

Tensor Networks for Noninvertible Symmetries in 3 + 1D and Beyond

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Tensor networks provide a natural language for noninvertible symmetries in general Hamiltonian lattice models. We use ZX-diagrams, which are tensor network presentations of quantum circuits, to define a noninvertible operator implementing the Wegner duality in 3 + 1D lattice $\mathbb{Z}_2$ gauge theory. The noninvertible algebra, which mixes with lattice translations, can be efficiently computed using ZX-calculus. We further deform the $\mathbb{Z}_2$ gauge theory while preserving the duality and find a model with nine exactly degenerate ground states on a torus, consistent with the Lieb-Schultz-Mattis-type constraint imposed by the symmetry. Finally, we provide a ZX-diagram presentation of the noninvertible duality operators (including noninvertible parity and reflection symmetries) of generalized Ising models based on graphs, encompassing the 1 + 1D Ising model, the three-spin Ising model, the Ashkin-Teller model, and the 2 + 1D plaquette Ising model. The mixing (or lack thereof) with spatial symmetries is understood from a unifying perspective based on graph theory.

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I. INTRODUCTION

The symmetry principle in theoretical physics has been generalized in many different directions in recent years. In particular, it has become increasingly clear that symmetries in quantum systems need not be invertible. These “noninvertible symmetries” are not described by group theory and go beyond the paradigm set by Wigner’s theorem. Nonetheless, they exist ubiquitously in many familiar quantum field theories and lattice models, leading to new conservation laws and selection rules. See Refs. [1–8] for reviews.

The simplest example of noninvertible symmetries is the Kramers-Wannier (KW) symmetry of the 1 + 1D Ising lattice model [9–13]. In particular, it has been realized recently that, in the quantum Hamiltonian Ising lattice model, the operator algebra of this noninvertible symmetry mixes with the lattice translation [14–16]. This symmetry implies a Lieb-Schultz-Mattis (LSM) type constraint [15,19],

forbidding a trivially gapped phase. See, for examples, Refs. [20–37] for recent discussions of noninvertible symmetries in 1 + 1D lattice systems and Refs. [38–47] for 2 + 1D examples.

The most natural generalization of the Kramers-Wannier duality in 1 + 1D is the Wegner duality in 3 + 1D lattice $\mathbb{Z}_2$ gauge theory [48,49]. While the Kramers-Wannier duality exchanges the broken and unbroken phases of an ordinary $\mathbb{Z}_2^{(0)}$ symmetry, the Wegner duality exchanges those of a $\mathbb{Z}_2^{(1)}$ 1-form global symmetry. At the self-dual point, the Wegner duality leads to a noninvertible symmetry. The topological defect associated with this symmetry was constructed in the Euclidean lattice gauge theory in Ref. [52], and its continuum counterpart was found in Refs. [53,54].

In this work, we introduce the corresponding noninvertible duality “operator” in the quantum Hamiltonian lattice $\mathbb{Z}_2$ gauge theory in 3 + 1D on a tensor product Hilbert space, generalizing the KW operator of the 1 + 1D transverse-field Ising model. In 1 + 1D, there are several different presentations of the KW operator, including as a sequential circuit, bosonization of the Majorana translation, and as a matrix product operator (MPO). However, it is not clear how to generalize the first two constructions to higher dimensions.

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To this end, we use the “tensor network” formalism [55,56] to present these noninvertible operators in general dimensions, generalizing the MPO [15,57–66] description. On the lattice, tensor networks are a natural tool to describe nonunitary operators. In this work, we use a particular formulation of tensor networks called “ZX-calculus” [67,68] (see Ref. [69] for a review). ZX-calculus is a graphical language to reason about linear maps between qubits. It is a concrete realization of a more abstract program of diagrammatic reasoning in quantum physics [70,71]. It has roots in quantum foundations, but has recently gained popularity in practical applications in quantum information ranging from optimizing quantum circuits [72–74] to concepts in quantum error correction and fault-tolerant quantum computation [75–81].

The central object of ZX-calculus is a “ZX-diagram.” It is a string diagram–tensor network representation of a quantum circuit or, more generally, a linear map between qubits. One of the most important features of a ZX-diagram is that it can be deformed arbitrarily, while maintaining the connectivity, without changing the linear map it represents. In other words, only topology of the diagram matters. Furthermore, a ZX-diagram can be modified to another topologically inequivalent diagram using a simple (and finite) set of local moves known as “rewrite rules.” Much of the power of ZX-calculus derives from the completeness of these rewrite rules, i.e., any reasoning about linear maps between qubits can be done entirely graphically using a finite sequence of rewrite rules. This is in contrast to the most general formulation of tensor networks, where a general product of tensors increases the bond dimension and does not have an obvious algorithm to simplify the tensors (especially in the case of high connectivity).

The Kramers-Wannier duality operator in 1 + 1D and the Wegner duality operator in 3 + 1D admit a simple

presentation as ZX-diagrams: First, we construct the duality maps represented by the ZX-diagrams shown in Fig. 1, and then, we compose them with half-translation maps (not shown in the figure). In both cases, the duality map takes the original lattice to the dual lattice, and the half-translation map takes it back to the original lattice. In the end, we get an operator on the original Hilbert space.

Using ZX-calculus, we reproduce the noninvertible operator algebra for the 1 + 1D KW duality symmetry and derive a new lattice algebra for the 3 + 1D Wegner duality symmetry. In particular, we find that the algebra of the Wegner duality operator D involves a lattice translation in the (1, 1, 1) direction, generalizing the 1 + 1D observation in Refs. [15,19]. See Table I. The noninvertible operator D exchanges the product state with the 3 + 1D toric code ground state. We also define a condensation operator C on the lattice, which creates a 3 + 1D toric code ground state from a product state.

We further discuss a deformation of the lattice $\mathbb{Z}_2$ gauge theory while preserving the noninvertible Wegner duality symmetry. This is a direct generalization of the deformation of the 1 + 1D Ising model in Ref. [82]. We argue for an LSM-type constraint for the noninvertible symmetry, which implies that the deformed model cannot be trivially gapped.

Finally, our tensor network presentation of the noninvertible symmetry can be readily extended to generalized transverse-field Ising models (TFIMs) based on arbitrary bipartite graphs. We present the condition under which such a model has a noninvertible duality symmetry, and derive a condition for when the noninvertible operator algebra mixes with the spatial symmetry. We then discuss several examples of noninvertible symmetries in this unifying presentation. In some examples, we find a noninvertible parity (or reflection) symmetry.

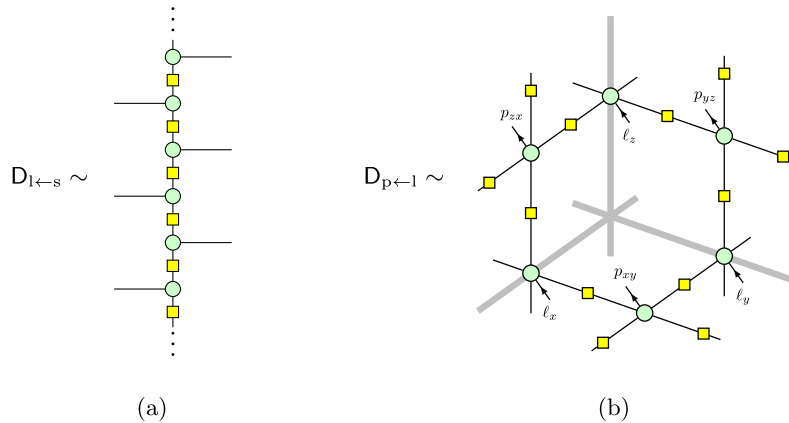


FIG. 1. The tensor network representations, or the ZX-diagrams, of (a) the Kramers-Wannier duality map in 1 + 1D and (b) the Wegner duality map in 3 + 1D. In (a), time flows from left to right, so the input legs are on the sites of the periodic chain and the output legs are on the links. In (b), the flow of time is indicated by the arrows, so the input legs are on the links (thick gray lines) of the periodic cubic lattice and the output legs are on the plaquettes. (We show only the part of the diagram in a unit cell; the full diagram is obtained by repeating and gluing this diagram periodically in the three spatial directions.).

TABLE I. The operator algebras of the Kramers-Wannier duality operator in 1 + 1D and the Wegner duality operator in 3 + 1D with themselves in the continuum and on the lattice. Here, η is the ordinary $\mathbb{Z}_2$ symmetry operator in 1 + 1D, while $\eta(\Sigma)$ is a 1-form symmetry operator on a two-cycle Σ in 3 + 1D. Finally, M is a spatial three-manifold, T is the generator of spatial translation in 1 + 1D, and $T_{1,1,1}$ is the generator of spatial translation in the (1, 1, 1) direction in 3 + 1D.

	1 + 1D	3 + 1D
Continuum	$\mathcal{D}^2 = 1 + \eta$	$\mathcal{D}(M)^2 = \frac{1}{ H^0(M, \mathbb{Z}_2) } \sum_{\Sigma \in H_2(M, \mathbb{Z}_2)} \eta(\Sigma) =: \mathcal{C}(M)$
Lattice	$\mathcal{D}^2 = (1 + \eta)T$	$\mathcal{D}^2 = \frac{1}{2^3} T_{1,1,1} \sum_{\Sigma} \eta(\hat{\Sigma}) =: \mathcal{C} T_{1,1,1}$

The rest of this paper is organized as follows. Section II reviews some of the essential elements of the ZX-calculus. Section III demonstrates the utility of tensor network representation and ZX-calculus in constructing and analyzing the 1 + 1D Kramers-Wannier duality operator. Section IV gives a brief review of the 3 + 1D lattice $\mathbb{Z}_2$ gauge theory, gauging the $\mathbb{Z}_2$ 1-form symmetry, and Wegner duality. Section V gives the explicit tensor network representation of the 3 + 1D Wegner duality operator and the derivation of its operator algebra using ZX-calculus. Section VI presents the tensor network representation of the noninvertible duality operator and its algebra in the generalized TFIM based on an arbitrary bipartite graph.

Section VII concludes with a discussion of the results and potential future directions. Appendix A gives the details of several derivations involving the Wegner duality operator and the condensation operator using ZX-calculus. And finally, Appendix B gives the tensor network representation of the higher quantum symmetry operators associated with higher gauging the $\mathbb{Z}_2$ 1-form symmetry on the lattice.

II. REVIEW OF ZX-CALCULUS

A ZX-diagram is a graphical representation of a linear map of qubits [83]. It is built out of the following generators:

- (1) Z-spider: $n \text{ --- } \alpha \text{ --- } m := |0\rangle^{\otimes m} \langle 0|^{\otimes n} + e^{i\alpha} |1\rangle^{\otimes m} \langle 1|^{\otimes n}$.
- (2) X-spider: $n \text{ --- } \alpha \text{ --- } m := |+\rangle^{\otimes m} \langle +|^{\otimes n} + e^{i\alpha} |-\rangle^{\otimes m} \langle -|^{\otimes n}$.
- (3) Hadamard, H : $\text{---} \square \text{---} := \frac{1}{\sqrt{2}} (|0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| - |1\rangle \langle 1|)$.
- (4) Identity: $\text{---} := |0\rangle \langle 0| + |1\rangle \langle 1|$.
- (5) SWAP: $\text{---} \bowtie \text{---} := |00\rangle \langle 00| + |01\rangle \langle 10| + |10\rangle \langle 01| + |11\rangle \langle 11|$.
- (6) A Bell pair (ket): $\text{---} \text{---} \text{---} \text{---} := |00\rangle + |11\rangle$.
- (7) A Bell pair (bra): $\text{---} \text{---} \text{---} \text{---} := \langle 00| + \langle 11|$.

Here, $n, m \in \mathbb{N} = \{0, 1, 2, \dots\}$ and $\alpha \sim \alpha + 2\pi$. It is customary to omit the phase α inside the Z/X-spiders if $\alpha = 0$. The following are elementary examples of ZX-diagrams of some familiar states and operators.

- (1) Eigenstates of Pauli Z: $\text{---} \text{---} \text{---} \text{---} = \sqrt{2} |0\rangle$ and $\text{---} \text{---} \text{---} \text{---} = \sqrt{2} |1\rangle$.
- (2) Eigenstates of Pauli X: $\text{---} \text{---} \text{---} \text{---} = \sqrt{2} |+\rangle$ and $\text{---} \text{---} \text{---} \text{---} = \sqrt{2} |-\rangle$.
- (3) A Bell pair (ket): $\text{---} \text{---} \text{---} \text{---} = |00\rangle + |11\rangle$.
- (4) Greenberger-Horne-Zeilinger (GHZ) state: $\text{---} \text{---} \text{---} \text{---} = |000\rangle + |111\rangle$ [84].
- (5) Pauli Z: $\text{---} \text{---} \text{---} \text{---} = |0\rangle \langle 0| - |1\rangle \langle 1|$.
- (6) Pauli X: $\text{---} \text{---} \text{---} \text{---} = |+\rangle \langle +| - |-\rangle \langle -|$.
- (7) Controlled-NOT gate, $\text{CNOT}_{i,j}$: $\sqrt{2} \text{---} \text{---} \text{---} \text{---} = |0\rangle \langle 0|_i \otimes I_j + |1\rangle \langle 1|_i \otimes X_j$.
- (8) Controlled-z gate, $\text{CZ}_{i,j}$: $\sqrt{2} \text{---} \text{---} \text{---} \text{---} = \frac{1}{2} (1 + Z_i + Z_j - Z_i Z_j)$.

Note that the ZX-diagrams of CNOT and CZ gates involve more than one generator. Indeed, one can build more general maps by combining ZX-diagrams as follows: let D_1 and D_2 be two diagrams with associated matrices denoted as $[D_1]$ and $[D_2]$, respectively.

- (1) The diagram associated with the tensor product $[D_1] \otimes [D_2]$ is obtained by placing them together. For example,

$$\text{---} \text{---} \text{---} \text{---} = (\sqrt{2})^2 |0\rangle_i \otimes |1\rangle_j. \quad (2.1)$$

- (2) The diagram associated with the product or composition $[D_2][D_1]$, provided it is well defined, is obtained by joining the outputs of D_1 to the inputs of D_2 [85]. For example,

finite sequence of rules that transform D_1 to D_2 . The “if” direction is known as “soundness” and the “only if” direction is known as “completeness” [86].

In many cases, a diagram contains several copies of a smaller diagram. In such cases, it is convenient to use the annotated !-box (read as “bang box”). For example,

$$(2.4)$$

Here, the part of the diagram inside the annotated !-box (the blue box) is repeated three times, and each copy, labeled by $i = 1, 2, 3$, is connected to the part of the diagram outside the annotated !-box in the same way.

It is also convenient to define a “periodic” version of the annotated !-box. For example,

$$(2.5)$$

It is very similar to the annotated !-box except that the wires that end on the dashed blue lines are glued together in a periodic way. Note that, when there are no wires that end on the dashed lines, there is no difference between the two boxes, so we will use the periodic version only when there are wires that end on the dashed lines. Moreover, in what follows, it is always understood that i labels the sites of a periodic 1D chain with L sites, so we omit the annotation “ $i = 1, \dots, L$ ” above the box.

Finally, for completeness, we mention the following. Given a ZX-diagram D representing a map $[D]$, we can obtain the ZX-diagram of

- (1) its conjugate $[D]^*$ by flipping the signs of all the phases in the spiders in D ;
- (2) its transpose $[D]^T$ by flipping the diagram about the vertical axis (or reversing the flow of time); and
- (3) its adjoint $[D]^\dagger$ by doing both.

III. NONINVERTIBLE KRAMERS-WANNIER OPERATOR IN 1+1D

In this section, we review the noninvertible Kramer-Wannier operator of the critical Ising lattice model in 1 + 1D using the language of ZX-calculus.

Consider a periodic chain of L sites, labeled by $i = 1, \dots, L$, with a qubit at each site, i.e., $\mathcal{H}_i \cong \mathbb{C}^2$. The total Hilbert space is the tensor product $\mathcal{H} = \otimes_i \mathcal{H}_i$. The Pauli operators acting on the site i are labeled as Z_i and X_i . An eigenstate of the Z_i ’s is denoted as $|\{m_i\}\rangle := \otimes_i |m_i\rangle$ with $m_i \in \{0, 1\}$. The corresponding ZX-diagram is

$$|\{m_i\}\rangle = \frac{1}{2^{L/2}} \begin{array}{c} \vdots \\ \text{red circle } m_{i-1}\pi \\ \text{red circle } m_i\pi \\ \text{red circle } m_{i+1}\pi \\ \vdots \end{array} = \frac{1}{2^{L/2}} \boxed{\text{red circle } m_i\pi - i} . \quad (3.1)$$

Similarly, an eigenstate of the X_i ’s is denoted as $|\{(-)^{m_i}\}\rangle := \otimes_i |(-)^{m_i}\rangle$ with $m_i \in \{0, 1\}$. The corresponding ZX-diagram is

$$|\{(-)^{m_i}\}\rangle = \frac{1}{2^{L/2}} \begin{array}{c} \vdots \\ \text{green circle } m_{i-1}\pi \\ \text{green circle } m_i\pi \\ \text{green circle } m_{i+1}\pi \\ \vdots \end{array} = \frac{1}{2^{L/2}} \boxed{\text{green circle } m_i\pi - i} . \quad (3.2)$$

While most of our discussions below are concerned with only the operators on the Hilbert space $\mathcal{H}$, it is useful to have in mind a specific example of a Hamiltonian. We start with the Hamiltonian of the 1 + 1D transverse-field Ising model

$$H = -J \sum_{i=1}^L Z_i Z_{i+1} - h \sum_{i=1}^L X_i, \quad (3.3)$$

but our discussion applies to more general Hamiltonians with the same symmetries of interest. [See Eq. (3.31) for an example.] The Ising Hamiltonian is translationally invariant, i.e., H commutes with the generator of translation

$$T = \begin{array}{c} \vdots \\ \text{curly brace} \\ \vdots \end{array} = \boxed{\begin{array}{c} \text{curly brace} \\ i \end{array}} . \quad (3.4)$$

It has a global $\mathbb{Z}_2$ symmetry generated by the operator

$$\eta = \prod_i X_i = \begin{array}{c} \vdots \\ \text{red circle } \pi \\ \text{red circle } \pi \\ \text{red circle } \pi \\ \vdots \end{array} = \boxed{\text{red circle } \pi - i} . \quad (3.5)$$

A. Noninvertible Kramers-Wannier duality operator

When $J = h$, known as the critical Ising model, it also has a noninvertible Kramers-Wannier duality symmetry, generated by the operator D , which exchanges the Ising and transverse-field terms in the Hamiltonian, i.e.,

$$DX_i = Z_i Z_{i+1} D, \quad DZ_i Z_{i+1} = X_{i+1} D. \quad (3.6)$$

There are several equivalent representations of the duality operator. We list several as follows:

- (1) $D = (1/\sqrt{2}) e^{-2\pi i L/8} (1 + \eta) e^{\frac{i\pi}{4} X_1} e^{\frac{i\pi}{4} Z_1 Z_2} e^{\frac{i\pi}{4} X_2} e^{\frac{i\pi}{4} Z_2 Z_3} \times e^{\frac{i\pi}{4} X_3} \dots e^{\frac{i\pi}{4} X_{L-1}} e^{\frac{i\pi}{4} Z_{L-1} Z_L} e^{\frac{i\pi}{4} X_L}$ [14,87],
- (2) $D = \frac{1}{2} (1 + \eta) H_1 CZ_{1,2} H_2 CZ_{2,3} H_3 \dots H_{L-1} CZ_{L-1,L} L H_L (1 + \eta)$ [15],
- (3) $D = \text{Tr}(\mathbb{U}^1 \mathbb{U}^2 \dots \mathbb{U}^L)$, where $\mathbb{U}^i = \begin{pmatrix} |0\rangle\langle +|_i & |0\rangle\langle -|_i \\ |1\rangle\langle -|_i & |1\rangle\langle +|_i \end{pmatrix}$ [15,88], and
- (4) $D = \sum_{\{m_i\}} | \{(-)^{m_{i-1}+m_i}\} \rangle \langle \{m_i\} | = (1/2^{L/2}) \times \sum_{\{m_i\}, \{\hat{m}_i\}} (-1)^{\sum_i \hat{m}_i (m_{i-1}+m_i)} | \{\hat{m}_i\} \rangle \langle \{m_i\} |$ [13,89].

