

# Beyond Stellar Rank: Control Parameters for Scalable Optical Non-Gaussian State Generation

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Advanced quantum technologies rely on non-Gaussian states of light, essential for universal quantum computation, fault-tolerant error correction, and quantum sensing. Their practical realization, however, faces hurdles: Simulating large multimode generators is computationally demanding, and benchmarks such as the *stellar rank* do not capture how effectively photon detections yield useful non-Gaussianity. We address these challenges by introducing the *non-Gaussian control parameters* ( $s_0, \delta_0$ ), a continuous and operational measure that goes beyond stellar rank. Leveraging these parameters, we develop a universal optimization method that reduces photon-number requirements and greatly enhances success probabilities while preserving state quality. Applied to the Gottesman-Kitaev-Preskill state generation, for example, our method cuts the required photon detections by a factor of 3 and raises the preparation probability by nearly  $10^8$ . Demonstrations across cat states, cubic phase states, Gottesman-Kitaev-Preskill states, and even random states confirm broad gains in experimental feasibility. Our results provide a unifying principle for resource-efficient non-Gaussian state generation, charting a practical route toward scalable optical quantum technologies and fault-tolerant quantum computation.

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## I. INTRODUCTION

Optical quantum information processing holds strong promise for scalable quantum computing, owing to its intrinsic compatibility with traveling-wave architectures and potential for large-scale integration. Considerable progress has been made in the preparation and manipulation of Gaussian states [1–4], which are relatively easy to generate and control. Yet non-Gaussian states are indispensable for unlocking the full power of optical platforms, enabling universal quantum computation [5–8], quantum error correction [9–12], and advanced quantum sensing [13–15].

A common approach to generating non-Gaussianity is to prepare multimode Gaussian states and conditionally project selected modes using photon-number-resolving measurements [16–18]. The recent optical realization of

Gottesman-Kitaev-Preskill (GKP) states [19] represents a major milestone, yet scaling toward fault tolerance demands larger photon numbers, higher success probabilities, and more complex architectures. Meeting these requirements makes optimization essential, but the search space expands exponentially with system size. In fact, its underlying structure is equivalent to Gaussian boson sampling, a problem known to be classically intractable [20–22], highlighting the fundamental difficulty of systematic optimization. The central challenge, therefore, is to identify and optimize the true resources that drive non-Gaussianity.

To benchmark such resources, the *stellar rank* [23–27] is often employed. Determined by the total detected photon number, it provides a simple measure directly tied to experimental resources. However, stellar rank does not provide the complete picture: It offers only an upper bound and fails to capture how effectively the non-Gaussianity of the photon-number measurement is harnessed into the generated state. As a result, states with the same rank can exhibit significantly different levels of operational non-Gaussianity, such as non-Gaussian squeezing [28–30].

To overcome this limitation, we introduce the *non-Gaussian control parameters*, which quantify how efficiently

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photon-number measurements harness non-Gaussianity. They offer a continuous and operational description of non-Gaussian resources, forming the basis for systematic optimization and scalable state generation beyond stellar rank.

Building on this framework, we develop an optimization algorithm that systematically improves state generators by reducing photon detections and boosting success probabilities, while preserving high fidelity and essential non-Gaussian features such as squeezing. For GKP states, the method lowers the photon-number requirement by a factor of 3 and enhances the success probability by nearly  $10^8$ . Comparable gains are obtained for Schrödinger cat and cubic phase states, with photon-number requirements again reduced threefold and success rates improved by up to 7 orders of magnitude. We further demonstrate applicability to random states, consistently finding substantial gains in experimental feasibility. Together, these results establish a new design principle for scaling state-of-the-art architectures [19,31,32] toward fault-tolerant quantum computation.

The structure of the paper is as follows. In Sec. II, we provide a preliminary definition of the non-Gaussian control parameters, giving intuition for their physical meaning. A rigorous derivation is presented in Sec. III, where we focus on two-mode non-Gaussian state generators [Fig. 3(b)] and introduce the *control-mode representation*, a description of the generator via the first and second moments of the control mode. The non-Gaussian control parameters then naturally emerge from this framework. In Sec. IV, we investigate the resource-theoretic aspect of these parameters, showing that they quantify how efficiently the non-Gaussianity characterized by stellar rank is utilized by the state generator. In Sec. V, we extend the formalism to general multimode generators, and in Sec. VI, we apply these results to formulate the optimization algorithm. Although our primary focus is on pure non-Gaussian states, the framework also extends to mixed states with Gaussian errors such as loss, as discussed in Appendix A.

## II. OVERVIEW OF NON-GAUSSIAN CONTROL PARAMETERS AND TWO-MODE EXAMPLES

Non-Gaussian state generators can produce a wide variety of quantum states, whose properties may appear disparate when described in terms of Fock-state expansions or Wigner functions. A central unifying idea of this work is that these states can instead be characterized by a small set of continuous *non-Gaussian control parameters*, denoted  $(s_0, \delta_0)$ . These parameters complement the discrete stellar-rank measure—defined as the maximum photon number in a finite Fock expansion followed by a Gaussian unitary, typically equal to the total number of detected photons [16]. The non-Gaussian control parameters govern the symmetry and structure of the output states in phase space, providing a common language to compare distinct non-Gaussian resources.

To make their role concrete, we first describe the simplest yet general setting in which the control parameters naturally arise: a two-mode non-Gaussian state generator, shown in Fig. 1(a). This serves as the fundamental building block for analyzing and optimizing the more general multimode non-Gaussian state generators introduced later. An entangled pure two-mode Gaussian state  $|G\rangle$  is prepared. One mode, the *control mode*, is measured in the photon-number basis, and, conditional on the outcome  $n$ , a non-Gaussian state is probabilistically generated in the other mode, the *signal mode*, through their entanglement. This setting already encompasses a wide range of known schemes, including photon subtraction [33,34], generalized photon subtraction [35,36], photon addition [37,38], coherent superpositions of subtraction and addition [39,40], heralded Fock-state generation from two-mode squeezed states [41,42], and CPS generation using displaced two-mode squeezed states [10,43].

In this setting, the non-Gaussian control parameters are defined as quantities that capture the structure of the output state and depend solely on the control mode, as formalized in the following proposition (restated and proven as Theorem 5 in Sec. III C).

*Proposition 1.* The output state of any two-mode non-Gaussian state generator can be expressed in the following particle form:

$$|\psi\rangle_{s_0, \delta_0, n} \propto_G (\hat{a}^\dagger + s_0 \hat{a} + \delta_0)^n |0\rangle, \quad (1)$$

where  $\hat{a}$  ( $\hat{a}^\dagger$ ) denotes the annihilation (creation) operator satisfying  $[\hat{a}, \hat{a}^\dagger] = 1$ ,  $|0\rangle$  is the corresponding vacuum state, and  $\propto_G$  denotes proportionality up to a Gaussian unitary operation. The measured photon number  $n$  sets the stellar rank [23–27] of the output state. We introduce the parameters  $s_0 \geq 0$  and  $\delta_0 \in \mathbb{C}$ , determined solely by the properties of the control mode, called *non-Gaussian phase sensitivity* and *non-Gaussian asymmetry*, respectively. Together, they are referred to as the *non-Gaussian control parameters*.

By tuning these two parameters, even at a fixed stellar rank  $n$ , a surprisingly diverse class of non-Gaussian states can be engineered within this simple setup, as shown in Fig. 1(b). The case  $(s_0, \delta_0) = (0, 0)$  yields a Fock state. Increasing  $s_0$  gradually transforms it into a phase-sensitive state, while a nonzero  $\delta_0$  biases the state, breaking the mirror symmetry in phase space. Notably, the regimes  $s_0 > 1$  and  $0 \leq s_0 < 1$  correspond to photon-subtracted and photon-added states, respectively (Sec. III D). In addition, two important classes, approximations of Schrödinger cat states and cubic phase states (CPS), lie within this spectrum, providing intuition for the roles of the two parameters, as described below.

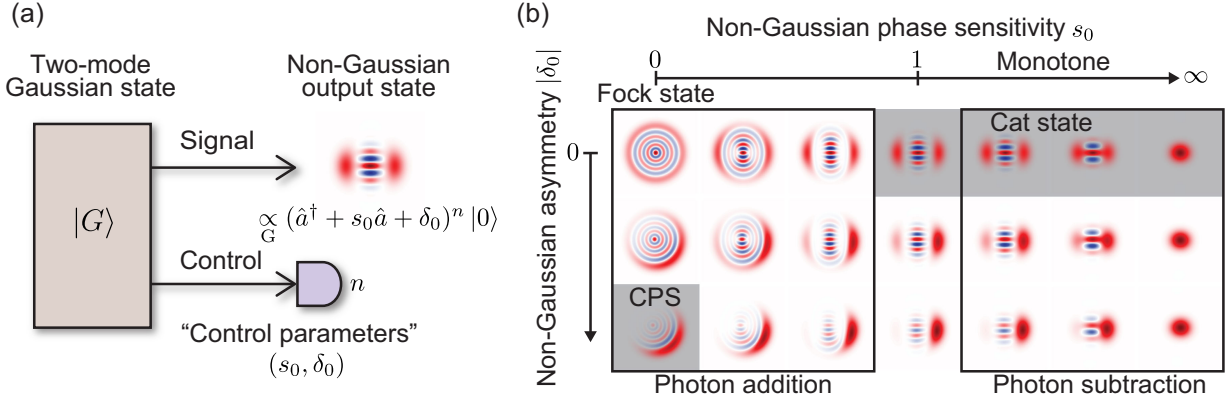


FIG. 1. Non-Gaussian state generation via photon-number measurements and non-Gaussian control parameters. (a) Schematic of a two-mode non-Gaussian state generator, consisting of a two-mode Gaussian state  $|G\rangle$  followed by a photon-number measurement. A measurement on the control mode heralds a non-Gaussian state in the signal mode, which can be characterized by the non-Gaussian control parameters  $s_0$  and  $\delta_0$ , determined solely by the properties of the control mode. (b) Wigner functions of states generated by the two-mode state generator, organized by the values of the non-Gaussian phase sensitivity  $s_0$  and the non-Gaussian asymmetry  $\delta_0$ . Although the output states are defined only up to Gaussian unitaries, we show the core state [25], corresponding to the “particle form” of Eq. (1). The columns correspond to  $s_0 = 0, 0.2, 0.5, 1, 1.5, 3, 100$ , while the rows correspond to  $\delta_0 = 0, 0.2 + 0.2i, 0.5 + 0.5i$ , with the measured photon number fixed at  $n = 6$ . This simple setup already generates a diverse class of non-Gaussian states:  $s_0 > 1$  corresponds to photon-subtracted states, while  $0 \leq s_0 < 1$  corresponds to photon-added states (Sec. III D). Notably, the case  $\delta_0 = 0, s_0 > 1$  yields cat states, whereas  $s_0 = 0$  and  $|\delta_0| \sim 1$  yields cubic phase states (CPS). For  $s_0 \geq 1$ , the phase sensitivity parameter  $s_0$  is nondecreasing under Gaussian maps (Sec. IV).

### A. Two distinctive examples: Cat and cubic phase states

The two control parameters each carry a distinct physical meaning. The parameter  $s_0$  controls the balance between creation and annihilation operators, capturing phase-sensitive interference effects between separated components in phase space, as exemplified by cat states. By contrast, the parameter  $\delta_0$  introduces an asymmetry that induces nonlinear structure in phase space, naturally leading to the CPS as an illustrative case. Examining these two states side by side reveals how the abstract parameters  $(s_0, \delta_0)$  manifest in concrete, experimentally relevant non-Gaussian resources.

#### 1. Non-Gaussian phase sensitivity $s_0$ : Cat state

When  $s_0 > 1$  and  $\delta_0 = 0$ , the output state approximates a Schrödinger cat state [11,44,45]

$$|\text{cat}_{\pm}(\alpha)\rangle \propto |\alpha\rangle \pm |-\alpha\rangle, \quad (2)$$

with the correspondence

$$|\psi\rangle_{s_0,0,n} \sim |\text{cat}_{\text{sgn}[(-1)^n]}(\sqrt{(n+1/2)/s_0})\rangle. \quad (3)$$

(See Sec. III D for the derivation.)

The key feature of cat states is the coherent superposition of two well-separated, localized components in phase space, which produces the characteristic interference

fringes in the Wigner function. This separation can be captured quantitatively by the so-called  $x^2$  squeezing [29], illustrated in Fig. 2(a), defined as

$$\xi_{\text{cat}} = \min_{\lambda>0} \left\langle \left( \frac{\hat{x}^2}{\lambda^2} - 1 \right)^2 \right\rangle, \quad (4)$$

where we define the quadrature operators as  $\hat{x} = \hat{a} + \hat{a}^\dagger$  and  $\hat{p} = -i\hat{a} + i\hat{a}^\dagger$  (see Sec. III A for our convention). This measure captures the concentration of the quadrature distribution around two symmetric peaks in phase space. It is strictly positive and tends to zero for large-amplitude cat states, whereas Gaussian states satisfy the bound  $\xi_{\text{cat}} \geq 2/3$ . For small  $s_0$ , the output resembles a Fock state, showing no visible double-peak structure, while for very large  $s_0$  the peaks merge and coherence is lost. Between these extremes lies an optimal  $s_0$ , as shown in Figs. 2(b) and 2(c), where the state exhibits maximum catlike character and approaches the stellar-rank limit, nearly saturating the generator’s ability to produce cat states.

#### 2. Non-Gaussian asymmetry $\delta_0$ : Cubic phase state

This example clarifies the role of non-Gaussian phase asymmetry  $\delta_0$ . When  $s_0 = 0$  and  $|\delta_0| > 0$ , the output state approximates a CPS [10,28,43,46]:

$$|\text{CPS}\rangle = e^{-i\hat{x}^3} |p=0\rangle, \quad (5)$$

as derived in Sec. III D.

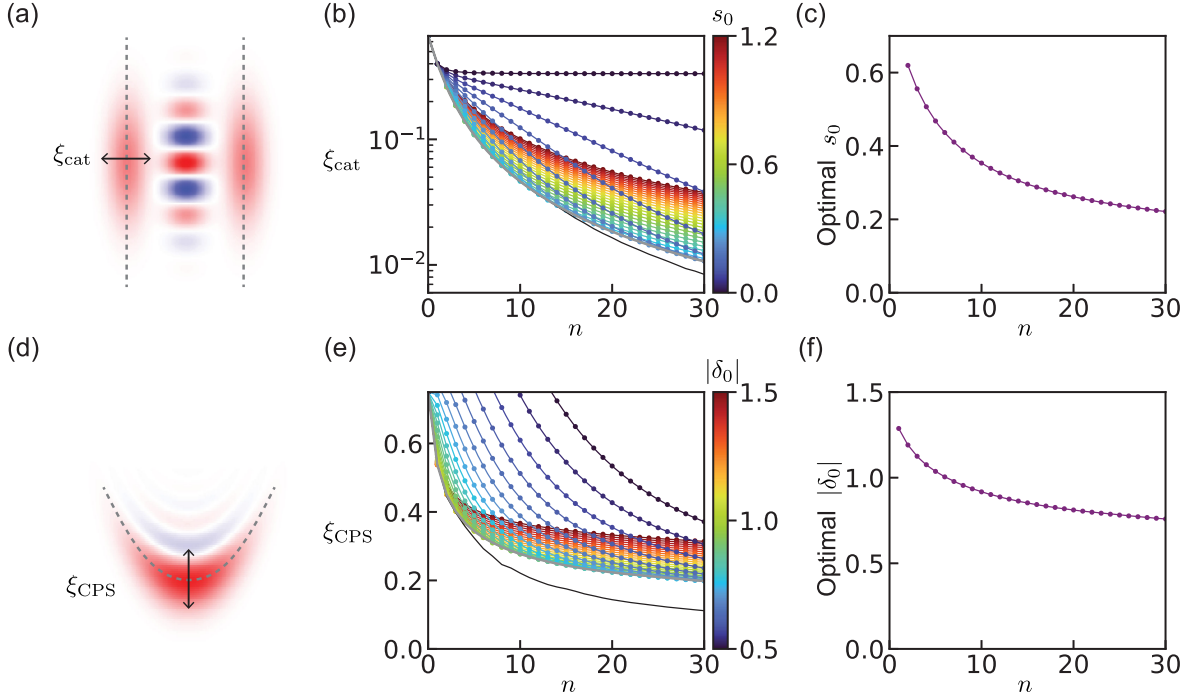


FIG. 2. Characterization of cat states and CPS generated by two-mode non-Gaussian state generators. (a)–(c) Cat states ( $s_0 > 0$ ,  $\delta_0 = 0$ ). (a) Illustration of  $x^2$  squeezing [29],  $\xi_{\text{cat}}$  [Eq. (4)], used as a metric for cat states. This quantifies the degree of localization into two symmetric peaks in phase space. (b)  $\xi_{\text{cat}}$  of the state  $|\psi\rangle_{s_0,0,n}^{(p)}$  as a function of measured photon number  $n$ , for different values of the non-Gaussian phase sensitivity  $s_0$ . Colored lines, different  $s_0$ ; gray, optimized over  $s_0$  for each  $n$ ; black, theoretical lower bound imposed by  $n$  [29]. (c) Optimal  $s_0$  yielding the minimum  $\xi_{\text{cat}}$  for each  $n$ . Values for  $n = 0, 1$  are not shown, as the state is independent of  $s_0$  up to Gaussian unitaries. (d)–(f) CPS ( $s_0 = 0$ ,  $|\delta_0| > 0$ ). (d) Illustration of cubic nonlinear squeezing [28],  $\xi_{\text{CPS}}$  [Eq. (6)], used as a metric for CPS. This quantifies the degree of localization along a parabolic curve in phase space. (e)  $\xi_{\text{CPS}}$  of  $|\psi\rangle_{0,\delta_0,n}^{(p)}$  as a function of  $n$ , for different values of the non-Gaussian asymmetry  $\delta_0$ . Colored lines, different  $|\delta_0|$ ; gray, optimized over  $|\delta_0|$ ; black, stellar-rank bound. (f) Optimal  $|\delta_0|$  yielding the minimum  $\xi_{\text{CPS}}$  for each  $n$ . The case  $n = 0$  is not shown, as the state is independent of  $\delta_0$  up to Gaussian unitaries.

The hallmark of CPS is the cubic nonlinearity of its wave function. This feature can be quantified by the *cubic nonlinear squeezing* [28], illustrated in Fig. 2(d), defined as

$$\xi_{\text{CPS}} = \min_{\lambda > 0, d \in \mathbb{R}} \left\langle \left( \lambda \hat{p} - \frac{\hat{x}^2}{4\sqrt{2}\lambda^2} - d \right)^2 \right\rangle. \quad (6)$$

This quantity measures the extent to which the phase-space distribution is concentrated along a parabolic curve, reflecting the cubic structure of the state. For an ideal CPS,  $\xi_{\text{CPS}}$  approaches zero, while Gaussian states satisfy  $\xi_{\text{CPS}} \geq 3/4$  [28,43]. For small  $|\delta_0|$ , the output resembles a symmetric, Fock-like state with little cubic character, whereas for very large  $|\delta_0|$  the cubic coherence is washed out. Between these regimes lies an optimal  $|\delta_0|$ , highlighted in Figs. 2(e) and 2(f), where the generated state exhibits its strongest cubic nonlinearity.

Together, the cat and cubic phase states demonstrate how the control parameters  $(s_0, \delta_0)$  map directly onto qualitatively distinct non-Gaussian resources. A more detailed classification of the states produced by two-mode generators is presented in Sec. III D.

## B. Non-Gaussian control parameters as quantifier of non-Gaussianity

The key insight from above examples is that the degree of non-Gaussianity is not determined by the stellar rank alone but depends crucially on the control parameters  $(s_0, \delta_0)$ . While all states share the same stellar rank  $n$ , their nonclassical features vary widely depending on  $(s_0, \delta_0)$ . For instance, when  $\delta_0 = 0$  and  $s_0$  is too large, the cat state collapses to a trivial Gaussian, erasing non-Gaussianity. This shows that a state generator does not necessarily exploit the full non-Gaussian potential implied by its stellar rank  $n$ . We explore the resource-theoretic role of this limitation in Sec. IV.

Building on this observation, we develop a general optimization framework. In certain regimes, by tailoring  $(s_0, \delta_0)$  appropriately, one can approximate the same target state using fewer detected photons, thereby lowering the required stellar rank (Sec. IV B). Moreover, we identify transformations that preserve  $(s_0, \delta_0)$  while modifying the success probability, enabling optimization of state generation rates without changing the output (Sec. III B). Generalizing these tools to multiple control modes,

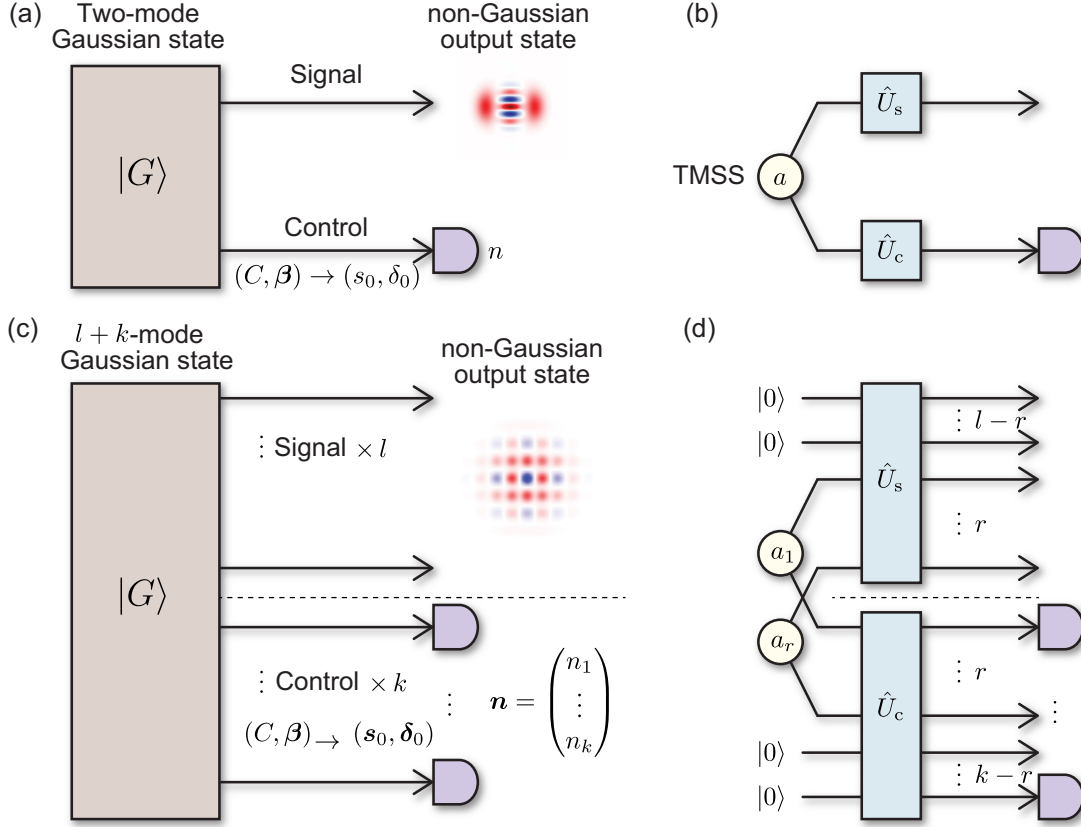


FIG. 3. Two-mode and multimode non-Gaussian state generators and their canonical forms. (a) A two-mode non-Gaussian state generator, consisting of a control mode (measured in the photon-number basis) and a signal mode (where the non-Gaussian output state is generated). The output state and its success probability are determined solely by the control moments  $(C, \beta)$  and the measured photon number  $n$  (Corollary 1). (b) Canonical form of the two-mode generator, equivalent to the system in (a). The squeezing strength  $a$  of the two-mode squeezed state and the Gaussian unitary  $\hat{U}_c$  on the control mode depend only on the control moments  $(C, \beta)$  (Theorem 1). (c) A general multimode non-Gaussian state generator with  $k$  signal modes and  $l$  control modes (Sec. V). The  $k$ -mode output state and its success probability are determined solely by the control moments  $(C, \beta)$  and the measured photon numbers  $\mathbf{n}$  (Corollary 2). (d) Canonical form of the multimode generator, equivalent to the system in (c). The parameters  $a_1, \dots, a_k$  and the Gaussian unitary  $\hat{U}_c$  depend only on the control moments  $(C, \beta)$  (Theorem 9).

we propose in Sec. VI an algorithm that systematically optimizes non-Gaussian state generators, achieving nearly the same output states with reduced stellar rank and enhanced probability of success.

