

Morse Theory and Meron-Mediated Interactions Between Disclination Lines in Nematic Materials

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The topological understanding of nematic liquid crystals has traditionally centered on defects and their classification via homotopy invariants. Here, we draw upon several aspects of Morse theory to describe a finer approach to topology that characterizes the isotopy invariants within a given homotopy class. This approach unifies various disparate ideas in liquid crystal physics, including the homotopy and Volterra classifications of defects, the winding and self-linking of disclinations, and the Pontryagin-Thom construction. Specifically, we use singularity theory and dividing curves to characterize nematic textures via tomography, as well as to provide a full topological classification of the rewiring and crossing of disclination lines. This process reveals the central role that nonsingular but nontrivial topological solitons—so-called merons—play in mediating interactions between disclination lines. We also clarify the notion of the charge of a disclination line, give a simple method for computing it, and apply this to examples drawn from experiment. In particular, this method shows that restricting a disclination line's charge modulo 2 is incorrect; we construct examples of disclinations with point defect charges 0 and -2 , which have the same local structure but very different global structures, underscoring the need for a more nuanced topological understanding of nematic systems.

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I. INTRODUCTION

Nematic ordering is a ubiquitous concept in physics, with applications ranging from molecular liquid crystals [1] to morphogenesis in biological tissues [2,3]. The topological understanding of such materials is traditionally centered on singularities, or defects, which have proved fundamental to many aspects of their physics. As centers of strong elastic distortion, defects are the essential drivers of dynamics in both passive and active systems, and the behavior of the system as a whole can often be approximated by the behavior of the defects alone. For example, defects control the material's strength [4], mediate elastic interactions [5,6] and phase transitions [7,8], and govern the coarsening dynamics of nematic order [9]. As a result, the physics of nematic defects has been studied extensively

for many decades [10–33]. In recent years, the focus has shifted to active liquid crystals [34,35], where the active flows cause $+1/2$ defects to self-propel, inducing complex, defect-driven dynamics that have sparked a great deal of interest [36–61].

The classification of defects in terms of homotopy theory is a cornerstone of our understanding of these behaviors [1,12,23]. In addition to classifying the local structure of defects, homotopy theory provides a coarse means of describing local minimizers of the energy: Two textures that correspond to distinct homotopy classes can only be deformed into one another by a process requiring the creation and annihilation of defects, which imposes an energy barrier between them. However, it has long been recognized that homotopy theory methods fail to fully describe materials with a spatially modulated ground state, such as smectics and cholesterics [13,14]. Extensions of the homotopy theory using more advanced topological methods that analyze finer topological structure have proved invaluable in understanding the behavior of these materials [62–72]. Topological invariants arising from homotopy theory do not fully capture the complexity of defect textures in complex geometries—for example, in colloidal systems [73]—which has

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led to the development of a more geometric theory [20–22,24,25,27]. More generally, characterizing the complex spatiotemporal structures that emerge in both passive and active systems is a major ongoing challenge motivating a more detailed analysis [26,42,47,53,57,59].

Many of these issues manifest more prominently in three dimensions than two. For example, in two dimensions, the distinction between $+1/2$ -winding and $-1/2$ -winding defects is fundamental—one cannot be converted into the other. However, homotopy theory tells us that converting one into the other *is* possible in three dimensions and that there is no topological distinction between $+1/2$ and $-1/2$ winding around a disclination line [17,23,31]. As a result, the topology of disclination lines has been downplayed due to a belief in its relative unimportance. However, we argue that this is an artifact of a homotopy-centric approach, which overlooks a notion that is very important to the underlying physics: In addition to topological defects, three-dimensional materials exhibit additional topological structures that, although nonsingular, are nevertheless nontrivial. These structures are topological solitons, called merons or half-skyrmions, which were originally introduced to describe the interactions resulting in quark confinement [74], and they also occur in magnetic systems [75] and in phases of superfluid helium [76–78].

In modulated phases such as smectics and cholesterics, merons are, much like defects, stable and topologically protected regions of strong elastic distortion with a finite and well-defined core size. Textures such as skyrmion lattices and cholesteric blue phases—in which these solitons are the central features—are well studied both theoretically and in experiment, and the fundamental role played by merons in mediating phase transitions and stabilizing complex structures in these phases has been well established [79–90]. In ordinary achiral nematics, merons are still topologically protected in the sense that they cannot be removed by purely local perturbations to the director, but as there is no confining length scale other than the system size to constrain them, they are not fundamental defects or even necessarily centers of strong elastic distortion like they are in some modulated phases.

In this work, we describe an approach to topology that is more nuanced than traditional homotopy theory and that draws heavily on the mathematical field of Morse theory [91]. In doing so, we unify several disparate ideas from across liquid crystal physics, including the extended Volterra process [19], the geometric theory of disclination lines as “ribbons” [20,21,24,25], the homotopy theory description of disclination lines in both achiral and chiral nematics [17,23,31,70,71], and descriptions of the structure and orientation of disclination lines and solitons in terms of tensors that capture aspects of their geometry [29,30,32,86].

Specifically, we combine “dividing curves,” singularity theory, and tomography to capture topological changes due

to isotopy in addition to those that result from homotopy. This method allows us to give a topological classification of important changes of structure in a nematic—including dynamical crossing and rewiring of disclination lines—in terms of invariants that are easily computable in simulations and also from experimental data. A key insight is the way that merons mediate changes in defect structure and the interactions between defects, which has not been appreciated previously. We also revisit the problem of assigning a defect charge to a disclination line, where we argue that the conflation of skyrmion charge and defect charge—reasonable in an orientable system—leads to confusion in the presence of disclination lines. We show that, in addition to global topological constraints, there are local topological constraints coming from Morse theory that have been overlooked. As a result, there are some inconsistencies in previous works, which we correct. The tools we introduce provide a means of assigning defect charge that is both locally and globally consistent and can be readily applied to textures of arbitrary complexity.

As we focus on generic topological properties of nematic fields, without reference to a particular class of materials or energetics, our results are quite generic. We show that meron creation or annihilation is a topologically required part of a variety of complex phenomena and interactions between defects, including changes in defect winding, as well as fundamental dynamical processes such as the rewiring and crossing of defect lines. Similar complex, soliton-mediated phenomena are known to occur in superfluid helium-3 [92,93], where some of our results may also be applied and are fundamental to the understanding of interactions in particle physics. These constraints we highlight are especially physically relevant in modulated phases where solitons are indeed stable defects, and we give numerous examples of how our results apply to real systems throughout the text. Our results can also be relevant to ordinary, achiral nematics with a uniform ground state, under some assumptions on the boundary conditions that we elucidate. The structural changes and interactions between disclination lines we study in this work also occur frequently in active nematics, especially in the turbulent regime [47]. Merons and skyrmions certainly arise in active cholesterics, where they are stabilized by contractile activity but can be decomposed into pairs of disclinations by extensile activity [48]. More generally, the role of topological solitons in active flows has received very little attention in the literature.

The remainder of this paper is divided into two parts. The first part, encompassing Secs. II–IV, introduces our topological framework, in which director fields are studied through dividing curves. In Sec. II, we highlight the distinction between homotopy and isotopy for director fields, one aspect of which is the distinction between Volterra-type descriptions of crystalline defects and the homotopy theory description of liquid crystal defects.

Isotopy invariants are finer than homotopy invariants but still relate to energy barriers within the material. Changes in isotopy class can describe a process by which the material moves around within a homotopy class; while changes in homotopy class are mediated by defects, changes in isotopy class are mediated by merons. We highlight two key examples of the distinction: the conversion of a hedgehog point defect into a hyperbolic point defect, and the change in the winding of a disclination by the emission of a meron line. In Sec. III, we introduce the technique of tomography, in which a complex 3D director is studied through a family of surfaces that span the material, and the topology is captured by the dividing curves on those surfaces. Tomography reduces the topological content of the director down to a finite set of features—defects and merons—along with the structural changes (or “degeneracies”) in these features, several of which involve interactions between defects and merons. In Sec. IV, we introduce the second part of our method, which focuses on dividing curves on surfaces that enclose defects and solitons rather than cutting across them. This process provides a complementary perspective to tomography, and both are needed to fully analyze the topological aspects of dynamical processes such as rewirings. It also clarifies the notion of the extended Volterra process [19] by describing the homotopies that govern changes in the isotopy class of disclinations.

The second part of the paper, Secs. V–VIII, applies this framework to several fundamental aspects of the behavior of disclinations. In particular, our focus is on how dynamical processes involving disclinations, which include both changes in local structure via homotopies and also interactions between defects—rewirings and crossings—either introduce or remove structural degeneracies in the tomography. In Sec. V, we explain how disclination lines are oriented and inherit a self-linking number related to the Burgers vector, which helps connect our work to previous studies that have focused on the geometry of disclination lines [25,29,32] while also highlighting the fundamental topology that underlies their more geometric constructions. In Sec. VI, we use dividing curves to describe how disclination lines of arbitrary knot type and with arbitrary director field structure are endowed with a defect charge that is both locally and globally consistent: This method is important for a proper understanding of how they interact with other disclination lines and also with point defects. In Sec. VII, we give a topological classification of rewiring processes. While these processes have been studied previously, a complete topological classification is lacking. Indeed, we show there are several distinct types of rewiring, which has not previously been appreciated, and analyze the role merons play in mediating these processes through the formation of structural degeneracies in the tomography. Finally, in Sec. VIII, we describe the topological aspects of defect crossing processes. Throughout, we illustrate our results with references to known textures, explain how our

methodology applies and recapitulates known results, and provide a novel perspective on these structures.

We conclude in Sec. IX with a discussion of our work and prospects for future applications and developments.

II. DISTINCTION BETWEEN HOMOTOPY AND ISOTOPY UNDERPINS HIDDEN TOPOLOGICAL STRUCTURE

A key aspect of this work is in identifying the role played by topological solitons, merons, in mediating changes of topological structure in a material. We argue that, in order to properly account for merons, the traditional homotopy-centric approach to topological classification needs refinement. We instead focus on isotopy, a finer way of describing the material’s evolution. When a homotopy of the local structure of a defect is not an isotopy, then the homotopy must produce an additional feature in order to match the rest of the texture. Within a given homotopy class, there are multiple distinct isotopy classes, and just as moving between different homotopy classes changes the defect set—i.e., the singularities—of a material, moving between different isotopy classes (within a given homotopy class) involves changes to merons. As homotopy classes correspond to large-scale energy barriers between textures, so too may isotopy classes correspond to smaller energy barriers. This case is especially true in materials where an inherent material length scale stabilizes merons, but the distinction can still be relevant even in materials where this is not the case. In the latter case, merons may shrink or expand freely and are not likely to be part of equilibrium structures but may still be involved in transitory processes.

We now review the concepts of homotopies and isotopies. We describe the distinction using two key examples of relevance to real systems; we will return to these examples repeatedly throughout the remainder of the paper, as they illustrate the main notions we wish to convey.

A. Homotopies vs isotopies

Throughout this work, the state of the nematic material is captured by a unit vector $\mathbf{n}$ called the director, defined on the complement of some set of defects. The topological theory of nematic liquid crystals is based on homotopies, smooth paths $\mathbf{n}_t$ of director fields depending on time t , which typically move the defect set but not create, destroy, merge, or cross defects. Homotopy theory therefore concerns the computation and interpretation of quantities that are left invariant by homotopies, such as the total defect charge. More broadly, it has been used to classify the local structure of both point and line defects [12,17,23,31,94], and it classifies textures on the complement of the defects in terms of a Euler class (skyrmion charge) [85,86,95] and Hopf invariant [84,89].

Homotopies are the natural model for the evolution of a nematic material, where the “sticks” may move

independently of the material background. In other kinds of elastic media—for example, crystals, elastic solids, and fluids—the state of the material at time t is instead captured by a diffeomorphism ψ_t that encodes the deformation away from some reference configuration; the vector field $\mathbf{u}_t = \partial_t \psi_t \circ \psi_t^{-1}$ is the displacement field, whose symmetric gradient is the strain tensor [96]. The evolution of an elastic solid away from its initial state is captured not by a homotopy but rather an isotopy, which is a smooth path of diffeomorphisms ψ_t , with ψ_0 the identity map.

We may consider isotopies in a director field as well: A director field $\mathbf{n}_t$ is isotopic to another director field $\mathbf{n}_0$ if there exists an isotopy ψ_t with $\psi_t^* \mathbf{n}_t = \lambda_t \mathbf{n}_0$ for some scalar function λ_t . Informally, an isotopy between directors means they are related by a coordinate transformation of the material domain. Isotopies must therefore map integral curves of the director (lines to which the director is tangent) to other integral curves. As a consequence, isotopies may not create or destroy merons, while homotopies may create or annihilate these features freely so long as the Euler class and Hopf invariant are conserved. Thus, the role of isotopies in the study of nematics is to pick out finer changes in the topology structure of the material to which the homotopy theory is insensitive.

The distinction between isotopy and homotopy has been discussed previously by Kleman in the context of the Volterra process for nematic defects [10,11,16,19]. Defects (dislocations) in a crystal are described by the Volterra process [97], in which we imagine cutting a tube out from the material, slicing the tube open, displacing the sides of the cut relative to one another via some rotation and translation, and then replacing the tube. The transformation we use to perform the displacement provides a topological label for the type of dislocation that is an isotopy, but not a homotopy, invariant. As Kleman noted [10,11,16,19], this process has to be refined in a liquid crystal because of the possibility of performing homotopies. Our framework helps to further elucidate this distinction. Furthermore, isotopies are fundamental in chiral systems: Any evolution of the director field that maintains the material's sense of handedness is an isotopy, and this strongly constrains the behavior of the material in physically relevant ways [65].

B. Example 1: Conversion of a hedgehog point defect into a hyperbolic point defect

Up to homotopy, point defects are classified purely by their integer charge, generically ± 1 . This value is obtained by surrounding the point with a sphere S . The director field can be oriented across S , and it then determines a map $S \rightarrow S^2$ whose degree (winding) is the defect charge [12,23]. However, point defects can have geometrically distinct local structures: They may be radial hedgehogs, where the director points out everywhere from the singular point, or they may have a hyperbolic structure. The distinction is given by an isotopy invariant called the

Morse index, which we describe properly in Sec. III. The Morse index is not a homotopy invariant, so we may freely convert between a hedgehog and hyperbolic structure by homotoping the director in a local neighborhood of the defect. This process has been observed experimentally in cholesteric droplets with radial anchoring [98], where the material's chirality induces the conversion [66].

Because this homotopy is only a local change, there are nontrivial questions about how it extends globally. First, let us briefly examine the analogous situation in 2D, Fig. 1(a). In a 2D polar material a $+1$ -winding singularity can change its stability, with the polar field going from pointing out from the defect to pointing in, by emitting a limit cycle; this result can also occur to a $+1$ singularity in a nematic field, whether it is oriented or not. This case is the Hopf bifurcation, which we illustrate in Fig. 1(a). The limit cycle, shown in red, can (informally) be regarded as a nonsingular defect, a kind of topological soliton preserved by isotopies. It cannot be freely removed by local rearrangements of the director, and it can only be destroyed by collapsing it into a singularity or merging it with another limit cycle.

The 3D version of the Hopf bifurcation, shown in Fig. 1(b), shows how a radial hedgehog point defect converts into a hyperbolic point defect: In a cross section, the process is exactly a Hopf bifurcation. Such “twisted hedgehogs” are known to occur in chromonic liquid crystals, as well as other materials whose splay elastic constant is significantly greater than their twist elastic constant [99–101].

Imagine removing a small ball of material containing a hedgehog point defect and trying to glue a hyperbolic defect in its place. Because they are homotopic, it is possible to do this without inducing additional defects on the boundary of the gluing region, so long as the defects carry the same charge. However, we must introduce a limit cycle as in Fig. 1(b) that witnesses the fact that the hedgehog belongs to a different isotopy class to the hyperbolic defect—the presence of this feature is a strict topological constraint.

C. Example 2: Change in the winding of a disclination line

An even more important example of this topological constraint occurs for disclination lines. Unlike the situation for a 2D point defect, the winding of the director around the disclination line, generically either $\pm 1/2$, is not a homotopy invariant, and it can freely change along the line and also over time [12,23]. However, there is a subtlety here.

Consider the conversion of a $+1/2$ defect into a $-1/2$ defect in a 2D system, Fig. 1(c). The constraint of conserving topological charge—the Poincaré-Hopf theorem—illustrates that this requires the production of an additional $+1$ defect. This process is therefore not a

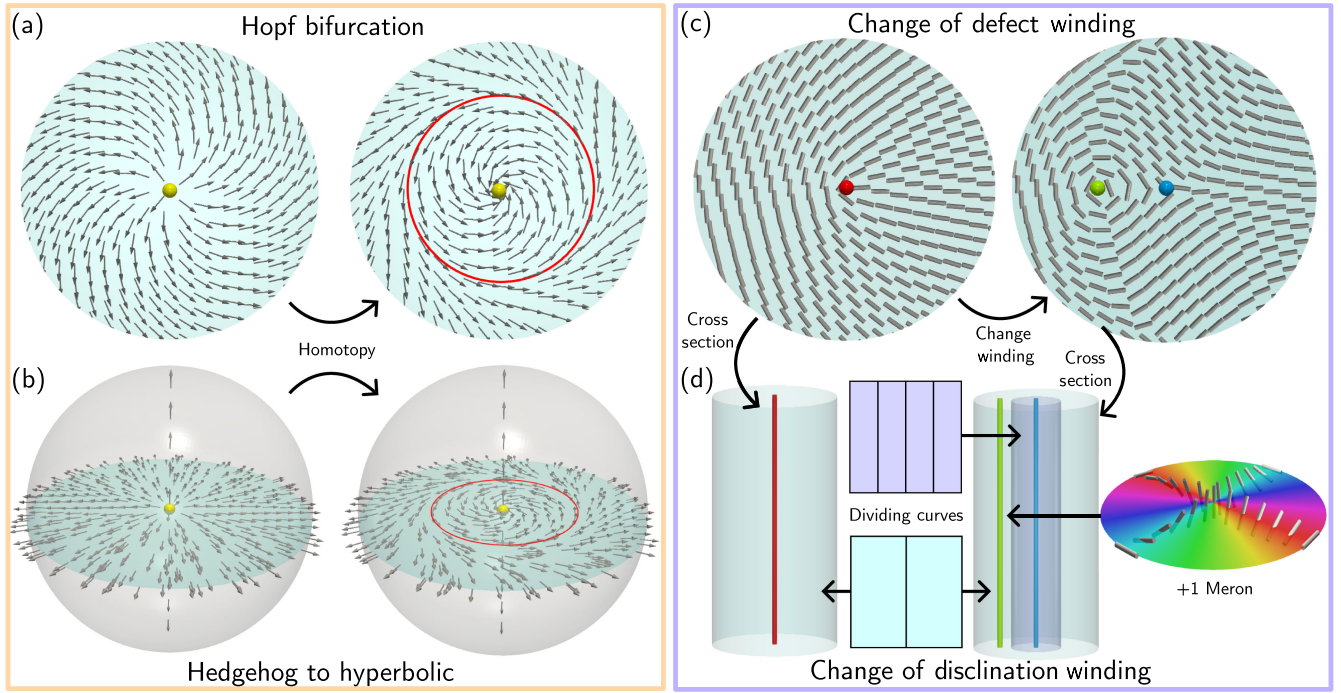


FIG. 1. Changes in a polar or nematic material that correspond to a homotopy that is not an isotopy resulting in the birth of nonsingular solitonlike topological features that are analogous to dislocations and that mediate dynamical behaviors. Understanding this finer topological constraint is fundamental to understanding wider phenomena that occur in 3D order materials, such as linking, rewiring, and crossing of defect lines. The essentials can be captured by two simple examples. In diagram (a), a two-dimensional polar material containing an aster (+1 point defect, yellow) can undergo a Hopf bifurcation, changing the stability of the singularity at the expense of expelling a limit cycle (red). In diagram (b), the analogous process in a 3D polar or nematic material is the conversion of a hedgehog defect to a hyperbolic defect, which occurs via a Hopf bifurcation in the director field on a cross section. This process is a homotopy, and it preserves the charge; however, as the structure of the director field around a hedgehog is different from a hyperbolic defect, we can distinguish them even when the orientation is removed. In diagram (c), we can change the winding of a point defect in a 2D nematic from $+1/2$ (red) to $-1/2$ (blue). The global constraint of defect charge conservation coming from the Poincaré-Hopf theorem implies we must emit a $+1$ singularity (green) in this process. In diagram (d), the analog in 3D is the conversion of a $+1/2$ -winding segment of a disclination into a $-1/2$ -winding segment. The conservation of charge on a cross section still applies: This process involves the emission of a $+1$ singular line. In 3D, integer-winding singular lines “escape,” becoming nonsingular defects known as merons. The fact that $\pm 1/2$ disclinations belong to different isotopy classes is witnessed by the dividing curve, introduced in Sec. III.

homotopy by our definition because it changes the defect set. Extending the 2D example into 3D, Fig. 1(d), shows that a $+1/2$ profile disclination line must still emit a $+1$ -winding singular line to convert to $-1/2$ winding. In a 3D material, integer winding singular lines are unstable and are well known to “escape into the third dimension” [71,102–105]. If the director is 2D in the far field, then this escape process will produce a meron. If the disclination line is of “twist” type and the director is parallel to the disclination tangent in the far field, then a meron is not necessary in the case of an achiral director field, although it will still be necessary if the director has to maintain chirality.