From the fourth expression, we derive the ZX-diagrammatic presentation of the duality operator,

$$D = 2^{L/2} \left[\text{diagram} \right] = 2^{L/2} \left[\text{diagram} \right] \quad (3.7)$$

$$\stackrel{\text{CC}'}{=} 2^{L/2} \left[\text{diagram} \right] = 2^{L/2} \left[\text{diagram} \right]$$

$$\stackrel{\text{CC}'}{=} 2^{L/2} \left[\text{diagram} \right] = 2^{L/2} \left[\text{diagram} \right],$$

where we have used the CC' rewrite rule to provide equivalent presentations. The top or the bottom (but not the middle) blue box gives the tensor $\mathbb{U}^i$ in the MPO [57,58] presentation equivalent to the third expression above.

To verify that these are indeed the ZX-diagrams of the matrix in the fourth expression, let us compute the matrix elements using ZX-calculus,

$$D|\{m_i\}\rangle = \frac{2^{L/2}}{2^{L/2}} \left[\text{diagram} \right] \quad (3.8)$$

$$\stackrel{\text{Pi,SC}}{=} \frac{1}{2^{L/2}} \left[\text{diagram} \right]$$

$$\stackrel{\text{SF}}{=} \frac{1}{2^{L/2}} \left[\text{diagram} \right]$$

$$\stackrel{\text{CC}}{=} \frac{1}{2^{L/2}} \left[\text{diagram} \right]$$

$$= | \{ (-)^{m_{i-1}+m_i} \} \rangle.$$

One can also modify the ZX-diagrams in Eq. (3.7) to make the equivalence to the other expressions manifest [90].

It is sometimes useful to define an auxiliary Hilbert space $\mathcal{H}_l$ on the links, in the same way as the Hilbert space on the sites, which we now denote as $\mathcal{H}_s$ for clarity, and define the following maps between $\mathcal{H}_s$ and $\mathcal{H}_l$:

$$D_{l \leftarrow s} = 2^{L/2} \left[\text{diagram} \right], \quad T_{l \leftarrow s} = \left[\text{diagram} \right],$$

$$D_{s \leftarrow l} = 2^{L/2} \left[\text{diagram} \right], \quad T_{s \leftarrow l} = \left[\text{diagram} \right], \quad (3.9)$$

where the sites are labeled by the integer i (modulo L), whereas the links are labeled by the half-integer $i - \frac{1}{2}$ (modulo L). These duality maps were studied in Refs. [13,61,65,88,89,91]—for instance,

$$D_{l \leftarrow s} \stackrel{\text{SF}}{=} 2^{L/2} \left[\text{diagram} \right]$$

$$= \left(\bigotimes_i \langle + |_i \right) \left(\prod_i \text{CZ}_{i,i-\frac{1}{2}} \text{CZ}_{i,i+\frac{1}{2}} \right) \left(\bigotimes_i | + \rangle_{i+\frac{1}{2}} \right), \quad (3.10)$$

which is the expression for the Kramers-Wannier duality map in terms of the cluster entangler [88,91]. Furthermore, changing the inputs on sites i to outputs in $\mathbf{D}_{1 \leftarrow s}$ gives the ZX-diagram of the cluster state associated with the nontrivial $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry protected topological (SPT) phase.

In terms of these maps, we can write

$$\begin{aligned}
 T &= \boxed{i \text{ --- } i} = T_{s \leftarrow 1} T_{1 \leftarrow s}, \\
 D &= 2^{L/2} \boxed{\begin{array}{c} \text{---} \text{green circle} \text{---} \\ | \\ \text{---} \text{yellow square} \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \\ | \\ \text{---} \text{yellow square} \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \end{array}} = T_{s \leftarrow 1} D_{1 \leftarrow s} \\
 &= 2^{L/2} \boxed{\begin{array}{c} \text{---} \text{green circle} \text{---} \\ | \\ \text{---} \text{yellow square} \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \\ | \\ \text{---} \text{yellow square} \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \end{array}} = D_{s \leftarrow 1} T_{1 \leftarrow s},
 \end{aligned} \tag{3.11}$$

which are operators on $\mathcal{H}_s$.

B. Action on operators

Let us compute the action of $\mathbf{D}$ on the $\mathbb{Z}_2$ -symmetric local operators. Such operators are generated by products of X_i and $Z_i Z_{i+1}$, so it suffices to understand the action on these operators. First, we have

$$\begin{aligned}
 DX_i &= 2^{L/2} \boxed{\begin{array}{c} \text{---} \text{red circle } \pi \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \\ | \\ \text{---} \text{yellow square} \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \\ | \\ \text{---} \text{yellow square} \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \end{array}} \stackrel{\text{Pi}}{=} 2^{L/2} \boxed{\begin{array}{c} \text{---} \text{red circle } \pi \text{---} \\ | \\ \text{---} \text{red circle } \pi \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \\ | \\ \text{---} \text{yellow square} \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \end{array}} \\
 &\stackrel{\text{CC}'}{=} 2^{L/2} \boxed{\begin{array}{c} \text{---} \text{green circle} \text{---} \\ | \\ \text{---} \text{red circle } \pi \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \\ | \\ \text{---} \text{yellow square} \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \\ | \\ \text{---} \text{yellow square} \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \end{array}} \stackrel{\text{SF}}{=} 2^{L/2} \boxed{\begin{array}{c} \text{---} \text{green circle} \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \\ | \\ \text{---} \text{yellow square} \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \\ | \\ \text{---} \text{yellow square} \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \end{array}} = Z_i Z_{i+1} D.
 \end{aligned} \tag{3.12}$$

Similarly, one can show that $DZ_i Z_{i+1} = X_{i+1} D$. Therefore, $\mathbf{D}$ indeed generates the KW duality transformation.

More generally, for $m_i \in \{0, 1\}$, we have

$$\begin{aligned}
 D \prod_i X_i^{m_i} &= 2^{L/2} \begin{array}{c} \text{Diagram: } i \text{ --- } (m_i \pi) \text{ --- } i \end{array} \\
 &\stackrel{\text{PI}}{=} 2^{L/2} \begin{array}{c} \text{Diagram: } i \text{ --- } (m_i \pi) \text{ --- } (m_i \pi) \text{ --- } i \end{array} \\
 &\stackrel{\text{SF}}{=} 2^{L/2} \begin{array}{c} \text{Diagram: } i \text{ --- } (m_i + m_{i-1}) \pi \text{ --- } i \end{array} \\
 &\stackrel{\text{SF, CC}'}{=} 2^{L/2} \begin{array}{c} \text{Diagram: } i \text{ --- } (m_i + m_{i-1}) \pi \text{ --- } i \end{array} \\
 &= \prod_i Z_i^{m_i + m_{i-1}} D = \prod_i (Z_i Z_{i+1})^{m_i} D,
 \end{aligned} \tag{3.13}$$

and similarly, $D \prod_i (Z_i Z_{i+1})^{m_i} = \prod_i X_i^{m_i} D$. Consider a special case when $m_1 = m_2 = \dots = m_{k-1} = 1$ for some $k > 2$, and all the other m_i 's are 0. We find that D maps a pair of order operators (Z) to the disorder operator (X),

$$DZ_1 Z_k = X_2 X_3 \dots X_k D. \tag{3.14}$$

Consider an even more special case where $m_i = 1$ for all i , we get

$$D\eta = \eta D = D. \tag{3.15}$$

C. Operator algebra

In this subsection, we compute the fusion rules of η , D , and T using ZX-calculus. In particular, we will rederive

$$D^2 = (1 + \eta)T \tag{3.16}$$

of Refs. [14,15] from ZX-calculus.

The fusion rules involving η and T are

$$\begin{aligned}
 \eta^2 &= \begin{array}{c} \text{Diagram: } i \text{ --- } \pi \text{ --- } \pi \text{ --- } i \end{array} \stackrel{\text{SF}}{=} \begin{array}{c} \text{Diagram: } i \text{ --- } \pi \text{ --- } i \end{array} \stackrel{\text{I}}{=} \begin{array}{c} \text{Diagram: } i \text{ --- } i \end{array} = 1, \\
 T\eta &= \begin{array}{c} \text{Diagram: } i \text{ --- } \pi \text{ --- } i \end{array} = \begin{array}{c} \text{Diagram: } i \text{ --- } \pi \text{ --- } i \end{array} = \eta T, \\
 T^L &= \begin{array}{c} \text{Diagram: } i \text{ --- } \dots \text{ --- } i \end{array} = 1.
 \end{aligned} \tag{3.17}$$

To see the last line of this equation explicitly, consider $L = 3$ for concreteness,

$$T = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \Rightarrow T^3 = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} = 1, \tag{3.18}$$

where we colored the wires for clarity.

Let us now compute the fusion rules involving D using ZX-calculus. We already showed that $D\eta = \eta D = D$. Moreover, it is easy to see that D commutes with T . We are then left with the fusion of D with itself,

$$\begin{aligned}
 D^2 &= D_{s \leftarrow l} T_{l \leftarrow s} T_{s \leftarrow l} D_{l \leftarrow s} = 2^L \begin{array}{c} \text{Diagram: } i \text{ --- } \dots \text{ --- } i \end{array} \\
 &= 2^L \begin{array}{c} \text{Diagram: } i \text{ --- } \dots \text{ --- } i \end{array} = T D_{s \leftarrow l} D_{l \leftarrow s}.
 \end{aligned} \tag{3.19}$$

(As an aside, note that $T_{l \leftarrow s} T_{s \leftarrow l}$ is the generator of translation on $\mathcal{H}_l$.) Similarly, we have

$$D^2 = D_{s \leftarrow l} D_{l \leftarrow s} T. \tag{3.20}$$

We now show that $D_{s \leftarrow l} D_{l \leftarrow s} = 1 + \eta$ using ZX-calculus,

$$\begin{aligned}
 D_{s \leftarrow l} D_{l \leftarrow s} &= 2^L \begin{array}{c} \text{Diagram: } i \text{ --- } \dots \text{ --- } i \end{array} \stackrel{\text{CC}'}{=} 2^L \begin{array}{c} \text{Diagram: } i \text{ --- } \dots \text{ --- } i \end{array} \\
 &\stackrel{\text{SF}}{=} 2^L \begin{array}{c} \text{Diagram: } i \text{ --- } \dots \text{ --- } i \end{array} \stackrel{\text{B}}{=} \frac{2^L}{2^{L/2}} \begin{array}{c} \text{Diagram: } i \text{ --- } \dots \text{ --- } i \end{array} \\
 &\stackrel{\text{SF}}{=} 2^{L/2} \begin{array}{c} \text{Diagram: } i \text{ --- } \dots \text{ --- } i \end{array} =: C,
 \end{aligned} \tag{3.21}$$

where C is the condensation operator. It is convenient to define a more general operator

$$C_n := 2^{L/2} \begin{array}{c} \text{Diagram: } i \text{ --- } n\pi \text{ --- } i \end{array}, \tag{3.22}$$

where $n = 0$ gives the condensation operator $C = C_0$.

We want to show that $C_n = 1 + (-1)^n \eta$. We have

$$\begin{aligned}
 C_n &= 2^{L/2} \begin{array}{c} \text{Diagram: } n \text{ auxiliary wires (green circles) connected to } L \text{ main wires (red circles).} \end{array} \stackrel{\text{SF}}{=} 2^{L/2} \begin{array}{c} \text{Diagram: } n \text{ auxiliary wires (green circles) connected to } L \text{ main wires (red circles).} \end{array} \\
 &= \frac{2^{L/2}}{2^{L/2}} \cdot (\sqrt{2})^2 \langle (-)^n |_a \left(\prod_{i=1}^L \text{CNOT}_{a,i} \right) | + \rangle_a, \quad (3.23)
 \end{aligned}$$

where a represents the auxiliary wire above 1. Now, we have

$$\begin{aligned}
 \prod_{i=1}^L \text{CNOT}_{a,i} &= \prod_{i=1}^L (|0\rangle\langle 0|_a \otimes I + |1\rangle\langle 1|_a \otimes X_i) \\
 &= |0\rangle\langle 0|_a \otimes I + |1\rangle\langle 1|_a \otimes \eta. \quad (3.24)
 \end{aligned}$$

It follows that

$$\begin{aligned}
 C_n &= 2 \langle (-)^n |_a (|0\rangle\langle 0|_a \otimes I + |1\rangle\langle 1|_a \otimes \eta) | + \rangle_a \\
 &= 1 + (-1)^n \eta. \quad (3.25)
 \end{aligned}$$

Therefore, we proved using ZX-calculus that $C_n = 1 + (-1)^n \eta$, and hence $D^2 = (1 + \eta)T$. More concretely, for $L = 3$, we have

$$\begin{aligned}
 D^2 &= 2^{3/2} \begin{array}{c} \text{Diagram: } 3 \text{ auxiliary wires (green circles) connected to } 3 \text{ main wires (red circles).} \end{array} = T(1 + \eta) \\
 &= 2^{3/2} \begin{array}{c} \text{Diagram: } 3 \text{ auxiliary wires (green circles) connected to } 3 \text{ main wires (red circles).} \end{array} = (1 + \eta)T. \quad (3.26)
 \end{aligned}$$

It follows from Eq. (3.15) that $DC = CD = 2D$. Let us verify this using ZX-calculus,

$$\begin{aligned}
 D_{1 \leftarrow s} C &= 2^L \begin{array}{c} \text{Diagram: } 1 \text{ auxiliary wire (green circle) connected to } L \text{ main wires (red circles).} \end{array} \\
 &\stackrel{B}{=} 2^L \cdot 2^{L/2} \begin{array}{c} \text{Diagram: } 1 \text{ auxiliary wire (green circle) connected to } L \text{ main wires (red circles).} \end{array} \\
 &\stackrel{\text{SF}}{=} 2^{3L/2} \begin{array}{c} \text{Diagram: } 1 \text{ auxiliary wire (green circle) connected to } L \text{ main wires (red circles).} \end{array} \\
 &\stackrel{\text{SF}}{=} 2^{3L/2} \begin{array}{c} \text{Diagram: } 1 \text{ auxiliary wire (green circle) connected to } L \text{ main wires (red circles).} \end{array} \\
 &\stackrel{\text{SF}}{=} 2^{3L/2} \begin{array}{c} \text{Diagram: } 1 \text{ auxiliary wire (green circle) connected to } L \text{ main wires (red circles).} \end{array} \\
 &\stackrel{\text{Hf}}{=} \frac{2^{3L/2}}{2^L} \begin{array}{c} \text{Diagram: } 1 \text{ auxiliary wire (green circle) connected to } L \text{ main wires (red circles).} \end{array} \\
 &\stackrel{\text{SF}}{=} 2^{L/2} \begin{array}{c} \text{Diagram: } 1 \text{ auxiliary wire (green circle) connected to } L \text{ main wires (red circles).} \end{array} \\
 &\stackrel{S}{=} 2 \cdot 2^{L/2} \begin{array}{c} \text{Diagram: } 1 \text{ auxiliary wire (green circle) connected to } L \text{ main wires (red circles).} \end{array} = 2D_{1 \leftarrow s}. \quad (3.27)
 \end{aligned}$$

Multiplying $T_{s \leftarrow 1}$ on the left on both sides gives the desired fusion rule. Showing that $CD = 2D$ is similar.

For the sake of completeness, let us also compute $C^2 = 2C$ using ZX-calculus,

$$\begin{aligned}
 C^2 &= 2^L \begin{array}{c} \text{Diagram: } 2 \text{ auxiliary wires (green circles) connected to } L \text{ main wires (red circles).} \end{array} \stackrel{\text{SF}}{=} 2^L \begin{array}{c} \text{Diagram: } 2 \text{ auxiliary wires (green circles) connected to } L \text{ main wires (red circles).} \end{array} \\
 &\stackrel{\text{GB}}{=} \frac{2^L}{2^{(L-1)/2}} \begin{array}{c} \text{Diagram: } 2 \text{ auxiliary wires (green circles) connected to } L \text{ main wires (red circles).} \end{array} \stackrel{\text{SC, SF}}{=} \frac{2^{(L+1)/2}}{\sqrt{2}} \begin{array}{c} \text{Diagram: } 2 \text{ auxiliary wires (green circles) connected to } L \text{ main wires (red circles).} \end{array} \\
 &\stackrel{S}{=} 2 \cdot 2^{L/2} \begin{array}{c} \text{Diagram: } 2 \text{ auxiliary wires (green circles) connected to } L \text{ main wires (red circles).} \end{array} = 2C. \quad (3.28)
 \end{aligned}$$

D. Action on states

The phase diagram of the Ising model Eq. (3.3) is well understood [92]: The $\mathbb{Z}_2$ global symmetry is spontaneously

broken (ferromagnetic phase) for $J > h$, whereas it is preserved (paramagnetic phase) for $J < h$. At the critical point $J = h$, there is a second-order phase transition captured by the Ising conformal field theory (CFT) at low energies.

Let us comment on the ground states and the superselection sectors in these phases. In the limit $J \gg h$ (or when $h = 0$), the Ising term dominates. In finite volume, there are two low-lying states whose energy splitting is exponentially small in L . In infinite volume, the Hilbert space splits into two superselection sectors,

$$|0\dots 0\rangle, \quad |1\dots 1\rangle. \quad (3.29)$$

On the other hand, in the limit $J \ll h$ (or when $J = 0$), the transverse-field term dominates, so there is a unique ground state,

$$|+\dots+\rangle. \quad (3.30)$$

In fact, these three product states $|0\dots 0\rangle, |1\dots 1\rangle, |+\dots+\rangle$ are the exact ground states of a special case ($\lambda = 1$) of the lattice model introduced in Ref. [82],

$$H_\lambda = -J \left[\sum_i (Z_i Z_{i+1} + X_i) - \frac{\lambda}{2} \sum_i (X_{i-1} Z_i Z_{i+1} + Z_i Z_{i+1} X_{i+2}) \right]. \quad (3.31)$$

For any λ , this Hamiltonian preserves both the $\mathbb{Z}_2$ and the KW duality symmetries. In the thermodynamic limit, the model is in the $c = 1/2$ Ising CFT phase for $0 \leq \lambda \lesssim 0.856$, reaches the $c = 7/10$ tricritical Ising CFT point at $\lambda \sim 0.856$, and becomes gapped with three superselection sectors for $\lambda \gtrsim 0.856$. These phases are consistent with the LSM-type constraint from the noninvertible symmetry in Ref. [15]. In particular, this model was shown to have exact threefold degeneracy even at finite L when $\lambda = 1$.

Irrespective of the Hamiltonian, one can ask how the symmetry operators η and D act on these states [15],

$$\begin{aligned} \eta|0\dots 0\rangle &= |1\dots 1\rangle, & D|0\dots 0\rangle &= |+\dots+\rangle, \\ \eta|1\dots 1\rangle &= |0\dots 0\rangle, & D|1\dots 1\rangle &= |+\dots+\rangle, \\ \eta|+\dots+\rangle &= |+\dots+\rangle, & D|+\dots+\rangle &= |0\dots 0\rangle + |1\dots 1\rangle. \end{aligned} \quad (3.32)$$

This basis corresponds to the three superselection sectors of the Hamiltonian Eq. (3.31) in the thermodynamic limit. Alternatively, one can work in a basis that diagonalizes the symmetry operators,

$$\begin{aligned} \frac{1}{\sqrt{2}}|+\dots+\rangle \pm \frac{1}{\sqrt{2}}|\text{GHZ}^+\rangle, & \quad D = \pm\sqrt{2}, \quad \eta = 1, \\ |\text{GHZ}^-\rangle, & \quad D = 0, \quad \eta = -1, \end{aligned} \quad (3.33)$$

where

$$|\text{GHZ}^\pm\rangle := \frac{1}{\sqrt{2}}(|0\dots 0\rangle \pm |1\dots 1\rangle). \quad (3.34)$$

The duality operator exchanges $|\text{GHZ}^+\rangle$ with the product state $|+\dots+\rangle$,

$$\begin{aligned} D|+\dots+\rangle &= \sqrt{2}|\text{GHZ}^+\rangle, \\ D|\text{GHZ}^+\rangle &= \sqrt{2}|+\dots+\rangle. \end{aligned} \quad (3.35)$$

IV. REVIEW OF 3+1D LATTICE $\mathbb{Z}_2$ GAUGE THEORY

We denote the sites, links, plaquettes, and cubes of a 3D cubic lattice by s, ℓ, p, c , respectively. Let L_x, L_y, L_z be the number of sites in each direction and assume periodic boundary conditions. We place a qubit associated with a local Hilbert space $\mathcal{H}_\ell \cong \mathbb{C}^2$ on every link. It is the $\mathbb{Z}_2$ 1-form gauge field. We denote the Pauli operators acting on link ℓ as X_ℓ, Z_ℓ . The total Hilbert space is $\mathcal{H} = \otimes_\ell \mathcal{H}_\ell$. Here, we focus on the cubic lattice, which is self-dual under exchanging the links with the plaquettes, and the sites with the cubes. In Sec. VI, we will discuss other lattices (or even graphs).