### III. FORMAL DEFINITION OF CONTROL PARAMETERS IN THE TWO-MODE NON-GAUSSIAN STATE GENERATOR

Having seen through examples that the non-Gaussian features of a state depend crucially on  $(s_0, \delta_0)$ , we now turn to a formal derivation while still remaining in the simple two-mode case. For this, we introduce the *control-mode representation*, which characterizes both the output state and its generation probability entirely in terms of the properties of the control mode. Within this framework, the parameters  $s_0$  and  $\delta_0$  naturally arise as a minimal parametrization of the output state. This two-mode case analysis provides the formal foundation for the multimode

extension in Sec. V and the optimization framework developed in Sec. VI.

#### A. Control-mode representation

We start by giving a formal definition of a two-mode non-Gaussian state generator. A two-mode Gaussian state  $|G\rangle$  is prepared, with modes denoted as the *signal mode* and the *control mode*. The control mode is subjected to a photon-number measurement, projecting it onto a Fock state  $|n\rangle$  and thereby heralding a non-Gaussian state  $|\psi\rangle \propto {}_c\langle n|G\rangle$  in the signal mode, with success probability  $p_n = \|{}_c\langle n|G\rangle\|^2$ .

Let  $\hat{\mathbf{q}}_s = (\hat{x}_s, \hat{p}_s)^T$  and  $\hat{\mathbf{q}}_c = (\hat{x}_c, \hat{p}_c)^T$  denote the quadrature vectors of the signal and control modes, respectively. The full quadrature vector is then written as  $\hat{\mathbf{q}} = (\hat{\mathbf{q}}_s^T, \hat{\mathbf{q}}_c^T)^T$ . Each mode satisfies the canonical commutation relation  $[\hat{x}, \hat{p}] = 2i$ , corresponding to the convention  $\hbar = 2$ , which is used throughout the paper.

The Gaussian state  $|G\rangle$  is completely characterized by the mean  $\gamma \in \mathbb{R}^4$  and covariance  $\Sigma \in \mathbb{R}^{4 \times 4}$ , defined as [47]

$$\gamma := \langle G | \hat{q} | G \rangle, \quad (7)$$

$$\Sigma := \frac{1}{2} \langle G | \{ \hat{q} - \gamma, (\hat{q} - \gamma)^T \} | G \rangle, \quad (8)$$

where  $\{\cdot, \cdot\}$  denotes the anticommutator. These can be written in block form as

$$\Sigma = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}, \quad \gamma = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad (9)$$

with  $A, B, C \in \mathbb{R}^{2 \times 2}$  and  $\alpha, \beta \in \mathbb{R}^2$ . The covariance matrix must satisfy the uncertainty relation

$$\Sigma \geq i\Omega^{\otimes 2}, \quad (10)$$

where

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (11)$$

is the symplectic form.

We now focus on the properties of the control mode and denote the partial moments  $(C, \beta)$  as the *control moments*. From Eq. (10), the control moments must also satisfy the uncertainty constraint

$$C \geq i\Omega. \quad (12)$$

An important observation is that the state  $|G\rangle$  can be regarded as a purification of the control mode. Corresponding to the well-known fact that a purification of a quantum state is unique up to unitaries acting on the auxiliary system [48], an analogous result holds for Gaussian states, where the unitary can be restricted to Gaussian unitaries [47,49,50]. Tailored to our setting, we have the following theorem.

**Theorem 1 (canonical form).** Any pure two-mode Gaussian state  $|G\rangle$  can be expressed in the following *canonical form* [Fig. 3(b)]:

$$|G\rangle = (\hat{U}_s \otimes \hat{U}_c) |\text{TMSS}(a)\rangle, \quad (13)$$

where

$$|\text{TMSS}(a)\rangle = \frac{2\sqrt{a}}{a+1} \sum_{j=0}^{\infty} \left( \frac{a-1}{a+1} \right)^j |j\rangle |j\rangle \quad (14)$$

is a two-mode squeezed state and  $\hat{U}_s$  and  $\hat{U}_c$  are Gaussian unitaries acting on the signal and control modes,

respectively. The parameter  $a > 1$  depends only on  $C$ , and  $\hat{U}_c$  can be chosen depending only on  $(C, \beta)$ .

*Proof.* See Appendix D. ■

A two-mode non-Gaussian state generator is specified by  $|G\rangle$  together with the measurement outcome  $n$ , which sets the stellar rank of the output state. From Theorem 1,  $|G\rangle$  depends only on a Gaussian unitary  $\hat{U}_s$  and the control moments  $(C, \beta)$ . Hence, a two-mode non-Gaussian state generator is completely characterized by the set  $(C, \beta, n, \hat{U}_s)$ , as summarized below.

**Corollary 1 (control-mode representation).** For a two-mode non-Gaussian state generator with measured photon number  $n$  and control moments  $(C, \beta)$ ,

- (i) the output non-Gaussian state is determined by  $(C, \beta, n)$ , up to Gaussian unitaries;
- (ii) the success probability of the state generation is determined by  $(C, \beta, n)$ .

*Proof.* The first statement follows from Theorem 1. The second follows because the probability of measuring  $n$  photons depends only on the control moments. ■

Therefore, for a fixed stellar rank  $n$  (i.e., a fixed measurement outcome  $n$ ), the control moments  $(C, \beta)$  completely determine the non-Gaussian features of the output state—somewhat paradoxically, this information is contained entirely in the control-mode subsystem, even though the non-Gaussian state itself is generated in the signal mode. We henceforth identify a state generator with this set of parameters and simply write “non-Gaussian state generator  $(C, \beta, n, \hat{U}_s)$ ,” with the output state and the success probability denoted by  $|\psi\rangle_{C, \beta, n, \hat{U}_s}$  and  $p_n(C, \beta)$ , respectively. This representation in terms of the control moments is called the *control-mode representation*. Whenever the Gaussian unitary  $\hat{U}_s$  is clear from the context or not of primary interest, we simply write the state as  $|\psi\rangle_{C, \beta, n}$  and the generator as  $(C, \beta, n)$ , ignoring the Gaussian degree of freedom. Moreover, for states  $|\psi\rangle$  and  $|\psi'\rangle$ , we introduce the notations

$$|\psi\rangle \propto_G |\psi'\rangle, \quad (15)$$

$$|\psi\rangle \propto_G |\psi'\rangle \quad (16)$$

to denote that there exists a Gaussian unitary  $U$  such that  $|\psi\rangle \propto U|\psi'\rangle$ , either exactly or approximately, respectively.

## B. Rotation and damping

Although the control-mode representation  $(C, \beta, n, \hat{U}_s)$  uniquely specifies the non-Gaussian output state, the corresponding state generator is not necessarily unique for a given output state. In fact, let us consider the phase-rotation operator

$$\hat{R}(\theta) = e^{-i\theta\hat{n}} \quad (17)$$

with  $\theta \in \mathbb{R}$ , which corresponds to a Gaussian unitary operation, and the damping operator

$$\hat{\Gamma}(\lambda) = e^{-\lambda\hat{n}} \quad (18)$$

with  $\lambda \in \mathbb{R}$ , which is a purity-preserving Gaussian filter [51] also known as noiseless linear attenuation ( $\lambda > 0$ ) or amplification ( $\lambda < 0$ ) [52–58]. Since both operators are diagonal in the Fock basis, inserting either of these operators immediately before the photon-number measurement yields a new two-mode state generator that produces the same output state as the original state generator [Fig. 5(a)]. Importantly, this modification can be implemented in practice by appropriately adjusting the control moments  $(C, \beta)$ , without physically applying these operators in the laboratory. This observation leads to the following theorems.

**Theorem 2 (rotation transformation).** For a two-mode non-Gaussian state generator  $(C, \beta, n, \hat{U}_s)$ , the output state and the success probability remain invariant:

$$|\psi\rangle_{(C', \beta', n, \hat{U}_s)} \propto |\psi\rangle_{(C, \beta, n, \hat{U}_s)}, \quad (19)$$

$$p_n(C', \beta') = p_n(C, \beta), \quad (20)$$

under the transformation

$$C' = OCO^T, \quad (21)$$

$$\beta' = O\beta, \quad (22)$$

where  $O \in SO(2)$ .

*Proof.* This transformation of the control moments corresponds to inserting a rotation operator  $\hat{R}(\theta)$  on the control mode. The theorem then follows from

$$|\psi\rangle_{(C', \beta', n, \hat{U}_s)} \propto {}_c\langle n | \hat{I}_s \otimes \hat{R}(\theta) | G \rangle \quad (23)$$

$$= e^{-i\theta n} {}_c\langle n | G \rangle \quad (24)$$

$$\propto e^{-i\theta n} |\psi\rangle_{(C, \beta, n, \hat{U}_s)}. \quad (25)$$

**Theorem 3 (damping transformation).** Let  $I$  denote the identity matrix. For a two-mode non-Gaussian state generator  $(C, \beta, n, \hat{U}_s)$  and any  $t \in \mathbb{R}$  satisfying

$$t > 1 \quad \text{or} \quad tI < -C, \quad (26)$$

under the following *damping transformation* of the control moments:

$$C' = \mathcal{D}_t(C) := (tC + I)(C + tI)^{-1}, \quad (27)$$

$$\beta' = \mathcal{D}_t(\beta) := \sqrt{t^2 - 1}(C + tI)^{-1}\beta, \quad (28)$$

there exists a Gaussian unitary  $\hat{U}'_s$  such that

$$|\psi\rangle_{(C', \beta', n, \hat{U}'_s)} \propto |\psi\rangle_{(C, \beta, n, \hat{U}_s)}. \quad (29)$$

*Proof.* This follows in the same way as Theorem 2, from the invariance of the output state under insertion of the damping operator  $\hat{\Gamma}(\lambda)$  with  $\lambda = \coth^{-1} t$ . The complete proof is given in Appendix E. ■

Note that under the “Cayley transform”

$$\tilde{C} = (C + I)^{-1}(C - I), \quad (30)$$

$$\tilde{\beta} = (C + I)^{-1}\beta, \quad (31)$$

$$\tilde{t} = (t + 1)^{-1}(t - 1), \quad (32)$$

Eqs. (27) and (28) simplify to

$$\tilde{C}' = \tilde{t}\tilde{C}, \quad \tilde{\beta}' = \sqrt{\tilde{t}}\tilde{\beta}. \quad (33)$$

(See Appendix F for the proof.) While the damping transformation preserves the output state, it generally modifies the success probability of state generation. This degree of freedom plays a crucial role in enhancing the success probability, as discussed in Sec. VI.

### C. Non-Gaussian control parameters $s_0, \delta_0$

According to Corollary 1, for a fixed measured photon number  $n$ , the output state of a two-mode non-Gaussian state generator can be fully characterized (up to Gaussian unitaries) by the control moments  $(C, \beta)$ , which together have five real degrees of freedom. By applying the rotation transformation (Theorem 2) and the damping transformation (Theorem 3), these degrees of freedom can be reduced to three parameters that uniquely characterize the non-Gaussian features of the output state.

Specifically,  $C$  can always be diagonalized as

$$C = O^T \text{diag}(c, d) O, \quad (34)$$

with  $c \geq d$  and  $O \in SO(2)$ . Using the rotation transformation (Theorem 2),  $C$  can be brought to diagonal form  $C = \text{diag}(c, d)$ .

Then, we find two invariant parameters under the damping transformation (Theorem 3), namely, the *non-Gaussian phase sensitivity*

$$s_0 = \frac{c - d}{cd - 1} \quad (35)$$

and the *non-Gaussian asymmetry*

$$\delta_0 = \frac{2}{\sqrt{cd-1}} \left( \sqrt{\frac{d+1}{c+1}} \tilde{\beta}_x - i \sqrt{\frac{c+1}{d+1}} \tilde{\beta}_p \right), \quad (36)$$

where  $(\tilde{\beta}_x, \tilde{\beta}_p)^T$  is defined by

$$\begin{pmatrix} \tilde{\beta}_x \\ \tilde{\beta}_p \end{pmatrix} := O\beta, \quad (37)$$

using the same  $O$  as in Eq. (34).

These parameters  $(s_0, \delta_0)$  provide a description of the output state equivalent to the control-mode representation  $(C, \beta)$ , as summarized in the following theorem.

**Theorem 4 (non-Gaussian control parameters).** The output state of a two-mode non-Gaussian state generator  $(C, \beta, n)$  is uniquely determined up to Gaussian unitary operations by the triplet  $(s_0, \delta_0, n)$ . Furthermore, two sets of control moments  $(C, \beta)$  and  $(C', \beta')$  that yield the same  $(s_0, \delta_0)$  can be transformed into each other via rotation and damping transformations.

*Proof.* See Appendix J. ■

Figure 4 illustrates the relation between the eigenvalues  $c \geq d$  of  $C$  and the corresponding value of  $s_0$ , providing an intuitive visualization of this definition.

We call the pair  $(s_0, \delta_0)$  the *non-Gaussian control parameters*. These parameters fully characterize the output state up to a Gaussian unitary. When there is no risk of confusion, we further write  $|\psi\rangle_{C, \beta, n}$  simply as  $|\psi\rangle_{s_0, \delta_0, n}$ , again omitting the dependence on  $\hat{U}_s$ . The state can then be

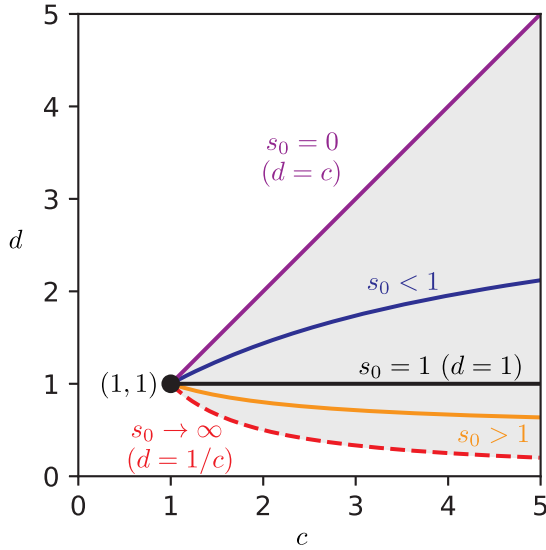


FIG. 4. Relation between the eigenvalues  $c \geq d$  of  $C$  and the non-Gaussian phase sensitivity  $s_0$  defined in Eq. (35). Solid curves show contours of constant  $s_0$ , while the gray shaded region marks the physically allowed domain satisfying Eq. (12) with  $c \geq d$ . The point (1,1) corresponds to the vacuum state in the control mode.

expressed explicitly in terms of  $(s_0, \delta_0, n)$  in several equivalent forms.

**Theorem 5 (particle form).** The output state of a two-mode non-Gaussian state generator with control parameters  $(s_0, \delta_0)$  and measured photon number  $n$  can be written as

$$|\psi\rangle_{s_0, \delta_0, n} \propto_G |\psi\rangle_{s_0, \delta_0, n}^{(p)} := (\hat{a}^\dagger + s_0 \hat{a} + \delta_0)^n |0\rangle. \quad (38)$$

*Proof.* See Appendix H. ■

**Theorem 6 (wave form).** The output state of a two-mode non-Gaussian state generator with control parameters  $(s_0, \delta_0)$  and measured photon number  $n$  can be written as

$$\begin{aligned} |\psi\rangle_{s_0, \delta_0, n} &\propto_G |\psi\rangle_{s_0, \delta_0, n}^{(w)} \\ &:= \exp\left(-\frac{1}{2\sqrt{s_0+1}} \delta_{0x} \hat{p}\right) \\ &\quad \cdot \exp\left(\frac{\sqrt{s_0+1}}{2} \delta_{0p} \hat{x}\right) \exp\left(-\frac{s_0}{4} \hat{x}^2\right) |n\rangle, \end{aligned} \quad (39)$$

where  $|n\rangle \propto \hat{a}^{\dagger n} |0\rangle$  is the Fock state.

*Proof.* See Appendix H. ■

We refer to  $|\psi\rangle_{s_0, \delta_0, n}^{(p)}$  and  $|\psi\rangle_{s_0, \delta_0, n}^{(w)}$  as the *particle form* and the *wave form*, respectively. By construction, these two forms are related by a Gaussian unitary, which is detailed in Appendix H. We remark that the relation between the particle form and the wave form is reminiscent of the wave-particle duality in quantum mechanics. The particle form corresponds to the “core state” [23–25], a superposition of Fock states up to  $n$  photons that explicitly displays its stellar rank  $n$ . This picture connects naturally to the intuition behind schemes such as photon subtraction [33,34], photon addition [37,38], and coherent superpositions of them [39,40], all of which fall within our general framework. Indeed, as we see in Sec. III D, the output states can be expressed as photon-subtracted, photon-added, or balanced superpositions of the two, applied to a Gaussian state, depending on the regime of  $s_0$ .

In contrast, the wave form represents the state as  $|n\rangle$  acted on by Gaussian filters [51] in the quadratures  $\hat{x}$  and  $\hat{p}$ , producing a “filtered” wave function in the quadrature basis. This representation is particularly suited for analyzing coherent structures in phase space. Hence, wavefunction engineering schemes [59,60] and breeding protocols based on homodyne conditioning [61–63] are naturally related to this picture and can be reduced to our framework by including the homodyne conditioning into the Gaussian state preparation (see Appendix M 3 for the example of GKP breeding [62]). The wave form picture, thus, provides a natural interpretation of these schemes and also serves as the basis for the approximation of the output state in Sec. IV B.

Moreover, for  $n \geq 2$ , the parameters  $(s_0, \delta_0)$  associated with a given output state are unique, up to the trivial transformation  $\delta_0 \rightarrow -\delta_0$ .

*Theorem 7 (uniqueness of control parameters).* For  $n \geq 2$ , if  $|\psi\rangle_{s_0, \delta_0, n} \propto |\psi\rangle_{s'_0, \delta'_0, n}$ , then either  $s_0 = s'_0 = 0$  with  $|\delta_0| = |\delta'_0|$  or  $(s_0, \delta_0) = (s'_0, \pm \delta'_0)$ .

*Proof.* See Appendix I. ■

This shows that, for  $n \geq 2$ , the control parameters  $(s_0, \delta_0)$  can be regarded as intrinsic properties of the output states, even though they were originally defined through state generators. By contrast, when  $n = 1$ , the particle form of the output state is

$$|\psi\rangle_{s_0, \delta_0, 1}^{(p)} \propto |1\rangle + \delta_0 |0\rangle \quad (40)$$

$$\propto \hat{R}(\theta)(|1\rangle + |\delta_0| |0\rangle), \quad (41)$$

where  $\hat{R}(\theta)$  is the phase-rotation operator. Thus, in this case, the state is specified by a single real parameter  $|\delta_0|$ .

#### D. Classification of the output state

We now classify the output states of two-mode non-Gaussian state generators according to the regime of the non-Gaussian control parameters  $(s_0, \delta_0)$ . Representative examples of non-Gaussian states for different values of  $(s_0, \delta_0)$  are shown in Fig. 1.

##### 1. $s_0 > 1$ : Photon-subtracted state

When  $s_0 > 1$ , there exists a Gaussian unitary  $\hat{U}$  such that

$$\hat{U}^\dagger \hat{a} \hat{U} \propto \hat{a}^\dagger + s_0 \hat{a} + \delta_0. \quad (42)$$

Hence, from Theorem 5, we obtain

$$|\psi\rangle_{s_0, \delta_0, n} \propto \hat{a}^n \hat{U} |0\rangle, \quad (43)$$

which shows that the state is, up to a Gaussian unitary, an  $n$ -photon-subtracted Gaussian state.

In particular, when  $\delta_0 = 0$ , the state reduces to a photon-subtracted squeezed vacuum state, which is well known to approximate the Schrödinger cat state

$$|\text{cat}_\pm(\alpha)\rangle \propto |\alpha\rangle \pm |-\alpha\rangle, \quad (44)$$

for large  $n$  [35,44]. This can be seen from the wave-function representation. For real  $\alpha$ , the  $x$ -wave function of the cat state  $|\text{cat}_\pm(i\alpha)\rangle$  is

$$\langle x | \text{cat}_+(i\alpha) \rangle \propto \cos(\alpha x) e^{-x^2/4}, \quad (45)$$

$$\langle x | \text{cat}_-(i\alpha) \rangle \propto \sin(\alpha x) e^{-x^2/4}. \quad (46)$$

On the other hand, from Theorem 6, the wave function of the generated state is

$$\langle x | \psi \rangle_{s_0, 0, n}^{(w)} \propto \phi_n(x) e^{-s_0 x^2/4}, \quad (47)$$

where  $\phi_n(x)$  denotes the wave function of the Fock state  $|n\rangle$ . Using the semiclassical approximation of  $\phi_n(x)$  around  $x = 0$  [64],

$$\phi_n(x) \propto \cos\left(\sqrt{n+1/2}x - \frac{n}{2}\pi\right), \quad (48)$$

we obtain

$$|\psi\rangle_{s_0, 0, n}^{(w)} \sim \hat{S} \left| \text{cat}_{\text{sgn}[(-1)^n]} \left( i\sqrt{(n+1/2)/s_0} \right) \right\rangle, \quad (49)$$

with the squeezing factor

$$\hat{S}^\dagger \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \hat{S} = \begin{pmatrix} 1/\sqrt{s_0} & 0 \\ 0 & \sqrt{s_0} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix}. \quad (50)$$

##### 2. $0 \leq s_0 < 1$ : Photon-added state

When  $0 \leq s_0 < 1$ , there exists a Gaussian unitary  $\hat{U}$  such that

$$\hat{U}^\dagger \hat{a}^\dagger \hat{U} = \hat{a}^\dagger + s_0 \hat{a} + \delta_0. \quad (51)$$

Hence, from Theorem 5, we obtain

$$|\psi\rangle_{s_0, \delta_0, n} \propto \hat{a}^{\dagger n} \hat{U} |0\rangle, \quad (52)$$

which shows that the state is, up to a Gaussian unitary, an  $n$ -photon-added Gaussian state. In particular,  $s_0 = \delta_0 = 0$  corresponds to the Fock state.

When  $s_0 = 0$  and  $|\delta_0|$  is sufficiently large, the state provides a good approximation of the CPS [10,43], defined as

$$|\text{CPS}\rangle = e^{-i\hat{x}^3} |p=0\rangle, \quad (53)$$

as illustrated in Fig. 1(b). As detailed in Appendix B, the asymptotic relation

$$|\psi\rangle_{0, \delta_0, n}^{(w)} \propto \hat{D}\left(\frac{i}{2}p_0\right) \hat{S}\left(\frac{1}{3}\ln \gamma\right) |\text{CPS}\rangle \quad (54)$$

holds for large  $n$ , where

$$p_0 = 2\sqrt{n + \frac{1}{2}}, \quad \gamma = \frac{1}{24\sqrt{n + \frac{1}{2}}}. \quad (55)$$

##### 3. $s_0 = 1$ : Critical state

At  $s_0 = 1$ , the state lies on the boundary between photon addition and subtraction:

$$|\psi\rangle_{1,\delta_0,n}^{(p)} \propto (\hat{x} + \delta_0)^n |0\rangle. \quad (56)$$

Its wave function is given by

$$\langle x | \psi_{1,\delta_0,n}^{(p)} \rangle \propto (x + \delta_0)^n e^{-x^2/4}. \quad (57)$$

In particular, when  $\delta_0 = 0$ , this state still provides a good approximation of a cat state for large  $n$ , while achieving a larger amplitude than the photon-subtracted cases with  $s_0 > 1$ :

$$|\psi\rangle_{1,0,n}^{(w)} \sim \left| \text{cat}_{\text{sgn}[-1]^n} \left( i\sqrt{n+1/2} \right) \right\rangle. \quad (58)$$

Reference [35] derived the condition  $s_0 = 1$  for the generation of large-amplitude cat states, referring to it as the *critical condition*, in the special case of two orthogonally squeezed vacuum states mixed by a phase-preserving beam splitter. They showed that Eq. (58) holds asymptotically for large  $n$ . Reference [18] later referred to this as the “monomial condition.” As a consequence of Theorem 4, we find that this condition is not restricted to that special case but, in fact, valid for any two-mode non-Gaussian state generator. In practice, however, Fig. 2(b) shows the corresponding optimal  $s_0$ , which is slightly shifted toward smaller values compared to the critical condition  $s_0 = 1$ .

