(\delta\mathbf{k}), \quad (48)$$

where $\delta\mathbf{k}$ is the variation of $\mathbf{p}_F - \mathbf{k}_F$ along the Fermi surface. For definiteness, we assume a Lorentzian form $\bar{\Gamma}_{2,\text{tot}}(\delta\mathbf{k}) = \bar{\Gamma}_{2,\text{tot}}(0)/[1 + (\delta\mathbf{k}/q_0)^2]$ [for simplicity, we use the same scale as the cutoff for the momentum

integration in Eq. (44)]. Substituting this form into Eq. (48) and approximating $\phi(\mathbf{k}_F + \delta\mathbf{k})$ by $\phi(\mathbf{k}_F)$, we reduce the equation for T^* to a conventional form:

$$1 = \lambda_{sc} L, \quad (49)$$

where, we remind, $L = \log \mu_0/T^*$ and

$$\lambda_{sc} = -\frac{1}{2} \nu \bar{\Gamma}_{2,\text{tot}}(0) \frac{q_0}{k_F}. \quad (50)$$

Superconductivity develops if $\bar{\Gamma}_{2,\text{tot}}(0) < 0$, i.e., if the magnon-mediated interaction with the constant part subtracted is negative. As is customary for an interaction peaked at small momentum transfer, the coupling λ_{sc} has almost the same value for all odd-parity channels (p -wave, f -wave, etc.) (see, e.g., Refs. [46,57]). To differentiate between channels, one has to analyze the full dynamical structure of $\bar{\Gamma}_{2,\text{tot}}(\delta\mathbf{k})$. This will most likely select p -wave as the leading instability.

We now compute $\bar{\Gamma}_{2,\text{tot}}(0)$. We have from Eq. (44)

$$\Gamma_{2,\text{tot}}(0) = -U^2 \int \frac{d^2 q}{4\pi^2} \frac{d\Omega_m}{2\pi} S[\mathbf{q}, \Omega_m, 0] \chi_{\uparrow\downarrow}^2(q, \Omega_m). \quad (51)$$

To get $\bar{\Gamma}_{2,\text{tot}}(0)$, we must remove the constant contribution from $\Gamma_{2,\text{tot}}(0)$. For this purpose, it is convenient to express S as the sum of two terms:

$$S[\mathbf{q}, \Omega_m, 0] = (S_a[\mathbf{q}, \Omega_m])^2 - (S_b[\mathbf{q}, \Omega_m])^2, \quad (52)$$

where [cf. Eq. (45)]

$$\begin{aligned} S_a[\mathbf{q}, \Omega_m] &= \gamma_{2\text{mag}}(\mathbf{q}, \Omega) \\ &+ \frac{1}{2} \left(G^\downarrow(\mathbf{k}_F - \mathbf{q}, -\Omega_m) + G^\downarrow(\mathbf{k}_F + \mathbf{q}, -\Omega_m) \right), \\ S_b[\mathbf{q}, \Omega_m] &= \frac{1}{2} \left(G^\downarrow(\mathbf{k}_F - \mathbf{q}, -\Omega_m) - G^\downarrow(\mathbf{k}_F + \mathbf{q}, -\Omega_m) \right). \end{aligned} \quad (53)$$

We note that both S_a and S_b vanish at $q = \Omega = 0$. Indeed, $S_b[0, 0]$ vanishes identically, while for $S_a[0, 0]$ the calculation of the convolution of two $G^\uparrow$ and one $G^\downarrow$ in Eq. (40) yields $\gamma_{2\text{mag}}(0, 0) = 1/(2\mu_0 c)$. Combining with $G^\downarrow(\mathbf{k}_F, 0) = -1/(2\mu_0 c)$, we find $S_a[0, 0] = \gamma_{2\text{mag}}(0, 0) + G^\downarrow(\mathbf{k}_F, 0) = 0$. This result is entirely expected [68], because the same two-magnon and two-fermion interaction can be used to analyze the renormalization of the magnon propagator $\chi_{\uparrow\downarrow}(\mathbf{q}, \Omega_m)$ by fermions. The vanishing of $S[0, 0, 0]$ then implies that the interaction with fermions vanishes in the long-wavelength limit and no mass is generated for the Goldstone boson. This is a ferromagnet-specific realization of the Adler principle (*Adler zero*) for a Goldstone boson [69,70].

For finite frequency and momentum, we expand S_a and S_b in Ω and q using our assumption that the low-energy contribution is confined to $q < k_F$ and $|\Omega| < \mu_0$, average over the angle between $\mathbf{k}_F$ and $\mathbf{q}$, and find

$$S_a^2[\mathbf{q}, \Omega_m] = \frac{\left(i\Omega_m + \frac{q^2}{2m} \frac{c-1}{c}\right)^2}{(2\mu_0 c)^4} = \frac{\chi_{\uparrow\downarrow}^{-2}(\mathbf{q}, \Omega_m)}{(2\mu_0 c)^2}, \quad (54)$$

$$S_b^2[\mathbf{q}, \Omega_m] = \frac{q^2}{2m} \frac{4\mu_0}{(2\mu_0 c)^4} \left(1 - \frac{2i\Omega_m}{\mu_0 c} + O(q^2)\right). \quad (55)$$