Next, we impose the Gauss law for the $\mathbb{Z}_2$ 1-form gauge field strictly as an operator equation at every site,

$$G_s = \prod_{\ell \ni s} X_\ell = 1. \quad (4.1)$$

We denote the Hilbert space subject to this constraint as $\tilde{\mathcal{H}}$, which is not a tensor product of local factors. We will later relax this operator equation and impose it energetically.

We start with the standard Hamiltonian for the lattice gauge theory,

$$\tilde{H} = -J \sum_p \prod_{\ell \in p} Z_\ell - h \sum_\ell X_\ell, \quad (4.2)$$

which acts on the constrained Hilbert space $\tilde{\mathcal{H}}$. The first term is the analog of the magnetic flux term, while the second term is the analog of the electric-field term in the continuum. (See Fig. 2 for an illustration of these terms.) We will later study deformations of this Hamiltonian while preserving all the symmetries of interest.

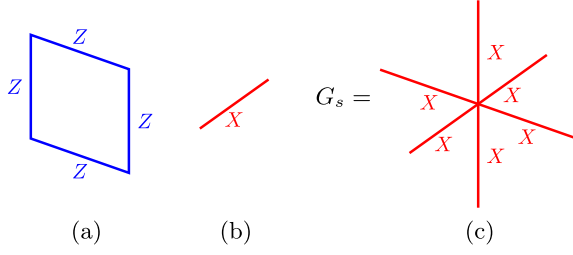


FIG. 2. The local terms and the Gauss law of the Hamiltonian Eq. (4.2) of the lattice $\mathbb{Z}_2$ gauge theory. (a) The magnetic flux term on a plaquette [first term of Eq. (4.2)], (b) the electric-field term on a link [second term of Eq. (4.2)], and (c) the Gauss law operator G_s at a site.

A. Topological versus nontopological 1-form global symmetries

The central global symmetry that will be important to us is a lattice global symmetry generated by

$$\eta(\hat{\Sigma}) = \prod_{\ell \in \hat{\Sigma}} X_\ell, \quad (4.3)$$

where $\hat{\Sigma}$ is a two-dimensional surface on the dual lattice. The product is over all the links orthogonal to $\hat{\Sigma}$ or, equivalently, over all the dual plaquettes on $\hat{\Sigma}$. In contrast to an ordinary global symmetry, this symmetry operator acts only on part of the space and commutes with the Hamiltonian $\tilde{H}$ for any $\hat{\Sigma}$. Since the symmetry operator is a product of the electric field X_ℓ on a codimension-one surface in the three-dimensional space, we refer to it as an electric 1-form global symmetry denoted by $\mathbb{Z}_2^{(1)}$. (The superscript stands for the codimension of the symmetry operator in space.)

When the Gauss law is imposed as a strict operator equation as in Eq. (4.1), this 1-form global symmetry is “topological” in the sense that

$$\eta(\hat{\Sigma}) = \eta(\hat{\Sigma}'), \quad \text{if } \hat{\Sigma} \sim \hat{\Sigma}', \quad (4.4)$$

where $\hat{\Sigma} \sim \hat{\Sigma}'$ means that they are in the same homology class on the dual lattice. That is, the symmetry operator depends only on the homological class of the two-cycle $\hat{\Sigma}$. Therefore, the $\mathbb{Z}_2$ 1-form symmetry is generated by the operators associated with the three fundamental noncontractible surfaces.

Similar to the continuum, this topological 1-form symmetry cannot be violated by any local operator deformation to the Hamiltonian. For instance, a term like Z_ℓ violates the Gauss law Eq. (4.1) and is excluded from the Hamiltonian as a valid deformation.

Alternatively, we can relax the Gauss law constraint Eq. (4.1), and impose it energetically in the Hamiltonian as

$$H = -J \sum_p \prod_{\ell \in p} Z_\ell - h \sum_{\ell} X_\ell - g \sum_s \prod_{\ell \ni s} X_\ell. \quad (4.5)$$

This Hamiltonian acts on the tensor product Hilbert space $\mathcal{H}$. When $g \rightarrow \infty$, it reduces to the previous case where the Gauss law is enforced strictly. When $h = 0$, this is the 3 + 1D toric code [93].

When the Gauss law is relaxed such as in H , the 1-form symmetry is no longer topological: $\eta(\hat{\Sigma})$ is a distinct operator acting on $\mathcal{H}$ for each dual surface $\hat{\Sigma}$. Now that the Gauss law is relaxed, the nontopological 1-form symmetry can be violated by local operators, such as Z_ℓ . The difference between topological and nontopological higher-form symmetries on the lattice has been previously discussed in Refs. [45,95–97].

This is a general feature: A higher-form global symmetry in the continuum or a topological higher-form symmetry on the lattice cannot be violated by local operators. On the other hand, a nontopological one on the lattice can be violated by local operator deformations.

B. Defects for the 1-form symmetry

Having discussed the conserved operator $\eta(\hat{\Sigma})$, we move on to the defect for the 1-form symmetry on the lattice. We work with the more general system where the Gauss law is only enforced energetically. The 1-form symmetry defect is represented as a modification of the Hamiltonian along a curve $\hat{\gamma}$ on the dual lattice. We refer to the Hamiltonian with a defect as the defect Hamiltonian, which is given by

$$H_\eta(\hat{\gamma}) = -J \sum_p (-1)^{\langle \hat{\gamma}, p \rangle} \prod_{\ell \in p} Z_\ell - h \sum_{\ell} X_\ell - g \sum_s \prod_{\ell \ni s} X_\ell, \quad (4.6)$$

where $\langle \hat{\gamma}, p \rangle$ is the intersection number of $\hat{\gamma}$ and p . The subscript and the superscript stand for the type and the location of the defect, respectively. The defect is topological in the sense that we can move its location by a product of local unitaries. Specifically, let $\hat{\gamma}'$ be another curve that is homological to $\hat{\gamma}$, and let $\hat{\Sigma}$ be the surface bounded by $\hat{\gamma} - \hat{\gamma}'$. Then,

$$\left(\prod_{\ell \in \hat{\Sigma}} X_\ell \right) H_\eta(\hat{\gamma}) \left(\prod_{\ell \in \hat{\Sigma}} X_\ell \right)^{-1} = H_\eta(\hat{\gamma}'), \quad (4.7)$$

where the product is over all dual faces or, equivalently, links ℓ , lying on the dual surface $\hat{\Sigma}$. Note that even though the 1-form symmetry operator $\eta(\hat{\Sigma})$ is not topological, the associated defect is.

C. Gauging the 1-form symmetry and the Wegner duality

The 1-form symmetry Eq. (4.3) is on site [98] and is free of 't Hooft anomaly. This means that it is compatible with a trivially gapped phase, and it can be gauged. Here, we gauge the 1-form global symmetry, following the discussion in Appendix B of Ref. [53] and Sec. III B of Ref. [99]. This generalizes the prescription for gauging a 0-form symmetry outlined in Refs. [15,100].

For concreteness, we focus on the Hamiltonian in Eq. (4.2), but most of the conclusions hold true for more general Hamiltonian invariant under the 1-form symmetry. The local term of such a general Hamiltonian is a product of X_ℓ and $\prod_{\ell \in p} Z_\ell$.

We gauge the 1-form global symmetry η by introducing a qubit with Pauli operators σ_p^a ($a = x, y, z$) on every plaquette p . In the continuum, they represent the 2-form $\mathbb{Z}_2$ gauge fields for the 1-form symmetry. The gauging consists of three steps:

- (1) Gauss law: Impose the Gauss law for the 2-form gauge field at every link

$$G_\ell = X_\ell \prod_{p \ni \ell} \sigma_p^z = 1. \quad (4.8)$$

With this new Gauss law, the original one $G_s = \prod_{\ell \ni s} X_\ell = 1$ is trivially satisfied, and we no longer have to impose the latter by hand.

- (2) Minimal coupling: Couple the gauge fields minimally to the Hamiltonian by replacing the plaquette term $\prod_{\ell \in p} Z_\ell$ by $\sigma_p^x \prod_{\ell \in p} Z_\ell$.
- (3) Flux term: We add a gauge-invariant term $\prod_{p \in c} \sigma_p^x$ of the 2-form gauge fields for every cube to the Hamiltonian. This term is analogous to the magnetic-field square term in the continuum.

The gauged Hamiltonian is then

$$\hat{H} = -J \sum_p \sigma_p^x \prod_{\ell \in p} Z_\ell - h \sum_\ell X_\ell - \hat{g} \sum_c \prod_{p \in c} \sigma_p^x, \quad (4.9)$$

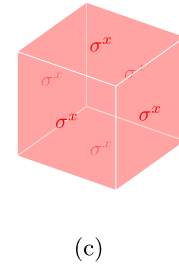
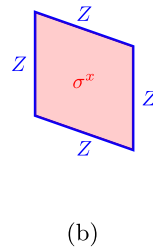
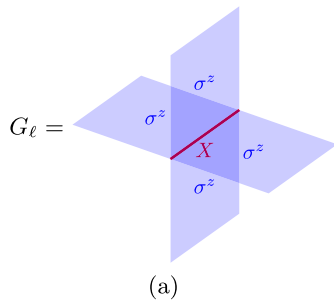


FIG. 3. The Gauss law and the local terms of the gauged Hamiltonian Eq. (4.9). (a) The Gauss law operator G_ℓ for the 2-form gauge field on a link, (b) the minimal coupling term on a plaquette [first term of Eq. (4.9)], and (c) the flux term of the 2-form gauge field on a cube [third term of Eq. (4.9)]. The second term of Eq. (4.9) is the same as Fig. 2(b).

subject to the Gauss law constraint for the 2-form gauge fields,

$$G_\ell = X_\ell \prod_{p \ni \ell} \sigma_p^z = 1. \quad (4.10)$$

See Fig. 3 for an illustration of these terms.

Next, we define a new set of gauge-invariant local operators that commute with the Gauss law G_ℓ ,

$$\hat{X}_p = \sigma_p^x \prod_{\ell \in p} Z_\ell, \quad \hat{Z}_p = \sigma_p^z. \quad (4.11)$$

The gauged Hamiltonian written in terms of the new variables is

$$\hat{H} = -J \sum_p \hat{X}_p - h \sum_\ell \prod_{p \ni \ell} \hat{Z}_p - \hat{g} \sum_c \prod_{p \in c} \hat{X}_p. \quad (4.12)$$

(See Fig. 4 for an illustration of these terms.) If we dualize the lattice, this becomes the Hamiltonian of a lattice $\mathbb{Z}_2$ gauge-field theory. The last term is interpreted as the Gauss law for the hatted gauge fields imposed dynamically. The gauged Hamiltonian has a dual, nontopological 1-form symmetry, generated by

$$\hat{\eta}(\Sigma) = \prod_{p \in \Sigma} \hat{X}_p, \quad (4.13)$$

where Σ is a closed two-cycle of the original lattice. Note that this dual symmetry exists whether we impose the Gauss law Eq. (4.1) for the 1-form gauge field exactly or energetically.

Finally, we consider the limit $\hat{g} \rightarrow \infty$, which sets the 2-form gauge fields to be topological. Furthermore, the dual 1-form symmetry $\hat{\eta}$ becomes topological, i.e., it depends on Σ only through its homological class. We refer to the gauging procedure with $\hat{g} \rightarrow \infty$ as the “topological gauging” of the 1-form symmetry.

In this limit, we retrieve the original Hamiltonian on the dual lattice before gauging with $J \leftrightarrow h$. We conclude that

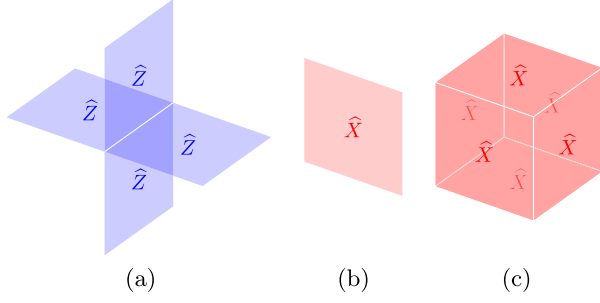


FIG. 4. The local terms and the Gauss law of the gauged Hamiltonian Eq. (4.9) in the new variables. (a) The dual magnetic flux term on a link [second term of Eq. (4.12)], (b) the dual electric-field term on a plaquette [first term of Eq. (4.12)], and (c) the dual Gauss law operator on a cube [third term of Eq. (4.12)].

the $\mathbb{Z}_2$ lattice gauge theory is self-dual under the topological gauging when $J = h$. This is the Wegner duality for the 3 + 1D $\mathbb{Z}_2$ lattice gauge theory [48], generalizing the Kramers-Wannier duality for the 1 + 1D Ising lattice model.

To summarize, we see that the topological gauging of the 1-form symmetry maps the local terms in the Hamiltonian as

$$X_\ell \rightsquigarrow \prod_{p \ni \ell} \hat{Z}_p, \quad \prod_{\ell \in p} Z_\ell \rightsquigarrow \hat{X}_p, \quad (4.14)$$

which is the Wegner duality transformation.

V. NONINVERTIBLE WEGNER OPERATOR IN 3 + 1D

When $J = h$, the Hamiltonian Eq. (4.2) for the lattice $\mathbb{Z}_2$ gauge theory

$$\tilde{H} = -J \left(\sum_p \prod_{\ell \in p} Z_\ell + \sum_\ell X_\ell \right), \quad (5.1)$$

is invariant under the Wegner duality transformation Eq. (4.14),

$$X_\ell \rightsquigarrow \prod_{\ell' \in \mathbf{t}(\ell)} Z_{\ell'}, \quad \prod_{\ell \in p} Z_\ell \rightsquigarrow X_{\mathbf{t}(p)}, \quad (5.2)$$

up to a half lattice translation $\mathbf{t}$,

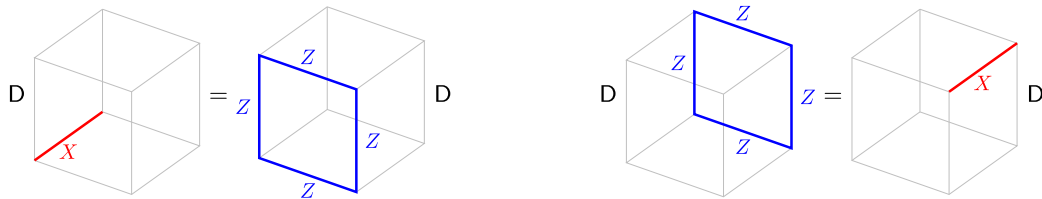


FIG. 5. Action of the duality operator D on the terms of the Hamiltonian $\tilde{H}$.

$$\mathbf{t}(x, y, z) = \left(x + \frac{1}{2}, y + \frac{1}{2}, z + \frac{1}{2} \right), \quad (5.3)$$

where $(x, y, z) \in (\mathbb{Z}/L_x\mathbb{Z}, \mathbb{Z}/L_y\mathbb{Z}, \mathbb{Z}/L_z\mathbb{Z})$ labels a site on the lattice, and so on.

However, this transformation Eq. (5.2) cannot be implemented by an invertible operator on a closed periodic torus. Suppose there were such an invertible operator U such that $UX_\ell U^{-1} = \prod_{\ell' \in \mathbf{t}(\ell)} Z_{\ell'}$. Then consider the conjugation of the 1-form symmetry operator $\eta(\hat{\Sigma})$ by U ,

$$U\eta(\hat{\Sigma})U^{-1} = U \left(\prod_{\ell \in \hat{\Sigma}} X_\ell \right) U^{-1} = \prod_{\ell \in \hat{\Sigma}} \prod_{\ell' \in \mathbf{t}(\ell)} Z_{\ell'} = 1. \quad (5.4)$$

This implies the 1-form symmetry operator η is trivial, which is a contradiction.

Rather, the transformation Eq. (5.2) is implemented by a noninvertible operator D in the following sense:

$$DX_\ell = \left(\prod_{\ell' \in \mathbf{t}(\ell)} Z_{\ell'} \right) D, \quad D \prod_{\ell \in p} Z_\ell = X_{\mathbf{t}(p)} D. \quad (5.5)$$

See Fig. 5 for an illustration of the action of D on the terms of the Hamiltonian Eq. (4.2). It follows from Eq. (5.5) that

$$D\eta(\hat{\Sigma}) = \eta(\hat{\Sigma})D = D, \quad (5.6)$$

which, in particular, implies $DG_s = G_s D = D$. This means that the operator D acts within the constrained Hilbert space $\tilde{\mathcal{H}}$, i.e., the subspace of $\mathcal{H}$ with $G_s = 1$.

From these relations, for $J = h$, we see that this operator D commutes with the Hamiltonian H in Eq. (4.5),

$$H = -J \left(\sum_p \prod_{\ell \in p} Z_\ell + \sum_\ell X_\ell \right) - g \sum_s \prod_{\ell \ni s} X_\ell, \quad (5.7)$$

even when the Gauss law is only imposed energetically. That is, $DH = HD$. We will henceforth view D as an operator acting on the tensor product Hilbert space $\mathcal{H} = \otimes_\ell \mathcal{H}_\ell$.

Moreover, we have

$$D^2 X_\ell = X_{t^2(\ell)} D^2, \quad D^2 \prod_{\ell \in p} Z_\ell = \left(\prod_{\ell \in t^2(p)} Z_\ell \right) D^2. \quad (5.8)$$

The algebra in Eqs. (5.6) and (5.8) suggests that D^2 is proportional to a projection onto the Gauss-law-preserving, $\mathbb{Z}_2^{(1)}$ -symmetric states times $T_{1,1,1}$, which is the generator of translation in the (1, 1, 1) direction.

In the following subsections, we will make the last statement more precise. Specifically, we give an explicit realization of the duality operator D in terms of ZX-diagrams and compute its fusion algebra with itself and other operators using ZX-calculus.

A. Noninvertible duality symmetry as a tensor network operator

In this subsection, we give a ZX-diagrammatic representation of the duality operator D . Our operator D in the 3 + 1D Hamiltonian lattice gauge theory is the counterpart of the duality defect for the 4D Euclidean lattice gauge theory of Ref. [52].

First, we define an auxiliary Hilbert space on the plaquettes $\mathcal{H}_p$ in the same way as the Hilbert space on the links defined above, which we now denote as $\mathcal{H}_l$ for clarity. Consider the following maps between $\mathcal{H}_l$ and $\mathcal{H}_p$:

$$D_{p \leftarrow l} := 2^{4V} \quad (5.9)$$

where $V = L_x L_y L_z$ is the number of unit cells in the lattice, the ingoing (outgoing) arrows represent inputs (outputs), the thick gray lines represent the links of the cubic lattice (in particular, the green dots are not the sites of the lattice),

and the diagram is repeated and glued in a periodic way in the three spatial directions. These maps have been considered in Ref. [101].

We also define the half-translation map $T_{p \leftarrow l}$ which maps a qubit in $\mathcal{H}_l$ on the link ℓ to a qubit in $\mathcal{H}_p$ on the plaquette $t(\ell)$. The other half-translation map $T_{l \leftarrow p}$ is defined similarly.