#### IV. NON-GAUSSIAN CONTROL PARAMETERS AS A RESOURCE

In this section, we show that the non-Gaussian phase sensitivity  $s_0$ , introduced in the previous section, serves as a non-Gaussian resource for two reasons.

First, in the regime  $s_0 > 1$ , we show that  $s_0$  is non-decreasing under Gaussian maps. This monotonicity is a fundamental property in resource theory [65]. Second, a state with large  $s_0 > 1$  can be approximated by a state with smaller  $s_0$  and a smaller photon number  $n$ . This demonstrates that  $s_0$  quantifies the reduction in non-Gaussianity relative to the amount expected from the stellar rank of the state.

The following subsections elaborate on these two points.

##### A. Monotonicity of $s_0$

To establish  $s_0$  as a proper non-Gaussian resource, it must satisfy the monotonicity property [65]: Namely,  $s_0$  cannot be reduced solely through Gaussian operations. In this subsection, we demonstrate that  $s_0$  indeed satisfies this property in the regime  $s_0 > 1$ .

We derive a condition under which  $|\psi\rangle_{s_0,\delta_0,n}$  can be converted into  $|\psi\rangle_{s'_0,\delta'_0,n}$  using only Gaussian maps. Such a transformation is impossible with Gaussian unitaries alone, as shown in Theorem 7; however, it becomes possible when allowing more general Gaussian maps.

For example, from Theorem 6 we have

$$e^{-a\hat{x}^2} |\psi\rangle_{s_0,0,n}^{(w)} \propto |\psi\rangle_{s'_0,0,n}^{(w)}, \quad (59)$$

with

$$s'_0 = s_0 + 4a. \quad (60)$$

Here, the operator  $e^{-a\hat{x}^2}$  with  $a > 0$  corresponds to a physically realizable Gaussian filter [51], which can be implemented, for example, by mixing with a vacuum mode on a beam splitter followed by homodyne conditioning. This shows that, for  $\delta_0 = 0$ , it is always possible to implement a transformation that increases  $s_0$ .

The necessary and sufficient conditions for general convertibility are summarized in the following theorem.

**Theorem 8.** For  $n \geq 2$ , a state  $|\psi\rangle_{s_0,\delta_0,n}$  can be (probabilistically) transformed into  $|\psi\rangle_{s'_0,\delta'_0,n}$  by a Gaussian map, if and only if at least one of the following conditions is satisfied:

$$\begin{cases} 0 \leq s_0 < 1, \\ s'_0 > s_0, \\ s_0 = s'_0 = 1 \quad \text{and} \quad \delta_{0x} = 0. \end{cases} \quad (61)$$

*Proof.* See Appendix G. ■

This structure of convertibility is depicted in Fig. 1. The region  $0 \leq s_0 < 1$  includes the Fock state  $s_0 = \delta_0 = 0$ , and, within this region, all states are mutually convertible. In contrast, when  $s_0 \geq 1$ , the state can only be converted into a state with an equal or larger value of  $s_0$ . In other words,  $s_0$  is monotonic with respect to Gaussian maps in the regime  $s_0 \leq 1$ . This irreversible structure characterizes the role of  $s_0$  as a non-Gaussian resource.

##### B. Approximation with lower stellar rank

When  $s_0 \geq 1$  and  $\delta_0 = 0$ , the state closely approximates the cat state, as shown in Sec. III D. The amplitude of the cat state is an indicator of non-Gaussianity, as it determines the number of interference fringes in the Wigner function. However, from Eq. (49), this amplitude depends on both the measured photon number  $n$  and  $s_0$ . This implies that the non-Gaussianity of the state cannot be fully captured by  $n$  alone, which represents the stellar rank [23] of the state. To accurately quantify non-Gaussianity, both  $s_0$  and  $n$  must be taken into account.

The following proposition states that a state with large  $s_0$  can be effectively approximated by a state with a smaller  $s_0$  and a smaller photon number  $n$ .

**Proposition 2.** For given non-Gaussian control parameters  $(s_0, \delta_0)$ , measured photon number  $n$ , and any  $n' < n$ , if  $s_0$  is sufficiently large, then there exist parameters  $s'_0 < s_0$  and  $\delta'_0$  such that

$$|\psi\rangle_{s'_0, \delta'_0, n'} \propto_{\hat{G}} |\psi\rangle_{s_0, \delta_0, n}; \quad (62)$$

that is,  $|\psi\rangle_{s'_0, \delta'_0, n'}$  provides a good approximation of  $|\psi\rangle_{s_0, \delta_0, n}$  up to a Gaussian unitary.

Although Proposition 2 does not provide an exact bound on the precision of the approximation, here we provide evidence to support this proposition, analytically in the case of  $\delta_0 = 0$  and numerically in the case of  $\delta_0 \neq 0$ .

First, we consider the cases without displacement ( $\delta_0 = 0$ ). From the approximation with the cat state Eq. (49), when  $n$  and  $n'$  share the same parity (i.e.,  $n - n'$  is even), using

$$s'_0 = \frac{2n' + 1}{2n + 1} s_0, \quad (63)$$

we have the approximation

$$|\psi\rangle_{s_0, 0, n}^{(w)} \sim \hat{S} |\psi\rangle_{s'_0, 0, n'}^{(w)}, \quad (64)$$

where the squeezing factor is

$$\hat{S}^\dagger \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \hat{S} = \begin{pmatrix} \sqrt{\frac{s'_0}{s_0}} & 0 \\ 0 & \sqrt{\frac{s_0}{s'_0}} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix}. \quad (65)$$

This result indicates that a state with large  $s_0$  can be approximated by smaller photon number by reducing  $s_0$ .

Using the wave form representation (Theorem 6), this approximation extends to states with displacement ( $\delta_0 \neq 0$ ). Figure 5(b) illustrates this mechanism. From the wave form, we have

$$|\psi\rangle_{s_0, \delta_0, n}^{(w)} \propto \exp\left(-\frac{1}{2\sqrt{s_0+1}}\delta_{0x}\hat{p}\right) \cdot \exp\left[-\frac{s_0}{4}\left(\hat{x} - \frac{\sqrt{s_0+1}}{s_0}\delta_{0p}\right)^2\right] |n\rangle. \quad (66)$$

In the  $x$  representation, the operator

$$\exp\left[-\frac{s_0}{4}\left(\hat{x} - \frac{\sqrt{s_0+1}}{s_0}\delta_{0p}\right)^2\right] \quad (67)$$

acts as a filter, effectively “cutting out” the wave function around

$$x = x_0 = \frac{\sqrt{s_0+1}}{s_0}\delta_{0p}. \quad (68)$$

Around this point, a semiclassical approximation [66] yields

$$\phi_n(x) \propto \phi_{n'}(kx - d), \quad (69)$$

where the parameters  $k$  and  $d$  are chosen so that the local behavior of  $\phi_n(x)$  near  $x_0$  is matched by  $\phi_{n'}$ . A heuristic construction of such  $k$  and  $d$  for arbitrary  $x_0$  is given explicitly in Appendix K. Using these values, we define the Gaussian unitary corresponding to the coordinate transformation as

$$\hat{U}_{s'_0, \delta'_0, n' \rightarrow s_0, \delta_0, n} = \hat{S}(\log k) \hat{D}(d), \quad (70)$$

where  $\hat{S}(\zeta) = \exp[\frac{1}{2}(\zeta^* \hat{a}^2 - \zeta \hat{a}^{\dagger 2})]$  and  $\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$  denote the squeezing and displacement operators, respectively.

After rescaling the envelope, we obtain the approximation

$$|\psi\rangle_{s_0, \delta_0, n}^{(w)} \propto \hat{U}_{s'_0, \delta'_0, n' \rightarrow s_0, \delta_0, n} |\psi\rangle_{s'_0, \delta'_0, n'}^{(w)}, \quad (71)$$

with transformed control parameters

$$s'_0 = \frac{1}{k^2} s_0, \quad (72)$$

$$\delta'_0 = \sqrt{\frac{s_0 + 1}{s_0 + k^2}} (k^2 \delta_{0x} + i \delta_{0p}) - i \frac{s_0 d}{k \sqrt{s_0 + k^2}}. \quad (73)$$

In Appendix K, we outline a general method for determining  $k$  and  $d$ . In the specific case where  $\delta_0 = 0$  and  $n = n' \bmod 2$ , they can be analytically derived as

$$k = \sqrt{\frac{2n+1}{2n'+1}}, \quad d = 0, \quad (74)$$

corresponding to the approximation Eq. (64).

This approximation can also be expressed as

$${}_c \langle n | G \rangle \propto {}_c \langle n' | (\hat{I} \otimes \hat{M}_{C, \beta, n \rightarrow s'_0, \delta'_0, n'}) | G \rangle, \quad (75)$$

using a Gaussian filter  $\hat{M}_{C, \beta, n \rightarrow s'_0, \delta'_0, n'}$ , whose detailed expression is derived in Appendix K. The corresponding transformation of the control moments can be computed via the Choi-Jamiołkowski isomorphism (Appendix C), which we denote concisely as

$$(C, \beta) \mapsto \hat{M}_{C, \beta, n \rightarrow s'_0, \delta'_0, n'}(C, \beta), \quad (76)$$

with a slight abuse of notation. This indicates that the necessary change of the control moments to achieve the approximation can be conveniently computed by virtually applying the Gaussian filter to the control mode. This representation is useful in Sec. V, where we discuss the general cases of multiple control modes.

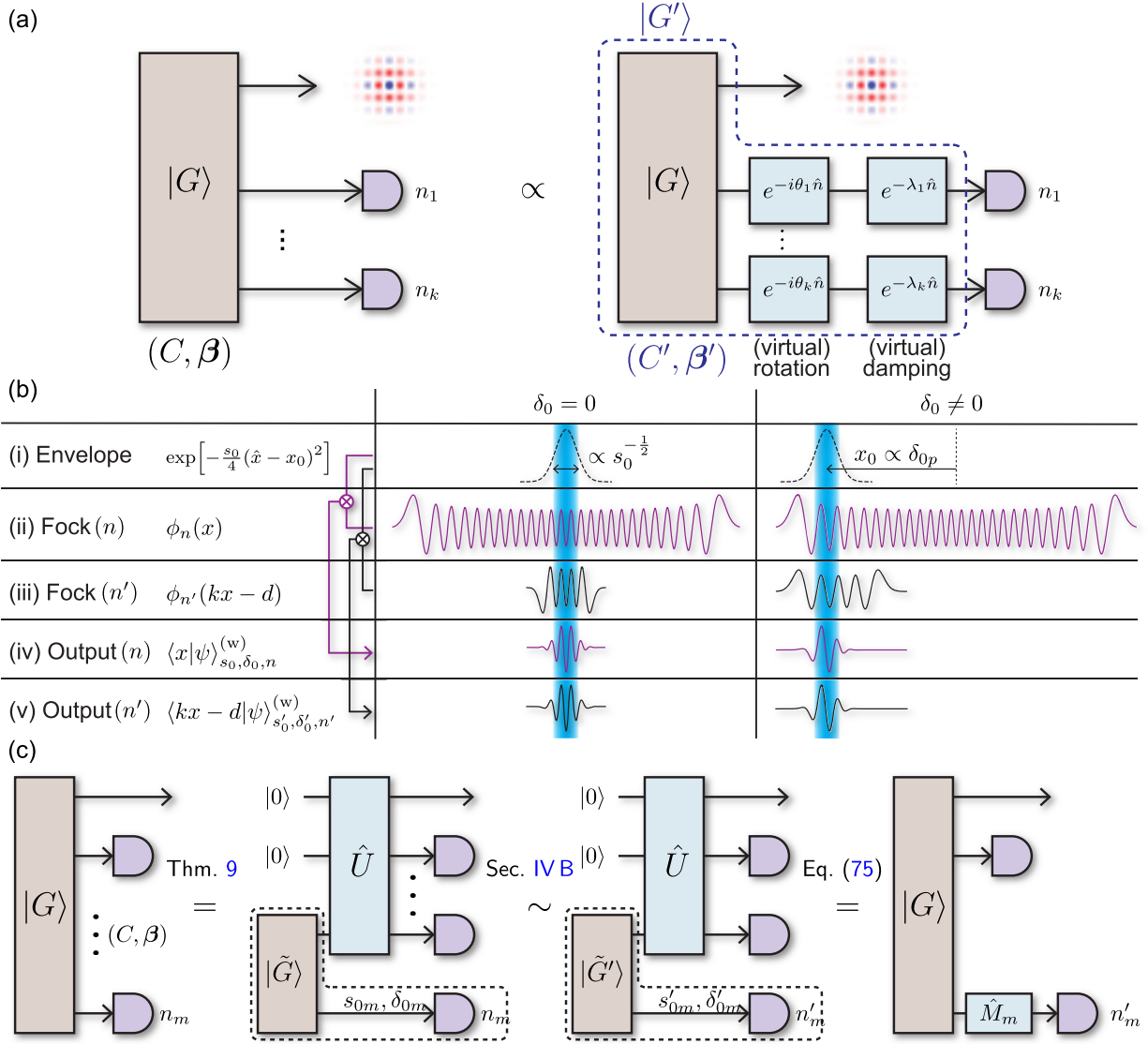


FIG. 5. Principles for optimizing non-Gaussian state generators. (a) Increasing the success probability of a multimode state generator without changing its output state (Sec. III B for the two-mode case and Sec. V B for the multimode case). A non-Gaussian state generator  $(C, \beta, n)$  can be transformed into another generator  $(C', \beta', n)$  by *virtually* applying rotation and damping operators (Theorems 2, 3, and 10). These transformations leave the output state invariant while potentially increasing the success probability. (b) Reducing the measured photon number for a two-mode state generator while approximately preserving the output state (Sec. IV B). In the wave-form representation (Theorem 6), the output wave function  $\langle x|\psi\rangle_{s_0, \delta_0, n}^{(w)}$  (iv) is given by the Fock-state wave function  $\phi_n(x)$  (ii) multiplied by a Gaussian envelope (i) determined by  $(s_0, \delta_0)$ . To approximate this with a lower photon number  $n'$ , the Fock-state wave function  $\phi_n(x)$  is replaced by  $\phi_{n'}(kx - d)$  (iii), which provides a good approximation near the center of the envelope. By rescaling the envelope through suitable choice of  $s'_0$  and  $\delta'_0$ , the resulting wave function  $\langle x|\psi\rangle_{s'_0, \delta'_0, n'}^{(w)}$  (v) closely approximates the original wave function (iv) up to extra Gaussian unitaries. (c) Reducing the measured photon number for a multimode state generator. For each control mode  $m$  with photon number  $n_m$  and partial control moment  $(C_m, \beta_m)$ , a corresponding two-mode generator  $(C_m, \beta_m, n_m)$  can be extracted as a subsystem (Theorem 11). By iteratively approximating each subsystem using the two-mode technique and applying the corresponding Gaussian filter  $\hat{M}_m$  [Eq. (97)], an approximation of the entire multimode state generator is obtained.

## V. MULTIMODE NON-GAUSSIAN STATE GENERATORS

Thus far, we have focused on two-mode non-Gaussian state generators. In this section, we generalize our framework to *multimode* non-Gaussian state generators including

multiple photon-number measurements, as illustrated in Fig. 3(c). This setup is sometimes referred to as Gaussian boson sampling [20] in the context of computational complexity, where the system exhibits exponential classical simulation cost.

In this scenario, an  $(l + k)$ -mode pure Gaussian state  $|G\rangle$  is prepared, where the last  $k$  modes (the control modes) are projected onto Fock states  $\otimes_m |n_m\rangle$ . This measurement probabilistically generates a non-Gaussian state in the first  $l$  modes (the signal modes), whose stellar rank is at most  $\sum_{m=1}^k n_m$  [16]. This scheme can produce a much broader range of non-Gaussian states compared to two-mode generators, enabling both multimode output states and single-mode states with more complex phase-space structures. An important single-mode example is the scheme for GKP state generation [17,19], which is discussed later in Sec. VI.

The control-mode representation and the non-Gaussian control parameters introduced in Sec. III extend naturally to this multimode setting, enabling systematic characterization and optimization of more complex non-Gaussian state generators.

### A. Control-mode representation

In the setup in Fig. 3(c), we define  $\mathbf{q}_s = (x_{1s}, p_{1s}, \dots, x_{ls}, p_{ls})^T$  and  $\mathbf{q}_c = (x_{1c}, p_{1c}, \dots, x_{kc}, p_{kc})^T$  as the quadrature-operator vectors of the signal and control modes, respectively, and set  $\mathbf{q} = (\mathbf{q}_s^T, \mathbf{q}_c^T)^T$ . The mean vector  $\boldsymbol{\gamma} \in \mathbb{R}^{2(l+k)}$  and covariance matrix  $\Sigma \in \mathbb{R}^{2(l+k) \times 2(l+k)}$  are then defined as in Eqs. (7) and (8). They admit the block decomposition

$$\Sigma = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}, \quad (77)$$

where  $A \in \mathbb{R}^{2l \times 2l}$ ,  $B \in \mathbb{R}^{2k \times 2l}$ ,  $C \in \mathbb{R}^{2k \times 2k}$ , and  $\boldsymbol{\alpha} \in \mathbb{R}^{2l}$ ,  $\boldsymbol{\beta} \in \mathbb{R}^{2k}$ . Here,  $C$  needs to satisfy the uncertainty relation

$$C \geq i\Omega^{\otimes k}. \quad (78)$$

Similarly to the case of two-mode non-Gaussian state generators, we can define the control-mode representation of this non-Gaussian state generator using the following theorem.

**Theorem 9.** An arbitrary pure  $l + k$ -mode Gaussian state  $|G\rangle$  can be expressed as the following *canonical form*:

$$|G\rangle = (\hat{U}_s \otimes \hat{U}_c) \cdot \left[ \left( \bigotimes_{m=1}^r |\text{TMSS}(a_m)\rangle_{m,l+m} \right) \otimes |0\rangle_s^{\otimes l-r} \otimes |0\rangle_c^{\otimes k-r} \right], \quad (79)$$

where  $a_1 \geq \dots \geq a_r > 1, \dots, 1$  are the symplectic eigenvalues [67] of  $C$ , which are unique for a given  $C$  [Fig. 3(d)].  $r$  is called the Schmidt rank of  $C$ . Here,

$$|\text{TMSS}(a)\rangle_{mm'} = \frac{2\sqrt{a}}{a+1} \sum_{j=0}^{\infty} \left( \frac{a-1}{a+1} \right)^j |j\rangle_m |j\rangle_{m'} \quad (80)$$

is a two-mode squeezed state, and  $\hat{U}_s$  and  $\hat{U}_c$  are Gaussian unitary operators acting on the signal modes and the control modes, respectively.  $\hat{U}_c$  can be chosen depending only on  $C, \boldsymbol{\beta}$ .

*Proof.* See Appendix D. ■

Theorem 9 allows us to represent the non-Gaussian state generator by the control-mode representation  $(C, \boldsymbol{\beta}, \mathbf{n}, \hat{U}_s)$ , similarly to the two-mode case. We have the following corollary, which is the multimode analog of Corollary 1.

**Corollary 2.** For a multimode non-Gaussian state generator  $(C, \boldsymbol{\beta}, \mathbf{n}, \hat{U}_s)$ ,

- (i) the output non-Gaussian state is determined by  $(C, \boldsymbol{\beta}, \mathbf{n})$ , up to Gaussian unitary operations;
- (ii) the success probability of the state generation is determined by  $(C, \boldsymbol{\beta}, \mathbf{n})$ , which we denote as  $p_n(C, \boldsymbol{\beta})$ .

*Proof.* It follows from Theorem 9. ■

Note that the number of output modes  $l$  is not solely determined by  $(C, \boldsymbol{\beta}, \mathbf{n})$ , but the states with the same  $(C, \boldsymbol{\beta}, \mathbf{n})$  and different  $l$  are convertible by adding vacuum ancillary modes and applying a Gaussian unitary operation.

### B. Damping transformation

As shown by Theorem 3, inserting the damping operator Eq. (81) before the photon-number measurement does not change the output state but alters the probability of success. By generalizing this to multiple control modes, it can be seen that inserting the damping operators to all modes [Fig. 5(a)] does not change the output state, only changing the success probability. The explicit form of the transformation of the state generator is given by the following theorem.

**Theorem 10.** For a real vector  $\mathbf{t} = (t_1, \dots, t_k)^T$ , the following *damping transformation* of the control-mode representation

$$\begin{aligned} (C, \boldsymbol{\beta}) &\rightarrow \mathcal{D}_t(C, \boldsymbol{\beta}) \\ &:= (T - \sqrt{T^2 - 1}(C + T)^{-1} \sqrt{T^2 - 1}, \\ &\quad \sqrt{T^2 - 1}(C + T)^{-1} \boldsymbol{\beta}) \end{aligned} \quad (81)$$

does not change the output state up to a unitary Gaussian operation while changing the success probability  $p_n(C, \boldsymbol{\beta})$ .

Here,  $T$  is a diagonal matrix with paired entries

$$T = \text{diag}(t_1, t_1, \dots, t_k, t_k). \quad (82)$$

The domain of  $\mathbf{t}$  is given in Appendix E.

*Proof.* It suffices to show that the damping operation transforms  $(C, \boldsymbol{\beta})$  according to Eq. (81). The detailed proof including the condition on  $\mathbf{t}$  is given in Appendix E. ■

Note that, under the Cayley transform

$$\tilde{C} = (C + I)^{-1}(C - I), \quad (83)$$

$$\tilde{\beta} = (C + I)^{-1}\beta, \quad (84)$$

$$\tilde{T} = (T + I)^{-1}(T - I), \quad (85)$$

Eq. (81) can be simplified as

$$\mathcal{D}_t(C, \beta) = (C', \beta'), \quad (86)$$

$$\tilde{C}' = \sqrt{\tilde{T}}\tilde{C}\sqrt{\tilde{T}}, \quad \tilde{\beta}' = \sqrt{\tilde{T}}\tilde{\beta}. \quad (87)$$

(See Appendix F for the proof.)

In Sec. VI, we use this degree of freedom to maximize the success probability for given photon numbers  $\mathbf{n}$ .

### C. Non-Gaussian control parameters

Finally, we extend the definition of non-Gaussian control parameters, previously defined for two-mode state generators in Sec. III C, to general multimode cases. Let  $C_m$  and  $\beta_m$  be the partial matrix and vector of  $C$  and  $\beta$ , respectively, corresponding to the  $m$ th control mode:

$$C_m = (C_{ij})_{i=2m-1, 2m, j=2m-1, 2m}, \quad (88)$$

$$\beta_m = (\beta_i)_{i=2m-1, 2m}. \quad (89)$$

The non-Gaussian control parameters are defined as vectors  $s_0 \in \mathbb{R}^l$  and  $\delta_0 \in \mathbb{C}^l$ , whose  $m$ th components  $s_{0m}$  and  $\delta_{0m}$  are defined similarly to the two-mode case [Eqs. (35) and (36)]:

$$s_{0m} = \frac{d_m - c_m}{c_m d_m - 1}, \quad (90)$$

$$\delta_0 = \frac{2}{\sqrt{c_m d_m - 1}} \left( \sqrt{\frac{d_m + 1}{c_m + 1}} \tilde{\beta}_{mx} - i \sqrt{\frac{c_m + 1}{d_m + 1}} \tilde{\beta}_{mp} \right). \quad (91)$$

Here,  $c_m \geq d_m$  are the eigenvalues of  $C_m$ , and we define  $\tilde{\beta}_{xm}$  and  $\tilde{\beta}_{pm}$  as

$$O\beta_m = \begin{pmatrix} \tilde{\beta}_{xm} \\ \tilde{\beta}_{pm} \end{pmatrix}, \quad (92)$$

using the orthogonal matrix  $O$  diagonalizing  $C_m$  as

$$C_m = O^T \begin{pmatrix} c_m & 0 \\ 0 & d_m \end{pmatrix} O. \quad (93)$$

Note that this definition of non-Gaussian control parameters is not invariant under the damping transformation.