We see that the contribution from S_a^2 to $\Gamma_{2,\text{tot}}(0)$ in Eq. (51) reduces to a constant which has to be subtracted from $\bar{\Gamma}(0)$ in Eq. (50). The contribution to $\bar{\Gamma}_{2,\text{tot}}(0)$ then comes only from S_b^2 . The leading term in S_b^2 scales as q^2 but is purely static. A static contribution to $\Gamma_{2,\text{tot}}(0)$ in Eq. (51) is confined to $\Omega \sim \Omega_0$, i.e., comes from outside of the low-energy region [the frequency vanishes if we extend the integration over Ω_m in infinite limits because of the double pole in $\chi_{\uparrow\downarrow}^2(\mathbf{q}, \Omega_m)$]. A nonzero low-energy contribution from the frequency integration comes from the subleading term that contains $i\Omega_m$. Substituting this term into Eq. (51) and using

$$\int \frac{d\Omega_m}{2\pi} \frac{1}{i\Omega_m + \frac{q^2}{2m} \frac{c-1}{c}} = \frac{1}{2} \quad (56)$$

independent on q , we obtain an *attractive* pairing interaction for odd-parity superconductivity in the form

$$\bar{\Gamma}_{2,\text{tot}}(0) = -\left(\frac{U}{\mu_0 c^2}\right)^2 \int_0^{q_0} \frac{q dq}{2\pi} \frac{q^2}{2m}. \quad (57)$$

We set the upper limit at the same q_0 as in Eq. (50). Using $U\nu = c$ and plugging into Eq. (50), we obtain the dimensionless coupling

$$\lambda_{sc} = \frac{4}{\pi c} \left(\frac{q_0}{k_F}\right)^5 \quad (58)$$

and

$$T^* \sim \mu_0 e^{-1/\lambda_{sc}}. \quad (59)$$

For $q_0 = k_F$, $\lambda = 1/(4c)$. This coupling is smaller than one but is quite sizable, particularly near $c = 1$ [71]. This implies that the attraction is rather strong; i.e., the pairing scale T^* in the ferromagnetically ordered state is a sizable fraction of the Fermi energy μ_0 . As the system moves deeper in to the ferromagnetic state, c increases, and the pairing scale gets smaller (i.e., superconducting $T_c \sim T^*$ falls). We show the dependence of T^* on c in Fig. 13. We expect the actual superconducting T_c to follow the same trend, although the transition may become first order [8].

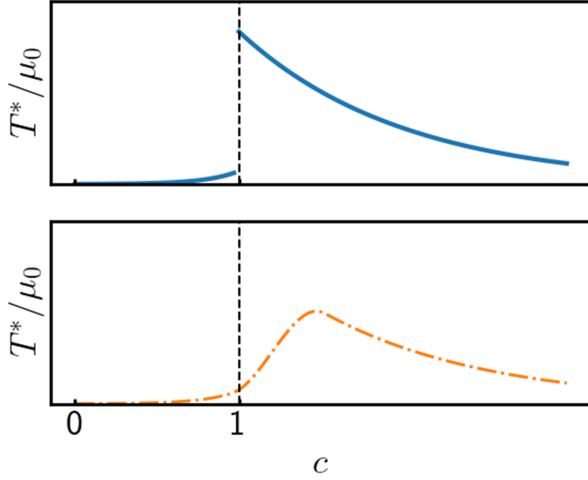


FIG. 13. Upper: schematic depiction of the pairing scale T^* (a proxy of superconducting T_c for spatially odd, spin-triplet pairing) as a function of $c = U\nu$ in both the paramagnetic ($c < 1$) and ferromagnetic ($c > 1$) phases for a two-dimensional system with a parabolic dispersion. T^* is small in a paramagnetic phase, because a static paramagnon propagator is almost flat for momentum transfers on the Fermi surface. At $c \leq 1$, the pairing state is topologically nontrivial: the pairing gap changes sign on the Matsubara frequency which leads to 4π phase slip on the real axis. T^* jumps to a larger value once the system enters into a ferromagnetic state and decreases as $1/c$ at larger c . Lower: heuristic depiction of the pairing scale T^* as a function of c in a three-dimensional system, assuming that the magnetization is continuous but quickly saturates inside the ferromagnetic state. The dashed vertical line shows the location of the Stoner transition $c = 1$.

The discontinuity in T^* at $c = 1$ is a consequence of the first-order nature of the Stoner transition in two dimensions. For a three-dimensional system, the Stoner transition is second order, and we expect the pairing scale T^* to be continuous at $c = 1$. Nonetheless, if the system rapidly saturates, there should still be a pronounced peak in T^* near $c = 1$ as shown in the lower panel in Fig. 13. We expect the same behavior in 2D, if the Stoner transition is second order with rapid increase of magnetization, which is the case for dispersions $k^{2\alpha}$ with $\alpha > 2$ (Ref. [38]) or when there is a first-order ferromagnetic transition before the susceptibility diverges [72]. The argument in both cases is that magnon-mediated superconductivity comes from magnons with energies comparable to the Fermi energy and, hence, is insensitive to changes of the system behavior at small energies. Replacing a Hubbard interaction by a more realistic screened Coulomb interaction also should not modify the results beyond numerical adjustments, because the structure of the magnon dispersion is set by the SU(2) spin symmetry.

V. CONCLUSION

In this work, we have considered magnon-mediated superconductivity in a two-dimensional itinerant electron

system near a ferromagnetic transition. We have shown that the nature of the pairing interaction is qualitatively different in the ferromagnetic and paramagnetic states, leading to a large discrepancy in pairing scales between the two. In the paramagnetic state, for purely parabolic electronic dispersion, T_c for pairing via paramagnon exchange is suppressed parametrically by a factor bk_F^2 , related to the weak dispersion of a paramagnon. A weak paramagnon dispersion also gives rise to a topologically nontrivial gap function with the sign change on the Matsubara axis.

The situation is qualitatively different in the ferromagnetic phase. The fully spin-polarized nature of the ground state, which emerges discontinuously at the Stoner transition, means that the pairing occurs between fermions with the same spin projection. The magnon-mediated pairing interaction involves the exchange of two transverse Goldstone magnons. Despite the fact that the vertex for fermion-magnon scattering vanishes in the long wavelength (as expected from the Adler principle), the pairing interaction remains attractive and sizable in the odd-parity channels, with the p -wave channel likely being the strongest. We estimated the corresponding dimensionless coupling [Eq. (58)] and found T_c as a fraction of E_F , much larger than in the paramagnetic phase. We, thus, expect a rapid increase in the pairing temperature as the Stoner transition is crossed into a ferromagnetic state (see Fig. 13) and a maximum of T_c near the transition.

