

Letting the Tiger out of Its Cage: Bosonic Coding without Concatenation

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Continuous-variable cat codes are encodings into a single photonic or phononic mode that offer a promising avenue for hardware-efficient fault-tolerant quantum computation. Protecting information in a cat code requires measuring the mode's occupation number modulo two, but this can be relaxed to a linear occupation-number constraint using the alternative two-mode pair-cat encoding. We construct multimode codes with similar linear constraints using any two integer matrices satisfying a Calderbank-Shor-Steane-like homological condition of a quantum rotor code. Just like the pair-cat code, syndrome extraction can be performed in tandem with stabilizing dissipation using current superconducting-circuit designs. The framework includes codes with various finite- or infinite-dimensional code spaces and codes with finite or infinite Fock-state support. It encompasses two-component cat, pair-cat, dual-rail, two-mode binomial, various bosonic repetition codes, and aspects of χ -squared encodings, while also yielding codes from homological products, lattices, generalized coherent states, and algebraic varieties. Among our examples are analogs of repetition codes, the Shor code, and a surface-code-like construction that is not a concatenation of a known cat code with the qubit surface code. Code words are coherent states projected into a Fock-state subspace defined by an integer matrix, and their overlaps are governed by Gelfand-Kapranov-Zelevinsky hypergeometric functions.

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I. INTRODUCTION

Quantum mechanical effects offer faster solutions to certain structured problems, secure communication from eavesdropping, and accurate simulation of natural processes [1–7]. The needed effects are difficult to preserve in an inherently classical world, and error correction is necessary in order to maintain them and realize useful applications.

The typical building block of a quantum system consists of only two low-lying quantized energy states of a natural system, such as an atom or molecule, or an engineered one, such as an electronic circuit. These building blocks have to be combined, or “scaled up,” to have enough redundancy so as to protect from noise.

Error-correcting codes have been extensively studied for such discrete-variable (DV) systems [8–11].

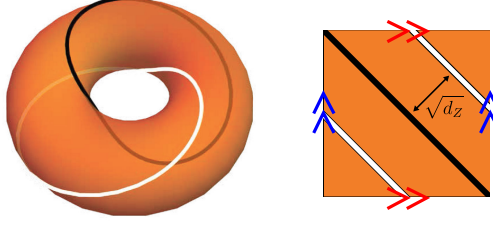
However, many quantum systems—atoms, molecules, and circuits included—have access to states of higher energy. Much research has focused on studying the potential of one or a few such systems, or “modes,” to encode information in their higher-lying state space [12–15]. This direction is bootstrapped by dramatic improvements in the control of such bosonic quantum systems (see, e.g., Refs. [16–21], among many other advancements).

It is tantalizing to consider error-correcting codes constructed intrinsically in a large number of modes, whose redundancy comes from both the large number of building blocks of the system (“breadth”) and a high energy per block (“depth”). One can consider concatenating a DV and a single-mode code, i.e., first encoding in a qubit stabilizer code and then encoding each physical qubit into a mode. However, since the scaling comes strictly from the DV part of such an encoding, more powerful codes that cannot be expressed as concatenated codes should be possible.

A recent work, Ref. [22], introduces a quantum spherical code construction for intrinsically multimode codes that

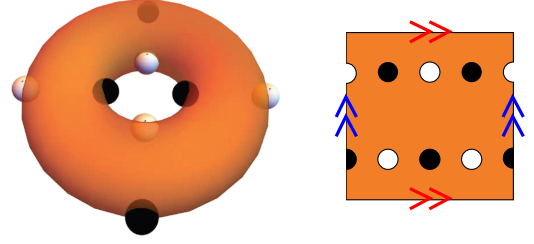
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(a) Tiger codes



$$|\overline{+}\rangle = \int_{\alpha \in \mathcal{C}_+} d\alpha |\alpha\rangle, \quad |\overline{-}\rangle = \int_{\alpha \in \mathcal{C}_-} d\alpha |\alpha\rangle$$

(b) Quantum spherical codes



$$|\overline{+}\rangle = \sum_{\alpha \in \mathcal{C}_+} |\alpha\rangle, \quad |\overline{-}\rangle = \sum_{\alpha \in \mathcal{C}_-} |\alpha\rangle$$

FIG. 1. The tiger code words of this work are superpositions of continuous sets of coherent-state “stripes”—circles or, more generally, tori. In contrast, quantum spherical code words are superpositions of a discrete set of coherent states. The use of continuous superpositions allows for linear Fock-state constraints to define the code, while discrete superpositions correspond to modular and, more generally, representation-theoretic constraints. Two-mode coherent states of values $\alpha = \alpha(e^{i\phi_1}, e^{i\phi_2})$ of fixed coordinate amplitude α but varying phases ϕ_1 and ϕ_2 form a torus, which is depicted either as itself or as a square with opposite boundaries identified. (a), (b) The subset of coherent states that form the logical code words of a tiger code and a quantum spherical code, respectively. The logical $+$ and $-$ states are defined as equal-weight superpositions of coherent states located at the black and white points. Therefore, the Z distance of the tiger code is given by the minimum squared Euclidean distance between points belonging to the constellations $\mathcal{C}_+ = \{\alpha(e^{i\phi}, e^{-i\phi}) \mid \forall \phi \in \mathbb{T}\}$ and $\mathcal{C}_- = \{\alpha(e^{i\phi}, e^{-i(\phi+\pi)}) \mid \forall \phi \in \mathbb{T}\}$. The tiger code example is the pair-cat code words defined by matrices $G = \begin{pmatrix} 2 & 2 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & -1 \end{pmatrix}$, while the quantum spherical code constellations are $\mathcal{C}_+ = \{(\alpha, \alpha), (i\alpha, -\alpha), (-\alpha, \alpha), (-i\alpha, -\alpha)\}$ and $\mathcal{C}_- = \{(\alpha, -\alpha), (i\alpha, \alpha), (-\alpha, -\alpha), (-i\alpha, \alpha)\}$.

cannot be split into a DV and a mode encoding. Spherical code words are superpositions of a handful of coherent states $|\alpha\rangle$ —classical multimode electromagnetic signals whose amplitude and phase are combined into a complex vector α —and are generalizations of the single-mode cat [14] and two-mode $2T$ -qutrit code [23] based on polytopes and spherical designs.

In this manuscript, we define “tiger” codes, whose code words are compact, yet continuous, coherent-state superpositions, with α taking values along particular “stripes”—circles or, more generally, tori—in coherent-state configuration space (see Fig. 1). Stabilizing operators are of Calderbank-Shor-Steane (CSS) rotor-code type [24], either linear combinations of occupation-number operators defined by rows of an integer matrix or products of annihilation-creation operators whose powers form rows of another such matrix. Any two integer matrices satisfying a CSS-type homological condition can define a tiger code.

Tiger codes come with several advantages over the cat code and its variants for protecting against loss of energy in each mode, as well as random rotations due to fluctuations of each mode’s frequency. (1) The syndrome required to detect losses corresponds to a linear combination of occupation-number operators—a simple multimode generalization of photon counting that relaxes the nonlinear constraints required for the cat code and its variants. (2) Protection against dephasing is autonomous, generated by engineered dissipation [25–28] whose stabilizing jump operators can have a lower degree and fewer terms than those

of concatenated-cat and quantum spherical codes. (3) Loss error syndromes can be extracted by appending an ancilla and evolving with an entangling Hamiltonian without turning off the autonomous dissipation, in contrast to multicomponent cat codes and similar to the pair-cat encoding [29], a special case of our codes. (4) The Z-type stabilizers and X-type dissipators commute with one another when the Fock-state support of the code words is infinite, making contact with the stabilizer formalism [9,10].

Our framework utilizes homology over the integers [24,30], giving rise to intrinsically multimode codes that are distinct from qubit stabilizer codes, which come from homology over $\mathbb{Z}_2$ [31–33]. The framework includes finite- and infinite-dimensional code spaces consisting of either a finite or infinite number of Fock states. We unite many seemingly isolated bosonic encodings under one roof, including two-component cat, pair-cat, various bosonic repetition codes [34,35], the dual-rail code [36,37], and two-mode binomial codes [38], as well as aspects of a $\chi^{(2)}$ code [39]. At the same time, we obtain new codes from lattices [40], generalized coherent states [41,42], and integer hypergraph products [24,43]. We show that circular superpositions of coherent states and tiger-code properties are intimately tied to Gelfand-Kapranov-Zelevinsky (GKZ) hypergeometric functions [44–50], yielding codes from algebraic varieties such as Calabi-Yau manifolds [51].

Tiger codes come with intrinsically linear syndromes and sport low-order autonomous protection. They are incredibly diverse, providing a framework for experimentally

viable, inherently multimode quantum information processing in the near future.

II. SUMMARY OF RESULTS

Tiger-code logical states, or code words, consist of a continuous superposition of coherent states along a circle ($\mathbb{T}$) or, more generally, a torus in coherent-state configuration space. Any state lying on the torus can be obtained from any other state by applying separate configuration-space rotations (also known as “phase shifters”) on every mode in a particular direction. The group of such rotations is the first ingredient in our code.

A. Projected coherent states

A single-mode rotation is an exponential of the occupation-number operator, $\hat{n} = \hat{a}^\dagger \hat{a}$, where $\hat{a}$ is the lowering (or annihilation) operator of a given mode [52,53]. The group is defined by integer vectors $\mathbf{h}$, which map out directions of rotation $\mathbf{h} \cdot \hat{\mathbf{n}}$, expressed in terms of the vector of occupation-number operators $\hat{\mathbf{n}} = (\hat{n}_1 \ \hat{n}_2 \ \dots)$. The vectors $\mathbf{h}$ generate the group and can be collected as rows of an integer “generator matrix” H .

For example, in the case of the two-mode pair-cat encoding, rotations are generated by the occupation-number difference, $\hat{n}_1 - \hat{n}_2$, corresponding to a generator matrix with one entry, $H = \mathbf{h} = (1 \ -1)$. The rotation group elements are $\exp[i\phi(\hat{n}_1 - \hat{n}_2)]$ for any angle $\phi \in [0, 2\pi)$.

Tiger code words consist of all possible rotations applied to some fiducial coherent state. We define “projected coherent states” (cf. Refs. [54,55]),

$$|\alpha\rangle_{\Delta}^H \propto \int d\phi e^{i\phi(H\hat{\mathbf{n}} - \Delta)} |\alpha\rangle, \quad (1)$$

as continuous superpositions of coherent states $|\alpha\rangle$ [56–58] whose coefficients are determined by H and an integer vector Δ . Here, the vector of angles ϕ iterates over all possible rotations. Pair-coherent [59–62], trio-coherent [63], two-mode binomial [41,42], and $SU(N)$ generalized coherent states [64–66] are all examples of such states.

Using the Fock-state expression for an ordinary coherent state, $|\alpha\rangle \propto \sum_{\mathbf{n}} \alpha^{\mathbf{n}} |\mathbf{n}\rangle / \sqrt{\mathbf{n}!}$ in multi-index notation. We see that the application of all possible rotations imposes a linear constraint on the state’s Fock states $|\mathbf{n}\rangle$,

$$H\hat{\mathbf{n}} = \Delta, \quad (2)$$

and rules out any Fock states that do not satisfy this constraint. The admissible set of Fock states can be finite or infinite, yielding tiger codes of finite or infinite Fock-state support. Tiger code words can be defined for any integer values of Δ for which there exist admissible Fock states.

The normalization function of a projected coherent state,

$$A_{\Delta}(\mathbf{y}) = \sum_{H\mathbf{n}=\Delta} \frac{\mathbf{y}^{\mathbf{n}}}{\mathbf{n}!} \quad (\text{GKZ}), \quad (3)$$

is a Gelfand-Kapranov-Zelevinsky (GKZ) hypergeometric function [44–50]—a generalization of many uni- and multivariate hypergeometric functions. Projected coherent states are in correspondence with such functions, and we utilize this connection to construct and analyze the resulting tiger codes.

B. Error model and protection

In the context of error correction, the vector Δ represents an error syndrome, and the constraint Eq. (2) divides the Fock space into disjoint sectors delineated by the vector’s possible values. Any loss operator $\hat{\mathbf{a}}^{\mathbf{p}} = \hat{a}_1^{p_1} \hat{a}_2^{p_2} \dots$, whose non-negative integer “loss vector” $\mathbf{p}$ denotes the losses in each mode, will change the value of the syndrome as long as the loss vector is not in the kernel of H . Detecting such loss errors is done by measuring each row in the vector $H\hat{\mathbf{n}}$ of linear constraint operators.

The other source of noise protected by tiger codes stems from modal frequency fluctuations [67], which elicit random single-mode rotations. Such “dephasing error” can rotate coherent states into each other, inducing logical errors.

Cat codes and more general quantum spherical codes [22] are well protected against dephasing error. Coherent-state overlaps are suppressed by the squared Euclidean distance between their configuration-space values,

$$|\langle \alpha | \beta \rangle|^2 = \exp(-\|\alpha - \beta\|^2), \quad (4)$$

and dephasing errors are suppressed exponentially with the energy and the minimum squared Euclidean distance between different constellations $\mathcal{C} \ni \alpha$ participating in each code word.

We conjecture that projected coherent states with infinite Fock-state support inherit this same degree of suppression. We prove this conjecture for a large family of tiger codes that contains all of our examples. We also provide example of codes that protect against dephasing at all nonzero energies, i.e., without requiring a large-energy limit at all.

Tiger codes with finite Fock-state support do not have a continuously tunable energy parameter. Nevertheless, we present examples of codes where exponential suppression of dephasing errors is achieved in the components of the discrete syndrome vector Δ , which determine the total energy of the code in this case.

Protection against dephasing error can be implemented autonomously by engineering stabilizing Lindbladians [25–28] that drive coherent states into the required code word constellations. Such Lindbladians are obtained using the fact that coherent states are right eigenstates of lowering operators, $\hat{\mathbf{a}}^{\mathbf{g}} |\alpha\rangle = \alpha^{\mathbf{g}} |\alpha\rangle$. Projected coherent states maintain

this property as long as $\mathbf{g}$ is in the kernel of H , yielding Lindbladian dissipators of the form $\hat{\mathbf{a}}^{\mathbf{g}} - \alpha^{\mathbf{g}}$.

The kernel constraint implies that such dissipators commute with the rotations in Eq. (1) for all values of ϕ , which means that a syndrome extraction circuit generated by a Hamiltonian in the syndrome generators $\mathbf{h} \cdot \hat{\mathbf{n}}$ commutes and can be performed in tandem, with the code's stabilizing dissipation.

Finite-support codes require dissipators whose vectors have negative entries, corresponding to monomials $\hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}}$, for a vector written as a difference of non-negative vectors, $\mathbf{g} = \mathbf{p} - \mathbf{q}$. We can still define dissipators for such codes using the identity $\hat{a}^{\dagger k} |\alpha\rangle = (\hat{n})_k |\alpha\rangle / \alpha^k$ for a single-mode coherent state $|\alpha\rangle$ and falling factorial $(u)_v = u(u-1) \cdots (u-v+1)$. Such dissipators no longer commute with each other, but the code space is still contained in their joint eigenspace.

Stabilizing dissipation allows one to track the syndrome Δ in software without necessarily correcting back to the original code space. For codes with infinite support, such tracking can continue indefinitely since there are an infinite number of syndromes. Tracking is also possible for finite-support codes, but only for a finite number of errors.

The vectors $\mathbf{g}$ corresponding to dissipators can be arranged as rows of a generator matrix G , whose span lies within the kernel of the “dual” generator matrix H , which specifies the linear occupation-number constraints. The two matrices satisfy the CSS-type condition

$$HG^T = \mathbf{0}, \quad (5)$$

and tiger codes can be defined using any such pair.

Definition—Tiger code: Let $G \in \mathbb{Z}^{r_x \times N}$ and $H \in \mathbb{Z}^{r_z \times N}$ be two integer matrices such that $HG^T = \mathbf{0}$. Let $\Delta \in \mathbb{Z}^{r_z}$ be r_z integers and $\alpha \in \mathbb{C}^N$ be N complex numbers.

The tiger code, $\mathcal{T}(G, H, \Delta, \alpha)$, consists in all rotations of the projected coherent state $|\alpha\rangle_{\Delta}^H$ defined by the kernel of G ,

$$\mathcal{T} = \text{Span} \left\{ e^{i\mu \cdot \hat{\mathbf{n}}} |\alpha\rangle_{\Delta}^H \mid \forall \mu \in \mathbb{T}^N, G\mu = 0 \pmod{2\pi} \right\}. \quad (6)$$

The code states, $|\bar{\Psi}\rangle \in \mathcal{T}$, furthermore satisfy the following constraints. For any $1 \leq j \leq r_z$, the j th row of H and the j th entry of Δ , denoted by $\mathbf{h}$ and $\Delta_{\mathbf{h}}$, respectively,

$$[\mathbf{h} \cdot \hat{\mathbf{n}} - \Delta_{\mathbf{h}}] |\bar{\Psi}\rangle = 0.$$

For all linear combination of the rows of G , $\mathbf{g} = \mathbf{s}G, \mathbf{s} \in \mathbb{Z}^{r_x}$, decomposed into $\mathbf{g} = \mathbf{p} - \mathbf{q}$, where $\mathbf{p}$ and $\mathbf{q}$ are non-negative integer vectors,

$$[\hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}} - \alpha^{\mathbf{g}} (\hat{\mathbf{n}})_{\mathbf{q}}] |\bar{\Psi}\rangle = 0.$$

In the Definition, $\mathbb{T} = [0, 2\pi)$. We use multi-index notation, e.g., $\mathbf{u}^{\mathbf{v}} \equiv \prod_j u_j^{v_j}$ for exponentiation and $(\mathbf{u})_{\mathbf{v}} \equiv \prod_j (u_j)_{v_j}$ for falling factorial. When the matrix G is non-negative, the second constraint simplifies to $[\hat{\mathbf{a}}^{\mathbf{g}} - \alpha^{\mathbf{g}}] |\bar{\Psi}\rangle = 0$ for any row $\mathbf{g}$ of G ; this corresponds to the case of infinite Fock-state support (see Secs. III, V, and VI). We explore the finite-support case in Sec. VII.

Recognizing the matrix constraint Eq. (5) as the defining equation of a chain complex (G, H) over the integers [24], we show that the dimension and structure of a tiger code space is given by the homology of this complex. The code space generally consists of finite- and infinite-dimensional factors, meaning that qudit or infinite-dimensional encodings are possible. Representatives of the homology classes define the logical- X operators, and representatives of the cohomology classes over $\mathbb{T} = [0, 2\pi)$ define logical- Z operators.

Loss and gain errors, $\hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}}$ for $\mathbf{p} \neq \mathbf{q}$, constitute X error for tiger codes. Undetectable X errors correspond to vectors $\mathbf{p} - \mathbf{q}$ that are in the kernel of H (not detectable by Z -type stabilizers), but are not in the image of G (are not X -type dissipators). The classes of logical- X operators formed by annihilation-creation monomials correspond to vectors in the quotient, $\ker H / \text{im} G$, similar to a qubit stabilizer code's logical Pauli operations forming a quotient of a normalizer by the code's stabilizer [68].

The X distance d_X is determined by the 1-norm of undetectable X error $\hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}}$ with the minimal 1-norm of $\mathbf{p} - \mathbf{q}$. A code can detect up to $d_X - 1$ and correct at least $\lfloor (d_X - 1)/2 \rfloor$ losses on any mode, granted that dephasing errors are suppressed. Certain codes can correct more losses due to the structure of the kernel of their H matrix.

We calculate the X distance using differences between creation and annihilation monomial coordinates in order to remove any dephasing error contribution, $\hat{\mathbf{a}}^{\dagger \mathbf{p}} \hat{\mathbf{a}}^{\mathbf{p}}$, which constitutes a Z error. We claim that such dephasing errors are suppressed exponentially with the Z distance of the code, which, for the infinite-support case, is the minimal square Euclidean distance between pairs of points in different coherent-state constellations making up each code word.

Logical- Z gates are tensor products of single-mode rotations which connect different code words via a Euclidean configuration-space path. More general quadratic and conditional phase gates can be realized with the same ingredients, yielding a universal gate set (see Table III).

Tiger codes can be defined for any pair of integer matrices satisfying the CSS-type condition Eq. (5). This exceedingly broad definition allows us to define codes using, e.g., generating matrices of lattices. The GKZ connection yields codes from matrices defining algebraic varieties. The contact with integer homology opens the door to using tessellations of topologically nontrivial surfaces to define intrinsically multimode codes. Altogether, we provide 17 examples of old and new tiger codes. The X and Z distances of several infinite-support examples are listed in Table I. The

TABLE I. List of distances for several tiger codes with infinite Fock-state support that are described in Sec. V. The stars $\star$ mark some of the new examples covered in this work. A code with X distance d_X can detect any X error $\hat{\mathbf{a}}^\dagger \mathbf{q} \hat{\mathbf{a}}^\mathbf{p}$ with 1-norm $|\mathbf{p} - \mathbf{q}| < d_X$ and correct at least $\lfloor (d_X - 1)/2 \rfloor$ losses on any mode, but several codes can correct more errors due to the structure of the kernel of their H matrix. For example, the pair-cat code can correct arbitrary losses on any one mode. The Z distance (58) d_Z for tiger codes encoding a logical qubit is the minimum squared Euclidean distance between coherent states in different tiger code word constellations; this distance governs the code's degree of dephasing suppression. For codes with infinite-dimensional logical encodings (rotors or modes), d_Z is still the square of minimum Euclidean distance, but listed as a function of the logical rotation angle φ , per Eq. (60). The lower bound on the tiger surface code is from Sec. VI; it becomes exact when the code is defined on a long strip, becoming the long-tiger, or “liger,” code.

Logical	Code	Modes	d_X	d_Z
Qubit	Two-component cat [26]	1	1	4
	Pair-cat [29]	2	2	4
	Four-mode tiger $\star$	4	2	8
	Coherent-state repetition [35]	N	1	$4N$
	Extended pair-cat [29]	N	N	$4N\sin^2(\pi/2N)$
	Tiger Shor $\star$	ML	M	$4ML\sin^2(\pi/2M)$
	Liger surface $\star$	$3r$	2	$4r$
Rotor	Tiger surface $\star$	$(2m - 1)r$	m	$\geq 4rms\sin^2(\pi/2m)$
	Pair-coherent state [59]	2	1	$4\sin^2(\varphi/2)$
	Three-mode tiger $\star$	3	3	$12\sin^2(\varphi/2)$
Mode	Fock-state repetition $\star$	N	N	$4N\sin^2(\varphi/2N)$

TABLE II. List of distances for several tiger codes with finite Fock-state support that are described in Sec. VII. The stars $\star$ mark some of the new examples covered in this work. The definition of X distance d_X is same as the one for infinite-support case. In lieu of the Z distance, we quantify finite-support codes' degree of dephasing suppression. Three examples can exactly detect dephasing errors, satisfying the Knill-Laflamme (KL) conditions $\langle \bar{\mu} | \hat{\mathbf{a}}^\dagger \mathbf{p} \hat{\mathbf{a}}^\mathbf{p} | \bar{\nu} \rangle = 0$ for code word with $\mu \neq \nu$ and $|\mathbf{p}| < \Delta$. Two of them suppress dephasing errors exponentially with Δ .

Logical	Code	Modes	d_X	Dephasing suppression
Qubit	Two-mode binomial [38]	2	2	Exact
	Four-mode binomial $\star$	4	2	Exact
	$\chi^{(2)}$ -like [39] $\star$	3	3	$\exp(-2\alpha\sqrt{\Delta})$
	Calabi-Yau $\star$	4	6	$(\sqrt{13}/4)^\Delta$
QuNit	Multinomial $\star$	N	2	Exact

parameters of several finite-support examples are summarized in Table II.

C. Infinite-support examples

The single-mode two-component cat code is a tiger code, admitting no occupation-number constraint ($H = 0$) and code words that are fixed-eigenvalue eigenstates of $\hat{a}^2$ ($G = 2$).

Coherent-state repetition codes, whose code words are tensor products of the same coherent state, correspond to the case where $H = 0$ and where G is the check matrix of a repetition code. Such codes can encode either a logical rotor or a logical qudit, depending on the parity of number of physical modes N —an interesting feature of homology. The duals of these codes are the Fock-state repetition codes, whose H matrices are generator matrices of the A_N root

lattice. Two-mode versions of both of these codes are special, yielding infinite-dimensional logical subspaces spanned by various pair-coherent states.

The two-mode pair-cat qubit is the simplest tiger code with nonzero generator matrices, but it does not improve over the Z distance of the two-component cat code. Its many-mode extension protects against arbitrary losses on all but one mode, but has a Z distance that vanishes with the number of modes.