When invariance is required, one can instead adopt the *invariant non-Gaussian control parameters*  $\tilde{s}_0$  and  $\tilde{\delta}_0$ , defined in Appendix L.

The following theorem is key to reducing measured photon numbers in multimode cases.

*Theorem 11.* The output non-Gaussian state of a multimode non-Gaussian state generator can be written as

$$\left( \bigotimes_{m' \neq m} \langle n_{m'} |_{m'} \right) \hat{U} \left( \bigotimes_{m' \neq m} |0\rangle_{m'} \right) \otimes |\psi\rangle_{c_{0m}, \beta_{0m}, n_m}^{(p)} \quad (94)$$

as illustrated in Fig. 5(c), where  $\hat{U}$  is a Gaussian unitary operator acting only on the modes other than the  $m$ th mode.

*Proof.* Let  $|G\rangle$  be the Gaussian state for the multimode non-Gaussian state generator. By interpreting only the  $m$ th control mode as the control mode and the remaining modes as signal modes and applying Theorem 9, we obtain

$$|G\rangle = (\hat{U}' \otimes \hat{U}_m) \otimes (|0\rangle^{\otimes k+l-2} \otimes |\text{TMSS}(a_m)\rangle), \quad (95)$$

where  $\hat{U}_m$  is a Gaussian unitary operator acting only on the  $m$ th control mode and  $\hat{U}'$  acts on the other modes. Since

$$\langle n_m | \hat{U}_m | \text{TMSS}(a_m) \rangle \propto \hat{U}_s |\psi\rangle_{c_{0m}, \beta_{0m}, n_m}^{(p)} \quad (96)$$

for some  $\hat{U}_s$ , by defining  $\hat{U} = \hat{U}' \hat{U}_s$ , Eq. (94) can be obtained. ■

Theorem 11 states that the two-mode state generator with non-Gaussian phase sensitivity parameters  $s_{0m}$  and  $\delta_{0m}$  can be extracted as a subsystem of the total state generator. Thus, applying the approximation of  $|\psi\rangle_{s_{0m}, \delta_{0m}, n_m}^{(p)}$  obtained in Sec. IV B, an approximation of  $|\psi\rangle_{C, \beta, \mathbf{n}, \hat{U}_s}$  can be obtained. Note that the closeness of the approximation of  $|\psi\rangle_{s_{0m}, \delta_{0m}, n_m}^{(p)}$  does not guarantee the closeness of  $|\psi\rangle_{C, \beta, \mathbf{n}, \hat{U}_s}$ , as the remaining system involves projections to Fock states, which are nonunitary processes. Nevertheless, this approximation is practically useful, as demonstrated in the examples in Sec. VI.

When the approximation for the  $m$ th control mode is applied, the corresponding transformation of the system can be described by a (virtual) application of the Gaussian filter

$$\hat{M}_m = \hat{M}_{C_m, \beta_m, n_m \rightarrow s'_{0m}, \delta'_{0m}, n'_m} \quad (97)$$

defined in Eq. (75) to the  $m$ th control mode, whose exact form can be computed via the Choi-Jamiołkowski isomorphism (Appendix C). Again, note that this change can be experimentally implemented simply by modifying the initial Gaussian state  $|G\rangle$  prior to the photon-number measurement, without actually performing the Gaussian filter operation [51]. By sequentially applying this “approximation filter” for all the control modes, we can

obtain a new non-Gaussian state generator with reduced measured photon numbers  $\mathbf{n}'$  but approximating the original output state. Note that the result of this process may depend on the order of the control modes in which the approximations are applied.

## VI. OPTIMIZATION OF NON-GAUSSIAN STATE GENERATORS

As an application of our formalism based on the non-Gaussian control parameters  $(s_0, \delta_0)$ , we propose a method to find a non-Gaussian state generator that produces approximately the same output state while requiring detection of fewer photons and achieving a higher success probability. Our method applies to general multimode non-Gaussian state generators as in Fig. 3(c). The optimization procedure is summarized in Fig. 6, and its key principles are illustrated in Fig. 5.

The algorithm takes as input an arbitrary non-Gaussian state generator  $(C, \beta, \mathbf{n}, \hat{U}_s)$  of the form in Fig. 3(c) together with target photon numbers  $\mathbf{n}'$  satisfying  $n'_m \leq n_m$ ,  $m = 1, \dots, k$ . It then outputs a new state generator  $(C', \beta', \mathbf{n}', \hat{U}'_s)$  with the target photon numbers and an improved success probability. The output state of the new generator closely approximates that of the original.

Our protocol consists of two parts: the reduction of detected photon numbers, based on the approximation in Sec. IV B, and the maximization of success probability, based on the damping operation in Sec. V B, as described below. Importantly, these procedure does not require a full simulation of the state generation process. Instead, it involves only manipulating the control moments, particularly the submatrix associated with a single control mode. This makes our approach significantly more tractable than

conventional brute-force methods that explore the full parameter space.

### A. Reduction of measured photon numbers

The first step reduces the measured photon numbers from  $\mathbf{n}$  to the target  $\mathbf{n}'$  while preserving the output state as faithfully as possible. For each control mode  $m = 1, \dots, k$ , we first compute its non-Gaussian control parameters  $(s_{0m}, \delta_{0m})$  as defined in Sec. V C. By Theorem 11, a two-mode non-Gaussian state generator with parameters  $(s_{0m}, \delta_{0m})$  can be extracted as a subsystem, as shown in Fig. 5(c).

Given this subsystem and the target photon number  $n'_m \leq n_m$  for the corresponding control mode, we choose new parameters  $(s'_{0m}, \delta'_{0m})$  such that the resulting wave function approximates the original one up to a Gaussian unitary:

$$|\psi\rangle_{s_{0m}, \delta_{0m}, n_m} \underset{G}{\propto} |\psi\rangle_{s'_{0m}, \delta'_{0m}, n'_m}, \quad (98)$$

using the correspondence established in Sec. IV B [see Fig. 5(b)]. The corresponding two-mode generator is then replaced by its reduced counterpart with  $(s'_{0m}, \delta'_{0m}, n'_m)$ , which amounts to updating the control moments via Eq. (75), thereby modifying  $(C, \beta)$ . The new control moment is calculated via the transformation

$$(C, \beta) \mapsto \hat{M}_m(C, \beta), \quad (99)$$

defined in Eq. (97). Iterating this procedure over all control modes yields a reduced state generator  $(C'', \beta'', \mathbf{n}', \hat{U}''_s)$  whose output state approximates that of the original generator but requires fewer detected photons.

### B. Maximization of success probability

The second step maximizes the success probability  $p_{\mathbf{n}'}$  of detecting the reduced photon number  $\mathbf{n}'$ . This is accomplished using the multimode damping transformation  $\mathcal{D}_t$  [Sec. V B, Fig. 5(a)]. We optimize over the damping parameters  $t$  to find

$$t^* = \arg \max_t p_{\mathbf{n}'}(\mathcal{D}_t(C'', \beta'')) \quad (100)$$

and apply the optimal transformation, yielding updated control moments

$$(C', \beta') = \mathcal{D}_{t^*}(C'', \beta''). \quad (101)$$

The resulting generator  $(C', \beta', \mathbf{n}', \hat{U}'_s)$  produces exactly the same output state as  $(C'', \beta'', \mathbf{n}', \hat{U}''_s)$  but with strictly higher success probability.

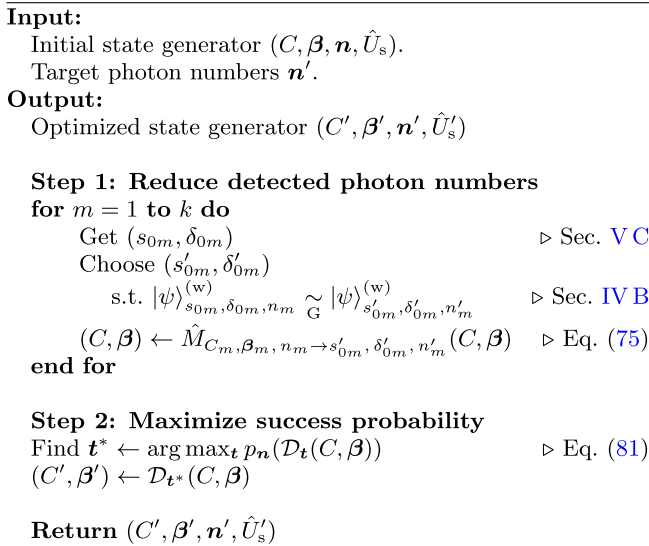


FIG. 6. Algorithm for optimizing non-Gaussian state generators.

|            | State generator | Output (Wigner) | Photon number $\mathbf{n}$ | Probability $p_{\mathbf{n}}$ | Control parameters $(s_0, \delta_0)$ | Non-Gaussian squeezing $\xi$ | Fidelity $F$ |
|------------|-----------------|-----------------|----------------------------|------------------------------|--------------------------------------|------------------------------|--------------|
| Cat (odd)  |                 | Bef.            | 15                         | $1.77 \times 10^{-6}$        | (3.12, 0)                            | 0.158                        | 0.9986       |
|            |                 | Aft.            | 5                          | $4.58 \times 10^{-2}$        | (1.11, 0)                            | 0.165                        |              |
| Cat (even) |                 | Bef.            | 16                         | $8.29 \times 10^{-7}$        | (3.12, 0)                            | 0.151                        | 0.9983       |
|            |                 | Aft.            | 6                          | $3.84 \times 10^{-2}$        | (1.23, 0)                            | 0.155                        |              |
| CPS        |                 | Bef.            | 20                         | $2.19 \times 10^{-8}$        | (0, 1.41i)                           | 0.315                        | 0.9964       |
|            |                 | Aft.            | 7                          | $7.43 \times 10^{-2}$        | (0, 1.19i)                           | 0.330                        |              |
| GKP        |                 | Bef.            | (18, 18, 18)               | $1.75 \times 10^{-12}$       | (3.37, 0)                            | 0.429                        | 0.993        |
|            |                 | Aft.            | (6, 6, 6)                  | $1.44 \times 10^{-4}$        | (1.14, 0)                            | 0.426                        |              |
| Random     |                 | Bef.            | (14, 14, 14, 14)           | $1.79 \times 10^{-30}$       | —                                    | —                            | 0.911        |
|            |                 | Aft.            | (9, 14, 2, 12)             | $4.50 \times 10^{-6}$        | —                                    | —                            |              |

FIG. 7. Examples of optimizing non-Gaussian state generators using the proposed algorithm (“bef.” = before optimization; “aft.” = after optimization). For the GKP case, the non-Gaussian control parameters  $(s_0, \delta_0)$  are shown only for one mode, since all control modes take the same value due to system symmetry. To quantify the quality of the states before and after optimization, suitable metrics are chosen for each case (except the random example):  $x^2$  squeezing [29] for cat states, cubic nonlinear squeezing [28] for CPS, and GKP nonlinear squeezing [30] for GKP states. These values are shown in the column labeled “non-Gaussian squeezing  $\xi$ .” The fidelity indicates the overlap between the initial and optimized states. See Appendix M for details of the system construction as well as the parameters and performance.

Thus, the algorithm faithfully preserves the target state while making the generation process experimentally more efficient.

### C. Examples

In this section, we demonstrate the performance of our optimization algorithm on several representative and practically important non-Gaussian state generation tasks. The results are summarized in Fig. 7. The key metrics include

the measured photon number  $\mathbf{n}$  and the success probability  $p_{\mathbf{n}}$  before and after optimization, the corresponding change in the non-Gaussian control parameters  $(s_0, \delta_0)$ , and the fidelity between the initial and optimized output states. For cat states, CPS, and GKP states, we also evaluate tailored metrics collectively referred to as the non-Gaussian squeezing  $\xi$  [28–30]. Numerical simulations of the state generation circuits were carried out using the open-source software MRMUSTARD [68].

In all cases, we observe a reduction of the measured photon number and an enhancement of the success probability, while maintaining a high fidelity ( $> 90\%$ ) and keeping the non-Gaussian squeezing nearly invariant. The details of the construction of the examples, the definitions of the metrics, and the optimization procedure are provided in Appendix M, which also includes the intermediate state generator obtained after the first step of photon-number reduction. Notably, as a general trend, we find that a noticeable enhancement of the success probability already occurs in this first step, almost without increasing the required squeezing. This behavior can be attributed to the exponentially decaying nature of the photon-number distribution of Gaussian states [69].

### 1. Cat state

Cat states, defined in Eq. (2), are widely used in quantum metrology and quantum error correction owing to their nonclassical features. Experimentally, they are commonly generated either by photon subtraction from squeezed vacuum states [31,33] or, more recently, through the generalized photon subtraction (GPS) scheme using two-mode squeezed states [36]. Both approaches fall within the class of schemes addressed by our method. In the following, we consider a GPS scheme as a starting point and examine whether further improvements can be achieved.

As in Eq. (49), different parity of the cat state is obtained depending on the parity of the detected photon number. Both odd and even cases are examined. As an input to the algorithm, we take a GPS system requiring 15- or 16-photon detection to generate a cat states. After optimization, we obtain a new configuration needing only five- or six-photon detection, achieving more than  $10^4$ -fold enhancement of probability, with the fidelity  $> 99.8\%$  for both cases. The  $x^2$  squeezing [29] defined in Eq. (4) is used as a tailored metric for evaluating the quality of the cat states, and only minor degradation is observed. Further details are provided in Appendix M. It is worth noting that in the first step of optimization, as well as the reduction of photon number, the success probability already improves by  $10^2$ , although the required squeezing level of the initial Gaussian state went down from (5.00 dB, 5.00 dB) to (3.46 dB, -4.22 dB), taking the odd-cat case as an example (see Appendix M, Fig. 10).

These improvements are attributed to the reduction in the non-Gaussian phase sensitivity parameter  $s_0$ , getting closer to the optimal value in Fig. 2(c).

### 2. Cubic phase state

CPS, defined in Eq. (5), is a crucial resource for universal continuous-variable quantum computation. We start from the CPS generation protocol proposed by Gottesman, Kitaev, and Preskill [10], which uses a displaced two-mode

squeezed state. With our algorithm, the photon number is reduced from 20 to 7, as well as enhancing the probability by  $10^6$ , while maintaining the fidelity  $> 99\%$ . The cubic nonlinear squeezing [28] defined in Eq. (6) is used as a tailored metric for evaluating the quality of the CPS, and only minor degradation is observed. Further details are provided in Appendix M.

This enhancement is attributed to the reduction of  $\delta_0$ , getting closer to the optimal value in Fig. 2(f).

### 3. GKP state

The GKP state [10], essential for bosonic quantum error correction, is defined as a superposition of displaced squeezed states:

$$|\text{GKP}\rangle = \sum_k \hat{D}(k\Delta) \hat{S}(r) |0\rangle. \quad (102)$$

It can be generated via the cat-breeding protocol [61,70], which combines multiple cat states using beam splitters and homodyne conditioning.

We start from a Gaussian breeding protocol that is also employed by the recent experimental demonstration [19], which can be obtained as a equivalent circuit of cat-breeding protocol. The detail of this reduction is found in Appendix M.

Applying our algorithm to this system, the photon number is reduced 3 times, and the probability enhances by  $10^8$ , while maintaining the fidelity  $> 99\%$ . GKP non-Gaussian squeezing [30] is used as a tailored metric for evaluating the quality of the GKP state, and only minor degradation is observed (see Appendix M for the definition).

This improvement can be attributed to the reduction of  $s_{0m}$ . A more detailed discussion of the trade-off between the control parameters and the photon number is provided in Appendix M.

### 4. Random state

To illustrate the general applicability of our method, we also optimize a randomly constructed non-Gaussian state generator. The details of the construction are provided in Appendix M. In this case as well, we observe a significant reduction in the required photon numbers together with an enhancement in the success probability, while maintaining the fidelity  $> 91\%$ .

## VII. CONCLUSION AND DISCUSSION

In summary, we have developed a framework to tackle the challenge of multimode non-Gaussian state generation, where brute-force exploration of the exponentially large parameter space is infeasible and traditional benchmarks such as stellar rank capture only part of the picture. To move beyond these limitations, we introduced the

non-Gaussian control parameters  $(s_0, \delta_0)$ , which reveal hidden inefficiencies in resource use and provide both a continuous description of non-Gaussianity beyond stellar rank and a practical handle for systematic optimization.

Using this framework, we constructed an optimization method that achieves striking gains in feasibility. Photon-number requirements are reduced by up to a factor of 3—depending on the quality of the initial state generator and potentially exceeding this value for less efficient cases—while heralding probabilities increase by several orders of magnitude across key resource states including Schrödinger cat states, CPS, and GKP states. The approach also generalizes to randomly generated targets, underscoring its universality. In particular, our framework provides a practical pathway to improving state-of-the-art experiments, such as the recent demonstration of GKP state generation [19], by reducing photon-number requirements and enhancing success probabilities in a tractable way. Taken together, these results establish non-Gaussian control parameters as a versatile tool for diagnosing and enhancing non-Gaussian state generators, paving the way for more resource-efficient experiments and scalable optical quantum technologies capable of supporting fault-tolerant computation.

Looking ahead, two natural extensions of this work appear particularly promising. First, generalizing the framework to account for errors such as loss, and thereby to mixed states, is essential for realistic experimental considerations on the path toward resource-efficient and fault-tolerant optical quantum computing. In Appendix A, we provide a preliminary analysis in this direction, showing that our scheme extends naturally to scenarios involving mixed Gaussian states.

Second, optimizing non-Gaussian state generators is not only experimentally valuable, but also closely related to questions of computational complexity [20,22,71,72]. While simulating their full quantum behavior is widely believed to be classically intractable, as exemplified by Gaussian boson sampling [20], our ability to optimize such generators efficiently highlights an important distinction between simulation and optimization. This perspective offers new insight into the relationship between physical resources and computational hardness, helping to clarify which features of a system drive intractability and where the boundary lies between classically simulable regimes and those that exhibit genuine quantum advantage.

Finally, we note that there are several complementary approaches sharing the goal of systematically optimizing non-Gaussian state generation [16,73], as well as refining stellar-rank-based characterizations of non-Gaussianity [74,75]. Exploring the fundamental connections between these approaches and our framework based on non-Gaussian control parameters represents an interesting direction for future research.

## ACKNOWLEDGMENTS

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## DATA AVAILABILITY

The data that support the findings of this article are openly available [76].

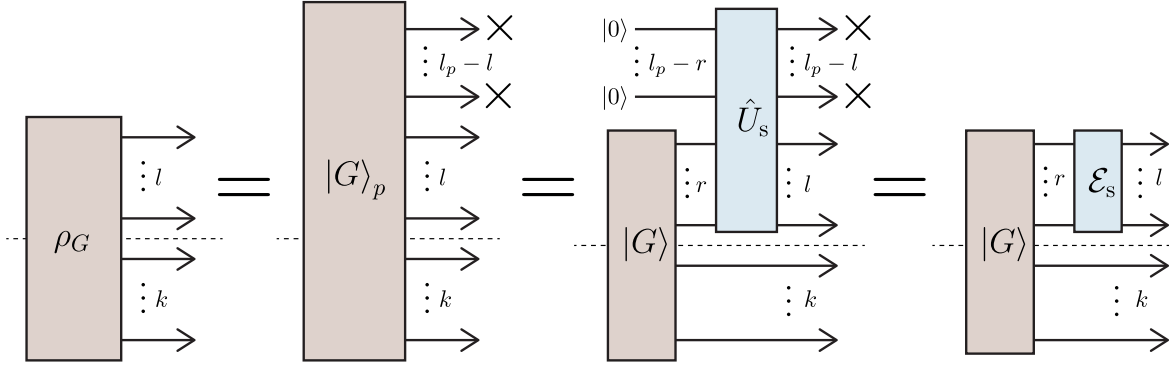
## APPENDIX A: MIXED STATE GENERATORS

In real-world experiments, non-Gaussian state generators inevitably suffer from imperfections such as loss, leading to mixed output states. The dominant errors in optical experiments, including loss, are Gaussian errors, which map a Gaussian state to another (generally mixed) Gaussian state. While we have focused on pure non-Gaussian state generators in the main text, the definition of the control-mode representation in Sec. VA is sufficiently general to also cover mixed non-Gaussian generators affected by Gaussian noise. In fact, the control moments  $(C, \beta)$  can be defined in exactly the same way, simply by dropping the assumption of a pure Gaussian state. This broader formulation enables the experimental evaluation of non-Gaussian control parameters for realistic state generators, which is crucial for applications. The following theorem is key for discussing mixed non-Gaussian state generators.

*Theorem A1.* For an arbitrary  $l + k$ -mode Gaussian state  $\hat{\rho}_G$  and a  $l' + k$ -mode pure Gaussian state  $|G\rangle$  having the same  $k$ -mode control moments  $(C, \beta)$ , there exists a Gaussian channel  $\mathcal{E}_s: \mathcal{D}(\mathcal{H}^{\otimes l'}) \rightarrow \mathcal{D}(\mathcal{H}^{\otimes l})$  acting only on the signal modes such that

$$\hat{\rho}_G = (\mathcal{E}_s \otimes \mathcal{I})(|G\rangle\langle G|). \quad (\text{A1})$$

*Proof.* Because all pure states with the same  $(C, \beta)$  are related by a Gaussian unitary transformation from Theorem 9, it suffices to consider the case  $l' = r$ , where

FIG. 8. Proof of Theorem A1. The symbol  $\times$  indicates tracing out the mode.

$r$  is the Schmidt rank of  $C$ . The proof is represented graphically in Fig. 8. Let  $|G\rangle_p$  be a Gaussian purification of  $\hat{\rho}_G$  with  $l_p + k$  modes. Since  $|G\rangle_p$  is also a purification of the control modes, Theorem 9 gives

$$|G\rangle_p = (\hat{U}_s \otimes \hat{I})(|0\rangle^{\otimes l_p - r} \otimes |G\rangle), \quad (\text{A2})$$

where  $\hat{U}_s: \mathcal{H}^{\otimes l_p} \rightarrow \mathcal{H}^{\otimes l_p}$  is a unitary operator acting on the signal modes. By tracing out the ancillary modes, we obtain Eq. (A1), where  $\mathcal{E}_s$  is defined as

$$\mathcal{E}_s(\rho) = \text{Tr}_a[\hat{U}(|0\rangle\langle 0|^{\otimes l_p - r} \otimes \rho)], \quad (\text{A3})$$

and  $\text{Tr}_a$  denotes the partial trace over  $l_p - l$  ancillary modes. ■

Mixed non-Gaussian output states can generally result from Gaussian noise acting on either the signal modes or the control modes. However, Theorem A1 shows that Gaussian noise on the control modes is equivalent to noise on the signal modes. Therefore, we can define the control-mode representation in the same way as in the pure-state case, as shown in the following corollary, corresponding to Corollary 2 for pure states.

**Corollary A1.** For a multimode (mixed) non-Gaussian state generator  $(C, \beta, n, \hat{U}_s)$ :

- (i) the output non-Gaussian state is determined by  $(C, \beta, n)$ , up to a Gaussian channel. Specifically, for an arbitrary output state  $\hat{\rho}_{(C, \beta), n}$  and a pure output state  $|\psi\rangle_{(C, \beta), n}$  with the same control moments, there exists a Gaussian channel  $\mathcal{E}_s$  such that

$$\hat{\rho}_{(C, \beta), n} = \mathcal{E}_s(|\psi\rangle_{(C, \beta), n}\langle\psi|_{(C, \beta), n}). \quad (\text{A4})$$

Accordingly, we denote the state as  $\hat{\rho}_{C, \beta, n, \mathcal{E}_s}$ .

- (ii) The success probability of the state generation is determined solely by  $(C, \beta, n)$ , which we denote by  $p_n(C, \beta)$ .

*Proof.* This follows directly from Theorem A1. ■

Furthermore, for the case of two-mode state generators, we have the following theorem, corresponding to Theorem 7.