This last result is our key testable experimental prediction. Other testable experimental prediction is insensitivity of T_c to changes in the excitation spectrum at low energies induced by, e.g., a spin-orbit coupling or a magnetic field. The reason is that superconductivity comes from fermions with energies comparable to Fermi energy. We also predict that superconducting order parameter is spatially odd (p -wave). While this by itself is not a “unique” prediction (several works [35] explored a Kohn-Luttinger-type scenario of p -wave pairing in multilayer graphene systems due to momentum dependence of screened Coulomb interaction), an experimental verification of a p -wave order parameter symmetry, combined with T_c peaked near the onset of a ferromagnetic order, would be a strong argument for magnon-mediated pairing.

The results of this study are also applicable to multilayer graphene structures BBG, RTG, and R5G, in which at least one superconducting region is inside a ferromagnetically ordered state, near its onset (see, e.g., Ref. [26]). In these two-valley systems, small Fermi pockets are located near K and K' points in the Brillouin zone. The pairing we consider here emerges due to intravalley Hubbard-like interaction, and the paired fermions are located on the opposite sides of the same Fermi surface near either K or K' . From a general point of view, this pairing is a pair-density-wave (PDW) phenomenon. A conventional superconductivity with zero total momentum of a pair is

a pairing between one fermion near K and one near K' . In BBG and RTG, superconductivity has been detected in a ferromagnetic half-metal state, where there exist Fermi surfaces for spin-up fermions in both valleys. In this situation, both a conventional superconductivity and a PDW order are possible. Our scenario, applied to these systems, describes a PDW order and is a competing scenario to a superconductivity with zero total momentum, made out of fermions from different valleys [34,73]. In R5G, superconductivity has been detected in a quarter-metal state which is believed to have both a FM order and valley polarization. In this situation, only spin-up fermions from one valley have a Fermi surface. Pairing of these fermions necessarily leads to a PDW order. The results of our analysis are applicable to this case in the sense that magnon exchange provides an attraction for superconductivity, but we caution that the coherence factors, associated with the Berry curvature, time-reversal symmetry breaking by valley polarization, and a steeper dispersion, are the factors that one should take into consideration in a more detailed analysis (see Ref. [74] for a recent study of the effects of coherence factors). Trigonal warping is another factor chiefly, because in its presence a ferromagnetic state may be a partially polarized one with both minority and majority carriers present [35,74]. This leads to two opposite trends: (i) a stronger magnon-mediated attraction, because already a single magnon exchange leads to pairing between fermions on the Fermi surface [34], and (ii) a cutoff of a Cooper logarithm, because the dispersion in the presence of a trigonal warping does not preserve the symmetry between k and $-k$. Which effect is stronger has to be determined in microscopic calculations.

Islands of superconductivity have been also detected in non-spin- and valley-ordered phases of two-valley graphene systems [21,26,29,75,76]. A possible explanation here is that the pairing glue in a paramagnetic phase is provided by strong valley fluctuations rather than spin fluctuations [50].

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: PARAMAGNON PROPAGATOR

In the paramagnetic state, the spin polarization bubble is given by

$$\begin{aligned}\Pi(q) &= -T \sum_k \frac{1}{i\omega_n + i\Omega_m - \xi_{\mathbf{k}+\mathbf{q}}} \frac{1}{i\omega_n - \xi_{\mathbf{k}}} \\ &= \sum_{\mathbf{k}} \frac{n_{\mathbf{k}+\mathbf{q}} - n_{\mathbf{k}}}{i\Omega_m - \frac{q^2}{2m} - \frac{\mathbf{k}\cdot\mathbf{q}}{m}},\end{aligned}\quad (\text{A1})$$

the two-dimensional Lindhard function on the Matsubara axis [77–79].

In the zero-temperature limit, shifting $\mathbf{k} \rightarrow \mathbf{k} - \mathbf{q}$ in the first term yields

$$\begin{aligned}\Pi(q) &= \int_0^{k_F} \frac{k dk}{2\pi} \oint \frac{d\theta}{2\pi} \\ &\times \left(\frac{1}{i\Omega_m + \frac{q^2}{2m} - \frac{\mathbf{k}\cdot\mathbf{q}}{m}} - \frac{1}{i\Omega_m - \frac{q^2}{2m} - \frac{\mathbf{k}\cdot\mathbf{q}}{m}} \right) \\ &= \int_0^{k_F} \frac{k dk}{2\pi} \oint \frac{d\theta}{2\pi} \left(\frac{1}{i\Omega + \frac{q^2}{2m} - \frac{kq}{m} \cos \theta} \right. \\ &\quad \left. - \frac{1}{i\Omega - \frac{q^2}{2m} - \frac{kq}{m} \cos \theta} \right).\end{aligned}\quad (\text{A2})$$

Factoring out (kq/m) from the denominators,

$$\begin{aligned}\Pi(q) &= \frac{\nu}{q} \int_0^{k_F} dk \oint \frac{d\theta}{2\pi} \\ &\times \left(\frac{1}{i \frac{m\Omega}{kq} + \frac{q}{2k} - \cos \theta} - \frac{1}{i \frac{m\Omega}{kq} - \frac{q}{2k} - \cos \theta} \right) \\ &= \frac{\nu}{q} \int_0^{k_F} dk \oint \frac{d\theta}{2\pi} \\ &\times \left(\frac{1}{i \frac{m\Omega}{kq} + \frac{q}{2k} - \cos \theta} + \frac{1}{-i \frac{m\Omega}{kq} + \frac{q}{2k} - \cos \theta} \right),\end{aligned}\quad (\text{A3})$$

where we have used $\theta \rightarrow \theta + \pi$ in the second term. Defining

$$a(\Omega) = \frac{q}{2} + i \frac{m\Omega}{q}, \quad (\text{A4})$$

we can perform the angular integration using Eq. (E1) to get

$$\Pi(q) = \frac{\nu}{q} \int_0^{k_F} dk \sum_{\pm} \frac{1}{\sqrt{\frac{q}{2k} \pm i \frac{m\Omega}{qk}} - 1} \sqrt{\frac{q}{2k} \pm i \frac{m\Omega}{qk}} + 1}. \quad (\text{A5})$$

