

Genuine and Spurious (Non-)Ergodicity in Single Particle Tracking

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In single-particle tracking experiments measuring anomalous diffusion dynamics, understanding ergodicity is crucial, as it ensures that the time average of an observable matches the ensemble average—and can thus be fitted with known ensemble-averaged observables. A commonly used criterion for assessing the ergodicity of a stochastic process is based on the comparison of the mean-squared displacement (MSD) with the time-averaged MSD (TAMSD). This approach has been widely applied and proves effective in cases of weak ergodicity breaking across various systems in both theoretical and experimental studies. However, there is relatively little discussion regarding the theoretical justification and limitations of this definition. Here, we demonstrate that this widely accepted criterion to some extent contradicts the classical definition of ergodicity as well as physical intuition, leading to spurious (non-)ergodicity results when applied to several well-known stochastic models. To address this limitation, we propose using the mean-squared increment (MSI) instead of the MSD for comparison of ensemble- and time-averaged observables. For processes with stationary increments, the results for MSI and MSD coincide. However, if the increments of the process reach stationarity at long times only, they differ, and the MSI-TAMSD comparison still provides the test for ergodicity. Several well-established examples demonstrate, through both trajectory simulations and theoretical analysis, that our MSI-TAMSD criterion provides a more accurate characterization of the genuine (non-)ergodicity of systems where the MSD-TAMSD method fails. Additionally, for systems exhibiting “ultra-weak” ergodicity breaking, the MSI can reveal the asymptotic stationarity and ergodic nature of the process’s increments. Our findings emphasize the important role of the MSI observable for SPT experiments and anomalous diffusion studies.

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I. INTRODUCTION

Single particle tracking (SPT) enables the precise recording of the motion of micron-sized or even smaller fluorescently labeled tracer particles such as biomolecules,

vesicles, virions, molecular complexes, or colloids. SPT has been established as a powerful tool to probe dynamic processes and transport in complex environments such as biomembranes or the cytoplasm of living biological cells [1], thus providing access to statistical characterization of such processes [2,3]. SPT has been extensively applied to investigate protein, DNA, and granule motion within cellular compartments [4–10], vesicle and viral particle transport [11–14], and protein dynamics within membranes [15–19].

One of the most central results obtained from SPT in such complex systems is the observation and measurement of the anomalous diffusion, i.e., diffusive motion deviating from ordinary Brownian motion (BM). Anomalous diffusion is quite ubiquitous and typically characterized by a

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power-law form of the mean-squared displacement (MSD) [3,20,21]

$$\text{MSD}(t) = \langle [x(t) - x(0)]^2 \rangle \sim 2K_\alpha t^\alpha, \quad (1)$$

where K_α is the generalized diffusion coefficient of physical dimension $[K_\alpha] = \text{length}^2/\text{time}^\alpha$, α is the anomalous diffusion exponent, $x(0)$ is the initial position of the particles, and $\langle \cdot \rangle$ represents the ensemble average. In this paper, for simplicity and without loss of generality, we consider random processes without drift, i.e., $\langle x(t) \rangle = 0$. When $0 < \alpha < 1$, the process is called subdiffusive, often arising from trapping, crowding, or obstructed environments. When $\alpha > 1$, the motion is referred to as superdiffusion, typically driven by positive, long-range correlations (long memory) or large jumps (Lévy walks). Ordinary Brownian motion corresponds to normal (Fickian) diffusion with $\alpha = 1$. Biologically relevant examples of subdiffusion at the micron-scale are usually associated with impeded motion of tracers in the crowded cytoplasm of living biological cells [7–10,22,23], in artificially crowded environments [24–27], in entangled actin networks [28–30], in concentrated protein solutions [31], as well as in lipid bilayer systems [32–36]. Superdiffusion has been observed, *inter alia*, for many active motor-driven transport processes in living cells [9,37–39], cell motion and cargo transport on carpets of active cells [40,41], for trajectories of serotonergic fibers in the brain [42,43], and for the motion of animals [44,45]. Outside of biology, enhanced diffusion is observed in cosmic rays [46,47], in soft Lorentz gases [48], and in quantum many-body systems [49].

Different stochastic mechanisms may give rise to the same MSD scaling while the other statistical characteristics are fundamentally distinct. For example, anomalous diffusion processes with similar MSD behavior may have different probability density functions (PDFs) and correlation functions, or may or may not show aging [20,50–52]. Therefore, a reliable interpretation of SPT data requires not only identifying the diffusion exponent, but also assessing other properties of the process. Model classification of an anomalous diffusion process is a central goal of the recent community-wide AnDi project [53,54].

One of the standard approaches for model classification uses a comparison of the MSD, Eq. (1), with its time-averaged counterpart called time-averaged mean-squared displacement (TAMSD),

$$\text{TAMSD}(\Delta, T) = \frac{1}{T - \Delta} \int_0^{T-\Delta} [x(t' + \Delta) - x(t')]^2 dt', \quad (2)$$

considered as a function of the “lag time” Δ and averaged along a single trajectory of length T . In this context, Δ is considered as an analog of t in Eq. (1). In SPT studies,

comparing the $\text{MSD}(t)$ with $\text{TAMSD}(t, T)$ in a long- T limit has been widely used to identify and distinguish different anomalous diffusion processes in both theoretical models and experimental systems [20,55]. The equivalence of both was considered as a sign of “ergodicity”, while the discrepancies between the two are often interpreted as signatures of ergodicity breaking [56,57]. Here we show that while this concept is an important concept in that it creates awareness that time and ensemble averaged quantities may differ from each other and thus care needs to be taken for parameter regression from data, a more careful approach is needed when talking about (non)-ergodicity of a given system.

Ergodicity is a fundamental concept in the theory of dynamical systems and stochastic processes. The term “ergodic” originates from the Greek words “ergon” (work) and “odos” (path), and was originally introduced by Boltzmann in his ergodic hypothesis within the kinetic theory of gases in statistical mechanics [58]. The formalization of this concept came through the ergodic theorem, developed by Birkhoff [59], Khinchin [60] and von Neumann [61], initiating a new field of mathematical research called ergodic theory—that has thrived ever since and was of great significance in various fields, including mathematics [60,62], physics [63], economics [64], and social sciences [65]. Essentially, ergodicity is an asymptotic property which asserts that for an asymptotically stationary stochastic process $x(t)$, the time average of an observable quantity $\mathcal{O}(x)$ over an infinite time span converges to its ensemble average,

$$\underbrace{\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathcal{O}(x(t')) dt'}_{\text{Time average } \bar{\mathcal{O}}} = \underbrace{\int_{-\infty}^{\infty} \mathcal{O}(x) P_{\text{st}}(x) dx}_{\text{Ensemble average } \langle \mathcal{O} \rangle}, \quad (3)$$

where $P_{\text{st}}(x)$ is the *stationary* PDF assigning weights to x . If the limit on the left-hand-side of Eq. (3) exists, it must be time independent, and so must be the right-hand side. Thus all ergodic processes are stationary processes (but not necessarily vice versa). For instance, Gaussian noise and Lévy noise qualify as both stationary and ergodic [66]. In contrast, certain special stochastic processes, e.g., $X(t) = C$, where C is a random variable, are stationary but not ergodic processes. If Eq. (3) holds, averaging over repeated experiments in an ensemble of similar systems yields the same results as a time average over a single, but prolonged experimental run, as shown by the schematic in Fig. 1.

Here we distinguish between ergodicity in dynamical systems and in stochastic processes. In dynamical systems theory, ergodicity refers to whether the system as a whole is ergodic. If it is, any observable (i.e., measure-preserving transformation) will satisfy Eq. (3). To assess this property, however, one must know both the state space and the underlying dynamics of the system. In contrast, in the case

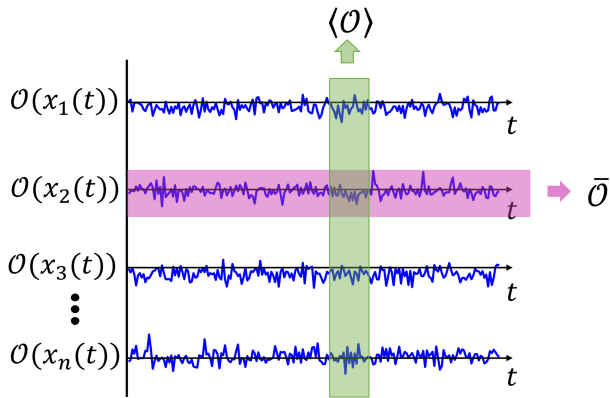


FIG. 1. Schematic for the ergodicity of an observable $\mathcal{O}$ of a stochastic process $x(t)$ with stationary PDF $P_{\text{st}}(x)$. The ensemble average $\langle \mathcal{O} \rangle = \int_{-\infty}^{\infty} \mathcal{O}(x) P_{\text{st}}(x) dx$ and the time average $\bar{\mathcal{O}} = (1/T) \int_0^T \mathcal{O}(x(t')) dt'$ of one single trajectory are highlighted by magenta and green, respectively. The process $x(t)$ is ergodic with respect to $\mathcal{O}$ if $\lim_{T \rightarrow \infty} \bar{\mathcal{O}} = \langle \mathcal{O} \rangle$.

of stochastic processes in experimental settings, little or nothing is typically known about the full state space of the system, and random processes provide only a reduced description of its dynamics. In this context, Eq. (3) serves as an operational definition of ergodicity for a specific observable. Since only measurable quantities can be analyzed, ergodicity must be defined separately for each statistical property of interest, for instance, whether a process is ergodic with respect to its mean, second moment, or correlation function. We also note that Eq. (3) may not hold when the long-time limit on the left-hand side is not defined. For instance, the corresponding integral may still fluctuate even in the limit of infinite data acquisition times. This case corresponds to the situation when the initial condition is never forgotten. Different scenarios may take place. One option is that the state space of the system may consist of several disconnected parts, and starting in each of them leads to a different value of the time average, a situation called “strong ergodicity breaking” [67]. Alternatively, the state space may be connected but the dynamics on it “freezes,” and a relevant trajectory cannot sample it fully even at infinite time. This situation corresponds to what is called “weak ergodicity breaking” [20,67,68]. Note that these situations cannot be resolved on the basis of single trajectories only, and there is not much difference from the point of view of the stochastic processes whether the system itself is nonergodic or ergodic but exhibiting an extremely slow relaxation [55]. Weak ergodicity breaking is observed in many experimental situations in which systems displaying aging behavior, traditionally included glassy systems such as spin glasses [67], gels [69], nanocrystals [70,71], or quantum many-body systems [72,73]. Scenarios of strong ergodicity breaking in similar systems are discussed in [74,75].

In SPT experiments involving diffusion phenomena studies, the main observable is the displacement $x(t + \Delta) - x(t)$, i.e., the increment of the process $x(t)$. Note that Eq. (3) typically defines the equality of the ensemble and time averages for an observable depending on the value of x at one single time, while the squared displacements considered in Eqs. (1) and (2) involve the values of x at two times. This fact is indeed of relevance, as we proceed to show. The notion of ergodicity is generalized to such two-time variables (by considering a set of increments for a fixed time lag), and most interest is paid to the behavior of the squared increment. The statistic corresponding to the left hand side of Eq. (3) is now the TAMSD, Eq. (2). The statistic corresponding to the right hand side of Eq. (3) can be defined in different ways. The standard way is using the MSD defined by Eq. (1), which explicitly depends on the initial condition. This approach was applied across various processes, both in theoretical studies and experiments [2,7,8,76–80], giving the possibility to distinguish between different processes. However, it is not suitable as the test for ergodicity. For instance, for BM, MSD-TAMSD equivalence holds, in spite of the fact that the process is nonstationary and should be classified as nonergodic. Moreover, the Ornstein-Uhlenbeck process starting at the minimum of the potential, is asymptotically stationary and ergodic, however it yields a TAMSD that is twice the MSD at long times, as we will see below.

In this paper, we show through several generic examples which are well-known random processes and commonly used in the description of SPT experiments, that the widely used criterion based on the comparison of the MSD and TAMSD is not suitable for determining the (non-)ergodic nature of a process. *In fact, the MSD represents the statistical property of the initial increment $x(t) - x(0)$, whereas, the definition of the TAMSD employs arbitrary increments $x(t + \Delta) - x(t)$ whose statistical properties may be different from those of the initial increment.* Using the mean-squared increment (MSI) instead of the MSD, that is

$$\text{MSI}(t, \Delta) = \langle [x(t + \Delta) - x(t)]^2 \rangle, \quad (4)$$

offers a robust and consistent framework for evaluating genuine (non-)ergodicity. Especially, when the process has stationary increments, the MSI reduces to the MSD. However, if the process reaches stationarity at long times only, the MSD is different from the MSI, with the latter being equal to the TAMSD if the increment process is ergodic. Based on the preceding discussion of ergodicity in stochastic processes, it is important to emphasize that, in the present work, the behaviors of the MSI and TAMSD are considered. The result for ergodicity of the increments of the process should not, however, be interpreted as demonstrating ergodicity of the stochastic process $x(t)$ with respect to other observables or higher-order statistics.

Comparing the behavior of the MSI and TAMSDs provides an additional statistic, which is useful in discriminating different classes of stochastic processes.

The MSI is essentially the structure function introduced by Kolmogorov in his study of Gaussian, self-similar processes with stationary, power-law-correlated increments—commonly referred to as Wiener spirals [81]. Such processes have been employed in the analysis of locally homogeneous and isotropic turbulence [82,83], providing a statistical characterization of turbulent flows. Kolmogorov’s pioneering work laid the foundation for later advancements, particularly the 1968 work by Mandelbrot and Van Ness in which they formulated an explicit integral representation of these processes—termed fractional Brownian motion (FBM) [84]. Today, FBM is recognized as a canonical example of a Gaussian process with long-range correlations, characterized by the Hurst exponent H [85]; for $H = 1/2$, FBM reduces to classical BM. Notably, FBM exhibits stationary and ergodic increments (FBM itself is nonstationary and thus nonergodic)—properties that can be effectively evaluated through the MSI and TAMSD. *In contrast, similar to BM, the equivalence of MSD and TAMSD traditionally used in the SPT framework, falsely suggests ergodicity for FBM* [20]. This example highlights the limitations of the traditional viewpoint and demonstrates the advantages of an MSI-based analysis for investigating ergodic behavior.

In particular, the MSI is instrumental in understanding the phenomenon of “ultraweak ergodicity breaking” [86–88], where a process demonstrates the same scaling behavior in both the MSD and TAMSD, yet with differing prefactors (see also [89]). While this mismatch breaks the MSD-TAMSD equivalence—signaling that the process itself is nonergodic—the increments can still exhibit asymptotic stationarity at long times, as revealed by the MSI for FBM in the form of a Riemann-Liouville fractional integral (RL-FBM) [90], and for Lévy walks [91].

The outline of this paper is as follows. Section II introduces the fundamental physical observables of stochastic processes frequently employed in SPT experiments and presents both the traditional MSD-TAMSD criterion and the proposed MSI-TAMSD criterion for evaluating (non-)ergodicity. Sections III to VI examine several well-established stochastic models frequently encountered in theoretical and experimental studies, demonstrating the limitations of the traditional MSD-TAMSD approach, which may lead to spurious (non-)ergodicity, and highlighting the robustness and effectiveness of our proposed MSI-TAMSD criterion. Specifically, Sec. III shows that the conventional MSD-TAMSD equivalence falsely indicates ergodicity in BM and FBM, while MSI-TAMSD accurately captures the ergodic nature of their increments. Section IV explores cases of spurious nonergodicity in the Ornstein-Uhlenbeck process (OUP) and diffusion processes under resetting—both are in fact stationary and ergodic processes.

Section V addresses the ultraweak ergodicity breaking in RL-FBM and Lévy walks—here the MSI-TAMSD criterion can reveal the asymptotic stationarity and ergodicity of the increments in these systems. In Sec. VI, the MSI-TAMSD criterion is applied to the continuous-time random walk (CTRW) model to assess the genuine ergodicity and nonergodicity in the cases with exponential waiting time PDF and power-law waiting time PDF, respectively. Finally, in Secs. VII and VIII, we summarize and discuss our results.

II. MAIN OBSERVABLES OF STOCHASTIC PROCESSES

SPT provides the entire trajectory of the tracked particle which allows one to evaluate different observables to quantify the dynamics of the system under observation. The most central measurable to quantify the averaged diffusive behavior of tracer particles is the MSD, defined through the mathematical expectation as

$$\langle \Omega^2(t) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - x_0)^2 P(x, t | x_0, 0) P_0(x_0) dx dx_0. \quad (5)$$

Here $P(x, t | x_0, 0)$ is the propagator (Green’s function) of the process and $P_0(x_0)$ is the PDF of the initial position x_0 .

The corresponding empirical MSD is obtained by averaging the time series $x_i(t)$ over ensembles of trajectories (labeled with the subscript i) at a given time t , with respect to each trajectory’s initial position,

$$\begin{aligned} \langle \Omega^2(t) \rangle_{\text{emp}} &= \langle [x(t) - x(0)]^2 \rangle_{\text{emp}} \\ &= \frac{1}{N} \sum_{i=1}^N [x_i(t) - x_i(0)]^2. \end{aligned} \quad (6)$$

Here N is the total number of trajectories. The number N of trajectories should be chosen as large as possible, but sometimes is limited by experimental conditions.