**Theorem A2.** For  $n \geq 2$ , if  $\hat{\rho}_{s_0, \delta_0, n, \mathcal{E}_s} = \hat{\rho}_{s'_0, \delta'_0, n, \mathcal{E}'_s}$ , at least one of the following holds:

$$s_0 = s'_0 = 0 \quad \text{and} \quad |\delta_0| = |\delta'_0|, \quad (\text{A5})$$

$$(s_0, \delta_0) = (s'_0, \pm\delta'_0), \quad (\text{A6})$$

$$(s_0, \delta_0) = (s'_0, \pm\delta'^*_0). \quad (\text{A7})$$

*Proof.* We show this by proving the uniqueness of the polynomial part of the characteristic function. For a state  $\rho$ , the characteristic function is defined as

$$\chi(\xi) = \text{Tr}(\rho \hat{D}(\xi)). \quad (\text{A8})$$

The characteristic function of a state with stellar rank  $n$  can be written as

$$\chi(\xi) = G(\xi, \xi^*)P(\xi, \xi^*), \quad (\text{A9})$$

where  $G$  is a Gaussian function and  $P$  is a polynomial in  $\xi, \xi^*$  of total degree at most  $2n$ . Since the characteristic function of  $|n\rangle\langle m|$  for  $n \leq m$  is

$$\chi_{|n\rangle\langle m|}(\xi) = \langle m|\hat{D}(\xi)|n\rangle \quad (\text{A10})$$

$$= e^{-(1/2)|\xi|^2} \sqrt{\frac{n!}{m!}} (-\xi^*)^{m-n} L_n^{(m-n)}(|\xi|^2), \quad (\text{A11})$$

where  $L_n^{(\alpha)}$  is the generalized Laguerre polynomial, the (pure) particle form  $|\psi\rangle_{s_0, \delta_0, n}^{(p)}$  has

$$G_{s_0, \delta_0, n}(\xi, \xi^*) \propto \exp\left(-\frac{1}{2}|\xi|^2\right), \quad (\text{A12})$$

$$P_{s_0, \delta_0, n}(\xi, \xi^*) = |\xi|^{2n} + \mathcal{O}(|\xi|^{2n-1}), \quad (\text{A13})$$

where  $\mathcal{O}(|\xi|^{2n-1})$  denotes terms of total degree at most  $2n - 1$ .

The action of a Gaussian map  $\mathcal{E}$  on the characteristic function is

$$\chi(\xi) \mapsto G_{\mathcal{E}}(\xi, \xi^*) \chi(\alpha_{\mathcal{E}} \xi + \beta_{\mathcal{E}} \xi^*), \quad (\text{A14})$$

where  $\alpha_{\mathcal{E}}, \beta_{\mathcal{E}} \in \mathbb{C}$  and  $G_{\mathcal{E}}(\xi, \xi^*)$  is a Gaussian prefactor [47]. Suppose  $\hat{\rho}_{s_0, \delta_0, n, \mathcal{E}} = \hat{\rho}_{s'_0, \delta'_0, n, \mathcal{E}'}$ . Then, the polynomial parts must match:

$$\begin{aligned} & P_{s_0, \delta_0, n}(\alpha_{\mathcal{E}} \xi + \beta_{\mathcal{E}} \xi^*, (\alpha_{\mathcal{E}} \xi + \beta_{\mathcal{E}} \xi^*)^*) \\ & \propto P_{s'_0, \delta'_0, n}(\alpha_{\mathcal{E}'} \xi + \beta_{\mathcal{E}'} \xi^*, (\alpha_{\mathcal{E}'} \xi + \beta_{\mathcal{E}'} \xi^*)^*). \end{aligned} \quad (\text{A15})$$

From Eq. (A13), comparing leading terms gives

$$\alpha_{\mathcal{E}'} \xi + \beta_{\mathcal{E}'} \xi^* = k(\alpha_{\mathcal{E}} \xi + \beta_{\mathcal{E}} \xi^*) \quad (\text{A16})$$

for some  $k \in \mathbb{C}$ , or

$$\alpha_{\mathcal{E}'} \xi + \beta_{\mathcal{E}'} \xi^* = k'(\alpha_{\mathcal{E}} \xi + \beta_{\mathcal{E}} \xi^*)^* \quad (\text{A17})$$

for some  $k' \in \mathbb{C}$ . In the first case,

$$P_{s_0, \delta_0, n}(\xi, \xi^*) \propto P_{s'_0, \delta'_0, n}(k\xi, (k\xi)^*), \quad (\text{A18})$$

so that

$$\begin{aligned} \chi_{s_0, \delta_0, n}(\xi, \xi^*) & \propto \chi_{s'_0, \delta'_0, n}(k\xi, k^* \xi^*) \\ & \cdot \exp\left[-\frac{1}{2}(1 - |k|^2)|\xi|^2\right]. \end{aligned} \quad (\text{A19})$$

This corresponds to the state  $|\psi\rangle_{s_0, \delta_0, n}^{(p)}$  after a phase rotation by  $\arg k$  and a loss channel with efficiency  $|k|^2$ . Since the left-hand side is a pure state and not the vacuum, we must have  $|k| = 1$ . Then, by Theorem 7, either  $s_0 = s'_0 = 0$  and  $|\delta_0| = |\delta'_0|$  or  $(s_0, \delta_0) = (s'_0, \pm \delta'_0)$ .

In the second case Eq. (A17),

$$P_{s_0, \delta_0, n}(\xi, \xi^*) \propto P_{s'_0, \delta'_0, n}(k' \xi^*, k'^* \xi) \quad (\text{A20})$$

and

$$\chi_{s_0, \delta_0, n}(\xi, \xi^*) \propto \chi_{s'_0, \delta'_0, n}(k' \xi^*, k'^* \xi) \exp\left[-\frac{1}{2}(1 - |k|^2)|\xi|^2\right] \quad (\text{A21})$$

$$= \chi_{s'_0, \delta'_0, n}(k'^* \xi, k' \xi^*) \exp\left[-\frac{1}{2}(1 - |k|^2)|\xi|^2\right]. \quad (\text{A22})$$

Here, we used

$$\chi_{s_0, \delta_0, n}(\xi^*, \xi) = \chi_{s_0, \delta_0, n}(\xi, \xi^*), \quad (\text{A23})$$

since conjugating  $\xi \mapsto \xi^*$  corresponds to transposition  $\rho \mapsto \rho^T$  in the Fock basis, which is equivalent to

$\delta_0 \mapsto \delta_0^*$  in the particle form. From Eq. (A22), as before we obtain either  $s_0 = s'_0 = 0$  and  $|\delta_0| = |\delta'_0|$  or  $(s_0, \delta_0) = (s'_0, \pm \delta'_0)$ . Combining both cases yields the condition stated in the theorem. ■

Note that the third condition (A7), absent in the pure-state case, arises because of the existence of Gaussian channels that implement time reversal with added noise, such as the phase-conjugating channel [77]. The proof of Theorem A1 shows that the polynomial part of the characteristic function acts as a “fingerprint” of the non-Gaussian state, allowing the extraction of the control parameters  $s_0$  and  $\delta_0$  from experimentally accessible data.

Theorem A1 demonstrates that the non-Gaussian control parameters, determined by the shape of the characteristic function, are invariant under Gaussian errors acting on the signal modes. By contrast, Gaussian errors on the control modes generally modify the control moments  $(C, \beta)$  and, thus, change  $(s_0, \delta_0)$ . However, if the error channel is known, one can precompensate the control mode so that it yields the desired  $(s_0, \delta_0)$  after the errors.

As an illustrative example, consider a loss channel acting on a control mode close to the vacuum state, with

$$C = \begin{pmatrix} 1 + \Delta c & 0 \\ 0 & 1 + \Delta d \end{pmatrix}, \quad (\text{A24})$$

$$\beta = \begin{pmatrix} \Delta \beta_x & 0 \\ 0 & \Delta \beta_p \end{pmatrix}, \quad (\text{A25})$$

where  $\Delta c, \Delta d, \Delta \beta_x, \Delta \beta_p \ll 1$ . In this regime, we have

$$s_0 \sim \frac{|\Delta c - \Delta d|}{\Delta c + \Delta d}, \quad (\text{A26})$$

$$\delta_0 \sim \frac{2}{\sqrt{\Delta c + \Delta d}}(\Delta \beta_x - i \Delta \beta_p). \quad (\text{A27})$$

The effect of a loss channel with efficiency  $\eta$  on  $(C, \beta)$  is

$$C \mapsto \eta C + (1 - \eta)I, \quad (\text{A28})$$

$$\beta \mapsto \sqrt{\eta} \beta. \quad (\text{A29})$$

Equivalently, in terms of the small deviations,

$$\Delta c \mapsto \eta \Delta c, \quad (\text{A30})$$

$$\Delta d \mapsto \eta \Delta d, \quad (\text{A31})$$

$$\Delta \beta_x \mapsto \sqrt{\eta} \Delta \beta_x, \quad (\text{A32})$$

$$\Delta \beta_p \mapsto \sqrt{\eta} \Delta \beta_p. \quad (\text{A33})$$

Thus, both  $s_0$  and  $\delta_0$  remain invariant under loss. This observation reproduces the well-known fact that the effect

of loss on the control mode can be mitigated in the “weak-pump” limit.

## APPENDIX B: APPROXIMATION OF CPS

When  $s_0 = 0$  and  $|\delta_0| > 0$ , the state  $|\psi\rangle_{0,\delta_0,n}^{(w)}$  is a good approximation of CPS [10,43], defined as

$$|\text{CPS}\rangle = e^{-i\hat{x}^3} |p = 0\rangle. \quad (\text{B1})$$

Because of the symmetry at  $s_0 = 0$ , we can assume  $\delta_{0x} > 0$  and  $\delta_{0p} = 0$  without loss of generality. As shown in Ref. [10], using the higher-order approximation of the Fock wave function  $\phi_n(x)$  valid for  $[x/(2\sqrt{n+1/2})] \ll 1$  [64], one has

$$\phi_n(x) \simeq \cos\left(\sqrt{n+1/2}x - \frac{x^3}{24\sqrt{n+1/2}} + \mathcal{O}(x^5) - \frac{n}{2}\pi\right). \quad (\text{B2})$$

Substituting this into the wave form Eq. (39) yields

$$\langle x|\psi\rangle_{0,\delta_{0x},n}^{(w)} \quad (\text{B3})$$

$$\propto \phi_n(x + i\delta_{0x}) \quad (\text{B4})$$

$$\simeq \exp\left(-i\frac{p_0}{2}x\right) \exp(-kx^2) \exp(-i\gamma x^3), \quad (\text{B5})$$

where

$$p_0 = 2\sqrt{n+1/2} + \frac{\delta_{0x}^2}{4\sqrt{n+1/2}}, \quad (\text{B6})$$

$$k = \frac{\delta_{0x}}{8\sqrt{n+1/2}}, \quad (\text{B7})$$

$$\gamma = \frac{1}{24\sqrt{n+1/2}}. \quad (\text{B8})$$

If  $\delta_{0x} \gg [1/(2\sqrt{n+1/2})]$ , this approximation is valid within the Gaussian envelope  $\exp(-kx^2)$ . In this regime, we obtain

$$|\psi\rangle_{0,\delta_{0x},n}^{(w)} \simeq \hat{D}\left(\frac{i}{2}p_0\right) \hat{S}\left(\frac{1}{3}\ln\gamma\right) \exp(-k\gamma^{-2/3}x^2) |\text{CPS}\rangle, \quad (\text{B9})$$

where the envelope coefficient is

$$k\gamma^{-2/3} = \frac{9^{1/3}}{2} (n+1/2)^{-1/6} \delta_{0x}. \quad (\text{B10})$$

This factor vanishes asymptotically as  $n \rightarrow \infty$ , which implies that

$$|\psi\rangle_{0,\delta_{0x},n}^{(w)} \simeq \hat{D}\left(\frac{i}{2}p_0\right) \hat{S}\left(\frac{1}{3}\ln\gamma\right) |\text{CPS}\rangle, \quad (\text{B11})$$

in the large- $n$  limit.

When we instead assume  $\delta_{0x} = 0$  and  $\delta_{0p} > 0$ , the same conclusion can be seen in terms of the Fourier-transformed wave function of the CPS [78]:

$$\langle p|\text{CPS}\rangle \propto \text{Ai}\left(\frac{p}{2 \cdot 3^{1/3}}\right), \quad (\text{B12})$$

where Ai is the Airy function of the first kind. Using the asymptotic formula for the Fock wave function around the turning point  $x = 2\sqrt{n+1/2}$  [64,79], we have

$$\phi_n(x) \simeq \text{Ai}\left(n^{1/6}(2\sqrt{n+1/2}-x)\right). \quad (\text{B13})$$

Then, from Eq. (39), within the range  $|2\sqrt{n+1/2}-x| \ll 1/\delta_{0p}$ ,

$$\langle x|\psi\rangle_{0,i\delta_{0p},n}^{(w)} \simeq \text{Ai}\left(n^{1/6}(2\sqrt{n+1/2}-x)\right). \quad (\text{B14})$$

This indicates

$$|\psi\rangle_{0,i\delta_{0p},n}^{(w)} \simeq \hat{D}(2\sqrt{n+1/2}) \hat{S}\left(-\frac{1}{3}\ln 24\sqrt{n}\right) \hat{R}(\pi/2) |\text{CPS}\rangle, \quad (\text{B15})$$

which is equivalent to Eq. (B11) up to a  $\pi/2$  rotation in the limit  $n \rightarrow \infty$ .

## APPENDIX C: CHOI-JAMIOŁKOWSKI ISOMORPHISM FOR GAUSSIAN MAPS

Any completely positive, but not necessarily trace-preserving Gaussian map  $M: \mathcal{D}(\mathcal{H}^{\otimes k}) \rightarrow \mathcal{D}(\mathcal{H}^{\otimes j})$  has one-to-one correspondence with a  $j+k$ -mode Gaussian state  $\rho_M$  (Choi-Jamiołkowski isomorphism [48]). The characteristic function of  $\rho_M$  is given by

$$\begin{aligned} \chi_M(\mathbf{q}_1, \mathbf{q}_2) &= \text{Tr}(\rho_M(\hat{D}(\mathbf{q}_1) \otimes \hat{D}(\mathbf{q}_2))) \\ &\propto \exp\left(-\frac{1}{2}(\mathbf{q}_1^T, \mathbf{q}_2^T) \Sigma_M \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{pmatrix} + i(\mathbf{q}_1^T, \mathbf{q}_2^T) \gamma_M\right), \end{aligned} \quad (\text{C1})$$

$$(\text{C2})$$

where  $\mathbf{q}_1$  and  $\mathbf{q}_2$  are  $2j$ - and  $2k$ -dimensional real quadrature vectors, respectively, and

$$\Sigma_M = \begin{pmatrix} J & L^T \\ L & K \end{pmatrix}, \quad \gamma_M = \begin{pmatrix} \epsilon \\ \delta \end{pmatrix}. \quad (\text{C3})$$

Now consider an  $l + k$ -mode Gaussian input state with covariance matrix and mean

$$\Sigma = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}, \quad \gamma = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \quad (\text{C4})$$

For convenience, we call the first  $l$  modes the *signal* modes and the remaining  $k$  modes the *control* modes and denote their quadrature vectors by  $\mathbf{q}_s$  and  $\mathbf{q}_c$ , respectively.

When the map  $\mathcal{M}$  acts on the signal modes, the characteristic function of the output state is given by

$$\chi'(\mathbf{q}_s, \mathbf{q}_c) = \int d\mathbf{q}'_s \chi_M(\mathbf{q}_s, Z^{\otimes l} \mathbf{q}'_s) \chi(\mathbf{q}'_s, \mathbf{q}_c). \quad (\text{C5})$$

Then, the integrand has covariance matrix and mean

$$\Sigma_{\text{all}} = \begin{pmatrix} A + Z^{\otimes l} K Z^{\otimes l} & Z^{\otimes l} L & B^T \\ L^T Z^{\otimes l} & J & 0 \\ B & 0 & C \end{pmatrix}, \quad (\text{C6})$$

$$\gamma_{\text{all}} = \begin{pmatrix} \alpha + Z^{\otimes l} \delta \\ \epsilon \\ \beta \end{pmatrix}, \quad (\text{C7})$$

with respect to  $(\mathbf{q}_s, \mathbf{q}_c)$ .

Since the integration is Gaussian, the output state's covariance matrix and mean can be computed analytically as

$$\Sigma' = \begin{pmatrix} A' & B'^T \\ B' & C' \end{pmatrix}, \quad \gamma' = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}, \quad (\text{C8})$$

$$A' = K - L^T Z^{\otimes l} (A + Z^{\otimes l} K Z^{\otimes l})^{-1} Z^{\otimes l} L, \quad (\text{C9})$$

$$B' = -B(A + Z^{\otimes l} K Z^{\otimes l})^{-1} Z^{\otimes l} L, \quad (\text{C10})$$

$$C' = C - B(A + Z^{\otimes l} K Z^{\otimes l})^{-1} B^T, \quad (\text{C11})$$

$$\alpha' = \epsilon - L^T Z^{\otimes l} (A + Z^{\otimes l} K Z^{\otimes l})^{-1} (\alpha + Z^{\otimes l} \delta), \quad (\text{C12})$$

$$\beta' = \beta - B(A + Z^{\otimes l} K Z^{\otimes l})^{-1} (\alpha + Z^{\otimes l} \delta). \quad (\text{C13})$$

A particularly important example is the *damping operator*:

$$\hat{\Gamma}(\lambda) = \bigotimes_m e^{-\lambda_m \hat{n}_m}, \quad (\text{C14})$$

which corresponds to a Gaussian map with covariance matrix and mean

$$\Sigma_{\Gamma} = \begin{pmatrix} T & \sqrt{T^2 - 1} Z^{\otimes k} \\ \sqrt{T^2 - 1} Z^{\otimes k} & T \end{pmatrix}, \quad \gamma_{\Gamma} = 0 \quad (\text{C15})$$

where

$$T = \text{diag}(t_1, t_1, \dots, t_k, t_k) \quad (\text{C16})$$

and

$$t_m = \coth \lambda_m. \quad (\text{C17})$$

This is equivalent to a tensor product of thermal states.

In the special case where  $\rho_M$  is pure, the map  $M$  is called a Gaussian filter [51], which maps pure states to pure states. Applying Theorem 9 to  $\hat{\rho}_M$ , we find that any Gaussian filter can be expressed as a transformation of the form

$$|\psi\rangle \rightarrow \hat{F}|\psi\rangle = \hat{U}_1 \hat{\Gamma}(\lambda) \hat{U}_2 |\psi\rangle, \quad (\text{C18})$$

where  $\hat{U}_1$  and  $\hat{U}_2$  are unitary operators, assuming  $j = k$ . Even in the case  $j \neq k$ , the same form can be obtained by adding auxiliary vacuum modes  $|0\rangle$ .

Using the Baker-Campbell-Hausdorff formula

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2} [\hat{A}, [\hat{A}, \hat{B}]] + \dots, \quad (\text{C19})$$

the damping operator  $\hat{\Gamma}(\lambda) = e^{-\lambda \hat{n}}$  acts on the annihilation and creation operators  $\hat{a}$  and  $\hat{a}^\dagger$  as

$$\hat{\Gamma}(\lambda) \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \hat{\Gamma}(\lambda)^{-1} = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}, \quad k = e^\lambda. \quad (\text{C20})$$

Therefore,  $\hat{F}$  also admits a matrix representation with respect to the operator vector  $\hat{\mathbf{a}} = (\hat{a}_1, \hat{a}_1^\dagger, \dots, \hat{a}_k, \hat{a}_k^\dagger)^T$  [51]:

$$\hat{F} \hat{\mathbf{a}} \hat{F}^{-1} = \hat{S} \hat{\mathbf{a}} + \mathbf{b}. \quad (\text{C21})$$

## APPENDIX D: DERIVATION OF THE CANONICAL FORMS

Here we give proofs of the Theorems 1 and 9 in the main text.

Note that, although Refs. [49,50] give a principle proof of Theorem 9, the uniqueness of  $\hat{U}_c$  for a given  $C, \beta$  has not been explicitly stated in these proofs. Here, we provide a complete proof for rigorosity.

Since Theorem 9 is a generalized version of Theorem 1, it is sufficient to give a proof for Theorem 9.

*Proof of Theorem 9.* By adding extra vacuum modes if necessary, we assume  $l = k$  without loss of generality. Because  $\alpha, \beta$  can be arbitrary controlled by including displacements in  $\hat{U}_s, \hat{U}_c$ , it is sufficient to prove the case of  $\alpha = \beta = 0$ .

From Williamson's theorem [67],  $C$  can be decomposed as

$$C = SDS^T, \quad (\text{D1})$$

where  $S$  is a symplectic matrix and the diagonal matrix with paired elements

$$D = \text{diag}(a_1, a_1, \dots, a_k, a_k) \quad (\text{D2})$$

is called symplectic eigenvalues. Using that the covariance matrix of the two-mode squeezed state  $|\text{TMSS}(a)\rangle$  is given by

$$\Sigma_a = \begin{pmatrix} aI & \sqrt{a^2 - 1}Z \\ \sqrt{a^2 - 1}Z & aI \end{pmatrix}, \quad (\text{D3})$$

and, using the symplectic matrices  $S_s, S_c$  corresponding to the Gaussian unitary operators  $\hat{U}_s, \hat{U}_c$ , the decomposition in Eq. (79) can be written as

$$\begin{aligned} \Sigma &= \begin{pmatrix} S_s^T & 0 \\ 0 & S_c^T \end{pmatrix} \begin{pmatrix} D & \sqrt{D^2 - 1}Z^{\otimes k} \\ \sqrt{D^2 - 1}Z^{\otimes k} & D \end{pmatrix} \\ &\times \begin{pmatrix} S_s & 0 \\ 0 & S_c \end{pmatrix}. \end{aligned} \quad (\text{D4})$$

We prove that this decomposition is always possible. By including the Gaussian unitary corresponding to the  $S$  in Eq. (D1) to  $S_c$ , we can assume  $C = D$ . In the same way,  $A$  can also be assumed to be diagonal. From Eq. (80), the symplectic eigenvalues  $a_j$  have one-to-one correspondence to the Schmidt coefficients. From general properties of purification, the signal modes should share the Schmidt coefficients with the control modes; hence,  $A$  also has a Williamson's decomposition with the same symplectic eigenvalues. Thus, we can assume

$$A = C = D. \quad (\text{D5})$$

From the condition that  $|G\rangle$  is a pure Gaussian state,  $\Sigma$  should be a symplectic matrix; hence, the block matrices  $A, B, C$  should satisfy

$$A\Omega^{\otimes k}A + B^T\Omega^{\otimes k}B = \Omega^{\otimes k}, \quad (\text{D6})$$

$$C\Omega^{\otimes k}C + B\Omega^{\otimes k}B^T = \Omega^{\otimes k}, \quad (\text{D7})$$

$$B\Omega^{\otimes k}A + C\Omega^{\otimes k}B = 0, \quad (\text{D8})$$

where  $\Omega$  is the symplectic form

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (\text{D9})$$

and

$$\Omega^{\otimes k} = \begin{pmatrix} \Omega & & \\ & \ddots & \\ & & \Omega \end{pmatrix}. \quad (\text{D10})$$

From Eqs. (D5) and (D7), we have

$$B\Omega^{\otimes k}B^T = -(D^2 - 1)\Omega^{\otimes k}. \quad (\text{D11})$$

By multiplying  $(D^2 - 1)^{-1/2}Z^{\otimes k}$  and its transpose from both sides, where

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\text{D12})$$

we obtain

$$(D^2 - 1)^{-1/2}Z^{\otimes k}B\Omega^{\otimes k}((D^2 - 1)^{-1/2}Z^{\otimes k}B)^T = \Omega^{\otimes k}, \quad (\text{D13})$$

which means  $(D^2 - 1)^{-1/2}Z^{\otimes k}B$  is symplectic; thus,

$$B = Z^{\otimes k}(D^2 - 1)^{1/2}S \quad (\text{D14})$$

for some symplectic matrix  $S$ . Here, we used the relation  $Z\Omega Z = -\Omega$ .