Performing the momentum integration using Eq. (E2), we get

$$\begin{aligned}\Pi(q) &= -\frac{\nu}{q} \sum_{\pm} \sqrt{\frac{q}{2} \pm i \frac{m\Omega}{q}} - k \sqrt{\frac{q}{2} \pm i \frac{m\Omega}{q}} + k \Big|_0^{k_F} \\ &= -\frac{\nu}{q} \sum_{\pm} \left(k_F \sqrt{\frac{q}{2k_F} \pm \frac{i\Omega}{v_F q}} - 1 \sqrt{\frac{q}{2k_F} \pm \frac{i\Omega}{v_F q}} + 1 \right. \\ &\quad \left. - \frac{q}{2} \mp \frac{im\Omega}{q} \right) \\ &= \nu - \frac{2k_F \nu}{q} \text{Re} \left(\sqrt{\frac{q}{2k_F} + \frac{i\Omega}{v_F q}} - 1 \sqrt{\frac{q}{2k_F} + \frac{i\Omega}{v_F q}} + 1 \right).\end{aligned}\quad (\text{A6})$$

We can separate this into a static piece and a dynamic piece:

$$\Pi(q) = \Pi(0, \mathbf{q}) + \delta\Pi(\Omega, \mathbf{q}) \quad (\text{A7})$$

with

$$\Pi(0, \mathbf{q}) = \nu, \quad (\text{A8})$$

$$\delta\Pi(\Omega, \mathbf{q}) = -\frac{2k_F \nu}{q} \text{Re} \left(\sqrt{\frac{q}{2k_F} + \frac{i\Omega}{v_F q}} - 1 \sqrt{\frac{q}{2k_F} + \frac{i\Omega}{v_F q}} + 1 \right). \quad (\text{A9})$$

APPENDIX B: IRREDUCIBLE PAIRING INTERACTION

Here, we present the expression for the dressed antisymmetrized interaction $\Gamma_{\alpha\beta;\gamma\delta}(k, p)$, in the approximation in which we keep only the diagrams with the polarization bubbles which do not depend on the internal running momenta. There are two such polarizations, $\Pi(k-p)$ and $\Pi(k+p)$. Earlier works [49,50] kept only $\Pi(k-p)$ and explored the fact that the pairing gap is odd in momentum in the triplet channel and even in momentum in the singlet channel. For completeness, here we keep both $\Pi(k-p)$ and $\Pi(k+p)$ in the diagrammatic series.

The series for the dressed interaction Γ^{direct} before antisymmetrization are shown in Fig. 14. They contain maximally crossed diagrams and bubble diagrams with fully dressed vertices. The series of maximally crossed diagrams sum up into

$$\delta_{\alpha\gamma}\delta_{\beta\delta} \left(\frac{U}{1 - U\Pi(k+p)} - U \right). \quad (\text{B1})$$

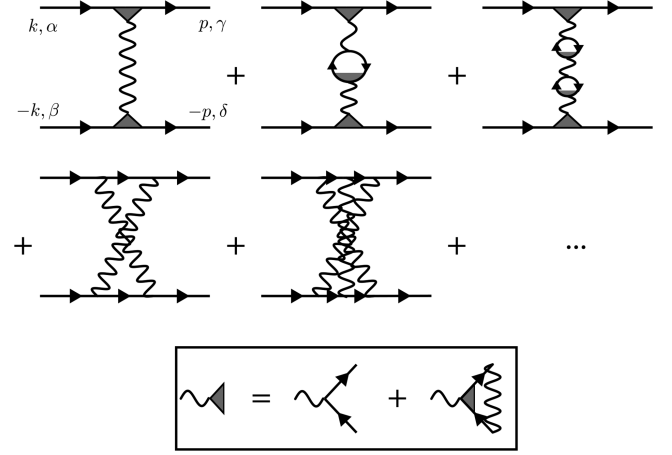


FIG. 14. The full set of ladder and bubble diagrams contributing to the irreducible interaction in the pairing channel. The antisymmetrized $\Gamma_{\alpha\beta;\gamma\delta}(k, p)$ is obtained by adding with the overall minus sign the same set of diagrams but with (p, γ) interchanged with $(-p, \delta)$.

The series of bubbles sum into

$$\delta_{\alpha\gamma}\delta_{\beta\delta} \frac{U\gamma^2}{1 + 2U\gamma\Pi(k-p)}. \quad (\text{B2})$$

The vertex γ is obtained by summing up the ladder series of vertex renormalizations:

$$\gamma = \frac{1}{1 - U\Pi(k-p)}. \quad (\text{B3})$$

Combining maximally crossed diagrams and bubbles, we obtain after a simple algebra

$$\Gamma^{\text{direct}} = \delta_{\alpha\gamma}\delta_{\beta\delta} U \left(\frac{1}{1 - U^2\Pi^2(k-p)} + \frac{1}{1 - U\Pi(k+p)} - 1 \right). \quad (\text{B4})$$

Adding the antisymmetrized piece, we obtain

$$\Gamma_{\alpha\beta;\gamma\delta}(k, p) = \Gamma^{\text{direct}}\delta_{\alpha\gamma}\delta_{\beta\delta} - \Gamma^{\text{asymm}}\delta_{\alpha\delta}\delta_{\beta\gamma}, \quad (\text{B5})$$

where

$$\begin{aligned}\Gamma^{\text{direct}} &= U \left(\frac{1}{1 - U^2\Pi^2(k-p)} + \frac{1}{1 - U\Pi(k+p)} - 1 \right), \\ \Gamma^{\text{asymm}} &= U \left(\frac{1}{1 - U^2\Pi^2(k+p)} + \frac{1}{1 - U\Pi(k-p)} - 1 \right).\end{aligned}\quad (\text{B6})$$