The MSI, qualifying the increment of displacement

$$\delta(\Delta; t) = x(t + \Delta) - x(t) \quad (7)$$

during the lag time Δ starting at physical time t , is defined as the mean-squared of the increment in the form [92]

$$\langle \delta^2(\Delta; t) \rangle = \langle [x(t + \Delta) - x(t)]^2 \rangle. \quad (8)$$

In analogy to the empirical MSD, the empirical MSI is obtained by averaging over N trajectories,

$$\langle \delta^2(\Delta; t) \rangle_{\text{emp}} = \frac{1}{N} \sum_{i=1}^N [x_i(t + \Delta) - x_i(t)]^2. \quad (9)$$

The MSI is equal to the MSD $\langle \Omega^2(\Delta) \rangle$ if the process has stationary increments. The definition of the MSI is identical to the structure function originally introduced by Kolmogorov and later employed by Yaglom in their works on locally homogeneous and isotropic turbulence [82,83,93–95].

The diffusive dynamics of an individual particle can be also quantified from a single particle trajectory $x(t)$ via the TAMSD [3,20,56]

$$\overline{\delta^2(\Delta; T)} = \frac{1}{T - \Delta} \int_0^{T-\Delta} [x(t' + \Delta) - x(t')]^2 dt', \quad (10)$$

where T is the length of the time series (measurement time) and Δ is the lag time. The TAMSD is typically used to evaluate few but long time series garnered in SPT experiments, e.g., in biological cells, of geo-tagged larger animals, or of financial time series [2,3,96,97]. We note that we use the term “TAMSD” for historical reasons. *In fact, the TAMSD also concentrates on increments and more precisely, this quantity represents the time-averaged squared-increment.*

Based on the TAMSD one can obtain the ensemble-averaged TAMSD (EA-TAMSD or mean TAMSD) by averaging over an ensemble of N individual trajectories in the form [3,20,56]

$$\langle \overline{\delta^2(\Delta; T)} \rangle_{\text{emp}} = \frac{1}{N} \sum_{i=1}^N \overline{\delta_i^2(\Delta; T)}. \quad (11)$$

A routine comparison criterion for a given time series in experiments is the equivalence of the MSD and the TAMSD in the limit of long trajectories $\Delta/T \ll 1$,

$$\overline{\delta^2(\Delta; T)} \sim \langle \Omega^2(\Delta) \rangle_{\text{emp}}. \quad (12)$$

This relation is frequently employed to assess the ergodicity of stochastic processes [3,20,56], and its violation is commonly interpreted as an indication of weak ergodicity breaking. While this MSD-TAMSD criterion (12) has been successfully applied in a large number of systems revealing weak ergodicity breaking phenomena, it contradicts the classical definition of ergodicity given by (3) and physical intuition. Especially, when applied to nonstationary processes with stationary increments, such as BM and FBM, the equivalence (12) of MSD and TAMSD may lead to spurious ergodicity. Conversely, it may wrongly indicate nonergodicity in systems that are actually stationary and ergodic, such as the OUP, as shown below. Moreover, unlike the classical definition (3), the MSD-TAMSD criterion (12) compares different physical observables on each side of the equation, further weakening its theoretical justification. The difference between the ensemble average (6) and the moving window average (10), is in fact due to the nonstationarity of the process under consideration.

We note that in the definition (10) of the TAMSD, the moving time window method essentially tests the properties of the increments of the process at different times, and ergodicity breaking can have its source either in the nonstationarity of the increments, or in the fact that different trajectories of the process, whether with stationary increments or not, are intrinsically very different from one another. Instead, in line with the classical definition of ergodicity given by relation (3), we propose to employ a different comparison criterion to test the ergodic nature of a process: the equivalence between the MSI and the TAMSD in SPT experiments for sufficiently long trajectories (T large) in the long-time limit (t large), that is,

$$\overline{\delta^2(\Delta; T)} \sim \langle \delta^2(\Delta; t) \rangle_{\text{emp}}. \quad (13)$$

This relation, central to this work, describes the ergodicity of the increments $\delta(\Delta; t) = x(t + \Delta) - x(t)$. In theoretical calculations, we should begin by checking the equivalence

$$\lim_{\Delta/T \rightarrow 0} \overline{\delta^2(\Delta; T)} = \lim_{\Delta/t \rightarrow 0} \langle \delta^2(\Delta; t) \rangle \quad (14)$$

between the ensemble average of the TAMSD (10) and the MSI (8). In the next step we then evaluate the ergodicity breaking (EB) parameter [3,20,56,98]

$$\text{EB}(\Delta; T) = \frac{\langle (\overline{\delta^2(\Delta; T)})^2 \rangle - \langle \overline{\delta^2(\Delta; T)} \rangle^2}{\langle \overline{\delta^2(\Delta; T)} \rangle^2}, \quad (15)$$

which measures the variance of the TAMSD amplitudes for a given combination of Δ and T . The EB parameter vanishes as $T \rightarrow \infty$ for processes whose increments are ergodic. As detailed below in the example of the sub-diffusive CTRW with a scale-free waiting time PDF, EB converges to a finite value in the limit $T \rightarrow \infty$. For further details on the EB parameter in both BM and FBM, we refer the reader to Appendix A. We also note that in some literature instead of EB parameter, the closely related relative standard deviation of the TAMSD or the magnitude correlation function are employed [99–101].

In the following, by examining several well-established stochastic models frequently encountered in theoretical and experimental studies, we demonstrate that the MSI-TAMSD criterion (13) offers broader applicability for assessing the (non-)ergodic nature of stochastic processes, effectively eliminating the spurious (non-)ergodicity indicated by the MSD-TAMSD criterion (12).

III. SPURIOUS ERGODICITY

Let us start by considering FBM, a long-range dependent Gaussian process that can be defined in terms of the Langevin equation driven by an external random force exhibiting power-law correlations (memory),

$$\frac{d}{dt}x(t) = \sqrt{2K_{2H}}\xi_H(t). \quad (16)$$

Here $\xi_H(t)$ is fractional Gaussian noise, which is a stationary Gaussian process with zero-mean and long-time power-law decay of the ACVF, $\langle \xi_H(t)\xi_H(t') \rangle \sim H(2H-1)|t-t'|^{2H-2}$ ($|t-t'| \rightarrow \infty$). Here we used the Hurst exponent H . The latter is connected to the anomalous diffusion exponent via $\alpha = 2H$, and its range is $H \in (0, 1]$. Ordinary BM corresponds to the limiting case $\alpha = 1$ in which the correlations vanish [84,102].

Unconfined FBM itself is nonstationary and thus non-ergodic, but its sequence of increments is stationary and ergodic [103]. The ergodicity of fractional Gaussian noise is a consequence of Khinchin's theorem stating that any stationary Gaussian process is ergodic if its ACVF decays to zero at infinity [104].

For FBM, the MSD, MSI, and EA-TAMSD, are identical, i.e., [87]

$$\langle \Omega^2(\Delta) \rangle = \langle \delta^2(\Delta; t) \rangle = \overline{\langle \delta^2(\Delta; T) \rangle} = 2K_{2H}\Delta^{2H}. \quad (17)$$

The equivalence between the MSD and EA-TAMSD incorrectly suggests ergodicity of the process itself, since FBM is nonstationary and thus nonergodic. Instead, examining the equivalence between the MSI and the EA-TAMSD, together with the vanishing variance of the TAMSD in the limit $T \rightarrow \infty$ —that is, the EB parameter approaching zero [103]—confirms the validity of Eq. (13). This demonstrates that the increments of these processes exhibit ergodic behavior. Further details on the EB parameter for FBM are provided in Appendix A.

Memory effects can also originate from the viscoelastic properties of the medium. For example, in the cytoplasm of a biological cell, a particle moves through a crowded environment containing elastic structures, where the cytoplasm “pushes back”, generating long-time correlations in the particle's trajectory [105]. In this case, the particle exhibits subdiffusive FBM, which can be described by the fractional Langevin equation (FLE) [106]. Analogous to FBM, the FLE has identical MSD, MSI and EA-TAMSD. Therefore it has ergodic increments [103], which can be revealed by the equivalence between MSI and EA-TAMSD, even though the process itself governed by the FLE is nonergodic.

IV. SPURIOUS ERGODICITY BREAKING

Consider a process $x(t)$ with a specific initial condition $x(0) = 0$ that becomes asymptotically stationary after a characteristic time τ . In the stationary state, the second moment of $x(t)$ attains a constant value denoted by $\langle x^2 \rangle_{\text{st}}$. The EA-TAMSD can then be rewritten in the form

$$\begin{aligned} \overline{\langle \delta^2(\Delta; T) \rangle} &= \frac{1}{T-\Delta} \int_0^\tau \langle [x(t'+\Delta) - x(t')]^2 \rangle dt' \\ &\quad + \frac{1}{T-\Delta} \int_\tau^{T-\Delta} \langle [x(t'+\Delta) - x(t')]^2 \rangle dt' \\ &\approx 2\langle x^2 \rangle_{\text{st}}, \end{aligned} \quad (18)$$

where in the last step we assumed that $\tau \ll \Delta \ll T$. In this limit the first term in the top line of Eq. (18) vanishes, and the second term corresponds to the stationary state. This result indicates that the TAMSD becomes twice the MSD, which equals the second moment under the condition $x(0) = 0$ in the stationary state, thereby leading to a spurious nonergodic behavior, according to the MSD-TAMSD criterion (12). In this section, we study two prototype ergodic systems: the OUP and diffusion with stochastic resetting. Both the OUP and diffusion with stochastic resetting converge to a stationary state. We demonstrate that the MSI can reproduce the ergodicity of such processes, via the comparison (13) with the EA-TAMSD. The corresponding EB parameter is discussed in Appendix A.

A. Ornstein-Uhlenbeck process

1. Normal Ornstein-Uhlenbeck process

The OUP is one of the most fundamental stochastic processes, with widespread application in financial mathematics and the physical sciences [107,108]. Originally proposed to describe the velocity distribution and relaxation of a Brownian particle subject to a friction force, the OUP is a Gaussian and Markovian process. It exhibits a stationary state and is ergodic. The process is governed by the overdamped Langevin equation [109,110]

$$\frac{dx}{dt} = -\lambda x + \sigma \xi(t), \quad (19)$$

with the initial condition x_0 . $\xi(t)$ is white Gaussian noise—the time derivative of Brownian motion (or Wiener process)—with zero mean and ACVF $\langle \xi(t)\xi(t') \rangle = \delta(t-t')$. The strength of the restoring force defines the characteristic relaxation time λ^{-1} , and σ quantifies the noise strength. When $\lambda \rightarrow 0$, the OUP reduces to the free BM with diffusion coefficient $K_1 = \sigma^2/2$.

Solving the stochastic differential equation (19), $x(t)$ can be formally expressed as

$$x(t) = e^{-\lambda t} \left(x_0 + \sigma \int_0^t e^{\lambda s} \xi(s) ds \right). \quad (20)$$

When $t \gg \lambda^{-1}$, the stationary state of the OUP follows the Gaussian stationary distribution

$$P_{\text{st}}(x) = \sqrt{\frac{1}{2\pi\langle x^2 \rangle_{\text{st}}}} \exp\left(-\frac{x^2}{2\langle x^2 \rangle_{\text{st}}}\right), \quad (21)$$

with the stationary value

$$\langle x^2 \rangle_{\text{st}} = \frac{\sigma^2}{2\lambda} \quad (22)$$

of the second moment.

The ACVF of the OUP is given by [111,112]

$$\langle x(t_1)x(t_2) \rangle = \left(\langle x_0^2 \rangle - \frac{\sigma^2}{2\lambda} \right) e^{-\lambda(t_1+t_2)} + \frac{\sigma^2}{2\lambda} e^{-\lambda|t_1-t_2|}. \quad (23)$$

Notably, in the long-time limit $t_1, t_2 \gg \lambda^{-1}$, the process becomes asymptotically stationary and thus ergodic. In the case when the initial condition x_0 is drawn from the stationary distribution (21), i.e., $\langle x_0^2 \rangle = \sigma^2/(2\lambda)$, the process is ergodic starting from $t = 0$ with the ACVF depending solely on the time difference $|t_1 - t_2|$.

From expression (23) for the ACVF, we can derive the MSD,

$$\langle \Omega^2(t) \rangle = \langle x_0^2 \rangle (1 - e^{-\lambda t})^2 + \frac{\sigma^2}{2\lambda} (1 - e^{-2\lambda t}), \quad (24)$$

the MSI

$$\langle \delta^2(\Delta; t) \rangle = \langle \Delta x^2 \rangle_{\text{st}} (1 - e^{-\lambda \Delta})^2 e^{-2\lambda t} + \frac{\sigma^2}{\lambda} (1 - e^{-\lambda \Delta}), \quad (25)$$

and the EA-TAMSD

$$\begin{aligned} \overline{\langle \delta^2(\Delta; T) \rangle} &= \langle \Delta x^2 \rangle_{\text{st}} (1 - e^{-\lambda \Delta})^2 \frac{1 - e^{-2\lambda(T-\Delta)}}{2\lambda(T-\Delta)} \\ &+ \frac{\sigma^2}{\lambda} (1 - e^{-\lambda \Delta}). \end{aligned} \quad (26)$$

In Eqs. (25) and (26), the factor $\langle \Delta x^2 \rangle_{\text{st}} = (\langle x_0^2 \rangle - \sigma^2/[2\lambda])$ measures the difference between the variance of the initial condition and the stationary state.

In the limit $\lambda \rightarrow 0$, the OUP reduces to free BM with stationary increments. In this scenario, the MSD, MSI, and EA-TAMSD are all equivalent, as given by Eq. (17) with $H = 1/2$ and $K_1 = \sigma^2/2$.

When $\lambda \neq 0$, the system behavior depends on the choice of the initial conditions. If the process is initialized with the stationary distribution (21), i.e., $\langle x_0^2 \rangle = \sigma^2/(2\lambda)$, the MSD, MSI, and EA-TAMSD all coincide,

$$\langle \Omega^2(\Delta) \rangle = \langle \delta^2(\Delta; t) \rangle = \overline{\langle \delta^2(\Delta; T) \rangle} = \frac{\sigma^2}{\lambda} (1 - e^{-\lambda \Delta}). \quad (27)$$

At long times $\Delta \gg \lambda^{-1}$, all three measures converge exponentially to the stationary value σ^2/λ , which corresponds to twice the second moment (22) in the stationary state. The equivalence (27) indicates that, under stationary initial conditions, both MSI and MSD reflect the ergodic behavior in the normal OUP.

This is no longer true when the initial condition is nonstationary, i.e., $\langle x_0^2 \rangle \neq \sigma^2/(2\lambda)$. In that case the system gradually relaxes towards its stationary state. After the time $t \gg \lambda^{-1}$, the MSD converges to

$$\langle \Omega^2(t) \rangle \sim \langle x_0^2 \rangle + \langle x^2 \rangle_{\text{st}}, \quad (28)$$

which explicitly depends on the initial condition. In contrast, in the same limit $t \gg \lambda^{-1}$ and with $T \gg \lambda^{-1}$, both the MSI (25) and EA-TAMSD (26) exhibit the same asymptotic behavior

$$\langle \delta^2(\Delta; t) \rangle \sim \overline{\langle \delta^2(\Delta; T) \rangle} \sim \frac{\sigma^2}{\lambda} (1 - e^{-\lambda \Delta}), \quad (29)$$

which thus differ from the MSD (28). We see that both MSI and EA-TAMSD are independent of the initial condition and identical to expression (27) with stationary initial conditions. In addition, when $\Delta \gg \lambda^{-1}$, we have

$$\langle \delta^2(\Delta; t) \rangle \sim \overline{\langle \delta^2(\Delta; T) \rangle} \sim 2\langle x^2 \rangle_{\text{st}}, \quad (30)$$

corresponding to twice the stationary value (22), as already seen above.

Interestingly, the stationary value (28) of the MSD contradicts the usual intuition that once the process reaches stationarity, all memory of the initial condition is lost. In contrast, the initial condition variance $\langle x_0^2 \rangle$ does not appear in the limiting value (29) of TAMSD and MSI. Figure 2(a) presents the MSD, MSI, and TAMSDs of 50 individual trajectories of the normal OUP (19). Although the OUP is stationary (asymptotically) and ergodic, the fact that the TAMSD (29) does not match the MSD (28) when the process arrives at the stationary state at long times, may lead to a spurious nonergodicity, as shown in Fig. 2(a). This discrepancy indicates that the MSD is not a suitable observable for testing ergodicity in this context. However, as seen from Eqs. (29) and (30), the TAMSD and MSI exhibit the same asymptotic behavior when $t, T \gg \lambda^{-1}$ in both the short and long lag time regime, making the MSI a more reliable measure for verifying ergodicity in the OUP. Moreover, the negligible amplitude variation among the TAMSDs of individual trajectories indicates a high reproducibility, as shown in Fig. 2(a), reinforcing the ergodic nature of the process.