With these preparations, the duality operator is defined as

$$D := T_{l \leftarrow p} D_{p \leftarrow l} = D_{l \leftarrow p} T_{p \leftarrow l}. \quad (5.10)$$

We can also write the translation generator in the (1, 1, 1) direction as

$$T_{1,1,1} = T_{l \leftarrow p} T_{p \leftarrow l}. \quad (5.11)$$

Similarly, there are auxiliary versions of these operators that act on $\mathcal{H}_p$,

$$\hat{T}_{1,1,1} = T_{p \leftarrow l} T_{l \leftarrow p}, \quad \hat{D} := T_{p \leftarrow l} D_{l \leftarrow p} = D_{p \leftarrow l} T_{l \leftarrow p}. \quad (5.12)$$

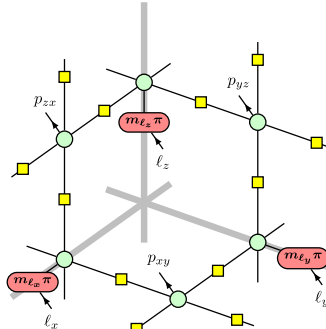
Equation (5.10) expresses the Wegner duality operator as a tensor network operator, also known as the projected entangled pair operator [56]. This is a higher-dimensional generalization of the MPO presentation of the 1 + 1D Kramers-Wannier operator [102].

B. Action on operators

Let us show that D defined in Eq. (5.10) is indeed the duality operator, i.e., it satisfies Eq. (5.5), using ZX-calculus. It turns out to be more convenient to consider the action of D on a more general operator,

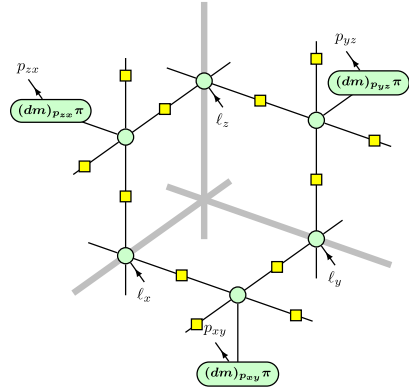
$$\prod_{\ell} X_{\ell}^{m_{\ell}} = \quad , \quad m_{\ell} \in \{0, 1\}. \quad (5.13)$$

We have



$$D_{p \leftarrow 1} \prod_{\ell} X_{\ell}^{m_{\ell}} = 2^{4V} \quad (5.14)$$

Using π commutation through the green dots on the links or identity removal, followed by CC', and finally SF through the green dots on the plaquettes, we get



$$D_{p \leftarrow 1} \prod_{\ell} X_{\ell}^{m_{\ell}} = 2^{4V} = \prod_p \hat{Z}_p^{(dm)_p} D_{p \leftarrow 1}, \quad (5.15)$$

where $(dm)_p := \sum_{\ell \in p} m_{\ell}$ is the sum of m_{ℓ} over the links ℓ around the plaquette p and $\hat{Z}_p$ is a Pauli operator on the auxiliary Hilbert space $\mathcal{H}_p$. Multiplying $T_{1 \leftarrow p}$ on the left on both sides of this equation, we get

$$\begin{aligned} D \prod_{\ell} X_{\ell}^{m_{\ell}} &= T_{1 \leftarrow p} \prod_p \hat{Z}_p^{(dm)_p} D_{p \leftarrow 1} \\ &= \prod_{\ell'} Z_{\ell'}^{(dm)_{t^{-1}(\ell')}} D \\ &= \prod_{\ell} \left(\prod_{\ell' \in t(\ell)} Z_{\ell'} \right)^{m_{\ell}} D, \end{aligned} \quad (5.16)$$

where, in the second equality, we used $T_{1 \leftarrow p} \hat{Z}_p = Z_{t(p)} T_{1 \leftarrow p}$, and in the last equality, we used $\ell \in t^{-1}(\ell') \Leftrightarrow \ell' \in t(\ell)$ [104]. This, in particular, implies the first equation of Eq. (5.5). Similarly, one can show that $D \prod_{\ell \in p} Z_{\ell} = X_{t(p)} D$ or, more generally,

$$D \prod_p \left(\prod_{\ell \in p} Z_{\ell} \right)^{\hat{m}_p} = \prod_p X_{t(p)}^{\hat{m}_p} D. \quad (5.17)$$

Therefore, D is indeed the duality operator that satisfies Eq. (5.5).

Let us look at some special cases of the action of D .

- (i) Let γ be a closed contractible curve bounding the open surface Σ . In the special case where $\hat{m}_p = 1$ for $p \in \Sigma$ and 0 otherwise, Eq. (5.17) reduces to

$$D \prod_{\ell \in \gamma} Z_{\ell} = \prod_{\ell \in t(\Sigma)} X_{\ell} D, \quad (5.18)$$

where $t(\Sigma)$ is an open dual surface with boundary $t(\gamma)$. This is illustrated in Fig. 6. We recognize the operator on the left-hand side as the Wilson line operator along the curve γ [105],

$$W(\gamma) := \prod_{\ell \in \gamma} Z_{\ell}, \quad (5.19)$$

whereas the operator on the right-hand side is a truncated $\mathbb{Z}_2$ 1-form symmetry operator along the open dual surface $t(\Sigma)$ with boundary $t(\gamma)$. This equation is, therefore, the lattice counterpart of the continuum result in Refs. [53,106]. In particular, see Fig. 1 of Ref. [106].

- (ii) In the special case where m_{ℓ} is such that $(dm)_p = 0 \bmod 2$ for all p , i.e., if m_{ℓ} is a flat $\mathbb{Z}_2$ gauge-field configuration, then Eq. (5.16) reduces to $D \prod_{\ell} X_{\ell}^{m_{\ell}} = D$. This is indeed the case for G_s and $\eta(\hat{\Sigma})$, so we have $D G_s = D \eta(\hat{\Sigma}) = D$. Similarly, one can show that $G_s D = \eta(\hat{\Sigma}) D = D$. These reproduce the algebra in Eq. (5.6) using ZX-calculus.

C. Operator algebra and the condensation operator

Here, we summarize the operator algebra of the Wegner duality operator D and the 1-form symmetry operator η . The most interesting one is

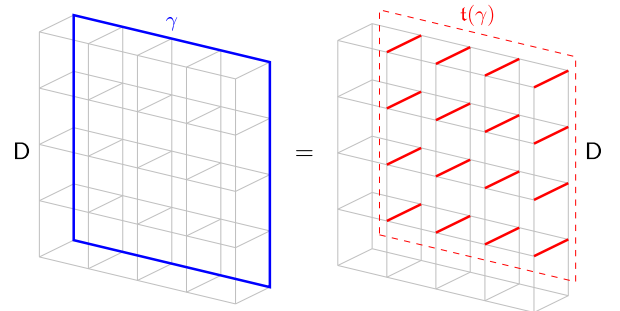


FIG. 6. The duality operator D maps a (contractible) Wilson loop along γ to a truncated $\mathbb{Z}_2$ 1-form symmetry operator on the open dual surface $t(\Sigma)$ with boundary $t(\gamma)$. Here, the blue (red) links correspond to the Pauli Z (X) operators.

$$\mathbf{D}^2 = \frac{1}{2^V} T_{1,1,1} \sum_{\hat{\Sigma}} \eta(\hat{\Sigma}), \quad (5.20)$$

where the sum is over all (contractible and noncontractible, and not necessarily connected) two-cycles $\hat{\Sigma}$ on the dual lattice. This is the 3 + 1D generalization of $\mathbf{D}^2 = (1 + \eta)T$ Eq. (3.16) found in Refs. [14,15]. We derive this algebra in Appendix A 2 using ZX-calculus.

The other equations are [107]

$$\begin{aligned} \mathbf{D}^\dagger &= \mathbf{D} T_{1,1,1}^{-1} = T_{1,1,1}^{-1} \mathbf{D}, \\ \eta(\hat{\Sigma})^2 &= 1, \quad \eta(\hat{\Sigma}) \mathbf{D} = \mathbf{D} \eta(\hat{\Sigma}) = \mathbf{D}. \end{aligned} \quad (5.21)$$

It is useful to define the condensation operator

$$\mathbf{C} = \frac{1}{2^V} \sum_{\hat{\Sigma}} \eta(\hat{\Sigma}). \quad (5.22)$$

We can decompose the sum over all two-cycles into a sum over the noncontractible two-cycles times a sum over all contractible surfaces,

$$\frac{1}{2^V} \sum_{\hat{\Sigma}} \eta(\hat{\Sigma}) = \frac{1}{2} \prod_{i < j} (1 + \eta_{ij}) \prod_{s \neq s_0} \left(\frac{1 + G_s}{2} \right), \quad (5.23)$$

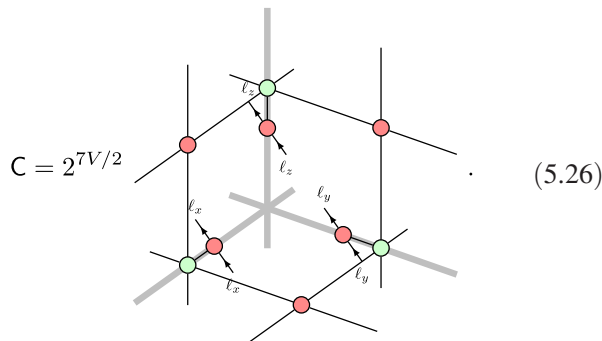
where s_0 is an arbitrary site and $G_s = \prod_{\ell \ni s} X_\ell$. Here, η_{ij} is the 1-form symmetry operator on the ij plane, defined as

$$\eta_{ij} = \eta(\hat{\Sigma}_{ij}) = \prod_{\ell \in \hat{\Sigma}_{ij}} X_\ell. \quad (5.24)$$

Using $\prod_s G_s = 1$, we can include the factor $(1 + G_{s_0})/2$ for free to obtain

$$\mathbf{C} = \frac{1}{2} (1 + \eta_{xy})(1 + \eta_{yz})(1 + \eta_{zx}) \prod_s \left(\frac{1 + G_s}{2} \right). \quad (5.25)$$

In Appendix A 1, we further show that $\mathbf{C}$ can be diagrammatically represented as



$$\mathbf{C} = 2^{7V/2} \quad (5.26)$$

In terms of the condensation operator, Eq. (5.20) can be rewritten as

$$\mathbf{D}^2 = \mathbf{C} T_{1,1,1}. \quad (5.27)$$

Using Eqs. (5.6) and (5.25), we also find

$$\mathbf{C}^2 = 4\mathbf{C}, \quad \mathbf{D}\mathbf{C} = \mathbf{C}\mathbf{D} = 4\mathbf{D}. \quad (5.28)$$

Furthermore,

$$\mathbf{C} \eta(\hat{\Sigma}) = \eta(\hat{\Sigma}) \mathbf{C} = \mathbf{C} \quad (5.29)$$

follows immediately from writing $\mathbf{C} = \mathbf{D}^2 T_{1,1,1}^{-1}$ and Eq. (5.6). (Here, we have used the fact that $T_{1,1,1}$ commutes with all the operators.) This, in particular, implies $\mathbf{C} G_s = G_s \mathbf{C} = \mathbf{C}$.

Let us now compare the lattice algebra with the continuum one. In the thermodynamic limit, $T_{1,1,1} \sim 1$ becomes the identity operator on the low-lying states, and Eq. (5.20) reduces to the algebra of the duality operator $\mathcal{D}(M)$ in continuum field theory [53,54,108,109]

$$\mathcal{D}(M)^2 = \frac{1}{|H^0(M; \mathbb{Z}_2)|} \sum_{\Sigma \in H_2(M; \mathbb{Z}_2)} \eta(\Sigma). \quad (5.30)$$

Here, M is the spatial three-manifold, which in our case is T^3 .

Next, we compare the condensation operators in the continuum and on the lattice. When restricted to the constrained Hilbert space $\mathcal{H}$ where the Gauss law is strictly enforced $G_s = 1$, the lattice condensation operator Eq. (5.25) reduces to $\frac{1}{2}(1 + \eta_{xy})(1 + \eta_{yz})(1 + \eta_{zx})$. This agrees with the definition of the condensation operator $\mathcal{C}(M)$ in continuum field theory [108] (see also Ref. [110]),

$$\mathcal{C}(M) = \frac{1}{|H^0(M; \mathbb{Z}_2)|} \sum_{\Sigma \in H_2(M; \mathbb{Z}_2)} \eta(\Sigma). \quad (5.31)$$

Finally, the full algebra of the condensation and duality operators in the continuum is [108]

$$\begin{aligned} \mathcal{D}(M)^2 &= \mathcal{C}(M), \quad \mathcal{C}(M)^2 = (\mathcal{Z}_2)_0 \mathcal{C}(M), \\ \mathcal{D}(M) \times \mathcal{C}(M) &= \mathcal{C}(M) \times \mathcal{D}(M) = (\mathcal{Z}_2)_0 \mathcal{D}(M), \end{aligned} \quad (5.32)$$

where $(\mathcal{Z}_2)_0$ stands for the partition function of the untwisted 2 + 1D $\mathbb{Z}_2$ gauge theory (i.e., the low-energy limit of the 2 + 1D toric code) on M . In our case, $M = T^3$ and $(\mathcal{Z}_2)_0 = 4$, reflecting the four anyon lines. This algebra agrees with the lattice operator algebra in Eqs. (5.27) and (5.28) in the thermodynamic limit, where the lattice translation $T_{1,1,1} \sim 1$ becomes the identity operator on the low-lying states. See Ref. [7] for a review of the duality and condensation operators and defects in continuum field theory.

D. Lattice symmetries

The presence of the lattice translation $T_{1,1,1}$ in Eq. (5.20) leads to an interesting action of lattice symmetries on the duality operator $\mathbf{D}$. The ZX-diagram of $\mathbf{D}$ makes the translation invariance manifest, i.e.,

$$\mathbf{D}T_a = T_a\mathbf{D}, \quad a \in \mathbb{Z}^3. \quad (5.33)$$

Similarly, $\mathbf{C}$ is translationally invariant.

The remaining lattice symmetries correspond to the point group of the cubic lattice [111], which is the octahedral group, denoted as O_h . It is isomorphic to $S_4 \times \mathbb{Z}_2$, where S_4 is associated with orientation-preserving symmetries (rotations) and $\mathbb{Z}_2$ is associated with the reflection (parity). It is easy to see that the ZX-diagrams of the duality maps $\mathbf{D}_{p \leftarrow 1}$ and $\mathbf{D}_{1 \leftarrow p}$ and the condensation operator $\mathbf{C}$ are invariant under rotations and reflections. So the nontrivial action on $\mathbf{D}$ comes from the transformation of the half-translation maps $T_{p \leftarrow 1}$ and $T_{1 \leftarrow p}$.

The parity operator P acts as

$$PT_aP = T_a^\dagger = T_{-a}, \quad P\mathbf{D}P = \mathbf{D}T_{1,1,1}^{-1} = \mathbf{D}^\dagger, \quad (5.34)$$

where the latter follows from the fact that the half-translation map $T_{p \leftarrow 1}$, associated with the half-translation $\mathbf{t}$, transforms into $(T_{1 \leftarrow p})^\dagger = T_{p \leftarrow 1}^{-1}T_{1,1,1}^{-1}$, associated with $\mathbf{t}^{-1}$, under parity $(x, y, z) \mapsto (-x, -y, -z)$. A similar statement holds for the other half-translation map $T_{1 \leftarrow p}$.

Let us consider rotations now. It is well known that S_4 acts freely and transitively on the space diagonals of a cube. It follows that, given a rotation R , there is a unique $\epsilon = (\epsilon_x, \epsilon_y, \epsilon_z) \in \{\pm 1\}^3$ such that $RT_{1,1,1}R^{-1} = T_\epsilon$ [112]. We define $\mathbf{D}_\epsilon := R\mathbf{D}R^{-1}$; in particular, $\mathbf{D}_{1,1,1} = \mathbf{D}$. Then,

$$\mathbf{D}_\epsilon^2 = R\mathbf{D}^2R^{-1} = RT_{1,1,1}\mathbf{C}R^{-1} = T_\epsilon\mathbf{C}, \quad (5.35)$$

where we used the fact that the ZX-diagram of $\mathbf{C}$ is rotationally invariant. In fact, $\mathbf{D}_\epsilon$ is related to $\mathbf{D}$ by a translation [113],

$$\mathbf{D}_\epsilon = \mathbf{D}T_{\frac{1}{2}(\epsilon-1)}, \quad (5.36)$$

where $\mathbf{1} = (1, 1, 1)$. In particular, $\mathbf{D}_{-1,-1,-1} = \mathbf{D}^\dagger$.

Consider the special case where the rotation R fixes the space diagonal along $(1, 1, 1)$ —such rotations form the S_3 subgroup of S_4 . This can happen in two ways:

- (1) R is the $2\pi/3$ rotation about the $(1, 1, 1)$ direction, which generates the $\mathbb{Z}_3$ subgroup of the stabilizer group S_3 . Then, $RT_{1,1,1}R^{-1} = T_{1,1,1}$ and $R\mathbf{D}R^{-1} = \mathbf{D}$.
- (2) R is the π rotation about $(1, -1, 0)$, $(0, 1, -1)$, or $(-1, 0, 1)$ direction. Then, $RT_{1,1,1}R^{-1} = T_{-1,-1,-1}$ and $R\mathbf{D}R^{-1} = \mathbf{D}^\dagger$.

Note that $\mathbf{D}$ is mapped to $\mathbf{D}^\dagger$ by both the parity P , which is an orientation-reversing operation, and the rotation R of the

second type above, which is an orientation-preserving operation. This is the lattice counterpart of the continuum fact that $\mathcal{D}(M)$ is the same as its orientation reversal.

E. Action on the toric code ground states

The phase diagram of the lattice $\mathbb{Z}_2$ gauge theory Eq. (4.2) is well understood [114]: The $\mathbb{Z}_2$ 1-form symmetry is spontaneously broken (Higgs phase) for $J > h$, whereas it is preserved (confining phase) for $J < h$. The broken phase is captured by the 3 + 1D topological $\mathbb{Z}_2$ gauge theory at low energies. At the self-dual point $J = h$, there is a first-order phase transition.

Let us say a few words about the ground states in these phases. In the limit $J \gg h$ (or when $h = 0$), the magnetic flux term dominates. On a cubic lattice with periodic boundary conditions, this leads to $2^3 = 8$ ground states,

$$|\xi\rangle := 2^{(V-1)/2} \prod_{i < j} \eta_{ij}^{\xi_{ij}} \prod_s \left(\frac{1 + G_s}{2} \right) |0 \dots 0\rangle, \quad (5.37)$$

where $\xi = (\xi_{xy}, \xi_{yz}, \xi_{zx}) \in \{0, 1\}^3$ with ξ_{ij} defined modulo 2. We recognize that $|\xi\rangle$ are the ground states of the 3 + 1D toric code [115]. On the other hand, in the limit $J \ll h$ (or when $J = 0$), the electric-field term dominates, so there is a unique ground state,

$$|+\dots+\rangle. \quad (5.38)$$

At the self-dual point $J = h$, there are nine ground states in the thermodynamic limit. These ground states are not the above nine states, but are related to them by a finite depth local unitary. Incidentally, these eight toric code ground states $|\xi\rangle$ and the product state $|+\dots+\rangle$ are the exact ground states of the Hamiltonian in Eq. (5.45) at $\lambda = 1$ discussed below.

Irrespective of the model, one can ask how the symmetry operators η_{ij} and $\mathbf{D}$ act on these ground states. Clearly, the 1-form symmetries η_{ij} permute the toric code ground states among themselves and leave the product state invariant,

$$\prod_{i < j} \eta_{ij}^{\xi'_{ij}} |\xi\rangle = |\xi + \xi'\rangle, \quad \eta_{ij} |+\dots+\rangle = |+\dots+\rangle. \quad (5.39)$$

More interestingly, the noninvertible symmetry exchanges the toric code ground states with the product state,

$$\mathbf{D}|\xi\rangle = \frac{1}{\sqrt{2}} |+\dots+\rangle, \quad \mathbf{D}|+\dots+\rangle = \frac{1}{\sqrt{2}} \sum_{\xi \in \{0,1\}^3} |\xi\rangle. \quad (5.40)$$

We interpret the above equations as the spontaneous breaking of the noninvertible Wegner symmetry $\mathbf{D}$ and the 1-form symmetry η . We derive Eq. (5.40) using ZX-calculus in Appendix A 3.