Assume the director field is perpendicular to the disclination in the far field. Consider removing a region of material containing a $+1/2$ disclination, indicated by the blue tube in Fig. 1(d)(i). Then, suppose we glue in a $-1/2$ -winding disclination, as in the purple tube in Fig. 1(d)(ii). Because these textures are homotopic, this

process can be performed without introducing additional defects, but because they are not isotopic, a meron must appear. As with the point defects, the presence of this feature is required by a strong topological constraint, which cannot be violated. Showing this feature, and describing the consequences for structural changes in the material, is the goal of Secs. III and IV.

III. TOMOGRAPHIC ANALYSIS REVEALS TOPOLOGICAL STRUCTURE OF NEMATIC TEXTURES

We now describe a tomographic approach to understanding 3D textures. This approach is designed to capture merons by analyzing 2D planes in which their projected topology appears to be singular and the constraints coming from the Poincaré-Hopf theorem apply. The result of this analysis is to break the material up into regions in which the structure can be described purely by an isotopy, separated

by surfaces in which the isotopy class changes due to the birth or death of a topological feature.

In brief, our process is as follows. We choose a family of surfaces that spans the material, say, surfaces of constant z coordinate. The structure of the director along each surface is captured by its projection, and the topology is unchanged, provided the topology of the projection does not change. Changes occur suddenly in jumps and signify fundamental topological changes in the material, which we characterize using singularity theory. Between the jumps, one may apply a diffeomorphism to bring the director into a form in which it is independent of the z coordinate—these forms can be viewed as fundamental building blocks of the structure, and the structure of the director can then be described coarsely by these blocks and the transitions between them. This approach is close to current experimental methods for the analysis and reconstruction of director fields using fluorescent confocal polarization microscopy (FCPM) [106–108].

We summarize our approach graphically in Fig. 2. There are four parts to the theory: (1) the analysis of the structure of the director near a surface and the detection of topological features, the disclinations and the merons; (2) how

this changes as we move the surface through the material; (3) a label, derived from singularity theory, that encodes local structure of disclinations and merons as seen by the projections; and (4) a classification of changes in this local structure, again in terms of singularity theory.

A. Projections and the dividing curve determine the local topology of the director along a surface

Let Σ be an oriented surface inside the material. Along Σ , the director defines a line field, the projection $\mathbf{n}^\perp$ of the director into the surface. As $\mathbf{n}$ is a unit vector, the projection completely determines the director along Σ , a fact that is used to reconstruct experimental directors from light microscopy data [106,108,109]. The projection has singularities in two places: $\pm 1/2$ singularities at points where the surface intersects disclination lines, and $+1$ singularities where the director is orthogonal to Σ .

The director can be homotoped so that it is tangent to Σ around all disclinations and indeed everywhere on the surface except possibly on a set of disks D_j around the points p_j where the director is orthogonal to the surface [71,85]. These singularities represent the local

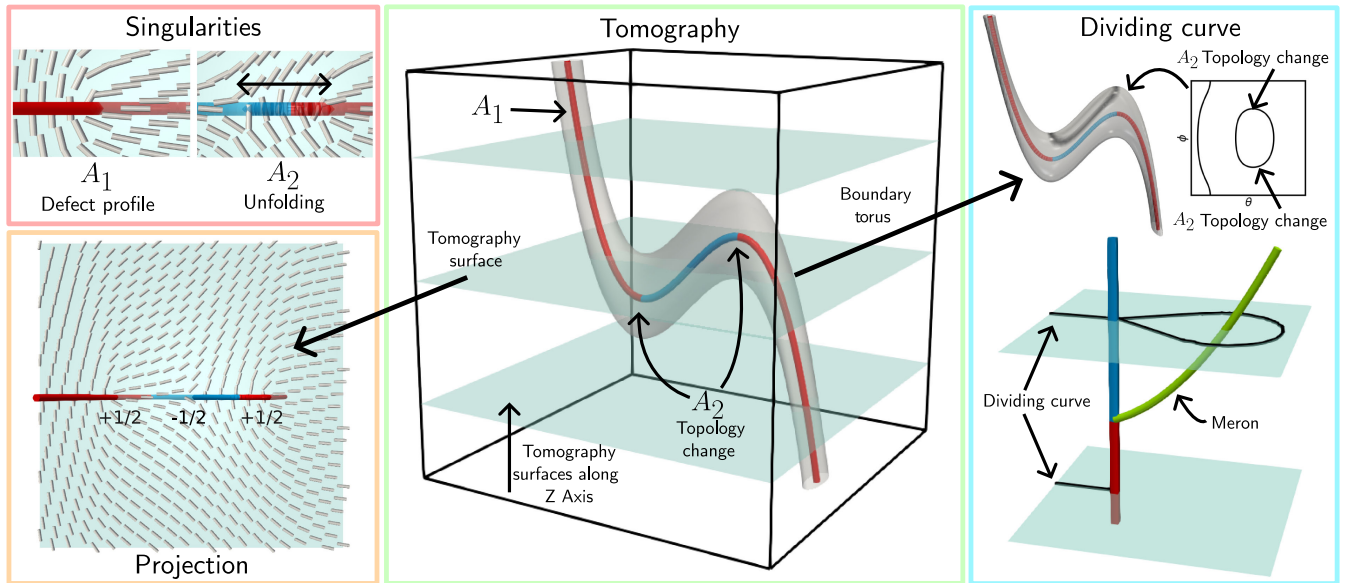


FIG. 2. Tomographic analysis of a director field, identifying and characterizing not only disclinations, which are singular, but also merons, which are nonsingular. We consider sliding a surface (pale blue) across a 3D director field that contains a disclination line. The projection of the director field encodes the topology of the director along that surface. As we slide the surface across the material, we obtain a family of projections that determine the entire topological structure of the director field. Crucially, the projection has singularities where it intersects not only disclinations but also merons, which are nonsingular in 3D. We color the disclination according to whether or not the singularity it induces in the projection has winding $+1/2$ (red) or $-1/2$ (blue). The isotopy class changes as the tomography traverses surfaces on which the number and type of singularities in the projection change. The changes in topological structure of the disclination line in the central panel can be seen in the dividing curve on a tube surrounding the disclination (top right). The isotopy class of the dividing curve associated with the projections into the tomography surfaces also changes as the tomography traverses surfaces across which the isotopy class of the director changes. This case is shown schematically on the bottom right, for a scenario in which a *chiral* disclination changes its winding from $+1/2$ to $-1/2$ by the emission of a meron line (green). The dividing curve (black) grows an additional loop, witnessing the change. For a nonchiral system, the extra loop may not appear; it will depend on boundary conditions.

obstruction to homotoping the director to make it tangent to Σ . They are merons [71,85]—the more familiar skyrmions are components of pairs of merons—and they define a topological invariant of the director called the Euler class. They are the minimal representation of the homotopy class of the director close to Σ , capturing the local topology. While they have a clear topological meaning, the physical and energetic relevance of these features will depend on their precise geometry, the boundary conditions, and the particular material under consideration. Notably, merons are (by definition) singularities in an axis orthogonal to the director field, so they are associated with pitch axis defects (λ lines) in a cholesteric and similar features (β lines) in twist-bend and splay-bend materials.

Suppose there are no disclinations intersecting Σ . Then, the director can be oriented in a neighborhood of S . Let p_j be the points where the director is orthogonal to Σ , which have winding w_j and orientation $s_j = \pm 1$ according to whether the director points out of or into the surface—that is, whether the singular line has escaped “up” or “down.” On a disk surrounding each meron point, the director determines a class in the relative homotopy group $\pi_2(\mathbb{RP}^2, \mathbb{RP}^1) \cong \mathbb{Z} \times \mathbb{Z}$ [71,85]. For $s_j = +1$, this class is $(w_j, 0)$, and for $s_j = -1$, it is $(0, -w_j)$. The value of the Euler class on Σ , equivalently the skyrmion charge Q , is equal to [70,71,110]

$$2Q = \sum_j s_j w_j. \quad (1)$$

The topology of the director can be encoded in an even simpler object than the projection. Consider the set of points Γ where the director is tangent to S . After perhaps performing a small homotopy of the director, Γ is a collection of disjoint curves referred to as the “dividing curve” [70,71,111]. Generically, the dividing curve consists of a finite number of disjoint, one-dimensional curves; if this is not the case, then we may perturb the surface slightly to make it so.

When Σ does not intersect disclinations, then the components of the dividing curves will either be closed loops or will have end points on the boundary of Σ in the case where Σ has a boundary. When Σ intersects disclinations, there will also be components of the dividing curve with end points on the disclinations—for a generic director field, there will be an odd number of such components. For an achiral nematic, there will generically be only one, and thus the dividing curve does not always distinguish between $\pm 1/2$ -winding profiles; however, the special topology of a chiral nematic ensures that a $+1/2$ singularity will generically have one component of the dividing curve terminate on it, while a $-1/2$ singularity will have three in order to maintain chirality of the director field near the defect [112]. Even in an achiral material, it is possible to

use dividing curves to distinguish between the types of winding around disclinations, but this requires us to examine the dividing curve on tubes enclosing the disclination and not cutting across it—we discuss this case later in the paper.

If the director can be oriented along Σ , the dividing curves separate regions where the director points into Σ (the set Σ^-) from regions where the director points out from Σ (the set Σ^+), which is the origin of the terminology. This process leads to an alternative formula for the skyrmion charge [70,71,110],

$$2Q = \chi(\Sigma^+) - \chi(\Sigma^-), \quad (2)$$

where χ denotes the Euler characteristic. Note that if Σ is a sphere, then the value of Q computed by Eq. (2) is exactly the defect charge contained within that sphere—in particular, this finding implies that the sphere is pierced by some number of merons and, indeed, that merons mediate interactions between point defects [66]. When Σ is not a sphere, a nonzero Q does not imply that there are any defects in the material—the simplest counterexample being a surface Σ cutting across a skyrmion lattice.

We see how the 2D Poincaré-Hopf theorem (which applies not just to vector but also line fields in 2D [113]) manifests itself here: It applies directly to the projection $\mathbf{n}_\perp$ on Σ , requiring that the sum of the windings of the singularities be equal to the Euler class of Σ , modulo the boundary condition. For example, if Σ is a cross-sectional disk of a sphere or cylinder with radial or planar anchoring, the winding of the singularities of $\mathbf{n}_\perp$ must add up to $+1$. Locally, this constraint applies to a disk D around a singularity, too: If a homotopy localized to the disk changes the winding of the singularity, say, from $+1/2$ to $-1/2$ as illustrated in Fig. 1(c), then it must produce a $+1$ singularity in the projection that witnesses the meron emission.

A key aspect of the theory of dividing curves is that they allow for “gluing” constructions. Suppose we separate a material domain into two pieces, M_0 and M_1 , with a surface Σ between them. Then, suppose we define a director $\mathbf{n}_0$ on M_0 and $\mathbf{n}_1$ on M_1 . It is possible to glue these two pieces together along the common boundary Σ without introducing any form of defect or soliton if and only if $\mathbf{n}_0$ and $\mathbf{n}_1$ induce the same dividing curve on Σ [111]. Observe that the dividing curve already shows that the void left by removing a $+1/2$ defect cannot be filled by just a $-1/2$ defect—the dividing curves do not line up, as illustrated in Fig. 1(d)—and the gluing process must be mediated by a meron.

Finally, for completeness, we explain how projections and dividing curves can be defined in the Q -tensor framework. Let Q be the tensor defining the nematic order. In the neighborhood of a point on a surface Σ , we can find a system of local coordinates (x, y, z) so that the z axis is always normal to the surface. We have an operation

$J = \mathbf{e}_z \times$ that effects an anticlockwise rotation. The space of traceless, symmetric 3×3 tensors along S is spanned by

$$\begin{aligned} D_1 &= \mathbf{e}_x \otimes \mathbf{e}_x - \mathbf{e}_y \otimes \mathbf{e}_y, \\ D_2 &= JD_1 = \mathbf{e}_x \otimes \mathbf{e}_y + \mathbf{e}_y \otimes \mathbf{e}_x, \\ D_3 &= -\mathbf{e}_x \otimes \mathbf{e}_x + \mathbf{e}_z \otimes \mathbf{e}_z, \\ F_1 &= \mathbf{e}_x \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_x, \\ F_2 &= \mathbf{e}_y \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_y. \end{aligned} \quad (3)$$

These matrices are all orthogonal with respect to the Euclidean inner product. It is clear that the matrices D_1 and D_2 span the subspace of 2D traceless and symmetric tensors, which therefore define a line field on Σ . Using this decomposition, we may then define an orthogonal projection that sends a 3D tensor Q to a 2D tensor $Q^\perp$, which acts by dropping the terms involving D_3 , F_1 , and F_2 . Note that, while a director field can be reconstructed from the projection, the Q tensor generally cannot be because it encodes an entire frame, and we require more information to reconstruct a frame than just the projection [114,115].

We have written this case in terms of a local coordinate basis, but it is clear this construction can be generalized to give a projection operator Π_Σ that acts on Q in the 3D space to produce the 2D tensor $Q_\perp$ over the entire surface Σ , as well as an operator J that affects a rotation about the unit normal to Σ . Zeros of $Q^\perp$ correspond to both disclinations and merons, just like singularities in the projection. The line field $\mathbf{n}^\perp$ corresponding to the positive eigenvalue of $Q^\perp$ defines the projection.

B. Tomography reduces a complex 3D director to a family of 2D directors that encode the global topology

To capture the full structure of a 3D director, we choose a family of surfaces Σ that span the material. A natural choice is to take surfaces Σ_z of constant z in a Cartesian coordinate frame x, y, z , but, at times, we will also consider a family of torus tubes S_r around a disclination line K of varying radius r ; these torus tubes will generally only span a neighborhood of the disclination not the entire space.

A natural physical justification for a particular choice of surfaces is that they are tangent to the far-field director, which, in our case, we take to be aligned with the y direction. If this is the case, then the merons have a clearer physical meaning even in an achiral system because they cannot expand to the boundary of the material and must remain localized. Even if this boundary condition is not a hard constraint and the director is free to rotate at the boundaries of the system, merons may still exist as transitory phenomena associated with structural changes, although, over time, they may expand to arbitrary sizes or be “pushed out” from the boundary. More generally, we note that the director must be constrained on the boundary

in some way in order for any topological analysis to make sense—for example, if there is no boundary condition, then even the defects can be pushed out of the sample.

The family Σ_z is indicated by the blue surfaces in Fig. 2. On each surface, we consider the projection $\mathbf{n}_z^\perp$ of the director $\mathbf{n}$ into the surface, which defines a family of 2D line fields parametrized by the coordinate z , as in Fig. 2. If we have dynamics, then we have a family $\mathbf{n}_t$ of director fields evolving in time, which leads to a two-parameter family of 2D line fields $\mathbf{n}_{z,t}^\perp$ on our family of surfaces that fully captures both the spatial and temporal structure of the director, referred to as the “tomography” of the director.

The tomography defines a family of sets of points $\Gamma_{z,t}$, where the director is tangent to the surface Σ_z at time t . As we have remarked, for a generic choice of surface, $\Gamma_{z,t}$ will be a dividing curve, consisting of a finite number of disjoint closed loops and lines. However, we have a parametrized family of surfaces: The generic condition is now that each $\Gamma_{z,t}$ is a dividing curve, except for specific isolated values of z and t where a component of the dividing curve undergoes a birth or death, merging, or crossing event. Between parameter values where these events take place, the director can be isotoped to a quasi-2D director field that is independent of z and t . Thus, the essential topological features of the time-dependent director $\mathbf{n}_t$ boil down to the behavior at specific values of z and t where these events occur.

Jumps in the isotopy class correspond to crossing, either spatially or temporally, a feature such as a disclination, point defect, or meron line. Tomography shows a duality between two processes: Moving the surface spatially through the material, we cross through a meron line; keeping the surface fixed but having the director change over time, a meron is emitted. When the dividing curves on two surfaces are not related by an isotopy, there has to be a defect between them—this defect may either be another disclination or a meron. Equivalently, if we view the transition as occurring in time rather than in space, the same argument shows that a defect must be created in the transition process. This duality between spatial and temporal changes provides a novel perspective on the structure and dynamics of a material.

At any point where the dividing curve changes, there will also be a change in the structure of the singularities in the projection $\mathbf{n}_{z,t}^\perp$. In a chiral material, changes in the singularities always correspond to changes in the structure of the dividing curve. For example, a change from a $+1/2$ - to a $-1/2$ -winding singular line in a cholesteric requires the growth of a “loop” in the dividing curve, as indicated in the bottom-right panel of Fig. 2. The loop may not be present in an achiral material, as it is related to the requirement of extra twisting in the director field. In what follows, we focus on the singular set of the tomography rather than the dividing curve—although we continue to make use of the idea of the dividing curve in other contexts—so that

our constructions are appropriate to general liquid crystal systems.

It is possible that some changes in the singular set (or dividing curve) are artifacts of the family of surfaces chosen and not topological changes in the material. Whether or not this is the case can be determined by examining the dividing curves on tubes surrounding the disclinations, which we discuss more in Sec. IV. For example, consider the structure shown in Fig. 2. This disclination “bends back” on itself. We might imagine that we can simply “straighten it out” by a homotopy. However, this structure is *not* homotopic to a straight disclination line, which does not bend back on itself, as can be seen from the dividing curve on the encircling torus: The only way this curve can deform to a straight line is by having the central $-1/2$ -winding section merge with one of the $+1/2$ sections.

Finally, we comment on the relationship between tomography and the Pontryagin-Thom (PT) construction [63]. In the latter, a surface is chosen where one component of the director field vanishes, and it is colored according to the angle between the remaining two components. Thus, we have a helpful visualization of the structure, and analysis of PT surfaces is now a standard technique in liquid crystal physics.

Both tomography and the PT construction capture all relevant topological information in a nematic, and in this sense, they are equivalent. Indeed, if one considers the PT surface $n_z = 0$, where the z component of the director vanishes, and then intersects it with a surface of constant z coordinate, then one obtains the dividing curve on that surface—thus, we may reconstruct the tomography from an appropriately chosen PT surface. From the tomography, one may correspondingly reconstruct the PT surface $n_z = 0$ as the union of dividing curves.

Both techniques have their advantages and disadvantages and should be seen as complementary for the analysis of complex textures. For very complex textures, the PT surface can be correspondingly complex and difficult to visualize. The dividing curve may be more flexible, and it facilitates comparison with experiments via its relationship with the bright and dark regions in FCPM images. It is easier to track how dividing curves change under changes to the topology of the system.

C. Singularity theory determines the local structure of disclinations and merons

A singularity is a zero of a vector or line field. We consider only point singularities here, which may be either point defects in the director (see Ref. [66]) or the singularities in the projection of the director into a surface S . Line singularities will be described via tomography, in terms of the point singularities in the family of 2D projections.

As we discussed in Sec. II, a point singularity in either a vector field or a line field has a single homotopy invariant, the defect charge [12,23]. Besides this homotopy invariant,

point singularities also have diffeomorphism invariants that provide a more refined description of their local structure. Around a zero p of a vector field $\mathbf{u}$, we can consider local diffeomorphisms ψ that fix the origin and map $\mathbf{u}$ to $\psi^*\mathbf{u}$. A “singularity” is an equivalence class of such maps [116]. Much as isotopy classes refine homotopy classes, singularity theory refines the usual homotopy theory of defects, allowing us to detect aspects of their geometry. For example, singularity theory determines the orientation of a $+1/2$ defect and can distinguish between radial (Neel) and rotational (Bloch) merons, which are topologically equivalent.

Basic orientable singularities are Morse type [91]. These are associated with vector fields $\mathbf{u} = \nabla f$, where f is a “Morse function” [66,91,116] of the form

$$f = \pm x^2 \pm y^2 \pm z^2. \quad (4)$$

The number of minus signs in this function is the “Morse index” (MI) of the defect, which is invariant under diffeomorphisms. These models define the local structure of generic 3D point defects [66]: Those with MI 0 or 2 have defect charge $+1$, and those with MI 1 or 3 have defect charge -1 . MI 0 and 3 defects have a “hedgehog” structure, with the director field pointing radially out from the defect everywhere, while those with MI 1 and 2 have a geometrically distinct “hyperbolic structure.” In 2D, we drop the z^2 term. The Morse index therefore varies from 0 to 2: Singularities with MI 0 or 2 are radial and have winding $+1$, and those with MI 1 are hyperbolic and have winding -1 .

Morse singular points are generic in the sense that any singular point can be made Morse by a small perturbation. However, there are two subtleties. First, the conversion may require breaking the singularity apart into multiple other singularities; second, if we have a one-parameter family of vector fields—for example, in a tomography of a director or along a disclination line—it is *not* the case that the singularities can be made Morse for *every* value of this parameter. In other words, while a perturbation can make the singularities of a given, single plane Morse, it necessarily alters neighboring planes since the texture must be smooth, effectively just moving the obstruction elsewhere.

In this context, singularity theory gives a framework for describing the local structure of singularities and the ways they change and break apart as we slide our surface through the material. The classification of Arnold *et al.* [116] describes all possible singularities in vector fields that are of the form ∇f , for f a polynomial function—different local structures that are not gradients of functions are of the form $R_\alpha \nabla f$, where R_α is a rotation about the surface normal by an angle α . The classification scheme also enumerates the possible “unfoldings” of the singularities, which describe the ways in which they split apart into singularities of lower complexity (e.g., a number of

Morse-type singularities with varying Morse indices) as a parameter is varied.