2. Fractional Ornstein-Uhlenbeck process

The fractional OUP, i.e., the process (19) driven by fractional Gaussian noise instead of white Gaussian noise,

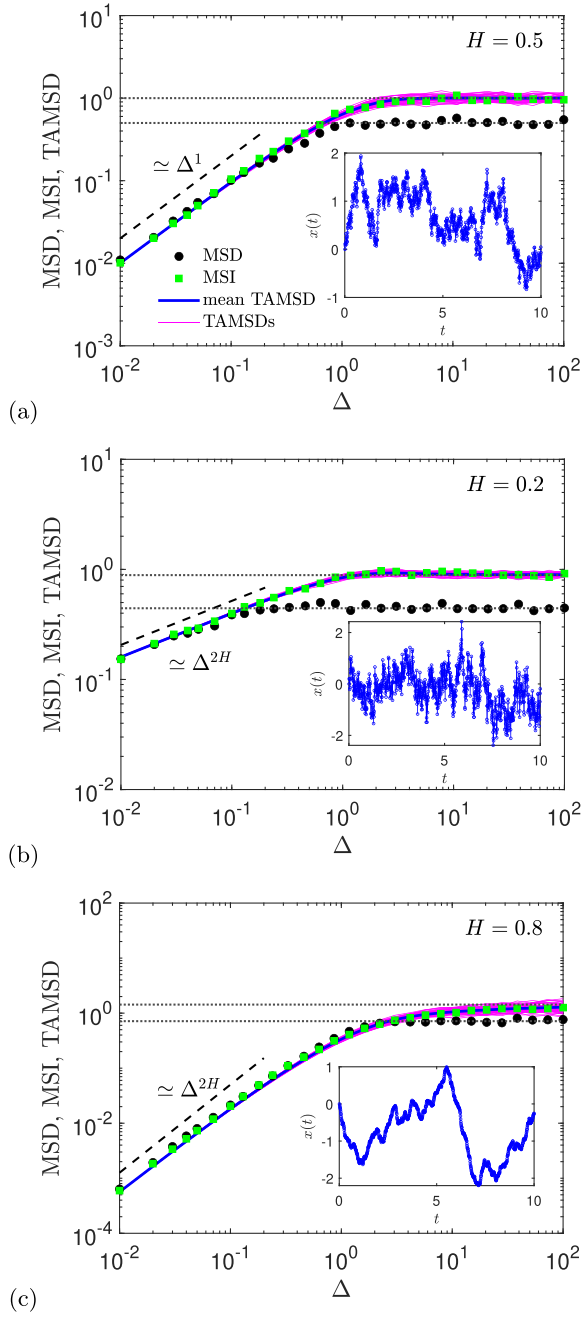


FIG. 2. Simulation results for fractional OUP (31) with initial condition $x_0 = 0$: MSD (black circles), MSI (green rectangles), and TAMSDs of 50 individual trajectories (magenta curves) for (a) normal diffusion with $H = 0.5$, (b) subdiffusion with $H = 0.2$, and (c) superdiffusion with $H = 0.8$. When $H = 0.5$, the fractional OUP reduces to the normal OUP (19) with initial condition $x_0 = 0$. The EA-TAMSD is represented by solid blue curves. The variation (scatter) of the amplitudes among individual TAMSDs is negligible, indicating that individual trajectories are completely reproducible. The stationary values of the MSI and MSD are indicated by dotted lines. Single trajectories are shown in the insets. Parameters: time step $dt = 0.01$, measurement time $T = 500$, starting time for the MSI $t = 10$, noise strength $\sigma_{2H} = 1$, relaxation rate $\lambda = 1$, and number of trajectories $N = 200$.

is an extension of the normal OUP [111,113,114]. The fractional OUP is a Gaussian yet non-Markovian process and has been applied in finance for the description of a “rough volatility” [115,116], and in fluid dynamics to reflect the temporal structure of turbulence [117,118]. The fractional OUP can be described by the Langevin equation [111,114]

$$\frac{dx}{dt} = -\lambda x + \sigma_{2H} \xi_H(t), \quad (31)$$

where $x_0 = x(t = 0)$ defines the initial condition, characteristic relaxation time is λ^{-1} , σ_{2H} quantifies the noise strength, and $\xi_H(t)$ denotes the fractional Gaussian noise introduced above. When $\lambda \rightarrow 0$, the fractional OUP reduces to the free FBM with generalized diffusion coefficient $K_{2H} = \sigma_{2H}^2/2$. In particular, when $H = 1/2$, the fractional OUP reduces to the normal OUP [119].

Similar to the normal OUP, at long times $t \gg \lambda^{-1}$, the fractional OUP becomes stationary and ergodic [114], and its PDF follows the Gaussian distribution

$$P_{\text{st}}(x) = \frac{1}{\sqrt{2\pi \langle x_{2H}^2 \rangle_{\text{st}}}} \exp\left(-\frac{x^2}{2 \langle x_{2H}^2 \rangle_{\text{st}}}\right), \quad (32)$$

where

$$\langle x_{2H}^2 \rangle_{\text{st}} = \frac{\sigma_{2H}^2}{2\lambda^{2H}} \Gamma(2H + 1). \quad (33)$$

We note that this stationary value explicitly depends on the Hurst index, a signature that the fractional OUP is governed by a Langevin equation driven by fractional Gaussian noise, in which the noise is treated as external, as is typical for open systems [122].

The formal solution to the Langevin equation (31) takes the form

$$x(t) = e^{-\lambda t} \left(x_0 + \sigma_{2H} \int_0^t e^{\lambda s} \xi_H(s) ds \right). \quad (34)$$

The ACVF can be expressed in terms of special functions, including the incomplete gamma function and the Kummer function, as detailed in Appendix B. Then, the MSD is given by

$$\begin{aligned} \langle \Omega^2(t) \rangle &= \langle x_0^2 \rangle (1 - e^{-\lambda t})^2 + \frac{\sigma_{2H}^2}{2\lambda^{2H}} \gamma(2H + 1, \lambda t) \\ &+ \sigma_{2H}^2 t^{2H} e^{-\lambda t} - \frac{\lambda \sigma_{2H}^2}{2(2H + 1)} t^{2H+1} e^{-2\lambda t} \\ &\times \mathcal{M}(2H + 1, 2H + 2, \lambda t), \end{aligned} \quad (35)$$

where

$$\gamma(z, x) = \int_0^x e^{-t} t^{z-1} dt \quad (36)$$

is the lower incomplete gamma function and

$$\mathcal{M}(a, b, z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt \quad (37)$$

is the Kummer function (the confluent hypergeometric function of the first kind) [123]. In particular, for $H = 1/2$, as $M(2, 3, \lambda t) = 2[e^{\lambda t}(\lambda t - 1) + 1]/(\lambda t)^2$ and $\gamma(2, \lambda t) = 1 - e^{-\lambda t}(\lambda t + 1)$, the MSD reduces to Eq. (24) of the normal OUP.

As we have seen above, the normal OUP is always stationary given the stationary initial conditions. In contrast, the fractional OUP only attains stationarity asymptotically in the long-time limit [111], as indicated by its ACVF (B1).

At long times $t \gg \lambda^{-1}$, the MSD (35) of the fractional OUP approaches a value that depends on the initial conditions,

$$\langle \Omega^2(t) \rangle \sim \langle x_0^2 \rangle + \langle x_{2H}^2 \rangle_{\text{st}}. \quad (38)$$

In contrast, when both $t, T \gg \lambda^{-1}$, the MSI and EA-TAMSD exhibit the same asymptotic behavior,

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &\sim \overline{\langle \delta^2(\Delta; T) \rangle} \\ &\sim \frac{\sigma_{2H}^2}{\lambda^{2H}} \Gamma(2H+1) + \sigma_{2H}^2 \Delta^{2H} \\ &\quad - \frac{\lambda \sigma_{2H}^2}{2(2H+1)} \Delta^{2H+1} e^{-\lambda \Delta} \mathcal{M}(2H+1, 2H+2, \lambda \Delta) \\ &\quad - \frac{\sigma_{2H}^2}{2\lambda^{2H}} \{ e^{\lambda \Delta} \Gamma(2H+1, \lambda \Delta) + e^{-\lambda \Delta} \Gamma(2H+1) \}. \end{aligned} \quad (39)$$

In addition, at long lag times $\Delta \gg \lambda^{-1}$, both the MSI and TAMSD converge to a value that is independent of the initial condition,

$$\langle \delta^2(\Delta; t) \rangle \sim \overline{\langle \delta^2(\Delta; T) \rangle} \sim 2\langle x_{2H}^2 \rangle_{\text{st}}. \quad (40)$$

Figure 2 depicts simulation results for the MSD, MSI, and TAMSDs of the fractional OUP (31) for different Hurst exponents. The TAMSDs (40) approach stationary values different from the stationary MSD (38), and from the MSD-TAMSD criterion (12) it may lead to the spurious non-ergodicity of the systems. However, the TAMSDs closely follow the behavior of the MSI across both short and long lag times, indicating the underlying ergodic nature.

A second key difference between normal and fractional OUPs concerns the manner in which the MSD, MSI, and TAMSD approach their stationary values. In the case of the normal OUP, all these observables exhibit exponential relaxation, as shown in Eqs. (24) and (29). For the

fractional OUP, however, the dynamics differ significantly, as reported in [124] and tested experimentally in [26]. Specifically, the MSD with respect to the initial condition $x(0) = 0$ relaxes exponentially (with power-law correction) towards its stationary value, as shown in Eq. (B7) of Appendix B,

$$\langle \Omega^2(\Delta) \rangle \sim \langle x_{2H}^2 \rangle_{\text{st}} - \frac{2\sigma_{2H}^2}{\lambda^2} H(2H-1) \Delta^{2H-2} e^{-\lambda \Delta}. \quad (41)$$

In contrast, the TAMSD and MSI exhibit the same asymptotic behavior [see Eq. (39)], and both follow a power-law relaxation toward stationarity, as shown in Eq. (B9) in Appendix B,

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &\sim \overline{\langle \delta^2(\Delta; T) \rangle} \\ &\sim 2\langle x_{2H}^2 \rangle_{\text{st}} - \frac{2\sigma_{2H}^2}{\lambda^2} H(2H-1) \Delta^{2H-2}. \end{aligned} \quad (42)$$

This result reinforces the conclusion that the MSI is a more reliable observable than the MSD for assessing ergodicity in both the normal and the fractional OUP.

B. Fractional Brownian motion under stochastic resetting

The simplest form of stochastic resetting involves the diffusion of a single particle that intermittently returns to its initial position at a constant, nonzero rate [125]. One of the most prominent effects of resetting is observed in the position distribution of the particle. Without resetting, the diffusion process is nonstationary; however, introducing a nonzero resetting rate leads to a nonequilibrium stationary state (NESS) at long times [126]. Resetting also significantly influences the associated first-passage time properties—it has been shown to accelerate search processes, and an optimally chosen resetting rate can minimize the mean first-passage time to a target [127,128]. Beyond search problems, stochastic resetting has a broad range of applications, including stochastic thermodynamics [129], chemical reaction kinetics [130], extreme-value statistics [131], and the dynamics of complex networks [132]. An experimental implementation of resetting dynamics was demonstrated in [133]. The (non-)ergodic properties of systems under stochastic resetting have been explored in recent studies, for instance, for BM [134,135], FBM [136], and geometric BM [137,138]. Partial resetting, when the position is reset to a fraction of its distance to the origin, and incomplete resetting, where the dynamics of the underlying process remain only partly affected by the reset (e.g., when the memory of the process prior to the resetting event is not fully erased), have also been investigated [139–144].

Here we consider FBM under Poissonian resetting with the rate r to the origin. Although this model has been shown

to reach stationarity, with its ACVF vanishing at long times [145], we demonstrate that the process is indeed ergodic with respect to the MSI-TAMSD criterion (13). Moreover, our findings can be extended to a broad class of anomalous diffusion processes under Markovian resetting—where each reset event renews the system’s memory and the waiting times between resets have a finite mean [136]. We further show that relying solely on the MSD, its comparison to the TAMSD may lead to misleading conclusions regarding ergodicity. In contrast, when the MSI is employed instead of the MSD, the ergodic nature of the process is clearly recovered.

The PDF of FBM under a constant resetting rate r is given by [146]

$$P(x, t) = e^{-rt} P_0(x, t) + \int_0^t r e^{-r\tau} P_0(x, \tau) d\tau, \quad (43)$$

where $P_0(x, t) = \exp[-x^2/(4K_{2H}t^{2H})]/\sqrt{4\pi K_{2H}t^{2H}}$ is the free-space propagator describing the probability density of an FBM particle, initially at the origin $x(0) = 0$, to reach position x at time t in the absence of resetting. The first term on the right-hand side of Eq. (43) corresponds to trajectories, in which no resetting occurs up to time t . The second term accounts for trajectories, that experience at least one reset before time t , with the last reset occurring at time $t - \tau$, followed by a period of free FBM evolution over the remaining duration τ . The factor $r e^{-r\tau}$ is the probability density that the last reset occurred at time $t - \tau$, followed by no further resets during the interval of length τ .

Multiplying Eq. (43) by x^2 and performing the integration with respect to x , we obtain the MSD

$$\langle \Omega^2(t) \rangle = 2K_{2H} \left[e^{-rt} t^{2H} + \frac{\gamma(2H+1, rt)}{r^{2H}} \right], \quad (44)$$

where $\gamma(a, t)$ is the lower incomplete gamma function [123]. When the process becomes stationary at long times $t \gg r^{-1}$, the MSD saturates to the value

$$\begin{aligned} \langle \Omega^2(t) \rangle &\sim 2K_{2H} \left[\frac{\Gamma(2H+1)}{r^{2H}} - \frac{2H}{r} t^{2H-1} e^{-rt} \right] \\ &\sim \frac{2K_{2H}}{r^{2H}} \Gamma(2H+1). \end{aligned} \quad (45)$$

The exact MSI and EA-TAMSD are presented in Appendix C using the renewal approach. For $t \gg r^{-1}$ and $T \gg r^{-1}$, the MSI and TAMSD exhibit the same asymptotic behavior

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &\sim \overline{\langle \delta^2(\Delta; T) \rangle} \\ &\sim 2K_{2H} \left[e^{-r\Delta} \Delta^{2H} + \frac{\gamma(2H+1, r\Delta)}{r^{2H}} \right. \\ &\quad \left. + (1 - e^{-r\Delta}) \frac{\Gamma(2H+1)}{r^{2H}} \right]. \end{aligned} \quad (46)$$

In particular, when $\Delta \gg r^{-1}$ the process approaches the stationary state exponentially,

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &\sim \overline{\langle \delta^2(\Delta; T) \rangle} \\ &\sim 4K_{2H} \left[\frac{\Gamma(2H+1)}{r^{2H}} - \frac{H}{r} \Delta^{2H-1} e^{-r\Delta} \right]. \end{aligned} \quad (47)$$

Thus, both MSI and EA-TAMSD approach twice the stationary value of the MSD (45),

$$\overline{\langle \delta^2(\Delta; T) \rangle} \sim \langle \delta^2(\Delta; t) \rangle \sim 2\langle \Omega^2(\Delta) \rangle. \quad (48)$$

It is worth noting that, in the long-time limit $\Delta \gg r^{-1}$, the MSD, MSI, and EA-TAMSD all approach their stationary values exponentially with a power-law correction, following the form $\Delta^{2H-1} e^{-r\Delta}$, as shown in Eqs. (45) and (47).

Additionally, we verify the EB parameter through simulations, which vanishes in the long-time limit of the trajectory length T (see Fig. 7). This again emphasizes a key point: the stationary MSD (45) is not a reliable indicator of ergodicity when compared with the TAMSD (48). By contrast, the MSI faithfully captures the ergodic property, as shown in Eq. (48).

Figure 3 displays the simulation results for reset FBM and is in perfect agreement with our theory. We note that for $H \geq 1/2$, the resetting events are clearly visible in the trajectories, whereas for $H < 1/2$, these events are obscured by the strong fluctuations.

For both normal and fractional OUPs, at short lag times $\Delta \ll \lambda^{-1}$, the processes are dominated by free BM and FBM, so that the MSD and MSI (or TAMSD) coincide. However, as shown in Fig. 3(a) and 3(c), for $H \geq 1/2$, noticeable deviations between the MSD and MSI (TAMSD) arise at short times $\Delta \ll r^{-1}$. This discrepancy can be explained from the short-time approximation (C5) for the MSI and TAMSD,

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &\sim \overline{\langle \delta^2(\Delta; T) \rangle} \\ &\sim 2K_{2H} \left(\Delta^{2H} + \frac{\Gamma(2H+1)}{r^{2H-1}} \Delta^1 \right). \end{aligned} \quad (49)$$

Unlike the MSD, which remains unaffected by resetting at short times and follows $\langle \Omega^2(\Delta) \rangle \sim 2K_{2H} \Delta^{2H}$, for MSI and TAMSD, however, the linear term in Eq. (49) dominates when $H > 1/2$, resulting in a different scaling behavior. When $H = 1/2$, both terms contribute equally, leading the MSI and TAMSD to be approximately twice the value of the MSD at short times.

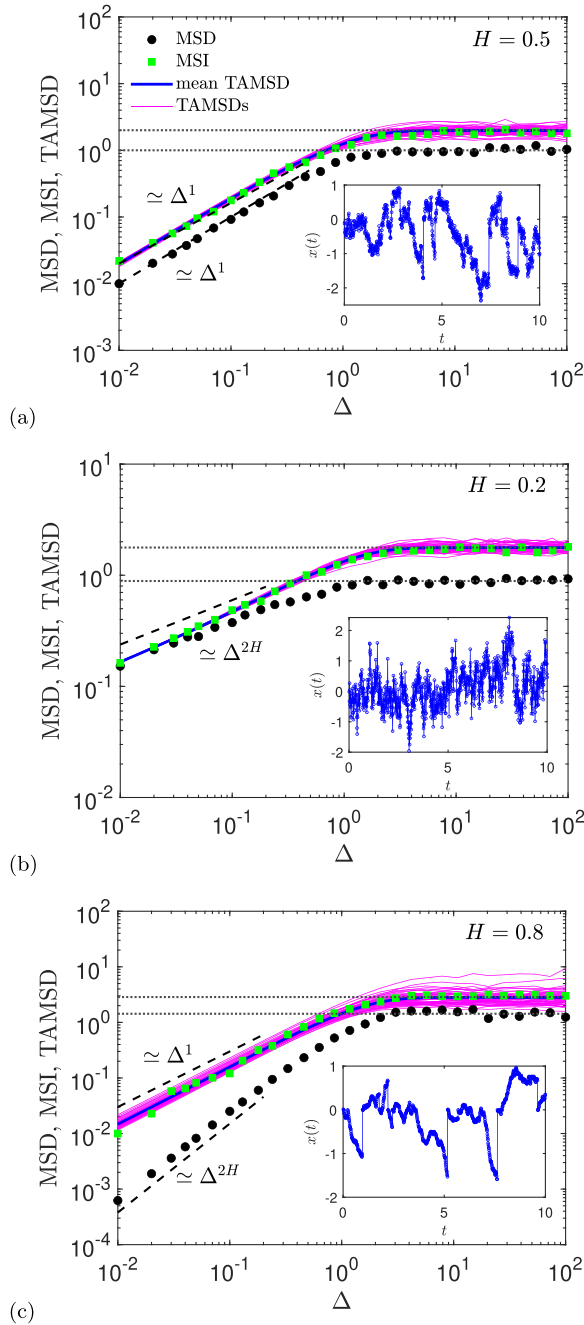


FIG. 3. Simulation results for FBM under Poissonian resetting with the rate r to the initial position $x(0) = 0$, showing MSD (black circles), MSI (green rectangles), and TAMSDs of individual trajectories (magenta curves) with (a) $H = 0.5$, (b) $H = 0.2$, and (c) $H = 0.8$. The EA-TAMSD is represented by solid blue curves. Other parameters: time step $dt = 0.01$, measurement time $T = 500$, starting time of the MSI $t = 10$, and resetting rate $r = 1$.