The basis in Eqs. (5.37) and (5.38) corresponds to the ground states in the nine superselection sectors of the model in Eq. (5.45) in the thermodynamic limit. Alternatively, for $\zeta = (\zeta_{xy}, \zeta_{yz}, \zeta_{zx}) \in \{0, 1\}^3$, we define

$$|\zeta\rangle := \frac{1}{2\sqrt{2}} \sum_{\xi \in \{0,1\}^3} (-1)^{\sum_{i<j} \zeta_{ij} \xi_{ij}} |\xi\rangle. \quad (5.41)$$

Then, the basis that diagonalizes the symmetry operators can be written as [116]

$$\begin{aligned} \frac{1}{\sqrt{2}} |+\cdots+\rangle \pm \frac{1}{\sqrt{2}} |\zeta=0\rangle, \quad D = \pm 2, \quad \eta_{ij} = 1, \\ |\zeta \neq 0\rangle, \quad D = 0, \quad \eta_{ij} = (-1)^{\zeta_{ij}}. \end{aligned} \quad (5.42)$$

See Appendix A 4 for more discussions on the two bases $|\xi\rangle$ and $|\zeta\rangle$ for the toric code ground states.

Using Eq. (5.25), we see that the toric code ground state $|\zeta=0\rangle$ can be obtained by acting the condensation operator on a product state [117],

$$|\zeta=0\rangle = \frac{2^{V/2}}{2} C |0\dots 0\rangle. \quad (5.43)$$

This generalizes the relation $|\text{GHZ}^+\rangle = \frac{1}{\sqrt{2}} (1 + \eta) |0\dots 0\rangle$ in 1 + 1D. The duality operator D exchanges this toric code ground state $|\zeta=0\rangle$ and the product state $|+\cdots+\rangle$,

$$D|\zeta=0\rangle = 2|+\cdots+\rangle, \quad D|+\cdots+\rangle = 2|\zeta=0\rangle. \quad (5.44)$$

This is the 3 + 1D analog of the relation Eq. (3.35) in 1 + 1D.

F. LSM-type constraint and duality-preserving deformations

Here, we argue that, in the thermodynamic limit $L_i \rightarrow \infty$, any Hamiltonian that commutes with the noninvertible operator D cannot be in a trivially gapped phase with a unique ground state without long-range entanglement. It is either gapless, gapped with multiple superselection sectors, or gapped with topological order.

This constraint is reminiscent of the LSM theorem [118] and is a 3 + 1D generalization of the constraint for the 1 + 1D Kramers-Wannier symmetry in Refs. [15, 19]. Our lattice argument largely follows from the corresponding theorem in the 3 + 1D continuum field theory proven in Ref. [53], which was interpreted as a 't Hooft anomaly of the noninvertible global symmetry [119][127–129].

Our setup is a 3D spatial cubic lattice with a qubit on each link. We assume that the Gauss law $G_s = \prod_{\ell \ni s} X_\ell = 1$ is enforced strictly as an operator equation on every site, so that the total Hilbert space is the constrained Hilbert space $\tilde{\mathcal{H}}$, which is a subspace of the tensor product Hilbert space $\mathcal{H}$. We consider a general translationally invariant

Hamiltonian $\tilde{H}$ on $\tilde{\mathcal{H}}$ that is invariant under the topological $\mathbb{Z}_2$ 1-form symmetry $\eta(\hat{\Sigma}) = \prod_{\ell \in \hat{\Sigma}} X_\ell$.

We, furthermore, assume that the Hamiltonian commutes with the noninvertible operator D , i.e., $D\tilde{H} = \tilde{H}D$. Up to a lattice translation, D implements the gauging of $\mathbb{Z}_2^{(1)}$. [See Eqs. (5.5) and (4.14).] Therefore, any Hamiltonian that commutes with D must be invariant under gauging the $\mathbb{Z}_2^{(1)}$ symmetry.

As a concrete example, consider a one-parameter deformation of the lattice $\mathbb{Z}_2$ gauge theory Hamiltonian Eq. (5.1) [130],

$$\tilde{H}_\lambda = -J \left(\sum_p \prod_{\ell \in p} Z_\ell + \sum_\ell X_\ell - \frac{\lambda}{8} \sum_{\ell, p: \ell \perp p} X_\ell \prod_{\ell' \in p} Z_{\ell'} \right), \quad (5.45)$$

where $\ell \perp p$ (or $p \perp \ell$) means the link ℓ is orthogonal to the plaquette p and they meet at a site. See Fig. 7 for an illustration of this deformation. This deformation is analogous to the deformations $X_{i-1}Z_iZ_{i+1}$ and $Z_{i-1}Z_iX_{i+1}$ in the 1 + 1D model Eq. (3.31) in Ref. [82]. For any λ , this deformed Hamiltonian preserves the $\mathbb{Z}_2$ 1-form symmetry and the Wegner duality symmetry D [131]. It will be shown in Ref. [130] that, when $\lambda = 1$, the deformed model has an exact ninefold ground-state degeneracy even in finite volume.

We are now ready to argue for the main statement. Let us assume that the thermodynamic limit of this Hamiltonian is trivially gapped with no long-range entanglement. Then, in the low-energy limit it must be described by a bosonic invertible-field theory for the topological phase protected by a $\mathbb{Z}_2^{(1)}$ global symmetry. However, it was shown in Ref. [53] that there is no such $\mathbb{Z}_2^{(1)}$ SPT phase that is invariant under gauging the $\mathbb{Z}_2^{(1)}$ symmetry [132]. This proves that the Hamiltonian cannot be in a trivially gapped phase.

What are the gapped phases that are compatible with the noninvertible Wegner symmetry D ? A trivially gapped

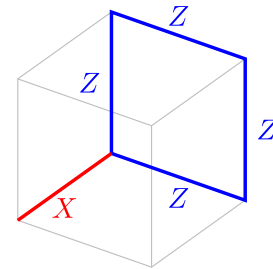


FIG. 7. The deformation term in $\tilde{H}_\lambda$ Eq. (5.45). As indicated, the red (blue) links represent the Pauli X (Z) operators on links. We add such a term for every link and plaquette that are orthogonal to each other and meet at a site. For example, each cube contains 24 such terms.

phase is exchanged with a deconfined $\mathbb{Z}_2$ gauge theory phase under gauging $\mathbb{Z}_2^{(1)}$; therefore, a first-order transition point of these two phases is one such example. This phase is realized at the self-dual point $J = h$ for the lattice $\mathbb{Z}_2$ gauge theory Eq. (4.2), and it has $1 + 8$ superselection sectors on a spatial three-torus in the infinite volume limit. It is also realized in the deformed Hamiltonian Eq. (5.45) at $\lambda = 1$, where these nine states are the exact ground states even in finite volume. Finally, in the continuum, this phase is realized by the pure $\text{SO}(3)$ Yang-Mills gauge theory with $\mathcal{N} = 1$ supersymmetry [135], where the anomalous R symmetry is resurrected as a noninvertible symmetry [54, 136].

There can be more general gapped phases compatible with $\mathcal{D}$, which we leave for future investigations.

VI. NONINVERTIBLE SYMMETRIES BASED ON GRAPHS

In this section, we extend the discussion of noninvertible duality operator to a large class of lattice models based on arbitrary bipartite graphs. We refer to them as generalized transverse-field Ising models due to their resemblance to the usual transverse-field Ising model. (See Refs. [48, 137–144] for related usages of the term “generalized Ising model.” See also Ref. [145] for duality maps between different models.) We note that, in the absence of a transverse-field term, these models reduce to the classical low-density parity-check codes. We give the condition under which there is a duality symmetry and derive the condition for which the duality operator does not mix with spatial symmetry. We also discuss several well-known lattice models as examples of this general model.

A. Hamiltonian of the generalized transverse-field Ising model

We consider an arbitrary tensor product Hilbert space of qubits. Denote the set of all qubits V such that the total Hilbert space is $\mathcal{H} := \bigotimes_{v \in V} \mathbb{C}^2$ [146]. Note that, for the current discussion, we do not need to make any assumption yet about geometric locality.

Consider an additional set $\hat{V}$ and define a bipartite graph $\mathcal{G}$ on $\hat{V} \sqcup V$, with edge set $E \subseteq \hat{V} \times V$. Without loss of generality, we assume that $\mathcal{G}$ is connected. Its adjacency matrix takes the form

$$\begin{pmatrix} 0_{|\hat{V}| \times |\hat{V}|} & \sigma_{|\hat{V}| \times |V|} \\ \sigma_{|V| \times |\hat{V}|}^\top & 0_{|V| \times |V|} \end{pmatrix}, \quad (6.1)$$

where σ is called the biadjacency matrix [147], given by $\sigma_{\hat{v}v} = 1$ if $(\hat{v}, v) \in E$ and 0 otherwise, and its incidence matrix takes the form

$$\begin{pmatrix} \hat{\tau}_{|\hat{V}| \times |E|} \\ \tau_{|V| \times |E|} \end{pmatrix}, \quad (6.2)$$

where $\hat{\tau}_{\hat{v}e} = 1$ if e is incident to $\hat{v}$ and 0 otherwise, and τ is defined similarly. Note that

$$\sigma = \hat{\tau} \tau^\top. \quad (6.3)$$

Moreover, the entries in all these matrices are 0 or 1.

We may use σ to construct a generalized transverse-field Ising Hamiltonian as follows. First, we define generalized Ising terms

$$B_{\hat{v}} = \prod_{v \in V} Z_v^{\sigma_{\hat{v}v}}, \quad (6.4)$$

for $\hat{v} \in \hat{V}$ (i.e., for each row $\hat{v}$ of σ , the positions of the Pauli Z 's are given by the entries in that row). Then, we can define the Hamiltonian

$$H = -J \sum_{\hat{v} \in \hat{V}} B_{\hat{v}} - h \sum_{v \in V} X_v. \quad (6.5)$$

To summarize, an element of V represents a physical qubit, while an element of $\hat{V}$ represents a generalized Ising interaction term.

The invertible symmetries of this Hamiltonian are $\mathbb{Z}_2$ symmetries. Let us define the operator

$$\eta_a = \prod_{v \in V} X_v^{a_v}, \quad (6.6)$$

for a vector a with components $a_v \in \{0, 1\}$ and see what choices of a give symmetries of this Hamiltonian. For this, we only need to check whether it commutes with the Ising terms. The commutation of a Z -type and an X -type Pauli operator is $\prod_w Z_w^{b_w} \prod_v X_v^{a_v} = (-1)^{\sum_v a_v b_v} \prod_w Z_w^{b_w} \prod_v X_v^{a_v}$. Therefore, the commutation between $B_{\hat{v}}$ and η_a is $(-1)^{\sum_v \sigma_{\hat{v}v} a_v}$, and we see that η_a is a symmetry of the Hamiltonian exactly when $a \in \ker \sigma$. To conclude, the total invertible symmetry group is $\mathbb{Z}_2^{\dim \ker \sigma}$.

B. Spatial symmetries from preserving automorphisms

A spatial symmetry of the generalized TFIM on the bipartite graph $\mathcal{G}$ is associated with a preserving automorphism π of $\mathcal{G}$ [148], which satisfies

$$\sigma_{\pi(\hat{v})\pi(v)} = \sigma_{\hat{v}v}. \quad (6.7)$$

Note that $\pi|_V$ is a permutation of the vertices in V . This spatial symmetry is implemented by a permutation operator T_π that maps the qubit at vertex v to the qubit at vertex $\pi(v)$. In particular,

$$T_\pi X_v T_\pi^{-1} = X_{\pi(v)}, \quad T_\pi Z_v T_\pi^{-1} = Z_{\pi(v)}. \quad (6.8)$$

$$D_{\hat{v} \leftarrow v} H = \hat{H} D_{\hat{v} \leftarrow v}, \quad (6.14)$$

It follows that

$$\begin{aligned} T_\pi B_{\hat{v}} T_\pi^{-1} &= T_\pi \prod_v Z_v^{\sigma_{\hat{v}v}} T_\pi^{-1} = \prod_v Z_{\pi(v)}^{\sigma_{\hat{v}v}} \\ &= \prod_v Z_{\pi(v)}^{\sigma_{\pi(\hat{v})\pi(v)}} = \prod_v Z_v^{\sigma_{\pi(\hat{v})v}} = B_{\pi(\hat{v})}. \end{aligned} \quad (6.9)$$

Therefore, the Hamiltonian commutes with T_π , i.e.,

$$T_\pi H = HT_\pi. \quad (6.10)$$

Since the composition of two preserving automorphisms is also a preserving automorphism, we have

$$T_\pi T_{\pi'} = T_{\pi \circ \pi'}. \quad (6.11)$$

The operator T_π acts on η_a as

$$\begin{aligned} T_\pi \eta_a T_\pi^{-1} &= T_\pi \prod_v X_v^{a_v} T_\pi^{-1} = \prod_v X_{\pi(v)}^{a_v} \\ &= \prod_v X_v^{a_{\pi^{-1}(v)}} = \eta_{\pi a}, \end{aligned} \quad (6.12)$$

where we defined $(^{\pi}a)_v = a_{\pi^{-1}(v)}$. As a check, note that $^{\pi}a \in \ker \sigma$ because $\sum_v \sigma_{\hat{v}v} (^{\pi}a)_v = \sum_v \sigma_{\pi^{-1}(\hat{v})v} a_v = 0$.

C. Noninvertible duality operators from reversing automorphisms

Let us now construct a map $D_{\hat{V} \leftarrow V}: \mathcal{H} \rightarrow \hat{\mathcal{H}}$, where $\hat{\mathcal{H}} := \bigotimes_{\hat{v} \in \hat{V}} \mathbb{C}^2$, as a ZX-diagram where we replace vertices of $\mathcal{G}$ with green spiders, place Hadamards on the edges, and add input legs into V and output legs out of $\hat{V}$. After a CC', the ZX-diagram representing this map is given by

$$D_{\hat{V} \leftarrow V} := 2^\kappa \quad \begin{array}{c} v \in V \quad \hat{v} \in \hat{V} \\ \text{---} \text{green circle} \quad \text{---} \text{red circle} \text{---} \text{yellow square} \text{---} \\ \vdots \quad \sigma \quad \vdots \\ \text{---} \text{green circle} \quad \text{---} \text{red circle} \text{---} \text{yellow square} \text{---} \end{array}, \quad (6.13)$$

where κ is a real number to be fixed later and the middle part of the diagram is understood as follows: For each green dot labeled by $v \in V$ and each red dot labeled by $\hat{v} \in \hat{V}$, the number of wires connecting them is $\sigma_{\hat{v}v}$. One can similarly define the map $D_{V \leftarrow \hat{V}}$, whose ZX-diagram is obtained by flipping the ZX-diagram of $D_{\hat{V} \leftarrow V}$ horizontally, i.e., $D_{V \leftarrow \hat{V}} = (D_{\hat{V} \leftarrow V})^\dagger$.

The map $D_{\hat{v} \leftarrow v}$ implements the gauging of the $\mathbb{Z}_2$ symmetries generated by η_a 's. (Such a map was studied in Refs. [88,91,101,137–140,143,145,149–157].) Indeed, commuting the Hamiltonian H through $D_{\hat{v} \leftarrow v}$ gives the gauged Hamiltonian $\hat{H}$, i.e.,

where

$$\hat{H} = -h \sum_{v \in V} \hat{B}_v - J \sum_{\hat{v} \in \hat{V}} \hat{X}_{\hat{v}}, \quad \hat{B}_v = \prod_{\hat{v} \in \hat{V}} \hat{Z}_{\hat{v}}^{\sigma_{\hat{v}v}}. \quad (6.15)$$

Evidently, the gauged Hamiltonian $\hat{H}$ is also a generalized TFIM, albeit on a different Hilbert space $\hat{\mathcal{H}}$.

So far, we have defined a map between two Hilbert spaces $\mathcal{H}$ and $\hat{\mathcal{H}}$. To turn it into an operator that acts within one Hilbert space, we need to identify these two Hilbert spaces in some way.

More precisely, this is achieved by choosing a reversing automorphism ρ of $\mathcal{G}$ [158], which satisfies

$$\sigma_{\rho(v)\rho(\hat{v})} = \sigma_{\hat{v}v}. \quad (6.16)$$

Define the map $T_{\hat{v} \leftarrow v}^\rho: \mathcal{H} \rightarrow \hat{\mathcal{H}}$ that maps the physical qubit at vertex v to the auxiliary qubit at vertex $\rho(v)$. One can similarly define the map $T_{v \leftarrow \hat{v}}^\rho$. Since ρ is reversing, ρ^2 is preserving, and we have

$$T_{\mathbf{v} \leftarrow \hat{\mathbf{v}}}^\rho T_{\hat{\mathbf{v}} \leftarrow \mathbf{v}}^\rho = T_{\rho^2}. \quad (6.17)$$

More generally, we have

$$T_{\mathbf{v} \leftarrow \hat{\mathbf{v}}}^{\rho} T_{\hat{\mathbf{v}} \leftarrow \mathbf{v}}^{\rho'} = T_{\rho \circ \rho'}, \quad (6.18)$$

which is well defined because the composition of two reversing automorphisms is a preserving automorphism. Similarly, we have

$$T_{\pi} T_{\mathbf{v} \leftarrow \hat{\mathbf{v}}}^{\rho} = T_{\mathbf{v} \leftarrow \hat{\mathbf{v}}}^{\pi \circ \rho}, \quad T_{\hat{\mathbf{v}} \leftarrow \mathbf{v}}^{\rho} T_{\pi} = T_{\hat{\mathbf{v}} \leftarrow \mathbf{v}}^{\rho \circ \pi}, \quad (6.19)$$

which are well defined because the composition of a reversing automorphism and a preserving automorphism is a reversing automorphism.

For every reversing automorphism ρ of $\mathcal{G}$, we define a noninvertible duality operator $\mathsf{D}_\rho: \mathcal{H} \rightarrow \mathcal{H}$ as

$$D_\rho := T^\rho_{v \leftarrow \hat{v}} D_{\hat{v} \leftarrow v} = D_{v \leftarrow \hat{v}} T^\rho_{\hat{v} \leftarrow v}, \quad (6.20)$$

where the equality of the two factorizations follows from Eq. (6.16). From Eq. (6.19), we have

$$T_\pi D_\rho = D_{\pi \circ \rho}, \quad D_\rho T_\pi = D_{\rho \circ \pi}. \quad (6.21)$$

D. Action on operators

D_ρ acts on the $\mathbb{Z}_2$ -symmetric operators as

$$\begin{aligned} D_\rho X_v &= T_{v \leftarrow \hat{v}}^\rho \prod_{\hat{v} \in \hat{V}} \hat{Z}_{\hat{v}}^{\sigma_{\hat{v}v}} D_{\hat{v} \leftarrow v} = \prod_{\hat{v} \in \hat{V}} Z_{\rho(\hat{v})}^{\sigma_{\hat{v}v}} D_\rho \\ &= \prod_{\hat{v} \in \hat{V}} Z_{\rho(\hat{v})}^{\sigma_{\rho(v)\hat{v}}} D_\rho = B_{\rho(v)} D_\rho, \end{aligned} \quad (6.22)$$

and similarly,

$$D_\rho B_{\hat{v}} = X_{\rho(\hat{v})} D_\rho. \quad (6.23)$$

This can be seen easily using ZX-calculus by pushing a red spider with a π phase through a vertex on either side of the ZX-diagram of $D_{\hat{v} \leftarrow v}$. Thus, it is clear that D_ρ commutes with the Hamiltonian when $J = h$. Moreover, we have

$$\begin{aligned} D_\rho \eta_a &= D_\rho \prod_{v \in V} X_v^{a_v} = \prod_{v \in V} B_{\rho(v)}^{a_v} D_\rho = \prod_{v \in V} \prod_{u \in V} Z_u^{\sigma_{\rho(v)u} a_v} D_\rho \\ &= \prod_{v \in V} \prod_{u \in V} Z_u^{\sigma_{\rho^{-1}(u)v} a_v} D_\rho = D_\rho, \end{aligned} \quad (6.24)$$

where we used $a \in \ker \sigma$. Similarly, we have $\eta_a D_\rho = D_\rho$.