To describe changes in the tomography of a liquid crystal, we need the classification of singularities and their unfoldings in line fields rather than vector fields, which are not explicitly covered by Arnold *et al.* [116]. However, in 2D, this classification is simple to obtain from the vector-field classification using the decomposition of the space of Q tensors in Eq. (3). We see that there is an isomorphism between Q tensors and vector fields in 2D, obtained by sending the tensor $Q = q_x D_1 + q_y D_2$ to the vector field $\mathbf{q} = q_x \mathbf{e}_x + q_y \mathbf{e}_y$. Defect geometry is captured by the observation that rotating the vector of $\mathbf{q}$ at a given point by an angle 2α rotates the “stick” of the corresponding eigendirections of Q by α . Thus, if $\mathbf{q}$ has a singularity of winding k at a point, then the eigendirections of Q will have singularities of winding $k/2$ —this allows us to use singularity theory to describe singularities of line fields as well as vector fields. For example, generic $\pm 1/2$ defects in a director field $\mathbf{n}$ correspond to Morse singularities in the associated vector field $\mathbf{q}$, with a local structure given by the Q tensor,

$$Q_\alpha^\pm = R_\alpha \begin{bmatrix} x & \pm y \\ \pm y & -x \end{bmatrix}, \quad (5)$$

where the sign sets the winding $\pm 1/2$ of the singularity and R_α denotes the matrix encoding a rotation by an angle α , which sets the orientation.

The families of singularities that occur in the tomography of a director are the A_k ($k \geq 1$) and $D_k^\pm$ ($k \geq 4$) families [116], defined in 2D by functional forms:

$$A_k: f = \pm x^{k+1} \pm y^2, \quad D_k^\pm: f = x^2 y \pm y^{k-1}. \quad (6)$$

Different choices of the signs that are not related by a permutation of the coordinates correspond to different (i.e., nondiffeomorphic) singularities. While there are many other, more complex families of singularities, these simple families and their unfoldings will suffice to describe the generic structure of disclination lines, merons, and the crossing and rewiring processes of disclination lines. The A_1 singularities are the Morse-type singularities. An A_k singularity is comprised of k Morse singularities, although we emphasize that a general unfolding does not need to split the singularity into k parts because it is real valued rather than complex valued. The $D_{2k}^\pm$ singularities have winding ± 2 and generically split into pairs of Morse singularities with the same winding.

When discussing disclinations in line fields throughout the remainder of this text, we will always consider one- or two-parameter families of 2D line fields arising from a tomography of the structure in question, and we will always describe these via the associated family of vector fields $\mathbf{q}$, which will need to be packaged into a Q tensor for a

realization of the structure. In this way, the singularities A_1 and D_4^- , respectively, describe disclinations, Fig. 3(a), and merons, Fig. 3(b), in the tomography of a director field.

D. Unfoldings of singularities classify the fundamental changes of topological structure in a 3D nematic

The topology of the singular set changes when we pass over surfaces in which the projection of the director field has a singularity that changes its class. The changes are in the isotopy class, and between these surfaces, the isotopy class is fixed. We identify four basic local structural changes that occur in the disclinations of liquid crystals, all of which also occur in a variety of other physical systems, including optics, hydrodynamics, cosmology, geology, and biology [117]. We refer to these as “structural changes” or “structural degeneracies” in the tomography.

We now give one-parameter families of vector fields $\mathbf{q}$ that describe the topography in the local neighborhood of these changes. These analytic models are shown in Fig. 3. We assume the tomography surfaces are given by level sets of the z coordinate, and there are x, y coordinates on the surfaces. The z coordinate is then the unfolding parameter of the singularity class [66,116]. When $\mathbf{q}$ contains a singularity of winding ± 2 —i.e., the projection of the director field it is describing has a winding ± 1 singularity—then this must be escaped, and it may escape either up or down.

We stress that these models are the simplest analytic models that capture the topology and may not be the most optimal from an energetic standpoint—discussing the energetics of these structural changes is beyond the scope of this paper and would be an interesting extension of our work. We remark that singularity theory models can be used to ensure that director fields satisfy certain properties such as being harmonic or chiral, which are relevant for energetics and have been shown to give good approximations to energy minima for point defects [66].

The most basic degeneracy is a change in the winding of a disclination that does not involve an interaction with a meron. It corresponds to an A_2 singularity (the “fold” bifurcation [117]) in the projection, Fig. 3(c),

$$\mathbf{q}^{\text{fold}} = (x^2 - z)\mathbf{e}_x + y\mathbf{e}_y. \quad (7)$$

Geometrically, this singularity arises exactly when a disclination line has a tangency with the tomography surfaces, as illustrated in the central panel of Fig. 2. It is possible that such tangencies can be removed by a homotopy of the structure without creating, destroying, or merging any defects or merons, just as some singularities in a Morse function may be superfluous—we give an example presently. For the structure shown in Fig. 2, this is not the case, which can be seen by an examination of the dividing curve

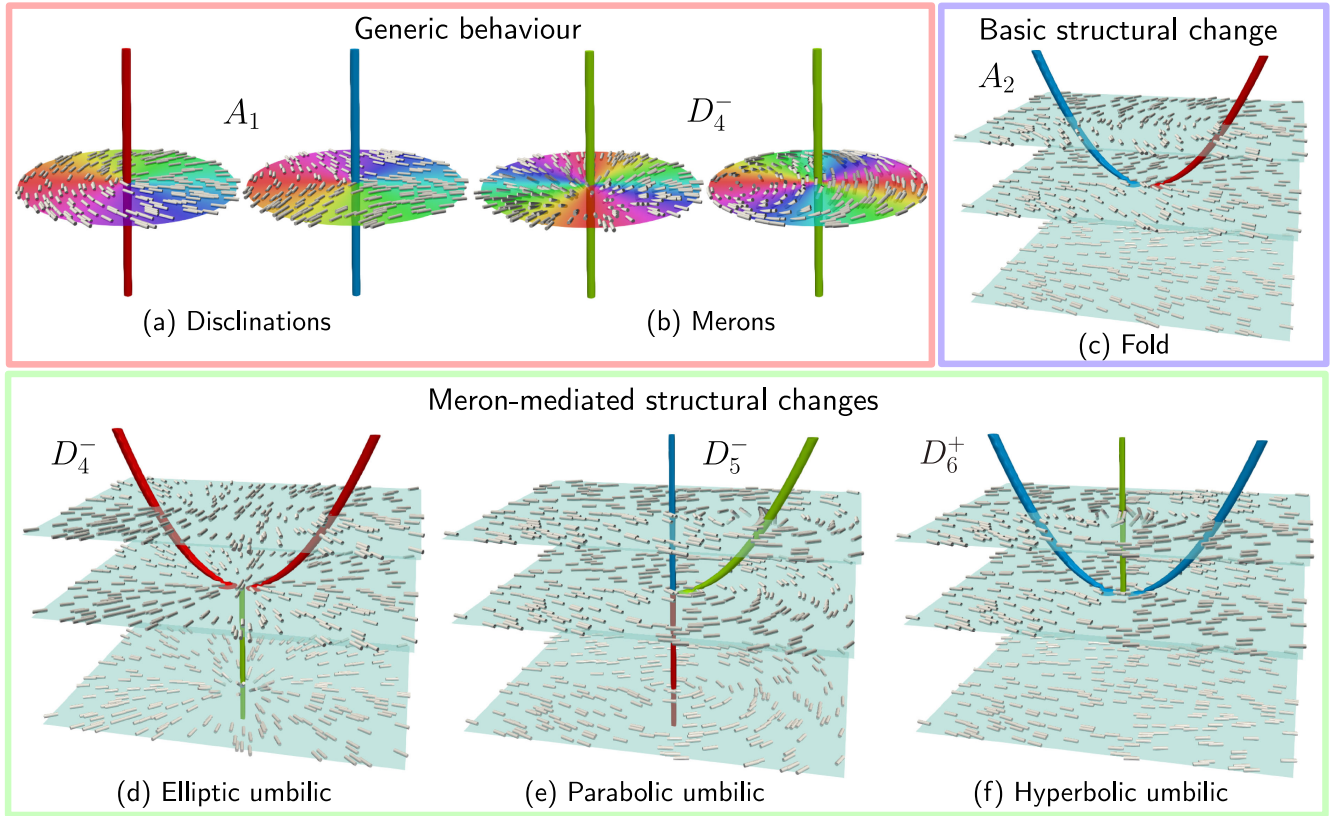


FIG. 3. Singularity theory providing a classification for possible structural changes in the topology of a director and identifying the role of merons in mediating these changes. At a generic point, a singularity in the projection corresponds either to (a) a disclination of winding $+1/2$ (red) or $-1/2$ (blue), or (b) a meron line with winding ± 1 ; illustrated are two cases of $+1$ winding, radial (left) and rotational (right). In terms of the associated vector field $\mathbf{q}$ defined in the main text, disclinations and merons belong, respectively, to the singularity classes A_1 and D_4^- . The cross section is colored according to the angle between the x and y components of the director. Panels (c)–(f) show that, at isolated nongeneric points, the director corresponds to a different singularity class, which describes a change in structure, and away from these points, the structure is given by an unfolding of the singularity. (c) Simplest structural change, which involves the meeting of disclination strands of opposite winding $+1/2$ (red) and $-1/2$ (blue), as given by the A_2 singularity. Then, there are three structural changes mediated by merons. (d) Meeting of strands of the same winding given by the D_4^- singularity. When the strands merge, there is a $+1$ singularity, which will generically escape to form a meron (green). (e) Emission of a meron line that remains tethered to the disclination, described by the D_5^- singularity. (f) More complex structure involving a cusp point in a disclination that meets a meron line, given by the D_6^+ singularity.

on a tube around the disclination line itself, as we will discuss in Sec. IV below.

There are three additional “umbilic” degeneracies, which describe structural changes mediated by meron lines. In what follows, we take the parameter $\sqrt{z}$ to be equal to 0 when $z \leq 0$. A meron line can split into a pair of disclinations, Fig. 3(d), which is described by an unfolding of the D_4^- singularity (the “elliptic umbilic” bifurcation [117]):

$$\mathbf{q}^{\text{eu}} = ((x + \sqrt{z})(x - \sqrt{z}) - y^2)\mathbf{e}_x + 2xy\mathbf{e}_y. \quad (8)$$

A disclination can change its winding along part of its length by the emission of a meron line, Fig. 3(e), which is described by the $D_5^\pm$ singularity (the “parabolic umbilic” bifurcation [117]), where the sign determines the winding ± 1 of the meron:

$$\mathbf{q}^{\text{pu}} = (y^2 - x(x - \sqrt{z})^2)\mathbf{e}_x - 2y(x - \sqrt{z})\mathbf{e}_y. \quad (9)$$

The change in the dividing curve as a result of this change is illustrated in the bottom right of Fig. 2. An analogous transition may occur in the layer structure of a smectic material [67].

Finally, there is another case where the disclination’s tangent vector lies in the tomography planes, but the profile around disclinations does not change its winding, Fig. 3(f). This case implies that there is a meron tether attached to the disclination, which is described by an unfolding of the D_6^+ singularity (the “hyperbolic umbilic” bifurcation [117]) and the vector field

$$\mathbf{q}^{\text{hu}} = (y^2 + x^2(x - \sqrt{z})(x + \sqrt{z}))\mathbf{e}_x - 2xy\mathbf{e}_y. \quad (10)$$

There is an instability here, in which the D_6^+ degeneracy can be replaced by a D_4^- degeneracy, as well as a pair of A_2 degeneracies. Geometrically, this instability is caused by an undulation of the disclination, so the disclination initially bends away from the meron (as in the elliptic umbilic case) before bending back via an A_2 fold structural change.

Our models exactly realize the director fields shown in Fig. 3. Nondiffeomorphic variants of each class are obtained by changing the sign of either the x component or the y component, which will flip the winding, or changing the sign of $\mathbf{q}$ to rotate each director “stick” by $\pi/2$ —the fact that these give distinct classes will be important when discussing rewiring events in Sec. VII. For example, in the hyperbolic umbilic, Fig. 3(f), we have illustrated $-1/2$ disclinations with a rotational meron, but we can also have $-1/2$ disclinations with a radial meron by taking $-\mathbf{q}^{\text{hu}}$, and $+1/2$ disclinations with a -1 meron by changing the sign of either the x or y component of $\mathbf{q}^{\text{hu}}$.

We have listed these degeneracies in order of complexity—the fold is less complex than the elliptic umbilic, etc. Informally, the more complex the singularity, the more energetically costly it is and the less likely it is to occur. This is borne out by examinations of the probability distribution of degeneracies that occur during defect coarsening processes [9]. Parabolic umbilics, in particular, must arise as a result of defect crossings due to the necessity of a meron tether connecting the disclination after a crossing [19,31,94]. All four degeneracies occur in observed stable textures of real materials, and we give several examples of each throughout the remainder of this work. Notably, the three classes of meron-mediated changes all occur in chiral nematics due to the energetic preference for merons in the form of “double-twist” cylinders. Recent experimental work has produced stable disclination structures that are similar to the elliptic umbilic and parabolic umbilic classes in a chiral nematic and analyzed them using simulation, homotopy theory, and energetic calculations [118]. Examples of the hyperbolic umbilic structure have recently been observed in ferroelectric nematics [119]—they are related to focal conic domains.

We also note that there is a change that can occur exclusively in the time domain, which is the conversion of a disclination line that has a constant profile $\pm 1/2$ into a disclination with constant profile $\mp 1/2$ by the emission of a meron line along the *entire* length, without it remaining tethered. This change is illustrated in Fig. 1(d).

IV. DIVIDING CURVES ON SURFACES THAT ENCLOSE DEFECTS

In order to fully understand the relationship between isotopy, homotopy, and global and local structure, we also study dividing curves on surfaces that enclose defects. Unlike the tomography, these dividing curves will always distinguish between $+1/2$ and $-1/2$ profiles. As we will

describe in following sections, this will underpin an understanding of various structural features of disclination lines, such as self-linking and charge, as well as dynamical processes such as the rewiring or crossing of defect lines, nontrivial examples of which typically involve mediation by merons. It also illustrates how homotopies localized in a neighborhood of a disclination line may introduce structural degeneracies into the tomography.

We describe the most basic topological features up to isotopy in terms of dividing curves. For a complete understanding, it is necessary to discuss not just point defects, merons, and disclinations, but also nonsingular textures that are either uniform or contain a cholesteric helix. Thinking of parts of the material in this way closely parallels the perspective offered by the Volterra process, and the ideas of this section offer a new take on results of Kleman [10,11,18]. Once we have described the isotopy classification, we can exhibit the two homotopies that may change the local structure and how these impact the dividing curves: These homotopies effect the “extended Volterra process” of Kleman and Friedel [19].

A. Dividing curves for point defects

In 3D, a generic point defect corresponds to a Morse singularity in the director field [66], as in Eq. (4). The defect can be surrounded by a sphere, and the director can be given an orientation on that sphere. The dividing curve Γ induced on that sphere will consist of some number of closed curve bounding disks, and their count determines the topological charge through the formula (2) [70,71,110]. Even if we do not choose an orientation for the director, the dividing curve still allows us to distinguish a radial hedgehog (MI 0 or 3) from a hyperbolic defect (MI 1 or 2): The former has no dividing curve, while the latter has a dividing curve with two components, Fig. 4(a).

A sphere whose dividing curve contains a single component—more generally, an odd number of curves that separate the sphere into exactly two disks and an even number of annuli—encloses a net zero defect charge; thus, we can detect the absence of charge in a region from the dividing curve without assigning a consistent orientation over the boundary of that region, a fact that we will return to in Sec. VI.

B. Dividing curves for merons and twist domains

We describe meron lines and nonsingular twist domains on solid torii (or cylinders) by their dividing curve. A texture on the solid torus T^2 induces a dividing curve Γ on the boundary, which, up to isotopy, determines a homotopy class in the fundamental group $\pi_1(T^2) \cong \mathbb{Z}^2$ of the torus. By choosing a meridian curve m and longitude ℓ on the torus, any class $[\Gamma]$ can be represented as the straight line $\Gamma = a\ell + bm$ for integers (a, b) that define the rational slope $b/a \in \mathbb{Q}$ of the line, and the tangent vector to the line

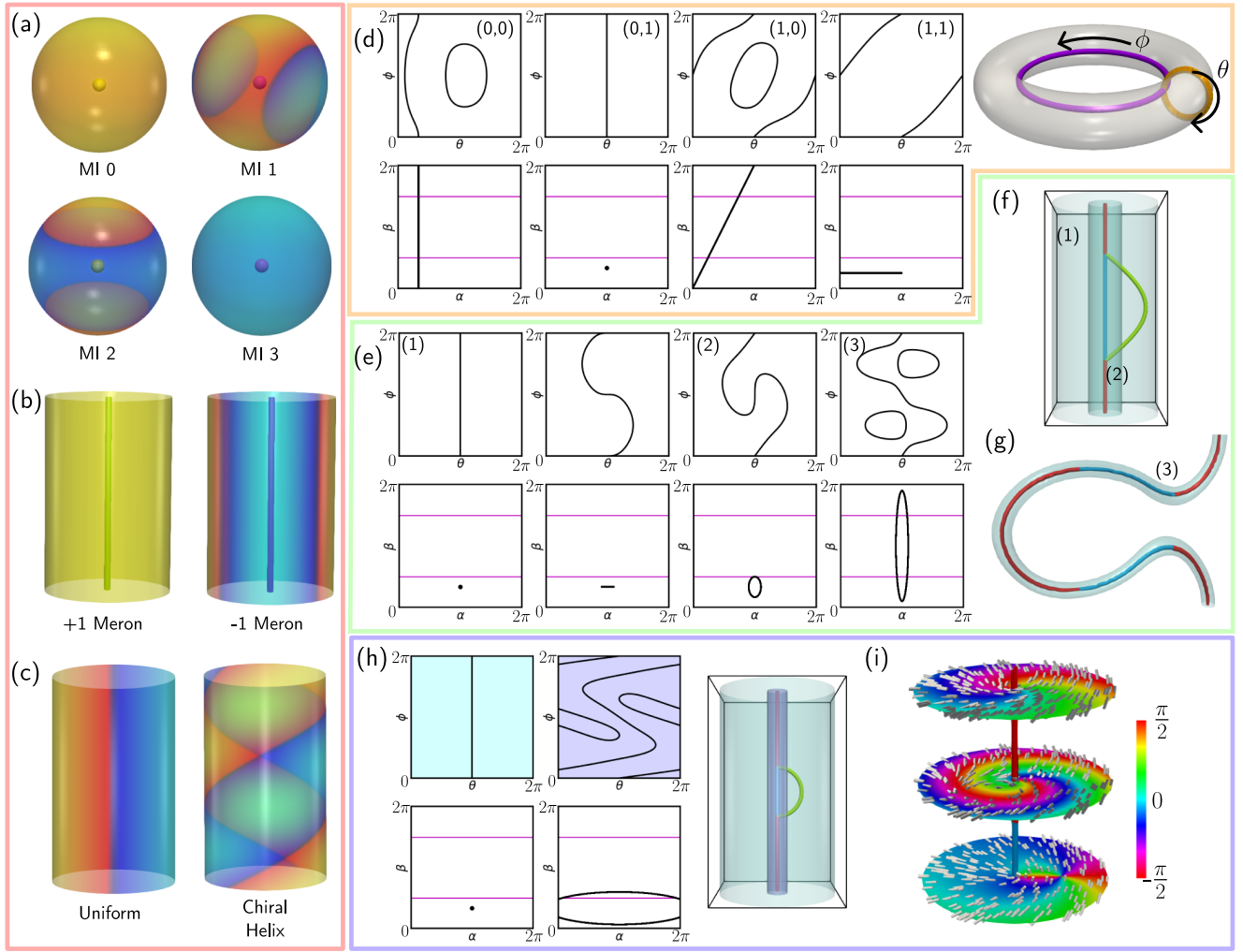


FIG. 4. Examining the dividing curves on a surface surrounding a topological feature provides a complementary perspective to tomography. Homotopies of the local structure of defects may change the isotopy class of this dividing curve, and doing so will, in turn, introduce structural changes into the tomography. Orientable features are either (a) point defects (generically, of Morse type) or (b) meron lines, and we can assume a local orientation is given. The dividing curve separates the surface enclosing the defect into regions S^+ (yellow/orange) and S^- (blue/purple) where the director points, respectively, out of and into the surface—the surfaces are colored according to the dot product between the director and the outward-facing surface normal. In panel (c), we also consider the dividing curves on torus tubes in which the director has no defect but may wind along the longitude of the torus as in a cholesteric—these curves are described by the director field $\mathbf{n}_{0,b}$ in Eq. (13), and we illustrate $b = 0$ (a uniform state) and $b = 1$ (a chiral helix). In panel (d), there are four homotopy classes of disclination lines, as described in the main text. The simplest possible dividing curves on a boundary torus associated with the class $(l, q) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ are shown in the top row, for a torus parametrized by a meridian angle $\theta \in [0, 2\pi]$ and longitudinal angle $\phi \in [0, 2\pi]$ —they are indicated on a schematic torus in the top right, with the meridian line shown in orange and the longitude in purple. In the bottom row, we illustrate that parameters $\alpha(\phi), \beta(\phi) \in [0, 2\pi]$, which are used in the model (18) to construct director fields with these specific dividing curves, as described in the text. (e) “Protrusion growth process,” a homotopy of the director field that induces a transformation of the dividing curve of a $(0, 1)$ disclination. The homotopy produces two null-homotopic components in the dividing curve. This process may occur in (f) the emission of a meron line segment, which introduces two parabolic umbilic degeneracies to the tomography, and also during (g) the growth of a protrusion in a disclination, an integral part of the emission of a disclination loop from a line segment, which introduces four fold degeneracies. In both panels (f) and (g), we show only a segment of the full loop—the full homotopy shown in panel (e) results in the emission of two meron loops, resulting in a disclination that is made of two segments like the picture in (f). The dividing curves on the blue surfaces labeled (1), (2), and (3) in panels (f) and (g) are as shown in the like-labeled panels in (e). (h) Second homotopy of the local structure, which changes the winding of the director along a longitude, a “Hopf process.” This process changes the longitudinal component of the dividing curve by introducing two additional intersections between it and the longitude, and it adds an additional null-homotopic component. The dividing curves on the blue/purple torii are those shown in the panels of the same color. This process involves the emission of a meron tether and hence the introduction of parabolic umbilic degeneracies. In panel (i), we show the director on slices across half of the loop—the other half is a mirror image. The slices are colored according to the angle between the x and y components of the director. The bottom slice contains an additional $+1$ singularity, which is the meron position.

is $b\mathbf{e}_\theta + a\mathbf{e}_\phi$ for θ the coordinate on the meridian and ϕ the coordinate on the longitude. The dividing curve may have multiple components, which is codified in the homotopy class—for example, a two-component dividing curve, where each component is parallel to a longitude, belongs to class $(2, 0)$, which is induced by a uniform director field on the solid torus. In general, the number of components $|\Gamma|$ of the dividing curve in the class (a, b) is the largest integer n such that a/n and b/n are still both integers. An empty dividing curve is also a possibility, corresponding to the class $(0, 0)$. All cases may be realized by distinct isotopy classes of texture, either by a defect-free texture or a texture containing a disclination or a meron; in turn, all *homotopy* classes of defects can be represented by one of these examples—there are additional isotopy classes that contain null-homotopic components to the dividing curve.