Meanwhile, from Eqs. (D5) and (D8), we have

$$BD\Omega^{\otimes k} = -D\Omega^{\otimes k}B. \quad (\text{D15})$$

By multiplying  $Z^{\otimes k}$  from right, we have

$$(BZ^{\otimes k})(D\Omega^{\otimes k}) = (D\Omega^{\otimes k})(BZ^{\otimes k}). \quad (\text{D16})$$

This indicates that  $BZ^{\otimes k}$  has the same eigenspace as  $D\Omega^{\otimes k}$ . Writing degeneracy of the symplectic eigenvalues  $a_1, \dots, a_k$  as  $d_1, \dots, d_m$  ( $d_1 + \dots + d_m = k$ ), we can write  $B$  in a block-diagonal form as

$$B = \text{diag}(B_1, \dots, B_m), \quad (\text{D17})$$

where, from Eq. (D15), each  $B_j$  is a  $2d_j \times 2d_j$  matrix satisfying

$$\Omega^{\otimes d_j}B_j + B_j\Omega^{\otimes d_j} = 0. \quad (\text{D18})$$

Thus,  $S$  in Eq. (D14) is also block diagonal:

$$S = \text{diag}(S_1, \dots, S_m) \quad (\text{D19})$$

and each  $S_j$  satisfies

$$\Omega^{\otimes d_j} S_j Z^{\otimes d_j} + S_j Z^{\otimes d_j} \Omega^{\otimes d_j} = 0. \quad (\text{D20})$$

By multiplying  $S_j^T$  from the left and  $\Omega^{\otimes d_j} Z^{\otimes d_j}$  from the right, we obtain

$$S_j^T S_j = I, \quad (\text{D21})$$

which means  $S_j$  is an orthogonal symplectic matrix; thus, we have  $S^T D S = D$ . Therefore, from Eq. (D14), we obtain

$$\begin{aligned} \Sigma &= \begin{pmatrix} S^T & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} D & \sqrt{D^2 - 1} Z^{\otimes k} \\ \sqrt{D^2 - 1} Z^{\otimes k} & D \end{pmatrix} \\ &\times \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix}. \end{aligned} \quad (\text{D22})$$

This proves the possibility of the decomposition in Eq. (D4). From the construction,  $\hat{U}_c$  depends only on  $(C, \beta)$ . ■

## APPENDIX E: DERIVATION OF DAMPING TRANSFORMATION

Here, we give proofs of Theorems 3 and 10 in the main text.

The domain of  $t$  is given by the following equivalent conditions:

(i)

$$T_+ > 1, \quad T_- < -\{\Pi_-[C + (T_+ \oplus 0)]^{-1}\Pi_-\}^{-1}.$$

(ii)

$$|T| > 1, \quad \Pi_-(C + T)^{-1}\Pi_- < 0.$$

(iii)

$$|T| > 1, \quad \mathcal{D}_t(C) > 0.$$

Here,  $T$  is divided into a positive part and a negative part as

$$T = T_+ \oplus T_-, \quad T_+ > 0, \quad T_- < 0, \quad (\text{E1})$$

and  $\Pi_-$  is a projection operator to the support of  $T_-$ .

Theorem 10 is a generalized version of Theorem 3. Thus, it is sufficient to give a proof for Theorem 10.

*Proof of Theorem 10.* We first show that the damping operation transforms  $(C, \beta)$  according to Eq. (81) and then derive the condition on  $T$ . For  $\lambda_j \geq 0$ , suppose the tensor product of damping operators

$$\bigotimes_j e^{-\lambda_j \hat{n}_j} \quad (\text{E2})$$

is applied to the control modes. By putting  $t_m = \coth \lambda_m$  and using the Choi-Jamiołkowski representation of

damping operation in Appendix C [Eqs. (C9)–(C13) and (C15)], we obtain the transformation of  $(C, \beta)$  as

$$\begin{aligned} (C, \beta) &\rightarrow \mathcal{D}_T(C, \beta) = [\mathcal{D}_T(C), \mathcal{D}_T(\beta)] \\ &:= \left( T - \sqrt{T^2 - 1}(C + T)^{-1}\sqrt{T^2 - 1}, \right. \\ &\quad \left. \sqrt{T^2 - 1}(C + T)^{-1}\beta \right). \end{aligned} \quad (\text{E3})$$

If  $T > I$ , the transformation corresponds to a physical operation satisfying  $\lambda_j > 0$ . The range of  $T$  can be further extended by using that, for arbitrary  $|T_1|, |T_2| > I$  and  $(C, \beta)$ , there is a relation

$$D_{T_1}(D_{T_2}(C, \beta)) = D_{T_1 \circ T_2}(C, \beta) \quad (\text{E4})$$

$$T_1 \circ T_2 := D_{T_1}(T_2) = \frac{T_1 T_2 + 1}{T_1 + T_2}. \quad (\text{E5})$$

For an arbitrary  $T$  satisfying  $|T| > I$ , we can find a sufficiently small  $\epsilon > 0$  and  $T' = (1 + \epsilon)I$  such that

$$T \circ T' > I. \quad (\text{E6})$$

Therefore, if  $\mathcal{D}_T(C, \beta)$  represents a physical control-mode representation satisfying Eq. (78),  $(C, \beta)$  and  $\mathcal{D}_T(C, \beta)$  both have the same output state as  $D_{T \circ T'}(C, \beta)$ .

Let us derive a closed-form condition for  $\mathcal{D}_T(C, \beta)$  satisfies the physical condition Eq. (78). The case  $T < -I$  corresponds to unphysical damping operations where  $\lambda_j < 0$ . From the condition  $\mathcal{D}_T(C) > 0$ ,  $T < -C$  is necessary. Conversely, when  $T < -C$ , because  $C$  satisfies Eq. (78), we have

$$0 < -C - T \leq -i\Omega^{\otimes k} - T; \quad (\text{E7})$$

thus,

$$\mathcal{D}_T(C) \geq T - \sqrt{T^2 - 1}(i\Omega^{\otimes k} + T)^{-1}\sqrt{T^2 - 1} \quad (\text{E8})$$

$$= i\Omega^{\otimes k}. \quad (\text{E9})$$

In summary, under the assumption  $T > 0$  or  $T < 0$ , the condition on  $T$  is  $T > I$  or  $T < -C$ .

Now, we consider the condition on  $T$  for general cases.  $T$  can be divided into a positive part and a negative part as

$$T = T_+ \oplus T_-, \quad (\text{E10})$$

where  $T_+ > 0$  and  $T_- < 0$ . We use that

$$T = (-\infty \oplus T_-) \circ (T_+ \oplus \infty). \quad (\text{E11})$$

This corresponds to applying the physical and unphysical damping operators separately. The control-mode

representation after applying the physical part can be written as

$$C_- := \Pi_- D_{T_+ \oplus \infty}(C) \Pi_- \quad (\text{E12})$$

$$= \{\Pi_- [C + (T_+ \oplus 0)]^{-1} \Pi_-\}^{-1}, \quad (\text{E13})$$

where we write projection matrices to the support of  $T_\pm$  as  $\Pi_\pm$ . From the discussion in the last paragraph,

$$T_- < -C_- \quad (\text{E14})$$

is the necessary and sufficient condition for the physical  $\mathcal{D}_T(C)$ . By using the formula for inversion of a block matrix, this is equivalent to

$$\Pi_-(C + T)^{-1} \Pi_- < 0. \quad (\text{E15})$$

It is clear from Eq. (E3) that Eq. (E15) needs to be satisfied for having  $\mathcal{D}_T(C) > 0$ . This shows that a weaker condition  $\mathcal{D}_T(C) > 0$  is enough for verifying the physicality condition. ■

#### APPENDIX F: DERIVATION OF THE CAYLEY-TRANSFORMED FORM OF THE DAMPING OPERATION

Here, we show that under the Cayley transform [Eqs. (83)–(85)], the damping transformation [Eq. (81)] can be written as Eqs. (86) and (87).

We have

$$C' \pm I = T \pm I - \sqrt{T^2 - I}(C + T)^{-1} \sqrt{T^2 - I} \quad (\text{F1})$$

$$= \sqrt{T^2 - I}[(T \mp I)^{-1} - (C + T)^{-1}] \sqrt{T^2 - I} \quad (\text{F2})$$

$$= \sqrt{T^2 - I}(C + T)^{-1}(C \pm I)(T \mp I)^{-1} \sqrt{T^2 - I} \quad (\text{F3})$$

$$= \sqrt{T^2 - I}(C + T)^{-1}(C \pm I) \sqrt{\frac{T \pm I}{T \mp I}}. \quad (\text{F4})$$

Using this, we obtain

$$\tilde{C}' = (C' + I)^{-1}(C' - I) \quad (\text{F5})$$

$$= \sqrt{\frac{T - I}{T + I}} \frac{C - I}{C + I} \sqrt{\frac{T - I}{T + I}} \quad (\text{F6})$$

$$= \sqrt{\tilde{T}} \tilde{C} \sqrt{\tilde{T}}. \quad (\text{F7})$$

Also,

$$\tilde{\beta}' = (C' + I)^{-1} \beta' \quad (\text{F8})$$

$$= (C' + I)^{-1} \sqrt{T^2 - I}(C + T)^{-1} \beta \quad (\text{F9})$$

$$= \sqrt{\frac{T - I}{T + I}}(C + I)^{-1} \beta \quad (\text{F10})$$

$$= \sqrt{\tilde{T}} \tilde{\beta}. \quad (\text{F11})$$

#### APPENDIX G: PROOF OF THEOREM 8 AND ITS GENERALIZATION

Here, we give a proof for Theorem 8 in the main text. For proving Theorem 8, we first show the following more general theorem.

*Theorem G1.* A pure Gaussian state with a control-mode representation  $(C, \beta)$  can be transformed to another Gaussian state with  $(C', \beta')$  via a Gaussian map acting on the signal modes, if and only if

$$C' \leq C \quad (\text{G1})$$

and

$$\beta - \beta' \in \text{Im}(C - C'). \quad (\text{G2})$$

*Proof of Theorem G1.* Let the number of signal and control modes be  $l$  and  $k$ , respectively. The covariance matrix and mean vector of the total system are given by

$$\Sigma = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}, \quad \gamma = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \quad (\text{G3})$$

Suppose we apply a Gaussian map  $M$  described by

$$\Sigma_M = \begin{pmatrix} J & L^T \\ L & K \end{pmatrix}, \quad \gamma_M = \begin{pmatrix} \epsilon \\ \delta \end{pmatrix} \quad (\text{G4})$$

(see Appendix C) to the signal mode and obtain a new control-mode representation  $(C', \beta')$  as a result. Here, we show that the necessary and sufficient condition for the existence of such an  $M$  is given by

$$C' \leq C \quad (\text{G5})$$

and

$$\beta - \beta' \in \text{Im}(C - C'). \quad (\text{G6})$$

From Theorem 9, by adding extra vacuum modes if necessary, we can assume without loss of generality

that  $l = k$  and

$$A = D, \quad (\text{G7})$$

$$B = S(D^2 - 1)^{1/2} Z^{\otimes k}, \quad (\text{G8})$$

$$\alpha = 0. \quad (\text{G9})$$

Here,

$$C = SDS^T \quad (\text{G10})$$

is the Williamson decomposition of  $C$ . Then, the transformation of the control-mode representation by the map  $M$  is expressed as

$$C - C' = S\sqrt{D^2 - 1}(D + K)^{-1}\sqrt{D^2 - 1}S^T, \quad (\text{G11})$$

$$\beta - \beta' = S\sqrt{D^2 - 1}(D + K)^{-1}\delta \quad (\text{G12})$$

(see Appendix C).

Suppose the map  $M$  exists. From Eq. (G11), the first condition (G5) clearly holds. Also, since  $(D + K)^{-1} > 0$  and  $\sqrt{D^2 - 1} \geq 0$ , we have

$$\begin{aligned} & \text{Im}\left(\sqrt{D^2 - 1}(D + K)^{-1}\sqrt{D^2 - 1}\right) \\ &= \text{Im}\left(\sqrt{D^2 - 1}(D + K)^{-1}\right). \end{aligned} \quad (\text{G13})$$

Since  $S$  is invertible, the second condition (G6) also holds.

Conversely, suppose Eqs. (G5) and (G6) hold. We define

$$\tilde{C}' = S^{-1}C'S^{-T}. \quad (\text{G14})$$

Then,

$$\tilde{C}' \leq S^{-1}CS^{-T} = D. \quad (\text{G15})$$

Also, Eq. (G11) can be written as

$$\tilde{C}' = D - \sqrt{D^2 - 1}(D + K)^{-1}\sqrt{D^2 - 1} \quad (\text{G16})$$

$$= D_D(K), \quad (\text{G17})$$

where  $\mathcal{D}_T(\cdot)$  is the damping transformation (Theorem 10). We want to a  $K$  satisfying this. Assume  $D$  can be written as

$$D = \text{diag}(t_1, t_1, \dots, t_m, t_m, 1, 1, \dots, 1, 1) \quad (\text{G18})$$

$$= \text{diag}(D_+, I_{2(k-m)}), \quad (\text{G19})$$

where  $D_+ > 1$ . Using the same division, we write  $\tilde{C}'$  in block form as

$$\tilde{C}' = \begin{pmatrix} \tilde{C}'_+ & \tilde{C}'_{+1}{}^T \\ \tilde{C}'_{+1} & \tilde{C}'_1 \end{pmatrix}. \quad (\text{G20})$$

From  $\tilde{C}' \leq D$ , it follows that  $\tilde{C}'_1 \leq I_{2(k-m)}$ . Because  $\tilde{C}'_1$  must satisfy the uncertainty principle, we conclude  $\tilde{C}'_1 = I_{2(k-m)}$ , which is the covariance matrix of a pure state. This implies  $\tilde{C}'_{+1}{}^T = 0$ , and, thus,

$$\tilde{C}' = \text{diag}(C'_+, I_{2(k-m)}), \quad (\text{G21})$$

$$C'_+ \leq D_+. \quad (\text{G22})$$

Therefore,

$$K = \text{diag}(K_+, I_{2(k-m)}), \quad (\text{G23})$$

$$\begin{aligned} K_+ &= \sqrt{D_+^2 - 1}(D_+ - C'_+)^{-1}\sqrt{D_+^2 - 1} - D_+ \\ &= D_{-D_+}(C_+) \end{aligned} \quad (\text{G24})$$

satisfies Eq. (G17) and, hence, Eq. (G11). From Theorem 10, this  $K$  represents a physical Gaussian map.

Moreover, from Eq. (G6),

$$(C - C')\tilde{\delta} = \beta - \beta' \quad (\text{G25})$$

holds for some  $\tilde{\delta}$ ; thus, we can set

$$\delta = \sqrt{D^2 - 1}S^T\tilde{\delta} \quad (\text{G26})$$

to satisfy Eq. (G12). Thus, Eqs. (G5) and (G6) are necessary and sufficient conditions. ■

*Proof of Theorem 8.* Suppose  $n \geq 2$  and consider the conversion from  $|\psi\rangle_{s_0, \delta_0, n, \hat{U}_s}$  to  $|\psi\rangle_{s'_0, \delta'_0, n, \hat{U}'_s}$ . From Theorems 7 and G1, this conversion is possible if and only if there exist control-mode representations  $(C, \beta)$  and  $(C', \beta')$  corresponding to the non-Gaussian phase-sensitivity parameters  $(s_0, \delta_0)$  and  $(s'_0, \delta'_0)$ , respectively, such that Eqs. (G1) and (G2) are satisfied.

Without loss of generality, we restrict our attention to diagonal control matrices  $C = \text{diag}(c, d)$  with  $c > d$  and  $C' = \text{diag}(c', d')$  with  $c' > d'$ . Figure 4, which illustrates the relation between  $c$ ,  $d$ , and  $s_0$ , provides an intuitive understanding of the convertibility conditions.

From Eq. (G1), we require  $c' \leq c$  and  $d' \leq d$ . Therefore:

- (i) When  $s_0 \geq 1$  and  $s'_0 < s_0$ , this condition cannot be satisfied.
- (ii) When  $0 \leq s_0 < 1$ , it is always possible to choose control parameters such that  $1 < d < c$ . By selecting  $c' = d' = 1 + \epsilon$  with sufficiently small  $\epsilon > 0$  and  $\beta' = 0$ , both Eqs. (G1) and (G2) are satisfied. This implies that the state can be converted to a Fock state (corresponding to  $s_0 = 0$ ,  $\delta_0 = 0$ ). Since

$c = d = N$  for arbitrarily large  $N$  also represents the same Fock state, this Fock state can further be converted to any other  $(C', \beta')$ .

- (iii) When  $s'_0 > s_0$ , since the damping transformation brings  $(c, d)$  to  $(\infty, 1/s_0)$ , we can always find  $C$  and  $C'$  corresponding to  $s_0$  and  $s'_0$  that satisfy  $C' < C$ . Hence, both Eqs. (G1) and (G2) can be satisfied.
- (iv) When  $s_0 = s'_0 > 1$ , Eq. (G1) requires  $C = C'$ . Then, Eq. (G2) requires  $\beta = \beta'$ , which implies that the two states are identical:  $(s_0, \delta_0) = (s'_0, \delta'_0)$ .
- (v) When  $s_0 = s'_0 = 1$ , Eq. (G1) requires  $C = \text{diag}(c, 1)$ ,  $C' = \text{diag}(c', 1)$  with  $c' \leq c$ . In this case, Eq. (G2) requires  $\delta_{0x} = 0$ .

In summary, the necessary and sufficient condition for the convertibility is that one of the following holds:

$$\begin{cases} 0 \leq s_0 < 1, \\ s'_0 > s_0, \\ s_0 = s'_0 = 1 \quad \text{and} \quad \delta_{0x} = 0. \end{cases} \quad (\text{G27})$$

## APPENDIX H: DERIVATION OF PARTICLE AND WAVE FORMS

Here we give proofs of Theorems 5 and 6 in the main text.

In addition, we show the following theorem, describing the relation between these forms.

*Theorem H1 (particle form  $\rightarrow$  wave form).* The particle form and the wave form are related by a Gaussian unitary operator  $\hat{U}_{s_0, \delta_0, n}^{(\text{p} \rightarrow \text{w})}$ :

$$|\psi\rangle_{s_0, \delta_0, n}^{(\text{w})} = \hat{U}_{s_0, \delta_0, n}^{(\text{p} \rightarrow \text{w})} |\psi\rangle_{s_0, \delta_0, n}^{(\text{p})}. \quad (\text{H1})$$

The action of  $\hat{U}_{s_0, \delta_0, n}^{(\text{p} \rightarrow \text{w})}$  on the quadratures is given by

$$\begin{aligned} & \hat{U}_{s_0, \delta_0, n}^{(\text{p} \rightarrow \text{w}) \dagger} \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \hat{U}_{s_0, \delta_0, n}^{(\text{p} \rightarrow \text{w})} \\ &= \begin{pmatrix} 0 & \frac{1}{\sqrt{s_0+1}} \\ -\sqrt{s_0+1} & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} + \begin{pmatrix} -\delta_{0p} \\ \delta_{0x} \end{pmatrix}. \end{aligned} \quad (\text{H2})$$

*Proof of Theorem 5.* Applying Choi-Jamiołkowski isomorphism (Appendix C) to the standard form (Theorem 1), the output state of a two-mode non-Gaussian state generator can be represented as

$$|\psi\rangle_{C, \beta, n, \hat{U}_s} = \hat{K}|n\rangle, \quad (\text{H3})$$

using a Gaussian filter

$$\hat{K} = \hat{U}_p \hat{\Gamma}(\lambda) \hat{S}(r) \hat{D}(\beta). \quad (\text{H4})$$

Here,  $\hat{S}(r) = \exp[\frac{1}{2}(r^* \hat{a}^2 - r \hat{a}^{\dagger 2})]$ ,  $\hat{D}(\beta) = \exp(\beta \hat{a}^\dagger - \beta^* \hat{a})$ , and  $\hat{\Gamma}(\lambda) = \exp(-\lambda \hat{n})$  are the squeezing, displacement, and damping operators, respectively, and  $\hat{U}_p$  is a Gaussian unitary operator. Without loss of generality, we assume  $C$  is diagonal, with eigenvalues  $c \geq d$ . Then, the parameters  $\lambda, r$  are determined as

$$\lambda = \coth^{-1} \sqrt{cd}, \quad r = \frac{1}{4} \log\left(\frac{c}{d}\right). \quad (\text{H5})$$

If  $\hat{U}_p$  can be chosen such that

$$\hat{K} \hat{a} \hat{K}^{-1} \propto \hat{a}, \quad (\text{H6})$$

then

$$\hat{K}|0\rangle \propto |0\rangle, \quad (\text{H7})$$

and, hence, the output state can be expressed as

$$|\psi\rangle_{C, \beta, n, \hat{U}_s} \propto \hat{K} \hat{a}^{\dagger n} |0\rangle \quad (\text{H8})$$

$$= (\hat{K} \hat{a}^\dagger \hat{K}^{-1})^n \hat{K} |0\rangle \quad (\text{H9})$$

$$\propto (\hat{K} \hat{a}^\dagger \hat{K}^{-1})^n |0\rangle, \quad (\text{H10})$$

which is a superposition of Fock states up to photon number  $n$ . In what follows, we determine such a  $\hat{U}_p$  and derive the particle form Eq. (38).

We first consider the case  $\beta = 0$ . In this case, we can choose a squeezing operator  $\hat{S}(r')$  as  $\hat{U}_p$ . Then, the action of  $\hat{K}$  becomes

$$\hat{K} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \hat{K}^{-1} \quad (\text{H11})$$

$$= \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} \begin{pmatrix} \cosh r' & \sinh r' \\ \sinh r' & \cosh r' \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \quad (\text{H12})$$

$$= \begin{pmatrix} k \cosh r & k^{-1} \sinh r \\ k \sinh r & k^{-1} \cosh r \end{pmatrix} \begin{pmatrix} \cosh r' & \sinh r' \\ \sinh r' & \cosh r' \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}, \quad (\text{H13})$$

where

$$k = e^\lambda. \quad (\text{H14})$$

By choosing  $r'$  such that

$$\tanh r' = -k^{-2} \tanh r, \quad (\text{H15})$$

we get

$$\hat{K} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \hat{K}^{-1} \quad (\text{H16})$$

$$= \frac{1}{\sqrt{\eta}} \begin{pmatrix} k \cosh r & k^{-1} \sinh r \\ k \sinh r & k^{-1} \cosh r \end{pmatrix} \cdot \begin{pmatrix} k \cosh r & -k^{-1} \sinh r \\ -k^{-1} \sinh r & k \cosh r \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \quad (\text{H17})$$

$$= \begin{pmatrix} \eta^{1/2} & 0 \\ \eta^{-1/2} s_0 & \eta^{-1/2} \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}, \quad (\text{H18})$$

where

$$s_0 = \frac{c-d}{cd-1} \quad (\text{H19})$$

is the non-Gaussian phase sensitivity and

$$\eta = k^2 \cosh^2 r - k^{-2} \sinh^2 r \quad (\text{H20})$$

$$= \frac{(c+1)(d+1)}{cd-1}. \quad (\text{H21})$$

Since this satisfies condition Eq. (H6), we obtain

$$|\psi\rangle_{C,n} \propto \hat{K} \hat{a}^{\dagger n} |0\rangle \quad (\text{H22})$$

$$\propto (\hat{K} \hat{a}^\dagger \hat{K}^{-1})^n |0\rangle \quad (\text{H23})$$

$$\propto (\hat{a}^\dagger + s_0 \hat{a})^n |0\rangle. \quad (\text{H24})$$

This gives the particle form for  $\beta = 0$ .