V. SPURIOUS ULTRAWEAK ERGODICITY BREAKING

“Ultraweak ergodicity breaking” refers to the phenomenon in which the MSD shares the same scaling behavior

with the EA-TAMSD, but the two differ by a constant prefactor, thus violating the equivalence (12) of the MSD-TAMSD. In this section, we provide two examples: RL-FBM and Lévy walks, both widely used stochastic models for biophysical systems. Despite the nonergodicity of the process itself, the increments are asymptotically stationary and thus ergodic at long times.

A. Riemann-Liouville fractional integral representation of fractional Brownian motion

Recently, another variant of the fractional Langevin equation, namely, the fractional Langevin equation far from equilibrium (FLEFE)

$$\frac{d^\alpha x(t)}{dt^\alpha} = \sqrt{2K_\alpha} \xi(t) \quad (50)$$

with $\alpha > 0$ was studied [87,147]. The initial conditions are specified as $x^{(k)}(0) = 0$ for $k = 0, \dots, [\alpha]$, where $[\cdot]$ denotes the integer part of a positive real number. $\xi(t)$ here represents white Gaussian noise with zero mean and ACVF $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$. The fractional derivative is to be interpreted in the Caputo sense with order $n - 1 < \alpha < n$, where $n \in \mathbb{N}^+$ [148,149],

$$\frac{d^\alpha x(t)}{dt^\alpha} \equiv {}_0^C D_t^\alpha [x(t)] = \int_0^t x^{(n)}(u) \frac{(t-u)^{-\alpha+n-1}}{\Gamma(n-\alpha)} du. \quad (51)$$

The conventional fractional Langevin equation originally suggested by Mori and Kubo [106,108,150,151] provides a stochastic description of thermalized systems near equilibrium and thus obeys the fluctuation-dissipation theorem (FDT) [150]. In contrast, the FLEFE does not obey the FDT, making it a more suitable candidate for evaluating the observed stochastic dynamics in systems out of equilibrium, including the movement of birds and other animals, or active particles such as bacteria and amoeba; or financial market fluctuations and climate dynamics.

The solution of the FLEFE is identical to FBM defined in terms of a Riemann-Liouville fractional integral (RL-FBM) with an initial condition $x(0) = 0$ and with Hurst exponent $H = \alpha - 1/2$, given by [90,147]

$$x(t) = \sqrt{2K_\alpha} \int_0^t \frac{(t-t')^{\alpha-1}}{\Gamma(\alpha)} \xi(t') dt'. \quad (52)$$

The RL-FBM was first introduced by Lévy [84,152].

Unlike FBM, RL-FBM has nonstationary increments. Recently, a variant of RL-FBM has found application in disordered systems with heterogeneous environments, e.g., when the diffusion coefficient and Hurst exponent are functions of time or space or even randomly chosen from certain distributions [153–156]. RL-FBM with time-dependent exponents has been successfully validated in terms of the MSD and the power spectral density through

recent experimental measurements of quantum dot motion within the cytoplasm of live mammalian cells, as observed by SPT [157].

The MSD of RL-FBM (52) reads

$$\langle \Omega^2(t) \rangle = \frac{2K_\alpha}{(2\alpha - 1)\Gamma(\alpha)^2} t^{2\alpha-1}, \quad (53)$$

and thus one can see that the MSD is physically meaningful for all $\alpha > 1/2$.

The exact MSI and EA-TAMSD can be expressed in terms of Fox H -functions [87],

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &= \frac{2K_\alpha}{(2\alpha - 1)\Gamma(\alpha)^2} [(t + \Delta)^{2\alpha-1} + t^{2\alpha-1}] \\ &\quad - \frac{4K_\alpha t^\alpha \Delta^{\alpha-1}}{\Gamma(\alpha)^2 \Gamma(1-\alpha)} H_{2.2}^{2.1} \left[\frac{\Delta}{t} \left| \begin{matrix} (1, 1), (1 + \alpha, 1) \\ (1 - \alpha, 1), (\alpha, 1) \end{matrix} \right. \right], \end{aligned} \quad (54)$$

and

$$\begin{aligned} \overline{\langle \delta^2(\Delta; T) \rangle} &= \frac{K_\alpha}{\alpha(2\alpha - 1)\Gamma(\alpha)^2} \\ &\quad \times \left[\frac{T^{2\alpha} - \Delta^{2\alpha}}{T - \Delta} + (T - \Delta)^{2\alpha-1} \right] \\ &\quad - \frac{4K_\alpha}{\Gamma(\alpha)^2 \Gamma(1-\alpha)} \frac{(T - \Delta)^\alpha}{\Delta^{1-\alpha}} \\ &\quad \times H_{3.3}^{1.3} \left[\frac{T - \Delta}{\Delta} \left| \begin{matrix} (-\alpha, 1), (\alpha, 1), (1 - \alpha, 1) \\ (0, 1), (-1 - \alpha, 1), (-\alpha, 1) \end{matrix} \right. \right]. \end{aligned} \quad (55)$$

From the exact expression (54) for the MSI, we see that the increments are nonstationary; however, at long times, the increments of RL-FBM become asymptotically stationary and moreover ergodic. This is because both the MSI and EA-TAMSD share the same asymptotic form for $1/2 < \alpha < 3/2$ and for both $t, T \gg \Delta$ [87],

$$\overline{\langle \delta^2(\Delta; T) \rangle} \sim \langle \delta^2(\Delta; t) \rangle \sim \frac{2K_\alpha}{\Gamma(2\alpha) |\cos(\pi\alpha)|} \Delta^{2\alpha-1}. \quad (56)$$

We also check by simulations the scatter of the TAMSDs characterized by the EB parameter with fixed lag time equal to the time step $\Delta = \delta t$ in Fig. 8. The results are the same as for FBM with Hurst exponent $H = \alpha - 1/2$, i.e., for $1/2 < \alpha < 1$, $EB(\Delta = \delta t; T) \simeq T^{-1}$ and for $1 < \alpha < 3/2$, $EB(\Delta = \delta t; T) \simeq T^{4\alpha-6}$. Therefore, the increments are ergodic.

It is notable that in this regime $1/2 < \alpha < 3/2$, in the long time limit the EA-TAMSD (56) differs from the MSD (53) in the prefactor, as shown in Fig. 4. This effect was

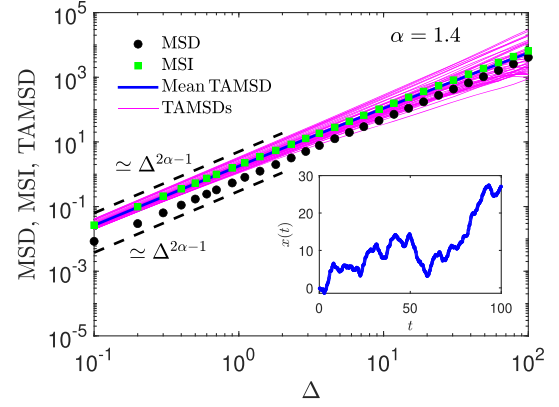


FIG. 4. Simulation results for RL-FBM (52) showing MSD (black circles), MSI (green rectangles), and TAMSDs of individual trajectories (magenta curves) with $\alpha = 1.4$. The EA-TAMSD is represented by solid blue curves. When $1/2 < \alpha < 3/2$, the EA-TAMSD of RL-FBM (52) is identical to the MSI rather than the MSD, leading to a spurious non-ergodicity. Other parameters: time step $dt = 0.01$, measurement time $T = 500$, starting time of MSI $t = 10$. The algorithm of numerical simulations is presented in Appendix C of [87].

referred to as ultraweak ergodicity breaking in Ref. [86]. Instead, we see that the EA-TAMSD converges to the MSI.

For $\alpha \geq 3/2$, the increments of FLEFE motion (RL-FBM) are nonergodic at all times $t > 0$; however, higher order increments are asymptotically ergodic with the $(n + 1)$ th order MSI ($n \geq 1$) restoring stationarity in the regime $(2n + 1)/2 < \alpha < (2n + 3)/2$ [87].

B. Lévy walks

The Lévy walk is a well-established stochastic model originally devised in [158] (see also [159]), combining two key features, the ability to generate anomalously faster motion than BM and a finite velocity of the random walker, with broad applications in physics and biology [38,160–167]. In the simplest form the Lévy walks can be formulated as follows [168] (see also [169] for a more general case): Consider a particle that alternates its velocity $v(t)$ randomly between the values $+v_0$ and $-v_0$ with independent and identically distributed random times τ ($0 < \tau < \infty$) between these velocity changes. The switching times are governed by the common PDF $\psi(\tau)$. To generate a sample of a Lévy walk, one can proceed as follows: The particle starts at $t = 0$ with velocity $+v_0$, travels a distance $v_0\tau_1$ where τ_1 is drawn from $\psi(\tau)$, before it is displaced by $-v_0\tau_2$, and so on. The position of the particle becomes $x(t) = \int_0^t v(t') dt'$. Lévy walks in confined domains and in external harmonic potentials were considered [170,171], including “soft” resetting in switching harmonic confinement [172].

The Lévy walk is inherently a nonstationary process and therefore nonergodic [86,91,173] and especially, the ultraweak ergodicity breaking was observed in Lévy walks with

power-law waiting time PDF. Here, our focus is to examine the equivalence between the MSI and EA-TAMSD in Lévy walks with exponential and power-law distributed waiting times. In particular, this comparison may indicate that the increments of the process become asymptotically stationary and ergodic even if the process itself remains nonergodic.

1. Exponentially distributed waiting times

We first examine Lévy walks with exponentially distributed waiting times, characterized by the waiting time PDF

$$\psi(\tau) = \lambda \exp(-\lambda\tau), \quad (57)$$

where λ denotes the rate parameter. In this scenario, utilizing the ACVF (D5) of the position in Appendix D, the MSD, MSI and EA-TAMSD all coincide,

$$\begin{aligned} \langle \Omega^2(\Delta) \rangle &= \langle \delta^2(\Delta; t) \rangle = \langle \overline{\delta^2(\Delta; T)} \rangle \\ &= \frac{v_0^2}{2\lambda^2} [2\lambda\Delta + e^{-2\lambda\Delta} - 1], \end{aligned} \quad (58)$$

indicating that the increments of Lévy walks are stationary and, furthermore, are ergodic. This is supported by Fig. 5(a), which shows consistent, reproducible TAMSDs.

At short times $\Delta \ll \lambda^{-1}$, we obtain

$$\langle \Omega^2(\Delta) \rangle = \langle \delta^2(\Delta; t) \rangle = \langle \overline{\delta^2(\Delta; T)} \rangle \sim v_0^2 \Delta^2, \quad (59)$$

thus revealing ballistic motion. In the long-time limit $\Delta \gg \lambda^{-1}$, the second moments become

$$\langle \Omega^2(\Delta) \rangle = \langle \delta^2(\Delta; t) \rangle = \langle \overline{\delta^2(\Delta; T)} \rangle \sim \frac{v_0^2}{\lambda} \Delta, \quad (60)$$

indicating normal-diffusive behavior.

Figure 5(a) presents the identical MSI, MSD, and EA-TAMSD for Lévy walks with exponentially distributed waiting times. Notably, the dynamics exhibit a distinct crossover from ballistic diffusion at short times to normal diffusion at long times.

2. Power-law distributed waiting times

We now turn to the case when the random waiting times τ follow an asymptotic power-law distribution as given by

$$\psi(\tau) = \frac{\beta/\tau_0}{(1 + \tau/\tau_0)^{1+\beta}}, \quad (61)$$

where τ_0 is a constant. For $0 < \beta < 1$, the mean waiting time $\langle \tau \rangle$ diverges, leading to ballistic motion of the process;

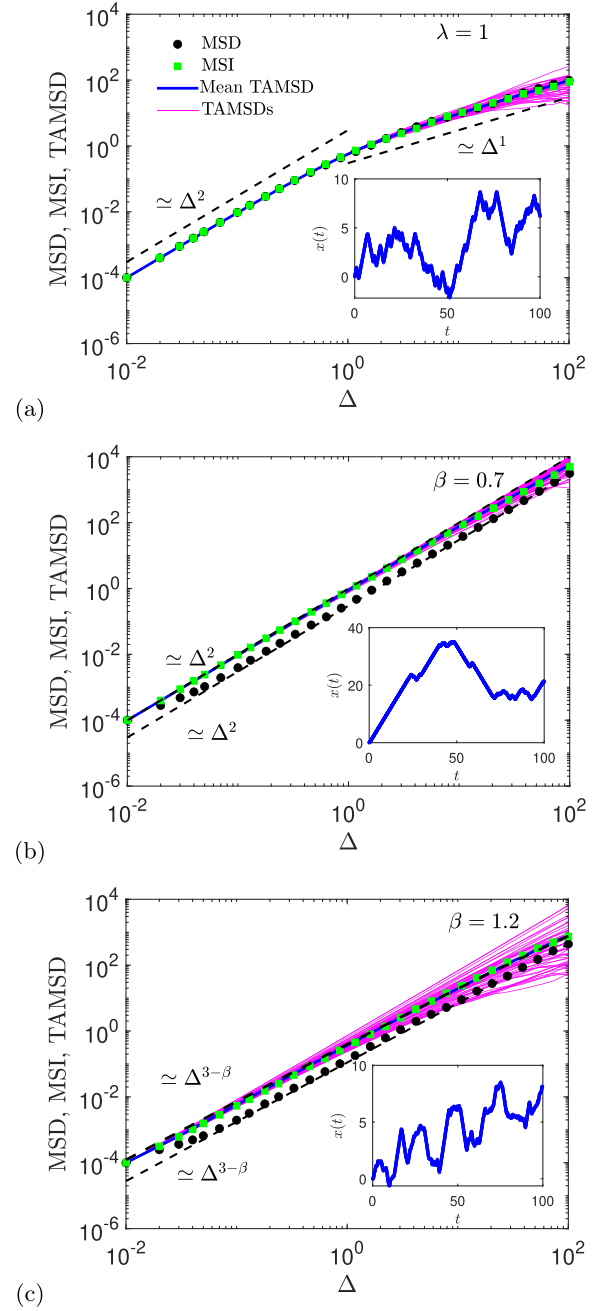


FIG. 5. Simulation results for Lévy walks showing the MSD (black circles), MSI (green rectangles), and TAMSDs of 50 individual trajectories (magenta curves) for (a) exponential waiting time PDF (57) with $\lambda = 1$ and a power-law waiting time PDF (61) with (b) $\beta = 0.7$, and (c) $\beta = 1.2$. The EA-TAMSD is represented by solid blue curves. The variation (scatter) of the amplitudes among individual TAMSDs is negligible, indicating that individual trajectories are completely reproducible. Other parameters: time step $dt = 0.01$, measurement time $T = 500$, starting time of MSI $t = 100$, and $\tau_0 = 0.01$. It is important to note that discrepancies in the MSD arise as the power-law density cannot be accurately generated in the short waiting time regime using the inverse transform sampling method.

for $1 < \beta < 2$, the mean becomes finite, $\langle \tau \rangle = \tau_0/(\beta - 1)$, and the process is superdiffusive [159,161].

In the ballistic motion regime $0 < \beta < 1$, the ACVF of the position can be expressed in terms of the beta function [91,173], as shown in Eq. (D7) in Appendix D. Consequently, the MSD is given by

$$\langle \Omega^2(t) \rangle = (1 - \beta)v_0^2 t^2, \quad (62)$$

which characterizes ballistic motion.

In Appendix D, we utilize the connection between the beta function and the Fox H function [174], along with the small-argument expansion of the H function, to derive the MSI for $t \gg \Delta$ as

$$\langle \delta^2(\Delta; t) \rangle \sim v_0^2 \Delta^2 \left[1 + \frac{2 \sin(\pi\beta)}{\pi(\beta - 3)(\beta - 2)(\beta - 1)} \left(\frac{\Delta}{t} \right)^{1-\beta} \right], \quad (63)$$

and the EA-TAMSD for $T \gg \Delta$ as

$$\langle \overline{\delta^2(\Delta; T)} \rangle \sim v_0^2 \Delta^2 \left[1 + \frac{2 \sin(\pi\beta)}{\pi(\beta - 3)(\beta - 2)(\beta - 1)\beta} \left(\frac{\Delta}{T} \right)^{1-\beta} \right]. \quad (64)$$

These results indicate that in the long-time limit, the increments of the process become approximately stationary and depend predominantly on the lag time. Moreover, the leading terms of the MSI and EA-TAMSD coincide and the following relation holds,

$$\langle \overline{\delta^2(\Delta; T)} \rangle \sim \langle \delta^2(\Delta; t) \rangle \sim \frac{\langle \Omega^2(\Delta) \rangle}{1 - \beta} = v_0^2 \Delta^2, \quad (65)$$

where the leading term of the EA-TAMSD (64) differs from the MSD (62) by the prefactor $1/(1 - \beta)$, as also reported in [86,89,91], demonstrating ultraweak ergodicity breaking behavior according to the MSD-TAMSD criterion (12). In contrast, we see that the leading term of the EA-TAMSD matches that of the MSI (63), reinforcing the notion that the increments become asymptotically stationary at long times. Additionally, the ergodicity of the increments is further supported by the observation that the fluctuations of the TAMSDs vanish in the long time limit, albeit slowly [91,173]. Interestingly, in the ballistic Lévy walk, non-ergodicity in terms of amplitude fluctuations of time averaged moments is not evident in the (decaying) fluctuations of the TAMSD itself, but rather in the persistent fluctuations of the shifted and rescaled dimensionless variable [173]

$$\xi = \frac{v_0^2 \Delta^2 - \overline{\delta^2(\Delta; T)}}{v_0^2 \Delta^2 - \langle \delta^2(\Delta; T) \rangle}. \quad (66)$$

Figure 5(b) presents simulation results illustrating the relationship between MSD, MSI, and TAMSD in the ballistic Lévy walk case. It is important to note that discrepancies observed in the simulated MSD stem from numerical inaccuracies in generating the power-law waiting time PDF (61) at short waiting time values using the inverse transform sampling method [175].