Even small examples of tiger codes, like the three- and four-mode tiger codes, show intriguing behavior. For instance, the three-mode tiger code encodes a logical rotor and triples the code distances compared to a bare rotor or a logical rotor encoded in pair-coherent state, respectively. A four-mode code achieves twofold enhancement in both X and Z distances compared to the two-component cat codes.

It belongs to a family of tiger-code analogs of the qubit Shor [8,69,70] (also known as quantum parity-check) family. Its dephasing protection holds for *all* values of the energy and does not require a large-energy limit—an effect that, to our knowledge, is a first among codes based on coherent states or their concatenations. This implies that code words for any nonzero Δ are exactly orthogonal. Moreover, the code admits a logical- X operation in the form of a quadratic (i.e., beam-splitter) Hamiltonian for such cases.

Our framework yields integer-homology analogs of many established $\mathbb{Z}_2$ -homology (read: qubit) CSS codes, realized using projected coherent states. We study the integer-homology analog of the Kitaev surface code [71–75], whose linear constraints are sums of occupation numbers around a cross of a two-dimensional (2D) lattice and whose dissipators contain products of four annihilation operators. This code inherits the surface code’s stringlike minimal logical operators, geometric locality of stabilizing operators, and scaling of code distances. It is less resource intensive than the surface-cat code, which concatenates a single-mode cat code with the surface code [76]. It requires

only linear occupation-number measurements, rather than nonlinear joint parity measurements, and does not rely on individual mode stabilization. Part of this code’s dephasing protection does not require a large-energy limit.

D. Finite-support examples

Many infinite-support codes can be converted to finite-support codes by multiplying columns of generator matrices by -1 , which corresponds to mathematically rotating the Fock-space sector used to define the code (see figure in Sec. VII).

Multiplying the second column of the pair-cat generator matrix by -1 , $(1 \ -1) \mapsto (1 \ 1)$, yields a constraint on the total occupation number of the two modes. Pairing this with $G = (2 \ -2)$ yields the two-mode binomial qubit, whose code words are two-mode binomial states [41,42]. This code can be extended to a “multinomial” N -mode quNit, whose code words are $SU(N)$ generalized coherent states [64–66].

A simple three-mode code with $G = (2 \ 2 \ -2)$ and occupation-number constraints on the sums of two mode pairs admits the same Fock-state support as a $\chi^{(2)}$ code [39].

TABLE III. Table comparing the pair-cat qubit to a general quKit tiger code that is infinitely supported in Fock space. The energy density per mode is α^2 , with $\alpha > 0$ assumed real for simplicity. Here, H is an integer matrix and G is an integer matrix with non-negative entries. Code words are labeled by indices $\mu, \nu \in (2\pi/K)\mathbb{Z}_K = (2\pi/K)\{0, 1, \dots, K-1\}$. We conjecture that the ability of a dephasing operator to distinguish the code space is suppressed exponentially with the product of the code distance d_Z and energy density, and we prove this conjecture for all tiger codes that satisfy $\mathbf{1} = (1 \ 1 \ \dots \ 1) \in \ker H$ (see Sec. IV C).

	Pair-cat code	Tiger code (quKit)
Generator matrices	$G = (2 \ 2); \ H = (1 \ -1)$	$G = \begin{pmatrix} \vdots \\ \mathbf{g} \\ \vdots \end{pmatrix}; \ H = \begin{pmatrix} \vdots \\ \mathbf{h} \\ \vdots \end{pmatrix}$
X -type dissipators	$\hat{a}_1^2 \hat{a}_2^2 - \alpha^4$	$\hat{\mathbf{a}}^{\mathbf{g}} - \alpha^{ \mathbf{g} }$ for all $\mathbf{g} \in \text{row}(G)$
Z -type stabilizers	$\hat{n}_1 - \hat{n}_2$	$\mathbf{h} \cdot \hat{\mathbf{n}}$ for all $\mathbf{h} \in \text{row}(H)$
X -type code words $ \bar{\mu}\rangle$	Pair-coherent states	Projected coherent states
Detectable losses	$\{\hat{a}_1^{p_1} \hat{a}_2^{p_2} p_1 \neq p_2 \geq 0\}$	$\{\hat{\mathbf{a}}^{\mathbf{p}} \mathbf{p} \notin \ker H\}$
Undetectable X error	$\hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}}$ for $\mathbf{p} - \mathbf{q} \in \ker H \setminus \text{im} G$	$\hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}}$ for $\mathbf{p} - \mathbf{q} \in \ker H \setminus \text{im} G$
Logical- X operator	$\hat{a}_1 \hat{a}_2 / \alpha^2$	$\hat{\mathbf{a}}^{\mathbf{p}} / \alpha^{ \mathbf{p} }$ for $\mathbf{p} \in \ker H / \text{im} G$
X distance d_X	2	$\min_{\mathbf{p}, \mathbf{q} \neq \mathbf{0}} \mathbf{p} - \mathbf{q} $
Hamiltonian X gate	$\hat{a}_1 \hat{a}_2 + \text{H.c.}$	$\hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}} + \text{H.c.}$
Hamiltonian XX gate	$(\hat{a}_1 \hat{a}_2)^{\otimes 2} + \text{H.c.}$	$(\hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}})^{\otimes 2} + \text{H.c.}$
Dephasing suppression	$ \langle \bar{\mu} \hat{a}_j^{\dagger p} \hat{a}_j^p \bar{\nu} \rangle ^2 \sim \alpha^{4p} \exp(-4\alpha^2)$	$ \langle \bar{\mu} \hat{\mathbf{a}}^{\dagger \mathbf{p}} \hat{\mathbf{a}}^{\mathbf{p}} \bar{\nu} \rangle ^2 \lesssim \text{poly}(\alpha) \exp(-d_Z \alpha^2)$
Logical- Z operators	$(-1)^{\hat{n}_1}$	$e^{i\mu \mathbf{z} \cdot \hat{\mathbf{n}}}$ for $\mu \mathbf{z} \in \ker_{2\pi} G / \text{im}_{2\pi} H$
Z distance d_Z	4	$\min_{\mu \in (2\pi/K)\mathbb{Z}_K} \min_{\phi} \ \mathbf{1} - e^{i(\phi H + \mu \mathbf{z})} \ ^2$
Phase gate	$(-1)^{\hat{n}_1^2}$	$e^{i(2\pi/K)(\mathbf{z} \cdot \hat{\mathbf{n}})^2}$
SUM gate	$(-1)^{\hat{n}_1 \otimes \hat{n}_1}$	$e^{i(2\pi/K)(\mathbf{z} \cdot \hat{\mathbf{n}}) \otimes (\mathbf{z} \cdot \hat{\mathbf{n}})}$

We extend this code to a larger family and show that dephasing errors decrease exponentially with the square root of a component of the syndrome vector Δ .

Certain GKZ functions encode algebraic curves and, more generally, algebraic varieties [51]. We highlight this connection by defining a code family based on surfaces of Calabi-Yau type. This family also admits some Δ -induced dephasing suppression and can detect arbitrary losses on any mode. We utilize the powerful theory of polynomial ideals to define a logical- X operator for this and the $\chi^{(2)}$ -like codes.

III. CODE CONSTRUCTION: INFINITE SUPPORT

We go through our construction for the case of infinite Fock-state support, where the two types of stabilizing operators (X -type dissipators and Z -type stabilizers) commute. Unless otherwise specified, throughout the remainder of the text, the term “stabilizers” refers collectively to both the X -type dissipators and the Z -type stabilizers. We cover both finite- and infinite-dimensional logical encodings. For reference, the features of a general quKit tiger code are compared side by side to the two-mode pair-cat qubit in Table III.

A. Stabilizers and dissipators

Tiger codes are defined using X - and Z -type generator matrices, which we call, respectively, G and H ,

$$G = \begin{pmatrix} \vdots \\ \mathbf{g} \\ \vdots \end{pmatrix} \begin{matrix} \uparrow \\ r_x \\ \downarrow \end{matrix} \quad \text{and} \quad H = \begin{pmatrix} \vdots \\ \mathbf{h} \\ \vdots \end{pmatrix} \begin{matrix} \uparrow \\ r_z \\ \downarrow \end{matrix}, \quad (7)$$

where $G \in \mathbb{N}^{r_x \times N}$ is an $r_x \times N$ integer matrix with non-negative entries whose row vectors we generically label by $\mathbf{g}$, and where $H \in \mathbb{Z}^{r_z \times N}$ is an $r_z \times N$ integer matrix whose row vectors we generically label by $\mathbf{h}$. We cover the case when G has negative entries in Sec. VII. The two generator matrices are required to satisfy the CSS-type condition

$$GH^T = \mathbf{0} \Leftrightarrow \mathbf{h} \cdot \mathbf{g} = 0, \quad \forall \mathbf{g} \in \text{row}(G), \quad \forall \mathbf{h} \in \text{row}(H), \quad (8)$$

where $\cdot^T$ is the transpose operation and where $\mathbf{0}$ is a zero matrix with size $r_x \times r_z$.

The matrix H defines the subspace of Fock states that supports the code. This subspace consists of all Fock states $|\mathbf{n}\rangle$ whose labels satisfy the following linear constraints:

$$H\mathbf{n} = \Delta \Leftrightarrow \mathbf{h} \cdot \mathbf{n} = \Delta_{\mathbf{h}} \quad \text{for } \Delta = \begin{pmatrix} \vdots \\ \Delta_{\mathbf{h}} \\ \vdots \end{pmatrix} \begin{matrix} \uparrow \\ r_z \\ \downarrow \end{matrix}, \quad (9)$$

for all rows $\mathbf{h}$ of H and for some fixed r_z -dimensional integer vector Δ . When the explicit value of Δ does not affect code properties, it will be set to zero, while in some cases it will be beneficial to choose it to be nonzero.

Since G consists of non-negative entries, the constrained Fock subspace is infinite dimensional. Indeed, taking any initial vector $\mathbf{n} \in \mathbb{N}^N$ satisfying the constraints (9), one can add to it any positive integer multiple of any row of G and obtain another valid Fock state with higher total occupation number, showing that there is an infinite number of solutions. Correspondingly, the code words will be of infinite Fock-state support.

For example, the two-mode pair-cat code, developed in Sec. VA, corresponds to $H = \begin{pmatrix} 1 & -1 \end{pmatrix}$, with the integer scalar Δ fixing the occupation-number difference $\hat{n}_1 - \hat{n}_2$ between the modes.

The operator $\mathbf{h} \cdot \hat{\mathbf{n}} - \Delta_{\mathbf{h}}$ for each row $\mathbf{h}$ and each corresponding entry $\Delta_{\mathbf{h}}$ annihilates all code words and generates a family of unitary Z -type stabilizers,

$$\exp[i\phi(\mathbf{h} \cdot \hat{\mathbf{n}} - \Delta_{\mathbf{h}})] \quad \text{for } \phi \in \mathbb{T} = [0, 2\pi). \quad (10a)$$

Each Z -type stabilizer is a tensor product of single-mode Fock-space rotations, and the code space lies in the $+1$ eigenspace of all stabilizers for all ϕ .

The non-negative-entry matrix G corresponds to a nonunitary X -type dissipator that is a monomial in the annihilation operators $\hat{a}_j$. We call these dissipators and define them using multi-index notation,

$$\hat{\mathbf{a}}^{\mathbf{g}} \equiv \hat{a}_1^{g_1} \hat{a}_2^{g_2} \cdots \hat{a}_N^{g_N}. \quad (10b)$$

Code words are eigenstates of such dissipators. For example, the two-mode pair-cat code words are eigenstates of $\hat{a}_1^2 \hat{a}_2^2$, corresponding to $G = \begin{pmatrix} 2 & 2 \end{pmatrix}$.

All Z -type stabilizers commute with all X -type dissipators due to the CSS-type condition Eq. (8), which implies that

$$(\mathbf{h} \cdot \hat{\mathbf{n}})(\hat{\mathbf{a}}^{\mathbf{g}}) = (\hat{\mathbf{a}}^{\mathbf{g}})(\mathbf{h} \cdot \hat{\mathbf{n}} - \mathbf{h} \cdot \mathbf{g}) = (\hat{\mathbf{a}}^{\mathbf{g}})(\mathbf{h} \cdot \hat{\mathbf{n}}). \quad (11)$$

The above relation holds more generally for any orthogonal pair of vectors $\mathbf{g}$ and $\mathbf{h}$. It can be obtained by remembering that $f(\hat{n})\hat{a} = \hat{a}f(\hat{n} - 1)$ for any analytic function f of the occupation-number operator.

B. Projected coherent states

The projection of a Z -type stabilizer can be expressed in terms of a projection Π_{Δ} onto admissible Fock states $|\mathbf{n}\rangle$,

which can be defined in terms of the Z -type stabilizers; i.e., the projection can be thought of as an (integral) averaging over the entire Z -type stabilizer group. The integral and sum forms of the projection correspond to the last two equalities below,

$$\Pi_{\Delta} \equiv \prod_{\mathbf{h}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi(\mathbf{h} \cdot \hat{\mathbf{n}} - \Delta_{\mathbf{h}})}, \quad (12a)$$

$$= \int_{\phi \in \mathbb{T}^{r_z}} \frac{d^{r_z} \phi}{(2\pi)^{r_z}} e^{i\phi \cdot (H\hat{\mathbf{n}} - \Delta)}, \quad (12b)$$

$$= \sum_{\substack{\mathbf{n} \in \mathbb{N}^N: \\ H\mathbf{n} = \Delta}} |\mathbf{n}\rangle \langle \mathbf{n}|. \quad (12c)$$

In the first expression, $\mathbf{h}$ runs over the rows of H , and its corresponding $\Delta_{\mathbf{h}}$ is the coordinate of the vector Δ at the same position. For each row $\mathbf{h}$, we integrate over an angle $\phi \in [0, 2\pi)$, i.e., the group $\mathbb{T}$. In the second expression, these angles are all combined into one vector ϕ which runs over the stabilizer group $\mathbb{T}^{r_z}$. The sum form in the third expression is obtained from the integral by applying the orthogonality relation $\int_0^{2\pi} (d\phi/2\pi) e^{i\phi(\hat{n} - \Delta)} = \delta_{\hat{n}, \Delta}$ for each coordinate of ϕ , where δ is the Kronecker delta function.

Projected coherent states, $|\alpha\rangle_{\Delta}^H$, are obtained by applying this projection to the ordinary coherent states $|\alpha\rangle$, for $\alpha \in \mathbb{C}^N$, and normalizing. Using the integral and sum equivalent forms of the projection yields two complementary representations (cf. Refs. [54,55]),

$$|\alpha\rangle_{\Delta}^H \equiv \frac{e^{\|\alpha\|^2/2}}{\sqrt{A_{\Delta}(\alpha^* \alpha)}} \Pi_{\Delta} |\alpha\rangle, \quad (13a)$$

$$= \frac{e^{\|\alpha\|^2/2}}{\sqrt{A_{\Delta}(\alpha^* \alpha)}} \int_{\phi \in \mathbb{T}^{r_z}} \frac{d^{r_z} \phi}{(2\pi)^{r_z}} e^{-i\phi \cdot \Delta} |\alpha e^{i\phi H}\rangle, \quad (13b)$$

$$= \frac{1}{\sqrt{A_{\Delta}(\alpha^* \alpha)}} \sum_{H\mathbf{n} = \Delta} \frac{\alpha^{\mathbf{n}}}{\sqrt{\mathbf{n}!}} |\mathbf{n}\rangle, \quad (13c)$$

where the coherent-state value in the second expression,

$$\alpha e^{i\phi H} \equiv \left(\alpha_1 e^{i(\phi H)_1}, \alpha_2 e^{i(\phi H)_2}, \dots, \alpha_N e^{i(\phi H)_N} \right), \quad (14)$$

is obtained from the identity $e^{i\phi \hat{n}} |\alpha\rangle = |\alpha e^{i\phi H}\rangle$, and where the notions $\mathbf{n}! \equiv n_1! n_2! \dots n_N!$, $\alpha^{\mathbf{n}} \equiv \prod_{j=1}^N \alpha_j^{n_j}$, and $\sum_{H\mathbf{n} = \Delta} \equiv \sum_{\mathbf{n} \in \mathbb{N}^N: H\mathbf{n} = \Delta}$ are used in the third expression. In Eq. (13), $\|\alpha\|^2 = \sum_{j=1}^N \alpha_j^* \alpha_j$ denotes the squared 2-norm of the complex vector α . The shorthand notations introduced above will be used throughout the remainder of this work.

The normalization A_{Δ} is a GKZ hypergeometric (also known as the H -hypergeometric) function. For any complex $\mathbf{y} = (y_1, y_2, \dots, y_N)$,

$$A_{\Delta}(\mathbf{y}) \equiv \int_{\phi \in \mathbb{T}^{r_z}} \frac{d^{r_z} \phi}{(2\pi)^{r_z}} e^{-i\phi \cdot \Delta} \exp \left(\sum_{j=1}^N y_j e^{i(\phi H)_j} \right), \quad (15a)$$

$$= \sum_{H\mathbf{n} = \Delta} \frac{\mathbf{y}^{\mathbf{n}}}{\mathbf{n}!}, \quad (15b)$$

with each form obtained from the corresponding expression in Eq. (13) using the formula for the overlap between ordinary coherent states and Fock states, respectively. This function completely determines the normalization and overlap between projected coherent states,

$${}^H \langle \alpha | \beta \rangle_{\Delta}^H = \frac{A_{\Delta}(\alpha^* \beta)}{\sqrt{A_{\Delta}(\alpha^* \alpha) A_{\Delta}(\beta^* \beta)}}, \quad (16)$$

where the argument $\alpha^* \beta = (\alpha_1^* \beta_1, \alpha_2^* \beta_2, \dots, \alpha_N^* \beta_N)$. We discuss its features in the Appendix. Since vectors are used to parametrize both operators and coherent states, we include commas when the vector appears in bra-ket notation or as an argument of GKZ hypergeometric functions, but omit them when the vector represents an operator.

All projected coherent states satisfy the stabilizer conditions

$$e^{i\phi(\mathbf{h} \cdot \hat{\mathbf{n}} - \Delta_{\mathbf{h}})} |\alpha\rangle_{\Delta}^H = |\alpha\rangle_{\Delta}^H, \quad \forall \phi \in [0, 2\pi) \quad (17a)$$

for any row $\mathbf{h}$ of H since, by definition, they are superpositions of Fock states that satisfy the corresponding constraint.

Projected coherent states also inherit a subset of properties of the original coherent states. Ordinary coherent states are eigenstates of any dissipator $\hat{\mathbf{a}}^{\mathbf{g}}$, and projected coherent states retain this property for any $\mathbf{g}$ in the kernel of H ,

$$\hat{\mathbf{a}}^{\mathbf{g}} |\alpha\rangle_{\Delta}^H = \alpha^{\mathbf{g}} |\alpha\rangle_{\Delta}^H, \quad \forall \mathbf{g} \in \ker H. \quad (17b)$$

The above equation can be derived by using the first expression [Eq. (13a)] for the projected coherent state, noting that the dissipator commutes with all stabilizers due to Eq. (11), and using $\hat{a} |\alpha\rangle = \alpha |\alpha\rangle$.

More generally, any combination of loss and gain operators that commute with all $\mathbf{h} \cdot \hat{\mathbf{n}}$ operators act the same way on projected coherent states as they do on ordinary coherent states. In some cases, this allows for quadratic (i.e., beam-splitter) logical gates for tiger codes and also allows for a simple treatment of the damping operator $\exp(-\hat{n})$, irrespective of the Fock-state support of the code.

Tiger code words form a subset of the projected coherent states for certain values of α . For simplicity, we chose to always include in this subset the all-ones coherent state, with $\alpha \propto \mathbf{1} = (1, 1, \dots, 1)$,

$$|\alpha = \alpha \mathbf{1}\rangle_{\Delta}^H \quad \text{for } \alpha > 0 \quad (\text{fiducial state}), \quad (18)$$

with total energy $N\alpha^2$ and energy density α^2 . (We keep α real to avoid burdensome notation.) The remaining code words can be obtained by applying logical operators—particular tensor products of single-mode rotations—which preserve the total energy and energy density.

All logical states $|\bar{\Psi}\rangle$ —superpositions of tiger code words—satisfying constraints from Eq. (17), equivalent to

$$(\mathbf{h} \cdot \hat{\mathbf{n}})|\bar{\Psi}\rangle = \Delta_{\mathbf{h}}|\bar{\Psi}\rangle \quad \forall \mathbf{h} \in \text{row}(H), \quad (19a)$$

$$\hat{\mathbf{a}}^{\mathbf{g}}|\bar{\Psi}\rangle = \alpha^{|\mathbf{g}|}|\bar{\Psi}\rangle \quad \forall \mathbf{g} \in \text{row}(G), \quad (19b)$$

where $|\mathbf{g}| = \sum_{j=1}^N |g_j|$ denotes the 1-norm of vector $\mathbf{g}$. If $\mathbf{h}$ is the i th row of H , then $\Delta_{\mathbf{h}}$ denotes the i th entry of the vector Δ . The joint eigenspace defined by these constraints defines the logical subspace, but its structure can be determined from the properties of G and H alone.

C. Logical subspace from homology

With the Z -type stabilizers imposed, the X -type matrix G can be used to determine the structure of the code space. As shown in Eq. (17b), the kernel of H permits one to define the annihilation operators that still act as such on the projected coherent states. The rows of G , which are X -type dissipators, span a strict subspace of this kernel and further constrain the code space. The code space structure is then determined by the remaining part of the kernel of H . We can write

$$\ker H = \text{im} G + \mathcal{L}_X, \quad (20)$$

where we denote this remaining part $\mathcal{L}_X$ and also call it the logical- X group. Here H and G acts on vectors with integer entries; hence, the kernel and the image should be thought of as (sub)groups under addition. The logical- X group is then the quotient group formed by the kernel of H modulo the image of G ,

$$\mathcal{L}_X \cong \frac{\ker H}{\text{im} G}, \quad (21a)$$

which is the homology group of the integer chain complex formed by G and H . Such a homology group can contain infinite factors, i.e., $\mathbb{Z}$, as well as finite factors of the form $\mathbb{Z}_K = \mathbb{Z}/K\mathbb{Z} = \{0, 1, \dots, K-1\}$. The former are referred to as the “free” factors and the latter as the “torsion” factors. We can compute a generating matrix for $\mathcal{L}_X$: It is an integer matrix that we denote by L_X .

In the example of the pair-cat code, where $H = \begin{pmatrix} 1 & -1 \end{pmatrix}$ and $G = \begin{pmatrix} 2 & 2 \end{pmatrix}$, the kernel of H is generated by the vector $\begin{pmatrix} 1 & 1 \end{pmatrix}$, while the image of G consists of only even-integer multiples of said vector. The quotient group thus corresponds to a qubit,

$$\mathcal{L}_X = \frac{\text{im} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}}{\text{im} \begin{pmatrix} 2 & 2 \end{pmatrix}} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \cong \mathbb{Z}_2 \quad (\text{pair cat}),$$

which is an example of torsion.