Splitting this Γ into charge and spin parts, as we did in the main text,

$$\Gamma_{\alpha\beta;\gamma\delta}(k, p) = \Gamma^{ch}(k, p)\delta_{\alpha\gamma}\delta_{\beta\delta} + \Gamma^{sp}(k, p)\sigma_{\alpha\beta} \cdot \sigma_{\gamma\delta}. \quad (\text{B7})$$

we obtain

$$\begin{aligned}\Gamma^{ch}(k, p) &= \frac{U}{2} \left(\frac{1}{1 + U\Pi(k-p)} + \frac{1 + 2U\Pi(k+p)}{1 - U^2\Pi^2(k+p)} - 1 \right), \\ \Gamma^{sp}(k, p) &= -\frac{U}{2} \left(\frac{1}{1 - U\Pi(k-p)} + \frac{1}{1 - U^2\Pi^2(k+p)} - 1 \right).\end{aligned}\quad (\text{B8})$$

If we were to neglect $\Pi(k+p)$, we would obtain

$$\begin{aligned}\Gamma^{ch}(k, p) &= \frac{U}{2} \left(\frac{1}{1 + U\Pi(k-p)} \right), \\ \Gamma^{sp}(k, p) &= -\frac{U}{2} \left(\frac{1}{1 - U\Pi(k-p)} \right).\end{aligned}\quad (\text{B9})$$

Near a ferromagnetic transition, $U\Pi(0) \approx 1$. Then, the charge component of Γ is irrelevant, and the spin one gives a magnetically mediated pairing interaction. Near a ferromagnetic transition, where small $|k-p|$ are relevant, this interaction is repulsive in the spin-singlet channel and attractive in the spin-triplet channel. In the first case, the spin component of the pairing vertex is $i\sigma_y$, the order parameter (the gap function) is spatially even, and the effective pairing interaction U_s^p , where p stands for pairing and s for singlet, is, after spin summation

$$U_s^p = \frac{3}{2} \frac{U}{1 - U\Pi(k-p)}. \quad (\text{B10})$$

In the second case, the spin component of the pairing vertex is σ_x , the order parameter is spatially odd, and the effective interaction U_t^p , where t stands for the triplet, is

$$U_t^p = -\frac{1}{2} \frac{U}{1 - U\Pi(k-p)}. \quad (\text{B11})$$

The full analysis, with $\Pi(k+p)$ present, yields the same results, but one has to keep both spin and charge

components of the vertex function. For the singlet channel, the effective interaction is

$$U_s^p = \frac{U}{2} \left(\frac{2 + U\Pi(k-p)}{1 - U^2\Pi^2(k-p)} + \frac{2 + U\Pi(k+p)}{1 - U^2\Pi^2(k+p)} - 2 \right). \quad (\text{B12})$$

Because the gap function $\Delta(k)$ is spatially even, the terms with $\Pi(k-p)$ and $\Pi(k+p)$ contribute equally. This simplifies Eq. (B12) to

$$U_s^p = U \left(\frac{1 + U\Pi(k-p) + U^2\Pi^2(k-p)}{1 - U^2\Pi^2(k-p)} \right). \quad (\text{B13})$$

For $U\Pi(k-p) \approx 1$, this reduces to Eq. (B10).

For the triplet channel, the effective interaction is

$$U_t^p = -\frac{U}{2} \left(\frac{U\Pi(k-p)}{1 - U^2\Pi^2(k-p)} - \frac{U\Pi(k+p)}{1 - U^2\Pi^2(k+p)} \right). \quad (\text{B14})$$

The gap function $\Delta(k)$ is now spatially odd; hence, the terms with $\Pi(k+p)$ contribute with the opposite sign comparing to the term with $\Pi(k-p)$. This simplifies Eq. (B14) to

$$U_t^p = -U \left(\frac{U\Pi(k-p)}{1 - U^2\Pi^2(k-p)} \right). \quad (\text{B15})$$

For $U\Pi(k-p) \approx 1$, this reduces to Eq. (B11).

APPENDIX C: PAIRING INTERACTION IN THE p -WAVE CHANNEL AT SMALL bk_F^2

In this appendix, we present the expressions for the pairing interaction $\tilde{\Gamma}_{l=1}^{sp}(\Omega_m)$ in Eq. (22) in the p -wave channel at the smallest bk_F^2 .

The generic expression for this interaction, valid for arbitrary bk_F^2 and arbitrary $c < 1$, is

$$\tilde{\Gamma}_{l=1}^{sp}(\Omega_m) = -\frac{c}{2\pi} \int_0^\pi \theta d\theta \frac{\cos \theta \sin \theta/2}{1 - c + 4bk_F^2 \sin^3 \theta/2 - c \text{Im} \sqrt{\cos^2 \frac{\theta}{2} + \frac{\Omega_m^2}{4v_F^2 k_F^2 \sin^3(\theta/2)} - i \frac{\Omega_m}{k_F v_F}}}. \quad (\text{C1})$$

Below, we set $c = 1$, i.e., consider the pairing interaction immediately before a FM instability.