For the superdiffusive motion regime $1 < \beta < 2$, the MSD can be obtained from the ACVF provided in Eq. (D24) of Appendix D, resulting in

$$\langle \Omega^2(t) \rangle = 2K_{3-\beta} t^{3-\beta}, \quad (67)$$

where $K_{3-\beta}$ is the generalized diffusion coefficient given by

$$K_{3-\beta} = \frac{\beta - 1}{(3 - \beta)(2 - \beta)} v_0^2 \tau_0^{\beta-1}. \quad (68)$$

The exact expressions for the MSI and the EA-TAMSD are provided in Eqs. (D26) and (D28) in Appendix D. In the limit $t \gg \Delta$, the MSI (D26) can be approximated as

$$\langle \delta^2(\Delta; t) \rangle \sim \frac{2K_{3-\beta}}{\beta - 1} \Delta^{3-\beta} \left[1 - \frac{(3 - \beta)(2 - \beta)}{2} \left(\frac{\Delta}{t} \right)^{\beta-1} \right], \quad (69)$$

indicating that the increments become asymptotically stationary in the long-time limit, while the EA-TAMSD (D28) in the regime $T \gg \Delta$, takes on the form

$$\langle \overline{\delta^2(\Delta; T)} \rangle \sim \frac{2K_{3-\beta}}{\beta - 1} \Delta^{3-\beta} \left[1 - \frac{3 - \beta}{2} \left(\frac{\Delta}{T} \right)^{\beta-1} \right]. \quad (70)$$

Analogous to the ballistic regime, in the superdiffusive case and when $t, T \gg \Delta$ we find that MSI and EA-TAMSD converge to the same expression, but differ from the MSD (67) by the prefactor $1/(\beta - 1)$,

$$\langle \overline{\delta^2(\Delta; T)} \rangle \sim \langle \delta^2(\Delta; t) \rangle \sim \frac{\langle \Omega^2(\Delta) \rangle}{\beta - 1} = \frac{2K_{3-\beta}}{\beta - 1} \Delta^{3-\beta}. \quad (71)$$

This behavior is also confirmed by numerical simulations, as shown in Fig. 5(c). The EB parameter (15) decays with measurement time T , scaling as $T^{1-\beta}$ [173].

In summary, Lévy walks with power-law distributed waiting times, both in the ballistic diffusion regime ($0 < \beta < 1$) and for superdiffusion ($1 < \beta < 2$), exhibit asymptotically stationary increments at long times, as evidenced by the asymptotic forms (63) and (69) of the MSI. Furthermore, the convergence of the EA-TAMSD to the MSI in the long-time limit points to the ergodic nature of the increments. In contrast, comparisons between the MSD and TAMSD fail to capture this fundamental property of the increments.

VI. GENUINE ERGODICITY BREAKING

The MSI-TAMSD criterion (13) not only reveals spurious (non-)ergodicity in systems where the MSD-TAMSD criterion (12) fails, but is also effective for systems exhibiting genuine ergodicity breaking. In this section, we consider the CTRW, one of the most fundamental models describing anomalous diffusion [158,159,176–180].

For the sake of simplicity, we consider a one-dimensional uncoupled CTRW, where random steps are separated by random waiting times. The jump lengths and waiting times are independent random variables, each drawn identically from their respective distributions, and the jump lengths are assumed to have finite variance. In the subordination framework [181,182], the CTRW can be interpreted as the interplay of two distinct random processes, $x(t) = x[n(t)]$. The parent process $x(n)$ represents a simple random walk with discrete steps, while the directing process (or subordinator) $n(t)$ acts as a renewal counting process that determines the random number of steps taken up to the physical time t . The associated MSD and MSI satisfy [57,183,184]

$$\langle \Omega^2(t) \rangle = a^2 \langle n(t) \rangle, \quad (72)$$

and

$$\langle \delta^2(\Delta; t) \rangle = a^2 [\langle n(t + \Delta) \rangle - \langle n(t) \rangle], \quad (73)$$

where a^2 is the mean-squared displacement in a single step.

A. Exponentially-distributed waiting times

When the waiting time follows the exponential PDF (57) with rate λ , the mean of the counting process $n(t)$ takes the form [183,185]

$$\langle n(t) \rangle = \lambda t. \quad (74)$$

Using this result, we obtain the equivalence between MSD, MSI, and EA-TAMSD,

$$\langle \Omega^2(\Delta) \rangle = \langle \delta^2(\Delta; t) \rangle = \overline{\langle \delta^2(\Delta; T) \rangle} = 2K_1\Delta, \quad (75)$$

with the diffusion coefficient $K_1 = \lambda a^2/2$. Figure 6(a) depicts simulation results for the MSD, MSI, and TAMSDs of the CTRW with exponential waiting times. We note that the MSD, MSI, and TAMSD indeed coincide with each other, underlining its ergodic nature. It is important to note that in the continuum limit such a CTRW—with exponential waiting times and finite-variance jump lengths—shares the same statistical properties as standard Brownian motion.

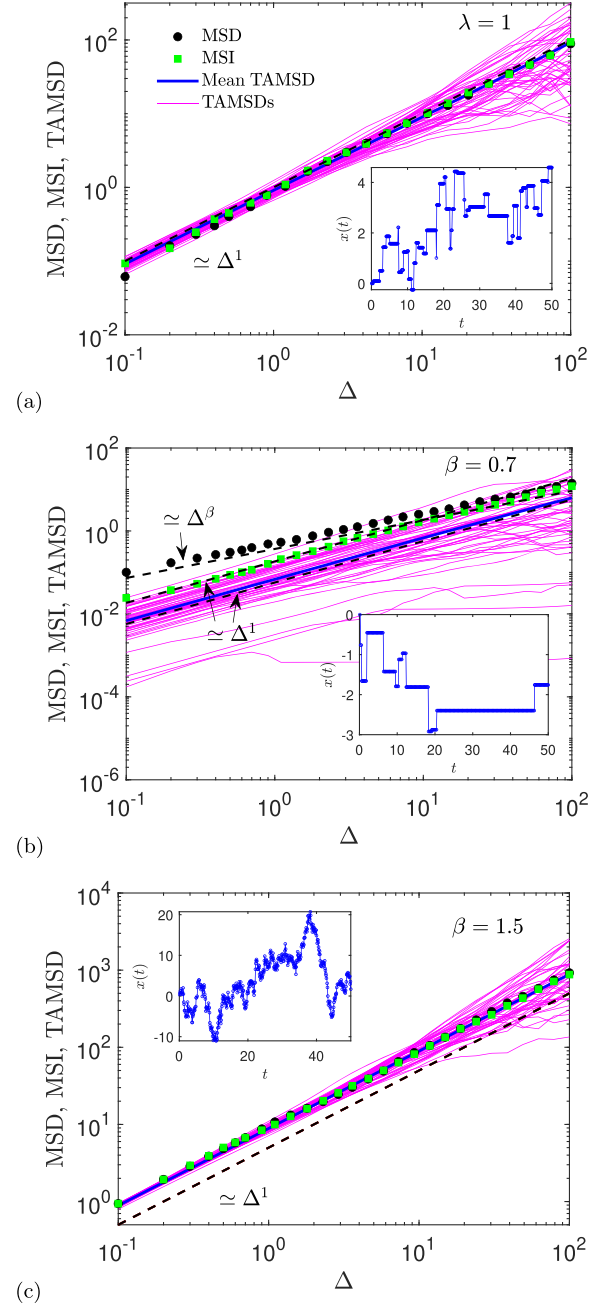


FIG. 6. Simulation results for CTRW showing MSD (black circles), MSI (green rectangles), and TAMSDs from 50 individual trajectories (magenta curves), for (a) exponential waiting time PDF (57) with $\lambda = 1$ and power-law waiting time PDF (61) with (b) $\beta = 0.7$ and (c) $\beta = 1.5$. The EA-TAMSD is represented by solid blue curves. For the ergodicity breaking in panel (b), the amplitude variation (scatter) among the TAMSDs of individual trajectories becomes significantly pronounced, indicating a complete lack of reproducibility between trajectories. Other parameters: time step $dt = 0.1$, measurement time $T = 500$, starting time of MSI $t = 10$, dimensionless time scale $\tau_0 = 0.01$, MSI in a single step $a^2 = 1$.

B. Power-law distributed waiting time

We now consider the case when the waiting time follows the power-law PDF (61). For $0 < \beta < 1$, the mean waiting time $\langle \tau \rangle$ diverges, while for $1 < \beta < 2$, it becomes finite, $\langle \tau \rangle = \tau_0/(\beta - 1)$.

In the scale-free case $0 < \beta < 1$, the mean of the counting process $n(t)$ with $t \gg \tau_0$ is given by [185,186]

$$\langle n(t) \rangle \sim \frac{\sin(\pi\beta)}{\pi\beta} \left(\frac{t}{\tau_0} \right)^\beta. \quad (76)$$

Accordingly, the MSD scales as

$$\langle \Omega^2(\Delta) \rangle \sim 2K_\beta \Delta^\beta, \quad (77)$$

where the generalized diffusion coefficient is given by $K_\beta = a^2 \sin(\pi\beta)/(2\pi\beta\tau_0^\beta)$.

In the long time limit $t \gg \Delta$ and $T \gg \Delta$ the MSI and EA-TAMSD are, respectively,

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &\sim 2K_\beta [(t + \Delta)^\beta - t^\beta] \\ &\sim 2\beta K_\beta \frac{\Delta}{t^{1-\beta}}, \end{aligned} \quad (78)$$

and

$$\begin{aligned} \overline{\langle \delta^2(\Delta; T) \rangle} &\sim \frac{2K_\beta}{\beta + 1} \frac{T^{\beta+1} - \Delta^{\beta+1} - (T - \Delta)^{\beta+1}}{T - \Delta} \\ &\sim 2K_\beta \frac{\Delta}{T^{1-\beta}}. \end{aligned} \quad (79)$$

Results from simulations for the CTRW with power-law waiting times are shown in Fig. 6(b), illustrating that the MSD, MSI, and EA-TAMSD are all distinct. However, both MSI and EA-TAMSD exhibit asymptotically linear scaling with the lag time, with prefactors that decay over time. This behavior indicates weak ergodicity breaking in CTRWs with scale-free, power-law distributed waiting times. Furthermore, the higher-order increments remain nonstationary in this regime; details can be found in Appendix E.

For $\beta > 1$, the mean of the counting process $n(t)$ for $t \gg \tau_0$ becomes [144]

$$\langle n(t) \rangle \sim (\beta - 1) \frac{t}{\tau_0}. \quad (80)$$

In this case, the MSD, MSI, and EA-TAMSD all coincide,

$$\langle \Omega^2(\Delta) \rangle \sim \langle \delta^2(\Delta; t) \rangle \sim \overline{\langle \delta^2(\Delta; T) \rangle} \sim 2K_1 \Delta, \quad (81)$$

where the diffusion coefficient is $K_1 = a^2(\beta - 1)/(2\tau_0)$. These results are also demonstrated by the simulations in Fig. 6(c). Compared with the CTRW with exponentially distributed waiting times, the equivalence between MSD, MSI, and EA-TAMSD holds only at times longer than τ_0 .

Moreover, a CTRW with heavy-tailed waiting times subject to an external force is described by the fractional Fokker-Planck equation [176,187] and exhibits weak ergodicity breaking [188].

VII. OVERVIEW OF THE RESULTS AND OUTLOOK

Anomalous diffusion has been widely deduced from SPT experiments and large scale computer simulations. While normal, Brownian diffusion is universal in its convergence to a Gaussian displacement PDF and the linear time dependence of the MSD, anomalous diffusion may have multiple origins and is described in terms of a multitude of stochastic processes [20,50,55]. Starting from measured or simulated data, an important aspect of stochastic time series analysis is the task of classification, determining which anomalous stochastic model process is indeed encoded in the data, as well as the regression of the associated model parameters. The classification can be achieved by statistical observables such as the MSD and TAMSD [3,20,55], the p -variation method [189], higher order moments, or the mean maximal excursion method [190], or by the scaling exponents J , L , and M introduced by Mandelbrot (see below) [76,191]. Moreover, dedicated Bayesian methods [192,193] and different machine learning approaches [53,54,194–201] have been developed for this classification and regression tasks. Here we proposed to include the MSI as a central observable and studied its performance for different anomalous stochastic processes.

In SPT studies, the measured positional time series provides the relevant information on the system, and the MSD remains the key physical observable to characterize anomalous diffusion. The discrepancy between MSD and TAMSD in the limit of long trajectories has been widely and effectively employed to measure nonergodicity (or weak ergodicity breaking) in numerous such systems. However, as argued here, the typically employed MSD-TAMSD criterion (12) may lead to incorrect conclusions when applied to some particular systems.

For real data, finite trajectory lengths and limited sampling constitute additional practical limitations. We illustrate these effects using simulated OUP trajectories in Appendix F. A finite measurement time primarily affects the TAMSD: when the lag time Δ becomes comparable to the trajectory length T , the number of available increments for time averaging decreases, leading to stronger trajectory-to-trajectory fluctuations and possible deviations of the mean TAMSD from the MSI, as shown in Fig. 9. Limited sampling, on the other hand, mainly affects ensemble-averaged quantities, such as the MSD and MSI, producing visible statistical fluctuations when only a small number of trajectories is available, as shown in Fig. 10. By contrast, the mean TAMSD is usually less sensitive to limited sampling, because each single-trajectory TAMSD already involves a time average over many increments.

TABLE I. Comparison of the statistical properties of stochastic models and the (non-)ergodic behavior of the processes themselves and their increments. Here, “Ergod.” denotes “Ergodicity.”

Process	MSD	MSI	EA-TAMSD	Ergod. of process	Ergod. of increment
FBM	$2K_{2H}\Delta^{2H}$, Eq. (17)	$2K_{2H}\Delta^{2H}$, Eq. (17)	$2K_{2H}\Delta^{2H}$, Eq. (17)	No	Yes
OUP	$\langle x_0^2 \rangle + \frac{\sigma^2}{2\lambda}$, Eq. (28)	$\frac{\sigma^2}{\lambda}$, Eq. (30)	$\frac{\sigma^2}{\lambda}$, Eq. (30)	Yes	Yes
Fractional OUP	$\langle x_0^2 \rangle + \frac{\sigma_{2H}^2 \Gamma(2H+1)}{2\lambda^{2H}}$, Eq. (38)	$\frac{\sigma_{2H}^2}{\lambda^{2H}} \Gamma(2H+1)$, Eq. (40)	$\frac{\sigma_{2H}^2}{\lambda^{2H}} \Gamma(2H+1)$, Eq. (40)	Yes	Yes
Reset FBM	$\frac{2K_{2H}}{r^{2H}} \Gamma(2H+1)$, Eq. (45)	$\frac{4K_{2H}}{r^{2H}} \Gamma(2H+1)$, Eq. (48)	$\frac{4K_{2H}}{r^{2H}} \Gamma(2H+1)$, Eq. (48)	Yes	Yes
RL-FBM	$\frac{2K_\alpha}{(2\alpha-1)\Gamma(\alpha)^2} \Delta^{2\alpha-1}$, Eq. (53)	$\frac{2K_\alpha}{\Gamma(2\alpha)\Gamma(\alpha)} \Delta^{2\alpha-1}$, Eq. (56)	$\frac{2K_\alpha}{\Gamma(2\alpha)\Gamma(\alpha)} \Delta^{2\alpha-1}$, Eq. (56)	No	Yes
Lévy walk	short time: $v_0^2 \Delta^2$, Eq. (59)	short time: $v_0^2 \Delta^2$, Eq. (59)	short time: $v_0^2 \Delta^2$, Eq. (59)	No	Yes
exponential w.t.	long time: $\frac{v_0^2}{\lambda} \Delta$, Eq. (60)	long time: $\frac{v_0^2}{\lambda} \Delta$, Eq. (60)	long time: $\frac{v_0^2}{\lambda} \Delta$, Eq. (60)		
Lévy walk power-law w.t. $0 < \beta < 1$	$(1-\beta)v_0^2 \Delta^2$, Eq. (62)	$v_0^2 \Delta^2$, Eq. (65)	$v_0^2 \Delta^2$, Eq. (65)	No	Yes
Lévy walk power-law w.t. $1 < \beta < 2$	$2K_{3-\beta} \Delta^{3-\beta}$, Eq. (67)	$\frac{2K_{3-\beta}}{\beta-1} \Delta^{3-\beta}$, Eq. (71)	$\frac{2K_{3-\beta}}{\beta-1} \Delta^{3-\beta}$, Eq. (71)	No	Yes
CTRW exponential w.t.	$2K_1 \Delta$, Eq. (75)	$2K_1 \Delta$, Eq. (75)	$2K_1 \Delta$, Eq. (75)	No	Yes
CTRW power-law w.t. $0 < \beta < 1$	$2K_\beta \Delta^\beta$, Eq. (77)	$2\beta K_\beta \frac{\Delta}{1-\beta}$, Eq. (78)	$2K_\beta \frac{\Delta}{1-\beta}$, Eq. (79)	No	No
CTRW power-law w.t. $1 < \beta < 2$	$2K_1 \Delta$, Eq. (81)	$2K_1 \Delta$, Eq. (81)	$2K_1 \Delta$, Eq. (81)	No	Yes

In addition, heterogeneity may further complicate the interpretation of real data. For instance, different cells or nanoparticles may have different intrinsic properties, such as size, shape, surface characteristics, local environments, or mobilities. Such heterogeneity can generate an additional scatter of individual TAMSDs and make signatures of nonergodicity harder to identify. From the perspective of superstatistics [202], the measured ensemble may be viewed as a mixture of subensembles with different effective diffusivities or dynamical parameters. Even if each subensemble is ergodic, the combined population can display broad distributions of single-trajectory observables and apparent discrepancies between ensemble- and time-averaged quantities. Thus, an observed mismatch between ensemble- and time-averaged observables in real data may originate not only from the stochastic dynamics itself, but also from heterogeneity and experimental variability.