This description is entirely analogous to the Volterra construction. The deviation of the dividing curve from the curve $(2, 0)$ is a measure of distortion away from a uniform state, and different homotopy classes in $\pi_1(T^2)$ correspond to the displacements in the Volterra process [10,11,16,18,19]—adding a component means introducing an extra “wedge” of material, removing a component means removing a wedge of material, and converting from straight line to helices involves a screw transformation. The constructions of this section can be generalized to materials with symmetry groups that differ from the nematic, but then certain classes may not be realizable due to not meeting the symmetry constraints (e.g., a polar material cannot have disclinations).

We consider a texture on the solid torus with a singular line at the core with nonzero winding $k \in \frac{1}{2}\mathbb{Z}$ around the meridian and winding $b \in \frac{1}{2}\mathbb{Z}$ around the longitude. For integer k , these lines can be escaped up or down to yield a meron, and on a cross-sectional disk, this process will either correspond to the element $(k, 0) \in \pi_2(\mathbb{RP}^2, \mathbb{RP}^1) \cong \mathbb{Z} \times \mathbb{Z}$ if it is escape up or to the element $(0, -k)$ if it is escape down; the dividing curve does not detect the difference, and one would have to examine a tomography to see it. All possible singular lines can be described by

$$\mathbf{n}_{k,b} = \cos(k\theta + b\phi)\mathbf{e}_x + \sin(k\theta + b\phi)\mathbf{e}_y, \quad (11)$$

where $\theta \in [0, 2\pi]$ is the polar angle and $\phi \in [0, 2\pi]$ is the coordinate along the axis of the torus. The dividing curve on the boundary torus is given by the set

$$\mathbf{n} \cdot \mathbf{e}_r = \sin((k-1)\theta + b\phi) = 0, \quad (12)$$

which implies that the dividing curve realizes the class $(2(1-k), 2b) \in \pi_1(T^2)$. Examples of the basic merons are shown in Fig. 4(b).

Defect-free textures have two intersections between the dividing curve and a meridian. They can be described by

Eq. (11) with $k = 0$ or, equivalently, by a tube cut through a cholesteric helical state:

$$\mathbf{n}_{0,b} = \cos(b\phi)\mathbf{e}_x + \sin(b\phi)\mathbf{e}_y + \mathbf{e}_\phi, \quad (13)$$

where $b \in \frac{1}{2}\mathbb{Z}$ is half-integer, x, y are coordinates in the cross-sectional disks of the torus, and $\phi \in [0, 2\pi]$ is the coordinate along the axis of the solid torus. Along a longitude, the director turns by b full 2π rotations, which allows us to have nonorientable textures when b is fractional and yields the classes $(2, 2b)$. Two examples are shown in Fig. 4(c).

For completeness, we note these local models (11) and (13) can also be described in the Q -tensor picture by

$$Q_{k,b} = \cos(b\phi)Q_k + \sin(b\phi)JQ_k, \quad (14)$$

where Q_k is the 2D Q tensor that encodes a defect of winding k and J is the $\pi/2$ anticlockwise rotation around the ϕ direction.

C. Dividing curves and homotopy classes of disclination lines

Generic disclination lines are realized by setting $k = \pm 1/2$ in Eq. (11). In this case, the dividing curve again realizes the class $(2(1-k), 2b) \in \pi_1(T^2)$. We therefore see that all homotopy classes of curves can be realized by some feature. Using the addition in this group, we can write this class as $(2, 0) + (-2k, 2b)$. Then, $(-2k, 2b)$ is a topological measure of the way that the material is distorted away from a uniform state by an isotopy. For a disclination or meron, we can define a “Burgers vector” $\mathbf{B}$ to be orthogonal to the dividing curve. For a straight disclination of winding k with no profile winding along the longitude, this process results in $\mathbf{B} = k\mathbf{e}_\theta$, the same as an edge dislocation in a crystal, while a disclination with winding b around the longitude will have $\mathbf{B} = k\mathbf{e}_\theta - b\mathbf{e}_\phi$, akin to a screw dislocation in a crystal. The Burgers vector defined by the strain—see, for example, Refs. [16,29]—is $k\mathbf{e}_z$ for both types, which makes *all* nematic disclinations akin to screw dislocations in a crystal. It would be interesting to compare the analysis of disclination dynamics in Refs. [29,32,33], which make use of the Peach-Koehler force defined using the Burgers vector, with a similar analysis based on dividing curves.

Different choices of k and b in Eq. (11) yield distinct isotopy classes [70] but not necessarily distinct homotopy classes of disclination. To understand the distinction and its consequences for the structure and dynamics of the defects, we now recall Jänich’s classification of the homotopy classes of disclination lines [17].

Here and in the next section, we make use of the theory of surface diffeomorphisms and the mapping class group [120], whose elements consist of diffeomorphisms of a surface up to isotopy. The mapping class group of the

torus is isomorphic to the special linear group $SL(2, \mathbb{Z})$, the set of 2×2 matrices with determinant 1. For surfaces, the Dehn-Lickorish theorem [120] shows that mapping class groups are generated by a special type of map called a Dehn twist. On the torus, Dehn twists can be described by their action on the meridian and longitude as follows. Writing the tangent to the meridian as a vector $\mathbf{e}_\theta = (1, 0)^T$ and the tangent to the longitude as a vector $\mathbf{e}_\phi = (0, 1)^T$, the action of a Dehn twist around the meridian $d_m^\pm$ and around a longitude $d_\ell^\pm$ can be written as matrices in $SL(2, \mathbb{Z})$,

$$\begin{aligned} d_m^- &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, & d_m^+ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \\ d_\ell^- &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, & d_\ell^+ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}. \end{aligned} \quad (15)$$

The sign $\pm$ corresponds to left-handed ($-$) and right-handed ($+$) Dehn twists. There is also a $\pi/2$ anticlockwise rotation J , given by the matrix

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = d_m^- d_\ell^+ d_m^-. \quad (16)$$

These maps are illustrated in Fig. 5. To classify the local structure of a disclination up to homotopy, Jänich considered the map $f: T^2 \rightarrow \mathbb{RP}^2$ induced by the director on a torus tube surrounding the disclination. This map must be nonorientable around the meridian to describe a disclination. We choose a basic map f_0 with this property and consider the family of maps

$$f_j := f_0 \circ (d_m^+)^j. \quad (17)$$

If the map f_0 corresponds to $(k, 0)$ in Eq. (11), then f_j arises from choosing $(k, j/2)$ in Eq. (11). Because these director fields are related by composition with Dehn twists, they are, by definition, not isotopic. However, Jänich showed that the map f_4 is homotopic to f_0 [17].

Additional perspectives on this result have been given by Chen in terms of the Pontryagin-Thom construction [63]; by Čopar in terms of the quaternion group [25]; and by Alexander and Kamien in terms of the Whitehead construction [31]. We give a proof of this result in terms of manipulations of the dividing curve on a torus surrounding the disclination, which helps us to connect it to physical processes that may occur in a real material—this proof is spread over both the remainder of this section and Sec. V.

Jänich's result then establishes an isomorphism between homotopy classes of directors around a disclination and the set $\mathbb{Z}_2 \times \mathbb{Z}_2$. We identify the pair (l, q) classifying the line as follows: l is the number of π turns the director field makes as it moves along a longitude of the torus, modulo 2; and $q + 1 \bmod 2$ is the evaluation of the “twisted Euler

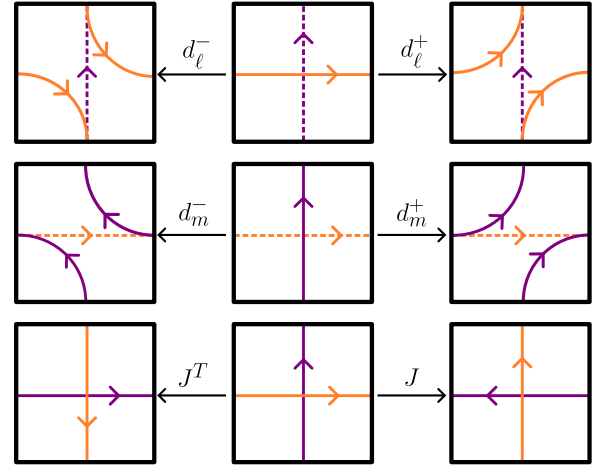


FIG. 5. Dehn twists on the torus allow us to enumerate the isotopy classes of disclination lines, and we will use them to illustrate how rewirings effect the dividing curve. We visualize the effect of a Dehn twist by showing its action on the meridian line (orange) and the longitude (purple) on the torus. The central column illustrates the meridian and longitudes on the initial torus, oriented so that the normal to the torus points out of the page, while the left and right columns show the result of applying the indicated transformations, defined in Eqs. (15) and (16). Top row: Left-handed (d_ℓ^-) and right-handed (d_ℓ^+) Dehn twists around the longitude (dashed) leave the longitude fixed but add a twist to the meridian. Middle row: Likewise for the left-handed (d_m^-) and right-handed (d_m^+) Dehn twists around the meridian. Bottom row: The map J effects a $\pi/2$ anticlockwise rotation of the entire torus, and its inverse J^T effects a rotation in a clockwise sense. We keep the orientation of the meridian and longitude consistent under the rotations J, J^T .

class” [85,95] of the nonorientable director field on a torus around the disclination, which can be identified with a $\mathbb{Z}_2$ skyrmion charge. The value q (rather than $q + 1$) has been identified with a defect charge [23,31], but one must be a little more careful; we describe the distinction between the twisted Euler class and defect charge in more detail in Sec. VI below.

The simplest possible dividing curves achievable for the four classes $(l, q) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ are shown in Fig. 4(d). We produce explicit directors realizing these dividing curves with a model for the local structure of a disclination introduced by Friedel and de Gennes [121], which has been used in many recent studies of disclination motion [29,32,53]. We use this approach in order to help connect our work to the previous studies, although we emphasize that the description here is a topological result and does not depend on a particular realization. We denote by $\mathbf{t}$ the tangent vector to K , and we define Cartesian coordinates u, v on the disks orthogonal to K in some small toroidal neighborhood. Associated with these Cartesian coordinates are polar coordinates r, θ on the orthogonal disks and a coordinate ϕ along K . We choose a “rotation vector” Ω along K with the property that the director lies everywhere

in the plane orthogonal to Ω ; then, we define $\mathbf{n} = \cos(\theta/2)\mathbf{m} + \sin(\theta/2)\mathbf{m}^\perp$, where

$$\begin{aligned}\mathbf{m} &= \cos \alpha \mathbf{e}_u + \sin \alpha \mathbf{e}_v, \\ \Omega &= \cos \beta \mathbf{t} - \sin \beta \mathbf{t} \times \mathbf{m}, \\ \mathbf{m}^\perp &= \cos \beta \mathbf{t} \times \mathbf{m} + \sin \beta \mathbf{t}.\end{aligned}\quad (18)$$

The angles α and β are functions that parametrize the model and control the shape of the director. A profile (the projection of the director onto a disk of constant ϕ) is called “splay” if $\beta = 0$ or π and “twist” if $\beta = \pi/2$ or $3\pi/2$. A twist profile is “radial twist” if $\alpha = 0$ or π , and “tangential twist” if $\alpha = \pi/2$ or $3\pi/2$. In general, α and β are functions of the coordinates r, θ, ϕ in a neighborhood of the line. We consider an arbitrarily small neighborhood of the line and hence make a simplification where they are functions of ϕ alone. Thus, the pair $\alpha(\phi), \beta(\phi)$ gives a map $S^1 \rightarrow T^2$; we visualize the image of these maps in the lower row of Fig. 4(d), which illustrates the connection between the structure of the dividing curve and the more common characterization of the profile in terms of splay and twist regions.

In each case, there exists a component of the dividing curve that has a nonzero intersection with any meridian on the torus: We refer to this as the “longitudinal component.” There may be additional components that bound disks on the torus—called “null-homotopic components.” They can be shrunk down to a point on the torus, as opposed to curves that wrap around a meridian or longitude and cannot be shrunk to a point, and they are associated with the presence of regions of a twist profile. The number of times the dividing curve intersects the boundary of a meridional disk D determines the winding of the projection of the director into D . One intersection corresponds to a $+1/2$ profile and three intersections to a $-1/2$ profile, and the absence of null-homotopic components implies the profile is constant, independent of the disk D we use to measure it [70].

D. Two fundamental homotopies that change the isotopy class of the dividing curve

We now describe how homotopies of the local structure of a disclination line change the dividing curve and how these relate to global topological and geometric changes. There are two fundamental homotopies, from which more complex changes in structure are obtained by performing these basic homotopies multiple times.

The first kind of homotopy does not change the homotopy class of the longitudinal component of the dividing curve in $\pi_1(T^2)$ but either adds or subtracts a pair of null-homotopic loops, Fig. 4(e). This process can occur in one of two ways. First, it is possible for the disclination to emit a meron line along part of its length, Fig. 4(f). Alternatively, the disclination can undergo a

change in its geometry by developing a protrusion, Fig. 4(g). This second case is part of the process by which a disclination line emits a loop, which we discuss in more detail in Sec. VII below. We refer to this homotopy as a protrusion growth process because it is more naturally associated with the geometric change illustrated in Fig. 4(g) than the emission of a meron tether.

The second kind of homotopy applies a pair of Dehn twists, either left d_m^- or right d_m^+ , to the longitudinal part of the dividing curve, and either adds or subtracts a single null-homotopic loop. This change is illustrated in Fig. 4(h). It cannot arise from a change in the geometry of the line; it must result from the emission of a meron tether or the emission of an entire meron loop from the line—a similar change can occur by rewiring disclination lines, but it is not a homotopy of the local structure. Locally, changing the self-linking results from a rotation of the profile and, for a $+1/2$ line, must occur in a series of Hopf bifurcations. In Fig. 4(i), we show cross sections across one half of this disclination loop. The slices are colored according to the angle between the x and y components of the director, which is shown as white sticks.

Hopf bifurcations in a cross section of a director field require a “twisting up” or “entangling” of features of the texture, for example, the looping of a point defect around a disclination. We expect the spontaneous occurrence of such processes to be energetically disfavored, and thus the Hopf process is unlikely to be observed in a passive nematic. It is possible that active flows could cause such processes to occur, perhaps by propelling one small defect ring through another larger ring, but we do not know of such an observation. Otherwise, anomalously small twist elastic constants may allow for this process to happen, as for the case of a point defect in a chromonic liquid crystal [100,101].

V. DIVIDING CURVE DETERMINES THE ORIENTATION AND SELF-LINKING OF DISCLINATION LINES

Now that we understand the two fundamental types of local homotopies that change the isotopy class, we can discuss the geometric and dynamical aspects of these transitions and the relationship with Jänich’s theorem. This discussion involves changes in the self-linking number of a disclination, which we describe now in terms of Dehn twists applied to dividing curves. This section has much overlap with the work of Čopar and Žumer on nematic braids [20,21,24], where the topological self-linking of disclinations was originally defined, and the observed relationship between self-linking in colloidal systems and the actual geometry of the disclination lines that appear in these systems is described in some detail. See also the review in Ref. [25]. There are still many open questions about the relationship between topology and geometry in

nematics, which we do not discuss here but represent an important extension of this work.

To properly describe linking and self-linking of disclination lines, it is first necessary to orient them in their tangent direction. While one may, in principle, assign any orientation we wish to the disclination lines, for proper analysis, it is necessary to orient them in a way that is globally consistent with the orientation on the manifold M on which the director is defined. The manifold has an orientation defined by a choice of a (local) ordered frame field for its tangent space—generally, we take it to be right handed. Any submanifold can then be assigned an orientation, coming from a choice of frame along the surface that is also right handed. Given a solid torus tube enclosing a disclination, this torus tube is then oriented as a submanifold of M , and its central line (the disclination), in turn, receives an orientation. Alternatively, we may choose a Seifert surface for the set of disclinations and orient them as the boundary of the Seifert surface, which either gives the same orientation or the opposite orientation to that given by the manifold. The tube around a meron can be oriented by the manifold in this way, and then it is possible to consistently assign merons elements of the homotopy group $\pi_2(\mathbb{RP}^2, \mathbb{RP}^1)$ [71,85], which realizes them as escape up or escape down relative to the manifold-induced orientation of the line, even when the director itself is not oriented.

The dividing curve Γ on a boundary torus then endows each disclination K with a framing in the following way. Take a component γ_ℓ of Γ that is longitudinal and not null-homotopic—if there are multiple such components, then they all belong to the same homotopy class in $\pi_1(T^2)$, and the choice is irrelevant. Since γ_ℓ is an element of $\pi_1(T^2)$, it can be viewed as an oriented curve in the complement of K , which means we have a well-defined self-linking number $\text{SL}(K) := \text{Lk}(K, \gamma_\ell)$. We then endow K with a framing corresponding to this self-linking number, which can be seen simply as a choice of normal vector that points at γ_ℓ everywhere. This method is a purely topological way of making K into a ribbon, and it is entirely consistent with the geometric methods introduced by Čopar and Žumer [20,21,24,25]; their definition involves a constant multiplicative factor that is superfluous from a topological point of view, but otherwise our definitions agree.

Other tensors derived from the director gradients can be used to define a dividing curve and the self-linking. Long *et al.* [29] produced a collection of orientation tensors related to directors of the form prescribed by Eq. (18). The orientation vector $\mathbf{p}$ is defined by

$$\mathbf{p} = \frac{1 + \cos \beta}{2} (\mathbf{m} \cos \alpha + \mathbf{m}_\perp \sin \alpha). \quad (19)$$

For a $+1/2$ disclination everywhere, pushing the disclination a short distance along $\mathbf{p}$ produces a curve on the

boundary torus that is isotopic (i.e., equivalent) to the dividing curve. For example, consider a director of the form (18) with $\alpha = \phi$ and $\beta = \pi/3$, for ϕ the longitudinal coordinate. This disclination has winding $+1/2$ and self-linking $+2$. The orientation vector is

$$\mathbf{p} = \frac{3}{4} (\mathbf{m} \cos \phi + \mathbf{m}_\perp \sin \phi), \quad (20)$$

which traces out a curve on a boundary torus that is isotopic to the dividing curve. We can define the self-linking by taking the linking number between the disclination and this curve, and of course, we would obtain the same result as when the dividing curve is used. Long *et al.* also defined an orientation tensor T , which has components

$$\begin{aligned} T_{ijk} = \frac{1 - \cos \beta}{2} & [(m_i m_j m_k - m_i m_j^\perp m_k^\perp - m_i^\perp m_j m_k^\perp \\ & - m_i^\perp m_j^\perp m_k^\perp) \cos \alpha + (m_i m_j m_k^\perp + m_i m_j^\perp m_k \\ & + m_i^\perp m_j m_k - m_i^\perp m_j^\perp m_k^\perp) \sin \alpha]. \end{aligned} \quad (21)$$

This tensor vanishes for a pure wedge $+1/2$ disclination with $\beta = 0$ and instead captures the geometry of $-1/2$ -winding profiles. As a symmetric third rank tensor, T defines a triad of associated directions that lie in the planes orthogonal to Ω . As with $\mathbf{p}$, pushing the disclination a short way along each of the directions results in the three curves on a boundary torus that are parallel to the dividing curves for a $-1/2$ profile and that also allow us to define the self-linking if the profile rotates—the rotation of the curves when the profile rotates is a consequence of the way the associated directions of T depend on α .