E. Operator algebra

Let us look at the action of D_ρ^2 on symmetric operators,

$$\begin{aligned} D_\rho^2 X_v &= D_\rho B_{\rho(v)} D_\rho = X_{\rho^2(v)} D_\rho^2, \\ D_\rho^2 B_{\hat{v}} &= D_\rho X_{\rho(\hat{v})} D_\rho = B_{\rho^2(\hat{v})} D_\rho^2. \end{aligned} \quad (6.25)$$

So one might expect that

$$D_\rho^2 \propto T_{\rho^2} \sum_{a \in \ker \sigma} \eta_a. \quad (6.26)$$

Indeed, we verify this using ZX-calculus. First, we have

$$\begin{aligned} D_\rho^2 &= D_{v \leftarrow \hat{v}} T_{\hat{v} \leftarrow v}^\rho D_{v \leftarrow \hat{v}} T_{\hat{v} \leftarrow v}^\rho \\ &= D_{v \leftarrow \hat{v}} D_{\hat{v} \leftarrow v} T_{\rho^2} = T_{\rho^2} D_{v \leftarrow \hat{v}} D_{\hat{v} \leftarrow v}. \end{aligned} \quad (6.27)$$

Now,

$$D_{v \leftarrow \hat{v}} D_{\hat{v} \leftarrow v} \stackrel{\text{HC, SF}}{=} 2^{2\kappa} \quad (6.28)$$

where in the second equality, we used Eq. (6.3). Now, for each $v \in V$, we apply the GB on the following part of the diagram:

$$v \text{ --- } \text{green circle} \text{ --- } \text{red circle} \text{ --- } v \stackrel{\text{GB}}{=} \left(\frac{1}{\sqrt{2}} \right)^{d_v-1} v \text{ --- } \text{green circle} \text{ --- } \text{red circle} \text{ --- } v, \quad (6.29)$$

where E_v is the set of edges incident to v and $d_v = |E_v|$ is the degree of the vertex v . Using this and noting that $E = \bigsqcup_{v \in V} E_v$, we get [160]

$$D_{v \leftarrow \hat{v}} D_{\hat{v} \leftarrow v} \stackrel{\text{GB}}{=} 2^{2\kappa} \left(\frac{1}{\sqrt{2}} \right)^{\sum_{v \in V} (d_v-1)} \quad (6.30)$$

where we used $|E| = \sum_{v \in V} d_v$ and defined the condensation operator $\mathbf{C}$. Therefore, we have

$$\mathbf{D}_\rho^2 = T_{\rho^2} \mathbf{C}. \quad (6.31)$$

More generally, for two reversing automorphisms ρ and ρ' , one can show that

$$\mathbf{D}_\rho \mathbf{D}_{\rho'} = T_{\rho \circ \rho'} \mathbf{C}. \quad (6.32)$$

Note that $\mathbf{D}_\rho$ mixes with the spatial symmetry generated by T_{ρ^2} , generalizing the observation in Refs. [14,15]. In the special case when the reversing automorphism ρ is an involution—i.e., $\rho^2 = 1$, where 1 is the identity automorphism—then we see that the duality operator $\mathbf{D}_\rho$ does not mix with any spatial symmetry. We will see examples of this below.

We can find the Pauli representation of $\mathbf{C}$ as follows. Using Eq. (A1) for all the green dots, we have

$$\begin{aligned} \mathbf{C} &= 2^{2\kappa} \left(\frac{1}{\sqrt{2}} \right)^{|E|-|V|} \frac{(\sqrt{2})^2 \sum_{v \in V} 1}{(\sqrt{2}) \sum_{v \in V(d_v+1)} 1} \left(\bigotimes_{\hat{v} \in \hat{V}} \langle 0 |_{\hat{v}} \right) \\ &\times \left[\prod_{v \in V} \left(1 + X_v \prod_{\hat{v} \in \hat{V}} \hat{X}_{\hat{v}v} \right) \right] \left(\bigotimes_{\hat{v} \in \hat{V}} |0\rangle_{\hat{v}} \right) \\ &= \frac{2^{2\kappa}}{2^{|E|-|V|}} \sum_{a \in \{0,1\}^{|V|}} \prod_{v \in V} X_v^{a_v} \prod_{\hat{v} \in \hat{V}} \delta_{\sum_{v \in V \sigma_{\hat{v}v} a_v}, 0} \\ &= \frac{2^{2\kappa}}{2^{|E|-|V|}} \sum_{a \in \ker \sigma} \eta_a, \end{aligned} \quad (6.33)$$

and hence,

$$\mathbf{D}_\rho^2 = \frac{2^{2\kappa}}{2^{|E|-|V|}} T_{\rho^2} \sum_{a \in \ker \sigma} \eta_a. \quad (6.34)$$

It follows that

$$\mathbf{D}_\rho \mathbf{C} = \mathbf{C} \mathbf{D}_\rho = \frac{2^{2\kappa} \times 2^{\dim \ker \sigma}}{2^{|E|-|V|}} \mathbf{D}_\rho. \quad (6.35)$$

We will say more about the choice of κ in the examples below. It turns out that, in all the examples that are translationally invariant, there is a natural choice of κ such

that all extensive factors cancel in the fusion of $\mathbf{D}_\rho$ and $\mathbf{C}$. The remaining coefficient is precisely the fusion coefficient expected from the continuum.

The following is a summary of the algebra of all symmetry operators:

$$\begin{aligned} \eta_a \eta_{a'} &= \eta_{a+a'}, & T_\pi T_{\pi'} &= T_{\pi \circ \pi'}, & T_\pi \eta_a &= \eta_{\pi_a} T_\pi, \\ \mathbf{D}_\rho \eta_a &= \eta_a \mathbf{D}_\rho = \mathbf{D}_\rho, & T_\pi \mathbf{D}_\rho &= \mathbf{D}_{\pi \circ \rho}, & \mathbf{D}_\rho T_\pi &= \mathbf{D}_{\rho \circ \pi}, \\ \mathbf{D}_\rho \mathbf{D}_{\rho'} &= T_{\rho \circ \rho'} \mathbf{C}, & \mathbf{C} &= \frac{2^{2\kappa}}{2^{|E|-|V|}} \sum_{a \in \ker \sigma} \eta_a, \end{aligned} \quad (6.36)$$

where $a, a' \in \ker \sigma$, π, π' are preserving automorphisms, $(\pi a)_v = a_{\pi^{-1}(v)}$, and ρ, ρ' are reversing automorphisms. The relation between the symmetries of the Hamiltonian H and the properties of the bipartite graph $\mathcal{G}$ are summarized in Table II.

F. Examples

1. 1+1D Ising model

In this case, V is the set of L sites on a periodic chain and $\hat{V}$ is the set of links. We use an integer i to label a site and a half-integer $i + \frac{1}{2}$ to label the link between sites i and $i + 1$. The Hamiltonian is

$$H = -J \sum_i Z_i Z_{i+1} - h \sum_i X_i. \quad (6.37)$$

The associated bipartite graph is shown in Fig. 8(a). Its biadjacency matrix is given by

$$\sigma_{i(j+\frac{1}{2})} = \delta_{i,j} + \delta_{i,j+1}. \quad (6.38)$$

The reversing automorphism ρ , shown as blue dashed arrow in Fig. 8(a), corresponds to a half-translation. This leads to the KW duality operator $\mathbf{D}$, which mixes with lattice translation as in Eq. (3.16). Since $|V| = L$, $|E| = 2L$, and $\dim \ker \sigma = 1$, we choose $\kappa = L/2$ to cancel all the extensive factors. This gives $\mathbf{D}\mathbf{C} = \mathbf{C}\mathbf{D} = 2\mathbf{D}$, which is precisely what we obtained in Sec. III C.

The half-translation ρ and all its odd powers are not the only reversing automorphisms of the bipartite graph in Fig. 8(a). In fact, there is a reversing involution ρ' , shown as red arrow in Fig. 8(a), that corresponds to a reflection. This

TABLE II. The one-one correspondence between the symmetries of the generalized TFIM and the properties of the associated bipartite graph. When ρ is a reversing involution, i.e., $\rho^2 = 1$, then $\mathbf{D}_\rho$ does not mix with spatial symmetries.

Symmetry	Graph-theoretic property	Fusion rule
Internal $\mathbb{Z}_2$'s (η_a)	$a \in \ker \sigma$	$\eta_a \eta_{a'} = \eta_{a+a'}$
Spatial (T_π)	Preserving automorphism (π)	$T_\pi T_{\pi'} = T_{\pi \circ \pi'}$
Duality ($\mathbf{D}_\rho$)	Reversing automorphism (ρ)	$\mathbf{D}_\rho \mathbf{D}_{\rho'} \sim T_{\rho \circ \rho'} \sum_{a \in \ker \sigma} \eta_a$

leads to a duality operator $D' = DP$ that does not mix with lattice translation. (Here, P is the parity operator that maps the qubit at i to the qubit at $-i$.) Indeed, using $DP = PD^\dagger$ and $TP = PT^\dagger$ (it is easy to see this using ZX-calculus), we have

$$(D')^2 = DPDP = DD^\dagger = 1 + \eta. \quad (6.39)$$

While D' commutes with η , it does not commute with T ,

$$D'T = T^\dagger D'. \quad (6.40)$$

Moreover, D' maps local symmetric operators around the site i to those around the site $-i$. Therefore, we refer to D' as a noninvertible parity (or reflection) operator. (See Refs. [54,122] for noninvertible parity and time-reversal symmetries in continuum field theory.)

2. 3+1D lattice $\mathbb{Z}_2$ gauge theory

In this case, V is the set of links on an $L_x \times L_y \times L_z$ periodic cubic lattice and $\hat{V}$ is the set of plaquettes. We use ℓ to label a link and p to label a plaquette. The Hamiltonian is

$$H = -J \sum_p \prod_{\ell \in p} Z_\ell - h \sum_{\ell} X_\ell. \quad (6.41)$$

Note that we do not impose Gauss law here, or equivalently, we set $g = 0$ in Eq. (4.5). The biadjacency matrix in this case is given by

$$\sigma_{\ell p} = \sum_{\ell' \in p} \delta_{\ell, \ell'}. \quad (6.42)$$

Since $|V| = 3L_x L_y L_z$, $|E| = 12L_x L_y L_z$, and $\dim \ker \sigma = L_x L_y L_z + 2$, we choose $\kappa = 4L_x L_y L_z$ to cancel all the extensive factors. This gives $\text{DC} = \text{CD} = 4\text{D}$, which is precisely what we obtained in Eq. (5.28).

There is a reversing automorphism given by the half-translation $\mathbf{t}$ in the $(1, 1, 1)$ direction. This leads to the Wegner duality operator D Eq. (5.10), which mixes with the lattice translation $T_{1,1,1}$.

There is also a reversing involution given by parity or, equivalently, a reflection followed by a π rotation. This gives a duality operator $D' = DP$ that does not mix with lattice translation. [Here, P is the usual parity operator that maps the qubit at link ℓ to the qubit at link $\mathbf{p}(\ell)$, where

$$\mathbf{p}(x, y, z) = (-x, -y, -z) \mod (L_x, L_y, L_z) \quad (6.43)$$

is the parity map on the lattice.] Since D' maps local symmetric operators around the link ℓ to those around the link $\mathbf{p}(\ell)$, we refer to it as a noninvertible parity operator.

3. 1+1D Ashkin-Teller model

The Hamiltonian is [161–165]

$$H = -J \sum_i (Z_{2i-1} Z_{2i+1} + Z_{2i} Z_{2i+2} + \lambda Z_{2i-1} Z_{2i} Z_{2i+1} Z_{2i+2}) - h \sum_i (X_{2i} + X_{2i+1} + \lambda X_{2i} X_{2i+1}). \quad (6.44)$$

Assuming $L \in 2\mathbb{Z}$, this model has two $\mathbb{Z}_2$ symmetries generated by

$$\eta_{\text{odd}} = \prod_i X_{2i-1}, \quad \eta_{\text{even}} = \prod_i X_{2i}. \quad (6.45)$$

When $J = h$, it is self-dual under the exchange

$$Z_{2i-1} Z_{2i+1} \leftrightarrow X_{2i}, \quad Z_{2i} Z_{2i+2} \leftrightarrow X_{2i+1}. \quad (6.46)$$

When $\lambda = 0$, this model is equivalent to two copies of the TFIM on the odd and even sites, respectively [166]. It is also equivalent to the $\mathbb{Z}_2$ dipole Ising model [167], so one can interpret this duality as the dipole KW duality [47,168]. The phase diagram of the quantum Ashkin-Teller (AT) model is well studied [164,169]. Along $J = h$, for a range of λ , there is a critical line with continuously varying critical exponents described by the orbifold branch of the $c = 1$ CFT. In particular, at $J = h$ and $\lambda = 0$, the critical point is described by the Ising² point on the orbifold branch of the $c = 1$ CFT, as expected from the above equivalence. Related discussions can be found in Refs. [45,170].

In this case, V and $\hat{V}$ are both the set of all sites, labeled by an integer i . The corresponding bipartite graph is shown in Fig. 8(b). Its biadjacency matrix is given by

$$\sigma_{ij} = \delta_{i,j-1} + \delta_{i,j+1}. \quad (6.47)$$

The generalized TFIM associated with this graph corresponds to the $\lambda = 0$ case.

There is a reversing involution ρ , shown as red arrow in Fig. 8(b), that flips the graph about the red vertical line. Therefore, the duality operator D that implements the above exchange does not mix with lattice translation,

$$\text{D}^2 = \text{C} = (1 + \eta_{\text{odd}})(1 + \eta_{\text{even}}). \quad (6.48)$$

Here, since $|V| = L$, $|E| = 2L$, and $\dim \ker \sigma = 2$, we chose $\kappa = L/2$ to cancel all the extensive factors, and hence, $\text{DC} = \text{CD} = 4\text{D}$.

We remark that the operators D , η_{odd} , and η_{even} together form the fusion category $\text{Rep}(D_8)$. Apart from the AT model, there are also other Hamiltonians that enjoy this symmetry, such as the 1+1D cluster Hamiltonian [28]. This noninvertible symmetry in the continuum $c = 1$ orbifold CFT was discussed in Refs. [129,171–173].

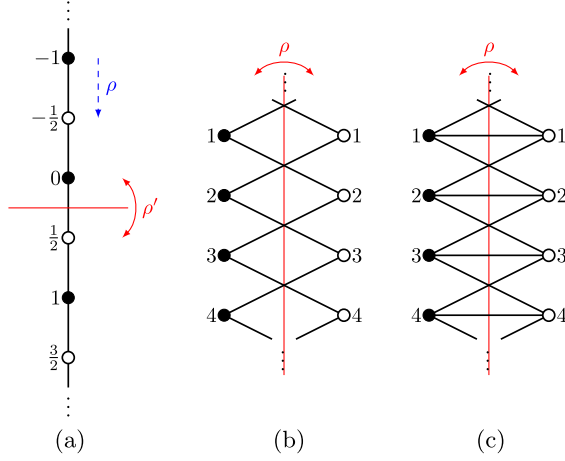


FIG. 8. The bipartite graphs for the 1 + 1D examples: (a) the transverse-field Ising model, Eq. (6.37), (b) the Ashkin-Teller model at $\lambda = 0$, Eq. (6.44), and (c) the three-spin Ising model, Eq. (6.49). The vertex sets V and $\hat{V}$ correspond to the solid and hollow vertices, respectively. The blue (dashed) arrow indicates a reversing automorphism that is not an involution and the red arrows indicate reversing involutions.

4. 1 + 1D three-spin Ising model

The Hamiltonian is

$$H = -J \sum_i Z_{i-1} Z_i Z_{i+1} - h \sum_i X_i. \quad (6.49)$$

Assuming $L \in 3\mathbb{Z}$, this model has two $\mathbb{Z}_2$ symmetries generated by

$$\eta_1 = \prod_{i \notin 3\mathbb{Z}+1} X_i, \quad \eta_2 = \prod_{i \notin 3\mathbb{Z}+2} X_i. \quad (6.50)$$

When $J = h$, it is self-dual under the exchange

$$Z_{i-1} Z_i Z_{i+1} \leftrightarrow X_i. \quad (6.51)$$

The three-spin Ising model, as well as the duality under the Kramers-Wannier action, has been pointed out in Refs. [174–176], as well as recently constructed as a noninvertible symmetry operator in Ref. [26]. The Hamiltonian at $J = h$ has been studied numerically where the central charge was found to be approximately 1, and the value of the critical exponents suggests that it lies on the orbifold branch of the $c = 1$ CFT moduli space [177–180].

In this case, V and $\hat{V}$ are both the set of all sites, labeled by an integer i . The corresponding bipartite graph is shown in Fig. 8(c). Its biadjacency matrix is given by

$$\sigma_{ij} = \delta_{i,j-1} + \delta_{i,j} + \delta_{i,j+1}. \quad (6.52)$$

There is a reversing involution ρ , shown as red arrow in Fig. 8(c), that flips the graph about the red vertical line.

Therefore, the duality operator D that implements the above exchange does not mix with lattice translation,

$$D^2 = C = (1 + \eta_1)(1 + \eta_2). \quad (6.53)$$

Here, since $|V| = L$, $|E| = 3L$, and $\dim \ker \sigma = 2$, we chose $\kappa = L$ to cancel all the extensive factors, and hence, $DC = CD = 4D$. These symmetries form a $\text{Rep}(D_8)$ fusion category.

5. 2 + 1D plaquette Ising model

Consider a 2D $L_x \times L_y$ periodic square lattice with a qubit on each site labeled by (i, j) . The Hamiltonian is

$$H = -J \sum_{i,j} Z_{i,j} Z_{i+1,j} Z_{i,j+1} Z_{i+1,j+1} - h \sum_{i,j} X_{i,j}. \quad (6.54)$$

This model has a $\mathbb{Z}_2$ subsystem symmetry generated by

$$\eta_i^x = \prod_j X_{i,j}, \quad \eta_j^y = \prod_i X_{i,j}. \quad (6.55)$$

They satisfy the constraint $\prod_i \eta_i^x = \prod_j \eta_j^y$ so there are only $L_x + L_y - 1$ independent generators. When $J = h$, this model is self-dual under the map

$$X_{i,j} \rightsquigarrow Z_{i,j} Z_{i+1,j} Z_{i,j+1} Z_{i+1,j+1} \rightsquigarrow X_{i+1,j+1}, \quad (6.56)$$

which already hints at the duality operator mixing with lattice translation in the (1,1) direction. This duality was studied in Refs. [41,43,44].

In this case, V is the set of all sites and $\hat{V}$ is the set of all plaquettes. We use (i, j) to label a site and $(i + \frac{1}{2}, j + \frac{1}{2})$ to label a plaquette bounded by the sites (i, j) , $(i + 1, j)$, $(i, j + 1)$, and $(i + 1, j + 1)$. The corresponding bipartite graph is shown in Fig. 9. Its biadjacency matrix is given by

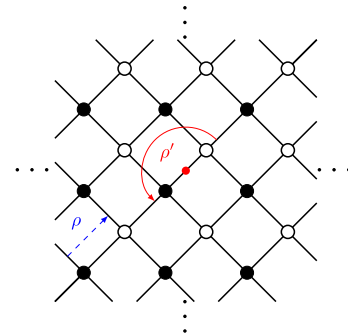


FIG. 9. The bipartite graphs for the 2 + 1D plaquette Ising model, Eq. (6.54). The vertex sets V and $\hat{V}$ correspond to the solid and hollow vertices, respectively. The blue (dashed) arrow indicates a reversing automorphism that is not an involution, whereas the red arrow indicates a reversing involution.

$$\begin{aligned} \sigma_{(i,j)(i'+\frac{1}{2},j'+\frac{1}{2})} &= \delta_{i,i'}\delta_{j,j'} + \delta_{i,i'+1}\delta_{j,j'} \\ &+ \delta_{i,i'}\delta_{j,j'+1} + \delta_{i,i'+1}\delta_{j,j'+1}. \end{aligned} \quad (6.57)$$

The reversing automorphism ρ , shown as blue dashed arrow in Fig. 9, corresponds to the half-translation in (1, 1) direction. It leads to the subsystem KW duality operator $\mathbf{D}$, which mixes with the lattice translation in the (1, 1) direction as follows:

$$\mathbf{D}^2 = T_{1,1} \mathbf{C}, \quad \text{where } \mathbf{C} = \frac{1}{2} \prod_i (1 + \eta_i^x) \prod_j (1 + \eta_j^y). \quad (6.58)$$

Since $|V| = L_x L_y$, $|E| = 4L_x L_y$, and $\dim \ker \sigma = L_x + L_y - 1$, we choose $\kappa = 3L_x L_y / 2$ to cancel all the extensive factors. This gives $\mathbf{D}\mathbf{C} = \mathbf{C}\mathbf{D} = 2^{L_x + L_y - 1} \mathbf{D}$.