Similar to the dueling sum and integral representations of projected coherent states, the powerful framework of homology yields a dual method of calculating the code space structure. The homology condition, $GH^T = \mathbf{0}$, implies that the rows of H also span a strict subspace of the kernel of G . The remaining subspace of this kernel, denoted $\mathcal{L}_Z$ and called the logical- Z group, provides another characterization of the code space structure. The kernel of G splits as

$$\ker_{2\pi} G = \text{im}_{2\pi} H + \mathcal{L}_Z,$$

where the matrices G and H are now considered to act on vectors with entries in $\mathbb{T} = [0, 2\pi)$, as signaled by the subscripts. The logical- Z group $\mathcal{L}_Z$ can then be determined by the quotient

$$\mathcal{L}_Z \cong \frac{\ker_{2\pi} G}{\text{im}_{2\pi} H}, \quad (21b)$$

which is the cohomology group of the (G, H) chain complex, defined over $\mathbb{T}$ coefficients. Because the cohomology is taken over $\mathbb{T}$, it is guaranteed to have the same structure as that of the homology group $\mathcal{L}_X$ (see Theorem 1 of Ref. [24]). Namely, for each factor $\mathbb{Z}$ in $\mathcal{L}_X$, there is a factor $\mathbb{T}$ in $\mathcal{L}_Z$, and for each factor $\mathbb{Z}_K$ in $\mathcal{L}_X$, there is a factor $(2\pi/K)\mathbb{Z}_K$ in $\mathcal{L}_Z$, where $(2\pi/K)\mathbb{Z}_K$ is the subgroup of $\mathbb{T}$ isomorphic to $\mathbb{Z}_K$,

$$\frac{2\pi}{K}\mathbb{Z}_K = \left\{ 0, \frac{2\pi}{K}, \dots, \frac{2\pi(K-1)}{K} \right\}. \quad (22)$$

In the case of the pair-cat code, the “ 2π kernel” of $G = \begin{pmatrix} 2 & 2 \end{pmatrix}$ contains all real multiples of $\begin{pmatrix} 1 & -1 \end{pmatrix}$, modulo 2π . The “ 2π image” of H precisely consists of these same multiples, suggesting (incorrectly) that the homology group, and thus the code space, is trivial. In reality, the 2π periodicity ensures that there is another vector, namely, $\pi \begin{pmatrix} 1 & 0 \end{pmatrix}$, that is in the 2π kernel of G but *not* in the 2π image of H . Other vectors such as $\frac{\pi}{2} \begin{pmatrix} 1 & 1 \end{pmatrix}$ and $\pi \begin{pmatrix} 0 & 1 \end{pmatrix}$ are also in $\ker G$, but these can be obtained from $\pi \begin{pmatrix} 1 & 0 \end{pmatrix}$ by adding an element in the 2π image of H . Since doubling $\pi \begin{pmatrix} 1 & 0 \end{pmatrix}$ yields the zero vector modulo 2π , and since there is no other vector like it, we obtain the same result as that of the previous homology calculation,

$$\mathcal{L}_Z = \frac{\mathbb{T} \times \pi\mathbb{Z}_2}{\mathbb{T}} \cong \pi\mathbb{Z}_2 \quad (\text{pair-cat code}).$$

We can also determine the generating matrix for $\mathcal{L}_Z$, which is an integer matrix that we denote by L_Z , and it

generates the Z -logical group by multiplying it with vectors with entries from $\mathcal{L}_Z$, i.e., $\mathbb{T}$ or $(2\pi/K)\mathbb{Z}_K$.

1. Computing (co)homology with torsion

Torsion occurs every time integer multiples of an integer vector are in the image of G without the vector itself being in the image. In other words, the m th multiple of a vector $\mathbf{w}$ not in the image satisfies

$$m\mathbf{w} = \mathbf{s}G \quad \text{for some } \mathbf{s} \in \mathbb{Z}^{r_x}. \quad (23)$$

Torsion can occur in homology over rings that are not fields, such as our case of the integers, and allows us to design finite-dimensional logical subspaces. The presence of torsion is also witnessed by the presence of “discrete vectors,” such as that in the pair-cat example, in the 2π kernel of G .

Since both the integer-homology and $\mathbb{T}$ -cohomology calculations yield the same answer (Theorem 1 of Ref. [24]), one can choose the more straightforward homology calculation to compute the code space structure. Computing L_X amounts to computing the Smith normal forms of G and H ,

$$V_G G = D_G W_G, \quad H W_H = V_H D_H, \quad (24)$$

where $V_{G/H} \in \mathbb{Z}^{r_{x/z} \times r_{x/z}}$ and $W_{G/H} \in \mathbb{Z}^{N \times N}$ are some unimodular matrices, and where $D_{G/H} \in \mathbb{Z}^{r_{x/z} \times N}$ have only nonzero integer entries on the diagonal. The rows of W_G corresponding to nonzero and nonunity diagonal elements of D_G are the generators for the torsion part of L_X , whose order is given by the corresponding diagonal entries. The generators for the free part are given by the columns of W_H corresponding to zero entries on the diagonal of D_H and that are not in the span of the rows of W_G corresponding to nonzero diagonal entries of D_G .

Knowing the code space structure, the computation of L_Z becomes easier as we can do the computation restricted to coefficients of the homology group. For instance, when we have logical quKits in the torsion part, we just have to do linear algebra over $\mathbb{Z}_K$ to compute the corresponding generators in L_Z as rows over $\mathbb{Z}_K$ and then interpret them as a rows over $\mathbb{Z}$. If we have a free part, we can do linear algebra over $\mathbb{Z}$ to find the corresponding generators.

D. Logical operators and code words

As described above, the homology (respectively, cohomology) group [Eq. (21)] determining the code space is, in general, a product of infinite factors $\mathbb{Z}$ (respectively, $\mathbb{T}$) and, due to torsion, possible finite factors $\mathbb{Z}_K$ (respectively, $(2\pi/K)\mathbb{Z}_K$), i.e., logical qubit ($K = 2$) or quKit subsystems. Each subsystem is generated by its own integer vector $\mathbf{z}$, and such generating vectors make up the rows of a generating matrix L_Z of logical- Z operators. Similarly, the

matrix L_X , of integer vectors $\mathbf{x}$, determines the logical- X operators,

$$L_X = \begin{pmatrix} \vdots \\ \mathbf{x} \\ \vdots \end{pmatrix} \quad \text{and} \quad L_Z = \begin{pmatrix} \vdots \\ \mathbf{z} \\ \vdots \end{pmatrix}, \quad (25)$$

and the number of rows of both matrices is equal the number of factors in the homology group. The inner product between a vector $\mathbf{x}$ and its partner $\mathbf{z}$ in the same row is 1, while all other inner products are 0, yielding

$$L_X L_Z^T = \mathbb{I} \quad (26)$$

and underpinning the commutation relations between the distinct subsystems.

Both matrices generate subspaces that are in the kernels of their corresponding generator matrices, ensuring that their associated logical operators, introduced below, commute with all stabilizers. Moreover, the generators for the logical operators can always be redefined by adding stabilizers,

$$L'_X \cong L_X + \mathbf{s}G, \quad L'_Z \cong L_Z + \mathbf{t}H, \quad (27)$$

where $\mathbf{s} \in \mathbb{Z}^{r_x}$ and $\mathbf{t} \in \mathbb{Z}^{r_z}$ are any integer vectors. Notably, since G is non-negative, we can always make L_X non-negative as well. When implementing logical- Z operators, as shown below, we can, furthermore, use the full stabilizer group, i.e., linear combinations of rows of H with coefficients in $\mathbb{T}$. By adding rows of G to L_X , one can generate a deformed X operator with negative entries. These correspond to operators of the form $\hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}}$, which have the X action within the code space but do not preserve it, thereby inducing leakage. To determine the X distance of the code, we must account for undetectable X error arising from both monomials solely by $\hat{a}$ and those involving both $\hat{a}^\dagger$ and $\hat{a}$.

Tiger code words can be obtained by applying logical- Z operators corresponding to vectors $\mathbf{z}$ to the all-ones projected coherent state $\boldsymbol{\alpha} = \alpha \mathbf{1}$. Code words are obtained in this way for both the finite- and infinite-dimensional logical subspace cases. From now on, we focus on codes that consist of a single logical subsystem of either finite or infinite dimension.

1. Logical qudit

Suppose that a given row vector $\mathbf{z} \in \mathbb{Z}^N$ of L_Z and its partner $\mathbf{x} \in \mathbb{Z}^N$ in L_X correspond to a logical qudit encoding of dimension K . Logical- Z operators are tensor-product rotations generated by $\mathbf{z} \cdot \hat{\mathbf{n}}$, multiplied by elements μ of the logical- Z group,

$$\bar{Z}(\mu) = \exp(-i\mu\mathbf{z} \cdot \hat{\mathbf{n}}) \quad \text{for } \mu \in \frac{2\pi}{K}\mathbb{Z}_K. \quad (28)$$

We can add any combination of the rows of H with arbitrary coefficients and get an equivalent Z -logical operator, i.e., for $\phi \in \mathbb{T}^{r_z}$,

$$e^{-i\mu\mathbf{z} \cdot \hat{\mathbf{n}}} \cong e^{-i\mu\mathbf{z} \cdot \hat{\mathbf{n}}} e^{-i\phi \cdot (H\hat{\mathbf{n}} - \Delta)} = e^{-i[(\mu\mathbf{z} + \phi H) \cdot \hat{\mathbf{n}} - \phi \cdot \Delta]}, \quad (29)$$

where we use “ $\cong$ ” to relate operators with equivalent logical action.

Applying the above to the all-ones state [Eq. (18)] yields an X -type code word of a logical qudit indexed by $\mu \in (2\pi/K)\mathbb{Z}_K$,

$$|\bar{\mu}\rangle = \bar{Z}(\mu)|\alpha\mathbf{1}\rangle_{\Delta}^H, \quad (30a)$$

$$= |\alpha e^{-i\mu\mathbf{z}}\rangle_{\Delta}^H, \quad (30b)$$

$$= \frac{e^{Na^2/2}}{\sqrt{\mathbf{A}_{\Delta}(\alpha^2\mathbf{1})}} \Pi_{\Delta} |\alpha e^{-i\mu\mathbf{z}}\rangle, \quad (30c)$$

where $\alpha e^{i\mu\mathbf{z}} = \alpha(e^{i\mu z_1}, e^{i\mu z_2}, \dots, e^{i\mu z_N})$, where the rightmost state on the last line is an ordinary coherent state and where the normalizing GKZ function $\mathbf{A}$ is $\mathbf{z}$ independent since all code words have the same vector of component norms, $\alpha^* \alpha = \alpha^2 \mathbf{1}$. The second equality can be obtained by using expression (13a) for the projected coherent state and observing that the logicals commute with all stabilizers due to Eq. (11).

Logical- X operators are monomials of annihilation operators whose powers are entries in $\mathbf{x}$,

$$\bar{X} = (\hat{\mathbf{a}}/\alpha)^{\mathbf{x}} = \hat{a}_1^{x_1} \hat{a}_2^{x_2} \dots \hat{a}_N^{x_N} / \alpha^{|\mathbf{x}|}. \quad (31)$$

We normalize each component in the product by α so that the operator more closely resembles a qudit Pauli operator. We can append various stabilizing operators without affecting the logical action, and there can exist other variants that contain $\hat{a}^{\dagger}$ terms.

The two types of logical operators satisfy the qudit commutation relation

$$\bar{X} \bar{Z} = e^{-i\frac{2\pi}{K}} \bar{Z} \bar{X}, \quad (32)$$

where $\bar{Z}$ without argument is just the generator of the group, $\bar{Z} = \bar{Z}(2\pi/K)$. Equation (32) is obtained from the logical relation Eq. (26) and the commutation relation Eq. (11). This commutation relation ensures that the X -type operator acts on the code words as $\bar{X}|\bar{\mu}\rangle = e^{-i\mu}|\bar{\mu}\rangle$. All logical operators commute with the stabilizing operators since their associated vectors $\mathbf{x}$ and $\mathbf{z}$ are in the kernels of H and G , respectively.

A dual basis of Z -type code words indexed by $\ell \in \mathbb{Z}_K$ can be defined by taking the logical Fourier transform,

$$|\bar{\ell}\rangle \propto \sum_{\mu \in \mathbb{Z}_K} e^{i\frac{2\pi}{K}\ell\mu} |\bar{\mu}\rangle, \quad (33a)$$

$$= \sum_{\mu \in \mathbb{Z}_K} e^{i\frac{2\pi}{K}\ell\mu} \bar{Z}^{\mu} |\alpha\mathbf{1}\rangle_{\Delta}^H, \quad (33b)$$

$$\propto \sum_{H\mathbf{n}=\Delta \wedge \mathbf{z} \cdot \mathbf{n}=\ell \bmod K} \frac{\alpha^{|\mathbf{n}|}}{\sqrt{\mathbf{n}!}} |\mathbf{n}\rangle, \quad (33c)$$

where $\mathbf{n}! \equiv n_1! n_2! \dots n_N!$ in multi-index notation, and where $|\mathbf{n}| \equiv |n_1| + |n_2| + \dots + |n_N|$. The second constraint in the last sum is modular, dividing the H -constrained Fock space further into K sectors corresponding to the K values of the inner product $\mathbf{z} \cdot \mathbf{n}$ modulo K . Each sector houses a Z -type code word, making this basis orthonormal. These code words satisfy

$$\bar{Z}(\mu)|\bar{\ell}\rangle = e^{-i\ell\mu}|\bar{\ell}\rangle, \quad (34a)$$

$$\bar{X}|\bar{\ell}\rangle \propto |\bar{\ell} - 1 \bmod K\rangle, \quad (34b)$$

which is obtained from the commutation relation (32) and Eq. (33b). Note that, in the second equation, the normalization of the code states is not perfectly conserved by $\bar{X}$, which comes from the nonorthogonality of $|\bar{\mu}\rangle$ states.

Given this orthonormal basis, we can define the code space projector as

$$P = \sum_{\ell \in \mathbb{Z}_K} |\bar{\ell}\rangle \langle \bar{\ell}|. \quad (35)$$

2. Infinite-dimensional logical subsystem

Suppose now that a given row vector $\mathbf{z}$ of L_Z and its partner $\mathbf{x}$ in L_X correspond to an infinite-dimensional subsystem. In this case, both the logical- Z operators and the X -type code words are parametrized by angles. Logical- Z operators then take the form

$$\bar{Z}(\varphi) = \exp(-i\varphi\mathbf{z} \cdot \hat{\mathbf{n}}), \quad \forall \varphi \in \mathbb{T}. \quad (36)$$

The X -type code words are obtained by applying logical- Z operators to the fiducial state

$$|\bar{\varphi}\rangle = \bar{Z}(\varphi)|\alpha\mathbf{1}\rangle_{\Delta}^H, \quad (37a)$$

$$= |\alpha e^{-i\varphi\mathbf{z}}\rangle_{\Delta}^H, \quad (37b)$$

$$= \frac{e^{Na^2/2}}{\sqrt{\mathbf{A}_{\Delta}(\alpha^2\mathbf{1})}} \Pi_{\Delta} |\alpha e^{-i\varphi\mathbf{z}}\rangle. \quad (37c)$$

The logical- X operators stay the same as those in Eq. (31) for this case, and the commutation relation becomes

$$\bar{X} \bar{Z}(\varphi) = e^{-i\varphi} \bar{Z}(\varphi) \bar{X}. \quad (38)$$

The logical- X operator acts as $\bar{X}|\bar{\varphi}\rangle = e^{-i\varphi}|\bar{\varphi}\rangle$.

A dual basis of integer-labeled Z -type code words $|\bar{\ell}\rangle$ can be defined by taking the Fourier transform of the X -type code words $|\bar{\varphi}\rangle$,

$$|\bar{\ell}\rangle \propto \int d\varphi e^{i\varphi\ell} |\bar{\varphi}\rangle, \quad (39a)$$

$$= \int d\varphi e^{i\varphi\ell} \bar{Z}(\varphi) |\alpha\mathbf{1}\rangle_{\Delta}^H, \quad (39b)$$

$$\propto \sum_{H\mathbf{n}=\Delta \wedge \mathbf{z}\cdot\mathbf{n}=\ell} \frac{\alpha^{|\mathbf{n}|}}{\sqrt{\mathbf{n}!}} |\mathbf{n}\rangle. \quad (39c)$$

The second constraint divides the H -constrained Fock space into sectors labeled by the values ℓ of the inner product $\mathbf{z} \cdot \mathbf{n}$. This inner product may yield no solution for some values of ℓ , but those sectors that do admit at least one Fock state will house one Z -type code word. Since the code words are supported on nonoverlapping sets of Fock states, this basis is orthonormal. We provide examples of logical-rotor ($\ell \in \mathbb{Z}$) and logical-mode ($\ell \geq 0$) encodings in Sec. V C.

The Z -type code words are eigenstates of $\bar{Z}(\varphi)$ with eigenvalue $\exp(-i\varphi\ell)$ and action of the logical- X operator decreases the value of ℓ by 1,

$$\bar{X}|\bar{\ell}\rangle \propto |\bar{\ell}-1\rangle. \quad (40)$$

The adjoint of this logical operator can raise the value of ℓ by 1. The code space projector is defined in the same way as that in Eq. (35).

E. Logical gates

The following two types of gates, together with the logical- X and $-Z$ gates described above, constitute a universal gate set for tiger codes with infinite Fock-state support and finite-dimensional logical encoding.

Continuous rotations generated by $\bar{X}$ and its tensor powers can be realized as Hamiltonian perturbations to the engineered dissipation induced by the dissipators $\hat{\mathbf{a}}^{\mathbf{h}} - \alpha^{|\mathbf{h}|}$. It is known that the leading-order contribution of a Hamiltonian perturbation to the steady-state subspace of a Lindbladian is the projection of the Hamiltonian into the subspace [28]. Applying the perturbation long enough relative to the dissipation strength yields the desired rotation [27]. This Zeno-type effect has been outlined in detail for the two-component cat [26] and pair-cat [29] qubits, and tiger codes follow the same recipe.

Higher-order polynomials of occupation-number operators provide simple realizations of higher-order Z -type gates for both finite- and infinite-dimensional tiger encodings. Their actions are most easily understood in terms of

Z -type code words $|\bar{\ell}\rangle$, which are constrained to consist of Fock states $|\mathbf{n}\rangle$ satisfying $\mathbf{z} \cdot \hat{\mathbf{n}} \equiv \ell$, either modulo K for the quKit case [Eq. (33c)] or exactly for the infinite-dimensional case [Eq. (39c)].

A quKit phase gate is implemented by the following “Kerr-effect” rotation [77]:

$$e^{i\frac{2\pi}{K}(\mathbf{z}\cdot\hat{\mathbf{n}})^2} |\bar{\ell}\rangle = e^{i\frac{2\pi}{K}\ell^2} |\bar{\ell}\rangle. \quad (41)$$

This equation can be proven for the quKit case by substituting $\mathbf{z} \cdot \hat{\mathbf{n}} = mK + \ell$ for some non-negative integer m . A similar operation yields a quadratic phase gate [30,78] for the infinite-dimensional case.

An entangling SUM gate of similar form acts as

$$e^{i\frac{2\pi}{K}(\mathbf{z}\cdot\hat{\mathbf{n}})\otimes(\mathbf{z}\cdot\hat{\mathbf{n}})} |\bar{\ell}_1, \bar{\ell}_2\rangle = e^{i\frac{2\pi}{K}\ell_1\ell_2} |\bar{\ell}_1, \bar{\ell}_2\rangle, \quad (42)$$

with its cousin for the infinite-dimensional case being a conditional phase gate [30].

IV. ERROR MODEL AND CODE DISTANCE

Tiger codes can protect against occupation-number loss and gain errors, and we consider losses for simplicity in this work. They also are able to suppress dephasing error, and we observe that the degree of suppression is related to the minimum Euclidean distance between coherent states in different (continuous) tiger constellations. This distance is the same as that of the quantum spherical codes [22], up to a factor of the total energy.

A. Loss error detection

In multi-index notation, occupation-number loss operators on N modes,

$$\mathbf{p} \in \mathbb{N}^n \Leftrightarrow \hat{\mathbf{a}}^{\mathbf{p}} \quad (\text{loss errors}), \quad (43)$$

are parametrized by non-negative integer vectors $\mathbf{p}$, which we will use interchangeably with the actual operator.

The Z -type stabilizers $\mathbf{h} \cdot \hat{\mathbf{n}}$ serve as check operators that can detect loss errors. The code space lies in the Fock-state subspace satisfying the constraint $\mathbf{h} \cdot \hat{\mathbf{n}} = \Delta_{\mathbf{h}}$ for all syndrome generators $\mathbf{h}$ in matrix H and corresponding entries $\Delta_{\mathbf{h}}$ of the syndrome vector Δ .

The action of a loss operator on a code state can map the code into a Fock-state subspace satisfying a different constraint. Permuting the loss operator through a Z -type stabilizer and applying the code space constraint yields

$$(\mathbf{h} \cdot \hat{\mathbf{n}})(\hat{\mathbf{a}}^{\mathbf{p}}) = (\hat{\mathbf{a}}^{\mathbf{p}})(\mathbf{h} \cdot \hat{\mathbf{n}} - \mathbf{h} \cdot \mathbf{p}) = (\hat{\mathbf{a}}^{\mathbf{p}})(\Delta_{\mathbf{h}} - \mathbf{h} \cdot \mathbf{p}). \quad (44)$$

If $\mathbf{h} \cdot \mathbf{p} \neq 0$, then the error has mapped the code space into an error space inside the Fock-state subspace corresponding to the syndrome $\Delta_{\mathbf{h}} - \mathbf{h} \cdot \mathbf{p}$. In that case, the loss error $\hat{\mathbf{a}}^{\mathbf{p}}$ can be detected by measuring $\mathbf{h} \cdot \hat{\mathbf{n}}$.

Combining all syndromes and using the definition of the code space projector [Eq. (12)],

$$\hat{\mathbf{a}}^{\mathbf{p}} \Pi_{\Delta} = \Pi_{\Delta-H\mathbf{p}} \hat{\mathbf{a}}^{\mathbf{p}}, \quad (45)$$

we see that the syndrome vector is shifted by $-H\mathbf{p}$. Any vectors $\mathbf{p}$ that are in the kernel of H will not change the syndrome vector. This fact yields a numerically tractable condition,

$$\mathbf{p} \notin \ker H \Leftrightarrow \hat{\mathbf{a}}^{\mathbf{p}} \text{ detectable}, \quad (46)$$

for a loss error to be detectable. More specifically, the Knill-Laflamme condition [79] for loss error has following property:

$$P \hat{\mathbf{a}}^{\mathbf{p}} P \propto \Pi_{\Delta} \hat{\mathbf{a}}^{\mathbf{p}} \Pi_{\Delta} = \Pi_{\Delta} \Pi_{\Delta-H\mathbf{p}} \hat{\mathbf{a}}^{\mathbf{p}} \propto \delta_{\mathbf{p} \in \ker H}, \quad (47)$$

where P is the code space projector in Eq. (35), and $\delta_{\mathbf{p} \in \ker H} \equiv \delta_{H\mathbf{p}, \mathbf{0}}$ vanishes if $\mathbf{p} \notin \ker H$. Consequently, the error-detection condition for loss is automatically satisfied for any $\hat{\mathbf{a}}^{\mathbf{p}}$ such that $\mathbf{p} \notin \ker H$.

If a loss vector $\mathbf{p}$ is in the kernel of H , then it is either an X -type dissipator or a logical operator. Dissipators are defined to be those operators that are also in the image of G , so $\mathbf{p}$ is a dissipator if it is in the span of the generators $\mathbf{g} \in \text{row}(G)$. Errors that are dissipators, like stabilizer Pauli strings for qubit stabilizer codes, are harmless since they apply the identity gate to the code space. For convenience, in the analysis of code distance, we use the term “undetectable X error” specifically for those that induce a logical- X action.

The last category of loss errors consists of errors that are undetectable and that cause a logical error. By definition, their error vectors are in the kernel of H and not in the image of G ,

$$\mathbf{p} \in \ker H \setminus \text{im} G \Leftrightarrow \hat{\mathbf{a}}^{\mathbf{p}} \text{ undetectable}. \quad (48)$$

This set can be organized into distinct logical gates by taking the quotient of the two groups. Namely, any such loss vector $\mathbf{p}$ can be expressed as a product of a coset representative $\mathbf{x} \in \ker H / \text{im} G$ and a dissipator. This decomposition is an analog of the logical Pauli group for stabilizer codes.

For example, the kernel of $H = \begin{pmatrix} 1 & -1 \end{pmatrix}$ for the pair-cat code consists of all integer multiples of $\begin{pmatrix} 1 & 1 \end{pmatrix}$. Vectors $\begin{pmatrix} p & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & p \end{pmatrix}$ for $p > 0$ are not in the kernel, meaning that the code detects all single-mode losses, $\hat{a}_1^p$ and $\hat{a}_2^p$. The operator $\hat{a}_1^2 \hat{a}_2^2$ is in the kernel, but it is the code’s sole dissipator since its corresponding vector is equal to $G = \begin{pmatrix} 2 & 2 \end{pmatrix}$. The operator $\hat{a}_1 \hat{a}_2$ is in the kernel of H but not in the image of G , making it a logical operator. Any odd power of $\hat{a}_1 \hat{a}_2$ has the same logical action and can

be expressed as a product of $\hat{a}_1 \hat{a}_2$ and a power of the code’s dissipator.

B. Loss error correction and X distance

Per the Knill-Laflamme conditions [79], correcting two nontrivial loss errors $\hat{\mathbf{a}}^{\mathbf{p}}$ and $\hat{\mathbf{a}}^{\mathbf{q}}$ requires the ability to detect their product $\hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}}$. It is useful to treat the adjoint portion as a negative vector and associate the product with the error vector $\mathbf{p} - \mathbf{q}$.