For $b = 0$, we have

$$\tilde{\Gamma}_{l=1}^{sp}(\Omega_m) = \frac{1}{2\pi} \int_0^\pi \theta d\theta \frac{\cos \theta \sin \frac{\theta}{2}}{\text{Im} \sqrt{\cos^2 \frac{\theta}{2} + \frac{\Omega_m^2}{4v_F^2 k_F^2 \sin^2 \frac{\theta}{2}} - i \frac{\Omega_m}{k_F v_F}}}. \quad (\text{C2})$$

At the smallest Ω_m , the dominant contribution comes from $\theta \approx \pi$, where $\cos(\theta/2)$ is small. Expanding to the leading order in $\epsilon = \pi - \theta$ and evaluating the integral over ϵ , we obtain $\tilde{\Gamma}_{l=1}^{sp}(0) = 1/4$. At large Ω_m ,

$$\begin{aligned} \text{Im} \sqrt{\cos^2 \frac{\theta}{2} + \frac{\Omega_m^2}{4v_F^2 k_F^2 \sin^2 \frac{\theta}{2}}} - i \frac{\Omega_m}{k_F v_F} \\ = \sin \frac{\theta}{2} + O[(k_F v_F / \Omega_m)]^2. \end{aligned} \quad (\text{C3})$$

Substituting into Eq. (C2) and integrating over θ , we obtain $\tilde{\Gamma}_{l=1}^{sp}(\Omega_m) = 0.5(2v_F k_F / \Omega_m)^2$. We see that, in the limits of small and large frequencies, the p -wave component of the pairing interaction is positive, i.e., repulsive. We show the full $\tilde{\Gamma}_{l=1}^{sp}(\Omega_m)$ at $b = 0$ in Fig. 15. We see that it is repulsive at all frequencies.

The situation changes at a finite b . Now, at large Ω_m , $\tilde{\Gamma}_{l=1}^{sp}(\Omega_m) = -(bk_F^2)/2$ is negative, i.e., attractive. The same happens at the smallest Ω_m . The two leading contributions here come from θ near zero and near π . Combining these contributions, we find that $\tilde{\Gamma}_{l=1}^{sp}(\Omega_m)$ is again negative and scales as $(bk_F^2)/\Omega_m^2$ at Ω_m in between bk_F^2 and $(bk_F^2)^{1/2}$ and as $1/[(bk_F^2)^2 |\Omega_m|]^{1/3}$ at $|\Omega_m| < (k_F^2)$. We show the full $\tilde{\Gamma}_{l=1}^{sp}(\Omega_m)$ in Figs. 16 and 17. We see that at small bk_F^2 it is attractive at small and large frequencies and repulsive at intermediate frequencies. As bk_F^2 increases, the range where $\tilde{\Gamma}_{l=1}^{sp}(\Omega_m)$ is positive (repulsive), shrinks, and eventually vanishes.

In Fig. 18, we show the gap function ϕ_n —the solution of the gap equation [Eq. (22)] at the onset temperature T^* , at different b . The temperature T^* is nonzero for any nonzero b and evolves with b as in Fig. 13 in the main text. We see that, at small bk_F^2 , ϕ_n changes sign twice as a function of Matsubara frequency. As we said in the main

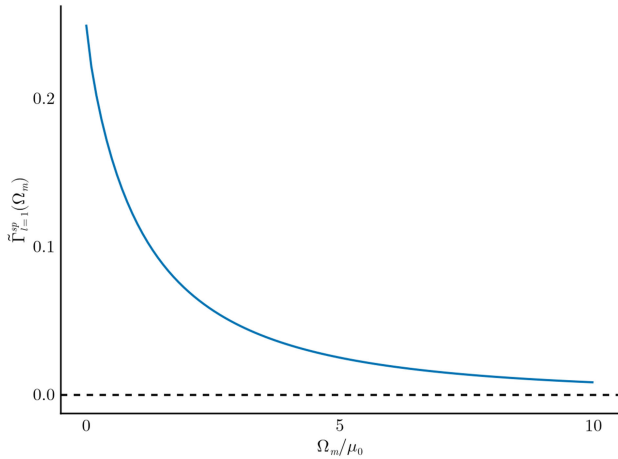


FIG. 15. The pairing interaction $\tilde{\Gamma}_{l=1}^{sp}(\Omega_m)$ for $b = 0$. It remains repulsive at all frequencies.

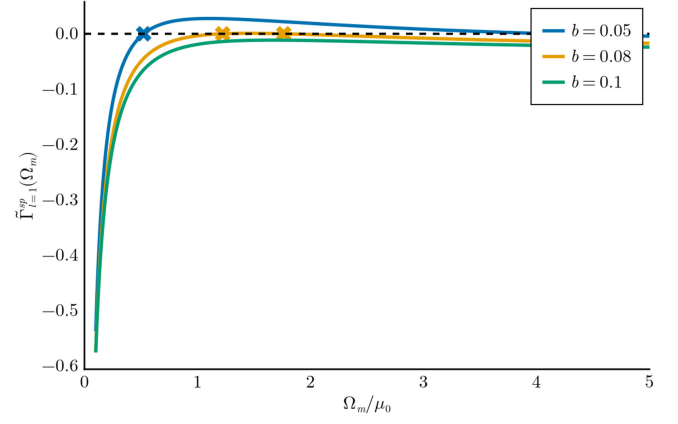


FIG. 16. The pairing interaction $\tilde{\Gamma}_{l=1}^{sp}(\Omega_m)$ for different values of bk_F^2 . The interaction is attractive at small and large frequencies and repulsive at intermediate frequencies. Cross marks indicate the points at which $\tilde{\Gamma}_{l=1}^{sp}(\Omega_m)$ changes sign.

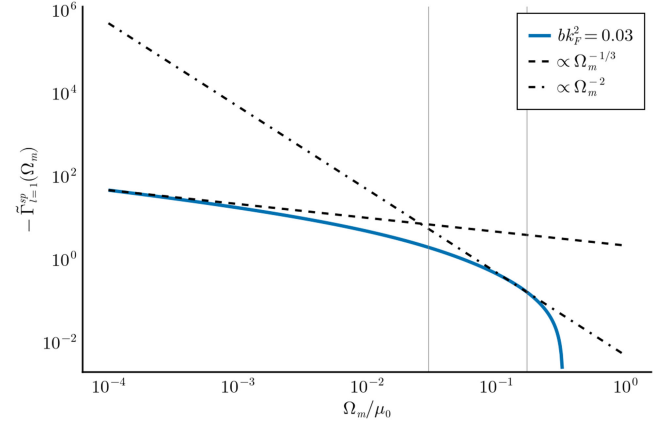


FIG. 17. Log-log plot of the pairing interaction $\tilde{\Gamma}_{l=1}^{sp}(\Omega_m)$ along with the power law scaling for the smallest and intermediate frequencies. Thin vertical lines indicate the crossover scales $(bk_F^2)^{1/2}$ and bk_F^2 .