There is also a reversing involution ρ' , shown as red arrow in Fig. 9, given by a π rotation about the red point. The corresponding duality operator $\mathbf{D}' = \mathbf{D}P$ does not mix with lattice translation. [Here, P is the usual parity operator that maps the qubit at site (i, j) to the qubit at site $(-i, -j)$. Note that P generates a π rotation in the spatial plane, which is orientation preserving.] We refer to it as a noninvertible parity operator because it maps local symmetric operators around (i, j) to those around $(-i, -j)$.

6. Tensor product models

A bipartite graph $\mathcal{G}$ has a reversing involution if and only if it is of the form $\mathcal{G}_0 \times K_2$ [181,182], where $\mathcal{G}_0$ is a graph on $|V|$ vertices and K_2 is the complete graph on two vertices (i.e., a “dumbbell”). [Given a reversing involution ρ of $\mathcal{G}$, we can construct the graph $\mathcal{G}_0$ as follows: Its vertex set is V and there is an edge between v and w if and only if $\sigma_{\rho(v)w} = 1$. This definition is symmetric in v and w because $\sigma_{\rho(v)w} = \sigma_{\rho(w)v}$. Conversely, the map $\rho: (v_0, \epsilon) \mapsto (v_0, 1 - \epsilon)$, where $\epsilon = 0, 1$ labels the vertices of K_2 , gives a reversing involution of $\mathcal{G}_0 \times K_2$.]

All the examples above have reversing involutions, so they fall into this category. Below, we consider a special case of this type, which leads to a tensor product of decoupled generalized TFIMs.

Consider the case where $\mathcal{G}_0$ is itself a bipartite graph with vertex set $\hat{V}_0 \sqcup V_0$ and edge set E_0 . Let $\tilde{\mathcal{G}}_0$ be the bipartite graph obtained by exchanging the two parts of $\mathcal{G}_0$, i.e., $\bar{V}_0 = \hat{V}_0$ and $\hat{V}_0 = V_0$. Let H_0 be the Hamiltonian of the generalized TFIM based on $\mathcal{G}_0$. It is easy to see that the Hamiltonian $\bar{H}_0$ of the generalized TFIM based on $\tilde{\mathcal{G}}_0$ corresponds to the gauged Hamiltonian $\hat{H}_0$, Eq. (6.15).

Now, the bipartite graph $\mathcal{G} := \mathcal{G}_0 \times K_2$ is canonically isomorphic to the disjoint union of $\mathcal{G}_0$ and $\tilde{\mathcal{G}}_0$, denoted as $\mathcal{G} + \tilde{\mathcal{G}}_0$ [183]. Therefore, the generalized TFIM based on $\mathcal{G}$ is the tensor product of the generalized TFIMs based on $\mathcal{G}_0$ and $\tilde{\mathcal{G}}_0$. In other words, it is the product of a generalized

TFIM and its gauged version. More concretely, the total Hilbert space is $\mathcal{H} = \mathcal{H}_0 \otimes \bar{\mathcal{H}}_0$, and the Hamiltonian is

$$H = H_0 + \bar{H}_0. \quad (6.59)$$

As explained above, $\mathcal{G} = \mathcal{G}_0 \times K_2$ has a reversing involution, and here, it exchanges the two components $\mathcal{G}_0$ and $\tilde{\mathcal{G}}_0$. So there is a duality symmetry that swaps the two Hamiltonians H_0 and $\bar{H}_0$ in H . The corresponding “noninvertible swap operator” $\mathbf{D}$ does not mix with any spatial symmetry; it satisfies

$$\mathbf{D}^2 = \mathbf{C} = \mathbf{C}_0 \bar{\mathbf{C}}_0, \quad (6.60)$$

where $\mathbf{C}_0$ and $\bar{\mathbf{C}}_0$ are the condensation operators of the generalized TFIMs based on $\mathcal{G}_0$ and $\tilde{\mathcal{G}}_0$, respectively [184].

The continuum field theory analog of this model is clear [45,46,185]. Let $\mathcal{Q}_0$ be a general quantum field theory (QFT) with a nonanomalous, generalized symmetry G_0 . Then the tensor product QFT $\mathcal{Q}_0 \times \mathcal{Q}_0 / G_0$ has a noninvertible swap symmetry since it is invariant under gauging $G_0 \times \hat{G}_0$, where $\hat{G}_0$ is the dual symmetry of G_0 .

An example of this type is the Ashkin-Teller model of Sec. VIF 3 with $J = h$ and $\lambda = 0$. In this case, the bipartite graphs $\mathcal{G}_0$ and $\tilde{\mathcal{G}}_0$ correspond to the Ising models on even and odd sublattices, respectively. One can also see from Fig. 8(b) that there are two connected components, each of which is isomorphic to the bipartite graph in Fig. 8(a).

Here is another example in $2 + 1\text{D}$ that was the focus of Ref. [45]. Consider a periodic $L_x \times L_y$ square lattice with qubits on sites and links, labeled by s and ℓ , respectively. The Hamiltonian is

$$H = -J \sum_{\ell} \prod_{s \in \ell} Z_s - h \sum_s X_s - \bar{J} \sum_s \prod_{\ell \ni s} \bar{Z}_{\ell} - \bar{h} \sum_{\ell} \bar{X}_{\ell}. \quad (6.61)$$

One can recognize the first two terms as the Hamiltonian of the $2 + 1\text{D}$ Ising model on the original lattice and the last two terms as the Hamiltonian of the $2 + 1\text{D}$ lattice $\mathbb{Z}_2$ gauge theory on the dual lattice (albeit without the Gauss law constraint). Therefore, it has a $\mathbb{Z}_2^{(0)} \times \mathbb{Z}_2^{(1)}$ symmetry generated by

$$\eta_0 = \prod_s X_s, \quad \bar{\eta}_0(\gamma) = \prod_{\ell \in \gamma} \bar{X}_{\ell}, \quad (6.62)$$

respectively, where γ is a curve along the links of the original lattice. This model also has a noninvertible swap symmetry that exchanges

$$X_s \leftrightarrow \prod_{\ell \ni s} \bar{Z}_{\ell}, \quad \prod_{s \in \ell} Z_s \leftrightarrow \bar{X}_{\ell}, \quad (6.63)$$

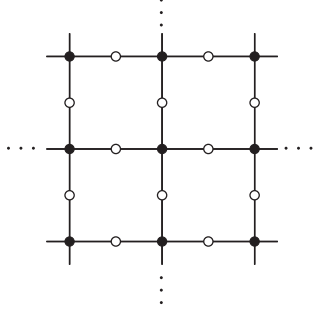


FIG. 10. The bipartite graph $\mathcal{G}_0$ for the 2 + 1D Ising model. The vertex sets V_0 and $\hat{V}_0$ correspond to the solid and hollow vertices, respectively. The bipartite graph $\tilde{\mathcal{G}}_0$ is obtained by replacing solid (hollow) vertices with hollow (solid) vertices.

when $J = \bar{h}$ and $\bar{J} = h$. The corresponding noninvertible swap operator $\mathbf{D}$ does not mix with lattice translation; it satisfies

$$\mathbf{D}^2 = \mathbf{C} = \mathbf{C}_0 \bar{\mathbf{C}}_0, \quad \mathbf{D}\mathbf{C} = \mathbf{C}\mathbf{D} = 4\mathbf{D}, \quad (6.64)$$

where

$$\mathbf{C}_0 = 1 + \eta_0, \quad \bar{\mathbf{C}}_0 = \frac{1}{2A} \sum_{\gamma} \bar{\eta}_0(\gamma), \quad (6.65)$$

and $A = L_x L_y$.

This example fits into the above model: the vertex sets V_0 and $\hat{V}_0$ are the sites and links, respectively, and there is an edge between s and ℓ if and only if $s \in \ell$. The resulting bipartite graph $\mathcal{G}_0$ is shown in Fig. 10. The generalized TFIM based on $\mathcal{G}_0$ is the 2 + 1D Ising model, whereas the one based on $\tilde{\mathcal{G}}_0$ is the 2 + 1D lattice $\mathbb{Z}_2$ gauge theory. The duality operator associated with the reversing involution of $\mathcal{G} = \mathcal{G}_0 \times K_2 \cong \mathcal{G}_0 + \tilde{\mathcal{G}}_0$ is precisely the noninvertible swap operator $\mathbf{D}$. As an aside, note that $|\hat{V}_0| = 2|V_0|$, so there is no duality symmetry in the 2 + 1D Ising model or the 2 + 1D lattice $\mathbb{Z}_2$ gauge theory separately.

To complete the example, we note that $|V| = |V_0| + |\hat{V}_0| = 3A$, $|E| = 2|E_0| = 8A$, and $\dim \ker \sigma = \dim \ker \sigma_0 + \dim \ker \bar{\sigma}_0 = 1 + (A + 1) = A + 2$. Therefore, choosing $\kappa = 2A$ to cancel all the extensive factors, we get $\mathbf{D}\mathbf{C} = \mathbf{C}\mathbf{D} = 4\mathbf{D}$, which agrees with the coefficient above.

7. On mixing with spatial symmetries

In all the examples above, the bipartite graph has a reversing involution, so there is a duality operator that does not involve a nontrivial permutation of the qubits. In fact, in any $(d + 1)$ -dimensional self-dual generalized TFIM with translation symmetry, there exists a duality operator that does not mix with translations (for instance, the noninvertible parity operator $\mathbf{D}'$ in the 1 + 1D Ising model). Therefore, to

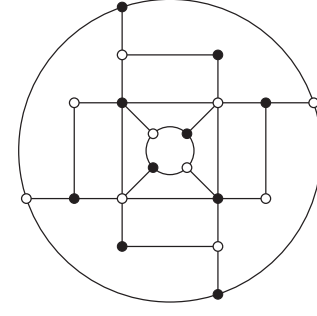


FIG. 11. A bipartite graph with a reversing automorphism (90° rotation) but no reversing involution.

find models where the duality operator necessarily mixes with a nontrivial permutation of the qubits, one must look beyond models with translation symmetry.

As an example, consider the bipartite graph shown in Fig. 11 [186]. It has a reversing automorphism given by a 90° rotation of the graph, but it does not have a reversing involution. So the generalized TFIM based on this graph has a duality symmetry that necessarily mixes with a nontrivial permutation of the qubits. Moreover, the duality operator is noninvertible because $\ker \sigma \neq 0$.

VII. DISCUSSIONS AND OUTLOOK

In this work, we used ZX-diagrams to present noninvertible duality operators in many Hamiltonian lattice models, including the Kramers-Wannier operator in the 1 + 1D Ising model, the Wegner operator in the 3 + 1D $\mathbb{Z}_2$ lattice gauge theory, and those in the generalized TFIMs based on graphs. In addition to the duality operator, we also provided graphical presentations for the condensation operators. Their noninvertible algebra was computed using ZX-calculus.

In the generalized TFIM based on a bipartite graph $\mathcal{G}$, we construct a noninvertible operator $\mathbf{D}_\rho$ associated with every reversing automorphism ρ of $\mathcal{G}$. In the special case when ρ is a reversing involution (i.e., if $\rho^2 = 1$), $\mathbf{D}_\rho$ does not mix with spatial symmetries. This is summarized in Table II.

There are several future directions:

- (i) The generalization to lattice models of n -dimensional qubits with $n > 2$ is particularly interesting since the 't Hooft anomaly of the noninvertible symmetry may depend on n in a nontrivial way, such as in Ref. [53]. To this end, one can use an extension of ZX-calculus to qubits, which is known to be universal [187,188] and was also recently shown to be complete [189], to study noninvertible symmetries in the generalized n -state clock model.
- (ii) In this work, we focused on the operators implementing the noninvertible symmetry. It is important to extend the discussion of Refs. [15,190,191] to study the corresponding defects, which involve local modifications of the Hamiltonian.

- (iii) It would be interesting to find the diagrammatic presentation for SPT states protected by (nonanomalous) noninvertible symmetries, generalizing the results of Ref. [25].
- (iv) Recently, there has been a growing interest in the Kennedy-Tasaki transformation [28,34,43,45,89,192,193], which is a twisted version of the Kramers-Wannier transformation. In the continuum, it corresponds to gauging with a discrete torsion. It is natural to consider extending our tensor network presentation to such maps.
- (v) What is the phase diagram for the duality-preserving deformed model in Eq. (5.45) in $3 + 1\text{D}$? Both the points $\lambda = 0$ and $\lambda = 1$ correspond to a first-order transition between the toric code phase and the product state, and it is plausible that there is no phase transition in between these two values of λ . This is unlike the situation for the analogous $1 + 1\text{D}$ model in Eq. (3.31) where the $\lambda = 0$ point is the Ising CFT, while the $\lambda = 1$ point is gapped, and there has to be a nontrivial phase transition in between [82].

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APPENDIX A: ZX-CALCULUS FOR THE DUALITY AND CONDENSATION OPERATORS

In this appendix, we provide more details of the ZX-calculus presentation of the duality and condensation operators in the $3 + 1\text{D}$ lattice $\mathbb{Z}_2$ gauge theory.

1. Condensation operator

Below, we show that the two expressions, Eqs. (5.22) and (5.26), for the condensation operator are equivalent. First, recall that

$$\begin{array}{c} \text{green dot} \\ \text{---} \text{red dot} \\ \vdots \\ \text{---} \text{red dot} \end{array} = \frac{1}{2^{n/2}} [1 + (-1)^m X_1 \cdots X_n], \quad (\text{A1})$$

which follows from the discussion below Eq. (3.22). Using this identity for each green dot on the link in the ZX-diagram Eq. (5.26) of $\mathcal{C}$, we can write

$$\mathcal{C}^{\text{SF}} \equiv 2^{7V/2} \left(\text{ZX-diagram} \right) = \underbrace{\frac{2^{7V/2} \cdot 2^{3V}}{2^{15V/2}}}_{1/2^V} \left(\bigotimes_p \langle 0|_p \right) \left[\prod_\ell \left(1 + X_\ell \prod_{p \ni \ell} \hat{X}_p \right) \right] \left(\bigotimes_p |0\rangle_p \right), \quad (\text{A2})$$

where we put a hat on the auxiliary Pauli operator $\hat{X}_p$ to distinguish it from the physical Pauli operator X_ℓ . Expanding the operator in the middle, we get

$$\mathcal{C} = \frac{1}{2^V} \sum_{\{m_\ell=0,1\}} \prod_\ell X_\ell^{m_\ell} \prod_p \delta_{(dm)_p,0} = \frac{1}{2^V} \sum_{\hat{\Sigma}} \eta(\hat{\Sigma}), \quad (\text{A3})$$

where, $\delta_{a,b}$ is the Kronecker delta function modulo 2, i.e., $\delta_{a,b} = 1$ if $a - b = 0 \bmod 2$ and 0 if

$a - b = 1 \bmod 2$. This is precisely the expression for $\mathcal{C}$ in Eq. (5.22).

2. Duality operator

Here, we derive the operator algebra involving $\mathcal{D}$ using ZX-calculus. We start with Eq. (5.20) or, equivalently, Eq. (5.27),

$$\mathcal{D}^2 = \mathcal{C} T_{1,1,1}. \quad (\text{A4})$$

First, using Eqs. (5.10), (5.12), and (5.11), we have

$$D^2 = D_{1 \leftarrow p} T_{p \leftarrow 1} D_{1 \leftarrow p} T_{p \leftarrow 1} = D_{1 \leftarrow p} D_{p \leftarrow 1} T_{1 \leftarrow p} T_{p \leftarrow 1} = D_{1 \leftarrow p} D_{p \leftarrow 1} T_{1,1,1}. \quad (A5)$$

The product $D_{1 \leftarrow p} D_{p \leftarrow 1}$ can be computed as follows. We have

$$D_{1 \leftarrow p} D_{p \leftarrow 1} = 2^{8V} \cdot \text{Diagram} \quad (A6)$$

Using color change and spider fusion, we can simplify this diagram to

$$D_{1 \leftarrow p} D_{p \leftarrow 1} = 2^{8V} \cdot \text{Diagram} \quad (A7)$$

$$D_{1 \leftarrow p} D_{p \leftarrow 1} = \frac{2^{8V}}{\underbrace{2^{9V/2}}_{2^{7V/2}}} \cdot \text{Diagram} = C. \quad (A8)$$

It follows that $D^2 = CT_{1,1,1}$.

It is also instructive to compute DC using ZX-calculus. We have

Now, we use the GB to simplify it further to

$$D_{p \leftarrow 1} C = \underbrace{2^{4V} \cdot 2^{7V/2}}_{2^{15V/2}} \cdot \text{Diagram} \quad (A9)$$

Applying the GB along the curved wires, along with SFs, we get

$$D_{p \leftarrow l} C = \underbrace{2^{15V/2} \cdot 2^{9V/2}}_{2^{12V}} \cdot \text{Diagram} \quad (\text{A10})$$

After a series of SFs and CC', we can modify the above diagram to

$$D_{p \leftarrow l} C = 2^{12V} \cdot \text{Diagram} \quad (\text{A11})$$

We now apply the GB again to get

$$D_{p \leftarrow l} C = \frac{2^{12V}}{\underbrace{2^{9V/2}}_{2^{15V/2}}} \cdot \text{Diagram} \quad (\text{A12})$$

Using SC, SF, and CC', we finally get

$$D_{p \leftarrow l} C = \frac{2^{15V/2}}{\underbrace{2^{3V/2}}_{2^{6V}}} \cdot \text{Diagram} \quad (\text{A13})$$

$$= \langle + \cdots + | C | + \cdots + \rangle D_{p \leftarrow l} ,$$

where, up to scalar factors, we identified the left diagram as the operator $D_{p \leftarrow 1}$ and the right diagram as the matrix element $\langle + \dots + | C | + \dots + \rangle$. The latter can be seen as follows:

$$\langle + \dots + | C | + \dots + \rangle = \frac{2^{7V/2}}{2^{3V}} \cdot \frac{2^{V/2} \cdot 2^{3V}}{2^{2V}} \cdot \text{Diagram} \quad (A14)$$

Multiplying $T_{1 \leftarrow p}$ on the left on both sides of Eq. (A13) gives $DC = \langle + \dots + | C | + \dots + \rangle D$ [194]. The matrix element here can be computed using Eq. (5.25),

$$\langle + \dots + | C | + \dots + \rangle = \frac{1}{2} \times 2^3 = 4, \quad (A15)$$

from which we get $DC = 4D$. Similarly, one finds $CD = 4D$.