The case $\mathbf{p} = \mathbf{q}$ corresponds to a dephasing error

$$\hat{\mathbf{a}}^{\dagger \mathbf{p}} \hat{\mathbf{a}}^{\mathbf{p}} \quad (\text{dephasing errors}), \quad (49)$$

which is undetectable since $\mathbf{p} - \mathbf{p} = \mathbf{0}$ is in the kernel of H for any $\mathbf{p}$. In other words, dephasing errors commute with all Z -type stabilizers $\mathbf{h} \cdot \hat{\mathbf{n}}$ since they can be expressed as functions of the occupation-number operators. These errors cannot induce logical- X errors since they are diagonal in Fock space. They may induce Z errors, which we tackle in the next subsection.

Error products for which $\mathbf{0} \neq \mathbf{p} \neq \mathbf{q} \neq \mathbf{0}$ can be organized into two classes, extending the two parts of Eq. (46),

$$\mathbf{p} - \mathbf{q} \notin \ker H \Leftrightarrow \hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}} \text{ detectable}, \quad (50)$$

$$\mathbf{p} - \mathbf{q} \in \ker H \setminus \text{im} G \Leftrightarrow \hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}} \text{ undetectable}. \quad (51)$$

Similar to the case of loss errors, a sufficient condition for detectability is for the error vector overlap with at least one row vector $\mathbf{h} \in \text{row}(H)$ such that $(\mathbf{p} - \mathbf{q}) \cdot \mathbf{h} \neq 0$, thereby mapping the code space to an error space with a different syndrome Δ . Similar to Eq. (47), we can show that the KL condition holds true for error $\hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}}$,

$$P \hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}} P \propto \Pi_{\Delta} \Pi_{\Delta-H(\mathbf{p}-\mathbf{q})} \hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}} \propto \delta_{\mathbf{p}-\mathbf{q} \in \ker H}. \quad (52)$$

The second class corresponds to undetectable errors that yield logical errors when projected into the code space. Any errors not falling into these classes correspond to operators that can be expressed as a product of dissipators (or their adjoints) and a dephasing error.

For the pair-cat code, the error $\hat{n}_1 \hat{a}_2$ with $\mathbf{p} = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 1 & 0 \end{pmatrix}$ is detectable since its error vector, $\mathbf{p} - \mathbf{q} = \begin{pmatrix} 0 & 1 \end{pmatrix}$, is not in the kernel of $H = \begin{pmatrix} 1 & -1 \end{pmatrix}$. This error maps the code space to the same Fock-space sector as $\hat{a}_2$ since the dephasing portion of the error does not affect the syndrome. On the other hand, $\mathbf{p} - \mathbf{q} = \begin{pmatrix} 1 & 1 \end{pmatrix}$ with $\mathbf{p} = \begin{pmatrix} 2 & 2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 1 & 1 \end{pmatrix}$, corresponding to $\hat{a}_1^{\dagger} \hat{a}_2^{\dagger} \hat{a}_1^2 \hat{a}_2^2 = \hat{n}_1 \hat{n}_2 \hat{a}_1 \hat{a}_2$, is an undetectable error since it is a product of a dephasing error and a logical operation.

We use the sum of the absolute values of net photon number changes in each mode involved during a loss-gain error event $\hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}}$ —the monomial degree or 1-norm $|\mathbf{p} - \mathbf{q}|$ —to define an X distance. Correcting up to t losses $\hat{\mathbf{a}}^{\mathbf{p}}$ for $(|\mathbf{p}| \leq t)$ requires detecting operator products $\hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}}$

with twice the total degree, $|\mathbf{p}| + |\mathbf{q}| \leq 2t$. Since dephasing errors do not cause logical- X action, we exclude them from this distance analysis. Therefore, we reduce the set of annihilation-creation products required for correction to those with $\mathbf{p} - \mathbf{q} \neq \mathbf{0}$ and allows us to substitute the 1-norm of the error vector, $|\mathbf{p} - \mathbf{q}| = \sum_{j=1}^N |p_j - q_j|$, for the sum of 1-norms of $\mathbf{p}$ and $\mathbf{q}$. The upper bound of $2t$ remains unchanged since this set includes errors such as $\hat{a}_j^{\dagger t} \hat{a}_k^t$ for $j \neq k$ whose support of creation and annihilation parts are disjoint, for which $|\mathbf{p} - \mathbf{q}| = |\mathbf{p}| + |\mathbf{q}| = 2t$.

The X distance is determined by the 1-norm of the undetectable X error with lowest weight,

$$d_X = \min_{\mathbf{p}, \mathbf{q}} |\mathbf{p} - \mathbf{q}|, \quad \forall \mathbf{0} \neq \mathbf{p} - \mathbf{q} \in \ker H \setminus \text{im} G, \quad (53)$$

which ensures that any products with 1-norm lower than d_X are combinations of detectable errors, dissipators, and dephasing errors. A tiger code with such a distance detects up to $d_X - 1$ losses, i.e., all errors $\hat{\mathbf{a}}^{\mathbf{p}}$ with $|\mathbf{p}| < d_X$. If $2t < d_X$ for some t and if dephasing errors are suppressed in the large- α limit, then a code can correct t losses on any mode. Neither condition is necessary, with codes like the pair-cat code detecting arbitrary single-mode losses at distance two and with a four-mode code that admits no dephasing errors for any α .

Each loss vector $\mathbf{p}$ is also characterized by the number of modes that lost at least one particle—the (Hamming) weight or support, $\text{wt} \mathbf{p}$. The 1-norm upper bounds the weight, $\text{wt} \mathbf{p} \leq |\mathbf{p}|$, with equality occurring when every mode lost at most one particle. The quantity $d_X - 1$ provides a tight upper bound on the number of modes that can undergo loss errors while preserving the logical information. This bound happens to be an equality for all of the examples we study. It can be shown as follows. The logical operator $\hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}}$ with $|\mathbf{p} - \mathbf{q}| = d_X$ necessarily has $\mathbf{p}$ and $\mathbf{q}$ of disjoint support; otherwise, it would contain a dephasing error that could be removed to decrease the distance. A code that admits such a logical cannot correct both $\hat{\mathbf{a}}^{\mathbf{p}}$ and $\hat{\mathbf{a}}^{\mathbf{q}}$. Assuming, without loss of generality, that $\text{wt} \mathbf{p} \geq \text{wt} \mathbf{q}$, the code can only correct loss errors acting on at most $\text{wt} \mathbf{p} - 1$ modes. Using the claim of disjoint support together with the 1-norm upper bound yields $\text{wt} \mathbf{p} \leq \text{wt}(\mathbf{p} - \mathbf{q}) \leq |\mathbf{p} - \mathbf{q}| = d_X$.

The pair-cat code can detect any single-mode errors, corresponding to vectors of the form $\mathbf{p} = (p \ 0)$ or $(0 \ p)$ for any $p \in \mathbb{N}$, whose 1-norm p can be arbitrarily high. But since logical operators such as $\hat{a}_1 \hat{a}_2$ have a 1-norm of two, $d_X = 2$. Such a distance implies that this code approximately corrects a single-mode loss in either mode in the limit of large α .

C. Dephasing errors and Z distance

Dephasing errors $\hat{\mathbf{a}}^{\dagger \mathbf{p}} \hat{\mathbf{a}}^{\mathbf{p}}$ cause only logical- Z errors since they are diagonal in Fock space. In other words, their expectation values with respect to X -type code words are all

equal, and such errors manifest as off-diagonal matrix elements between the code words. We conjecture that these matrix elements are suppressed exponentially in the energy density α^2 . We further conjecture that the coefficient multiplying this density can be lower bounded by the squared Euclidean distance between the coherent-state constellations present in each code word, which we define to be the Z distance of the code. We verify these behaviors for all qudit tiger codes satisfying a technical constraint, namely, that $\mathbf{1} \in \ker H$.

1. Logical qudit

We consider the dephasing matrix element between qudit code words [Eq. (30)], $|\bar{\mu}\rangle, |\bar{\nu}\rangle$ with $\mu, \nu \in (2\pi/K)\mathbb{Z}_K$. Plugging Eq. (45) into the projected coherent-state overlap Eq. (16), the matrix element reduces to a ratio of GKZ functions with different arguments,

$$\langle \bar{\nu} | \hat{\mathbf{a}}^{\dagger \mathbf{p}} \hat{\mathbf{a}}^{\mathbf{p}} | \bar{\mu} \rangle = \frac{e^{N\alpha^2} \langle \alpha e^{i\nu \mathbf{z}} | \Pi_{\Delta} \hat{\mathbf{a}}^{\dagger \mathbf{p}} \hat{\mathbf{a}}^{\mathbf{p}} \Pi_{\Delta} | \alpha e^{i\mu \mathbf{z}} \rangle}{A_{\Delta}(\alpha^2 \mathbf{1})}, \quad (54a)$$

$$= \alpha^{2|\mathbf{p}|} \frac{A_{\Delta-H\mathbf{p}}(\alpha^2 e^{i(\mu-\nu)\mathbf{z}})}{A_{\Delta}(\alpha^2 \mathbf{1})}. \quad (54b)$$

The case $\mathbf{p} = \mathbf{0}$ corresponds to the overlap between the nonorthogonal states, i.e., an intrinsic memory error. This element needs to be zero for all $\mathbf{p}$ in order for the code to exactly protect against dephasing. We show that, in the worst case, it goes to zero quickly in the large- α limit for a large class of codes, and we conjecture that this suppression holds for all codes.

Since each tiger code word is a continuous superposition of a coherent-state constellation, this overlap is a “sum” over all possible overlaps between points taken from the two constellations. The integral form [Eq. (15a)] of a GKZ function can be easily upper bounded by the overlap of the two closest points,

$$|A_{\Delta}(\alpha^2 e^{i\mu \mathbf{z}})| \leq e^{N\alpha^2} \int \frac{d^r \boldsymbol{\phi}}{(2\pi)^{r_z}} e^{-\frac{1}{2}\alpha^2 \|\mathbf{1} - e^{i(\boldsymbol{\phi}H + \mu \mathbf{z})}\|^2}. \quad (55)$$

The above integrand is the absolute value of the overlap (4) between two ordinary coherent states: the all-ones coherent state and another state whose value depends on $\boldsymbol{\phi}$. It is maximized at the angle vector for which the two coherent states are the closest, i.e., $\boldsymbol{\phi}_{\star} = \text{argmin}_{\boldsymbol{\phi}} \|\mathbf{1} - e^{i(\boldsymbol{\phi}H + \mu \mathbf{z})}\|^2$. Substituting this optimal value yields a bound for the numerator in Eq. (54b).

The special case of the above bound with $\mathbf{z} = \mathbf{0}$ and $\boldsymbol{\phi}_{\star} = \mathbf{0}$ applies to the denominator in Eq. (54b), yielding an upper bound of $\exp(N\alpha^2)$. We cannot use this to upper bound the dephasing matrix element since we would need a lower bound for the denominator. However, a careful asymptotic

analysis of the denominator in the large- α limit (see the Appendix) yields an exact expression with the same exponential dependence, provided that $\mathbf{1} \in \ker H$. For such cases, we obtain

$$|\langle \bar{\nu} | \hat{\mathbf{a}}^\dagger \mathbf{p} \hat{\mathbf{a}} \mathbf{p} | \bar{\mu} \rangle|^2 \stackrel{\alpha \rightarrow \infty}{\lesssim} \text{poly}(\alpha) e^{-\alpha^2 \|\mathbf{1} - e^{i(\phi \star H + (\mu - \nu) \mathbf{z})}\|^2}, \quad (56)$$

which omits polynomial prefactors to highlight the exponential suppression.

The above bound shows that all dephasing errors are suppressed at large α , with the strength of the suppression depending on the minimal Euclidean distance between code word constellations. The bound in Eq. (56) motivates a Euclidean definition of the Z distance.

Returning to the case of a logical qudit, we have to consider dephasing matrix elements between any two code words labeled by $\nu \neq \mu \in (2\pi/K)\mathbb{Z}_K$. Global rotations allow us to reduce this to K matrix elements between $|\bar{\mu}\rangle$ and the fiducial state $|\bar{0}\rangle$, with each element bounded by the minimal square Euclidean distance between a pair of constellations,

$$d_Z(\mu) = \min_{\phi \in \mathbb{T}^{r_z}} \|\mathbf{1} - e^{i(\phi H + \mu \mathbf{z})}\|^2, \quad (57a)$$

$$= 4 \min_{\phi \in \mathbb{T}^{r_z}} \sum_{j=1}^N \sin^2 \frac{1}{2} (\phi H + \mu \mathbf{z})_j, \quad (57b)$$

where the squared Euclidean distance between two complex vectors $\mathbf{u}, \mathbf{v}$ is defined as $\|\mathbf{u} - \mathbf{v}\|^2 = \sum_{j=1}^N (u_j^* - v_j^*)(u_j - v_j)$. Being a measure of closeness in the configuration space of coherent states, this distance also quantifies the “shortest” logical rotation that one can apply to map one X -type code word to another. The code’s Z distance is then the square of minimum Euclidean distance among all paths connecting pairs of distinct X -type code words,

$$d_Z = \min_{\mu \in \frac{2\pi}{K}\mathbb{Z}_K} d_Z(\mu). \quad (58)$$

We illustrate the Z distance of the pair-cat code in Fig. 1(a), which is the squared Euclidean distance between the black and white lines.

Plugging this distance into a generic dephasing matrix element for codes satisfying $\mathbf{1} \in \ker H$ and squaring the matrix element to rid ourselves of a factor of one-half yields

$$|\langle \bar{\nu} | \hat{\mathbf{a}}^\dagger \mathbf{p} \hat{\mathbf{a}} \mathbf{p} | \bar{\mu} \rangle|^2 \stackrel{\alpha \rightarrow \infty}{\lesssim} \text{poly}(\alpha) \exp(-d_Z \alpha^2). \quad (59)$$

For example, in the case of the two-component cat code with code words $|\pm \alpha\rangle$, the distance is 4—the squared Euclidean distance between the two coherent-state values at $\alpha = 1$.

The code distance yields a tight bound on the dephasing suppression coefficient for all but one of our examples—the

liger surface code—where a careful asymptotic analysis of the dephasing matrix elements yields a coefficient that is strictly larger than the squared Euclidean distance. The exact suppression coefficient may be dependent on H , number of modes N , and syndrome vector Δ . As such, we conjecture that the code distance serves as a lower bound on the dephasing suppression, with some codes exhibiting greater suppression. Proving this conjecture requires a derivation of the above asymptotic inequality for codes with $\mathbf{1} \notin \ker H$.

2. Infinite-dimensional logical subsystem

Infinite-dimensional logical encodings are parametrized by a continuous angle parameter ϕ . There is now a continuum of continuous constellations, one for each value of ϕ . The analogous rotor-code distance [Eq. (67) of Ref. [24]] was defined as a minimization of a relative distance—the squared Euclidean distance of the code divided by the squared Euclidean distance of the “bare” single-mode implementation $e^{i\phi}$ of the logical rotation. Here, we keep the Euclidean distance ϕ dependent to make contact with the coherent-state overlaps discussed above,

$$d_Z(\phi) = \min_{\phi \in \mathbb{T}^{r_z}} \|\mathbf{1} - e^{i(\phi H + \phi \mathbf{z})}\|^2. \quad (60)$$

We leave the question of relating the two distances a question for future work.

V. EXAMPLES OF TIGER CODES

In this section, we provide examples of new tiger codes and recast known codes in our framework. All codes here have infinite Fock-state support, but can encode logical qubits, qudits, rotors, or modes.

A. Pair-cat code

The pair-cat code [29] is a two-mode bosonic code that protects against arbitrary single-mode photon losses. Having used it as an example throughout the previous section, here we provide only a brief overview.

Its generator matrices are $G = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}$ and $H = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}$. We can verify that these follow the CSS condition, $GH^T = 0$. They correspond to one dissipator and one Z -type stabilizer, respectively,

$$\hat{a}_1^2 \hat{a}_2^2 - \alpha^4 \quad \text{and} \quad \hat{n}_2 - \hat{n}_1 - \Delta, \quad (61)$$

where α^2 is the energy density and where the integer syndrome Δ quantifies the difference between the occupation numbers in the two modes.

A simple homology calculation, done in Sec. III C, shows that the code encodes a qubit with logical generators $\mathbf{x} = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and $\mathbf{z} = \begin{pmatrix} 1 & 0 \end{pmatrix}$. These yield logical operator representatives $\bar{X} = \hat{a}_1 \hat{a}_2 / \alpha^2$ and $\bar{Z} = \exp(i\pi \hat{n}_1)$, respectively.

The X -type code words are the coherent states $|\alpha, \alpha\rangle$ and $\bar{Z}|\alpha, \alpha\rangle = |-\alpha, \alpha\rangle$, each projected into a fixed- Δ subspace via the projection $\Pi_\Delta = \frac{1}{2\pi} \int d\phi e^{i\phi(\hat{n}_2 - \hat{n}_1 - \Delta)}$. Explicitly, for $\Delta > 0$,

$$|\bar{\pm}\rangle = \frac{1}{\sqrt{I_\Delta(2\alpha^2)}} \sum_{n \geq 0} \frac{(\pm 1)^n \alpha^{2n+\Delta}}{\sqrt{n!(n+\Delta)!}} |n, n+\Delta\rangle, \quad (62)$$

where we have divided out constants upon normalization. The negative Δ case is related to the above by a swap of the modes. These are instances of pair-coherent [59–62] or fixed-charge [56,80] states, with the occupation-number difference referred to as the “charge” in this context.

The GKZ function associated with this code is bivariate since the code is on two modes, but the two arguments collapse into one after simplifying,

$$A_\Delta(y_1, y_2) = \sum_{n \geq 0} \frac{y_1^n y_2^{n+\Delta}}{n!(n+\Delta)!} = \left(\frac{y_2}{y_1}\right)^{\Delta/2} I_\Delta(2\sqrt{y_1 y_2}), \quad (63)$$

where I is the modified Bessel function of the first kind [81].

Using the definition from Eq. (58), we obtain

$$d_Z = \min_{\phi} \|(1 \ 1) - (-e^{i\phi} \ e^{-i\phi})\|^2 = 4, \quad (64)$$

which matches the exponent of an asymptotic calculation of the dephasing suppression coefficient (see Table II in Ref. [29]), making the bound Eq. (59) tight for this case.

We derived the X distance of this code, $d_X = 2$, in the previous section, indicating that the code can detect arbitrary single-photon losses. Furthermore, in the large- α limit, the pair-cat code can approximately correct single-photon losses $\{\hat{a}_1, \hat{a}_2\}$ in each mode. However, since $\hat{a}_1^\dagger \hat{a}_2$ is not a logical operator, and since all dephasing errors are exponentially suppressed, the code can approximately correct arbitrary photon losses in each mode $\{\hat{a}_1^p, \hat{a}_2^p\}_{p>0}$ in the large- α limit. Therefore, the pair-cat code is an approximate error-correcting code and an exact error-detecting code against single-mode losses.

The pair-cat code can be generalized to $H = (m_1 \ -m_2)$ for any coprime positive integers $m_1 \neq m_2$. The kernel of H is spanned by $L_X = (m_2 \ m_1)$, yielding a distance of $d_X = m_1 + m_2$. Such a code also detects up to $m_1 + m_2 - 1$ losses on any mode, in addition to detecting all single-mode losses.

B. Extended pair-cat code

An N -mode extension of the pair-cat code was shown to detect arbitrary losses in $N - 1$ modes [29,82] (cf. Ref. [63] for the $N = 3$ case). This code is also a tiger code, which we call the extended pair-cat code, with generator matrices

$$G = (2 \ 2 \ \cdots \ 2),$$

$$H = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & 1 & -1 \end{pmatrix} = A_N, \quad (65)$$

with the latter matrix generating the A_N root lattice [40]. These correspond to the dissipator $\hat{\mathbf{a}}^2 - \alpha^{2N}$, where $\mathbf{2} = (2 \ 2 \ \cdots \ 2)$, and $N - 1$ pairs of neighboring occupation-number differences, $\hat{n}_{j+1} - \hat{n}_j - \Delta_j$, for its syndromes.

Logical operators are given by

$$\bar{X} = (\hat{\mathbf{a}}/\alpha)^{\mathbf{1}} \quad \text{and} \quad \bar{Z} = (-1)^{\hat{n}_1}. \quad (66)$$

Logical- X code words are coherent states projected into the Fock subspace of all occupation-number differences fixed by Δ . We call these “extended pair-coherent states.” For $\Delta = \mathbf{0}$, these are expressed simply as

$$|\bar{\pm}\rangle = \frac{(\pm 1)^{\hat{n}_1}}{\sqrt{A_0(\alpha^2 \mathbf{1})}} \sum_{n \geq 0} \left(\frac{\alpha^n}{\sqrt{n!}} |n\rangle \right)^{\otimes N}. \quad (67)$$

The GKZ normalization for this case collapses all N arguments into a product,

$$A_0(\mathbf{y}) = {}_0F_{N-1}(\ ; \mathbf{1}; y_1 y_2 \cdots y_N), \quad (68)$$

where $\mathbf{1}$ is the all-ones vector of length $N - 1$, and where ${}_pF_q$ is the generalized hypergeometric function [81].

This code has an X distance $d_X = N$, but has the distinct feature of being able to approximately correct (and not just detect) losses on $N - 1$ modes in the large- α limit, which is due to the structure of the kernel of H , generated by the logical operator’s vector $\mathbf{1}$. Since vectors $\mathbf{p} - \mathbf{q}$, with $\mathbf{p}$ and $\mathbf{q}$ supported on up to $N - 1$ modes, are outside of the kernel, their corresponding operators $\hat{\mathbf{a}}^\dagger \mathbf{q} \hat{\mathbf{a}}^{\mathbf{p}}$ are detectable. Combined with the fact that dephasing errors are suppressed with large α yields correction of any losses up to one less the code distance.

The large- X distance of N comes at the price of a vanishing Z distance, $d_Z = 4N \sin^2(\pi/2N)$, because an alternative logical operator, $\exp(i \sum_j \pi \hat{n}_j / N)$, related to the operator $\exp(i\pi \hat{n}_1)$ by a Z -type stabilizer, rotates between code words by applying a vanishingly small rotation on each mode.

We confirm that the dephasing suppression coefficient of the code is given by the code distance. Since the estimate from Eq. (54) is independent of the power of the dephasing error, we can extract the suppression coefficient from the code word overlap,

$$\langle \bar{-} | + \rangle = \frac{A_0(\alpha^2(-1, 1, \dots, 1))}{A_0(\alpha^2 \mathbf{1})} = \frac{{}_0F_{N-1}(\cdot; \mathbf{1}; -\alpha^{2N})}{{}_0F_{N-1}(\cdot; \mathbf{1}; \alpha^{2N})}. \quad (69)$$

Using asymptotic properties of generalized hypergeometric functions [81], we can determine the coefficient in the large- α limit,

$$\left| \frac{{}_0F_{N-1}(\cdot; \mathbf{1}; -\alpha^{2N})}{{}_0F_{N-1}(\cdot; \mathbf{1}; \alpha^{2N})} \right|^2 \sim \text{poly}(\alpha) \exp\left(-4N\alpha^2 \sin^2 \frac{\pi}{2N}\right). \quad (70)$$

C. Pair-coherent states as tiger codes

Two mutually dual variants of the pair-cat code are simple examples of tiger codes with two types of infinite-dimensional logical spaces—a logical rotor and a logical mode. These are special cases of the coherent-state repetition code and the Fock-state repetition code, respectively.

1. Logical rotor

Let us pick the generator matrices

$$G = \begin{pmatrix} 1 & 1 \end{pmatrix} \quad \text{and} \quad H = 0, \quad (71)$$

which correspond to a single dissipator $\hat{a}_1 \hat{a}_2 - \alpha^2$.

A Z-type code word for this code consists of pair-coherent states with fixed argument α^2 . Such states are defined for any occupation-number difference, and there is exactly one state in each sector of fixed difference,

$$|\bar{\ell}\rangle = \frac{1}{\sqrt{I_\ell(\alpha)}} \sum_{n \geq 0} \frac{\alpha^{2n+\ell}}{\sqrt{n!(n+\ell)!}} |n, n+\ell\rangle, \quad (72)$$

for positive difference ℓ , with the negative- ℓ case related to the above by a swap of the modes. Since Z-type code words with different ℓ lie in separate Fock-state subspaces, they form an orthonormal basis. Such a code space is naturally associated with the angular momentum space of a (planar) rotor, also known as a particle on a circle.

The X-type code words, parametrized by an angle φ , can be defined by applying the logical operator $\bar{Z}(\varphi) = \exp[i\varphi(\hat{n}_1 - \hat{n}_2)]$ to the all-ones coherent state,

$$|\bar{\varphi}\rangle = \bar{Z}(\varphi)|\alpha\rangle = (1, 1)\rangle = |\alpha\rangle = (e^{i\varphi}, e^{-i\varphi})\rangle. \quad (73)$$

The X distance $d_X = 1$ since single-photon loss vectors are in the kernel of H , which is everything, but not in the image of G .