The self-linking of a disclination is not a homotopy invariant of a disclination line because it can be changed via the Hopf process illustrated in Fig. 4. However, its mod 2 reduction is a homotopy invariant because an even (respectively, odd) self-linking implies that the disclination line links with an even (respectively, odd) number of other disclinations, where merons are counted as a pairs of disclinations [17,23,25]. Rewirings may or may not change the self-linking number, which we discuss presently in Sec. VII. Crossing a disclination line through another disclination changes both of the self-linking numbers by ± 1 and changes the homotopy type accordingly.

We can use these observations about the self-linking to give a proof of Jänich's classification [17] based on manipulations of the dividing curve, which helps us to visualize this difficult topological result and understand its physical meaning. In Fig. 6(a), we show the result of repeatedly changing the self-linking of a disclination K via applied Dehn twists. We visualize this process using the Heegaard decomposition of S^3 [122,123]. This approach realizes the 3-sphere as the union of two solid tori H_1 and H_2 , which are glued together along their common boundary. The map that affects the gluing is the map J defined in

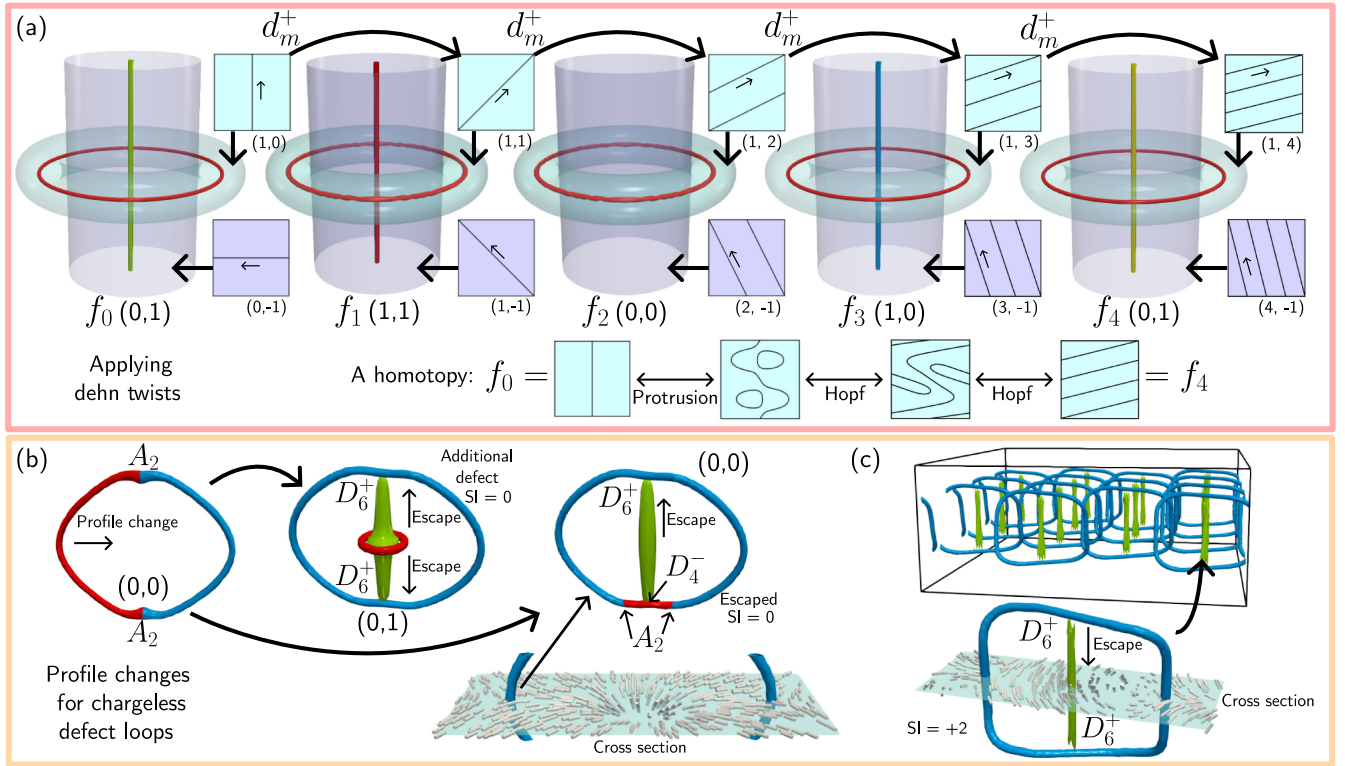


FIG. 6. Changes in the self-linking number of a disclination line resulting from interactions between the disclination and either other disclination lines or merons. These changes are captured by applying Dehn twists to the torus tube that encloses a given disclination, which can be seen to correspond to removing $+1/2$ disclination lines from the tube's complement. (a) Beginning from a $(0, 1)$ defect loop with a $+1/2$ profile everywhere (leftmost panel), we consider the distinct textures that result from applying Dehn twists to the dividing curve (moving rightwards). Our loop lives inside a solid torus tube (pale blue) and initially induces a map $f_0: T^2 \rightarrow \mathbb{RP}^2$ on the boundary of this tube. The complement of this tube in S^3 is another solid torus (purple) whose boundary is glued to that of the pale blue torus via the map J that associates a meridian on the pale blue torus with a longitude on the purple torus. This is the Heegaard decomposition [122]. By fixing a dividing curve on the blue torus, we therefore determine a dividing curve on the purple torus—in each panel, we show the minimal structure within the purple torus that achieves the required dividing curve. Each time we apply a Dehn twist D_m^+ to the dividing curve, we obtain a texture corresponding to a map f_j as in Eq. (17). This result changes the Jänich invariant from $(0, 1)$ to $(1, 1)$ for f_1 , $(0, 0)$ for f_2 , and $(1, 0)$ for f_3 ; finally, f_4 corresponds to the homotopy class $(0, 1)$ again but now in a distinct isotopy class from f_0 [17]. The homotopy that connects these textures is shown in the lower part of the panel and involves the two processes illustrated in Fig. 4. In each case, the dividing curves on the boundaries of the two torii are illustrated, along with the homotopy class in $\pi_1(T^2)$ that they represent. In the complement (purple), we show $+1$ -winding merons in green and -1 -winding merons in yellow. (b) Basic loop from Jänich class $(0, 0)$ with no self-linking, with half of its profile $+1/2$ winding (red) and half $-1/2$ winding (blue), described by an unfolding of A_2 . We can convert the $+1/2$ winding to $-1/2$ by pulling out a $+1$ radial meron line (green). Depending on whether this escapes up everywhere or escapes up in one half and down in the other, it may result in either another $(0, 0)$ loop or a $(0, 1)$ loop with an additional point defect. The singularity types classifying the structural degeneracies on the loop are indicated, and self-linking of all of these lines vanishes. In panel (c), by rotating the profile, we create a geometrically distinct kind of loop described by a different form of the A_2 singularity. Performing the same process here results in a $+1$ -rotational meron. The direction of escape sets the handedness [71]; we illustrate escape down for a right-handed texture. The loop then has a $-1/2$ profile everywhere and self-linking $+2$. Lattices of these loops have been observed in simulations of cholesterics in a thin domain [81–83,87], and the cross section is exactly that of a meron lattice.

Eq. (16), which attaches the longitude on one torus to the meridian on the other. We regard our disclination K (red) as being surrounded by H_1 , shown in pale blue in Fig. 6(a). Whatever structure exists in the complement can be reduced to a single curve by “squeezing” all of the defects down onto the core line of the solid torus H_2 , which we show in purple. The dividing curves on the boundaries of these solid torii must be related by the map J in order for

the textures to “glue” correctly. The defect shown at the center of H_2 is therefore the minimal defect structure necessary to complete the texture in H_1 to a smooth texture over S^3 .

We begin with a case where K belongs to the Jänich class $(0, 1)$, with a constant $+1/2$ profile, which corresponds to a map $f_0: T^2 \rightarrow \mathbb{RP}^2$. Its complement is a meron line of winding $+1$ (green). We show the dividing curves as insets,

labeled by the class in $\pi_1(T^2)$ they correspond to; these curves are also oriented (black arrows) so that $(1, 0)$ is different from $(-1, 0)$. Applying the Dehn twist d_m^+ to the dividing curve of K changes its Jänich class to $(1, 1)$, corresponding to a map $f_1 = f_0 \circ d_m^+$. This process is topologically equivalent to excising a $+1/2$ defect from the complement, which then converts a $+1$ meron into a $+1/2$ disclination. In this example, K (in the blue tube) is oriented anticlockwise and has self-linking $+1$; the other disclination (in the purple tube) is oriented “up” the page and has self-linking -1 .

Performing this process again applies a second d_m^+ to the dividing curve, resulting in the map f_2 . Now, K becomes a disclination with Jänich class to $(0, 0)$ and $\text{SL}(K) = +2$, which excises a second $+1/2$ disclination from the complement, which is then readily seen to be a chiral helix described by the director (13) with winding $b = -1/2$. In particular, we note that this disclination has no defect charge because its complement is defect-free. Applying a third Dehn twist to obtain the map f_3 is equivalent to removing another $+1/2$ disclination from the defect-free purple torus, so it leaves behind a $-1/2$ disclination with self-linking -1 . Now, K belongs to Jänich class to $(1, 0)$ and has $\text{SL}(K) = +3$.

Applying yet another Dehn twist results in K having a dividing curve $(1, 4) \in \pi_1(T^2)$, which corresponds to a map $f_4 = f_0 \circ d_m^{+4}$; the complement now contains a -1 -winding meron line (yellow). Jänich’s classification asserts that this is homotopic to f_0 , so this line must be of class $(0, 1)$ —it is a distinct isotopy class within the same homotopy class. We can see this clearly by using the visualization of the allowed homotopies in Fig. 4 and their effect on the dividing curve. The dividing curve has self-linking number $+4$. We can therefore apply two of the Hopf process homotopies, Fig. 4(h), to obtain a new dividing curve with vanishing self-linking but two additional null-homotopic components, exactly as shown in Fig. 4(e), panel (3). Applying the protrusion growth process then takes us back to a dividing curve with a single component parallel to a longitude, exactly the original map f_0 . This homotopy is shown schematically in the bottom part of Fig. 6(a). A similar process allows us to reduce each of the structures shown in Fig. 6(a) to their simplest representatives shown in Fig. 4(d), which belong to different isotopy classes within the same homotopy class.

The Hopf process is only a global homotopy if the singular line that is pulled out escapes one way—if it escapes in opposite directions, then this process is instead equivalent to pulling a point defect out of the disclination, changing its Jänich class. In Figs. 6(b) and 6(c), we visualize this process for a disclination belonging to Jänich class $(0, 0)$. This disclination carries no charge, and the director may be made fully planar, described exactly by a pair of fold structural degeneracies put together. We consider the director defined by

$$\mathbf{q}_{A_2} = (x^2 - \cos z)\mathbf{e}_x + y\mathbf{e}_y, \quad (22)$$

with $z \in [-\pi, \pi]$, which is shown in Fig. 6(b). Now, we pull a $+1$ singular line out of the $+1/2$ region to change its winding to $-1/2$. Two outcomes result. In the first case, we escape this singular line alternately up and down, which produces a point defect in the center; in simulations, it generically grows into a small ring, as shown in Fig. 6(b). The Jänich class of the original loop has now changed to $(0, 1)$. Alternatively, we can escape the singular line everywhere in the same direction. The Jänich class remains the same, and the loop still has a small $+1/2$ region. Now, the D_6^+ degeneracy at the bottom of the loop is instead replaced by a D_4^- degeneracy along with two A_2 degeneracies, a possibility we described in Sec. III. This method ultimately drives a buckling of the disclination loop, changing its shape from a circle to a cardioid.

In neither case does the self-linking change. In fact, whether the self-linking can change here is dependent on the orientation of the profile of the $(0, 0)$ loop, an interesting geometric constraint. Suppose instead that the local profile of our loop has a director defined by $-\mathbf{q}_{A_2}$, for $\mathbf{q}_{A_2}$ as in Eq. (22). Pulling out a $+1$ singular line from the $+1/2$ segment and escaping now results in a rotational meron, which can escape to give an everywhere $-1/2$ winding for the singular line, Fig. 6(c). Examining the dividing curve on a torus tube shows it has self-linking ± 2 , as in the disclination corresponding to the map f_2 in Fig. 6(a), with a sign according to the direction of escape. Rotational profile $+1$ singular lines have an innate chirality, whereby “escape up” corresponds to a left-handed texture and “escape down” to a right-handed texture [71]. Here, this chirality manifests itself in the sign of the self-linking number, equivalently, whether this process results in a left- or right-handed Dehn twist being applied to the dividing curve—we illustrate a right-handed texture, escape down, with self-linking $\text{SL} = +2$.

Disclinations with nonzero self-linking number can form entangled around colloidal particles [25,73,124]. They also arise naturally in confined cholesterics in thin cells, where the pitch length and the height of the cell are incommensurate [81–83,87]. We show a numerically simulated example of the second case, a lattice of disclination loops in a right-handed cholesteric cell [82], in Fig. 6(c). The simulation is performed by a finite-difference minimization of the Landau-de Gennes free energy in the one constant approximation with a chiral term in a cell of height h with fixed boundary conditions. We set the chirality so that the cholesteric pitch is $p = 0.75h$. Each disclination has an everywhere $-1/2$ profile and exhibits a $+1$ -meron tether with a rotational profile (a double-twist cylinder), which is exactly the structure we have just described. One of these disclinations consists of a pair of D_6^+ hyperbolic umbilic structural degeneracies, exactly as shown in Fig. 3(f).

The merons are necessary in order for the projection into surfaces of constant z to meet the constraints of the Poincaré-Hopf theorem. An alternative texture, not shown here but described in previous studies [83,87], has helices instead of rings—these helices also have nonzero self-linking and have $+1$ merons tethered to them, and they can be seen as the result of rewiring parts of the loops in the texture in Fig. 6(c).

VI. ASSIGNING DEFECT CHARGE TO DISCLINATIONS

In this section, we discuss the question of how to assign charge to disclination loops. There are two ways in which we can conceive of the charge of a disclination loop, according to whether we surround it with a measuring torus or with a measuring sphere. We argue that the defect charge should be, as for point defects, an element of the homotopy group $\pi_2(S^2) \cong \mathbb{Z}$, and we compare this number with the binary charge associated with a measuring torus, which we argue is really a skyrmion charge. An assignment of $\pi_2(S^2)$ charge to defects (both points and lines) must meet not only the global topological constraint imposed by the Poincaré-Hopf theorem but also a local consistency constraint that governs the interactions between defects. The charge of a single defect is determined from a sphere surrounding just that defect; local consistency requires examining a sphere that surrounds a pair of defects and ensuring that the signs of their individual charges are consistent with the charge we measure across the whole sphere. Using Morse theory, it is straightforward to assign charges to a system with only point defects so that those charges are both locally and globally consistent. However, up until now, how to properly extend this approach to defect loops has remained unclear. In particular, a lack of appreciation for the local consistency requirement has led to some errors in the literature. For example, while the conflation of skyrmion charge with defect charge is reasonable in an oriented system [viz. Eq. (2)], in the presence of disclination lines, these concepts need to be carefully disambiguated. In the following, we use Morse theory and the notion of dividing curves to set out a consistent notion of assigning charge to (arbitrary) disclination loops, correcting errors in the literature in the process.

A. Morse theory assigns charge to point defects in a manner that is consistent both locally and globally

First, we briefly recall the definition of the charge of a point defect and its relationship with the dividing curve. Suppose we have an oriented (or at least orientable) director field in some domain. It may have only point defects. Let W be the complement of small balls around each defect, regions that we interpret as the defect cores. The Euler class

$e(\mathbf{n})$ of the director field is a homotopy invariant associated with the oriented vector bundle consisting of the planes orthogonal to the director on W [70,71,86,95,110,111]. Its value on a closed surface Σ depends only on the homotopy class $[\Sigma] \in \pi_2(W)$. The value of the Euler class on a sphere is twice the defect charge within the sphere. As Eq. (2) shows, its value on a closed surface that is not a sphere is twice the skyrmion charge across that surface, that is, a signed count of the number of merons the surface intersects. The Poincaré-Hopf theorem imposes the global constraint that the defect charges must sum to zero (or to an appropriate value if M has a boundary, e.g., $+1$ for a sphere with outward-facing radial anchoring).

A sphere S that is not null-homotopic in W must surround some number of point defects, and the value $e(\mathbf{n})[S]$ of the Euler class on the homotopy class $[S]$ gives the defect charge contained within S . In particular, by using formula (2), we may compute the defect charge using the dividing curve, provided that an orientation is chosen for the director across the sphere S . However, the dividing curve tells us more than just the defect charge: It also tells us whether the Morse index of the defect is 0 or 3, or whether it is 1 or 2—in the former case, the sphere has an empty dividing curve; in the latter, it has two components, Fig. 4(a).

If the director field is not oriented, then the charge is only determined up to a sign, and we cannot tell the difference between MI 0 and MI 3. However, the requirement for both charge and MI to be locally consistent assures us that there are only two ways of assigning them across the material. Suppose a region of the material contains two generic point defects. We know each defect has charge ± 1 , but we do not know from just the local structure of a given defect which sign to assign it. However, suppose we now surround these two point defects with a sphere S and that the charge across this sphere is zero. We conclude that the two point defects within S must have charges of opposite sign, which is what we mean by local consistency of defect charge—it can be extended to all defects in the system by examining all pairwise interactions in this way. Once the sign of the charge for one defect is chosen, all the rest are fixed by the requirement of local consistency.

We can use Morse theory and dividing curves to simplify the computations. We illustrate this process in a 2D setting in Fig. 7(a), but it is equivalent in 3D. We are free to assign a MI to any point singularity, which is equivalent to choosing an orientation for one integral curve emanating from that singularity (red). This choice then propagates along integral curves of the director field (light green) and consistently assigns a MI to every other singularity, thus imposing a local consistency of Morse indices, which in 3D will lead to a local consistency of defect charges. We see that the only choice is the orientation of a single

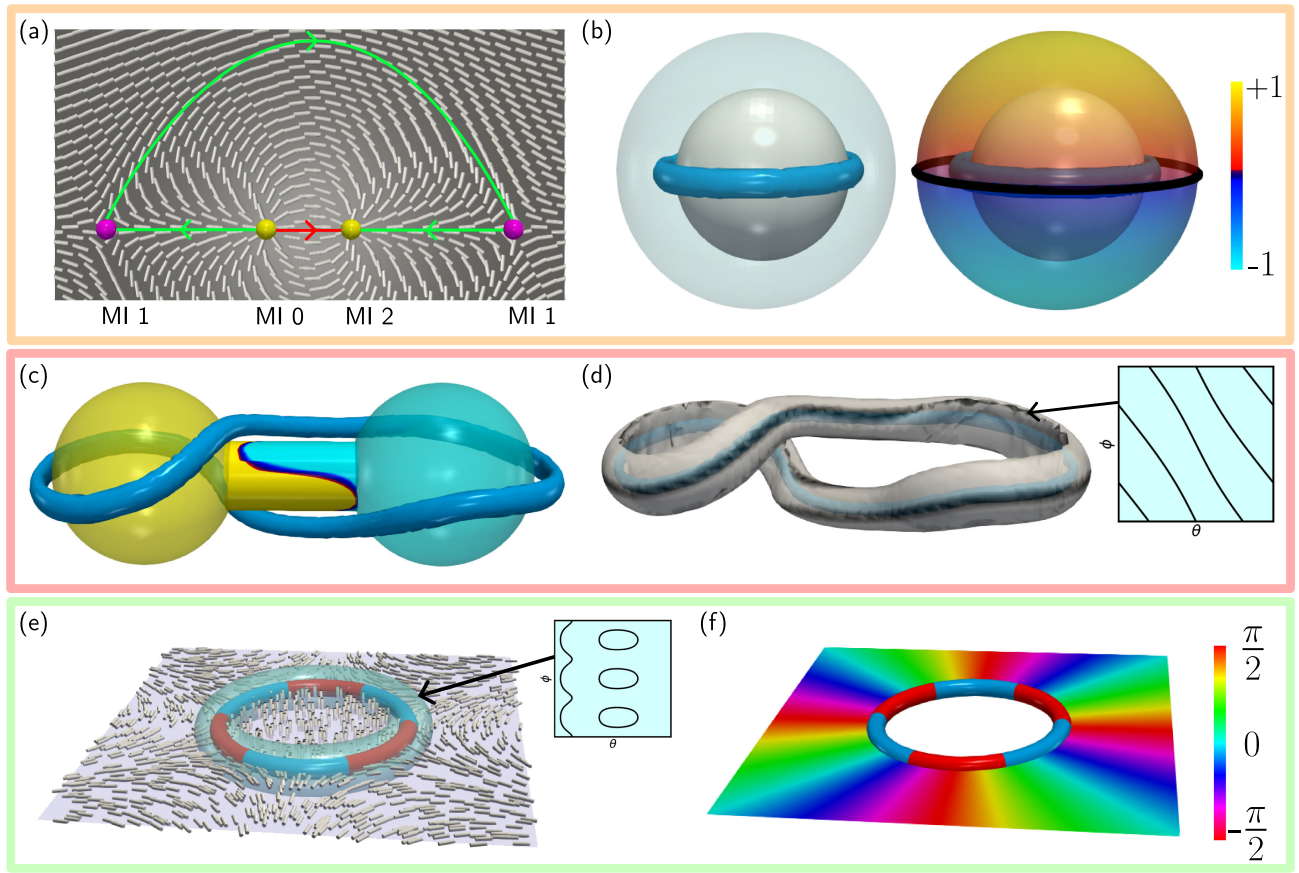


FIG. 7. Care must be taken when assigning charge and Morse indices to defects in order to ensure local as well as global consistency, which is essential for correctly describing the results of interactions between defects. Panel (a) shows that, in a 2D orientable line field, the choice of a MI for any of the singular points assigns a Morse index to every other singular point by propagating the choice along integral curves. Equivalently, we may orient a single integral curve and propagate this orientation along the other integral curves. Either case works and assigns a unique (up to a global sign flip) charge even in the presence of nonorientable singularities. In panel (b), a Saturn ring is a disclination with $-1/2$ winding encircling a colloidal particle with normal anchoring of the director field on the boundary. We surround this structure with a measuring sphere (left panel, pale blue). If we orient the director over the sphere and compute the dot product (right panel), we find that the dividing curve (black) is an equator. Using Eq. (2), we see the combined charge of the pair is zero. Thus, if we decide the colloid represents a hedgehog with MI 0 and charge $+1$, then the disclination has charge -1 and must contract to a hyperbolic point defect of MI 1. Panel (c) shows that, when more colloids are introduced to the system, more complex disclinations are nucleated. Here, a pair of colloidal particles are encircled by a figure-eight disclination loop (blue line). We show a surface, topologically a sphere, enclosing the pair of colloidal particles (not pictured for clarity). We may orient the director over this sphere, which is then colored according to the sign of the dot product between the director and the surface normal—yellow for positive, blue for negative. The line separating these regions is the dividing curve, which separates the sphere into two hemispheres. From Eq. (2), we see the combined charge of the colloids must be zero, as described in the main text. (d) Dividing curve on a torus tube around the disclination shown in panel (c). This curve can also be used to show that the charge of the disclination vanishes, as we describe in the main text. (e) Charge -2 disclination obtained by blowing up a -2 point defect, as described in the text. The line has three regions of $+1/2$ winding (red) and three of $-1/2$ winding (blue). The dividing curve is shown on a torus enclosing the line; the director (white sticks) is shown on a slice. In panel (f), we show the Pontryagin-Thom surface where the z component of the director is zero, colored according to the angle between the x and y components of the director. The color winding illustrates the -2 charge, which can also be obtained from the dividing curve as described in the main text.

integral curve; therefore, there are exactly two distinct possible assignments of Morse indices, which differ mod 2. In a 3D system, this approach still works to consistently assign a Morse index—and hence charges—to the point singularities, even if there are disclination lines in the system as well.