Next, we consider the case with finite displacement  $\beta \neq 0$ . We modify  $\hat{U}_p$  to include a displacement:

$$\hat{U}_p = \hat{D}(\beta') \hat{S}(r'). \quad (\text{H25})$$

Then,

$$\hat{K} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \hat{K}^{-1} = \begin{pmatrix} \eta^{1/2} & 0 \\ \eta^{-1/2} s_0 & \eta^{-1/2} \end{pmatrix} \begin{pmatrix} \hat{a} + \beta' \\ \hat{a}^\dagger + \beta'^* \end{pmatrix} + \begin{pmatrix} \beta \\ \beta^* \end{pmatrix}. \quad (\text{H26})$$

By choosing

$$\beta' = -\eta^{-1/2} \beta, \quad (\text{H27})$$

we get

$$\hat{K} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \hat{K}^{-1} = \begin{pmatrix} \eta^{1/2} & 0 \\ \eta^{-1/2} s_0 & \eta^{-1/2} \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} + \begin{pmatrix} 0 \\ \eta^{-1/2} \delta_0 \end{pmatrix}, \quad (\text{H28})$$

where

$$\delta_0 = -\eta^{-1/2} s_0 \beta + (\eta^{1/2} - \eta^{-1/2}) \beta^* \quad (\text{H29})$$

$$= \frac{2}{\sqrt{cd-1}} \left( \sqrt{\frac{d+1}{c+1}} \beta_x - i \sqrt{\frac{c+1}{d+1}} \beta_p \right). \quad (\text{H30})$$

Hence, the output state is

$$|\psi\rangle_{s_0, \delta_0, n} \propto (\hat{a}^\dagger + s_0 \hat{a} + \delta_0)^n |0\rangle. \quad (\text{H31})$$

■

*Proof of Theorem 6.* We consider the operator

$$\hat{K} = \exp(-b_p \hat{p}) \exp(-b_x \hat{x}) \exp(-k \hat{x}^2) \quad (\text{H32})$$

and calculate the wave-function representation of  $\hat{K}|0\rangle$ . Because

$$\begin{aligned} \langle x | \exp(-b_x \hat{x}) \exp(-k \hat{x}^2) | 0 \rangle \\ \propto \exp \left[ - \left( k + \frac{1}{4} \right) x^2 - b_x x \right], \end{aligned} \quad (\text{H33})$$

by a Fourier transform we obtain

$$\begin{aligned} \langle p | \exp(-b_x \hat{x}) \exp(-k \hat{x}^2) | 0 \rangle \\ \propto \exp \left[ - \frac{1}{4(4k+1)} p^2 + \frac{i b_x}{4k+1} p \right]. \end{aligned} \quad (\text{H34})$$

Thus,  $p$ -wave function of  $\hat{K}|0\rangle$  can be calculated as

$$\langle p | \hat{K} | 0 \rangle \propto \exp \left[ - \frac{1}{4(4k+1)} p^2 + \frac{i b_x}{4k+1} p - b_p p \right]. \quad (\text{H35})$$

This can be written as

$$\hat{K} | 0 \rangle \propto \hat{U}_w | 0 \rangle, \quad (\text{H36})$$

where

$$\hat{U}_w = \hat{S} \left( \frac{1}{2} \log(4k+1) \right) D \left( \frac{2b_x}{\sqrt{4k+1}}, 2\sqrt{4k+1} b_p \right). \quad (\text{H37})$$

If we define

$$\hat{K}' = \hat{U}_w^\dagger \hat{K}, \quad (\text{H38})$$

we have

$$\hat{K}'|0\rangle \propto |0\rangle; \quad (\text{H39})$$

thus,

$$\hat{K}'|n\rangle \propto (\hat{K}'\hat{a}^\dagger\hat{K}'^{-1})^n|0\rangle, \quad (\text{H40})$$

similarly to the case of the particle form [Eq. (H10)]. The transformation of the quadrature operators is calculated as

$$\hat{K}\begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix}\hat{K}^{-1} \quad (\text{H41})$$

$$= \begin{pmatrix} 1 & 0 \\ -4ik & 1 \end{pmatrix} \begin{pmatrix} \hat{x} + 2ib_p \\ \hat{p} - 2ib_x \end{pmatrix} \quad (\text{H42})$$

$$= \begin{pmatrix} 1 & 0 \\ -4ik & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} + \begin{pmatrix} 2ib_p \\ 8ab_p - 2ib_x \end{pmatrix}. \quad (\text{H43})$$

Therefore,

$$\hat{K}'\begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix}\hat{K}'^{-1} \quad (\text{H44})$$

$$= \begin{pmatrix} 1 & 0 \\ -4ik & 1 \end{pmatrix} \begin{pmatrix} (4k+1)^{-1/2}\hat{x} - \frac{2b_x}{4k+1} \\ (4k+1)^{1/2}\hat{p} - 2(4k+1)b_p \end{pmatrix} + \begin{pmatrix} 2ib_p \\ 8ab_p - 2ib_x \end{pmatrix} \quad (\text{H45})$$

$$= \begin{pmatrix} \frac{1}{\sqrt{4k+1}} & 0 \\ -\frac{4ik}{\sqrt{4k+1}} & \sqrt{4k+1} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} + \begin{pmatrix} -\frac{2b_x}{4k+1} + 2ib_p \\ i \end{pmatrix}. \quad (\text{H46})$$

Hence,  $\hat{a}^\dagger$  is transformed as

$$\hat{K}'\hat{a}^\dagger\hat{K}'^{-1} \quad (\text{H47})$$

$$= \frac{1}{2}(1-i) \begin{pmatrix} \frac{1}{\sqrt{4k+1}} & 0 \\ -\frac{4ik}{\sqrt{4k+1}} & \sqrt{4k+1} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} - \frac{2b_x}{4k+1} + 2ib_p \quad (\text{H48})$$

$$= \frac{1}{2} \left( \frac{1-4k}{\sqrt{1+4k}} \hat{x} - i\sqrt{4k+1} \hat{p} \right) - \frac{2b_x}{4k+1} + 2ib_p \quad (\text{H49})$$

$$= \frac{1}{\sqrt{1+4k}} \hat{a}^\dagger - \frac{4k}{\sqrt{1+4k}} \hat{a} - \frac{2b_x}{4k+1} + 2ib_p \quad (\text{H50})$$

$$\propto \hat{a}^\dagger - 4k\hat{a} - \frac{2b_x}{\sqrt{4k+1}} + 2i\sqrt{4k+1}b_p. \quad (\text{H51})$$

For comparing this to the particle form, we add an extra  $-\pi/2$  phase rotation, obtaining

$$\hat{R}\left(-\frac{\pi}{2}\right)\hat{K}'\hat{a}^\dagger\hat{K}'^{-1}\hat{R}^\dagger\left(-\frac{\pi}{2}\right) \propto \hat{a}^\dagger + 4k\hat{a} + 2\sqrt{4k+1}b_p + \frac{2ib_x}{\sqrt{4k+1}}. \quad (\text{H52})$$

From this, we have the correspondence

$$k = \frac{s_0}{4}, \quad (\text{H53})$$

$$b_x = -\frac{1}{2}\sqrt{s_0+1}\delta_{0p}, \quad (\text{H54})$$

$$b_p = \frac{1}{2\sqrt{s_0+1}}\delta_{0x}, \quad (\text{H55})$$

leading to the wave form Eq. (39). ■

*Proof of Theorem H1.* From the proof of Theorem 6, the conversion from the particle form to the wave form is given by the Gaussian unitary operator  $\hat{U}_w\hat{R}(\pi/2)$  [Eqs. (H37) and (H52)]. Redefining this operator as  $\hat{U}_{s_0,\delta_0,n}^{(p\rightarrow w)}$ , it can be written as

$$\hat{U}_{s_0,\delta_0,n}^{(p\rightarrow w)} = \hat{S}\left(\frac{1}{2}\log(s_0+1)\right)\hat{D}(-\delta_{0p},\delta_{0x})\hat{R}\left(\frac{\pi}{2}\right), \quad (\text{H56})$$

and its action is given by Eq. (H2). ■

## APPENDIX I: PROOF OF THEOREM 7

Here, we give a proof of Theorem 7.

*Proof of Theorem 7.* In the particle form (Theorem 5), the coefficients for  $|n\rangle$ ,  $|n-1\rangle$ , and  $|n-2\rangle$  can be explicitly written as

$$\langle n|\psi\rangle_{s_0,\delta_0,n}^{(p)} \propto \sqrt{n!}, \quad (\text{I1})$$

$$\langle n-1|\psi\rangle_{s_0,\delta_0,n}^{(p)} \propto \delta_0\sqrt{n\cdot n!}, \quad (\text{I2})$$

$$\langle n-2|\psi\rangle_{s_0,\delta_0,n}^{(p)} \propto \frac{1}{2}(s_0+\delta_0^2)\sqrt{n(n-1)\cdot n!}. \quad (\text{I3})$$

Hence,

$$\frac{\langle n-1|\psi\rangle_{s_0,\delta_0,n}^{(p)}}{\langle n|\psi\rangle_{s_0,\delta_0,n}^{(p)}} = \delta_0\sqrt{n}, \quad (\text{I4})$$

$$\frac{\langle n-2|\psi\rangle_{s_0,\delta_0,n}^{(p)}}{\langle n|\psi\rangle_{s_0,\delta_0,n}^{(p)}} = (s_0+\delta_0^2)\frac{\sqrt{n(n-1)}}{2}. \quad (\text{I5})$$

Now, suppose

$$|\psi\rangle_{s_0, \delta_0, n, \hat{U}_s} = |\psi\rangle_{s'_0, \delta'_0, n, \hat{U}'_s}. \quad (\text{I6})$$

We can assume one of the states is in the particle form, without loss of generality. Since phase rotation is the only Gaussian unitary operation leaving the state in finite superposition of Fock states [23], we have

$$|\psi\rangle_{s_0, \delta_0, n}^{(p)} = e^{i\theta\hat{n}} |\psi\rangle_{s'_0, \delta'_0, n}^{(p)}. \quad (\text{I7})$$

Thus, from Eqs. (I4) and (I5), we have

$$\delta_0 = \delta'_0 e^{i\theta}, \quad (\text{I8})$$

$$s_0 + \delta_0^2 = (s'_0 + \delta_0'^2) e^{2i\theta}. \quad (\text{I9})$$

The solutions to these equations are either  $s_0 = s'_0 = 0$  and  $|\delta_0| = |\delta'_0|$  or  $s_0 = s'_0 > 0$  and  $\delta_0 = \pm \delta'_0$ . ■

## APPENDIX J: PROOF OF THEOREM 4

Here, we provide the proof of Theorem 4 in the main text.

*Proof of Theorem 4.* Using the rotation transformation from Theorem 2, we can diagonalize  $C$  as

$$C = O^T \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} O, \quad (\text{J1})$$

where, without loss of generality, we assume  $c \geq d$ . Thus, it suffices to consider the case where  $C$  is diagonal.

Let us introduce the Cayley transform of the variables:

$$\tilde{c} = \frac{c-1}{c+1}, \quad \tilde{d} = \frac{d-1}{d+1}, \quad (\text{J2})$$

$$\tilde{\beta}_x = \frac{1}{c+1} \beta_x, \quad \tilde{\beta}_p = \frac{1}{d+1} \beta_p. \quad (\text{J3})$$

Then, from Eqs. (83)–(85), the damping transformation in Theorem 3 can be written as

$$(\tilde{c}, \tilde{d}, \tilde{\beta}_x, \tilde{\beta}_p) \rightarrow (\tilde{t} \tilde{c}, \tilde{t} \tilde{d}, \sqrt{\tilde{t}} \tilde{\beta}_x, \sqrt{\tilde{t}} \tilde{\beta}_p). \quad (\text{J4})$$

Since this is simply a scaling transformation, the following three parameters are invariant:

$$\tilde{s}_0 = \frac{\tilde{d}}{\tilde{c}}, \quad (\text{J5})$$

$$\tilde{\beta}_{0x} = \frac{\tilde{\beta}_x}{\sqrt{\tilde{c}}}, \quad (\text{J6})$$

$$\tilde{\beta}_{0p} = \frac{\tilde{\beta}_p}{\sqrt{\tilde{c}}}. \quad (\text{J7})$$

$s_0$  [Eq. (35)] and  $\delta_0$  [Eq. (36)] can be expressed in terms of these invariants:

$$s_0 = \frac{1 - \tilde{s}_0}{1 + \tilde{s}_0}, \quad (\text{J8})$$

$$\delta_0 = 2\sqrt{\frac{2}{1 + \tilde{s}_0}} (\tilde{\beta}_{0x} - i\tilde{\beta}_{0p}). \quad (\text{J9})$$

Thus,  $s_0$  and  $\delta_0$  are invariant under the damping transformation.

Conversely, if different sets of parameters  $(c, d, \beta_x, \beta_p)$  yield the same  $(s_0, \delta_0)$ , they necessarily share the same  $(\tilde{s}_0, \tilde{\beta}_{0x}, \tilde{\beta}_{0p})$ , which guarantees that there exists a  $\tilde{t}$  that transforms one parameter set into the other. ■

## APPENDIX K: DETAIL OF APPROXIMATION WITH LOWER STELLAR RANK

In Sec. IV B, we have shown how to obtain a two-mode non-Gaussian state generator with a smaller measured photon number  $n'$ , from the constants  $k, d$  satisfying

$$\phi_n(x) \lesssim \phi_{n'}(kx - d), \quad (\text{K1})$$

where  $\phi_n(x)$  is the wave function of Fock state  $|n\rangle$  and the approximation holds around

$$x = x_0 = \frac{\sqrt{s_0 + 1}}{s_0} \delta_{0p}. \quad (\text{K2})$$

In this section, we explain how to obtain such  $k, d$ . We also show that the transformation of the non-Gaussian state generator can be expressed by application of a Gaussian filter to the control mode.

### 1. When $x_0 = 0$ and $n \equiv n' \pmod{2}$

First, we consider the simplest case with no displacement ( $x_0 = 0$ ) and with  $n$  and  $n'$  of the same parity. In this case, we may set  $d = 0$ . Let the wave function of the Fock state  $|n\rangle$  be  $\phi_n(x)$ . This satisfies the Schrödinger equation

$$\phi_n''(x) = -(4n + 2 - x^2)\phi_n(x). \quad (\text{K3})$$

Using the “local momentum”

$$p(x) = \sqrt{4n + 2 - x^2}, \quad (\text{K4})$$

this can be rewritten as

$$\phi_n''(x) = -[p(x)]^2 \phi_n(x). \quad (\text{K5})$$

Near  $x = 0$ , and taking parity into account, the semi-classical approximation [63,64,66] gives

$$\phi_n(x) \propto \cos\left(\int_0^x p(x')dx' - \frac{n}{2}\pi\right). \quad (\text{K6})$$

Now consider the scaled wave function

$$\tilde{\phi}_{n'}(x) := \phi_{n'}(kx). \quad (\text{K7})$$

It satisfies

$$\tilde{\phi}_{n'}''(x) = -[\tilde{p}(x)]^2 \tilde{\phi}_{n'}(x), \quad (\text{K8})$$

with

$$\tilde{p}(x) = k^2(k^2x^2 - 4n' - 2). \quad (\text{K9})$$

Hence, choosing

$$k = \sqrt{\frac{2n+1}{2n'+1}} \quad (\text{K10})$$

ensures that, near  $x = 0$ ,

$$p(x) \sim \tilde{p}(x). \quad (\text{K11})$$

Since  $n$  and  $n'$  have the same parity, the initial conditions at  $x = 0$  also match:

$$[\phi_n(0), \phi_n'(0)] \propto [\tilde{\phi}_{n'}(0), \tilde{\phi}_{n'}'(0)]. \quad (\text{K12})$$

Therefore, from Eq. (K6), the approximation

$$\phi_n(x) \propto \tilde{\phi}_{n'}(x) \quad (\text{K13})$$

holds around  $x = 0$ .

## 2. Case of $x_0 \neq 0$

In the case with displacement ( $x_0 \neq 0$ ), the Schrödinger equation for the squeezed and displaced Fock-state wave function

$$\tilde{\phi}_{n'}(x) := \phi_{n'}(kx - d) \quad (\text{K14})$$

is

$$\tilde{\phi}_{n'}''(x) = -[\tilde{p}(x)]^2 \tilde{\phi}_{n'}(x), \quad (\text{K15})$$

$$\tilde{p}(x) = k^2[(kx - d)^2 - (4n' + 2)]. \quad (\text{K16})$$

Thus, the approximation

$$\phi_n(x) \propto \tilde{\phi}_{n'}(x) \quad (\text{K17})$$

holds near  $x = x_0$ , if the local momentum matches:

$$p(x_0) = \tilde{p}(x_0), \quad (\text{K18})$$

and the initial conditions also match:

$$[\phi_n(x_0), \phi_n'(x_0)] \propto [\tilde{\phi}_{n'}(x_0), \tilde{\phi}_{n'}'(x_0)]. \quad (\text{K19})$$

By numerically solving Eqs. (K18) and (K19) simultaneously, the values of  $(k, d)$  can be narrowed down to a finite set. Among these, selecting the one that minimizes the difference in the first derivative of the local momentum,

$$|p'(x_0) - \tilde{p}'(x_0)|, \quad (\text{K20})$$

provides a better approximation over a wider range (method 1).

This method gives good approximations when  $x_0$  lies inner of the turning point  $x_t$  [where  $p(x_t) = 0$ ], but when this is not the case, a better approximation can be obtained by solving the conditions

$$p(x_0) = \tilde{p}(x_0), \quad (\text{K21})$$

$$p'(x_0) = \tilde{p}'(x_0), \quad (\text{K22})$$

simultaneously to find  $k$  and  $d$ . In this case, the equations reduce to a cubic equation for  $k$ , which can be solved analytically (method 2). We employ a heuristic method based on numerical experiments as follows. Using the largest root  $x_z > 0$  of  $\phi_n(x)$ ,

- (i) when  $|x_0| < x_z$ , use method 1;
- (ii) when  $x_z \leq |x_0| < x_t$ , use method 2 with  $x_0 = x_t$ ;
- (iii) when  $x_t \leq |x_0|$ , use method 2.

## 3. Representation by a Gaussian filter

So far, we have shown that the output state of a two-mode non-Gaussian state generator can be approximated up to Gaussian unitary degrees of freedom. We now derive the explicit form of the transformation of the state generators, in the form of a Gaussian filter acting on the control mode.

The approximation Eq. (71) can be explicitly written as a transformation of non-Gaussian state generators:

$${}_c\langle n|G\rangle_{C,\beta}^{(w)} \propto {}_c\langle n'| \left( \hat{U}_{s'_0, \delta'_0, n' \rightarrow s_0, \delta_0, n} \right)_s |G\rangle_{C', \beta'}^{(w)}, \quad (\text{K23})$$

where  $|G\rangle_{C,\beta}^{(w)}$  is a Gaussian state having the control-mode representation  $(C, \beta)$  and corresponding to the wave form  $|\psi\rangle_{s_0, \delta_0, n}^{(w)}$  and  $\hat{U}_{s'_0, \delta'_0, n' \rightarrow s_0, \delta_0, n}$  is the Gaussian unitary operator defined in Eq. (70). Using the Choi-Jamiołkowski isomorphism, we define the operator corresponding to  $|G\rangle_{C,\beta}^{(w)}$  as  $\hat{G}_{C,\beta}^{(w)}$  (see Appendix C). Then, this transformation becomes

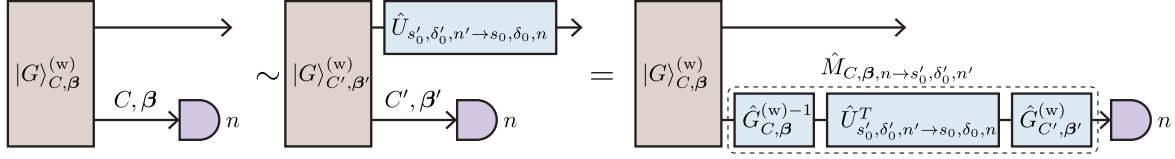


FIG. 9. Graphical illustration of how the approximation of a two-mode non-Gaussian state generator can be represented as a Gaussian filter acting on the control mode.

$$|G\rangle_{C,\beta}^{(w)} \rightarrow \left( \hat{U}_{s'_0, \delta'_0, n' \rightarrow s_0, \delta_0, n} \right)_s |G\rangle_{C',\beta'}^{(w)} \quad (\text{K24})$$

$$= \left[ \hat{G}_{C',\beta'}^{(w)} \hat{U}_{s'_0, \delta'_0, n' \rightarrow s_0, \delta_0, n}^T (\hat{G}_{C,\beta}^{(w)})^{-1} \right]_c |G\rangle_{C,\beta}^{(w)} \quad (\text{K25})$$

$$=: (\hat{M}_{C,\beta, n \rightarrow s'_0, \delta'_0, n'})_c |G\rangle_{C,\beta}^{(w)}. \quad (\text{K26})$$

Thus, the transformation can be expressed as applying a Gaussian filter  $\hat{M}_{C,\beta, n \rightarrow s'_0, \delta'_0, n'}$  on the control mode (see Fig. 9). Note that this operator is not necessarily be physical. Since an arbitrary non-Gaussian state generator  $|G\rangle_{C,\beta}$  can be expressed as

$$|G\rangle_{C,\beta} = \hat{U}_s^{(w)} |G\rangle_{C,\beta}^{(w)} \quad (\text{K27})$$

using a Gaussian unitary operator  $\hat{U}_s^{(w)}$  acting on the signal mode, the transformation of  $|G\rangle_{C,\beta}$  is also written as

$$|G\rangle_{C,\beta} \rightarrow (\hat{M}_{C,\beta, n \rightarrow s'_0, \delta'_0, n'})_c |G\rangle_{C,\beta} \quad (\text{K28})$$

using the same  $\hat{M}_{C,\beta, n \rightarrow s'_0, \delta'_0, n'}$ .

The choice of  $(C', \beta')$  corresponding to  $(s'_0, \delta'_0)$  is not unique and has the degree of freedom of applying a damping operation. One possible choice is the  $C'$  that has the same symplectic eigenvalues as  $C$ , which we employ for the algorithm implemented in Sec. VI.

## APPENDIX L: INVARIANT NON-GAUSSIAN CONTROL PARAMETERS

In Sec. V, we defined the non-Gaussian control parameters  $s_0, \delta_0$  for multimode non-Gaussian state generators. Although this definition is intuitive and plays a key role in the optimization algorithm proposed in this work, it suffers from the drawback that it is not invariant under the damping transformation. In this section, we introduce an alternative definition—the *invariant non-Gaussian control parameters*—which are invariant under damping. These quantities are useful for a quantitative analysis of the properties of the output states and are used in the discussion of the GKP state in Appendix M.

Previously, the partial moments  $C_m, \beta_m$  were defined as the  $2 \times 2$  submatrices and subvectors of  $C, \beta$  [Eqs. (88)

and (89)]. Instead, consider projecting all control modes except the  $m$ th mode onto vacuum:

$$|G\rangle_m := \left( \prod_{\substack{l \leq m' < l+k \\ m' \neq m}} m' \langle 0| \right) |G\rangle. \quad (\text{L1})$$

The *invariant partial moments*  $\tilde{C}_m, \tilde{\beta}_m$  are defined as the control moments of  $|G\rangle_m$ . By construction,  $\tilde{C}_m$  and  $\tilde{\beta}_m$  are invariant under any damping transformation applied to modes other than the  $m$ th mode. Consequently, the *invariant non-Gaussian control parameters*  $\tilde{s}_0, \tilde{\delta}_0$  are obtained by replacing  $C_m, \beta_m$  with  $\tilde{C}_m, \tilde{\beta}_m$  in the definitions of the standard control parameters. These parameters are, therefore, invariant under damping.

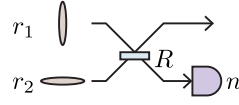
The physical interpretation of  $\tilde{s}_0, \tilde{\delta}_0$  is that they represent the values of  $s_0, \delta_0$  when all other control modes are projected onto the vacuum state ( $n = 0$ ). Heuristically,  $\tilde{s}_0$  tends to take larger values than  $s_0$ , due to the additional Gaussian filtering induced by these vacuum projections. These invariant quantities are particularly useful for quantitative comparisons of the output states of multimode non-Gaussian state generators, such as in the GKP state example discussed in Appendix M.

## APPENDIX M: DETAILS OF THE OPTIMIZATION EXAMPLES

We now provide details of the application of the algorithm in Sec. VI to the examples presented in Sec. VIC. The procedure consists of two steps: photon-number reduction and probability enhancement.

In Figs. 11, 12, and 14, we show the resulting state generators after both steps (“final”), together with the intermediate result after the first step (“reduced”). Each figure includes schematics of the state generator and its parameters, as well as the control-mode representation  $(C, \beta)$ . For clarity, we adopt an ordering convention for  $(C, \beta)$  given by  $x_1, \dots, x_k, p_1, \dots, p_k$ , instead of the  $x_1, p_1, \dots, x_k, p_k$  convention used in the main text.