Using the definition from Eq. (60), we obtain

$$d_Z = 4 \sin^2 \frac{\varphi}{2}, \quad (74)$$

the same as the squared Euclidean distance of the bare encoding $\|1 - e^{i\varphi}\|^2$. In other words, the code does no better

at protecting against dephasing error than an unencoded rotor.

2. Logical mode

The dual code is obtained by only imposing the H constraint that generates the kernel of the previous code's G matrix. The code's generator matrices are

$$G = 0 \quad \text{and} \quad H = \begin{pmatrix} 1 & -1 \end{pmatrix}, \quad (75)$$

which correspond to a single Z-type stabilizer, $\hat{n}_1 - \hat{n}_2 - \Delta$. The X-type code word consists of pair-coherent states of arbitrary argument but with fixed difference.

The Z-type code word consists of two-mode fixed-difference Fock states,

$$|\bar{\ell}\rangle = |\ell + \Delta, \ell\rangle, \quad \forall \ell \geq 0, \quad (76)$$

for positive Δ , with the negative- Δ case related by a swap of the modes. This basis is labeled by non-negative integers, so most naturally corresponds to a logical-mode encoding (as opposed to a rotor). One can also think of this as a rotor whose angular momenta are restricted to be non-negative [30].

The X distance is $d_X = 2$, since the lowest-degree undetectable X error is $\hat{a}_1 \hat{a}_2$, which gives the logical-X operator. This code is a special case of the Fock-state repetition code, whose Z distance is discussed in the next subsection.

D. Fock-state repetition code

A close relative of both the logical-mode code from the previous example and the extended pair-cat code is the N -mode Fock-state repetition code, with

$$G = 0 \quad \text{and} \quad H = A_N. \quad (77)$$

There are $N - 1$ Z-type stabilizers, $\hat{n}_{j+1} - \hat{n}_j - \Delta_j$ for all $1 \leq j \leq N - 1$, the same as those for the extended pair-cat code. The difference is that there is no dissipator, which enlarges the logical space from a qubit to a mode.

The mode's Z-type code word, for $\Delta = \mathbf{0}$, is made up of Fock states $|\bar{\ell}\rangle^{\otimes N}$. The mode's logical-X operator, $\bar{X} = (\hat{\mathbf{a}}/\alpha)^{\mathbf{1}}$, is the same as the logical operator of the extended pair-cat qubit from Sec. VB.

The X-type code word consists of extended pair-coherent states from Sec. VB for arbitrary argument but with all occupation-number differences fixed. The X distance is $d_X = N$.

As with the extended pair-cat code, the canonical logical-Z operator is $\bar{Z}(\varphi) = \exp(i\varphi \hat{n}_1)$, but an alternative logical, $\exp(i \sum_j \varphi \hat{n}_j / N)$, applies a rotation that vanishes with N to each mode. The Z distance from Eq. (60) is

$$d_Z = 4N \sin^2 \frac{\varphi}{2N}, \quad (78)$$

being of order $O(1/N)$ for large N and any fixed φ .

E. Coherent-state repetition code

Dual to the Fock-state repetition code from the previous example is the coherent-state repetition code, whose G matrix is that of a cyclic repetition code,

$$G = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = R_N \quad \text{and} \quad H = 0. \quad (79)$$

Its X distance is trivial with $d_X = 1$, while its Z distance increases as system size increases, $d_Z = 4N$.

Fixing N to be odd yields torsion [Eq. (23)], namely, a vector $\mathbf{e}_1$ that is not in the image of G but whose multiple is. It is easy to check that the alternating sum of all the rows of G gives $2\mathbf{e}_1$, with $\mathbf{e}_1$ itself not in the row image of G ,

$$\mathbf{f}_N = (1 \quad -1 \quad 1 \quad -1 \quad \cdots \quad 1), \quad (80a)$$

$$\mathbf{e}_1 = (1 \quad 0 \quad 0 \quad \cdots \quad 0), \quad (80b)$$

$$\mathbf{f}_N G = 2\mathbf{e}_1. \quad (80c)$$

The X -type code words are

$$|\pm\rangle = |\pm\alpha\rangle^{\otimes N}, \quad (81)$$

which can be viewed as a concatenation between the two-component cat code (in the coherent-state basis) and a qubit repetition code [34,35].

A set of logical vectors is $\mathbf{x} = \mathbf{e}_1$ and $\mathbf{z} = \mathbf{1}$,

$$\bar{X} = \frac{\hat{a}_1}{\alpha} \quad \text{and} \quad \bar{Z} = (-1)^{\sum_{j=1}^N \hat{n}_j}, \quad (82)$$

respectively. Other logical operators include $\bar{X} = \hat{a}_j/\alpha$ for any mode j , and these are related to each other via a dissipator and the condition that any $\hat{a}_i^2 \propto \bar{X}^2$ for all $1 \leq i \leq N$ acts as identity on the code space.

The strength of dephasing suppression is easily calculated from the code word overlap,

$$|\langle - | + \rangle|^2 = \frac{|\langle \alpha | - \alpha \rangle|^{2N}}{|\langle \alpha | \alpha \rangle|^{2N}} = \exp(-4N\alpha^2) \quad (83)$$

saturating the upper bound in Eq. (59) with $d_Z = 4N$. The GKZ function for this code is $\mathbf{A}(\mathbf{y}) = \exp(\sum_j y_j)$.

Other variants of this code arise when N is even, which yields only a logical rotor despite the closed or open

boundary. For an odd-length system, one can also obtain a logical rotor by removing the last row of G , corresponding to open boundaries on the repetition code.

F. Three-mode tiger code

In this subsection, we introduce a three-mode tiger code inspired by the $U(1)$ -covariant three-rotor code proposed in Refs. [83,84] in the context of holographic quantum information. The corresponding tiger code encodes a single logical rotor into three physical modes.

The code is defined by the following parity-check matrices:

$$G = \begin{pmatrix} 3 & 1 & 2 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} -4 & 6 & 3 \end{pmatrix}, \quad (84)$$

which correspond to dissipator $\hat{a}_3^3 \hat{a}_2 \hat{a}_3^2 - \alpha^6$ and Z -type stabilizer $-4\hat{n}_1 + 6\hat{n}_2 + 3\hat{n}_3 - \Delta$.

The Z basis of the logical rotor, for $\Delta = 0$, is

$$|\bar{\ell}\rangle \propto \sum_{n \geq \ell} \frac{\alpha^{6n-\ell}}{\sqrt{(3n)!(n+\ell)!(2n-2\ell)!}} |3n, n+\ell, 2(n-\ell)\rangle, \quad (85)$$

for all $\ell \in \mathbb{Z}$.

The logical operators are specified by the vectors $\mathbf{x} = \begin{pmatrix} 3 & 2 & 0 \end{pmatrix}$ and $\mathbf{z} = \begin{pmatrix} 1 & -1 & -1 \end{pmatrix}$, corresponding to logical operators

$$\bar{X} = \hat{a}_1^3 \hat{a}_2^2 / \alpha^5, \quad (86a)$$

$$\bar{Z}(\varphi) = e^{i\varphi(\hat{n}_1 - \hat{n}_2 - \hat{n}_3)}. \quad (86b)$$

The X distance of the code as $d_X = 3$ because $\begin{pmatrix} 0 & 1 & -2 \end{pmatrix} \in \ker H$ that gives an lowest-degree undetectable X error $\hat{a}_2^\dagger \hat{a}_3^2$.

To compute the Z distance, we examine the squared Euclidean distance between logical code words via Eq. (60). When φ is sufficiently small, the distance is

$$d_Z = 4 \times 3 \sin^2 \frac{\varphi}{2} = 12 \sin^2 \frac{\varphi}{2}. \quad (87)$$

Compared to a bare rotor or the logical rotor encoded in pair-coherent states, which have X and Z distances of 1 and $4\sin^2(\varphi/2)$, respectively, the three-mode tiger code achieves a threefold improvement in both X and Z distances.

G. Four-mode tiger code

Emulating a rotor code whose homology is based on a tessellation of the projective plane [24], we construct a minimal example whose distances, $d_X = 2$ and $d_Z = 8$, are both twice that of the two-component cat code.

This four-mode code is a combination of the logical-rotor pair-coherent-state code from Sec. V C and the pair-cat code

from Sec. VA, and there is no discrete-variable code participating in this construction. We first define two logical rotors, with each rotor encoded into two modes, and then build a pair-cat qubit on top of the two rotors. It can be regarded as a tiger-code (i.e., integer-homology) version of the $[[4, 1, 2]]$ qubit stabilizer code in Ref. [85]. This code exhibits a new feature not previously observed in coherent-state codes—dephasing errors stemming from detectable losses are detected exactly without the need for the large- α limit.

The code's generator matrices are

$$G = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix}. \quad (88)$$

The first two rows of G correspond to two copies of the logical-rotor pair-coherent-state encoding, while the last row is the X -type pair-cat constraint. These yield a set of three dissipators and one Z -type stabilizer,

$$\begin{aligned} \hat{a}_1 \hat{a}_2 - \alpha^2, \\ \hat{a}_3 \hat{a}_4 - \alpha^2, \quad \text{and} \quad \hat{n}_1 - \hat{n}_2 - \hat{n}_3 + \hat{n}_4 - \Delta. \\ \hat{a}_2^2 \hat{a}_4^2 - \alpha^4, \end{aligned} \quad (89)$$

A homology calculation yields logical operator vectors $\mathbf{x} = (0 \ 1 \ 0 \ 1)$ and $\mathbf{z} = (0 \ 0 \ 1 \ 1)$, corresponding to logical-qubit operators

$$\bar{X} = \frac{\hat{a}_2 \hat{a}_4}{\alpha^2} \quad \text{and} \quad \bar{Z} = (-1)^{\hat{n}_3 + \hat{n}_4}. \quad (90)$$

For convenience, we consider code words for the $\Delta = 0$ case. The X -type code word, in integral form, consists of orbits of the two coherent states $\boldsymbol{\alpha} = \alpha \mathbf{1}$ and $\boldsymbol{\alpha} = \alpha(1, 1, -1, -1)$ over the group generated by the Z -type stabilizer,

$$|\pm\rangle = \frac{e^{2\alpha^2}}{\sqrt{A_0(\alpha^2 \mathbf{1})}} \int \frac{d\phi}{2\pi} |\alpha(e^{i\phi}, e^{-i\phi}, \pm e^{-i\phi}, \pm e^{i\phi})\rangle. \quad (91)$$

The code's GKZ function simplifies to a modified Bessel function of the first kind [81] with nontrivial argument. We will need it for all values of Δ . For positive values, we obtain

$$A_\Delta(\mathbf{y}) = \sum_{\mathbf{n} \in \mathbb{N}^4} \delta_{n_1 - n_2 - n_3 + n_4, \Delta} \frac{\mathbf{y}^{\mathbf{n}}}{\mathbf{n}!}, \quad (92a)$$

$$= \sum_{j \geq \Delta} \frac{(y_1 + y_4)^j}{j!} \sum_{n_2, n_3 \geq 0} \delta_{j, n_2 + n_3 + \Delta} \frac{y_2^{n_2} y_3^{n_3}}{n_2! n_3!}, \quad (92b)$$

$$= \left(\frac{y_1 + y_4}{y_2 + y_3} \right)^{\Delta/2} I_\Delta \left(2\sqrt{(y_1 + y_4)(y_2 + y_3)} \right), \quad (92c)$$

with the case of negative Δ related by a swap of variables $y_1 + y_4 \leftrightarrow y_2 + y_3$. Above, we split the Kronecker delta function $\delta_{n_1 - n_2 - n_3 + n_4, \Delta}$ with a useful identity,

$$\delta_{n_1 - n_2 - n_3 + n_4, \Delta} = \sum_{j \geq \Delta} \delta_{n_1 + n_4, j} \delta_{n_2 + n_3 + \Delta, j} \quad (93)$$

and apply the binomial theorem twice.

The code detects any losses $\hat{\mathbf{a}}^{\mathbf{p}}$ with $\mathbf{p} \notin \ker H$. The kernel is spanned by the first two rows of G and by $(0 \ 1 \ 0 \ 1)$. The two rows are dissipators, while the third is a logical- X operator. No loss vectors of weight 1 are in the kernel, so the code exactly detects up to one loss on any mode— $d_X = 2$. However, there is a weight-2 operator in the kernel— $\hat{a}_1 \hat{a}_4^\dagger$ with error vector $(1 \ 0 \ 0 \ -1)$ —that is not detectable. In other words, both $\hat{a}_1$ and $\hat{a}_4$ errors yield the same Z -type syndrome, and we could not distinguish and correct either of them by checking the Z syndrome. Therefore, this code cannot correct single-photon losses.

Plugging the GKZ function into the dephasing matrix elements [Eq. (54b)] and using the fact that $I_j(0) = \delta_{j,0}$ yields

$$\begin{aligned} \langle \bar{-} | \hat{\mathbf{a}}^{\dagger \mathbf{p}} \hat{\mathbf{a}}^{\mathbf{p}} | \bar{+} \rangle &= \alpha^{2|\mathbf{p}|} \frac{A_{-H\mathbf{p}}(\alpha^2(1, 1, -1, -1))}{A_0(\alpha^2 \mathbf{1})}, \\ &= \delta_{\mathbf{p} \in \ker H} \frac{\alpha^{2|\mathbf{p}|}}{I_0(4\alpha^2)}, \end{aligned} \quad (94)$$

where the Kronecker delta function $\delta_{\mathbf{p} \in \ker H}$ is 1 when $\mathbf{p}$ is in the kernel of H and zero otherwise. The matrix element is identically zero for dephasing errors associated with any detectable losses, meaning that no large- α limit is needed to detect them. A similar feature is also present in a thin surface code—the liger surface code—described in the next section.

Intrinsic dephasing due to nonorthogonality ($\mathbf{p} = \mathbf{0}$) and dephasing associated with a logical operator (e.g., $\mathbf{p} = \mathbf{z}$) remain finite, but are suppressed in said limit with exponent $4\alpha^2$ since $I_0(4\alpha^2) \sim \exp(-4\alpha^2)$. The dephasing suppression for arbitrary dephasing is

$$|\langle \bar{-} | \hat{\mathbf{a}}^{\dagger \mathbf{p}} \hat{\mathbf{a}}^{\mathbf{p}} | \bar{+} \rangle|^2 \sim \delta_{\mathbf{p} \in \ker H} \times \alpha^{4|\mathbf{p}|} \exp(-8\alpha^2), \quad (95)$$

saturating the bound in Eqs. (59) with $d_Z = 8$. In other words, not only is the dephasing suppression twice that of the four-mode extended pair-cat code, dephasing errors $\hat{\mathbf{a}}^{\dagger \mathbf{p}} \hat{\mathbf{a}}^{\mathbf{p}}$ stemming from any nontrivial detectable losses $\hat{\mathbf{a}}^{\mathbf{p}}$ ($\mathbf{p} \neq \mathbf{0}$ and $\mathbf{p} \notin \ker H$) are corrected exactly.

The above dephasing calculation brings about more surprises. First, it guarantees that the logical- X and $-Z$

bases are both orthonormal if we pick a nonzero Δ parameter to define the code words. To our knowledge, coherent-state codes with both orthogonal X and Z bases at arbitrary energy, which is a desirable property previously thought to be impossible, have not been reported before.

Second, such a choice yields a logical- X operator that can be implemented with a beam splitter,

$$\bar{X} = \frac{\hat{a}_4^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_4 - \hat{a}_3^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_3}{\Delta}. \quad (96)$$

The technique to obtain this operator is presented in Sec. VII.

H. Tiger Shor code

Just as the $[[4, 1, 2]]$ qubit stabilizer code can be extended to the Shor code [8, 69, 70], also known as $[[m^2, 1, m]]$ quantum parity-check code, the four-mode code in the previous example is a special case of an analogous tiger family. We generalize it by combining the higher-mode versions of the logical-rotor pair-coherent-state code and the pair-cat code.

To increase d_Z , we take a length- L coherent state repetition code with generator matrices

$$H_{X,1} = T_L \quad \text{and} \quad H_{Z,1} = 0, \quad (97)$$

where T_L is defined as R_L in Eq. (79) without the last row. This code gives logical operators $\mathbf{x}_1 = (1 \ 0 \ \cdots \ 0)$, $\mathbf{z}_1 = (1 \ -1 \ 1 \ \cdots \ (-1)^{L-1})$.

On the other hand, to have an increasing d_X , we take a length- M extended pair-cat code with generator matrices

$$H_{X,2} = \mathbf{2}_M \quad \text{and} \quad H_{Z,2} = A_M, \quad (98)$$

and logical operators $\mathbf{x}_2 = \mathbf{1}_M$ and $\mathbf{z}_2 = (1 \ 0 \ 0 \ \cdots \ 0)$.

Combining those two blocks, we construct the $(L \times M)$ -mode tiger Shor code—encoding a logical qubit—with generator matrices

$$G = \begin{pmatrix} H_{X,1} \otimes \mathbb{1}_M \\ \mathbf{x}_1 \otimes H_{X,2} \end{pmatrix} \quad \text{and} \quad H = (\mathbf{z}_1 \otimes H_{Z,2}). \quad (99)$$

Here, $\mathbb{1}_M$ is the M -dimensional identity. Its logical operators are $\mathbf{x} = \mathbf{x}_1 \otimes \mathbf{1}_M$ and $\mathbf{z} = \mathbf{z}_1 \otimes \mathbf{z}_2$.

This code has distances $d_X = |\mathbf{x}| = M$ and $d_Z = 4ML\sin^2(\pi/2M)$. By letting $L \sim M^2$, both distances increase as we increase the system size. However, while all Z -type stabilizers have constant weight, one dissipator is nonlocal with weight $2M$. This nonlocal stabilizer also appears in the Shor code. In the next section, we provide a surface-code construction with constant-degree stabilizers and increasing X and Z distances.

VI. TIGER AND LIGER SURFACE CODES

We present a surface-code family of tiger codes with constant-weight stabilizers and X and Z distances that increase with the system size. This code is distinct from a concatenation of a single-mode cat code with a qubit surface code [76], and it is obtained by a hypergraph product of two repetition codes over the integers [24, 43]. This code inherits the stringlike logical operators, geometric locality of stabilizing operators (which we refer to as “checks” in this section), and scaling of code distances of the original surface code.

The code is laid out on a two-dimensional lattice of size $r \times (2m - 1)$, encoding a logical qubit into $2rm - r$ physical modes with distances $d_X = m$ and $d_Z \geq 4rms\sin^2(\pi/2m)$. We call the $m = 2$ case the liger (also known as long-tiger) surface code, whose distances reduce to $d_X = 2$ and $d_Z = 4r$. The liger surface code exhibits exact orthogonality in both logical- X and - Z bases for certain choices of Δ , in arbitrary energy.

The generator matrices R_m, T_m for the coherent state repetition code in Sec. VE are

$$R_m = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad T_m = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}, \quad (100)$$

where R_m is a $m \times m$ matrix as Eq. (79), and T_m is a $(m - 1) \times m$ matrix which is the same as R_m without the last row.

We input R_r and T_m into a hypergraph product, yielding the following generator matrices:

$$G = \left(\mathbb{1}_m \otimes R_r \mid T_m^T \otimes \mathbb{1}_r \right), \quad H = \left(T_m \otimes \mathbb{1}_r \mid -\mathbb{1}_{m-1} \otimes R_r^T \right), \quad (101)$$

where $r \geq 3$ is odd and where $m \geq 2$. Picking r odd ensures that the logical space is that of a logical qubit, and one gets an infinite-dimensional logical space if r is even.

As Eq. (101) shows, the R_m and T_m have only two nonzero entries per row and one or two per column; hence, both G and H have only three or four nonzero entries per row, which implies this code is a local stabilizer code consisting of local degree-3 or degree-4 X and Z stabilizing operators, as shown in Figs. 2(a) and 2(b).

The lattice layout of the tiger surface code is depicted in Fig. 2(a) where top and bottom boundaries of this lattice are open, and the left and right boundaries are periodic.

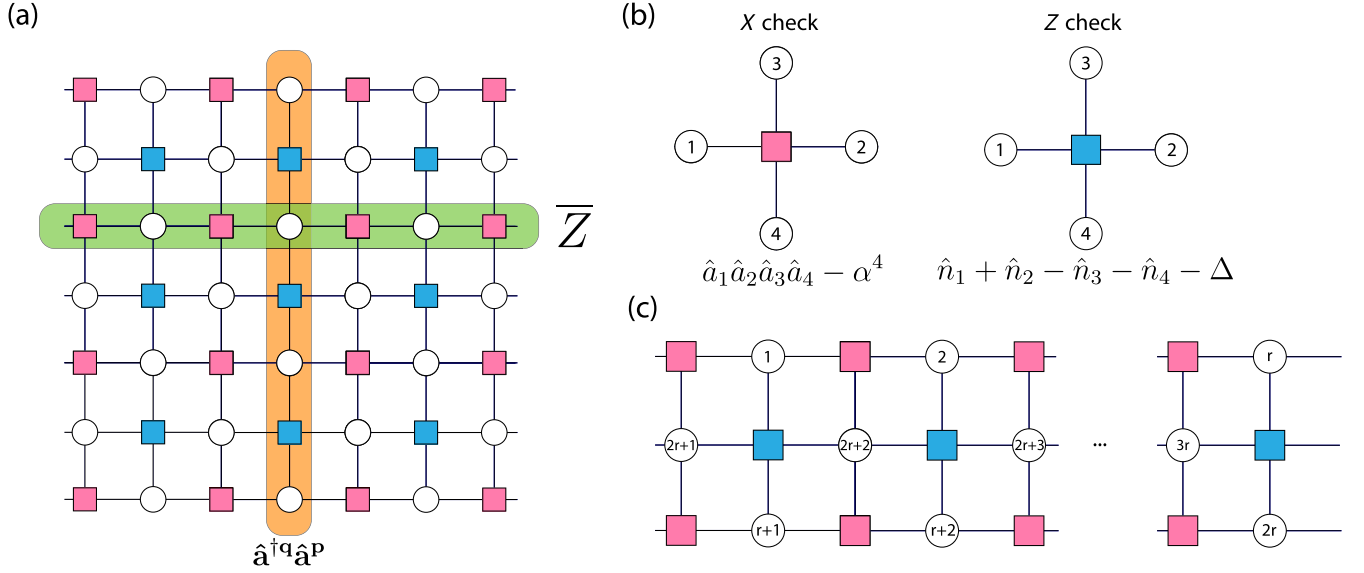


FIG. 2. (a) The structure of tiger surface code lattice and its stabilizers, where each white circle represents a physical mode, a pink square represents an X check acting on the physical modes it connects, and a blue square represents a Z check acting on the physical modes it connects. Each row contains r physical modes, and there are $2m - 1$ rows in total, forming a 2D lattice with $2mr - r$ physical modes. The undetectable X error $\hat{a}^\dagger \mathbf{q} \hat{\mathbf{a}}^{\mathbf{p}}$, with minimal 1-norm, and the logical- Z operator are string operators connecting the top-bottom and left-right boundaries, respectively. (b) The form of the X and Z checks. (c) The liger surface code with $d_X = 2$ and $d_Z = 4r$, encoding a logical qubit into a chain involving $3r$ physical modes.

Here the parameter r represents the diameter of the cross section of the tube, and m represents the length of the tube. Figure 2(c) represents the lattice structure of the liger surface code, where three-body X checks are put at the top and bottom rows, and four-body Z checks are in the middle row.

The general construction of logical operators for a hypergraph product is given in Sec. E.2 of Ref. [24]. When one of the two seed codes has torsion, as in Eq. (23), we use it to build the logical operators. The torsion we have here is described in Sec. V E, and we use the vector $\mathbf{f}_r$ from Eq. (80) to obtain

$$\mathbf{x} = \left(\mathbf{f}_m \otimes \mathbf{e}_1 \mid \mathbf{0} \right) \text{ and } \mathbf{z} = \left(\mathbf{e}_1 \otimes \mathbf{f}_r \mid \mathbf{0} \right). \quad (102)$$

One can verify that they indeed commute with stabilizers and have logical X and Z actions, respectively, using Eq. (80) and $T_m \mathbf{f}_m^T = \mathbf{0}$.