B. Charge of a disclination in a colloidal system is determined by dividing curves

Now, suppose the director contains some disclination loops as well as point defects. We may still define W to be the complement of the defect set, and the director defines a plane field everywhere on W . The definition of the Euler

class requires the vector bundle to be orientable, but if there are disclination lines, this is not the case. Instead, we may define a “twisted Euler class” [85,95], which, for a single disclination, lives in the $\mathbb{Z}_2$ cohomology group $H^2(W, \mathbb{Z}_2)$; hence, its value on any surface Σ is either 0 or 1, a $\mathbb{Z}_2$ skyrmion charge.

The twisted Euler class on a torus tube around a disclination has been identified with a defect charge [23,31], but we should be careful here: The defect charge of a disclination line should be understood as an integer assigned to a sphere around the disclination, not to a torus, and while the skyrmion charge in a nonorientable system is only a mod 2 integer in $\mathbb{Z}_2$, the defect charge is still an integer in $\pi_2(S^2) \cong \mathbb{Z}$ and should behave additively under rewiring. The conflation of the twisted Euler class with a defect charge has led to an erroneous claim that the defect charge is only defined mod 2. In the remainder of this section, we demonstrate that this is incorrect and give a proper method for computing the charge of a disclination loop using dividing curves. In particular, we show that disclinations can have charge 0 and -2 and that they have a different global, but not local, structure.

We first explain this finding in terms of a well-studied experimental system in which disclinations are nucleated around colloids [20,21,73,124,125] before turning to the general case. In this colloidal system, the process of assigning charge has caused some confusion: for example, in Ref. [125], a chargeless -1 -winding meron line is incorrectly identified as carrying defect charge -2 .

Consider the Saturn ring configuration shown in Fig. 7(b)—this is a numerical simulation, obtained by numerically minimizing the Landau-de Gennes free energy starting from a random Q tensor in a box, subject to the boundary condition of the director being aligned with the z director at the boundary of the box and being aligned radially on a fixed colloidal particle (gray). In the resulting structure, we may identify the colloid with a hedgehog point defect, and there is also a disclination loop with $-1/2$ winding everywhere (blue). Now, we surround the pair with a sphere S and compute the dividing curve. We can orient the director across the sphere, allowing us to identify regions S^+ and S^- and apply Eq. (2) to compute the defect charge [right panel of Fig. 7(b)] from the simulated director field. From this result, we see that, together, the pair must have zero charge. Therefore, assigning the colloidal particle charge $+1$ (MI 0) unambiguously assigns the defect loop charge -1 and vice versa. The two defects must be able to annihilate one another to produce a defect-free texture—if we shrink the loop down to a point defect, then it must be hyperbolic with MI 1, a texture that is also seen in experiments [5,73,126].

If we consider two such colloid-disclination pairs, then the requirement that the Morse indices be locally consistent with one another makes it clear that one colloid must be MI 0 with charge $+1$ while the other must have MI 3 and

charge -1 —taken together, the pair of colloids have charge zero, which we could also see by examining the dividing curve on a sphere that surrounds just the two colloids and not the disclinations. Additionally, each individual colloid-disclination pair is identical to the structure shown in Fig. 7(b), so the pair also has charge zero; it follows that the disclinations must have opposite charges to one another. Indeed, if these disclinations were collapsed to points, they would be hyperbolic point defects with MIs 1 and 2, and a cross section of the structure would be as shown in Fig. 7(a).

For a further example with two colloids, we simulate a texture containing a figure-eight disclination loop. This simulation is performed using the same methodology as the Saturn ring, except now we have two colloids with fixed radial directors. The resulting texture is shown in Figs. 7(c) and 7(d). The disclination line has vanishing defect charge, unambiguously—this structure has also been incorrectly assigned a charge of -2 . To see this case, note that Morse theory implies the two colloids must have opposite charge, which follows from propagating the assignment of MI 0 on the left colloid to the right colloid along an integral curve connecting them, which assigns it MI 3—as a pair, they have zero defect charge, which implies the disclination does as well. We can see this case explicitly by computing the defect charge on a surface, topologically a sphere, which surrounds the two colloids, again using our simulated director field and Eq. (2), as shown in Fig. 7(c). Indeed, if we begin from a pair of Saturn ring configurations, the figure-eight loop in Figs. 7(c) and 7(d) is obtained by a rewiring process that merges sections of the two disclinations. Thus, we see that the integer defect charge we assign here is properly additive, exactly as it is for point defects and as is required by homotopy theory. The -1 meron described in Ref. [125] is obtained by merging the two Saturn ring disclinations along their entire length and then escaping the resulting singular line; thus, it is clear that it must also have zero charge (merons, of course, are nonsingular and can never carry defect charge).

A disclination loop that genuinely has charge -2 can be constructed by “blowing up” a charge -2 point defect, similarly to how a disclination with charge ± 1 is obtained by blowing up a hedgehog point defect. The D_4^- singularity provides a vector field modeling a charge -2 point defect [66]:

$$\mathbf{m}_{-2} = 2xy\mathbf{e}_x + (x^2 - y^2)\mathbf{e}_y + z\mathbf{e}_z. \quad (23)$$

We normalize this formula to obtain a director field with the desired point defect. To blow it up into a disclination ring, we remove a cylindrical neighborhood of the z axis and replace it with a uniform field in the z direction. Starting from this configuration, we then numerically minimize the Landau-de Gennes energy (subject to fixing the director on a sphere enclosing the disclination) in order to smooth out

the discontinuity we introduced using the cutting process, resulting in a smooth director field containing a single disclination loop and no other defects. This case is illustrated in Figs. 7(e) and 7(f). Surrounding this loop with a sphere, computing the dividing curve, choosing an orientation for the director field, and then applying Eq. (2), we indeed measure a defect charge of -2 (or $+2$, if we choose the opposite orientation). We can also see this result from a Pontryagin-Thom surface, Fig. 7(f).

Our reasoning about the Saturn ring and figure-eight textures extends to general configurations with n colloidal particles and a uniform far field—if the far field is not uniform, we must regard it as an additional effective point defect with some charge. Any disclination entangled around some number n of colloids can be built from a basic texture consisting of n Saturn rings by successive rewiring processes, as described in Refs. [20,21,124]. To determine the charge of any disclination in this kind of system, we need only make use of the following three observations: (1) A Saturn ring always has opposite charge and complementary MI to the colloid it encircles, (2) the colloids must have alternating MIs 0 and 3 and corresponding alternating charges, and (3) the charge is additive under a rewiring process. Combined with the work of Refs. [20,21,124], we therefore have an algorithm for computing the charge of any system of disclinations tangled around colloids, up to a global sign flip.

The total charge q of these disclination loops is closely related to the linking and self-linking numbers defined in the previous section. Notably, for a link consisting of n disclination loops $K_1, \dots, K_n$, the total charge q satisfies

$$\sum_i \text{SL}(K_i) + \sum_{i \neq j} \text{Lk}(K_i, K_j) + n = q \bmod 2. \quad (24)$$

See Ref. [25] for a full derivation. Note that the formula given in Ref. [25] has an additional factor of $3/2$ that is not present in our formula; this is merely a rescaling due to the different convention we adopt for defining the self-linking number. In line with the perspective outlined in this section, the quantity on the right in Eq. (24) should be understood as the value of the twisted Euler class on a surface Σ consisting of a set of disjoint torus tubes enclosing the disclinations.

C. Charge of a general disclination is a nonlocal property

Now, we generalize this idea to arbitrary sets of defect loops in arbitrary systems, with or without point defects. To assign a $\pi_2(S^2)$ defect charge to a feature, we can only use a sphere as a measuring surface, and the director must be orientable over that sphere. Toroidal measuring surfaces can only measure the Euler class (skyrmion charge) in an oriented medium and the twisted Euler class in an unoriented medium. Thus, to assign charge to a single disclination loop, we must choose a sphere S enclosing it, and S

must not intersect any other defect. This case is a curious example of “nonlocality” in the following sense: The charge of a point defect is local because it is determined entirely by the director in an arbitrarily small neighborhood of the point defect, but the defect charge of a disclination line is nonlocal because it is *not* determined by the director in an arbitrarily thin tube around the disclination. Indeed, it is not possible, in general, to determine this charge purely from a torus tube enclosing the disclination because the dividing curve on this tube does not distinguish between charge 0 and charge ± 2 , as explained in Ref. [23]. The minimal amount of additional information required to compute the point defect charge is knowledge of the director across a Seifert surface. The Jänich class is local and cannot determine the charge; it only determines its mod 2 reduction, the twisted Euler class, which is what is shown in Ref. [23] and elsewhere in the literature.

This charge of a disclination is defined as follows. We can always excise the material in the interior of the sphere S from the medium and replace it with a new piece of material containing at most a single point defect; the disclination’s charge is the charge of that point defect. Alternatively, we can imagine continuously shrinking the disclination (the opposite of the blow up we described in the previous section) until it contracts to a point defect; the disclination’s charge is the charge of that point defect. The sign of this charge is ambiguous but, as we described for point defects, once the sign is chosen for one defect (or for the far field if it is not uniform), the signs of the charge for all other defects are fixed by consistency requirements. The disclinations can have any knot type as long as they are unlinked with other disclinations. Disclinations that are part of a nontrivial link (e.g., a Hopf link) cannot be individually assigned a charge, but we can assign a charge to the link as a whole by surrounding it with a measuring sphere.

The $\pi_2(S^2) \cong \mathbb{Z}$ charge defined in this way is then entirely unambiguous up to a global sign flip, it is locally consistent, and it globally obeys the Poincaré-Hopf theorem. It is invariant under all homotopies of the texture, even if they change the local structure of the disclination, so long as those homotopies do not involve the merging or splitting of defects or rewiring processes—it is additive under the latter. This charge does not require a consistent global orientation of the director.

We now describe the specifics of how this charge is calculated using dividing curves. Consider a solid torus tube H around a disclination K —physically, we interpret the interior of H as the core of the defect, which is our local neighborhood. Topologically, we can view H as a cylinder with a smaller cylinder cut out, as illustrated in Fig. 8. Filling this hole in with a plug—a cylinder C that is topologically a ball—will result in another topological ball B that contains the disclination, also shown in Fig. 8. The plug C must have a director field inducing a dividing curve on the boundary of C so that the two dividing curves line up

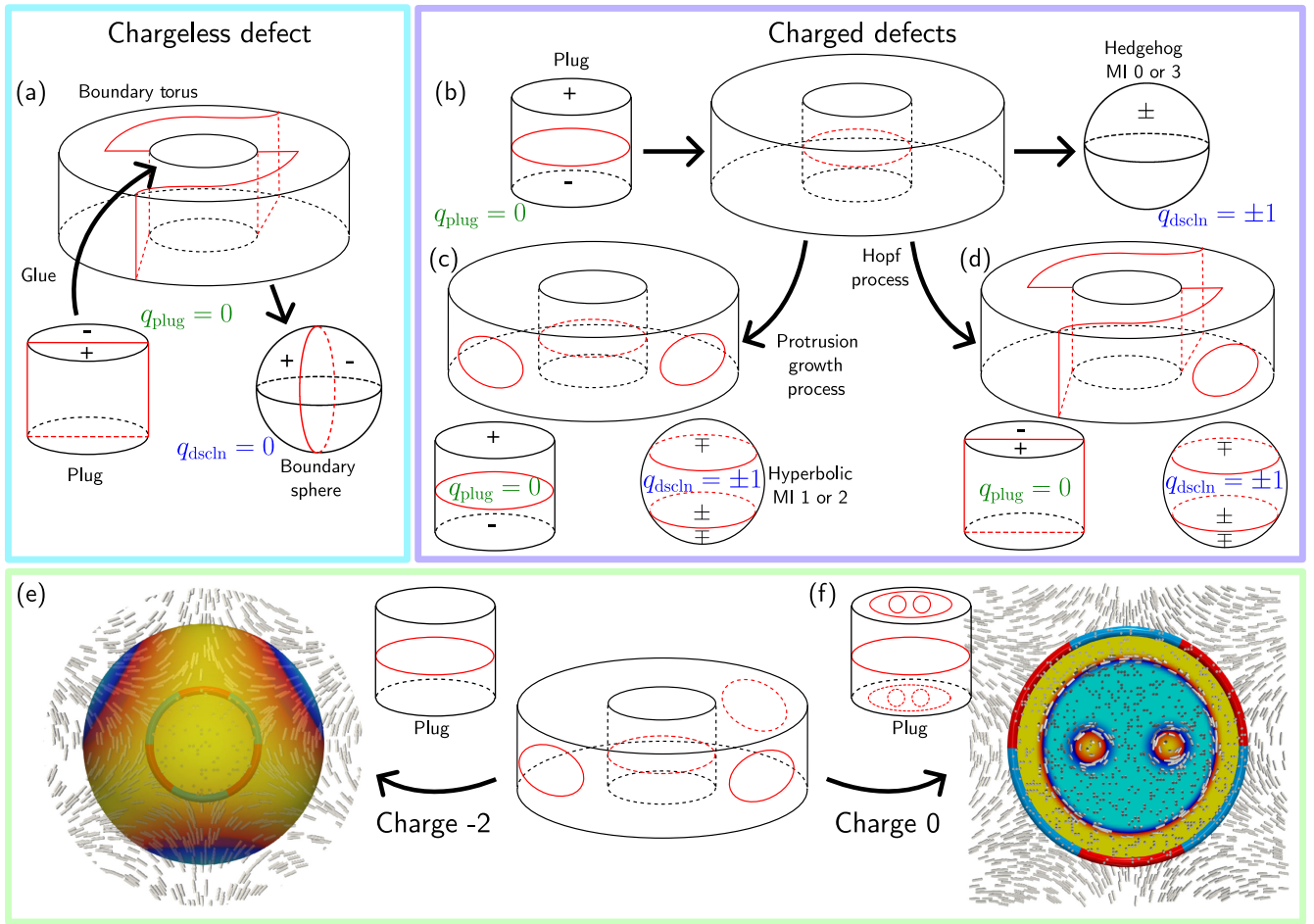


FIG. 8. Charge of a general disclination loop determined from the dividing curve. The associated point defect can then be assigned a Morse index consistent with the local and global constraints imposed by the other defects. (a) Disclination belonging to Jänich class $(0, 0)$ with self-linking -2 . A solid torus tube H with boundary torus T around this line is shown in black, with the dividing curve in red. Topologically, we can view the solid torus tube as a cylinder with a smaller cylinder cut out. To fill this hole, we can glue in a small cylinder C , which we call the “plug.” The texture on this cylinder is a global feature of the texture, not determined by a neighborhood of the disclination, but it must be such that the dividing curves line up across the surface where they are glued. For this disclination, the dividing curve on the plug has a single component. After gluing C into T , we obtain a (topological) solid ball with spherical boundary S containing the original disclination as well as whatever defects were in the plug. By imposing a consistent orientation on S that flips each time we cross the dividing curve and then applying Eq. (2), we obtain the $\pi_2(S^2)$ charge q_{dscln} of the disclination. In this case, we see the charge is zero. (b) Same process for a Hedgehog defect blown up into a ring shows that carries a charge $q_{\text{dscln}} = \pm 1$. For this example, we also illustrate the change in the dividing curve under the two homotopies: (c) the protrusion growth process and (d) the Hopf process. The charge is still $q_{\text{dscln}} = \pm 1$, but if we consider the material within this ball as an effective point defect of Morse type, the Morse index changes from 0 or 3 to 1 or 2 after these homotopies. In panels (e) and (f), we show that the material within the plug region may be different for two disclinations whose local structure in H is identical. This distinction is not local to the line, and it can affect the $\pi_2(S^2)$ charge of the disclination. We show (e) a -2 charge defect and (f) a chargeless defect with the same local structure but different global structure. In panel (e), we show a sphere enclosing the disclination, and in panel (f), we show a Seifert surface, both colored according to the dot product between the director and surface normal, with yellow being positive and blue negative. The disklike boundaries of the plug are essentially Seifert surfaces for the disclination.

along the hole in H . Thus, the director fields agree across the join, up to a homotopy [111]. The dividing curve on the “curved side” of the cylindrical plug C is therefore fixed by the dividing curve around H , and it is essentially a local property of the disclination. However, the dividing curve on the disklike “ends” of the cylinder C is not fixed by the director around K . The spherical surface S of this ball is

what allows us to measure the defect charge of K , which we do by constructing the dividing curve and applying Eq. (2). Part of the dividing curve on S comes from the dividing curve on H , which is a local property of the disclination K , but part of it comes from the disklike ends of the cylinder C —this is the nonlocal part, and constructing it assumes the director is known at least near the boundary of C as well as

around H . Note that the disklike ends of C are essentially Seifert surfaces of K , so computing the defect charge is possible provided we know the director around the disclination and also across a Seifert surface.

Suppose we have this information. We can exploit the freedom to move components of the dividing curve around via isotopies to ensure that C itself contains zero charge, and this is convenient to simplify the calculation. Physically, it corresponds to choosing a measuring sphere for the disclination that does not enclose any point defects—for example, the measuring sphere shown in Fig. 7(b) does not have this property. Then, we may compute the defect charge from the dividing curve on S .

In Fig. 8(a), we illustrate this computation for a defect belonging to Jänich class $(0, 0)$ with self-linking -2 . We obtain this computation by taking a basic loop of class $(0, 0)$, as illustrated in Fig. 6(b), and applying a Hopf bifurcation. The original loop is “quasi-2D,” meaning the director can be chosen so that it lies everywhere in a set of tomography planes—without loss of generality, planes of constant z . Necessarily, this defect must have zero charge, which we can see by examining the dividing curve around this disclination, as well as the dividing curve on the plug. We have labeled the pieces of the plug with $+$ and $-$ signs to indicate whether the director points into or out of the surface under a consistently chosen orientation. The result of gluing in the plug is a ball, and when the director is consistently oriented across the boundary S , the dividing curve is seen to separate it into hemispheres S^+ and S^- . Applying Eq. (2), we find that this disclination has vanishing defect charge, as we expect, and also that this is completely unambiguous. The figure-eight loop shown Figs. 7(c) and 7(d) has this self-linking and belongs to this Jänich class, and a plug for this disclination has the topological structure shown for the plug in Fig. 8(a)—this is an alternative way for us to see that it has vanishing defect charge.

In Fig. 8(b), we illustrate the construction for a $+1/2$ -winding disclination in Jänich class $(0, 1)$. This disclination is obtained by blowing up a hedgehog defect and therefore must have charge ± 1 . We can again see this case from the dividing curves. In Figs. 8(c) and 8(d), we show the construction again but after the $(0, 1)$ line has undergone the two fundamental homotopies of its local structure—respectively, a protrusion growth and a Hopf process. The former homotopy, Fig. 8(c), results in two additional components in the dividing curve on the boundary of B , so the disclination is now seen to be topologically equivalent to a hyperbolic point defect with MI 1 or MI 2. This process can then be seen as a disclination line version of the process by which a point defect changes from a hedgehog to a hyperbolic structure, shown in Figs. 1(a) and 1(b). The same is true for the Hopf bifurcation, although the structure is different. It would be interesting to study the different ways in which a point defect can expand into a disclination

and whether there is an energetic preference that might lead to these alternative structures.