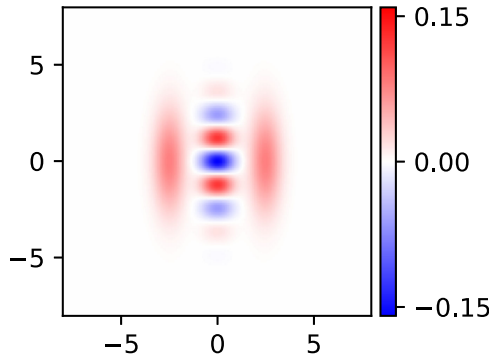
All squeezing values  $r$  are reported in decibel (dB) scale, according to the conversion formula



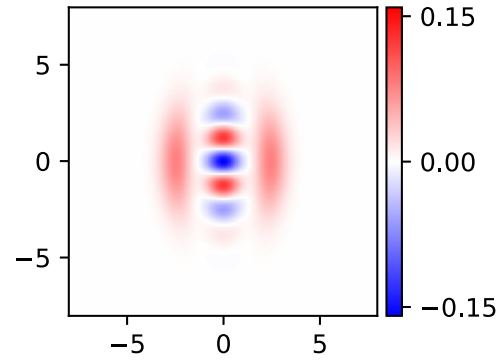
(a) Setup for cat state generation.

|          | Odd cat state |       |       |      |                       |                                                             |                                              | Even cat state |     |       |       |      |                       |                                                             |                                              |         |
|----------|---------------|-------|-------|------|-----------------------|-------------------------------------------------------------|----------------------------------------------|----------------|-----|-------|-------|------|-----------------------|-------------------------------------------------------------|----------------------------------------------|---------|
|          | $n$           | $r_1$ | $r_2$ | $R$  | $p$                   | $C$                                                         |                                              | $\beta$        | $n$ | $r_1$ | $r_2$ | $R$  | $p$                   | $C$                                                         |                                              | $\beta$ |
| Original | 15            | 5.0   | -5.0  | 0.10 | $1.77 \times 10^{-6}$ | $\begin{bmatrix} 0.60 & 0.00 \\ 0.00 & 2.88 \end{bmatrix}$  | $\begin{bmatrix} 0.00 \\ 0.00 \end{bmatrix}$ |                | 16  | 5.0   | -5.0  | 0.10 | $8.29 \times 10^{-7}$ | $\begin{bmatrix} 0.60 & 0.00 \\ 0.00 & 2.88 \end{bmatrix}$  | $\begin{bmatrix} 0.00 \\ 0.00 \end{bmatrix}$ |         |
| Reduced  | 5             | 3.46  | -4.22 | 0.24 | $3.55 \times 10^{-4}$ | $\begin{bmatrix} 0.97 & 0.00 \\ 0.00 & 1.78 \end{bmatrix}$  | $\begin{bmatrix} 0.00 \\ 0.00 \end{bmatrix}$ |                | 6   | 3.52  | -4.34 | 0.22 | $1.21 \times 10^{-4}$ | $\begin{bmatrix} 0.94 & 0.00 \\ 0.00 & 1.84 \end{bmatrix}$  | $\begin{bmatrix} 0.00 \\ 0.00 \end{bmatrix}$ |         |
| Final    | 5             | 14.33 | -5.96 | 0.22 | $4.58 \times 10^{-2}$ | $\begin{bmatrix} 0.91 & 0.00 \\ 0.00 & 21.09 \end{bmatrix}$ | $\begin{bmatrix} 0.00 \\ 0.00 \end{bmatrix}$ |                | 6   | 15.00 | -5.90 | 0.21 | $3.84 \times 10^{-2}$ | $\begin{bmatrix} 0.83 & 0.00 \\ 0.00 & 25.17 \end{bmatrix}$ | $\begin{bmatrix} 0.00 \\ 0.00 \end{bmatrix}$ |         |

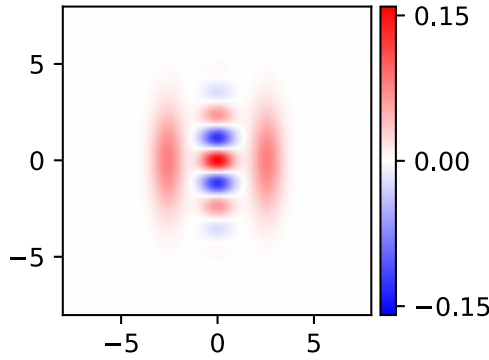
(b) Parameters of odd and even cat state generators. “Original”: before optimization; “Reduced”: after the first step (photon-number reduction); “Final”: after the second step (probability enhancement).



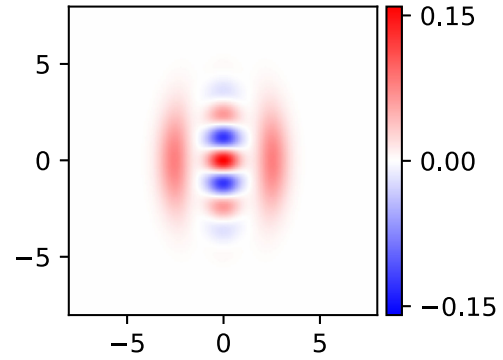
(c) Odd cat state before optimization.



(d) Odd cat state after optimization.



(e) Even cat state before optimization.



(f) Even cat state after optimization.

FIG. 10. Optimization of odd and even cat states.

$$r_{\text{dB}} = \frac{20}{\ln 10} r \approx 8.686r. \quad (\text{M1})$$

We remark that in all examples some probability enhancement is already observed in the first step, reflecting the general tendency that larger photon numbers correspond to lower success probabilities.

### 1. Cat state

We consider the GPS scheme [35] for generating cat states, illustrated in Fig. 10(a). The initial squeezing is set to  $r_1 = -r_2 = 5$  dB, and the beam splitter reflectance to 0.1, both of which are reasonable experimental assumptions. We begin with  $n = 15, 16$  photon detections for odd and even cat states, respectively, and set the target photon

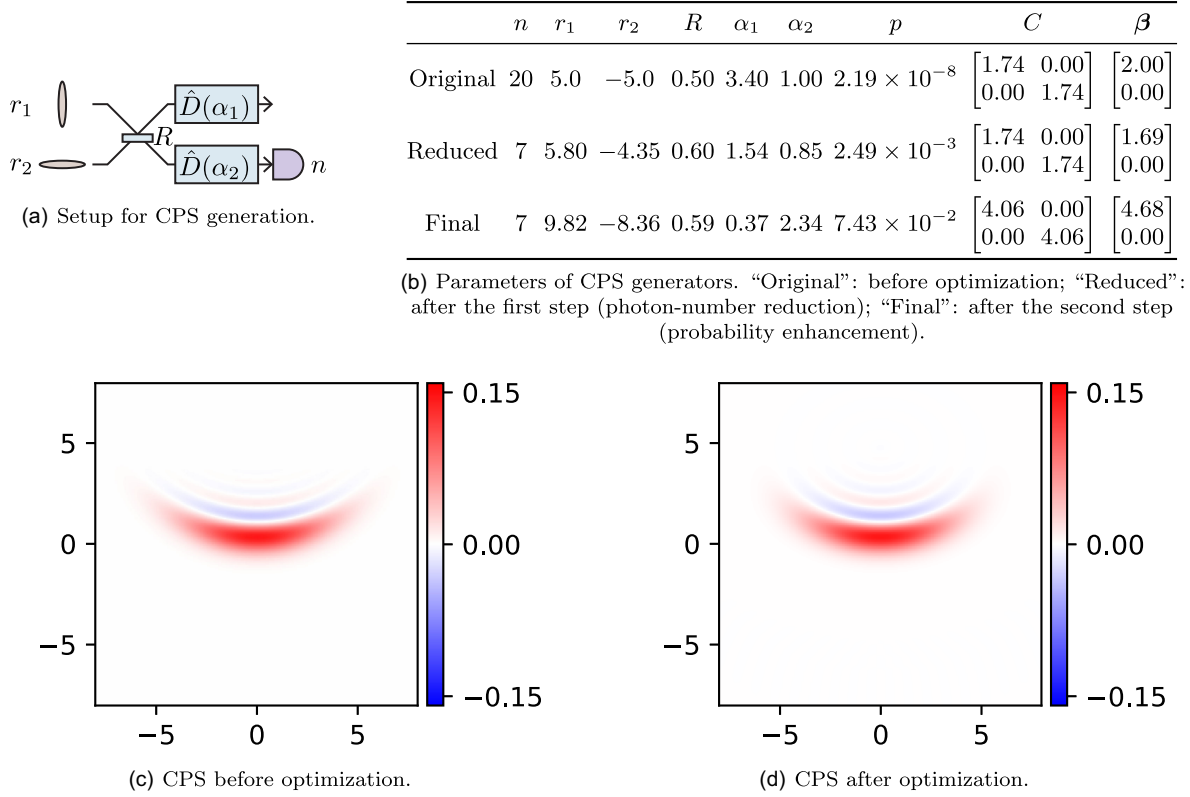


FIG. 11. Optimization of CPS.

numbers to  $n' = 5, 6$  so that the resulting states maintain high quality. To quantify the performance, we use the  $x^2$  squeezing [29] defined in Eq. (4) in the main text. The detailed results are presented in Fig. 10.

## 2. CPS state

We consider the scheme of Ref. [10] for generating CPS states using a displaced two-mode squeezed state, illustrated in Fig. 11(a). The initial squeezing is set to  $r_1 = -r_2 = 5$  dB, with a displacement  $\alpha_2 = 1$  applied to the control mode. A displacement is also applied to the signal mode, although not essential, so as to center the output state in phase space. We begin with  $n = 20$  and set the target photon number to  $n' = 7$ . To quantify the performance, we employ the cubic nonlinear squeezing [28] defined in Eq. (6) in the main text. The detailed results are presented in Fig. 11.

## 3. GKP state

We consider the cat-breeding protocol [62] with  $x = 0$  conditioning, shown in Fig. 12(a). Since homodyne detection is Gaussian, exchanging the order of photon-number measurement and conditioning maps this protocol to the Gaussian breeding circuit [17] in Fig. 12(b). A detailed derivation of this equivalence can be found in Supplemental Material of Ref. [18].

The circuit obtained from the breeding has the form

$$C = \begin{pmatrix} C_{xx} & 0 \\ 0 & dI \end{pmatrix}, \quad (\text{M2})$$

$$C_{xx} = \begin{pmatrix} a & a - \frac{1}{d} & \cdots & a - \frac{1}{d} \\ a - \frac{1}{d} & a & \cdots & a - \frac{1}{d} \\ \vdots & \vdots & \ddots & \vdots \\ a - \frac{1}{d} & a - \frac{1}{d} & \cdots & a \end{pmatrix}, \quad (\text{M3})$$

where  $d > 1$  and  $ad > 1$ . The invariant non-Gaussian phase sensitivity  $\tilde{s}_{0m}$  (see Appendix L) is given by

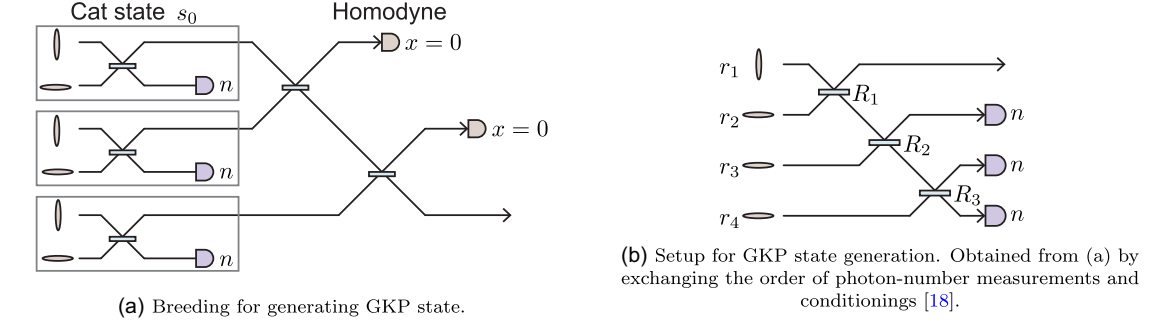
$$\tilde{s}_{0m} = \tilde{s}_0 = \frac{d - \tilde{c}}{\tilde{c}d - 1}, \quad (\text{M4})$$

with

$$\tilde{c} = \frac{1}{d} \cdot \frac{a + (k-1)(a-1/d)}{a + (k-2)(a-1/d)} - 1. \quad (\text{M5})$$

Breeding  $k$  cat states with non-Gaussian phase sensitivity  $s_0$  yields

$$\tilde{s}_0 = ks_0 + k - 1. \quad (\text{M6})$$



|          | $n$ | $r_1$ | $r_2$  | $r_3$  | $r_4$  | $R_1$ | $R_2$ | $R_3$ | $p$                    | $C$                                                                                                                                                                                                                                                                                               | $\beta$                                                                      |
|----------|-----|-------|--------|--------|--------|-------|-------|-------|------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------|
| Original | 20  | 6.53  | -8.29  | -7.38  | -7.38  | 0.80  | 0.33  | 0.50  | $1.75 \times 10^{-12}$ | $\begin{bmatrix} 0.45 & 0.27 & 0.27 & 0.00 & 0.00 & 0.00 \\ 0.27 & 0.45 & 0.27 & 0.00 & 0.00 & 0.00 \\ 0.27 & 0.27 & 0.45 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 5.47 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 5.47 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 5.47 \end{bmatrix}$    | $\begin{bmatrix} 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \end{bmatrix}$ |
| Reduced  | 7   | 5.29  | -6.31  | -4.17  | -4.17  | 0.58  | 0.33  | 0.50  | $5.54 \times 10^{-9}$  | $\begin{bmatrix} 0.77 & 0.39 & 0.39 & 0.00 & 0.00 & 0.00 \\ 0.39 & 0.77 & 0.39 & 0.00 & 0.00 & 0.00 \\ 0.39 & 0.39 & 0.77 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 2.62 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 2.62 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 2.62 \end{bmatrix}$    | $\begin{bmatrix} 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \end{bmatrix}$ |
| Final    | 7   | 8.09  | -16.20 | -14.00 | -14.00 | 0.60  | 0.33  | 0.50  | $1.44 \times 10^{-4}$  | $\begin{bmatrix} 0.89 & 0.85 & 0.85 & 0.00 & 0.00 & 0.00 \\ 0.85 & 0.89 & 0.85 & 0.00 & 0.00 & 0.00 \\ 0.85 & 0.85 & 0.89 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 25.12 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 25.12 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 25.12 \end{bmatrix}$ | $\begin{bmatrix} 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \end{bmatrix}$ |

(c) Parameters of GKP state generators. “Original”: before optimization; “Reduced”: after the first step (photon-number reduction); “Final”: after the second step (probability enhancement). The order of quadratures for  $C$  and  $\beta$  is  $x_1, \dots, x_k, p_1, \dots, p_k$ .

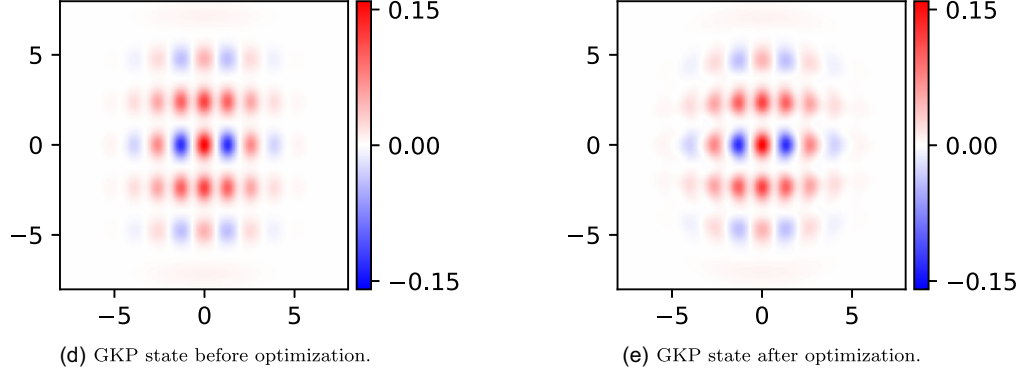


FIG. 12. Optimization of GKP state.

Using this  $\tilde{s}_0$ , the wave function of the generated state can be written explicitly as

$$\psi(x) \propto [\phi_n(x)]^n \exp \left[ -\frac{\tilde{s}_0 - k + 1}{4} x^2 \right]. \quad (\text{M7})$$

For the initial state generator in our optimization, we use the circuit corresponding to the critical condition  $s_0 = 1$  for each cat state, which is the maximum  $s_0$  achieved by the commonly used photon-subtraction scheme (see Sec. III D). The corresponding parameters for the cat-state

generator in Fig. 10(a) are  $r_1 = -r_2 = 8.00$  dB and  $R = 0.137$ , giving the invariant non-Gaussian phase sensitivity  $\tilde{s}_0 = 5$ .

We evaluate the performance using the GKP non-Gaussian squeezing [30], defined as

$$\xi_{\text{GKP}} = \min_{\lambda, \phi_1, \phi_2} \left\langle 2\cos^2 \left( \lambda \frac{\sqrt{\pi}}{2} \hat{x} + \phi_1 \right) + 2\cos^2 \left( \frac{1}{\lambda} \frac{\sqrt{\pi}}{2} \hat{p} + \phi_2 \right) \right\rangle. \quad (\text{M8})$$

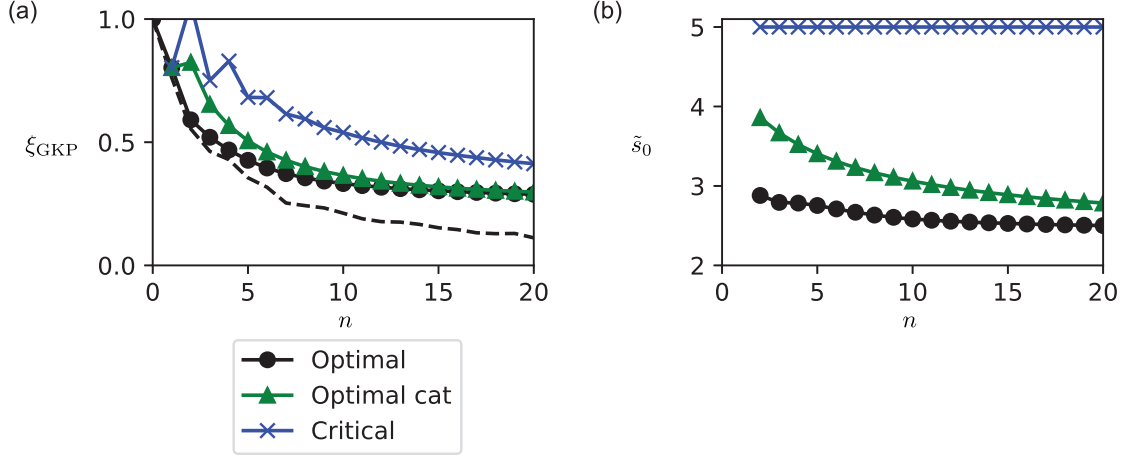


FIG. 13. (a) GKP squeezing  $\xi_{\text{GKP}}$  as a function of  $\tilde{s}_0$  for different  $n$ . “Optimal”: optimized over  $\tilde{s}_0$ . “Optimal cat”: breeding using cat states with  $s_0$  taken from Fig. 2(c). “Critical”: breeding using cat states under the critical condition  $s_0 = 1$ . Dashed line: stellar-rank limit [30]. (b) Corresponding values of  $\tilde{s}_0$  for (a). Values for  $n = 0, 1$  are not shown, as the state is independent of  $\tilde{s}_0$  up to Gaussian unitaries.

| $n$                       | $p$                    | $C$   | $\beta$ |
|---------------------------|------------------------|-------|---------|
| Original (14, 14, 14, 14) | $1.79 \times 10^{-30}$ | 1.51  | -0.27   |
|                           |                        | -0.14 | -0.06   |
|                           |                        | -0.15 | 0.38    |
|                           |                        | 0.20  | -0.11   |
|                           |                        | 0.26  | 0.22    |
|                           |                        | -0.22 | 0.02    |
|                           |                        | 0.12  | 0.30    |
|                           |                        | -0.28 | -0.17   |
| Reduced (9, 14, 2, 12)    | $1.25 \times 10^{-22}$ | 0.78  | -0.13   |
|                           |                        | 0.06  | -0.31   |
|                           |                        | 0.10  | 0.23    |
|                           |                        | -0.16 | 0.20    |
|                           |                        | -0.06 | -0.22   |
|                           |                        | -0.21 | 0.08    |
|                           |                        | 0.08  | -0.13   |
|                           |                        | 0.04  | 0.01    |
| Final (9, 14, 2, 12)      | $4.50 \times 10^{-6}$  | 2.27  | -1.40   |
|                           |                        | 1.06  | -1.79   |
|                           |                        | 0.79  | -0.20   |
|                           |                        | -0.88 | 0.16    |
|                           |                        | 2.87  | -1.70   |
|                           |                        | -6.24 | 4.02    |
|                           |                        | 1.55  | -1.54   |
|                           |                        | 6.02  | -3.82   |

(a) Parameters of Random state generators. “Original”: before optimization; “Reduced”: after the first step (photon-number reduction); “Final”: after the second step (probability enhancement). The order of quadratures for  $C$  and  $\beta$  is  $x_1, \dots, x_k, p_1, \dots, p_k$ .

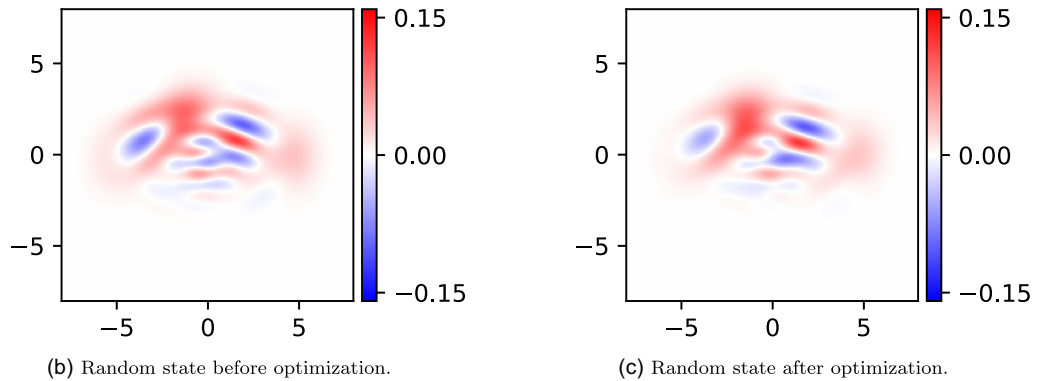


FIG. 14. Optimization of the random state.

As in the cat-state case in Sec. II, there exists an optimal  $\tilde{s}_0$  that yields the best GKP squeezing, as shown in Fig. 13. This optimal value is smaller than that obtained from cat breeding, whether under the critical condition ( $s_0 = 1$ , Sec. III D) or under the condition for optimal  $x^2$  squeezing [Fig. 2(c)].

In our case, after optimization,  $\tilde{s}_0$  decreases from  $\tilde{s}_0 = 5$  to  $\tilde{s}_0 = 3.05$ , approaching the optimal value. This reduction improves the photon number without degrading state quality, highlighting the nonoptimality of the unmodified breeding scheme based on conventional photon-subtracted cat states with  $s_0 < 1$ .

#### 4. Random state

Finally, we apply the algorithm to random Gaussian states. A random Gaussian unitary  $U$  is sampled using the `random_symplectic` function in MRMUSTARD [68], constructed as  $U = WS(r)V$  from two Haar-random symplectic orthogonal matrices  $W$ ,  $V$  and a random squeezing vector  $r \in [0, r_{\max}]$ . With a displacement  $d \in [0, d_{\max}]^{\otimes 2(k+1)}$ , the state is

$$|G\rangle = D(d)U|0\rangle. \quad (\text{M9})$$

We use  $r_{\max} = 1$  and  $d_{\max} = 0.5$ . Figure 14 shows one example, showing improvements in both probability and the measured photon numbers.

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