

Erratum: Unitary k -Designs from Random Number-Conserving Quantum Circuits **[Phys. Rev. X **15**, 021022 (2025)]**

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We correct a lower bound on the diamond norm and the corresponding lower bound on the convergence time τ for a number-conserving random circuit to an ε -approximate k -design as presented in our paper. The correct bound given by $\tau \gtrsim L^2 \ln(1/\varepsilon)$ is weaker by a $\ln L$ factor but does not change the main results of the paper.

The lower bound on the diamond norm [Eq. (80)] and the resulting lower bound on the convergence time τ [Eq. (85)] are incorrect. They should be replaced by Eqs. (E1) and (E2), respectively. The corrected bound on τ is weaker than Eq. (85). Nevertheless, the central result holds: The depth τ required for a local number-conserving random circuit to converge to an ε -approximate k -design is lower bounded by charge diffusion.

The mistake stems from a missing factor of Hilbert space dimension $\mathcal{D}$ in Eq. (80). The corrected inequality $(1/\mathcal{D})\|A\|_2^2 \leq \|A\|_\diamond^2$, however, does not provide a useful bound on τ .

We instead use

$$|\lambda| \leq \|A\|_\diamond \quad (\text{E1})$$

to derive the modified lower bound on the convergence time τ , where λ is an eigenvalue of the superoperator A . Choosing λ to saturate the bound on the spectral gap in Eq. (78), we obtain

$$\tau \gtrsim L^2 \ln \frac{1}{\varepsilon}. \quad (\text{E2})$$

Inequality (E2) corrects and replaces Eq. (5) in the main text. This timescale is consistent with a diffusion-limited process and is independent of spatial dimension d and replica count k .

Recall that the diamond norm is defined in terms of the 1-to-1 norm [Eq. (3.281) in Ref. [1]],

$$\|A\|_\diamond \equiv \|A \otimes \mathbb{1}\|_{1 \rightarrow 1} \equiv \sup_{\rho} \frac{\|(A \otimes \mathbb{1})(\rho)\|_1}{\|\rho\|_1}. \quad (\text{E3})$$

The diamond norm adds ancillary degrees of freedom to the superoperator A , with the identity channel $\mathbb{1}$ acting on as many degrees of freedom as A .

To prove Eq. (E1), we proceed in two steps. First, we prove that the diamond norm is lower bounded by the 1-to-1 norm $\|A\|_{1 \rightarrow 1} \leq \|A\|_\diamond$ (Lemma 3.50 in Ref. [1]). Next, we prove that the 1-to-1 norm is lower bounded by the absolute value of any of the superoperator eigenvalues $|\lambda| \leq \|A\|_{1 \rightarrow 1}$.

A lower bound to the right-hand side of Eq. (E3) is obtained by restricting the set of states ρ to only those which are fully mixed on the ancillary space $\rho = \rho' \otimes \mathbb{1}$,

$$\sup_{\rho'} \frac{\|A(\rho') \otimes \mathbb{1}(\mathbb{1})\|_1}{\|\rho' \otimes \mathbb{1}\|_1} \leq \sup_{\rho} \frac{\|(A \otimes \mathbb{1})(\rho)\|_1}{\|\rho\|_1}. \quad (\text{E4})$$

As

$$\|A\|_{1 \rightarrow 1} \equiv \sup_{\rho'} \frac{\|A(\rho')\|_1}{\|\rho'\|_1} = \sup_{\rho'} \frac{\|A(\rho') \otimes \mathbb{1}(\mathbb{1})\|_1}{\|\rho' \otimes \mathbb{1}\|_1}, \quad (\text{E5})$$

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the result follows. Intuitively, the ancillary degrees of freedom in the diamond norm can only increase the distinguishability of two channels so that $\|\Phi_\varepsilon - \Phi_H\|_{1 \rightarrow 1} \leq \|\Phi_\varepsilon - \Phi_H\|_\diamond$.

Next, consider the inequality $|\lambda| \leq \|A\|_{1 \rightarrow 1}$ with λ an eigenvalue of the superoperator: $A[\Lambda] = \lambda\Lambda$. The specific value of $\rho = \Lambda$ can be used to lower bound the sup in the definition of the 1-to-1 norm,

$$|\lambda| = \frac{\|A(\Lambda)\|_1}{\|\Lambda\|_1} \leq \sup_{\rho} \frac{\|A(\rho)\|_1}{\|\rho\|_1} \equiv \|A\|_{1 \rightarrow 1}. \quad (\text{E6})$$

The lower bound in Eq. (E1) provides a lower bound to τ . Recall that τ is defined as the smallest t such that $\|A(t)\|_\diamond \leq \varepsilon$. For any eigenvalue $\lambda(t)$ of $A(t)$, if $|\lambda(t)| > \varepsilon$, then $\|A(t)\|_\diamond \geq |\lambda(t)| > \varepsilon$. The spectral gap bound [Eq. (78)] restricts the maximal absolute value eigenvalue by

$$e^{-\frac{C}{L^2}t} \leq \max_{\lambda} |\lambda(t)|. \quad (\text{E7})$$

The circuit depth t must be larger than $(1/C)L^2 \ln(1/\varepsilon)$ in order for $\|A(t)\|_\diamond \leq \varepsilon$ to hold. Equation (E2) thus follows.

This correction has minor implications to the applications discussed in Sec. VIII of the main text. The highlighted applications implement finite-depth random circuits to approximate the Haar ensemble. This correction implies the depth of the random circuits, and associated circuit resource requirements may be reduced by a factor of $k \ln L$ compared to the original lower bound. This correction does not change the main result, which is that this convergence is significantly slower than in the case of circuits without charge conservation.

Complete list of changes:

- (i) The inequality $kL^2 \ln(L) \lesssim \tau$ is replaced by $L^2 \lesssim \tau$ in both the abstract and the Discussion section.
 - (ii) The inequality in Eq. (5) is replaced by Eq. (E2).
 - (iii) The lower bound in Eq. (80) is replaced by Eq. (E1).
 - (iv) The text between Eqs. (82) and (85) is replaced by the contents of this erratum.
- We thank Jeongwan Haah for noticing the error in Eq. (80).

[1] J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, Cambridge, England, 2018).