A. Tiger surface code X distance

The logical vector $\mathbf{x}$ gives the stringlike undetectable X error $\hat{a}^\dagger \mathbf{q} \hat{\mathbf{a}}^{\mathbf{p}}$, where $\mathbf{p} = (1 \ 0 \ 1 \ 0 \ \cdots) \otimes \mathbf{e}_1$ and $\mathbf{q} = (0 \ 1 \ 0 \ 1 \ \cdots) \otimes \mathbf{e}_1$, which is given by the decomposition of logical vector $\mathbf{x} = \mathbf{p} - \mathbf{q}$ as Eq. (51) with minimal 1-norm. It consists of alternating $\hat{a}$ and $\hat{a}^\dagger$ acting on neighboring modes along a vertical string connecting the top and bottom boundaries, as shown in Fig. 3(a). The minimal undetectable X error involves a nontrivial creation piece $\hat{a}^\dagger \mathbf{q}$ with $\mathbf{q} \neq \mathbf{0}$ —a first example of this kind.

The X distance is the 1-norm of $\mathbf{x}$,

$$d_X = |\mathbf{x}| = |\mathbf{f}_m| = m. \quad (103)$$

This code can correct at least $\lfloor (m-1)/2 \rfloor$ losses on any mode.

A longer logical- X string consisting of only annihilation operators runs diagonally between the top and bottom boundaries, with $\hat{a}$ acting on each of the $2m - 1$ modes inside the string [see Fig. 3(b)].

Since the 1-norm of this minimal pure-loss vector is $2m - 1$, the code can detect up to $2m - 2$ losses—nearly twice the code distance. For example, the liger surface code ($m = 2$) can detect two losses on any mode, but it cannot correct any losses.

B. Tiger surface code Z distance

In Fig. 2(a), the logical- Z operator defined by $\mathbf{z}$, forms a length- r horizontal string operator consisting of parities $(-1)^{\hat{n}}$ and connecting the left and right boundaries. However, the Z distance of a code is determined by the squared Euclidean distance of two neighboring code words [see Eq. (57b)], and rotating by the defining logical- Z string operator does not produce the shortest Euclidean path.

We obtain upper and lower bounds on the distance,

$$d_Z \geq 4rm \sin^2 \frac{\pi}{2m}, \quad (104a)$$

$$d_Z \leq 4rm \sin^2 \frac{\pi}{2m} + 4 \sum_{j=1}^{m-1} \sin^2 \frac{(m-j)\pi}{m}, \quad (104b)$$

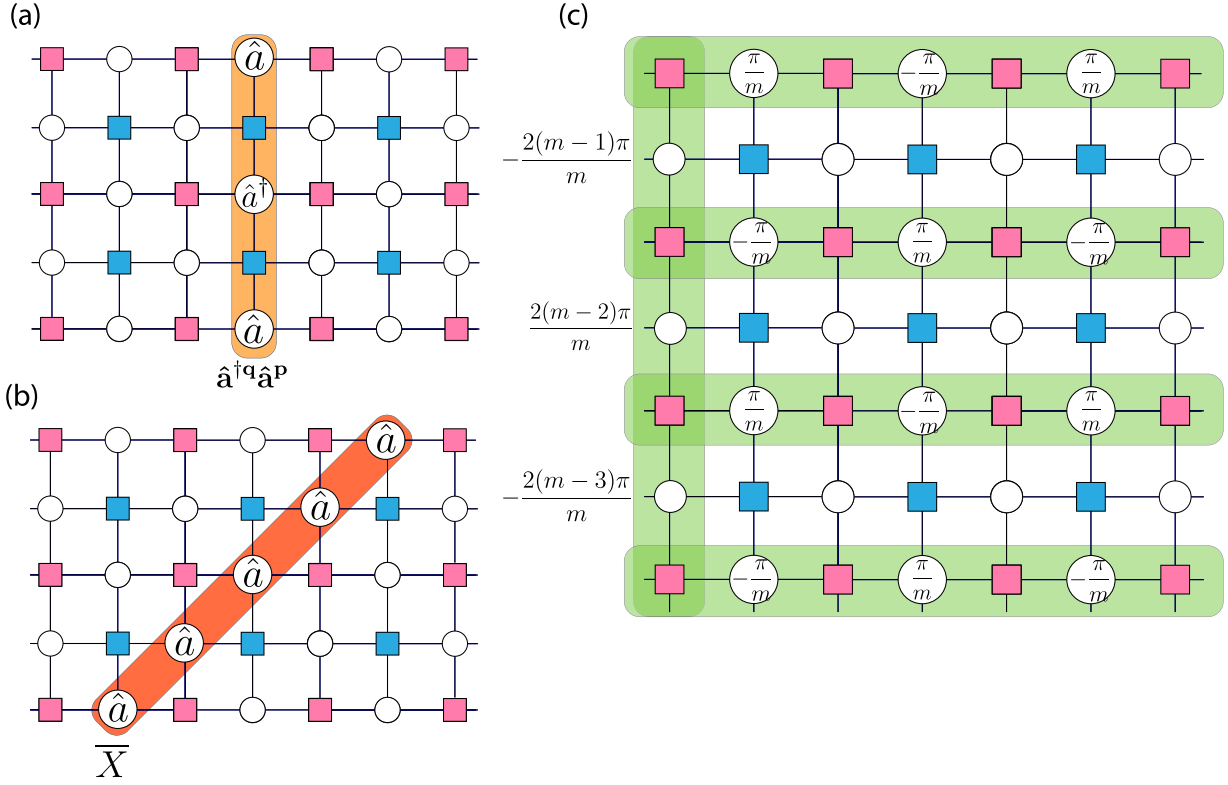


FIG. 3. In (a), the vertical string operator in the light orange region represents a lowest-weight undetectable X error $\hat{a}^\dagger \hat{q} \hat{a}^p$. Thus, the tiger surface code can correct loss errors $\hat{a}^v$ whose 1-norm is at least $\lfloor (m-1)/2 \rfloor$. In (b), the diagonal string operator in the dark orange region represents the lowest-weight logical- X operator. It implies that tiger surface code can detect loss error $\hat{a}^p$ on any $(2m-2)$ modes. (c) The error pattern that gives upper bound of the squared Euclidean distance between two code words in Eq. (58), which corresponds to a “smeared” version of the defining logical- Z operator of tiger surface code. In this error pattern, each physical mode inside the green horizontal strips undergoes a phase-space rotation by an angle of $\pm(\pi/m)$. The physical modes inside the green vertical strip in the first column undergo configuration-space rotations by angles $[-2(m-1)\pi/m, 2(m-2)\pi/m, \dots, (-1)^{m-1}2\pi/m]$, from top to bottom.

by choosing other logical- Z operators that act on more modes, but that rotate each mode by a smaller angle. These bounds differ by an order $O(m)$ term in the limit of large system size.

The intuition behind the upper bounds of d_Z is as follows. Using Z -type stabilizers, we can “smear” our original logical operator into one that acts on m rows of modes, rotating each mode by π/m . As Fig. 3(c) shows, each physical mode in the green-colored rows is acted on by an $\exp(\pm i\pi \hat{n}/m)$ phase shift, and physical modes in each green-colored row contribute $4r\sin^2(\pi/2m)$ to the squared Euclidean distance.

During the smearing process, we also generate phase shifts in the first column as shown in Fig. 3(c) because of the periodic boundary conditions between the left and right sides. The first column contributes a $4\sum_{j=1}^{m-1} \sin^2[(m-j)\pi/m]$ term in the upper bound of d_Z . Intuitively, for a fixed m , we can linearly increase d_Z by increasing r because we add more $m\sin^2(\pi/2m)$ terms to the lower and upper bounds.

The lower bound of d_Z is obtained by removing the phase shift acting on the first column induced during the

smearing. The general approach to obtain the d_Z lower bound can be found in Lemma 1 of Ref. [24].

C. Code words and dephasing suppression

Tiger surface code words can be obtained by applying its logical- Z string operator in Fig. 2 to the fiducial state,

$$|\pm\rangle = \frac{e^{\frac{(2rm-r)\alpha^2}{2}}}{\sqrt{A_0(\alpha^2 \mathbf{1})_{\text{tiger}}}} \Pi_0 |\pm\rangle^{\otimes r} \otimes |\alpha\rangle^{\otimes 2r(m-1)}, \quad (105)$$

where Π_0 is the collection of Z projectors given by H .

Combining our lower bound Eq. (104) on d_Z and Eq. (59), and noting that $\mathbf{1} \in \ker H$, we can bound the degree of dephasing suppression using the result from the Appendix,

$$|\langle - | \hat{a}^\dagger \hat{p} \hat{a}^p | + \rangle_{\text{tiger}}|^2 \stackrel{\alpha \rightarrow \infty}{\lesssim} \alpha^{4|p|} \exp\left(-4rm\alpha^2 \sin^2\left(\frac{\pi}{2m}\right)\right). \quad (106)$$

To achieve the same amount of dephasing suppression, we can either increase r or increase α^2 . To simultaneously increase both the X and Z distances, we can choose $r = m^2$ such that $d_X = m, d_Z \geq 4m^3 \sin^2(\pi/2m)$ increases as m increases. In general, we can take $r \sim \Omega(m^2)$ to ensure both distances increase with system size.

D. Liger (long-tiger) surface code

This simpler version of the tiger surface code has a tractable GKZ function with interesting properties. The liger (also known as long-tiger) surface code is the tiger surface code on a thin strip with $m = 2$. A length- r liger surface code contains $3r$ bosonic modes for odd r . It encodes a logical qubit with distances $d_X = 2$ and $d_Z = 4r$, with the latter known exactly because the lower and upper bounds from Eq. (59) are equal for this case. In other words, the defining logical- Z string formed by $(-1)^{\hat{n}}$ in Fig. 2(a) exactly gives us the shortest Euclidean path between code words.

Using code words shown in Eq. (105) and repeatedly applying the splitting identity of Eq. (93), we simplify the liger GKZ function to

$$A_{\Delta}(\mathbf{y})_{\text{liger}} = \sum_{\mathbf{n} \in \mathbb{N}^r} \sum_{\mathbf{j} \in \mathbb{N}^r} \prod_{i=1}^r \frac{(y_i + y_{r+i})^{j_i - \min\{0, \Delta_i\}} y_{2r+i}^{n_i}}{(j_i - \min\{0, \Delta_i\})! n_i!} \times \delta_{n_i + n_{i+1}, j_i + \max\{0, \Delta_i\}}, \quad (107)$$

where $y_{3r+1} = y_{2r+1}$ and $n_{r+1} = n_1$ due to the periodic boundary conditions. The indices follow the convention of Fig. 2(c).

Using the GKZ function, we can calculate the liger code word overlap to be

$$\langle \overline{-} | \overline{+} \rangle_{\text{liger}} = \frac{A_0(-\alpha^2 \mathbf{1}_r, \alpha^2 \mathbf{1}_{2r})_{\text{liger}}}{A_0(\alpha^2 \mathbf{1})_{\text{liger}}} = \frac{1}{A_0(\alpha^2 \mathbf{1})_{\text{liger}}}, \quad (108)$$

where the numerator is one for arguments $y_i = -y_{r+i}$ for $i = 1, 2, \dots, r$. In other words, the overlap depends only on the code word normalization.

The GKZ function of the liger code also yields another property, namely, that $A_{\Delta}(-\alpha^2 \mathbf{1}_r, \alpha^2 \mathbf{1}_{2r})_{\text{liger}} \propto \Theta(\Delta)$, where the Heaviside step function Θ is zero if the vector Δ has at least one negative entry.

Plugging Eq. (107) in Eq. (54b) yields the explicit expression of the dephasing matrix element

$$|\langle \overline{-} | \hat{\mathbf{a}}^\dagger \mathbf{p} \hat{\mathbf{a}} \mathbf{p} | \overline{+} \rangle_{\text{liger}}|^2 = \frac{\Theta(-H\mathbf{p}) \alpha^{4(|\mathbf{p}| + |H\mathbf{p}|)}}{A_0(\alpha^2 \mathbf{1})_{\text{liger}}^2} \left(\sum_{\mathbf{n} \in \mathbb{N}^r} \prod_{i=1}^r \frac{1}{n_i!} \delta_{n_i + n_{i+1}, (-H\mathbf{p})_i} \right)^2, \quad (109)$$

where the dephasing matrix element is exactly zero if $H\mathbf{p}$ has at least one positive entry. The nonzero dephasing

matrix elements are exponentially suppressed as $\exp(-6r|\alpha|^2)$, in the large- α limit, where the suppression coefficient $6r$ exceeds the code's Z distance $d_Z = 4r$ —our only example where the dephasing suppression surpasses the Z distance.

E. Open boundaries

Tiger surface codes can also be defined for open left and right boundaries. By replacing the last row R_r by $(K \ 0 \ 0 \ \dots \ 0)$ and keeping T_m unchanged, the hypergraph-product construction will yield a tiger surface code with open boundary condition that encodes a quKit.

This open boundary modification will increase the degree of the boundary- X checks and change the coefficient in front of $\hat{n}$ in the boundary- Z checks. It will not affect any checks in the bulk. This modification will not change the X distance and will only slightly decrease the Z distance. We leave further exploration of the boundaries of the tiger surface code for future investigation.

VII. FINITE-SUPPORT TIGER CODES

In previous sections, we introduce tiger codes with a non-negative integer matrix G and integer matrix H . Here, we study codes whose G has arbitrary integer entries, which turns out to imply that the Fock-state subspace fixed by H will be finite, yielding code words of finite support and complicating the construction of logical operators. This also implies that dissipators consist of not only annihilation operators, but also creation operators, and will thus not commute with each other. The good news is that homology still yields the code space, which still lies in a joint eigenspace of all stabilizing operators.

A known code encompassed by our finite-support framework is the two-mode binomial code [15,38], which includes the experimentally relevant [36,86–90] dual-rail $\{|10\rangle, |01\rangle\}$ code as a special case. We also make contact with a $\chi^{(2)}$ code [39] by defining a tiger code with the same Fock-state support.

General generator matrices for finite-support codes are both integer valued,

$$G \in \mathbb{Z}^{r_s \times N}, \quad H \in \mathbb{Z}^{r_z \times N}, \quad GH^T = \mathbf{0}. \quad (110)$$

To ensure that any negative entries in G are not an artifact of the basis used to define the generator matrix, we assume that there exists an all-positive vector $\mathbf{b}$ in the row image of H , i.e., $\mathbf{b} \in \text{im} H$ with $b_j > 0$ for all j . This condition forces the rows of G to have some negative entries in order to be orthogonal to $\mathbf{b}$. This condition is equivalent to the condition that there is a finite number of Fock states satisfying the H constraints. In other words, only a finite set of Fock states are in any particular eigenspace of the operator $\mathbf{b} \cdot \hat{\mathbf{n}}$. Typically, we will have $\mathbf{b} = \mathbf{1}$, corresponding to a constraint on the total occupation number, $\mathbf{1} \cdot \hat{\mathbf{n}} = |\hat{\mathbf{n}}|$.

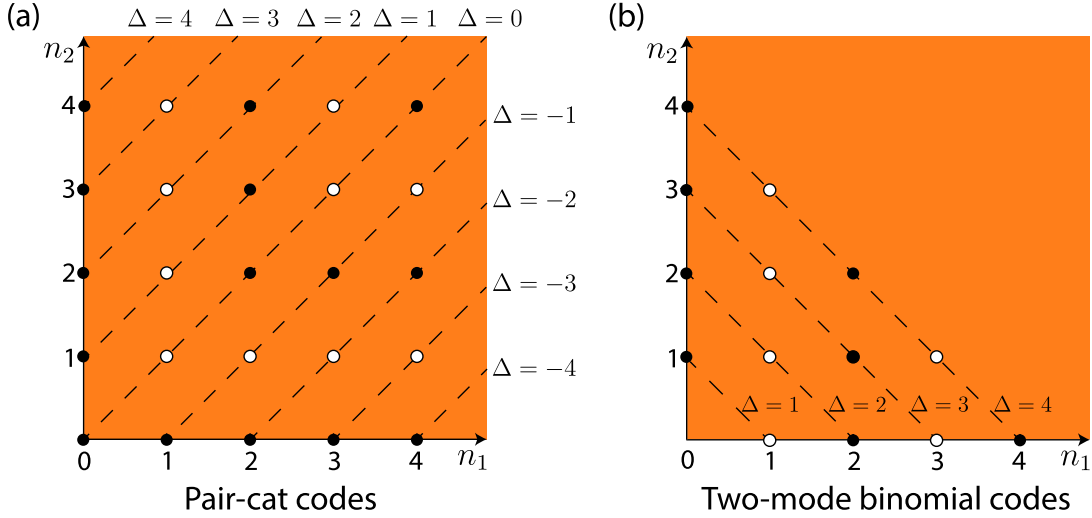


FIG. 4. Pictorial comparison between infinite-support pair-cat code and finite-support two-mode binomial code in the two-mode Fock space (n_1, n_2) . (a) The Fock-state structure of the pair-cat code for different choices of occupation-number difference, $\Delta = n_2 - n_1$. Each choice specifies a 1D semi-infinite lattice (dashed line with Δ) extending in the upper-right direction in the two-mode Fock space, and the code words are superpositions of the Fock states in the semi-infinite lattice. Inside each lattice, the black (white) circles are in the Fock-state support of code word $|\bar{0}\rangle_\Delta$ ($|\bar{1}\rangle_\Delta$). Since the 1D lattice with Δ involves infinite numbers of Fock states, the code words are superpositions of Fock states involving infinitely many terms. (b) The Fock-state structure of the two-mode binomial code for different choices of the total occupation number, $\Delta = n_1 + n_2$. Each choice specifies a finite-1D lattice in the two-mode Fock space, and the code words are superpositions of Fock states in the finite lattice. The finite lattices from left to right correspond to $\Delta = 1, 2, 3, 4$. Inside each lattice, the black (white) circles are in the Fock-state support of code word $|\bar{0}\rangle_\Delta$ ($|\bar{1}\rangle_\Delta$). The supports for both $|\bar{0}\rangle_\Delta$ and $|\bar{1}\rangle_\Delta$ are finite.

Code words are defined as projected coherent states, analogous to the infinite-support tiger codes. The Z -type stabilizers are also the same: $\mathbf{h} \cdot \hat{\mathbf{n}} - \Delta_{\mathbf{h}}$ for all rows $\mathbf{h}$ of H . The full Fock-state subspace supported by the code is defined by $H\hat{\mathbf{n}} = \Delta$ for integer syndrome vector Δ .

Logical- Z operators, corresponding to vectors $\mathbf{z}$, are obtained via the same homology calculation. Code words can still be obtained by applying logical- Z operators to the all-ones coherent state $|\alpha\mathbf{1}\rangle$ [Eq. (18)], but the dependence on the energy density α^2 drops out if the total occupation number is fixed by the code. In other words, finite-support code words cannot have arbitrary energy since they do not have an infinite runaway of Fock states. Any dephasing protection should thus either be exact or exponentially suppressed with increasing values of the syndrome vector Δ . We see both situations in our examples.

The X -type dissipators now contain creation and annihilation operator monomials, i.e., operators $\hat{\mathbf{a}}^{\dagger\mathbf{q}}\hat{\mathbf{a}}^{\mathbf{p}}$ for any row $\mathbf{g} = \mathbf{p} - \mathbf{q}$ of G . Here, we decompose the row into its positive part $\mathbf{p} \in \mathbb{N}^N$ and negative part $\mathbf{q} \in \mathbb{N}^N$. Acting with such operators on coherent states does not produce a constant since coherent states are not right eigenstates of creation operators. To develop dissipators that annihilate coherent states, we need an additional formula.

For a non-negative integer $k \geq 0$, we have that

$$\hat{a}^{\dagger k}|\alpha\rangle = \frac{(\hat{n})_k}{\alpha^k}|\alpha\rangle, \quad (111)$$

where the falling factorial

$$(\hat{n})_k = \hat{a}^{\dagger k}\hat{a}^k = \prod_{j=0}^{k-1} (\hat{n} - j) = \hat{n}(\hat{n} - 1) \cdots (\hat{n} - k + 1). \quad (112)$$

As a convention, we use $(\hat{n})_0 = 1$.

Applying the above formula, we can define the dissipators as follows:

$$[\hat{\mathbf{a}}^{\dagger\mathbf{q}}\hat{\mathbf{a}}^{\mathbf{p}} - \alpha^{\mathbf{g}}(\hat{\mathbf{n}})_{\mathbf{q}}]|\alpha\rangle_{\Delta}^H = 0, \quad (113)$$

where we use multi-index notation, e.g., $(\hat{\mathbf{n}})_{\mathbf{q}} = (\hat{n}_1)_{q_1}(\hat{n}_2)_{q_2} \cdots (\hat{n}_N)_{q_N}$.

These dissipators no longer commute with one another in general. They still commute with all Z -type stabilizers since $\mathbf{g} = \mathbf{p} - \mathbf{q}$ is in the kernel of H and since their second terms are polynomials in the occupation numbers.

Given a logical vector $\mathbf{x}$ obtained via homology, it is sometimes possible to construct logical- X operators by choosing appropriate compensating occupation-number polynomials. Let us focus on the case of a logical qubit to simplify the presentation, but other logical code spaces work similarly. For this, we need to pick a logical- X representative $\mathbf{x} \in \mathbb{Z}^N$, decomposed into $\mathbf{v} \in \mathbb{N}^N$ and $\mathbf{w} \in \mathbb{N}^N$ as $\mathbf{x} = \mathbf{v} - \mathbf{w}$, and denote its conjugated

logical- Z representative by $\mathbf{z}$. We thus have the following two facts:

$$\hat{\mathbf{a}}^{\dagger \mathbf{w}} \hat{\mathbf{a}}^{\mathbf{v}} |\alpha\rangle_{\Delta}^H = \alpha^{\mathbf{x}(\hat{\mathbf{n}})_{\mathbf{w}}} |\alpha\rangle_{\Delta}^H, \quad (114a)$$

$$(-1)^{\mathbf{z} \cdot \hat{\mathbf{n}}} \hat{\mathbf{a}}^{\dagger \mathbf{w}} \hat{\mathbf{a}}^{\mathbf{v}} = -\hat{\mathbf{a}}^{\dagger \mathbf{w}} \hat{\mathbf{a}}^{\mathbf{v}} (-1)^{\mathbf{z} \cdot \hat{\mathbf{n}}}. \quad (114b)$$

If the operator $\alpha^{\mathbf{x}(\hat{\mathbf{n}})_{\mathbf{w}}}$ was invertible, we would just multiply on the left in Eq. (114) by its inverse to obtain the logical X . This is indeed what happens in the infinite-support case where $\mathbf{w} = \mathbf{0}$, and so $\alpha^{\mathbf{x}(\hat{\mathbf{n}})_{\mathbf{w}}} = \alpha^{\mathbf{x}}$. However, here $\mathbf{x}$ has some negative entries, implying that $\alpha^{\mathbf{x}(\hat{\mathbf{n}})_{\mathbf{w}}}$ has some zero eigenstates, namely, all Fock states with an occupation number on the j th mode lower than w_j . Therefore, $\alpha^{\mathbf{x}(\hat{\mathbf{n}})_{\mathbf{w}}}$ is not invertible.

That said, there are several representatives for this logical X , obtained by adding vectors from the row image of G . In general, any linear combination,

$$\sum_{\mathbf{x}=\mathbf{v}-\mathbf{w}} \beta_{\mathbf{x}} \hat{\mathbf{a}}^{\dagger \mathbf{w}} \hat{\mathbf{a}}^{\mathbf{v}}, \quad (115)$$

where the sum is over some set of logical representatives $\mathbf{x}$, still satisfies Eq. (114b). Crucially, we can promote the coefficient to any polynomial in $\hat{\mathbf{n}}$, $\beta_{\mathbf{x}} \rightarrow \hat{\beta}_{\mathbf{x}}$, while still maintaining the logical relation. The operator $\sum_{\mathbf{x}} \hat{\beta}_{\mathbf{x}} \alpha^{\mathbf{x}(\hat{\mathbf{n}})_{\mathbf{w}}}$ on the right-hand side of the analog of Eq. (114a) now has more possibilities to be invertible and can even be constant on the code space in some cases, by taking into account H constraints. The latter case is more convenient since then we have a logical- X operator that is a polynomial in creation and annihilation operators instead of being a rational function, as in the former case.