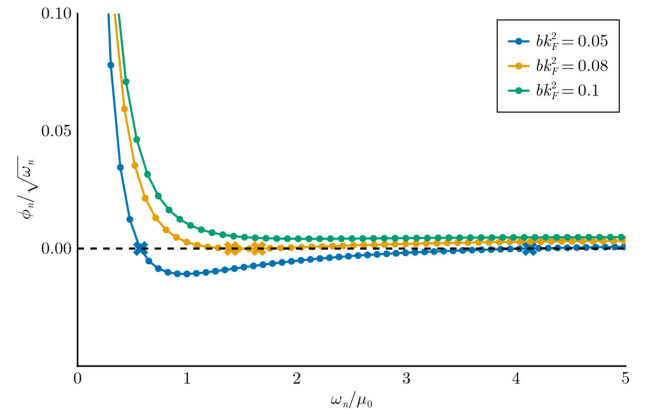


FIG. 18. The gap function ϕ_n for different values of bk_F^2 . Cross marks indicate the points at which ϕ_n changes sign. As b increases, grow closer together and at b_c collide and move into the complex plane.

text, the points at which ϕ_n changes sign is the center of a dynamical vortex. As bk_F^2 increases, the two vortices come close to each other and merge at some finite b . At larger b , they move in opposite directions away from the Matsubara axis and eventually leave the upper half-plane of frequency.

APPENDIX D: ACCURACY OF THE APPROXIMATIONS USED IN OUR ANALYSIS

In this appendix, we discuss the accuracy of the approximations used in both PM and FM states.

1. Paramagnetic state

In Fig. 19, we show diagrams for an (infinitesimally small) ferromagnetic order parameter. The dimensionless coupling $c = U\nu$, at which this graphic equation has a solution, is the instability point toward ferromagnetism. The upper line in the figure is the ladder series of diagrams. Keeping on this line, we obtain critical $c = 1$, which we used in the main text. Diagrams in the second line are the lowest-order contributions outside of the ladder approximation. The first three diagrams cancel out, but the last one with the crossed interaction lines remains. This last diagram renormalizes U into U_{eff} . The strength of this one-loop renormalization depends

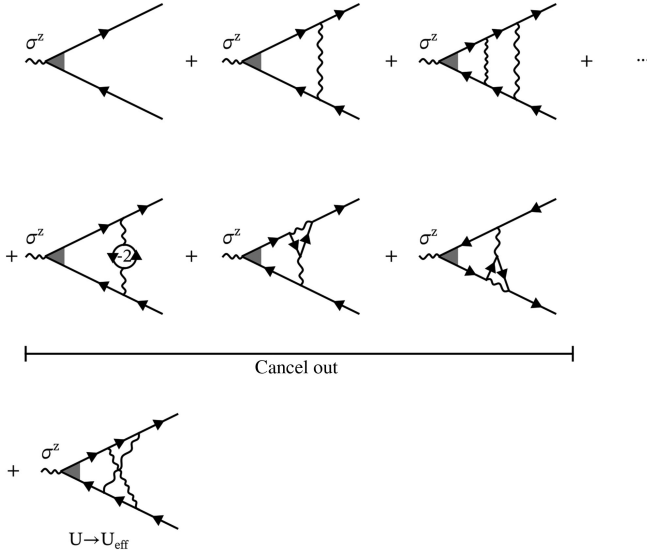


FIG. 19. Diagrams for the (infinitesimally small) ferromagnetic order parameter. The value of the dimensional coupling $c = U\nu$, at which this graphical equation has the solution, is the condition for a Stoner instability. The diagrams in the upper row form ladder series and, taken alone, yield a Stoner instability at $c = 1$. The diagram in the other two rows are other contributions. The bubble and vertex correction diagrams in the second row cancel each other; the third row diagram remains and accounts for the renormalization of U into U_{eff} (see the text). In the main text, we incorporated this renormalization into U .

logarithmically on the upper limit Λ set for the interaction U (Refs. [38,41]):

$$U_{\text{eff}} = U \left(1 - c \log \frac{\Lambda}{E_F} \right). \quad (\text{D1})$$

This renormalization is strong if $\Lambda \gg E_F$ but gets reduced if $\Lambda \leq E_F$, which we assume to hold. In physical terms, we assume that U is a screened Coulomb interaction with screening scale $\kappa \leq k_F$. Such an interaction can be approximated as a constant U at momenta below κ and get strongly reduced at larger momenta. In this situation, U_{eff} gets reduced compared to U but remains of the order of U . In the main text, we assumed that this renormalization is already absorbed into U .

2. Ferromagnetic state

In our derivation of magnon-mediated interaction, we summed up the ladder series of diagrams with polarization bubbles made out of Green's functions of fermions with opposite spin projection (graphically, each such bubble is a convolution of one solid and one dashed line). One may wonder whether one should also include terms with polarization bubbles made out of fermions with the same spin projection (bubbles with two solid or two dashed fermionic lines). We show the corresponding diagrams in Fig. 20. We argue that these terms vanish in our case of full polarization. The physical argument is that bubbles made out of fermions with equal spin projection contribute to longitudinal spin susceptibility, and this susceptibility vanishes for full polarization. Mathematically, this comes about for two reasons. First, for spin-down fermions, the bubble made of two dashed lines vanishes, because these fermions are gapped and have no Fermi surface. Second, for spin-up fermions, the polarization bubble is finite, because these fermions have a Fermi surface, but at each loop order there are two types of renormalizations involving the same bubble, which cancel each other (see Fig. 20).