This computation can be carried out for linked disclinations by enclosing the components with a surface of higher genus g (e.g., a genus 2 handlebody for a Hopf link) and performing the same construction with g plugs. The director must be known in a neighborhood of the disclinations, as well as across a Seifert surface for each individual component of the link.

We can use this methodology to analyze the distinction between the charge -2 loop we constructed in the previous subsection and a charge 0 loop belonging to the same Jänich class and with the same dividing curve. The charge -2 loop has a dividing curve with a single component parallel to a longitude and three null-homotopic components, illustrated schematically in Fig. 8(e). The plug in this case has the same director as for the blown-up hedgehog shown in Fig. 8(b). The dividing curve is shown on a sphere enclosing the defect, colored according to the dot product between the director and the outward-facing normal for a given choice of orientation. The region S^+ is a thrice-punctured sphere (with Euler characteristic $\chi = -1$), while S^- consists of three disks ($\chi = +1$ each). Applying Eq. (2), we see the charge is indeed -2 .

If we take a basic charge zero loop in the Jänich class $(0, 0)$ and apply a protrusion growth process, then the resulting dividing curve is the same as for the charge -2 loop. We can produce an analytic model for a disclination line that is equivalent to this texture by blowing up a -2 point defect as in the previous section but by gluing a different plug, as indicated in Fig. 8(f). Informally, the disclination loop has winding -2 around an “outside” meridian as a result of blowing up a -2 point defect, and to obtain an overall charge of 0, it must be canceled out by a winding of $+2$ in the plug. We emphasize that this is an analytic model; we do not know of explicit realizations of this highly complex, and energetically unfavorable, structure.

The plug we have chosen gives a disclination linked by two merons enclosed by another cylinder over which the director rotates by π , a complex structure reminiscent of the “skyrmion bags” studied in Ref. [127], as well as the cross sections of some disclination lines observed in cholesteric droplets [108]. We show a Seifert surface, colored according to the dot product between the normal and the director from our analytic construction. If we take a sphere S enclosing the disclination, then, near the poles, the picture of the regions S^+ and S^- is the same as for the Seifert surface, while near the equator, it looks like the -2 disclination shown in Fig. 8(e). Explicitly, the region S^+ consists of a 5-times punctured sphere (Euler characteristic $\chi = -3$) as well as four disks ($\chi = +1$ each). The region S^- consists of two twice-punctured disks ($\chi = -1$ each) as well as three disks ($\chi = +1$ each). Thus, we have $\chi(S^+) = 1$ and $\chi(S^-) = 1$; putting these results into

Eq. (2), we have a $\pi_2(S^2)$ charge of zero for the disclination, which nonetheless has the same structure in a small toroidal neighborhood as the charge -2 disclination.

Point defects with charge -2 and -3 have been observed in cholesteric droplets with radial anchoring [98,108], where they are stabilized by a combination of boundary conditions and chirality [66]. Interestingly, textures that appear to contain disclination lines with higher charge were also observed at high chirality in the experiments reported in Ref. [108]. These systems contain boundary hedgehog defects, each of which must be assigned MI 0 to be consistent with an orientation of the director pointing outwards from the boundary. Propagating this charge inwards in the manner we have described here allows for a consistent assignment of charge to each defect in the interior. Applying the method described in this section to certain textures illustrated in Ref. [108] suggests that disclination loops of charge -3 have indeed been observed in cholesteric droplets. These disclinations have $+1/2$ winding everywhere, but the director across a Seifert surface (which determines the structure of the plug) is punctured by two $+1$ -winding meron tubes, making it topologically distinct from an expanded hedgehog defect. Our prediction could be confirmed by a direct calculation of the charge from the experimental data. Further experiments on cholesteric droplets with radial anchoring could potentially find other examples of high-charge disclination loops.

D. Interactions between point defects and disclinations

Finally, we comment on aspects of the interaction between point defects and disclination lines. There are two such interactions that are of interest: (1) the absorption of a point defect into a disclination line (equivalently, the emission of a point defect from a disclination line) and (2) moving a point defect around a loop that encircles a disclination line.

We already showed an example of the latter in Fig. 6(b). A disclination strand can change its winding by the emission of a singular line of winding ± 1 . Instead of escaping this line to give a nonsingular texture, the line escapes alternately up and down, leaving a singular point defect behind. This process results in a point defect tethered to the disclination by merons, a structure observed in defect coarsening processes [9]. The point defect may blow up into a ring threaded by the meron tether; this is one process by which a disclination loop may nucleate a second disclination by a change of its local structure, which may potentially be of some interest in studies of active nematics.

The second interaction has been studied previously using homotopy theory [23]. We provide a Morse theoretic perspective on this behavior by means of a simple concrete example. In Fig. 9(a), we show a schematic of a Saturn ring texture. Suppose we move the colloid, representing a

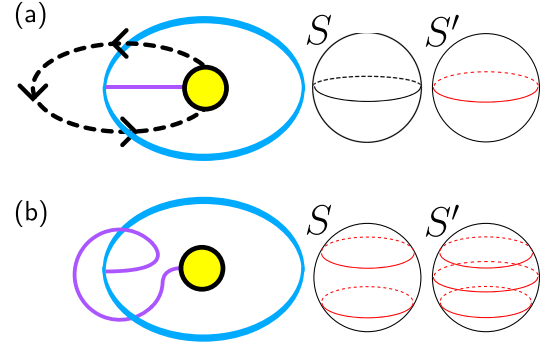


FIG. 9. Effect of moving a point defect on a loop that encircles a disclination, understood through dividing curves. (a) Schematic diagram of a Saturn ring texture, with a $-1/2$ disclination loop (blue) and either a hedgehog point defect or spherical colloid with radial anchoring (yellow). An integral curve of the director (purple) connects them. A sphere S that encloses the defect or colloid but not the disclination has an empty dividing curve due to the radial structure of the defect, while a sphere S' that encloses both defects has a dividing curve (red) with one component, as in Fig. 7—here, we assume both spheres intersect the path indicated by the dashed black line. (b) Moving the defect or colloid in a loop along the path indicated by the dashed black line in panel (a). The result is to entangle the integral curve around the disclination, in a manner similar to how curves of constant director angle are twisted by a Hopf bifurcation. The changes to the dividing curves on the spheres S and S' are as shown—each picks up two additional components to the dividing curve.

hedgehog defect of MI 0, in a circle around the disclination indicated by the black dashed line. If we fix a neighborhood H of K and B of the point defect, the homotopy indicated in Fig. 9(a) can be performed while leaving the director entirely unchanged in these regions, altering it only in the complement of H and B . Therefore, this process does not need to change the Jänich index of the disclination or the charge or MI of the point defect.

We show what happens to an integral curve of the director that connects the disclination to the colloid. The result is to introduce a winding into the profile of the disclination as we move radially outwards from the line. The result of this change would be similar to what is shown in Fig. 4(i): The local profile of the director undergoes two Hopf bifurcations, each of which rotates the profile by π .

We choose a Seifert surface for the disclination that does not intersect the colloid. For the Saturn ring, this disk has an empty dividing curve, with the director pointing everywhere outwards (or inwards, depending on the choice of orientation). After the homotopy, the dividing curve now contains a single null-homotopic component. We choose a sphere S that encloses the colloid but not the disclination and lies outside the fixed regions H and B . For the Saturn ring, the dividing curve on S is empty, a MI 0 structure as in Fig. 4(a). After the homotopy, the dividing curve has two disks, like the MI 2 structure in Fig. 4(a). This homotopy of

a director field is then conceptually similar to what is shown in Fig. 1(b) but inside out—if we take a small sphere enclosing the colloid and gradually increase it, we see a transition from an empty dividing curve while it lies inside the fixed region B to a dividing curve with two disks once it leaves this region.

If we consider a sphere S' enclosing both the disclination and colloid, before the homotopy, the dividing curve has a single component, as in Fig. 7(b); afterwards, it has three components, but after consistently orienting the director, we still measure a charge of zero.

VII. TOPOLOGICAL CLASSIFICATION OF REWIRING PROCESSES

Rewirings are a fundamental process by which segments of disclination lines merge and split, and they are the basic means by which disclination lines interact. They have been discussed extensively in the context of defects forming around colloidal particles, where the basic textures can be related to each other by these rewirings [20,21,25,27,73,124,128], and also in active nematics [47]. In this section, we give a full topological classification of rewiring processes in terms of tomography and singularity theory. This classification reveals that there are, in fact, several different classes of rewiring that are geometrically and topologically distinct; it also clearly elucidates the role merons play in mediating these interactions.

Formally, we define a rewiring as a process involving a topological ball B (which we picture as a cylinder, as in the previous section) inside the material. We imagine that B intersects two strands of disclination line, as illustrated in Fig. 10—these may come from two different disclinations, or they may be two segments of the same disclination. A rewiring involves either (i) cutting out this ball and

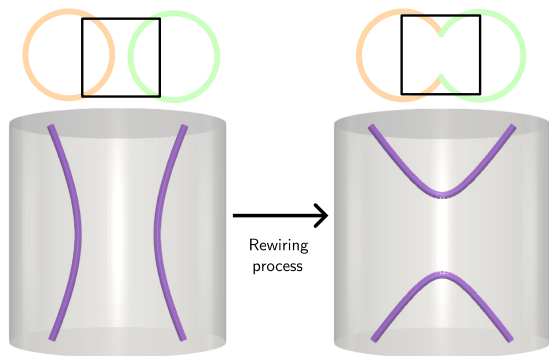


FIG. 10. Rewiring process that involves taking two segments of disclination lines (purple) inside a cylinder (left) and merging them in the core of the cylinder in order to change the connectivity (right). This process can be viewed either as excising the cylinder on the left from the material and replacing it with the texture in the cylinder on the right, or as a dynamical process that occurs in the material.

gluing in a replacement that changes the connectivity of the disclination strands, with the dividing curves on the boundary lining up with those on the original ball, or (ii) a dynamical process in which two strands merge in the middle and then separate in an orthogonal direction to change the connectivity. We adopt the more physical perspective of rewiring as a dynamical process that occurs over some normalized time $t \in [0, 1]$. We can find local coordinates x, y, z in B —for our models, we assume that x and y satisfy $x^2 + y^2 = 4$ and $z \in [-\pi, \pi]$ —and we perform a tomography using disks of constant z value. Locally, the director can be made tangent to all of these disks at the start of the rewiring process—though it may not remain so if the rewiring involves a meron—and thus can be described by a two-parameter family $\mathbf{q}_{z,t}$ of 2D vector fields, packaged into a t -parametrized family of Q tensors as described in Sec. III.

There are two classes of rewiring: One that involves strands of the opposite winding and is described by the A_2 singularity classes and another that involves strands of the same winding and is described by the D_4^- singularity classes—the fundamental distinction is then between the types of degeneracies that appear in the tomography and whether or not these involve merons (see Fig. 3). Within these classes are distinct forms that depend on the relative orientations of the singularities in the cross sections, analogous to what we have already described in Sec. III regarding the local structure of degeneracies and in Sec. V regarding changes to class $(0, 0)$ disclination loops.

A. Rewiring of strands of opposite winding

First, we describe the rewiring of strands of the opposite winding. A model of this process is illustrated in Fig. 11(a). Up to homotopy, the dividing curves on the strands are straight lines, panel (i). Once we rewire, panel (ii), the disclination segments are half $+1/2$ profile and half $-1/2$ profile, so we have replaced two nondegenerate segments of disclination with a pair of fold degeneracies. The dividing curve is as indicated in the insets of Fig. 11(a). There are two geometrically distinct types of the 2D form of the A_2 singularity: One where it splits into a pair of Morse singularities with MI 0 and MI 1, and another type where it splits into a pair of Morse singularities with MI 1 and MI 2. The distinction between the two cases is purely about whether the longitudinal component of the dividing curve veers to the left or to the right of the other component—we show the left-veering case in Fig. 11(a)(ii).

We give analytic models for these rewirings based on unfoldings of the A_2 singularity. As we increase the time parameter $t \in [0, 1]$, the disclination strands will merge in the region $z \in [-\pi/2, \pi/2]$. The two types of rewiring process, labeled $A_2^\pm$, can be described using tomography via the following two-parameter (space z and time t) families of vector fields, which can be identified with

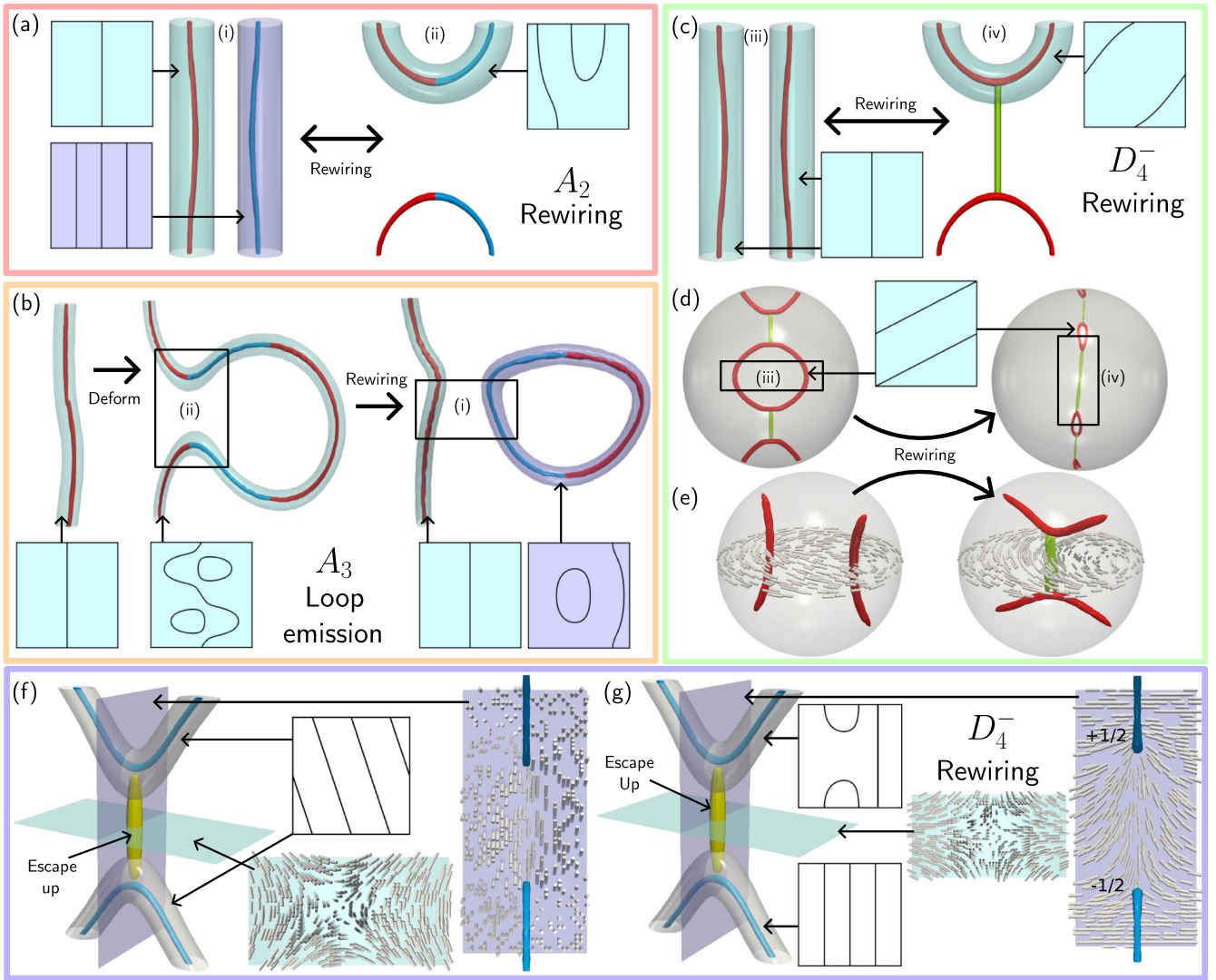


FIG. 11. Rewiring processes, which are the fundamental means by which disclination lines interact with one another, analogous to (but also far more complicated than) the creation and annihilation events of point defects. The result of a rewiring is to either introduce or remove structural degeneracies in the tomography, and the processes can be classified topologically by examining these degeneracies. (a) Rewirings of A_2 type involve strands of opposite winding. In diagram (i), we begin with two segments of disclination line, along which we may “straighten” the dividing curve, as winding is a global property. A rewiring process as described in the main text produces diagram (ii) and changes the dividing curve on each segment, as illustrated. (b) Process that occurs during the emission of a defect loop from another disclination. The disclination first undergoes a protrusion growth homotopy, which results in a region that is topologically the same as that shown in diagram (a)(ii). Rewiring then replaces this with diagram (a)(i), and the result is the separating off of a loop. The dividing curves are as shown; the loop is always of Jänich class $(0, 0)$. This entire spatiotemporal process can be described by tomography and an unfolding of the A_3 singularity. (c) D_4^- rewirings involving strands of the same winding; we illustrate $+1/2$, but the $-1/2$ case is analogous. These rewirings always result in a meron line. The winding depends on the exact profile and is important, as described in the main text—as illustrated in Eq. (28). (d) Texture in a droplet of an active cholesteric with planar anchoring, observed in the numerical simulations of Ref. [51]. The rewiring process describes the disintegration of a defect loop of Jänich class $(0, 0)$ and self-linking $+2$ into two loops of the same class and winding. (e) Another dynamical texture observed in the numerical simulations of Ref. [49] involves two disclination lines that “dance”—this is nothing more than another realization of the rewiring process. (f), (g) Two distinct versions of the D_4^- rewiring for each winding number $\pm 1/2$, which we illustrate here for the case of $-1/2$ winding. They depend on the relative orientations of the singularities in the cross section. In panel (f), the $-1/2$ analog of the process shown in panels (c)–(e) puts a Dehn twist into the dividing curves of the lines after the rewiring process, which is given by Eq. (29). The director field is shown on two cross sections (pale blue, purple), and we also show the dividing curve on two tubes around the disclinations (white). There is a -1 -winding meron (yellow). (g) An alternative process involving a $\pi/2$ rotation of each director stick, resulting in a different outcome that does not introduce winding to the profile of the dividing curve—given by Eq. (30). While the disclinations induce a $-1/2$ -winding singularity into each z surface, the projection into an x surface (orthogonal to the disclination) shows a $+1/2$ singularity in the winding of the projection. For $+1/2$ winding, the distinction between these disclinations is the distinction between a rotational and radial meron, as described in the main text.

two-parameter families of Q tensors as described in Sec. III:

$$\mathbf{q}_{A_2}^{\pm\pm} = \pm(x^2 - 1 + t(1 - \cos z))\mathbf{e}_x \pm y\mathbf{e}_y. \quad (25)$$

Making opposite choices of the signs simply swaps the position of the $+1/2$ and $-1/2$ singularities, while making the same choice rotates the profiles, switching between the case where the singularities have MI 0 and MI 1 and the case where they have MI 1 and MI 2. By means of a different model for an A_2 unfolding—for example, described in Refs. [41,45]—we may rotate the profiles of the defect strands independently of one another; while this process alters the energetics and ultimately the path that the defects follow in the merging process, it does not affect the topological aspects of the process.

This type of rewiring either introduces or destroys pairs of A_2 fold degeneracies in the tomography and therefore does not involve mediation by merons. Played out in reverse, the rewiring is part of the process by which a disclination line emits a loop. This process occurs commonly in active nematics, being the main mechanism by which new disclination loops are nucleated [47]. We simulate an active nematic using a lattice Boltzmann scheme to solve Stokes' equation, along with a finite-difference scheme for the Q tensor, in a box with periodic boundary conditions and an initially random Q tensor. We take the activity large enough so that the material is in a turbulent regime. In that regime, we observe many occurrences of this loop emission process. Snapshots from one such simulation are shown in Fig. 11(b)—for clarity, we show only the segment of a full disclination loop that undergoes this process. First, the disclination segment undergoes a protrusion growth process, illustrated in Figs. 4(e) and 4(g). This process introduces four fold degeneracies into the tomography. Then, a rewiring takes place to pinch off the protrusion, removing two of these fold degeneracies; the other two remain as part of the loop that has been emitted. We show schematic dividing curves for each stage in the process. An analytic model for this entire process can be described via tomography using a two-parameter unfolding of the A_3 singularity to capture both the protrusion growth and the rewiring.

B. Rewiring of strands of like winding

The topological and geometric distinction between the different rewirings corresponding to D_4^- is more fundamental than the distinction between the A_2 types. The D_4^- singularity has winding ± 2 according to the choice of signs, corresponding to rewirings of $\pm 1/2$ strands, respectively. For each winding, there are two distinct cases.