Formally, given a chosen logical $\mathbf{x}$, we can define the polynomial ideal, $I_{\mathbf{x}} \subset \mathbb{R}[n_1, n_2, \dots, n_N]$ generated by all the $(\mathbf{n})_{\mathbf{w}}$ for all logical representatives $\mathbf{x}' = \mathbf{v} - \mathbf{w}$ equivalent to $\mathbf{x}$ and all constraints from H ,

$$I_{\mathbf{x}} = \langle \{(\mathbf{n})_{\mathbf{w}} | \mathbf{m} \in \mathbb{Z}^{r_x}, \mathbf{v}, \mathbf{w} \in \mathbb{N}^N, \mathbf{x} + \mathbf{m}G = \mathbf{v} - \mathbf{w}\} \cup \{\lambda(H \cdot \mathbf{n} - \Delta) | \lambda \in \mathbb{Z}^{r_z}\} \rangle. \quad (116)$$

This ideal, $I_{\mathbf{x}}$, contains all operators that are polynomials in the n_i that can be generated on the right-hand side of Eq. (114) by applying on the left-hand side an operator of the type of Eq. (115). Additionally, since the logical X has to stabilize the state $|\alpha\rangle_{\Delta}^H$, this right-hand side needs to be 1. Then there exists an operator that is a polynomial of the annihilation and creation operators representing the logical X given by $\mathbf{x}$ as long as $1 \in I_{\mathbf{x}}$, and the coefficient in front of the generators of $I_{\mathbf{x}}$ generating 1 directly gives the coefficients $\beta_{\mathbf{x}}$ after division by $\alpha^{\mathbf{x}}$.

We present a few examples of interest to illustrate the framework.

A. Two-mode binomial code

The two-mode binomial code [15,36,38] is an example of a finite-support tiger code. Our framework reproduces what is known about these codes from the spin-coherent state framework [29], in which the code words are identified as two-mode binomial states [41,42,66] lying on the x axis of an effective Bloch sphere. In this way, our framework offers another way to look at, and generalize, two-mode binomial states.

The code is obtained by moving the sign from one pair-cat generator matrix to the other to yield

$$G = \begin{pmatrix} 2 & -2 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad (117)$$

where H gives occupation number constraint $\hat{n}_1 + \hat{n}_2 = \Delta$, and G yields the dissipator

$$\hat{\mathbf{a}}^{\dagger} \begin{pmatrix} 0 & 2 \end{pmatrix} \hat{\mathbf{a}} \begin{pmatrix} 2 & 0 \end{pmatrix} - \alpha^{2-2} (\hat{\mathbf{n}})_{\begin{pmatrix} 0 & 2 \end{pmatrix}} = \hat{a}_2^{\dagger 2} \hat{a}_1^2 - \hat{n}_2 (\hat{n}_2 - 1). \quad (118)$$

Another dissipator for $-G = \begin{pmatrix} -2 & 2 \end{pmatrix}$ gives rise to the above, but with modes swapped.

A homology calculation yields logical vectors

$$\mathbf{x} = \begin{pmatrix} 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{z} = \begin{pmatrix} 1 & 0 \end{pmatrix}. \quad (119)$$

We can solve for a logical- X operator, which is a linear combination of two representatives of the logical- X equivalence class, namely, $\begin{pmatrix} 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \end{pmatrix}$. The logical operators are

$$\bar{X} = \frac{\hat{a}_1^{\dagger} \hat{a}_2 + \hat{a}_2^{\dagger} \hat{a}_1}{\Delta} \quad \text{and} \quad \bar{Z} = (-1)^{\hat{n}_1} \cong (-1)^{\hat{n}_2 + \Delta}, \quad (120)$$

where $\cong$ denotes that two logical operators are equivalent via a Z -type stabilizer. Squaring the logical- X operator gives an additional X stabilizer.

Logical- X code words are coherent states $\alpha = (\alpha, \alpha)$ and $(-\alpha, \alpha)$, projected into the Fock subspace of fixed total occupation number Δ ,

$$|\pm\rangle = \frac{e^{2\alpha^2}}{\sqrt{A_{\Delta}(\alpha^2 \mathbf{1})}} \Pi_{\Delta} |\alpha(\pm 1, 1)\rangle, \quad (121a)$$

$$= \frac{1}{\sqrt{A_{\Delta}(\alpha^2 \mathbf{1})}} \sum_{n_1+n_2=\Delta} \frac{(\pm \alpha)^{n_1} \alpha^{n_2}}{\sqrt{n_1! n_2!}} |n_1, n_2\rangle, \quad (121b)$$

$$= \frac{1}{\sqrt{2^{\Delta}}} \sum_{n=0}^{\Delta} (\pm 1)^n \sqrt{\binom{\Delta}{n}} |n, \Delta - n\rangle, \quad (121c)$$

with dependence on α removed after normalizing by the GKZ function,

$$A_\Delta(y_1, y_2) = \sum_{n=0}^{\Delta} \frac{y_1^n y_2^{\Delta-n}}{n!(\Delta-n)!} = \frac{(y_1 + y_2)^\Delta}{\Delta!}. \quad (122)$$

These X code words are orthogonal, as is usually the case for finite-support tiger codes.

Equations (54) and (122) imply that this code can detect dephasing errors exactly up to power Δ ,

$$\langle \bar{-} | \hat{\mathbf{a}}^\dagger \mathbf{p} \hat{\mathbf{a}} \mathbf{p} | \bar{+} \rangle \propto (y_1 - y_2)^{\Delta-p_1-p_2} \big|_{y_1=y_2=\alpha^2} = 0 \quad (123)$$

whenever $\Delta > p_1 + p_2$ (cf. Ref. [15]).

The code can detect an arbitrary number of single-mode photon losses, just like the pair-cat code. It cannot correct single-photon losses since their product, $\hat{a}_1^\dagger \hat{a}_2$, is a logical operator (see Sec. IV B).

However, losing a photon just moves the system from Δ to $\Delta - 1$; hence, one can just keep track of the total occupation number and continue to operate this logical qubit until Δ photons have been lost, yielding an erasure. Indeed, if we pick $\Delta = 1$, this reduces to the dual-rail code.

B. Multinomial code

The code above can be generalized to three or more modes by defining $G \in \mathbb{Z}^{(N-1) \times N}$ and $H \in \mathbb{Z}^{1 \times N}$ as follows:

$$G = \begin{pmatrix} 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix}, \quad (124)$$

$$H = (1 \quad 1 \quad \cdots \quad 1).$$

Computing the homology, we find a logical quNit on N modes, with logical vectors $\mathbf{x} = (1 \quad -1 \quad 0 \quad \cdots \quad 0)$ and $\mathbf{z} = (0 \quad 1 \quad 2 \quad \cdots \quad N-1)$. The corresponding logical operators are given by

$$\bar{X} = \frac{1}{\Delta} \sum_{j=0}^{N-1} \hat{a}_j^\dagger \hat{a}_{j+1}, \quad (125a)$$

$$\bar{Z} = \exp \left(i \frac{2\pi}{N} \sum_{j=0}^{N-1} j \hat{n}_j \right). \quad (125b)$$

The projected coherent states in this case reduce to the $SU(N)$ coherent states [64–66],

$$|\alpha\rangle_\Delta^H \propto \sum_{|\mathbf{n}|=\Delta} \sqrt{\binom{\Delta}{\mathbf{n}}} \alpha^{\mathbf{n}} |\mathbf{n}\rangle, \quad (126)$$

which are parametrized by the states of a qudit, $\alpha \in \mathbb{C}P^{N-1} = SU(N)/\mathbb{T}$ (with $\mathbb{T}$ the group generated by H) for complex α , and whose Fock-state expansion contains a multinomial coefficient. The GKZ function is a multinomial expansion of the coordinate sum of α . For $N = 2$, these reduce to the two-mode binomial states [41,42,66] lying on the Bloch sphere of a qubit.

Any dephasing protection in this case is exact, holding due to the multinomial generalization of Eq. (123). The degree of loss protection is also the same.

C. Center-of-mass mode

Multiplying the second column of the G matrix of the logical-rotor pair-coherent-state code by -1 yields

$$G = \begin{pmatrix} 1 & -1 \end{pmatrix} \quad \text{and} \quad H = 0, \quad (127)$$

corresponding to dissipators

$$\hat{a}_2^\dagger \hat{a}_1 - \hat{n}_2 \quad \text{and} \quad \hat{a}_1^\dagger \hat{a}_2 - \hat{n}_1. \quad (128)$$

The logical space is a single-mode oscillator, which is infinite dimensional, with logical vectors $\mathbf{x} = (1 \quad 0) \cong (0 \quad 1)$ and $\mathbf{z} = (1 \quad 1)$ and related logical- X and - Z operators $\bar{X} = (\hat{a}_1 + \hat{a}_2/\sqrt{2})$, $\bar{Z}(\varphi) = e^{i\varphi(\hat{n}_1 + \hat{n}_2)}$. We see that the logicals need not commute with the dissipators in the case of finite support.

Applying the logical- Z operator to the all-ones coherent states yields logical- X -type code words,

$$|\bar{\varphi}\rangle = \bar{Z}(\varphi)|\alpha = \alpha \mathbf{1}\rangle = |\alpha e^{i\varphi}\rangle^{\otimes 2}, \quad (129)$$

which are coherent states of the center-of-mass mode.

Alternatively, adding the dissipators together, we can concoct a Z -type basis of eigenstates of $\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger$, i.e., two-mode binomial states for each value of the total occupation number ℓ ,

$$|\bar{\ell}\rangle = \frac{1}{\sqrt{2}^\ell} \sum_{n=0}^{\ell} \sqrt{\binom{\ell}{n}} |n, \ell - n\rangle, \quad \forall \ell \geq 0. \quad (130)$$

Each of these can be obtained by applying a 50-50 beam splitter U_{BS} to a Fock state, $|\bar{\ell}\rangle = U_{BS}|\ell, 0\rangle$.

The center-of-mass mode admits its own annihilation, creation, and occupation-number operators, respectively,

$$\bar{a} = \sqrt{2} \hat{a}_1 \cong \sqrt{2} \hat{a}_2 \cong \frac{\hat{a}_1 + \hat{a}_2}{\sqrt{2}}, \quad (131a)$$

$$\bar{a}^\dagger = \frac{\hat{a}_1^\dagger + \hat{a}_2^\dagger}{\sqrt{2}}, \quad (131b)$$

$$\bar{n} = \bar{a}^\dagger \bar{a} \cong \hat{n}_1 + \hat{n}_2, \quad (131c)$$

where we use $\cong$ for operators that are equivalent up to stabilizers or dissipators. Note that $\overline{a^\dagger} = \overline{a}^\dagger$ is satisfied by only one form of the logical lowering operator.

D. Four-mode binomial code

We can define a finite-support version of the four-mode tiger code described in Sec. V G using generator matrices

$$G = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \quad H = (1 \quad 1 \quad 1 \quad 1), \quad (132)$$

which are obtained by multiplying the second and third columns of the generator matrices in Eq. (88) by -1 and taking linear combinations. The logical vectors are $\mathbf{x} = (1 \ 0 \ 0 \ -1)$, $\mathbf{z} = (1 \ 1 \ 0 \ 0)$.

We can view this code as a two-mode binomial code constructed out of two center-of-mass modes $\bar{a}$ and $\bar{b}$, which allows us to succinctly write the code words as

$$|\bar{\pm}\rangle = \frac{1}{\sqrt{2^\Delta}} \sum_{n=0}^{\Delta} (\pm 1)^n \sqrt{\binom{\Delta}{n}} |\bar{n}, \overline{\Delta - n}\rangle, \quad (133)$$

where the overlined states on the right-hand side are the center-of-mass Fock states from Eq. (130). As with the two-mode binomial code, the code words are orthogonal.

Similarly, the center-of-mass mode operators in Eq. (131) yield four-mode stabilizing operators,

$$\left(\frac{\overline{a^\dagger b} + \overline{b^\dagger a}}{\Delta} \right)^2 \quad \text{and} \quad \overline{a^\dagger a} + \overline{b^\dagger b} - \Delta, \quad (134)$$

and logical operators,

$$\bar{X} = \frac{\overline{a^\dagger b} + \overline{b^\dagger a}}{\Delta} \quad \text{and} \quad \bar{Z} = (-1)^{\overline{a^\dagger a}} \equiv (-1)^{\overline{b^\dagger b}}. \quad (135)$$

We can construct another logical- X operator by noticing that $(1 \ 0 \ 0 \ -1)$, $(-1 \ 0 \ 0 \ 1)$, $(0 \ -1 \ 1 \ 0)$, and $(0 \ 1 \ -1 \ 0)$ are all logical- X representatives, meaning that

$$(\hat{a}_4^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_4 + \hat{a}_2^\dagger \hat{a}_3 + \hat{a}_3^\dagger \hat{a}_2) |\bar{\pm}\rangle \quad (136a)$$

$$= \pm(\hat{n}_4 + \hat{n}_1 + \hat{n}_2 + \hat{n}_3) |\bar{\pm}\rangle = \pm \Delta |\bar{\pm}\rangle. \quad (136b)$$

Hence, we can use a beam-splitter generator as a logical,

$$\bar{X} = \frac{\hat{a}_4^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_4 + \hat{a}_2^\dagger \hat{a}_3 + \hat{a}_3^\dagger \hat{a}_2}{\Delta}, \quad (137)$$

which contains less terms than the expressions obtained from the concatenation point of view. This code has $d_X = 2$ and can exactly suppress dephasing error $\hat{a}^\dagger \hat{p} \hat{a}^p$ for $|\mathbf{p}| < \Delta$, similar to the dephasing suppression of two-mode binomial code.

E. $\chi^{(2)}$ -like code

We define a three-mode code with

$$G = (2 \quad 2 \quad -2), \quad H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad \Delta = \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix}. \quad (138)$$

Picking $\Delta_1 = \Delta_2 = \Delta$ yields a code in the same Fock-state support as one of the $\chi^{(2)}$ codes from Ref. [39]. The general case gives rise to two Z -type stabilizers,

$$\hat{n}_2 + \hat{n}_3 - \Delta_1 \quad \text{and} \quad \hat{n}_1 + \hat{n}_3 - \Delta_2, \quad (139a)$$

and two dissipators,

$$\hat{a}_3^{\dagger 2} \hat{a}_1^2 \hat{a}_2^2 - \alpha^2 \hat{n}_3 (\hat{n}_3 - 1), \quad (139b)$$

$$\hat{a}_1^{\dagger 2} \hat{a}_2^{\dagger 2} \hat{a}_3^2 - \hat{n}_1 (\hat{n}_1 - 1) \hat{n}_2 (\hat{n}_2 - 1) / \alpha^2. \quad (139c)$$

The code space is a logical qubit, with logical vectors

$$\mathbf{x} = (1 \quad 1 \quad -1), \quad \mathbf{z} = (0 \quad 1 \quad 0). \quad (140)$$

The logical- Z operator is $\bar{Z} = (-1)^{\hat{n}_2}$.

For the logical X , we need to check if 1 is in the following ideal, $I \subset \mathbb{R}[n_1, n_2, n_3]$:

$$I = \langle n_2 + n_3 - \Delta_1, n_1 + n_3 - \Delta_2, n_1 n_2, n_3 \rangle. \quad (141)$$

This is indeed the case, and we can express the logical X as

$$\bar{X} = \frac{\alpha^2 \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_3 + (\hat{n}_2 - \Delta_1 - \Delta_2) \hat{a}_3^\dagger \hat{a}_1 \hat{a}_2}{\alpha \Delta_1 \Delta_2}. \quad (142)$$

We can write down code words for $\Delta_2 \geq \Delta_1$ as

$$|\bar{\pm}\rangle = \frac{\alpha^{\Delta_2}}{\sqrt{\mathbf{A}_\Delta(\alpha^2)}} \sum_{m=0}^{\Delta_1} \frac{(\pm \alpha)^m |\Delta_2 - \Delta_1 + m, m, \Delta_1 - m\rangle}{\sqrt{(\Delta_2 - \Delta_1 + m)! m! (\Delta_1 - m)!}}, \quad (143)$$

where $\mathbf{A}_\Delta(\mathbf{y})$, the GKZ function, is a generalized Laguerre polynomial $L_n^{(a)}$ [81],

$$\mathbf{A}_\Delta(\mathbf{y}) = \frac{y_1^{\Delta_2} \binom{y_3}{y_1}^{\Delta_1}}{\Delta_2!} \sum_{m=0}^{\Delta_1} \frac{\binom{y_1 y_2}{y_3}^m}{m!} \binom{\Delta_2}{\Delta_1 - m}, \quad (144a)$$

$$= \frac{y_1^{\Delta_2} (y_3/y_1)^{\Delta_1}}{\Delta_2!} L_{\Delta_1}^{(\Delta_2-\Delta_1)} \left(-\frac{y_1 y_2}{y_3} \right). \quad (144b)$$

The degree of dephasing protection depends on both Δ and α in this case. Taking the case of large $\Delta_2 \geq \Delta_1 \rightarrow \infty$, using Eqs. (54) and (144b), and using known behaviors of Laguerre polynomials [81,91], we obtain the following asymptotic behavior:

$$\langle \bar{-} | \hat{\mathbf{a}}^\dagger \mathbf{p} \hat{\mathbf{a}} \mathbf{p} | \bar{+} \rangle_{\Delta_2 \geq \Delta_1 \rightarrow \infty} \sim \exp(-2\alpha\sqrt{\Delta_1} + \alpha^2) \text{poly}(\alpha, \Delta), \quad (145)$$

where $\mathbf{p} = (p_1 \ p_2 \ p_3)$ is such that $p_2 + p_3 \leq \Delta_1$, $p_1 + p_3 \leq \Delta_2$, and $p_2 \geq p_1$. We see that dephasing is exponentially suppressed in $\sqrt{\Delta_1}$, and that there should be an optimal value for α^2 of the order of Δ_1 . In addition to the polynomial factors not displayed here, there are sweet spots (cf. Refs. [29,92]) when varying α^2 that can be used to further suppress the overlap.

F. Calabi-Yau code

GKZ functions can be used to define algebraic varieties, including those of the Calabi-Yau type (Table 1 of Ref. [51]). A simple example is the cubic curve lying in the projective plane, which corresponds to the generator matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \quad (146)$$

This matrix constrains the total occupation number, along with differences between the fourth mode and the second and third modes.

The kernel of H is generated by the vector $\mathbf{t} = (-3 \ 1 \ 1 \ 1)$, and we can define a tiger code that encodes a logical qubit by setting

$$G = 2\mathbf{t} = (-6 \ 2 \ 2 \ 2). \quad (147)$$

The minimal-weight logical operator corresponds to the vector $\mathbf{t}$, yielding a code distance $d_X = |\mathbf{t}| = 6$. In other words, the code detects at least five losses on any mode.

Similar to the extended pair-cat code, the kernel is quite constrained, and we can say more. In this case, all pure-loss vectors are outside of the kernel; i.e., the code can detect arbitrary losses on any mode.

Because the all-one vector is in $\text{im}H$, the dependency in α disappears in the code words. We can evaluate the behavior of the overlap

$$\langle \bar{-} | \bar{+} \rangle^{\Delta \rightarrow \infty} \sim \left(\frac{\sqrt{13}}{4} \right)^\Delta \text{poly}(\Delta). \quad (148)$$

Since $\sqrt{13}/4 < 1$, we infer that dephasing is exponentially suppressed in Δ .

Using the method presented at the beginning of Sec. VII, Eq. (116), we can solve for a logical- X using computer software such as SAGEMATH [93]. We did not find a general expression, but can find a solution for specific values of Δ . For instance, when $\Delta = 3$, we obtain

$$\bar{X} = \frac{765\hat{n}_4^2 - 1332\hat{n}_4 + 575}{8} \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_3^\dagger \hat{a}_4^\dagger + \frac{85\hat{n}_4^2 + 22\hat{n}_4 + 4}{24} \hat{a}_1^{\dagger 3} \hat{a}_2 \hat{a}_3 \hat{a}_4. \quad (149)$$

It is simple to generalize the H matrix to N modes. Such a generalization constrains the total occupation number along with differences between a central mode and all but one of the remaining modes. This corresponds to a family of hypersurfaces in projective space of dimension $N - 2$ [51]. We leave further study of these exotic codes to future work.

VIII. REALIZING TIGER CODES

Tiger codes are stabilized by X -type dissipators and Z -type stabilizers. The X -type dissipators are monomials of annihilation operators, which can be implemented via engineered dissipation, while the Z -type stabilizers are linear constraints on occupation numbers. The pair-cat code is a concrete example within this general framework. Experimental techniques developed for implementing pair-cat codes are expected to extend naturally to general tiger codes.

A. Autonomous quantum error correction (QEC) against dephasing

X -type dissipators for the infinite-support tiger codes are of the form $\hat{\mathbf{a}}^{\mathbf{g}} - \alpha^{|\mathbf{g}|}$, where $\mathbf{g}$ encodes the powers of the loss operators of each mode in the loss-operator monomial. These monomial-type jump operators are simpler than the polynomial-type jump operators of spherical codes [22], and there are several proposals to realize them.

The first idea (Sec. VI in Ref. [29]) proposes the use of an ancillary Josephson junction and lossy cavity to induce a cascade of a four-wave mixing process to realize the pair-cat jump operator, for which $\mathbf{g} = (2 \ 2)$. Extending this proposal would require a separate process for each nonzero coordinate of $\mathbf{g}$.

More recent techniques utilize an asymmetrically threaded superconducting quantum interference device (SQUID), or asymmetrically threaded SQUID [94], to realize one or more jump operators with the same ancilla. These techniques, originally proposed to generate the more

difficult polynomial dissipators, are outlined in the Supplement of Ref. [94], Appendix B.2 in Ref. [76], and the Supplement of Ref. [22].

B. Syndrome-based QEC against loss

Z-type stabilizers are of the form $\mathbf{h} \cdot \hat{\mathbf{n}}$, where $\mathbf{h}$ specifies the integer coefficients in the linear combination of occupation-number operators. The Z-type syndrome can be extracted by entangling the physical modes with an ancillary mode via a suitable unitary operation, followed by a measurement of the ancilla.

The entangling unitary between physical modes and ancillary mode is the optomechanical interaction $\exp[i(\mathbf{h} \cdot \hat{\mathbf{n}})\hat{p}_{\text{anc}}]$, which shifts the position of the ancillary mode by the syndrome value. The syndrome extraction circuit of Z-type syndrome $\mathbf{h} \cdot \hat{\mathbf{n}}$ is given by

$$\begin{aligned} \exp[i(\mathbf{h} \cdot \hat{\mathbf{n}})\hat{p}_{\text{anc}}]|\psi\rangle_{\mathbf{h} \cdot \hat{\mathbf{n}}=\Delta_{\mathbf{h}}} \otimes |q=0\rangle_{\text{anc}} \\ = |\psi\rangle_{\mathbf{h} \cdot \hat{\mathbf{n}}=\Delta_{\mathbf{h}}} \otimes |q=\Delta_{\mathbf{h}}\rangle_{\text{anc}}, \end{aligned} \quad (150)$$

where $|\psi\rangle_{\mathbf{h} \cdot \hat{\mathbf{n}}=\Delta_{\mathbf{h}}}$ denotes a noisy code state with syndrome $\Delta_{\mathbf{h}}$ and where $|q=0\rangle_{\text{anc}}$ is the position eigenstate of the ancillary mode. After Eq. (150), the ancillary mode can be measured in the position basis, with the measurement outcome directly yielding the corresponding Z-type syndrome $\Delta_{\mathbf{h}}$. The continuous-position measurement outcome can be rounded to yield the approximate syndrome value. This proposal has been detailed for the pair-cat code in Sec. VII of Ref. [29] and is generalizable to other tiger codes.

One could also substitute the ancillary momentum operator $\hat{p}_{\text{anc}}$ with the mode's occupation number $\hat{n}_{\text{anc}}$. Evolving with this interaction Hamiltonian up to a time $2\pi/q$ and appropriately measuring the ancillary mode would extract the syndrome modulo q , which is sufficient if done rapidly so that no more than q photons are lost between measurement rounds. Engineering this interaction requires tuning each occupation-number interaction, known as a cross-Kerr coupling [95,96], to the corresponding coordinate of $\mathbf{h}$. Tuning interactions to fixed values is difficult during fabrication, but there are several proposals for doing so dynamically (e.g., Refs. [97,98]). A proposal in this family was used to create a pair-coherent state in Ref. [21].