The three major results of this work are as follows. First, we demonstrated that the MSD-TAMSD criterion (12) may introduce a spurious ergodicity when applied to nonstationary and nonergodic processes but with stationary increments, e.g., BM and FBM, and in turn produce spurious weak ergodicity breaking in stationary and ergodic systems, e.g., the OUP and FBM under stochastic resetting. Second, we propose that the MSI is a more suitable observable than the MSD to assess the (non) ergodic properties of a stochastic process via comparison with the TAMSD. The new MSI-TAMSD criterion (13) in fact coincides with the classic definition (3) of ergodicity and reflects the true (non)ergodic nature of the above systems where the MSD fails. Third, we found that the MSI

can recognize long-time stationarity and ergodicity of increments of the process in systems exhibiting ultraweak ergodicity breaking, in which the MSD and TAMSD have the same scaling but a different prefactor. The behavior of MSD, MSI, and EA-TAMSD for the 11 specific models considered in the text and discussed above are summarized in Table I. These results are ready to be used in the statistical analysis of particular experimental situations. Our work has broad relevance beyond time-series analysis. The examples considered in the manuscript are therefore not intended as a collection of model-specific applications, but rather serve to establish the domains of validity of the MSD-TAMSD comparison and the circumstances under which the MSI-TAMSD comparison is required.

The distinction between the MSD-TAMSD and MSI-TAMSD comparisons can be summarized as follows. For processes with a stationary state, a mismatch between MSD and TAMSD may simply reflect a nonstationary preparation of the initial ensemble: the MSD retains information about the initial condition, whereas the TAMSD probes increments after relaxation. This is the mechanism behind the spurious nonergodicity found for the OUPs. For nonstationary processes, such as FBM, lacking a stationary distribution, the situation is different. The issue is no longer whether the initial distribution coincides with a stationary one, but whether the lag-time increment process is stationary, or asymptotically stationary. Since the TAMSD is a time average of squared increments, its natural ensemble counterpart is the MSI, not the MSD. The MSI-TAMSD comparison therefore provides a direct test of the ergodicity of increments, while the MSD-TAMSD comparison

remains a useful diagnostic of aging, initial-condition dependence, and nonstationarity.

Recent SPT studies revealed that intracellular transport of endo- and exogenous tracers of various sizes is often not only anomalous, but also heterogeneous in time and space. This implies that a single diffusion coefficient or scaling exponent of standard anomalous-diffusion models is insufficient to describe the underlying physical phenomena [203–209]. Two main classes of models have been proposed to address such complexity. The first is the diffusing diffusivity (DD) model [210–219], in which the diffusion coefficient is treated as a stochastic process with temporal correlations. Over short timescales, the diffusion coefficient remains relatively stable due to strong correlations, reflecting local behavior. Over longer timescales, this memory fades, and the particle motion is described by an effective diffusivity when exploring the entire available space. Consequently, the DD model typically yields stationary and ergodic increments. The second class assumes that either the diffusion coefficient or the anomalous diffusion exponent is a random variable drawn from a distribution, for example, FBM with a random diffusivity [220,221] or random exponent [155,156,222]. In these models, the increments are stationary but nonergodic, because each trajectory maintains a fixed, strongly correlated diffusivity or exponent. As a result, individual TAMSDs remain inherently irreproducible.

Beyond physical and biological systems, our findings may also find applications in finance, particularly in models such as the Black-Scholes [223] and Heston [224] frameworks. In these models, the asset price $S(t)$ follows $dS(t)/S(t) = \sqrt{\nu(t)}dW(t)$, where $\nu(t)$ denotes the price volatility. In the Black-Scholes model, the volatility is constant, whereas in the Heston model it follows a Cox-Ingersoll-Ross (CIR) process [225], which guarantees non-negativity, similar in spirit to the DD model. The Heston model reproduces well several important stylized facts of low-frequency price data, namely, leverage effect, time-varying volatility, and fat tails [226]. Recent high-frequency data analyses suggest that volatility exhibits rough behavior, with the log-volatility effectively following a fractional Brownian motion (FBM) across practical timescales [115]. Because asset prices grow exponentially, the standard MSI is not directly applicable; instead, a logarithmic MSI can be defined as

$$R^2(\Delta, t) = \langle (\log S(t + \Delta) - \log S(t))^2 \rangle. \quad (82)$$

For a nonstationary process, the origin of time can no longer be chosen arbitrarily. This raises the question of aging, that is, the explicit dependence of physical observables on the time span t_a between the original preparation of the system and the start of the recording of data. Aging is a key property of glassy systems [227] but was also rationalized in polymeric semiconductors [228], potassium

channel motion in the membrane of human embryonic kidney cells [79], living MIN6 insulinoma cells [8], the surface of hippocampal neurons [229], groundwater systems [230], quantum dots [70], drug molecules in silica slabs [231], as well as financial time series [97,232]. Aging effects can be studied via the aged MSD [183], the latter being identical to the MSI and aged TAMSD in temporally or spatially heterogeneous processes [233,234].

In both RL-FBM and Lévy walk, which exhibit the phenomenon of ultraweak ergodicity breaking, the asymptotic stationarity of the increments over long times is demonstrated by the MSI. One might wonder if this asymptotic stationarity of the increments is universal in ultraweak ergodicity breaking phenomena. The answer lies in the origin of anomalous diffusion. The EA-TAMSD and MSD can be expressed as [76,191] $\langle \delta^2(\Delta; T) \rangle \simeq T^{2L+2M-2} \Delta^{2J}$ and $\langle \Omega^2(\Delta) \rangle \simeq \Delta^{2H}$, where the exponent J represents long-range correlations (“Joseph effect”), L is related to the fat-tailed probability density of increments (“Noah effect”), and M reflects nonstationarity (the “Moses effect”). These exponents satisfy the relation $H = J + L + M - 1$ [76,235]. In ultraweak ergodicity breaking, only a constant factor between MSD and TAMSD is observed, leading to $H = J$, hence $L + M = 1$, suggesting that processes with aging ($M < 1/2$) and heavy-tailed PDF of increments ($L > 1/2$) could induce similar scaling of MSD and TAMSD. RL-FBM and Lévy walks are specific cases with asymptotically stationary increments ($M = 1/2$) and nonheavy tails ($L = 1/2$).

VIII. CONCLUSIONS

SPT approaches have found extensive applications in many domains of science, particularly in biological systems on subcellular, cellular and organism levels. Drawing reasonable conclusions from SPT measurements implies application of different statistical approaches to their analysis and classification, in order to place the process at hand into one of the broader classes of stochastic processes (or to show that it belongs to a yet unknown class) or for parameter regression. One of these approaches is based on the comparison of the MSD of a particle from its initial position at $t = 0$ with a moving time average of the squared displacement over a fixed time interval Δ (TAMSD). The beginning of this Δ -interval, at time t , slides over the entire available part of the trajectory of the process, $0 \leq t \leq T - \Delta$. Moreover, in some cases an additional averaging over the ensemble of trajectories is applied.

The equality of the two means above is often interpreted as a sign of the ergodicity of the process, while their difference is interpreted as a breaking thereof. While the difference or equality of these means can be used as a signature of the process, allowing to place the process at hand into the class of random processes with similar properties, the interpretation of the corresponding finding as a sign of ergodicity (breaking) is at best vague. The reason for this ambiguity is the

indiscriminate application of the definition of the ergodicity for a single-time observable [$\mathcal{O}(x(t))$ in Eq. (3)] to two-time observables [increments $x(t + \Delta) - x(t)$].

Ergodicity of a random process leads to the independence of the property of interest on initial conditions at long times, and is an asymptotic property. However, the MSD is an ensemble mean of the very first increment of the process, between $t = 0$ (which may be the preparation time of the system, when the initial condition applies) and $t = \Delta$. It may keep an explicit dependence on the initial condition, and be different from the time average even for a process which is ergodic according to a precise mathematical definition, as we have shown considering a particular example of the OUP. Moreover, a prerequisite of ergodicity of a random process is its stationarity, so that applying the criterion to a nonstationary process cannot lead to any strict conclusions on its ergodicity or breaking thereof.

In the present article we discuss the MSD-TAMSD criterion and propose a new ramification based on the comparison of the ensemble mean of the increments of the process (MSI) with the TAMSD. Different from the MSD, the MSI is averaged over both the ensemble of motions, and the initial time t of the measurement interval ($t, t + \Delta$). For stationary processes, and for nonstationary processes with stationary (or asymptotically stationary) increments, the equality of MSI and TAMSD signifies the ergodicity of the increments of the process.

The proposed statistical measure of the MSI offers a robust and consistent framework for the evaluation of genuine (non-)ergodicity of increments in both stationary and nonstationary processes.

To show this, we consider several examples covering a broad field of processes typically used as models for biological motions, see Table I in which we summarize the results about the behavior of the MSD, MSI, and TAMSD.

Our detailed discussion of (non-)ergodicity in the context of stochastic time series clarifies some ambiguities in the current literature and will be of immediate relevance for the statistical analysis of SPT data from experiment and simulations.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [236].

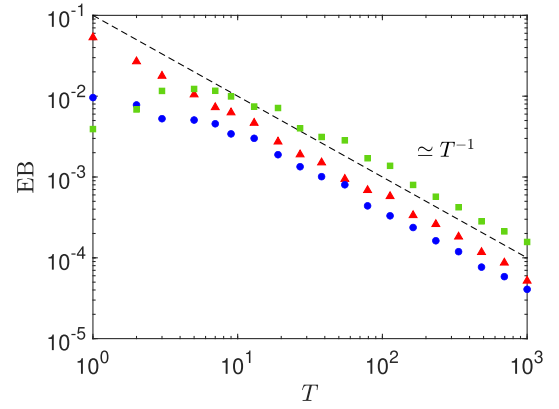


FIG. 7. Simulation results for the EB parameter of FBM under resetting for $H = 0.2$ (red triangles), for $H = 0.5$ (blue circles), and $H = 0.8$ (green rectangles). The lag time is fixed to the time step $\Delta = dt = 0.01$, and the resetting rate is $r = 1$.

APPENDIX A: ERGODICITY BREAKING PARAMETER

The randomness of each TAMSD of a stochastic process at a finite measurement time T gives rise to a certain amplitude scatter of the TAMSD around its trajectory-to-trajectory average, the EA-TAMSD. This scatter can be quantified by the EB parameter (15) [3,56,183]. For ordinary BM in the long time limit $T \rightarrow \infty$, one finds

$$\text{EB}(\Delta; T) \sim \frac{4\Delta}{3T}. \quad (\text{A1})$$

For FBM, the EB parameter for all H is given by [103,237]

$$\text{EB}(\Delta; T) \sim \begin{cases} C_1 \times \frac{\Delta}{T}, & 0 < H < 3/4 \\ C_2 \times \left(\frac{\Delta}{T}\right)^{4-4H}, & 3/4 < H < 1 \end{cases}, \quad (\text{A2})$$

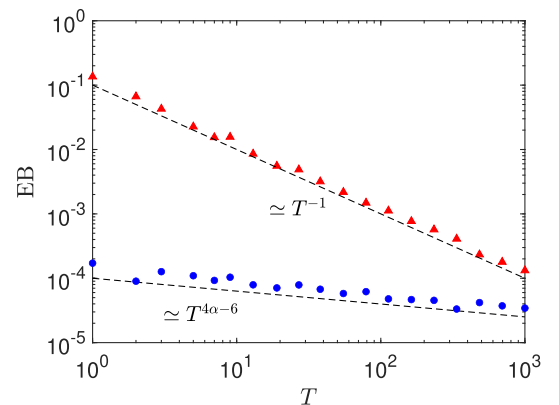


FIG. 8. Simulation results for the EB parameter $\text{EB}(\Delta; T)$ of RL-FBM with $\alpha = 1.4$ (blue circles), and $\alpha = 0.7$ (red triangles). The lag time is fixed to the time step $\Delta = dt = 0.01$.

where $C_1 = \int_0^\infty [(1+s)^{2H} + |1-s|^{2H} - 2s^{2H}]^2 ds$ and $C_2 = [2H(2H-1)]^2 [(4H-3)^{-1} - (4H-2)^{-1}]$. In the long-time limit $T \rightarrow \infty$, the EB parameter vanishes. Analytical expressions for the EB parameters of RL-FBM and reset FBM are currently unavailable. Therefore, we present simulation results instead. In these simulations, we fix the lag time as the time step, $\Delta = dt$, and vary the measurement time T . The analysis shows that for RL-FBM with exponent α , as shown in Fig. 8, the EB parameter

exhibits the same long-time scaling as that of FBM (A2) with Hurst exponent $H = \alpha - 1/2$. For reset FBM in Fig. 7, the EB parameter consistently shows an asymptotic behavior proportional to T^{-1} for all values of H .

APPENDIX B: FRACTIONAL ORNSTEIN-UHLENBECK PROCESS

The ACVF of the fractional OUP is given by [111,124]

$$\begin{aligned} \langle x(t_1)x(t_2) \rangle &= \langle x_0^2 \rangle e^{-\lambda(t_1+t_2)} + \frac{\sigma_{2H}^2}{2} \{ e^{-\lambda t_1} t_2^{2H} + e^{-\lambda t_2} t_1^{2H} - |t_1 - t_2|^{2H} \} \\ &+ \frac{\sigma_{2H}^2}{4\lambda^{2H}} \{ e^{-\lambda|t_2-t_1|} \gamma(2H+1, \lambda t_1) + e^{\lambda|t_2-t_1|} \gamma(2H+1, \lambda t_2) - e^{\lambda|t_2-t_1|} \gamma(2H+1, \lambda|t_2-t_1|) \} \\ &+ \frac{\lambda \sigma_{2H}^2}{4(2H+1)} \{ |t_2-t_1|^{2H+1} e^{-\lambda|t_2-t_1|} \mathcal{M}(2H+1, 2H+2, \lambda|t_2-t_1|) \\ &- t_1^{2H+1} e^{-\lambda(t_1+t_2)} \mathcal{M}(2H+1, 2H+2, \lambda t_1) - t_2^{2H+1} e^{-\lambda(t_1+t_2)} \mathcal{M}(2H+1, 2H+2, \lambda t_2) \}, \end{aligned} \quad (\text{B1})$$

where

$$\gamma(z, x) = \int_0^x e^{-t} t^{z-1} dt \quad (\text{B2})$$

is the lower incomplete gamma function and

$$\mathcal{M}(a, b, z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt. \quad (\text{B3})$$

is the Kummer function (the confluent hypergeometric function of the first kind) [123]. Unlike the ACVF (23) of the normal OUP which becomes stationary when the initial condition is drawn from its equilibrium distribution (21), the ACVF of the fractional OUP remains inherently non-stationary regardless of the initial condition chosen from the stationary distribution (32).

With the ACVF (B1), the MSD can be obtained as

$$\begin{aligned} \langle \Omega^2(t) \rangle &= \langle x_0^2 \rangle (1 - e^{-\lambda t})^2 + \frac{\sigma_{2H}^2}{2\lambda^{2H}} \gamma(2H+1, \lambda t) \\ &+ \sigma_{2H}^2 t^{2H} e^{-\lambda t} - \frac{\lambda \sigma_{2H}^2}{2(2H+1)} t^{2H+1} e^{-2\lambda t} \\ &\times \mathcal{M}(2H+1, 2H+2, \lambda t). \end{aligned} \quad (\text{B4})$$

For large z , the Kummer function can be expanded [123],

$$\begin{aligned} \mathcal{M}(a, b, z) &\sim \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} \left[1 + \frac{(1-a)(b-a)}{z} \right. \\ &\left. + \frac{(1-a)(2-a)(b-a)(b-a+1)}{2z^2} + \dots \right], \end{aligned} \quad (\text{B5})$$

and for the incomplete gamma function we have

$$\gamma(a, z) \sim \Gamma(a) - z^{a-1} e^{-z} \left[1 + \frac{a-1}{z} + \frac{(a-1)(a-2)}{z^2} + \dots \right]. \quad (\text{B6})$$

We obtain the following approximation for the MSD (B4),

$$\langle \Omega^2(t) \rangle \sim \langle x_{2H}^2 \rangle_{\text{st}} - \frac{2\sigma_{2H}^2}{\lambda^2} H(2H-1) t^{2H-2} e^{-\lambda t}, \quad (\text{B7})$$

which exponentially approaches the stationary value $\langle x_{2H}^2 \rangle_{\text{st}} = \sigma_{2H}^2 \Gamma(2H+1)/(2\lambda^{2H})$ with a power-law correction.