3. Action of D on the states

Below, we derive the action of D in Eq. (5.40) using ZX-calculus. First, using Eq. (5.6), we have

$$D|\xi\rangle = 2^{(V-1)/2} D|0 \dots 0\rangle. \quad (A16)$$

Now, we have

$$D_{p \leftarrow 1}|0 \dots 0\rangle = \frac{2^{4V}}{2^{9V/2}} \cdot \frac{1}{2^{V/2}} \cdot \text{Diagram} \quad (A17)$$

Applying SC through green dots on the links, CC', and SF into the green dots on the plaquettes, we get

$$D_{p \leftarrow 1}|0 \dots 0\rangle = \frac{1}{2^{V/2} \cdot 2^{9V/2}} \cdot \text{Diagram} \quad (A18)$$

Multiplying by $T_{1 \leftarrow p}$ on the left on both sides, we get

$$D|0 \dots 0\rangle = \frac{1}{2^{V/2}} | + \dots + \rangle \Rightarrow D|\xi\rangle = \frac{1}{\sqrt{2}} | + \dots + \rangle. \quad (A19)$$

Using Eqs. (5.20) and (5.23), we further have

$$\begin{aligned} D| + \dots + \rangle &= \sqrt{2} D^2 |\xi\rangle = \frac{1}{\sqrt{2}} \prod_{i < j} (1 + \eta_{ij}) \prod_s \left(\frac{1 + G_s}{2} \right) |\xi\rangle \\ &= \frac{1}{\sqrt{2}} \sum_{\xi \in \{0,1\}^3} |\xi\rangle. \end{aligned} \quad (A20)$$

Hence, we have derived Eq. (5.40).

4. Ground states of 3+1D toric code

Here, we relate the loop-gas and membrane-gas representations of the ground states of the 3 + 1D toric code.

Recall that the ground states of the 3 + 1D toric code are given by Eq. (5.37),

$$|\xi\rangle := 2^{(V-1)/2} \prod_{i < j} \eta_{ij}^{\xi_{ij}} \prod_s \left(\frac{1 + G_s}{2} \right) |0 \dots 0\rangle, \quad (A21)$$

where $\xi = (\xi_{xy}, \xi_{yz}, \xi_{zx}) \in \{0, 1\}^3$ with ξ_{ij} defined modulo 2. An alternative basis for these states is

$$|\xi\rangle := \frac{1}{2\sqrt{2}} \sum_{\xi \in \{0,1\}^3} (-1)^{\sum_{i<j} \xi_{ij} \xi_{ij}} |\xi\rangle, \quad (\text{A22})$$

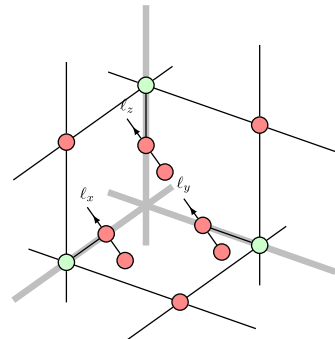
where $\xi = (\xi_{xy}, \xi_{yz}, \xi_{zx}) \in \{0,1\}^3$ with ξ_{ij} defined modulo 2.

It is instructive to write the state $|\xi\rangle$ as

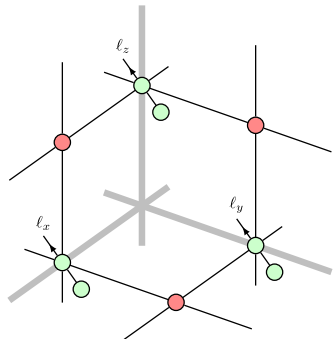
$$\begin{aligned} |\xi\rangle &= 2 \times 2^{V/2} \prod_{i<j} \left(\frac{1 + (-1)^{\xi_{ij} \eta_{ij}}}{2} \right) \prod_s \left(\frac{1 + G_s}{2} \right) |0\dots 0\rangle \\ &= 2^{V-1} \prod_{\substack{i,j,k \\ \text{cyclic}}} W_k^{\xi_{ij}} \prod_p \left(\frac{1 + \prod_{\ell \in p} Z_\ell}{2} \right) |+\dots+\rangle, \end{aligned} \quad (\text{A23})$$

where $W_k := W(\gamma_k)$ is the Wilson line operator Eq. (5.19) along the curve γ_k , which is a noncontractible cycle in the k th spatial direction [195]. One can recognize the first (second) line of Eq. (A23) as the membrane-gas (loop-gas) representation of the ground states of the 3 + 1D toric code.

We can use ZX-calculus to quickly see the equality of these two representations. Consider the state $|\xi = 0\rangle$ for simplicity (the other states are obtained by multiplying $\prod_{\substack{i,j,k \\ \text{cyclic}}} W_k^{\xi_{ij}}$). We have

$$\begin{aligned} |\xi = 0\rangle &= 2 \cdot 2^{V/2} \prod_{i<j} \left(\frac{1 + \eta_{ij}}{2} \right) \prod_s \left(\frac{1 + G_s}{2} \right) |0\dots 0\rangle \\ &= \frac{2^{V/2}}{2} C|0\dots 0\rangle = \frac{2^{V/2} \cdot 2^{7V/2}}{2 \cdot 2^{3V/2}} \cdot \frac{2^{5V/2}}{2} \cdot \text{Diagram} \end{aligned} \quad (\text{A24})$$


Applying SF, identity removal, and SF again, we get

$$|\xi = 0\rangle = \frac{2^{5V/2}}{2} \cdot \text{Diagram} \quad (\text{A25})$$


This diagram is called the 2-form representation of the state $|\xi = 0\rangle$ in Ref. [196] (see also Ref. [197] for related discussion). Using Eq. (A1), with Z/X-spiders exchanged, on all the red dots, we get

$$|\xi = 0\rangle = 2^{V-1} \prod_p \left(\frac{1 + \prod_{\ell \in p} Z_\ell}{2} \right) |+\dots+\rangle, \quad (\text{A26})$$

which proves the equality of the first and second lines of Eq. (A23). As a consequence, we also have

$$\begin{aligned} |\xi\rangle &= 2^{(V-1)/2} \prod_{i<j} \eta_{ij}^{\xi_{ij}} \prod_s \left(\frac{1 + G_s}{2} \right) |0\dots 0\rangle \\ &= \sqrt{2} \cdot 2^V \prod_{\substack{i,j,k \\ \text{cyclic}}} \left(\frac{1 + (-1)^{\xi_{ij}} W_k}{2} \right) \prod_p \left(\frac{1 + \prod_{\ell \in p} Z_\ell}{2} \right) |+\dots+\rangle. \end{aligned} \quad (\text{A27})$$

For the sake of completeness, we note that the state $|\xi = 0\rangle$ can be represented as

$$|\xi = 0\rangle = \frac{2^V}{\sqrt{2}} \quad \stackrel{\text{SF,I}}{=} \quad \frac{2^V}{\sqrt{2}} \quad (A28)$$

These diagrams correspond to the first and second lines of Eq. (A27), respectively. The first diagram is called the 1-form representation of the state $|\xi = 0\rangle$ in Ref. [196] (see also Ref. [197] for related discussion). On the other hand, the second diagram can be thought of as the action of the condensation operator $C^{(2)}$ for the 2-form symmetry of the 3 + 1D toric code on the product state $|+\cdots+\rangle$, i.e.,

$$|\xi = 0\rangle = \frac{2^V}{\sqrt{2}} C^{(2)} |+\cdots+\rangle, \quad (A29)$$

where

$$C^{(2)} := 2^{3V/2} \quad (A30)$$

$$= \frac{1}{2^{2V}} \sum_{\gamma} W(\gamma) = \frac{1}{4} \prod_k (1 + W_k) \prod_p \left(\frac{1 + \prod_{\ell \in p} Z_{\ell}}{2} \right).$$

APPENDIX B: HIGHER QUANTUM SYMMETRY

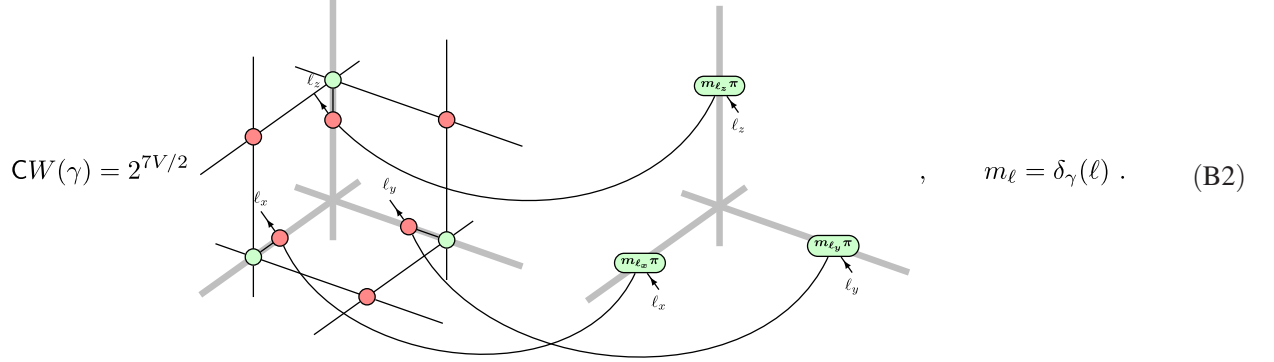
It is well known that gauging a finite global symmetry leads to a quantum (or dual) finite symmetry [198]. For example, gauging a $\mathbb{Z}_N$ p -form symmetry in d -dimensional spacetime gives a dual $\mathbb{Z}_N$ $(d - p - 2)$ -form symmetry [50,51]. Similarly, “higher gauging,” where the symmetry is gauged on a submanifold of positive codimension, results in a dual symmetry referred to as “higher quantum symmetry” [110]. One way to construct a higher quantum symmetry is by pushing a charged object from the bulk across a condensation defect. In this appendix, we use this to construct the higher quantum symmetry operators in the 3 + 1D lattice $\mathbb{Z}_2$ gauge theory.

The ZX-diagram for the Wilson line operator is

$$W(\gamma) = \prod_{\ell \in \gamma} Z_{\ell} = \quad , \quad m_{\ell} = \delta_{\gamma}(\ell) , \quad (B1)$$

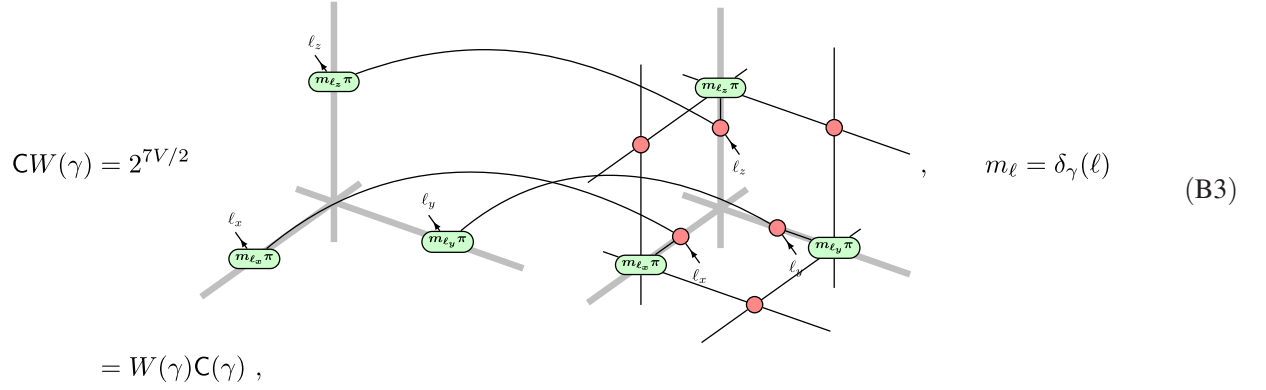
where γ is any (not necessarily connected) curve along the links of the lattice and $\delta_\gamma(\ell) = 1$ when $\ell \in \gamma$ and 0 otherwise, i.e., δ_γ is the indicator function for γ on the links. Recall that, when the Gauss laws are imposed exactly, $W(\gamma)$ is gauge invariant only when γ is closed. On the other hand, when the Gauss laws are imposed energetically, γ is allowed to be open and the end points of γ create Gauss law violations.

Consider the commutation of condensation operator $\mathbf{C}$ with Wilson line operator $W(\gamma)$,



$$CW(\gamma) = 2^{7V/2} \quad , \quad m_\ell = \delta_\gamma(\ell) . \quad (\text{B2})$$

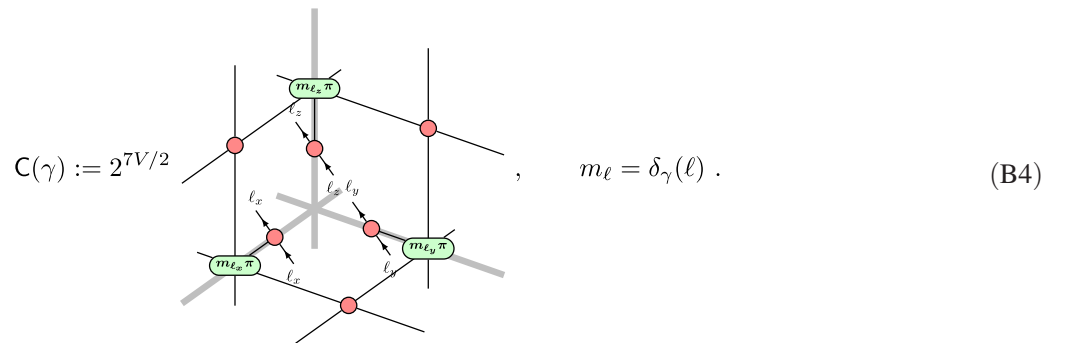
Using π commutation or identity removal followed by SF, we get [199]



$$CW(\gamma) = 2^{7V/2} \quad , \quad m_\ell = \delta_\gamma(\ell) \quad (\text{B3})$$

$$= W(\gamma)\mathbf{C}(\gamma) ,$$

where we defined



$$\mathbf{C}(\gamma) := 2^{7V/2} \quad , \quad m_\ell = \delta_\gamma(\ell) . \quad (\text{B4})$$

Note that a similar derivation as in Appendix A 1 gives

$$\mathbf{C}(\gamma) = \frac{1}{2} \prod_{i < j} [1 + (-1)^{\langle \gamma, \hat{\Sigma}_{ij} \rangle} \eta_{ij}] \prod_s \left(\frac{1 + (-1)^{\delta_{\partial\gamma}(s)} G_s}{2} \right), \quad (\text{B5})$$

where $\langle \gamma, \hat{\Sigma}_{ij} \rangle$ is the intersection number of γ and $\hat{\Sigma}_{ij}$, and $\delta_{\partial\gamma}(s) = 1$ when $s \in \partial\gamma$ and 0 otherwise, i.e., $\delta_{\partial\gamma}$ is the indicator function for $\partial\gamma$ the set of end points of γ .

Recall that higher gauging a $\mathbb{Z}_2$ 1-form symmetry on a three-dimensional submanifold gives a $\mathbb{Z}_2$ 1-form higher quantum (or dual) symmetry on that submanifold. The operator $\mathbf{C}(\gamma)$ is the higher quantum symmetry operator at a fixed time slice.

The higher quantum symmetry operator $\mathbf{C}(\gamma)$ depends only on the topology of γ . Indeed, using SF and π commutation, one can show that $\mathbf{C}(\gamma) = \mathbf{C}$ whenever γ is a closed contractible curve. For example, when γ is the boundary of a plaquette p , i.e., $\gamma = \partial p$, we have

$$\begin{aligned}
 \mathbf{C}(\partial p) &= 2^{7V/2} \stackrel{\text{SF}}{=} 2^{7V/2} \\
 &\stackrel{\text{Pi}}{=} 2^{7V/2} \stackrel{\text{SF}}{=} 2^{7V/2} = \mathbf{C},
 \end{aligned}
 \tag{B6}$$

where we show only a part of the ZX-diagram around the plaquette p . It follows that $\mathbf{C}(\gamma) = \mathbf{C}(\gamma')$ whenever γ and γ' have the same end points and $[\gamma - \gamma'] = 0 \in H_1(M, \mathbb{Z}_2)$, where M is the original lattice.

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- [84] Here, a helpful mnemonic is that the GHZ state is constructed from the green Z spider.
- [85] Unless otherwise specified, we place the inputs (outputs) on the left (right) so that “time” flows from left to right. Where this is not possible—for example, in higher-dimensional systems—we use ingoing (outgoing) arrows to represent the inputs (outputs).
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$$\text{Hadamard} = e^{-i\pi/4} \left(\text{CNOT} \right) = e^{-i\pi/4} \left(\text{CNOT} \right).$$

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- [104] Recall that $t(p)$ is the link obtained by translating from the plaquette p by $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and $t(\ell)$ is the plaquette obtained by translating from the link ℓ by $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.
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$$\begin{aligned} \mathcal{D}^\dagger &= (\mathcal{D}_{p \leftarrow 1})^\dagger (\mathcal{T}_{1 \leftarrow p})^\dagger = \mathcal{D}_{1 \leftarrow p} (\mathcal{T}_{1 \leftarrow p})^\dagger \\ &= \mathcal{D}_{1 \leftarrow p} \mathcal{T}_{p \leftarrow 1} (\mathcal{T}_{p \leftarrow 1})^\dagger (\mathcal{T}_{1 \leftarrow p})^\dagger = \mathcal{D} \mathcal{T}_{1,1,1}^{-1}. \end{aligned}$$

Similarly, it can be shown that $\mathcal{D}^\dagger = \mathcal{T}_{1,1,1}^{-1} \mathcal{D}$.

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- [112] One can think of ϵ as labeling the space diagonal along that direction. Note that ϵ and $-\epsilon$ label the same space diagonal.
- [113] The half-translation map $\mathcal{T}_{p \leftarrow 1}$ transforms into $\mathcal{T}_{p \leftarrow 1}^\epsilon$ under the rotation R . Here, $\mathcal{T}_{p \leftarrow 1}^\epsilon$ maps the physical qubit on link ℓ to the auxiliary qubit on plaquette $t_\epsilon(\ell)$, where $t_\epsilon(x) = x + \frac{1}{2}\epsilon$ is the half-translation in the ϵ direction. The desired relation then follows from the simple observation $\mathcal{T}_{p \leftarrow 1}^\epsilon = \mathcal{T}_{p \leftarrow 1} \mathcal{T}_{\frac{1}{2}(\epsilon-1)}$. Similar statements apply to the other half-translation map $\mathcal{T}_{1 \leftarrow p}$.
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- [116] Comparing with the discussion in Sec. III D, the basis $|\xi\rangle$ is the analog of $|0 \dots 0\rangle, |1 \dots 1\rangle$ in $1 + 1\text{D}$, while the basis $|\zeta\rangle$ is the analog of $|\text{GHZ}^\pm\rangle$.
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- [148] An automorphism α of a graph is a bijection from its vertex set to itself such that $\alpha(v)$ and $\alpha(w)$ have an edge between them if and only if v and w have an edge between them. For a bipartite graph, an automorphism is said to be preserving if it fixes the two parts of the vertex set, i.e., $\alpha(V) = V$ and $\alpha(\hat{V}) = \hat{V}$, whereas it is said to be reversing if it exchanges the two parts of the vertex set, i.e., $\alpha(V) = \hat{V}$ and $\alpha(\hat{V}) = V$.
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- [183] An isomorphism ϕ from $\mathcal{G}_1$ to $\mathcal{G}_2$ is a bijective map $\phi: V(\mathcal{G}_1) \rightarrow V(\mathcal{G}_2)$ such that v_1 and w_1 have an edge between them if and only if $\phi(v_1)$ and $\phi(w_1)$ have an edge between them. Disjoint union of graphs $\mathcal{G}_0 + \tilde{\mathcal{G}}_0$ is also known as graph sum, which explains this notation. The canonical isomorphism from $\mathcal{G}_0 + \tilde{\mathcal{G}}_0$ to $\mathcal{G}$ is given by the obvious inclusions,
- $$V_0 \hookrightarrow V_0 \times \{0\}, \quad \bar{V}_0 = \hat{V}_0 \hookrightarrow \hat{V}_0 \times \{0\},$$
- $$\hat{V}_0 \hookrightarrow \hat{V}_0 \times \{1\}, \quad \hat{\bar{V}}_0 = V_0 \hookrightarrow V_0 \times \{1\}.$$
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$$(1 + \eta)Z_i = 2^{L/2} \quad i \text{ --- } \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \vdots \end{array} \quad i \stackrel{\text{Pi,SF}}{=} 2^{L/2} \quad i \text{ --- } \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \vdots \end{array} = Z_i(1 - \eta),$$

where $C_0 = 1 + \eta$ can be viewed as the condensation operator for an ordinary $\mathbb{Z}_2$ symmetry. The higher quantum symmetry is $C_1 = 1 - \eta$.