It is useful to compare syndrome extraction circuits for cat and tiger codes. The Z-type syndrome extraction unitary for a four-component cat code is given by $\exp(i\pi\hat{n}\hat{\sigma}_x^{\text{anc}})$, which flips the Z eigenvalue of the ancillary qubit by the syndrome value. This unitary can be implemented by evolving the dispersive interaction $\hat{n}\hat{\sigma}_x^{\text{anc}}$ for a duration of π . The corresponding syndrome extraction circuit is

$$\begin{aligned} \exp(i\pi\hat{n}\hat{\sigma}_x^{\text{anc}})|\Phi\rangle_{\hat{n} \bmod 2=\Delta} \otimes |0\rangle_{\text{anc}} \\ = |\Phi\rangle_{\hat{n} \bmod 2=\Delta} \otimes |\Delta\rangle_{\text{anc}}, \end{aligned} \quad (151)$$

where $|\Phi\rangle_{\hat{n} \bmod 2=\Delta}$ denotes a noisy code state with parity syndrome Δ and where $|0\rangle_{\text{anc}}$ is the Z eigenstate of the ancillary qubit. After Eq. (151), the ancillary qubit is measured in its Z basis, whose measurement result gives the cat code's syndrome. Note that the dispersive interaction $\hat{n}\hat{\sigma}_x^{\text{anc}}$ used in the cat code commutes only with the dissipator when the evolution time is an integer multiple of π , which means that the X-type dissipator has to be turned off during the ramp-up time for any cat code with four or more components as well as any of its concatenations. This results in the system lacking dephasing protection during the Z-type syndrome measurement. In contrast, the time evolution [see Eq. (150)] generated by the linear constraints of a tiger code commutes with all X-type dissipators for all times, so those dissipators need not be turned off during syndrome extraction. Such commutation provides enhanced protection when syndrome extraction is implemented via cross-Kerr coupling.

In passing, we note that continuous (i.e., non-syndrome-based) protection against loss has been proposed for the pair-cat code (see Sec. VIII in Ref. [29]) and may be extendable to general tiger codes.

C. Realizing the four-mode tiger code

We outline the resource overhead required to realize dissipators and syndrome extraction for the four-mode tiger code.

All X-type dissipators in the code are two-body interactions, comprising two degree-2 terms and one degree-4 term. Although each dissipator acts on a different subset of modes, the degree-2 terms correspond to the same dissipator that stabilizes pair-coherent states, which is implemented in Ref. [21], while the degree-4 term matches the dissipator used to stabilize pair-cat codes. Each of these dissipators can be implemented by coupling the relevant physical modes to a three-level Josephson junction, driven by a multifrequency pulse to engineer the desired dissipation (see Sec. VI in Ref. [29]). Each Josephson junction is coupled to a low- Q cavity that performs entropy extraction. Specifically, realizing the three X-type dissipators requires three auxiliary Josephson junctions coupled respectively to the mode pairs $\{\hat{a}_1, \hat{a}_2\}$, $\{\hat{a}_3, \hat{a}_4\}$, $\{\hat{a}_2, \hat{a}_4\}$, with appropriate coupling and drive schemes.

The Z-type syndrome measurement, analogous to that in the pair-cat code, can be implemented by coupling each physical mode to a readout cavity via a Josephson junction. The effective interaction induces a conditional displacement of the readout cavity whose amplitude is proportional to the Z syndrome (Sec. VII in Ref. [29]).

The hardware cost of realizing each individual X-type dissipator and the Z-type stabilizer is comparable to that of the pair-cat code. The main experimental challenge lies in integrating these components coherently, which demands multiple simultaneous ancilla couplings and driving strengths. The circuit layout is shown in Fig. 5, where

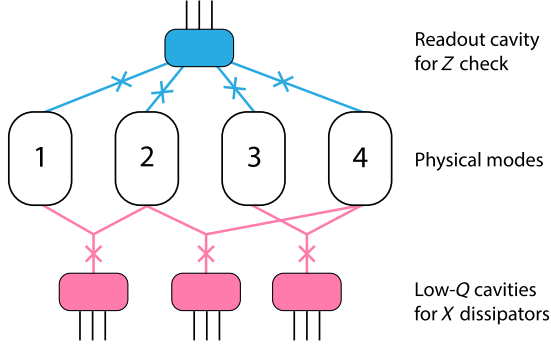


FIG. 5. Potential implementation of four-mode tiger codes is analogous to the realization of the pair-cat code in Ref. [29]. A line with a cross sign represents a Josephson junction ancilla, pink rounded rectangles represent low- Q cavities for entropy extraction, and the blue rounded rectangle at the top represents the readout cavity for Z -syndrome extraction.

the pink parts label ancillary systems for X -type dissipators and where the blue part corresponds to the Z -type syndrome extraction system.

IX. DISCUSSION AND OUTLOOK

We construct CSS-like intrinsically multimode tiger codes from two integer-valued generator matrices, G and H , satisfying a homological condition. The matrix G defines code dissipators for autonomous stabilization, while H defines linear constraints on the modes' occupations. Our framework yields bosonic codes that are not constructed via concatenation with a qubit stabilizer code.

Tiger code words maintain the features of the pair-cat code [29] that made it intrinsically amenable to realization. Namely, code syndromes are linear Fock-space constraints, and extracting them using Hamiltonian evolution is possible without having to turn off any dissipator-induced autonomous stabilization.

Tiger code words are superpositions of infinitely or finitely many Fock states and can encode logical qubits, qudits, rotors, or modes. Previously known codes, such as the pair-cat code, coherent- and Fock-state repetition codes, pair-coherent state codes, dual-rail codes, two-mode binomial codes, and aspects of $\chi^{(2)}$ codes can all be described as instances of a tiger code for specific choices of generator matrices. We break new ground with codes based on homology over the integers, the theory of GKZ hypergeometric functions, and algebraic varieties such as those of Calabi-Yau type.

For example, we construct a tiger surface code and its liger (long-tiger) version defined in a 2D strip, using a hypergraph product of two repetition codes over the integers. Its stabilizers are geometrically local and of constant weight, and the code has increasing distance against occupation-number loss and dephasing error. This is the first example of an intrinsically multimode

bosonic code with topological protection, opening a new avenue for constructing topological bosonic codes and dissipative topological phases of matter. It will be interesting to see what types of phases can be constructed in our framework and to make contact with phase classification of open systems [99–104]. It could also be interesting to see how to classify the wide landscape of tiger codes in the language of approximate QEC [105].

Tiger code words lie in a subspace of multimode Fock states $|\mathbf{n}\rangle$ defined by the constraint $H\mathbf{n} = \Delta$ for some vector Δ . While the value of Δ does not change the asymptotic behavior of the simpler pair-cat code, it does play an important role for other tiger codes. We show that a Shor-like four-mode tiger code as well as the liger surface code have both orthogonal X and Z bases for certain values of Δ , regardless of its energy. Such codes have a counter-intuitive property in that they are based on nonorthogonal coherent states, yet can protect against dephasing without the need of a large-energy limit.

We demonstrate that each tiger code is naturally associated with a corresponding GKZ hypergeometric function that describes the behavior of high-dimensional curves. Furthermore, we show that properties (such as code distances) of a given tiger code can be obtained by analyzing its corresponding GKZ hypergeometric functions. Informally, the GKZ hypergeometric function can be viewed as a “generating function” for the tiger code. The Knill-Laflamme condition can be readily obtained by taking partial derivatives of the corresponding GKZ function. Therefore, useful tiger codes should be obtainable by searching for GKZ functions with desired properties. GKZ functions may help to draw connections between codes with other fields in physics or mathematics. For example, they are widely discussed in the context of geometric analysis [51,106,107] and Feynman path integrals [108]. There may be connections between the integer homology behind tiger codes and related properties of GKZ functions [109], and established formulas for asymptotics of GKZ functions with Δ may prove useful (see Theorem 6 in Ref. [110]). There may also be connections with Grassmannian GKZ functions [50,111–114] and Grassmannian coherent states [115,116].

While we do provide a universal gate set, not all gates are fault tolerant. However, the homological framework allows for two avenues for developing such gates. One avenue is to extend qubit gates, e.g., for hypergraph-product codes over $\mathbb{Z}_2$ [117,118] or more general codes [119–124], to the setting of integer homology. The second is to extend existing few-mode bosonic protocols, e.g., fault-tolerant schemes for rotation symmetric bosonic codes [78,125,126] or bias-preserving gates for the pair-cat code [127]. Our higher-order phase gates in Table III suggest the presence of a rotor analog of the Clifford hierarchy [128–132], whose development and potential connection to the bosonic binary expansion may be interesting directions.

The fault tolerance and threshold properties of the tiger code remain important topics for future research. Prior works (Refs. [74,133]) have shown that the optimal threshold of discrete-variable stabilizer codes can be determined by analyzing phase transitions in their associated statistical-mechanical models. Extending this statistical-mechanical mapping to the continuous-variable setting, as in Ref. [134], for instance, and applying it to investigate the threshold behavior of tiger codes is an interesting future direction.

Hardware-efficient implementations of concatenated-cat codes have been proposed [76,135,136] and the repetition-cat code is recently implemented [20] in the superconducting-circuit platform. Pair-coherent states have also been realized with similar technology [21]. While requiring many modes, the hardware requirements per mode are arguably less for our codes, as nonconcatenated tiger codes avoid the need for individual mode stabilization, making them a viable candidate for future blueprints.

The t -design is widely used in the construction of quantum spherical codes [22], whose code words are discrete superpositions of coherent states. Its recent extension, the curve t -design [137], is a set of curves such that the average of a degree- t polynomial over the curves is equal to that over the entire sphere. In the case of tiger codes, whose code words are continuous superpositions of coherent states along curves, detecting up to degree- t photon loss implies that the expectation values of degree- t polynomials in $\hat{a}$ are indistinguishable between code words. This implies that curve t -designs may give rise to new tiger codes, and it remains to determine whether such curves may be stabilized by tiger-code check matrices.

Four- and higher-component cat codes [26] and their concatenations are not tiger codes since their code spaces are defined by nonlinear (parity) occupation-number constraints. Analogously, single- and two-mode binomial codes that contain a gap, or “spacing,” between Fock states in the support of different code words are not tiger codes. While codes with linear syndromes are less taxing in terms of resource overhead, they come at a price of requiring a higher number of modes for the same degree of protection. It will be interesting to develop hybrid codes that are tailored to particular architectures and that strike an appropriate balance between syndrome complexity and degree of protection.

Note added.—Recently, we learned of the preprint Ref. [138] that describes new families of topological bosonic stabilizer codes [which include analog (Gaussian) stabilizer codes and Gottesman-Kitaev-Preskill codes] using the tools from anyon theory. The stabilizers of those are discrete or continuous groups of quadrature displacements, and their code words consist of noncompact superpositions of coherent states. Both our work and the preprint utilize integer

homology [24] and develop intrinsically multimode codes with topological protection, but the classes of codes and the primary research focuses are, to our knowledge, completely distinct.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: GKZ HYPERGEOMETRIC FUNCTIONS

Tiger codes have a close connection to the theory of GKZ hypergeometric functions [44–50], and we collect the various connections and calculations associated with them in this section. We recommend Refs. [139–145] for clear introductions into GKZ theory.

Recall from Eq. (15) the integral and sum forms of the GKZ function, reproduced below. For $\mathbf{y} = (y_1, y_2, \dots, y_N) \in \mathbb{C}^N$,

$$A_{\Delta}(\mathbf{y}) = \int \frac{d^r \phi}{(2\pi)^{r_z}} e^{-i\phi \cdot \Delta} \exp \left(\sum_{j=1}^N y_j e^{i(\phi H)_j} \right), \quad (\text{A1a})$$

$$= \sum_{H\mathbf{n}=\Delta} \frac{\mathbf{y}^{\mathbf{n}}}{\mathbf{n}!}, \quad (\text{A1b})$$

where the index $\mathbf{n}$ goes over non-negative integer vectors and where we use multi-index notation, e.g., $\mathbf{y}^{\mathbf{n}} = y_1^{n_1} y_2^{n_2} \dots y_N^{n_N}$. The sum can be obtained by expanding the exponential in each y_j and integrating term by term. Infinite (finite) sums correspond to projected coherent states supported on an infinite-dimensional (finite-dimensional) Fock-state subspace.

A third form is possible for the infinite-support case if there exists a non-negative integer vector $\mathbf{s}$ satisfying $H\mathbf{s} = \Delta$ such that (see Lemma 1 in Ref. [47], as well as Ref. [46])

$$A_{\Delta}(\mathbf{y}) = \sum_{\mathbf{n} \in \ker H} \frac{\mathbf{y}^{\mathbf{n}+\mathbf{s}}}{(\mathbf{n}+\mathbf{s})!}. \quad (\text{A1c})$$

1. From projected coherent states to GKZ via the Segal-Bargmann mapping

Both GKZ functions and projected coherent states are defined by the same variables, H and Δ . Moreover, GKZ functions govern overlaps between their corresponding states, as evident by Eq. (16), which suggests a deep correspondence between appropriate GKZ functions and projected coherent states.

Here, we show that expressing projected coherent states in the Segal-Bargmann representation yields their associated GKZ functions, with arguments scaled by α . A similar correspondence holds between operator relations of projected coherent states and the defining relations of GKZ functions.

The Segal-Bargmann mapping is a representation of functions in $L^2(\mathbb{R}^N)$ by holomorphic functions F in N complex variables z_j [56–58, 146, 147]. Each Fock state $|\mathbf{n}\rangle$ maps to a monomial term $\mathbf{z}^{\mathbf{n}}$ such that a general state maps as follows:

$$|\Psi\rangle = \sum_{\mathbf{n} \in \mathbb{N}^N} \beta_{\mathbf{n}} |\mathbf{n}\rangle \mapsto F_{\Psi}(\mathbf{z}) = \sum_{\mathbf{n} \in \mathbb{N}^N} \frac{\beta_{\mathbf{n}}}{\sqrt{\mathbf{n}!}} \mathbf{z}^{\mathbf{n}}. \quad (\text{A2})$$

Un-normalized coherent states map to exponentials,

$$\sum_{\mathbf{n} \in \mathbb{N}^N} \frac{\alpha^{\mathbf{n}}}{\sqrt{\mathbf{n}!}} |\mathbf{n}\rangle \mapsto \sum_{\mathbf{n} \in \mathbb{N}^N} \frac{\alpha^{\mathbf{n}}}{\mathbf{n}!} \mathbf{z}^{\mathbf{n}} = \exp(\alpha \cdot \mathbf{z}). \quad (\text{A3})$$

Such states correspond to the trivial version of projected coherent states, i.e., with no projection. More generally, un-normalized projected coherent states map to GKZ functions whose arguments are rescaled by α ,

$$\sum_{H\mathbf{n}=\Delta} \frac{\alpha^{\mathbf{n}}}{\sqrt{\mathbf{n}!}} |\mathbf{n}\rangle \mapsto \sum_{H\mathbf{n}=\Delta} \frac{\alpha^{\mathbf{n}}}{\mathbf{n}!} \mathbf{z}^{\mathbf{n}} = A_{\Delta}(\alpha\mathbf{z}), \quad (\text{A4})$$

for argument $\alpha\mathbf{z} = (\alpha_1 z_1, \alpha_2 z_2, \dots, \alpha_N z_N)$.

GKZ functions satisfy certain defining relations, and the Segal-Bargmann representation can be used to obtain those from the coherent-state relations used in this work. Operators $\hat{a}_j$, $\hat{a}_j^{\dagger}$, and $\hat{n}_j$ have a simple interpretation,

$$\hat{a}_j \mapsto \partial_j, \quad \hat{a}_j^{\dagger} \mapsto z_j, \quad \hat{n}_j \mapsto z_j \partial_j, \quad (\text{A5})$$

where the partial derivative $\partial_j = \partial/\partial z_j$.

Expressing the H constraints of a tiger code in the Segal-Bargmann representation gives rise to a system of differential equations. For each row $\mathbf{h}$ of H , we have

$$(\mathbf{h} \cdot \hat{\mathbf{n}} - \Delta_{\mathbf{h}}) |\Psi\rangle \mapsto \left(\sum_{j=1}^N h_j z_j \partial_j - \Delta_{\mathbf{h}} \right) F_{\Psi}(\mathbf{z}) = 0. \quad (\text{A6})$$

Such a set of equations is one of the two defining constraints of GKZ functions, in which H is often called A .

We can also put creation and annihilation operator relations, Eq. (113), into Segal-Bargmann form. For each row, $\mathbf{g} = (\mathbf{p} - \mathbf{q}) \in \ker H$, and after a little rewriting,

$$(\hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{p}} - \alpha^{\mathbf{g}}(\hat{\mathbf{n}})_{\mathbf{q}}) |\Psi\rangle = 0, \quad (\text{A7a})$$

$$\hat{\mathbf{a}}^{\dagger \mathbf{q}} ((\hat{\mathbf{a}}/\alpha)^{\mathbf{p}} - (\hat{\mathbf{a}}/\alpha)^{\mathbf{q}}) |\Psi\rangle = 0, \quad (\text{A7b})$$

$$\mapsto \left(\prod_j \partial_j^{p_j} - \prod_j \partial_j^{q_j} \right) F_{\Psi}(\alpha\mathbf{z}) = 0. \quad (\text{A7c})$$

Above, prior to applying the mapping, we got rid of the prefactor $\hat{\mathbf{a}}^{\dagger \mathbf{q}}$ since it has no right zero eigenstate and so does not change the constraint in itself. The above mapped constraints form the second defining ingredient of a GKZ system. The above holds true for any vector in the kernel of H , which, in the case of tiger codes, is partitioned into dissipators generated by G and logicals generated by L_X (see Sec. III C).

We use the form of Eq. (A7a) for the dissipators in the main text and assume $\mathbf{q} = \mathbf{0}$ for codes with infinite support. In general, this is not necessary, and dissipators of the form $\hat{\mathbf{a}}^{\mathbf{p}} - \hat{\mathbf{a}}^{\mathbf{q}}$ [Eq. (A7)] also exist for both finite- or infinite-support codes.

2. Dephasing matrix elements of projected coherent states

Matrix elements of dephasing terms, $\hat{\mathbf{a}}^{\dagger \mathbf{p}} \hat{\mathbf{a}}^{\mathbf{p}}$, with respect to projected coherent states can be calculated using either their integral or sum form, reproducing a nontrivial identity on their corresponding GKZ functions. This identity leads to relations between expectations values for different $\mathbf{p}$.

Matrix elements for general projected coherent states $|\alpha\rangle_{\Delta}^H$ and $|\beta\rangle_{\Delta}^H$ can be obtained by a slight generalization of the calculation in Eq. (54), yielding

$${}_{\Delta}^H \langle \alpha | \hat{\mathbf{a}}^{\dagger \mathbf{p}} \hat{\mathbf{a}}^{\mathbf{p}} | \beta \rangle_{\Delta}^H \propto (\alpha^{\star} \beta)^{\mathbf{p}} A_{\Delta-H\mathbf{p}}(\alpha^{\star} \beta), \quad (\text{A8})$$

where the proportionality sign hides normalization factors in the denominator.

The same matrix elements can be obtained by converting Fock-state relations into differential form,

$${}_{\Delta}^H \langle \alpha | \hat{\mathbf{a}}^{\dagger \mathbf{p}} \hat{\mathbf{a}}^{\mathbf{p}} | \beta \rangle_{\Delta}^H \propto \sum_{H\mathbf{n}=\Delta} \frac{(\alpha^{\star} \beta)^{\mathbf{n}}}{\mathbf{n}!} \langle \mathbf{n} | \hat{\mathbf{a}}^{\dagger \mathbf{p}} \hat{\mathbf{a}}^{\mathbf{p}} | \mathbf{n} \rangle, \quad (\text{A9a})$$

$$= \sum_{H\mathbf{n}=\Delta} \frac{(\alpha^{\star} \beta)^{\mathbf{n}}}{\mathbf{n}!} \frac{\mathbf{n}!}{(\mathbf{n} - \mathbf{p})!}, \quad (\text{A9b})$$

$$= \sum_{H\mathbf{n}=\Delta} \mathbf{y}^{\mathbf{p}} \partial^{\mathbf{p}} \frac{\mathbf{y}^{\mathbf{n}}}{\mathbf{n}!} \Big|_{\mathbf{y}=\alpha^* \beta}, \quad (\text{A9c})$$

$$= \mathbf{y}^{\mathbf{p}} \partial^{\mathbf{p}} \mathbf{A}_{\Delta}(\mathbf{y}) \Big|_{\mathbf{y}=\alpha^* \beta}, \quad (\text{A9d})$$

where $\partial^{\mathbf{p}} = (\partial/\partial y_1)^{p_1} \dots (\partial/\partial y_N)^{p_N}$. Equating the above two equations yields an identity between GKZ functions for different Δ .

The above identity, together with the defining GKZ relation Eq. (A7c), $\partial^{\mathbf{p}} \mathbf{A}_{\Delta}(\mathbf{y}) = \partial^{\mathbf{q}} \mathbf{A}_{\Delta}(\mathbf{y})$ for differences of non-negative vectors, $\mathbf{p} - \mathbf{q} \in \ker H$, can be used to relate different dephasing matrix elements as

$$(\alpha^* \beta)^{\mathbf{q}} \Delta^H \langle \alpha | \hat{\mathbf{a}}^{\dagger \mathbf{p}} \hat{\mathbf{a}}^{\mathbf{p}} | \beta \rangle_{\Delta}^H = (\alpha^* \beta)^{\mathbf{p}} \Delta^H \langle \alpha | \hat{\mathbf{a}}^{\dagger \mathbf{q}} \hat{\mathbf{a}}^{\mathbf{q}} | \beta \rangle_{\Delta}^H. \quad (\text{A10})$$

This relation reduces the number of matrix elements one needs to calculate in order to determine the degree of dephasing protection of a tiger code.

3. Asymptotics of GKZ integrals

We can estimate some GKZ integrals for large arguments using the method of steepest descents and the saddle-point approximation, beautifully explained in Sec. 6.6 of Ref. [148]. This method yields the degree to which dephasing matrix elements are exponentially suppressed, allowing us to extract the exact value of the exponent and compare it to a given tiger code's Z distance.

Roughly, the method of steepest descents estimates an integral asymptotically in a limit by rewriting it as a complex contour integral, choosing a contour that goes through saddle points (at which the phase of the integrand is constant), and approximating the contour integral by Gaussian integrals centered at said points.

To simplify notation, we redefine variables as

$$\mathbf{A}(\mathbf{y}) = \int \frac{d^{r_z} \phi}{(2\pi)^{r_z}} \exp \left(-i \sum_{j=1}^{r_z} \phi_j \Delta_j + \sum_{k=1}^N |y_k| \rho_k \right), \quad (\text{A11})$$

where $y_k = |y_k| e^{i\theta_k}$ and where

$$\rho_k \equiv \exp \left[i \left(\sum_{j=1}^{r_z} (H)_{jk} \phi_j + \theta_k \right) \right]. \quad (\text{A12})$$

To find a saddle-point value of ϕ , we have to solve the following equations:

$$\begin{aligned} \vec{\nabla} \left(\sum_{k=1}^N |y_k| \rho_k - \sum_{j=1}^{r_z} \phi_j \Delta_j \right) &= 0 \\ \Rightarrow \sum_{k=1}^N \rho_k |y_k| (H)_{jk} &= \Delta_j, \quad \forall j = 1, 2, \dots, r_z. \end{aligned} \quad (\text{A13})$$

We can write $z_j = e^{i\phi_j}$, and the above becomes

$$\sum_{k=1}^N y_k \prod_{j=1}^{r_z} z_j^{(H)_{jk}} (H)_{lk} = \Delta_l, \quad \forall l = 1, 2, \dots, r_z. \quad (\text{A14})$$

There are r_z equations for r_z variables above, so solutions exist, in principle. While it is hard to solve these polynomial equations, in general, they can be simplified in some special cases.

The normalization function of tiger code words, $\mathbf{A}_{\Delta}(\alpha^2 \mathbf{1})$, is one such case, corresponding to equal GKZ arguments $y_k = \alpha^2$ for real α . Below, we work in the limit of large energy density, $\alpha^2 \gg \Delta_l$ for all rows l , but can also perform the opposite limit for the finite-support cases. For the former, the saddle-point equation can be approximated by

$$\sum_{k=1}^N \prod_{j=1}^{r_z} z_j^{(H)_{jk}} (H)_{lk} \approx 0, \quad \forall l = 1, 2, \dots, r_z. \quad (\text{A15})$$

Imposing the additional constraint that the all-ones vector $\mathbf{1} \in \ker H$, we see that $z_j = 1$ ($\phi_j = 0$) for all j is a valid solution to the saddle-point equations. In fact, this solution gives the dominant saddle because $N\alpha^2$ is the largest possible exponent. Because the normalization is real, and because $z_j = 1$ ($\phi_j = 0$) $\forall j$ is the only dominant saddle, we can expand the saddle and perform a Gaussian integration along the real direction of all ϕ_j 's to obtain the leading-order asymptotic behavior of the tiger-code normalization functions,

$$\mathbf{A}(\alpha^2 \mathbf{1}) \sim \frac{\exp(N\alpha^2)}{(2\pi\alpha^2)^{r_z/2}} \det(HH^T)^{-1/2}. \quad (\text{A16})$$

Performing a similar analysis for the more general case of $\theta_k \neq 0$ requires solving explicitly the saddle-point equations and the tangent direction of the constant phase hyperplane to perform the integration about a possibly complex contour in the enlarged parameter space of ϕ .

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