The exact expressions for the MSI and TAMSD for the fractional OUP are derived in [111]. These expressions are quite complex, but they share the same asymptotic form in the combined limit $t, T \gg \lambda^{-1}$, reading

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &\sim \overline{\langle \delta^2(\Delta; T) \rangle} \\ &\sim \frac{\sigma_{2H}^2}{\lambda^{2H}} \Gamma(2H+1) + \sigma_{2H}^2 \Delta^{2H} - \frac{\lambda \sigma_{2H}^2}{2(2H+1)} \\ &\times \Delta^{2H+1} e^{-\lambda \Delta} \mathcal{M}(2H+1, 2H+2, \lambda \Delta) \\ &- \frac{\sigma_{2H}^2}{2\lambda^{2H}} \{ e^{\lambda \Delta} \Gamma(2H+1, \lambda \Delta) + e^{-\lambda \Delta} \Gamma(2H+1) \}, \end{aligned} \quad (\text{B8})$$

where $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$ is the upper incomplete gamma function with the property $\Gamma(a, z) = \Gamma(a) - \gamma(a, z)$, where $\Gamma(a)$ is the complete gamma function [123]. With the expansions of the Kummer and the

incomplete gamma function for $\Delta \gg \lambda^{-1}$, we obtain

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &\sim \overline{\langle \delta^2(\Delta; T) \rangle} \\ &\sim 2 \langle x_{2H}^2 \rangle_{\text{st}} - \frac{2\sigma_{2H}^2}{\lambda^2} H(2H-1) \Delta^{2H-2}, \end{aligned} \quad (\text{B9})$$

approaching, in power-law form, the stationary value, when $H \neq 1/2$.

APPENDIX C: FRACTIONAL BROWNIAN MOTION UNDER RESETTING

The ACVF of FBM with resetting is derived in [145], from which the MSI can be obtained. Alternatively, the MSI of the reset FBM can be directly derived using a renewal approach [128,136], and it is given by

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &= 2K_{2H} \left[e^{-r\Delta} \Delta^{2H} + \frac{\gamma(2H+1, r\Delta)}{r^{2H}} \right. \\ &\quad \left. + (1 - e^{-r\Delta}) \left(e^{-rt} t^{2H} + \frac{\gamma(2H+1, rt)}{r^{2H}} \right) \right]. \end{aligned} \quad (\text{C1})$$

The EA-TAMSD can then be obtained by integrating the MSI

$$\begin{aligned} \overline{\langle \delta^2(\Delta; T) \rangle} &= 2K_{2H} \left[e^{-r\Delta} \Delta^{2H} + \frac{\gamma(2H+1, r\Delta)}{r^{2H}} \right] \\ &\quad + 2K_{2H} \frac{1 - e^{-r\Delta}}{r^{2H}} \left[\gamma(2H+1, r(T-\Delta)) \right. \\ &\quad \left. + \frac{1}{r(T-\Delta)} (\gamma(2H+1, r(T-\Delta)) \right. \\ &\quad \left. - \gamma(2H+2, r(T-\Delta))) \right]. \end{aligned} \quad (\text{C2})$$

For t and T much larger than r^{-1} , both the MSI and the TAMSD exhibit the same asymptotic behavior,

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &\sim \overline{\langle \delta^2(\Delta; T) \rangle} \\ &\sim 2K_{2H} \left[e^{-r\Delta} \Delta^{2H} + \frac{\gamma(2H+1, r\Delta)}{r^{2H}} \right. \\ &\quad \left. + (1 - e^{-r\Delta}) \frac{\Gamma(2H+1)}{r^{2H}} \right]. \end{aligned} \quad (\text{C3})$$

At long lag times $\Delta \gg r^{-1}$, expanding the incomplete gamma function according to Eq. (B6), both the MSI and EA-TAMSD approach the stationary value exponentially,

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &\sim \overline{\langle \delta^2(\Delta; T) \rangle} \\ &\sim 4K_{2H} \left[\frac{\Gamma(2H+1)}{r^{2H}} - \frac{H}{r} \Delta^{2H-1} e^{-r\Delta} \right]. \end{aligned} \quad (\text{C4})$$

At short lag times $\Delta \ll r^{-1}$, with the approximation $\gamma(2H+1, r\Delta) \sim (r\Delta)^{2H+1}/(2H+1)$ and $e^{-r\Delta} \sim 1 - r\Delta$, we have

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &\sim \overline{\langle \delta^2(\Delta; T) \rangle} \\ &\sim 2K_{2H} \left(\Delta^{2H} + \frac{\Gamma(2H+1)}{r^{2H-1}} \Delta^1 \right), \end{aligned} \quad (\text{C5})$$

when $H < 1/2$, the MSI and EA-TAMSD exhibit sub-diffusive behavior at short times, while for $H > 1/2$ the linear term dominates, resulting in normal diffusion. In the normal-diffusive case $H = 1/2$, both terms contribute equally, leading to the fact that MSI and TAMSD are approximately twice the MSD at short times.

APPENDIX D: LÉVY WALKS

The displacement-ACVF of Lévy walks is related to the velocity-ACVF through

$$\langle x(t)x(t+\Delta) \rangle = \int_0^t ds_1 \int_0^{t+\Delta} ds_2 \langle v(s_1)v(s_2) \rangle. \quad (\text{D1})$$

Since $v(t+\Delta)$ is either equal to $v(t)$ or $-v(t)$, the number n of directional changes within the time interval $[t, t+\Delta]$ must be even or odd, accordingly. Therefore, the velocity-ACVF is given by

$$\langle v(t)v(t+\Delta) \rangle = \sum_{n=0}^{\infty} (-1)^n v_0^2 p_n(t, t+\Delta), \quad (\text{D2})$$

where $p_n(t, t+\Delta)$ denotes the probability of observing n velocity reversals during the time interval $[t, t+\Delta]$.

1. Exponentially distributed waiting times

For the case of the exponential PDF (57) of waiting times, the probability of velocity reversals during $[t, t+\Delta]$ becomes [185]

$$p_n(t, t+\Delta) = \frac{(\lambda\Delta)^n}{n!} e^{-\lambda\Delta}. \quad (\text{D3})$$

Substituting Eq. (D3) into Eq. (D2), we get

$$\begin{aligned} \langle v(t)v(t+\Delta) \rangle &= v_0^2 \sum_{n=0}^{\infty} (-1)^n p_n(t, t+\Delta) \\ &= v_0^2 \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda\Delta)^n}{n!} e^{-\lambda\Delta} \\ &= v_0^2 e^{-2\lambda\Delta}. \end{aligned} \quad (\text{D4})$$

Applying Eq. (D4) to Eq. (D1), the displacement-ACVF is given by

$$\begin{aligned} \langle x(t)x(t+\Delta) \rangle &= \frac{v_0^2}{(2\lambda)^2} [4\lambda t + e^{-2\lambda(t+\Delta)} \\ &\quad + e^{-2\lambda t} - e^{-2\lambda\Delta} - 1]. \end{aligned} \quad (\text{D5})$$

Then MSD, MSI, and EA-TAMSD can then be derived directly, as given by Eq. (58) in the main text.

2. Power-law distributed waiting times

a. Scale-free regime $0 < \beta < 1$

For the ballistic motion regime with $0 < \beta < 1$, the velocity- and displacement-ACVFs are given by

$$\langle v(t)v(t+\Delta) \rangle = \frac{v_0^2 \sin \pi \beta}{\pi} B\left(\frac{t}{t+\Delta}; \beta, 1-\beta\right), \quad (\text{D6})$$

and

$$\begin{aligned} \langle x(t)x(t+\Delta) \rangle &= \frac{v_0^2 \sin \pi \beta}{2\pi} \left[2t(t+\Delta) B\left(\frac{t}{t+\Delta}; \beta, 1-\beta\right) - (t+\Delta)^2 B\left(\frac{t}{t+\Delta}; 1+\beta, 1-\beta\right) \right. \\ &\quad \left. - t^2 B\left(\frac{t}{t+\Delta}; -1+\beta, 1-\beta\right) \right] - \frac{\beta(v_0 t)^2}{2}, \end{aligned} \quad (\text{D7})$$

where $B(y; a, b) = \int_0^y u^{a-1}(1-u)^{b-1} du$ denotes the incomplete Beta function [123].

The MSI can be exactly expressed as

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &= \frac{v_0^2 \sin(\pi \beta)}{\pi} \left[-2t(t+\Delta) B\left(\frac{t}{t+\Delta}; \beta, 1-\beta\right) + (t+\Delta)^2 B\left(\frac{t}{t+\Delta}; 1+\beta, 1-\beta\right) \right. \\ &\quad \left. + t^2 B\left(\frac{t}{t+\Delta}; -1+\beta, 1-\beta\right) \right] + v_0^2 t^2 \left[1 + (1-\beta) \left(1 + \frac{\Delta}{t} \right)^2 \right]. \end{aligned} \quad (\text{D8})$$

In the first term of Eq. (D8), using the relation $B(z; a, b) = (z^a/a) {}_2F_1(a, 1-b; a+1; z)$ between the incomplete beta function and the hypergeometric function ${}_2F_1$ along with the transformation between the hypergeometric function and the Fox H -function, given by Eq. (1.131) in Ref. [174], we find

$$\begin{aligned} B\left(\frac{t}{t+\Delta}; \beta, 1-\beta\right) &= \frac{1}{\beta} \left(\frac{t}{t+\Delta}\right)^\beta \times {}_2F_1\left(\beta, \beta; \beta+1; \frac{t}{t+\Delta}\right) \\ &= \frac{1}{\beta} \left(\frac{t}{\Delta}\right)^\beta \times {}_2F_1\left(\beta, 1; \beta+1; -\frac{t}{\Delta}\right) \\ &= \left(\frac{t}{\Delta}\right)^\beta \times H_{2,2}^{1,2} \left[\frac{t}{\Delta} \middle| \begin{matrix} (1-\beta, 1), (0, 1) \\ (\beta, 1), (-\beta, 1) \end{matrix} \right], \end{aligned} \quad (\text{D9})$$

where in the second line, the Pfaff transformation ${}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1\{a, c-b; c; [z/(z-1)]\}$ [123] was applied. Then, using the identities (8.3.2.7) and (8.3.2.8) of the H function in [238], the incomplete Beta function can be expressed as a special case of the H function,

$$B\left(\frac{t}{t+\Delta}; \beta, 1-\beta\right) = H_{2,2}^{1,2} \left[\frac{t}{\Delta} \middle| \begin{matrix} (1, 1), (\beta, 1) \\ (\beta, 1), (0, 1) \end{matrix} \right] = H_{2,2}^{2,1} \left[\frac{\Delta}{t} \middle| \begin{matrix} (1-\beta, 1), (1, 1) \\ (0, 1), (1-\beta, 1) \end{matrix} \right]. \quad (\text{D10})$$

Applying a similar transformation in the second term in the MSI (D8), we obtain

$$B\left(\frac{t}{t+\Delta}; 1+\beta, 1-\beta\right) = H_{2,2}^{1,2} \left[\frac{t}{\Delta} \middle| \begin{matrix} (1, 1), (\beta, 1) \\ (1+\beta, 1), (0, 1) \end{matrix} \right] = H_{2,2}^{2,1} \left[\frac{\Delta}{t} \middle| \begin{matrix} (-\beta, 1), (1, 1) \\ (0, 1), (1-\beta, 1) \end{matrix} \right]. \quad (\text{D11})$$

The third term can be simplified via the identity ${}_2F_1(a, b; b; z) = (1-z)^{-a}$,

$$B\left(\frac{t}{t+\Delta}; \beta-1, 1-\beta\right) = \frac{1}{\beta-1} \left(\frac{t}{t+\Delta}\right)^{\beta-1} {}_2F_1\left(\beta-1, \beta; \beta; \frac{t}{t+\Delta}\right) = \frac{1}{\beta-1} \left(\frac{\Delta}{t}\right)^{1-\beta}. \quad (\text{D12})$$

Substituting Eqs. (D10) to (D12) into the MSI (D8), we find

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &= v_0^2 \frac{\sin(\pi\beta)}{\pi} t^2 \left\{ -2 \left(1 + \frac{\Delta}{t} \right) H_{2,2}^{2,1} \left[\frac{\Delta}{t} \middle| \begin{matrix} (1-\beta, 1), (1, 1) \\ (0, 1), (1-\beta, 1) \end{matrix} \right] + \left(1 + \frac{\Delta}{t} \right)^2 H_{2,2}^{2,1} \left[\frac{\Delta}{t} \middle| \begin{matrix} (-\beta, 1), (1, 1) \\ (0, 1), (1-\beta, 1) \end{matrix} \right] \right. \\ &\quad \left. + \frac{1}{\beta-1} \left(\frac{\Delta}{t} \right)^{1-\beta} \right\} + v_0^2 t^2 \left[1 + (1-\beta) \left(1 + \frac{\Delta}{t} \right)^2 \right]. \end{aligned} \quad (\text{D13})$$

Using the expansion (8.3.2.3) of H -function in Ref. [238] with $\Delta \ll t$, we have

$$\begin{aligned} H_{2,2}^{2,1} \left[\frac{\Delta}{t} \middle| \begin{matrix} (1-\beta, 1), (1, 1) \\ (0, 1), (1-\beta, 1) \end{matrix} \right] &= \sum_{k=0}^{+\infty} \frac{(-1)^k \Gamma(1-k-\beta) \Gamma(\beta+k)}{\Gamma(1-k) k!} \left(\frac{\Delta}{t} \right)^k + \sum_{k=0}^{+\infty} \frac{(-1)^k \Gamma(\beta-1-k) \Gamma(k+1)}{\Gamma(\beta-k) k!} \left(\frac{\Delta}{t} \right)^{1-\beta+k} \\ &= \frac{\pi}{\sin(\pi\beta)} + \frac{1}{\beta-1} \left(\frac{\Delta}{t} \right)^{1-\beta} - \frac{1}{\beta-2} \left(\frac{\Delta}{t} \right)^{2-\beta} + \frac{1}{\beta-3} \left(\frac{\Delta}{t} \right)^{3-\beta} + O\left(\left(\frac{\Delta}{t} \right)^{3-\beta} \right), \end{aligned} \quad (\text{D14})$$

where only the single term $k=0$ survives in the first summation since the Gamma function has simple poles at non-positive integers. Likewise,

$$\begin{aligned} H_{2,2}^{2,1} \left[\frac{\Delta}{t} \middle| \begin{matrix} (-\beta, 1), (1, 1) \\ (0, 1), (1-\beta, 1) \end{matrix} \right] &= \sum_{k=0}^{+\infty} \frac{(-1)^k \Gamma(1-k-\beta) \Gamma(1+\beta+k)}{\Gamma(1-k) k!} \left(\frac{\Delta}{t} \right)^k + \sum_{k=0}^{+\infty} \frac{(-1)^k \Gamma(\beta-1-k) \Gamma(k+2)}{\Gamma(\beta-k) k!} \left(\frac{\Delta}{t} \right)^{1-\beta+k} \\ &= \frac{\pi\beta}{\sin(\pi\beta)} + \frac{1}{\beta-1} \left(\frac{\Delta}{t} \right)^{1-\beta} - \frac{2}{\beta-2} \left(\frac{\Delta}{t} \right)^{2-\beta} + \frac{3}{\beta-3} \left(\frac{\Delta}{t} \right)^{3-\beta} + O\left(\left(\frac{\Delta}{t} \right)^{3-\beta} \right). \end{aligned} \quad (\text{D15})$$

Substituting Eqs. (D14) and (D15) into the MSI (D13), one can then see that the main order terms proportional to $t^2(\Delta/t)^0$, $t^2(\Delta/t)^1$, $t^2(\Delta/t)^{1-\beta}$, and $t^2(\Delta/t)^{2-\beta}$ cancel out. With the main remaining terms, we derive the MSI (63) for the long time limit $\Delta \ll t$,

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &\sim v_0^2 t^2 \left[\left(\frac{\Delta}{t} \right)^2 + \frac{2 \sin(\pi\beta)}{\pi(\beta-3)(\beta-2)(\beta-1)} \left(\frac{\Delta}{t} \right)^{3-\beta} + O\left(\left(\frac{\Delta}{t} \right)^{3-\beta} \right) \right] \\ &\sim v_0^2 \Delta^2 \left[1 + \frac{2 \sin(\pi\beta)}{\pi(\beta-3)(\beta-2)(\beta-1)} \left(\frac{\Delta}{t} \right)^{1-\beta} + O\left(\left(\frac{\Delta}{t} \right)^{1-\beta} \right) \right] \\ &\sim v_0^2 \Delta^2. \end{aligned} \quad (\text{D16})$$

Based on the MSI (D13), the exact EA-TAMSD can be expressed via the Fox H function as

$$\overline{\langle \delta^2(\Delta; T) \rangle} = \frac{v_0^2}{3} (T - \Delta)^2 + \frac{(1-\beta)v_0^2 T^3 - \Delta^3}{3(T - \Delta)} + \frac{v_0^2 \sin(\pi\beta)}{\pi} \frac{\Delta^2}{T - \Delta} \int_0^{T-\Delta} I(t, \Delta) dt, \quad (\text{D17})$$

where

$$I(t, \Delta) = -2 \frac{t}{\Delta} \left(1 + \frac{t}{\Delta} \right) H_{2,2}^{1,2} \left[\frac{t}{\Delta} \middle| \begin{matrix} (1, 1), (\beta, 1) \\ (\beta, 1), (0, 1) \end{matrix} \right] + \left(1 + \frac{t}{\Delta} \right)^2 H_{2,2}^{1,2} \left[\frac{t}{\Delta} \middle| \begin{matrix} (1, 1), (\beta, 1) \\ (1+\beta, 1), (0, 1) \end{matrix} \right] + \frac{1}{\beta-1} \left(\frac{t}{\Delta} \right)^{\beta+1}. \quad (\text{D18})$$