Finally, in deriving the interaction mediated by two magnons, we included in the main text the diagram with the

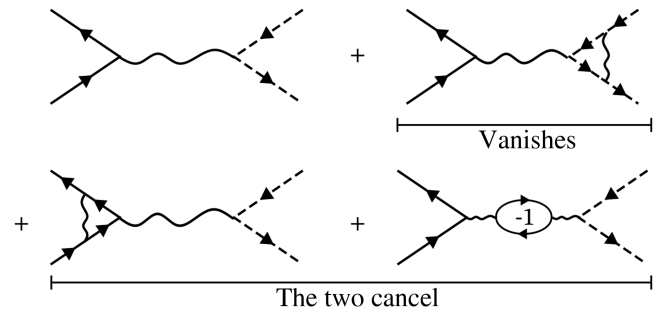


FIG. 20. Vertex corrections to ladder series for an effective four-fermion interaction mediated by a Goldstone magnon. We show that these corrections vanish in the case of full polarization.

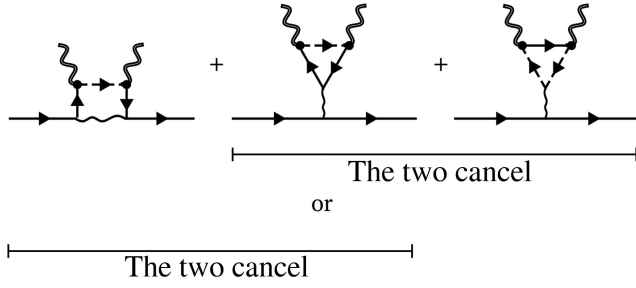


FIG. 21. Three distinct diagrams for the effective interaction between two spin-up fermions and two magnons. The first diagram was used in the main text (Fig. 10). The other two diagrams mutually cancel. Alternatively, one can use the fact that the first two diagrams also mutually cancel and express the effective two-magnon interaction via the Hubbard U between fermionic densities with opposite spin projections, which explicitly agrees with the Pauli principle.

four-fermion interaction between fermions with equal spin projection (Fig. 10). One may wonder whether this is consistent with the Pauli principle by which on-site Hubbard U can couple only fermions with opposite spin projection. Interaction of fermions with equal spin projection is an element of the diagrammatic theory, which requires only that spin projection does not change at each vertex, but still the Pauli principle has to be satisfied. We show in Fig. 21 that this is indeed the case. Namely, there are three distinct diagrams for the interaction between spin-up fermions and two magnons. The first one (the same as in Fig. 10) and the second one contain the interaction U between fermions with the same spin projection; the third diagram contains the interaction between spin-up and spin-down fermions. One can verify that the second and the third diagrams cancel each other; hence, the full interaction is indeed given by the first diagram in Fig. 21, which is the same as in Fig. 10 in the main text. Alternatively, one can verify that the first and the second diagrams in Fig. 21 cancel each other. Then, the full two-magnon interaction comes from U between the densities of spin-up and spin-down fermions, consistent with the Pauli principle.

This reasoning also shows why one does not have to include additional ladder renormalizations for two-magnon

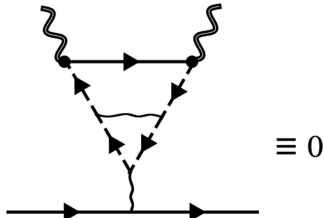


FIG. 22. The correction to the third diagram in Fig. 21. This diagram vanishes because spin-down fermions (dashed lines) have no Fermi surface.

interaction. If we use the convention in which we keep only the last diagram in Fig. 21, additional ladder series shown in Fig. 22 all vanish because, like we said, the polarization bubble made out of two dashed lines (two gapped spin-down fermions) vanishes.

APPENDIX E: USEFUL INTEGRALS

In the course of our calculations, we make use of the following integrals:

$$I[u] \equiv \oint \frac{d\theta}{2\pi} \frac{1}{u - \cos \theta} = \frac{1}{\sqrt{u+1}\sqrt{u-1}}, \quad (\text{E1})$$

$$J[r] \equiv \int \frac{kd k}{\sqrt{r-k}\sqrt{r+k}} = -\sqrt{r+k}\sqrt{r-k} + C, \quad (\text{E2})$$

which are derived in the following subsections.

1. $I[u]$

We can perform the integral Eq. (E1) by contour integration with the change of variables $z = e^{i\theta}$. Then,

$$\begin{aligned} I[u] &\equiv \oint \frac{d\theta}{2\pi} \frac{1}{u - \cos \theta} \\ &= \oint_{\gamma} \frac{dz}{2\pi i z} \frac{1}{u - \frac{z+z^{-1}}{2}} \\ &= -2 \oint_{\gamma} \frac{dz}{2\pi i} \frac{1}{z^2 - 2uz + 1}, \end{aligned} \quad (\text{E3})$$

where the path γ is the unit circle which has poles at $z = u \pm \sqrt{u^2 - 1}$. Clearly, the integral has branch cut when $\text{Im } u = 0$ and $\text{Re } u \in [-1, 1]$. We can, thus, perform the integration for a particular region of u and analytically continue around to branch cut. Taking $\sqrt{\cdot}$ to have a branch cut along the negative real axis, as is conventional, we therefore find that

$$I[u] = \frac{1}{\sqrt{u+1}\sqrt{u-1}}. \quad (\text{E4})$$

2. $J[r]$

For Eq. (E2), we rewrite

$$\begin{aligned} J[r] &= \int \frac{kd k}{\sqrt{r-k}\sqrt{r+k}} = \int dk \frac{r+k-(r-k)}{2\sqrt{r-k}\sqrt{r+k}} \\ &= \int dk \left(\frac{\sqrt{r+k}}{\sqrt{r-k}} - \frac{\sqrt{r-k}}{\sqrt{r+k}} \right) = -\sqrt{r+k}\sqrt{r-k} + C. \end{aligned} \quad (\text{E5})$$

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