We consider $+1/2$ first. The two $+1/2$ singularities must have distinct Morse indices 0 and 2. The rewiring merges them in the middle, creating a $+1$ -winding singular line between the rewired sections. There is a geometric

distinction according to whether the “heads” of the $+1/2$ singularities face each other or face away from each other. In the former case, the $+1$ singularity that stretches between the two arcs resulting from the rewiring is splay type. This case is described by the toy model

$$\mathbf{q}_{D_4^-}^{+-} = -(y^2 - x^2 - f(t, z))\mathbf{e}_x + 2xy\mathbf{e}_y, \quad (26)$$

where the unfolding is controlled by the function

$$f(t, z) = \begin{cases} \cos z & |z| > \pi/2, \\ 0 & \text{otherwise.} \end{cases} \quad (27)$$

Between $z = -\pi/2$ and $z = \pi/2$ at $t = 1$, there is a $+1$ -winding singular line that may either escape up or escape down, regardless of material chirality [71]. The result is to introduce a null-homotopic loop into the dividing curve of one of the line segments. This rewiring results in the creation of two D_4^- elliptic umbilic degeneracies. We do not illustrate this case here, but we illustrate the equivalent $k = -1/2$ structure. The other case is described by the vector field

$$\mathbf{q}_{D_4^-}^{+-} = (y^2 - x^2 - f(t, z))\mathbf{e}_x - 2xy\mathbf{e}_y. \quad (28)$$

Between the surfaces $z = -\pi/2$ and $z = \pi/2$ at $t = 1$, there is a $+1$ singular line with a rotational structure, which may either escape up or down; the former case is left handed and changes the self-linking of each component by -1 , and the latter is right handed and changes the self-linking by $+1$ [71]. This case is illustrated in Fig. 11(c), and we see that it again involves the creation of a pair of D_4^- elliptic umbilic degeneracies.

We give two examples of this rewiring, which have been observed in simulations of active chiral nematics in a spherical droplet with planar anchoring [49,51], Figs. 10(d) and 10(e). We simulate an active nematic using a lattice Boltzmann scheme to solve Stokes' equation, along with a finite-difference scheme for the Q tensor, the same methodology as in Refs. [49,51], and we choose the parameter values reported in those references. In one case, Fig. 11(d), there exist “boojum” defects at the poles of the sphere and a disclination ring in the interior. The ring is of Jänich class $(0, 0)$ with self-linking $+2$, and it is tethered to the boojums by meron lines, which are topologically necessary as the projection of the direction into any cross-sectional disk has winding $+1$ due to the boundary condition; the degeneracies here are both of elliptic umbilic type, and therefore this structure is somewhat different from the analogous loops shown in Fig. 6(c), where the degeneracies are of hyperbolic umbilic type. Over time, this loop undergoes a rewiring and splits into two loops of the same type with the same self-linking, which move towards—and are eventually absorbed into—the boojums. At higher activity, we observe a texture containing two $+1/2$ -winding strands

tethered to the boundary. These strands “dance” with one another, undergoing a series of rewiring processes that transition back and forth between panels (iii) and (iv) of Fig. 11(c).

We can also give models for the rewiring of disclination strands of $-1/2$ winding, and there are again two cases distinguished by the geometry of the profile and the effect on the dividing curve. We illustrate analytic models for these two cases in Figs. 11(f) and 11(g). While $+1$ -winding merons can be energetically favorable in chiral nematics and can also be potentially stabilized in other classes of material, such as twist-bend and splay-bend nematics, we do not know of a material that favors the production of -1 -winding merons. However, such merons can be forced to occur as a result of confinement—for example, in colloidal systems [124], a texture with a -1 -winding meron appears as an alternative to the figure-eight disclination loop shown in Figs. 7(c) and 7(d). It is possible that these rewiring processes and similar structures could be observed in this experimental system, as well as in systems involving more complex colloidal particles in the shape of handlebodies. We describe these two cases for completeness of our exposition and potential future application, even though we cannot immediately identify their experimental relevance.

The first case is

$$\mathbf{q}_{D_4^+}^{++} = (y^2 - x^2 - f(z, t))\mathbf{e}_x + 2xy\mathbf{e}_y. \quad (29)$$

Between the surfaces $z = -\pi/2$ and $z = \pi/2$ at $t = 1$, there is a -1 -winding singular line that may either escape up or down. These two cases correspond, respectively, to a left- and right-handed Dehn twist in the dividing curve, changing the self-linking number of the lines by ± 1 . Note that the distinction between a dividing curve that bends to the left and one that bends to the right is not dependent on the orientation of the disclination line itself (although the self-linking number is); rather, they are distinguished by the handedness of the Dehn twist. A numerical simulation of this case is shown in Fig. 11(f) for a left-veering dividing curve, with the director shown on a slice across the meron (light blue) and also on a slice that is orthogonal to the “cusp” point of the disclinations (purple).

The final case is

$$\mathbf{q}_{D_4^-}^{--} = -(y^2 - x^2 - f(z, t))\mathbf{e}_x - 2xy\mathbf{e}_y, \quad (30)$$

which is shown in Fig. 11(g). This case is especially interesting as it illustrates the distinction between the winding around the projection of the director field into a tomography and the winding around orthogonal disks, as seen by the dividing curve on a tube surrounding the disclination. For this case, the profile always has a total -1 winding, given by the meron in the central part [light blue surface in Fig. 11(g)] and splitting into two $-1/2$ disclination strands. However, if we take a cross section

that is orthogonal to the disclination line at the cusp point [purple surface in Fig. 11(g)], we see that it has $+1/2$ winding. Indeed, we may also see this winding from the dividing curve, which contains a null-homotopic component and clearly shows a region of $+1/2$ profile. This example illustrates why the tomography perspective on a material should always be supplemented by the dividing curve picture of the local structure of a disclination in order to obtain a full picture of the material structure—a disparity between the two relates to the actual geometry of the line and will be discussed in future work.

Thus, to summarize, there are four different types of rewiring involving strands of opposite winding, two for the $-1/2$ profile and two for the $+1/2$ profile. Each case corresponds to a different (nondiffeomorphic) form of the D_4^- singularity. For each kind of winding, one of the cases results in one of the lines picking up a small region of the opposite winding, with no change in the self-linking number. The other case has two subcases, corresponding to a left- or right-handed Dehn twist in the dividing curve; these subcases are energetically equivalent in an achiral material, but in a chiral material, there may be a preference depending on the material’s handedness [71].

These distinctions may manifest themselves in the Frank-Read mechanism in a liquid crystal [129]. A Frank-Read source corresponds to a disclination arc whose end points both lie on a boundary component. Such an arc can act as a source for the emission of disclination loops by a series of rewiring processes. This mechanism can be induced by passive elasticity or by active forces [129]. A Frank-Read source is “half” of a disclination loop, attached to the boundary of the material. It can be comprised of a $+1/2$ and a $-1/2$ segment in the form of half of a $(0, 0)$ -type loop—we call this an A_2 source—or it can have a constant $+1/2$ or $-1/2$ profile, being half of a $(0, 1)$ -type loop—a D_4^- source. The emitted loops have distinct Jänich invariants, linking numbers, charge, and very different geometric structure.

The A_2 -type Frank-Read source emits $(0, 0)$ loops with no self-linking. These loops can either have the longitudinal component on the inside (a loop with “radial twist”) or the outside (a loop with “tangential twist”) according to which form of the A_2 singularity the source corresponds to. The D_4^- -type sources can emit loops of class $(0, 0)$ with self-linking ± 2 [the case illustrated in Fig. 8(a)], or they can emit loops of type $(0, 0)$ with no self-linking, with the distinction being down to the relative orientations of the profile as in Figs. 11(f) and 11(g). In each case, there is a sharp distinction from the A_2 -type sources: The loop emitted remains tethered to the source by a meron line. The dynamical process illustrated in Fig. 11(d) is analogous to the emission of a loop by a Frank-Read source—one can see that it involves twisting of the sources that are tethered to the boundary—except played in reverse so that the sources absorb the loops rather than emit them.

VIII. CROSSING PROCESSES

In addition to rewirings, disclination lines may cross one another, a process that necessarily results in meron lines tethering the disclinations together [31,94]. Crossings cost more energy than rewirings and are therefore much less likely to be observed, although they can be seen in the defect coarsening process that follows the isotropic-nematic transition [9] and potentially in active nematics as well. Often what are called crossing processes in the literature are actually what we call rewiring processes [130]. Crossing processes may be more likely to occur in chiral nematics due to the energetic preference for merons, and they have also been discussed in recent experimental work on biaxial materials [118].

For completeness, and for a nice illustration of the tomography technique, we describe the crossing of strands of opposite windings using an analytical model, which is illustrated in Fig. 12. This analytical model describes the

only way by which this crossing process can occur. It corresponds to a complicated two-parameter unfolding of the D_6^+ singularity that we describe presently. Snapshots from this process are shown in Figs. 12(a)–12(c). We perform a tomography on the bottom half of the structure (the top half is simply the mirror image), illustrating the director on three slices labelled (i)–(iii) in Fig. 12(d). Part of the process is the merging of strands of the same winding, in this case $+1/2$, which then escape to form the meron tethers. The bending back that occurs in one of the disclination lines, Fig. 12(b), involves the protrusion growth process shown in Figs. 4(e) and 4(g) and results in the creation of four A_2 fold degeneracies in the tomography. The result of merging these strands is to replace the fold degeneracies with D_5^- parabolic umbilic degeneracies. We see the result of this process is, in each case, to change the self-linking of the disclinations by ± 1 and add a null-homotopic component to the dividing curve.

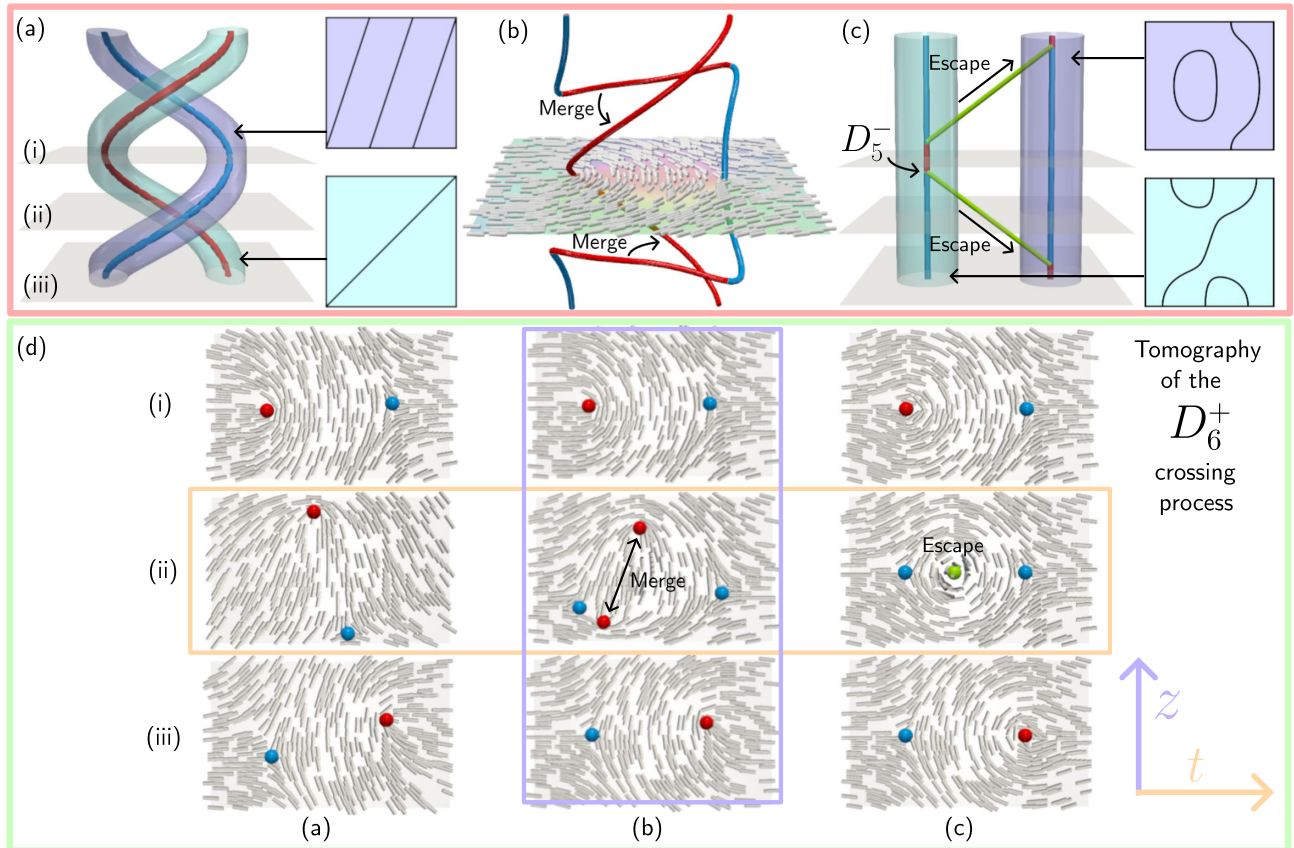


FIG. 12. Tomographic analysis of the crossing of disclination strands of opposite winding showing how this process results in the formation of meron tethers between the lines. (a)–(c) Snapshots from the crossing process described in the main text. In panel (a), we begin with a pair of linked disclination strands with opposite winding $+1/2$ (red) and $-1/2$ (blue). In panel (b), the crossing process proceeds by having the $-1/2$ strand deform by a protrusion growth process, resulting in the appearance of segments of $+1/2$ winding. These segments then merge with segments of the other disclination. In panel (c), the merged strands are $+1$ singular lines that escape to give meron tubes (green). The disclination strands are now unlinked, but the meron tethers remain. In panel (d), we examine this process spatiotemporally through the tomography. We illustrate the projection of the director on three surfaces, labeled (i)–(iii) in panel (a). Going from panel (a) to (c), we show how the structure of the projection changes over time.

We give an explicit functional form for the structure shown in Fig. 12. As with rewiring processes, we describe the crossing of two strands inside a small topological ball with some local coordinates $x^2 + y^2 = 4$ and $z \in [-\pi, \pi]$. The explicit functional form used to describe the structure shown in Fig. 5(c) is

$$\mathbf{q}^{\text{tethered}} = [y^2 + (x + \cos z)^2(x + 1)(x - 1)]\mathbf{e}_x - 2(x + \cos z)y\mathbf{e}_y. \quad (31)$$

At $\cos z = 1$, we have a singularity of winding $+1$ at $(-1, 0)$ and of winding -1 at $(+1, 0)$. These singularities remain fixed throughout. As we step through $\cos z$ from 1 to -1 , a new singularity of winding $+2$ is born and travels along the path $(-\cos z, 0)$ until it reaches the other singularity, which effects the change of the winding number. These singularities must be escaped to give the meron lines. Locally, each of the points where the merons and disclinations intersect is a parabolic umbilic degeneracy in the defect set, as shown in Fig. 3(e).

To describe a director with linked strands, define two functions by

$$\begin{aligned} u(x, y, z) &= x \cos z + y \sin z, \\ v(x, y, z) &= y \cos z - x \sin z, \end{aligned} \quad (32)$$

which represent a rotating coordinate system for the planes of constant z . It rotates in an anticlockwise (right-handed) sense as we increase z ; of course, we can also define a coordinate system that rotates in a clockwise (left-handed) sense, simply by replacing x with $-x$ or y with $-y$. Now, we consider the following one-parameter family of vector fields:

$$\mathbf{q}^{\text{linked}} = ((u^2 - 1)(4 - v^2) + v^2)\mathbf{e}_x + v(4 - v^2)\mathbf{e}_y. \quad (33)$$

We may construct a z -parametrized family of Q tensors using this vector field. The resulting texture has a pair of disclination lines at the points $u = \pm 1$ and $v = 0$, one with $+1/2$ and the other with $-1/2$ winding, and they are linked once: see Fig. 12(a).

We produce the full process shown in Figs. 5(a)–5(c) by interpolating linearly between these two vector fields; i.e., we take the two-parameter family $(1 - t)\mathbf{q}^{\text{linked}} + t\mathbf{q}^{\text{tethered}}$ for t a normalized time parameter. This two-parameter family of vector fields should be identified with a two-parameter family of Q tensors as described in Sec. III, which gives the tomography of the crossing process over time.

IX. DISCUSSION

The growing interest in the complex spatiotemporal defect patterns observed in 3D active nematics drives a need for new theories to describe and explain these

structures. In this work, we have introduced a powerful new analytic framework for nematic director fields based on ideas from Morse theory. Our analysis is applicable to both passive and active systems, and, by accounting for the extra topological structure, it can be applied to materials with broken symmetry in the ground state, such as smectics and cholesterics. For example, in a cholesteric, the energetic requirement of maintaining a consistent sense of handedness restricts us to the use of Dehn twists with the same sense of handedness. More constraints on the dividing curve in cholesterics are discussed in Refs. [70, 71, 111], and some of the restrictions that apply to smectics are described, albeit from a different perspective, in Ref. [67]. We have outlined the ways in which our method relates to experimental techniques and also to other theoretical studies of nematic materials. Our method unifies the various different ways in which the topology of nematic materials have been discussed—homotopy groups, Pontryagin-Thom surfaces, the Volterra construction, and the Čopar-Žumer ribbon construction, among many others—in one comprehensive framework in which the dividing curve is the key object. Our method also highlights the important, and often unappreciated, role that merons play in mediating interactions between defects in nematics.

We have given a topological classification of the kinds of degeneracies that can occur in nematic disclination lines, points where the winding of the disclination changes sign or where it attaches to a meron line tether. These degeneracies occur in real materials, most notably in cholesterics [118], and we have given a wealth of examples that can be compared with experiment. We have further classified the types of rewiring and crossing events that can occur in terms of singularity theory, illustrating that the winding and geometry of the profile have fundamental importance for the results of the rewiring process. Again, our models can be directly compared with experimental or simulated data, which allows for an assignment of topological labels to structural changes in passive and active nematic materials, facilitating further analysis.

Examinations of the dividing curve greatly clarify the homotopy classification of disclination lines and its relationship with an isotopy classification arising from the Volterra process. We have shown how changes in the structure of a disclination can occur via two fundamental homotopies and also how differences in the structure influence the topological objects that appear in the complement of the disclination. Morse theory methods also shed light on the question of the defect charge in a material, helping to bridge the gap between global and local constraints on the charge. We have carefully articulated the distinction between twisted Euler class and the $\pi_2(S^2)$ defect charge, and we have shown that the conflation of the two leads to the errors in the assignment of the latter in experimental systems. We have illustrated a charge -2 defect loop and carefully shown how it differs in global

(but not local) structure from a charge 0 defect loop. Our analysis leads to testable predictions, for example, that high-charge disclinations can be observed in cholesteric droplets [108]. We have also examined the process of moving a point defect on a loop around a disclination and visualized it in terms of changes to the dividing curve.

In this work, we have focused entirely on topological aspects of these changes in structure. Notably absent from our work is any detailed discussion of the geometry adopted by the disclination line or any computation of energetics. As part of our classification, we have provided model director fields based on singularity theory; these fields are designed to capture the topological aspects of structural changes and rewirings in the simplest possible manner, with no concern for the geometry or the energetic optimality. An analysis of disclinations in colloidal systems shows that there is a strong interplay between topology, geometry, and energetics, with topological properties of the disclination lines (e.g., self-linking) manifesting in the twisting and writhing of the lines themselves [25]. Furthermore, while the highly symmetric singularity-theory-based models of defects show a close correspondence with simulation and experiment in systems that likewise exhibit a high degree of symmetry [66], in the general case, the exact form adopted by a director field around a complex defect structure can be quite subtle. Recent studies of the mechanics of disclination lines [29,32,33] in terms of the Peach-Koehler force provide a promising avenue towards understanding how our topological constructions relate to the actual geometry of disclination lines. We can describe disclination loops directly in terms of blow ups of point defects, with a plug region that will then determine the charge—further exploring this construction and comparing more closely with experimental data would be an interesting extension of our work. An alternative approach describes disclinations directly in terms of their geometry via the solid-angle function [54,56,131]. The relationship between the solid angle and the topological structure of the director around the line is entirely unexplored.

Along these lines, we remark that our work also highlights how complex spatiotemporal structure in nematic textures has features reminiscent of particle physics, where changes to objects (disclination lines) are mediated by solitons (merons). This finding suggests that there may exist analogies to Feynman diagrams, with the structural changes and rewiring processes being informally similar to particle creation or annihilation and scattering events. Feynman diagrams notably allow for an approximation of the energy of a system: It would be interesting to examine whether a similar construction can be made for liquid crystals, allowing us to estimate the relative energies of different textures from tomography. There is also some interest in describing the transition pathways between stable, energy-minimizing states [69]. Using tomography,

the transition pathways between energy-minimizing states can be described, at least on a topological level, in a manner analogous to the calculus of surgery diagrams [123]. In materials such as cholesterics, the additional topological structure restricts the possibilities considerably; hence, such materials provide an especially convenient setting for exploring this idea [132].

Another useful extension of the tomography technique would be to apply it not just to the director but also to the velocity field in an active nematic. The velocity field is oriented and therefore does not have line defects, but it contains meron lines, which in a fluid are vortex tubes. Tomography allows us to analyze the linking of these lines, which is related to vorticity [133], as well as the changes in linking over time. Vortex lines can be related to the topological entropy of the flow, and a study of their development and motion over time may help with our understanding of active turbulence in 3D. For a 2D active nematic, a tomography in the time domain may reveal subtle changes in the structure of the material that can be connected to dynamics.

Taken together, we argue that the extension and further application of ideas rooted in Morse theory are likely to prove instrumental in the continued understanding of ordered fluids such as liquid crystals, and we therefore welcome further work in the area.

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