Using the integral (1.16.4.1) of the H function and the identities (8.3.2.7) and (8.3.2.8) in Ref. [238], we can integrate the first H function in Eq. (D18),

$$\begin{aligned}
& \frac{1}{T-\Delta} \int_0^{T-\Delta} \frac{t}{\Delta} \left(1 + \frac{t}{\Delta}\right) H_{2,2}^{1,2} \left[\frac{t}{\Delta} \middle| \begin{matrix} (1,1), (\beta,1) \\ (\beta,1), (0,1) \end{matrix} \right] dt \\
&= \frac{T-\Delta}{\Delta} H_{3,3}^{1,3} \left[\frac{T-\Delta}{\Delta} \middle| \begin{matrix} (-1,1), (1,1), (\beta,1) \\ (\beta,1), (0,1), (-2,1) \end{matrix} \right] + \left(\frac{T-\Delta}{\Delta} \right)^2 H_{3,3}^{1,3} \left[\frac{T-\Delta}{\Delta} \middle| \begin{matrix} (-2,1), (1,1), (\beta,1) \\ (\beta,1), (0,1), (-3,1) \end{matrix} \right] \\
&= H_{3,3}^{1,3} \left[\frac{T-\Delta}{\Delta} \middle| \begin{matrix} (0,1), (2,1), (1+\beta,1) \\ (1+\beta,1), (1,1), (-1,1) \end{matrix} \right] + H_{3,3}^{1,3} \left[\frac{T-\Delta}{\Delta} \middle| \begin{matrix} (0,1), (3,1), (2+\beta,1) \\ (2+\beta,1), (2,1), (-1,1) \end{matrix} \right] \\
&= H_{3,3}^{3,1} \left[\frac{\Delta}{T-\Delta} \middle| \begin{matrix} (-\beta,1), (0,1), (2,1) \\ (1,1), (-1,1), (-\beta,1) \end{matrix} \right] + H_{3,3}^{3,1} \left[\frac{\Delta}{T-\Delta} \middle| \begin{matrix} (-1-\beta,1), (-1,1), (2,1) \\ (1,1), (-2,1), (-1-\beta,1) \end{matrix} \right].
\end{aligned} \tag{D19}$$

Applying similar steps in the second H function, and using Eq. (8.3.2.6) to reduce to lower orders, one has

$$\begin{aligned}
& \frac{1}{T-\Delta} \int_0^{T-\Delta} \left(1 + \frac{t}{\Delta}\right)^2 H_{2,2}^{1,2} \left[\frac{t}{\Delta} \middle| \begin{matrix} (1,1), (\beta,1) \\ (1+\beta,1), (0,1) \end{matrix} \right] dt \\
&= H_{2,2}^{2,1} \left[\frac{\Delta}{T-\Delta} \middle| \begin{matrix} (-\beta,1), (2,1) \\ (0,1), (1-\beta,1) \end{matrix} \right] + 2H_{3,3}^{3,1} \left[\frac{\Delta}{T-\Delta} \middle| \begin{matrix} (-1-\beta,1), (0,1), (2,1) \\ (1,1), (-1,1), (-\beta,1) \end{matrix} \right] \\
&+ H_{3,3}^{3,1} \left[\frac{\Delta}{T-\Delta} \middle| \begin{matrix} (-2-\beta,1), (-1,1), (2,1) \\ (1,1), (-2,1), (-1-\beta,1) \end{matrix} \right].
\end{aligned} \tag{D20}$$

Substituting the above results into the third term of Eq. (D17) yields

$$\begin{aligned}
\text{the third term} &= \frac{\sin(\pi\beta)}{\pi} v_0^2 \Delta^2 \left\{ -2H_{3,3}^{3,1} \left[\frac{\Delta}{T-\Delta} \middle| \begin{matrix} (-\beta,1), (0,1), (2,1) \\ (1,1), (-1,1), (-\beta,1) \end{matrix} \right] - 2H_{3,3}^{3,1} \left[\frac{\Delta}{T-\Delta} \middle| \begin{matrix} (-1-\beta,1), (-1,1), (2,1) \\ (1,1), (-2,1), (-\beta-1,1) \end{matrix} \right] \right. \\
&+ H_{2,2}^{2,1} \left[\frac{\Delta}{T-\Delta} \middle| \begin{matrix} (-\beta,1), (2,1) \\ (0,1), (1-\beta,1) \end{matrix} \right] + 2H_{3,3}^{3,1} \left[\frac{\Delta}{T-\Delta} \middle| \begin{matrix} (-1-\beta,1), (0,1), (2,1) \\ (1,1), (-1,1), (-\beta,1) \end{matrix} \right] \\
&\left. + H_{3,3}^{3,1} \left[\frac{\Delta}{T-\Delta} \middle| \begin{matrix} (-2-\beta,1), (-1,1), (2,1) \\ (1,1), (-2,1), (-\beta-1,1) \end{matrix} \right] + \frac{1}{(\beta-1)(\beta+2)} \left(\frac{\Delta}{T-\Delta} \right)^{-1-\beta} \right\}.
\end{aligned} \tag{D21}$$

Finally, applying the expansion (8.3.2.3) of the H function in Ref. [238] with $\Delta \ll t$, and combining the first two terms of Eq. (D17), one can see that the main order terms proportional to $\Delta^2(\Delta/t)^{-2}$, $\Delta^2(\Delta/t)^{-1-\beta}$, $\Delta^2(\Delta/t)^{-1}$, and $\Delta^2(\Delta/t)^{-\beta}$ cancel out. With the main remaining terms, we can derive the TAMSD (D17) in the long time limit,

$$\begin{aligned}
\langle \overline{\delta^2(\Delta; T)} \rangle &\sim v_0^2 \Delta^2 \left[1 + \frac{2 \sin(\pi\beta)}{\pi(\beta-3)(\beta-2)(\beta-1)\beta} \left(\frac{\Delta}{T} \right)^{1-\beta} + O\left(\left(\frac{\Delta}{T} \right)^{1-\beta} \right) \right] \\
&\sim v_0^2 \Delta^2.
\end{aligned} \tag{D22}$$

b. Regime $1 < \beta < 2$

For the superdiffusive motion regime $1 < \beta < 2$, the velocity-ACVF is [91,173]

$$\langle v(t)v(t+\Delta) \rangle = \frac{v_0^2 \tau_0^\beta}{(\beta-1)\langle \tau \rangle} [\Delta^{1-\beta} - (t+\Delta)^{1-\beta}]. \tag{D23}$$

Integrating Eq. (D23) as in Eq. (D1), we obtain the displacement-ACVF

$$\langle x(t)x(t+\Delta) \rangle = \frac{K_{3-\beta}}{(\beta-1)} t^{3-\beta} \left[\beta + \left(1 + \frac{\Delta}{t}\right)^{3-\beta} - (3-\beta) \left(1 + \frac{\Delta}{t}\right)^{2-\beta} - \left(\frac{\Delta}{t}\right)^{3-\beta} \right], \tag{D24}$$

where

$$K_{3-\beta} = \frac{\beta-1}{(3-\beta)(2-\beta)} v_0^2 \tau_0^{\beta-1}. \quad (\text{D25})$$

The MSI can be obtained in the form

$$\langle \delta^2(\Delta; t) \rangle = \frac{2K_{3-\beta}}{\beta-1} t^{3-\beta} \left[-1 - (2-\beta) \left(1 + \frac{\Delta}{t} \right)^{3-\beta} + (3-\beta) \left(1 + \frac{\Delta}{t} \right)^{2-\beta} + \left(\frac{\Delta}{t} \right)^{3-\beta} \right]. \quad (\text{D26})$$

Expanding the expressions for $t \gg \Delta$, the MSI becomes

$$\begin{aligned} \langle \delta^2(\Delta; t) \rangle &\sim \frac{2K_{3-\beta}}{\beta-1} t^{3-\beta} \left[\frac{(3-\beta)(2-\beta)}{2} \left(\frac{\Delta}{t} \right)^2 + \left(\frac{\Delta}{t} \right)^{3-\beta} + O\left(\left(\frac{\Delta}{t} \right)^3 \right) \right] \\ &\sim \frac{2K_{3-\beta}}{\beta-1} \Delta^{3-\beta} \left[1 - \frac{(3-\beta)(2-\beta)}{2} \left(\frac{\Delta}{t} \right)^{\beta-1} \right]. \end{aligned} \quad (\text{D27})$$

Similarly, the exact EA-TMSD can be derived as

$$\begin{aligned} \langle \overline{\delta^2(\Delta; T)} \rangle &= \frac{2K_{3-\beta}}{\beta-1} T^{3-\beta} \left[-\frac{1}{4-\beta} \left(1 - \frac{\Delta}{T} \right)^{3-\beta} + \frac{1}{4-\beta} \frac{1 - (\Delta/T)^{4-\beta}}{1 - \Delta/T} - \frac{\Delta}{T} \frac{1 - (\Delta/T)^{3-\beta}}{1 - \Delta/T} + \left(\frac{\Delta}{T} \right)^{3-\beta} \right]. \end{aligned} \quad (\text{D28})$$

For $T \gg \Delta$, this can be approximated as

$$\langle \overline{\delta^2(\Delta; T)} \rangle \sim \frac{2K_{3-\beta}}{\beta-1} \Delta^{3-\beta} \left[1 - \frac{3-\beta}{2} \left(\frac{\Delta}{T} \right)^{\beta-1} \right]. \quad (\text{D29})$$

APPENDIX E: CTRW $(n+1)$ TH ORDER INCREMENT

We consider the MSD in Eq. (1) as the zeroth order increment. The MSI defined in Eq. (8) is then identified as the first order increment, namely,

$$\Delta^{(1)}x(t; \tau_1) = x(t + \tau_1) - x(t). \quad (\text{E1})$$

The second order increment is defined as the increment of the first-order increment,

$$\Delta^{(2)}x(t; \tau_1, \tau_2) = \Delta^{(1)}x(t + \tau_2; \tau_1) - \Delta^{(1)}x(t; \tau_1). \quad (\text{E2})$$

With these definitions, the second-order increment can be interpreted as the “velocity of the velocity.” By extension,

the $(n+1)$ th order increment is defined as the difference of the n th order increment during the lag time τ_{n+1} [239],

$$\begin{aligned} \Delta^{(n+1)}x(t; \tau_1, \dots, \tau_{n+1}) &= \Delta^{(n)}x(t + \tau_{n+1}; \tau_1, \dots, \tau_n) \\ &\quad - \Delta^{(n)}x(t; \tau_1, \dots, \tau_n), \end{aligned} \quad (\text{E3})$$

similar in spirit to the theory of the n th order increment by Yaglom [94,95]. Consequently, it is reasonable to assume that $\tau_n \gg \tau_{n-1} \gg \dots \gg \tau_2 \gg \tau_1$.

Using the position-ACVF derived in [240],

$$\langle x(t_1)x(t_2) \rangle = a^2 \left[\Theta(t_2 - t_1) \langle n(t_1) \rangle + \Theta(t_1 - t_2) \langle n(t_2) \rangle \right], \quad (\text{E4})$$

where $\Theta(z)$ is the Heaviside step function, we find that the ACVF of the first-order increment vanishes,

$$\langle \Delta^{(1)}x(t + \Delta; \tau_1) \times \Delta^{(1)}x(t; \tau_1) \rangle = 0. \quad (\text{E5})$$

Then we obtain that the second order MSI in the regime $0 < \beta < 1$ takes on the form

$$\begin{aligned} \langle [\Delta^{(2)}x(t; \tau_1, \Delta)]^2 \rangle &= \langle [\Delta^{(1)}x(t + \Delta; \tau_1) - \Delta^{(1)}x(t; \tau_1)]^2 \rangle \\ &= \langle [\Delta^{(1)}x(t + \Delta; \tau_1)]^2 \rangle + \langle [\Delta^{(1)}x(t; \tau_1)]^2 \rangle \\ &= a^2 [\langle n(t + \tau_1 + \Delta) \rangle - \langle n(t + \Delta) \rangle + \langle n(t + \tau_1) \rangle - \langle n(t) \rangle]. \end{aligned} \quad (\text{E6})$$

Considering the mean (76) of the counting process in the long time limit $t \gg \Delta \gg \tau_1$,

$$\begin{aligned} \langle [\Delta^{(2)}x(t; \tau_1, \Delta)]^2 \rangle &= 2K_\beta [(t + \tau_1 + \Delta)^\beta - (t + \Delta)^\beta + (t + \tau_1)^\beta - t^\beta] \\ &\sim 4\beta K_\beta (\tau_1 t^{\beta-1}), \end{aligned} \quad (\text{E7})$$

which is still nonstationary.

Moreover, the $(n+1)$ th MSI can be generalized as

$$\begin{aligned} \langle [\Delta^{(n+1)}x(t; \tau_1, \dots, \tau_n, \Delta)]^2 \rangle &= \langle [\Delta^{(n)}x(t + \Delta; \tau_1, \dots, \tau_n) - \Delta^{(n)}x(t; \tau_1, \dots, \tau_n)]^2 \rangle \\ &\sim 2n\beta K_\beta (\tau_1 t^{\beta-1}), \end{aligned} \quad (\text{E8})$$

in the long time limit $t \gg \tau_n \gg \dots \gg \tau_1$, and they are all nonstationary and dependent on $t^{\beta-1}$.

APPENDIX F: FINITE TRAJECTORY LENGTH AND LIMITED SAMPLING EFFECTS FOR THE OUP

First, we examine the effect of finite trajectory length by fixing the number of trajectories and varying the

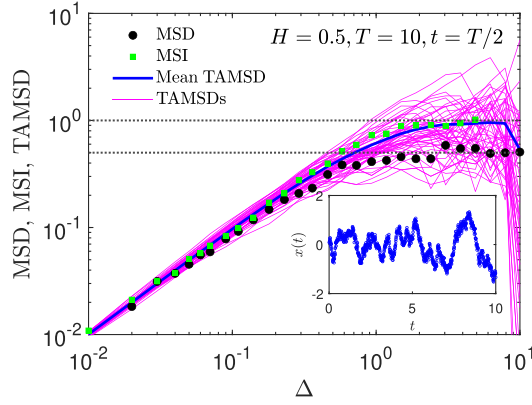


FIG. 9. Effects of finite trajectory length in the fractional OUP with $H = 0.5$, measurement time $T = 10$, and starting time for the MSI $t = T/2$. The MSD, MSI, ensemble-averaged TAMSD, and individual TAMSDs are shown by black circles, green squares, the thick blue curve, and thin magenta curves, respectively. The other parameters are the same as in Fig. 2(a): initial condition $x_0 = 0$, time step $dt = 0.01$, noise strength $\sigma_{2H} = 1$, relaxation rate $\lambda = 1$ and number of trajectories $N = 200$.

measurement time T . For the TAMSD, the number of increments available for the time average decreases as the lag time Δ increases. In the discrete case this number is $(T - \Delta)/dt$. Therefore, when Δ becomes comparable to the measurement time T , the TAMSD is no longer a well-converged time average. This leads to enhanced fluctuations of individual TAMSDs and may also bias the ensemble-averaged TAMSD away from the MSI. In the limiting case $\Delta \rightarrow T$, the TAMSD is essentially determined by only a few increments, and in the extreme case by the single displacement $x(T) - x(0)$. Consequently, its ensemble average becomes closer to the MSD rather than to the MSI. This behavior is illustrated for the OUP with $H = 0.5$ and measurement time $T = 10$ in Fig. 9. It explains the

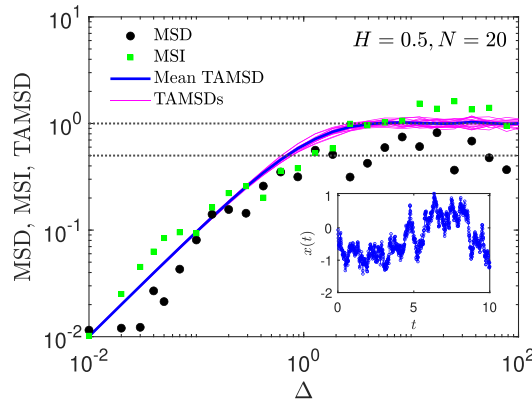


FIG. 10. Effects of limited sampling in the fractional OUP with $H = 0.5$ and number of trajectories $N = 20$. The other parameters are the same as in Fig. 2(a): initial condition $x_0 = 0$, time step $dt = 0.01$, noise strength $\sigma_{2H} = 1$, relaxation rate $\lambda = 1$, measurement time $T = 500$ and starting time for the MSI $t = 10$.

apparent downward bending of the mean TAMSD and the large scatter of individual TAMSDs at long lag times in finite trajectories. By contrast, the MSI is an ensemble-averaged increment observable and is not affected by the reduction of the time-averaging window in the same way. Its finite-time limitations are mainly that one must have $t + \Delta \leq T$, and that the starting time t should be sufficiently large for the process to have reached the (asymptotically) stationary regime. Thus, finite measurement time does not invalidate the MSI-TAMSD criterion, but it restricts its reliable application to lag times satisfying $\Delta/T \ll 1$.

We also examined the influence of limited sampling within the same OUP model. Limited sampling affects the MSD, MSI, and mean TAMSD in different ways. The MSD and MSI are direct ensemble averages over squared displacements or squared increments, and therefore they exhibit visible sampling fluctuations when the number of trajectories is small. In contrast, the mean TAMSD is less sensitive to limited sampling, because each individual TAMSD already involves a time average over many increments along a single trajectory. As a result, even for a relatively small ensemble, the mean TAMSD can remain rather smooth and close to the MSI at small and intermediate lag times, as shown in Fig. 10.

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