

Leading and Beyond Leading-Order Spectral Form Factor in Chaotic Quantum Many-Body Systems Across All Dyson Symmetry Classes

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We show the emergence of random matrix theory (RMT) spectral correlations in the chaotic phase of generic, periodically kicked, interacting, quantum many-body systems by analytically calculating the spectral form factor (SFF), $K(t)$, up to two leading orders in time, t . We explicitly consider the presence or absence of time-reversal ($\mathcal{T}$) symmetry to investigate all three Dyson symmetry classes, namely, the circular orthogonal ensemble (COE) for $\mathcal{T}$ -invariant systems with $\mathcal{T}^2 = 1$, the circular unitary ensemble (CUE) in the absence of $\mathcal{T}$ symmetry, and the circular symplectic ensemble (CSE) for $\mathcal{T}$ -invariant systems with $\mathcal{T}^2 = -1$. Our derivation applies the random phase approximation to perform ensemble averages, and we give its justification for generic many-body interactions. We express the SFF as a sum of contributions of certain diagrams. While the number of diagrams that contribute to the SFF in the first and second orders in t for the COE and CUE classes is finite, the number of such diagrams is exponentially large in t for the CSE class. Thus, we have developed a new diagrammatic technique that uses reduced diagrams to incorporate contributions from many diagrams. For the COE class, we show that beyond Thouless time t^* , the SFF takes the form $K(t) \simeq 2t - 2t^2/\mathcal{N}$ up to second order in time, where $\mathcal{N}$ is the Hilbert space dimension. For the CUE class, we show that $K(t) \simeq t$ beyond t^* , and there is no universal term in the second order, unlike the COE case. For the CSE class, we show that $K(t) \simeq 2t + 2t^2/\mathcal{N}$ up to two orders in time beyond t^* . The above findings of the SFF up to second order in time are in agreement with the RMT predictions for these classes. In all three cases, the scaling of t^* with system size L is determined by the eigenvalues of a doubly stochastic matrix $\mathcal{M}$. For strongly interacting fermionic chains, $\mathcal{M}$ is $SU(2)$ invariant in all three cases, leading to $t^* \propto L^2$ in the presence of $U(1)$ symmetry. In the absence of $U(1)$ symmetry, we find $t^* \propto L^0$, due to the gapped, nondegenerate, second-largest eigenvalue of $\mathcal{M}$, or $t^* \propto \ln(L)$, due to the gapped second-largest eigenvalue with degeneracy proportional to L^ζ , $\zeta > 0$. Our calculation of the SFF is plausible in higher space dimensions as well, where similar system-size scalings of t^* can be obtained.

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I. INTRODUCTION

The quantum origin of chaos is still very actively studied even after 50 years of intensive research. An epoch in the field started with observing universal statistical properties of the spectra of quantum systems whose classical analogs are chaotic. These spectral properties are excellently described by random matrix theory (RMT) [1–5], which was introduced in nuclear physics by Wigner to predict the

distribution of energy-level spacings of heavy nuclei [6–10]. The success of RMT in capturing the universal spectral properties has been so remarkable that the presence of RMT behavior in a physical system is investigated to diagnose quantum chaos. Further advancing our understanding of quantum chaos requires an explanation for the emergence of RMT behavior in physical systems.

A widely recognized and meaningful approach in this context involves analytically identifying statistical measures for random matrices and quantum systems with chaotic classical analogs. The results for the two cases are expected to match, but the steps leading to such a match would explain why such quantum systems behave identically to random matrices. The spectral form factor (SFF), a measure of correlation between energy levels, is

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extensively studied to diagnose quantum chaos. For time-reversal ($\mathcal{T}$)-invariant autonomous systems with $\mathcal{T}^2 = 1$ representing Gaussian orthogonal ensembles, the SFF takes the following form in RMT for $0 < t < t_H$ [6]:

$$K_{\text{GOE}}(t) = 2t - t \ln \left(1 + \frac{2t}{t_H} \right) = 2t - \frac{2t^2}{t_H} + \frac{2t^3}{t_H^2} - \dots, \quad (1)$$

where t is the time and t_H is the Heisenberg time, which is related to the Hilbert space dimension $\mathcal{N}$. Berry [11] calculated the SFF up to leading order in time using the semiclassical periodic orbit theory, which was later extended up to second order in time by Sieber and Richter [12,13]. Finally, the complete derivation, up to all orders in time, was performed by Müller *et al.* [14,15]. Additionally, a rigorous explanation has only been possible for quantum graphs [16,17]. These studies provided valuable information on the reasons for the effectiveness of RMT. However, the RMT spectral statistics has also been observed to emerge in quantum many-body systems without classical analogs, and it is used to diagnose chaos in such systems as well. Therefore, a proper explanation of the emergence of RMT statistics in quantum many-body systems without classical analogs requires approaches that extend beyond the semiclassical periodic orbit theory.

In recent years, the SFF has been calculated in different studies to understand quantum chaos in interacting many-body systems and its connection to RMT [18–39]. Some of these studies showed the emergence of RMT SFF in quantum circuits with random unitary gates within the limits of large local Hilbert space dimensions [18,21,25,27,31]. While quantum circuits are powerful theoretical tools, the requirement of a large local Hilbert space dimension leaves the question unanswered for physical systems with finite local Hilbert space dimensions like interacting fermions or qubits. Other studies involve an effective field theoretic description [33], but a general mechanism explaining the emergence of RMT behavior for generic many-body quantum systems is still under question. Kos *et al.* [19] made a significant advancement in this direction by deriving the SFF up to two leading orders in time. They considered a one-dimensional (1D) system of qubits with long-range Ising interactions and a periodically kicked transverse magnetic field. Within a random phase approximation (RPA), they expressed the leading-order contribution of the SFF in terms of a partition function of a classical 1D Ising model (on a ring of circumference t), which can be calculated via a 2×2 transfer matrix. This method was plausible as the transverse field in the driving Hamiltonian does not lead to any coupling among qubits during time evolution. This calculation was further generalized for the generic driving Hamiltonian in Ref. [26] to derive the SFF only in leading order in time. We now highlight the main challenges that remain to be addressed for extended many-body quantum systems without a classical analog.

- (i) The higher-order terms of SFF for generic many-body quantum systems have yet to be found.
- (ii) Furthermore, the models studied in Refs. [19,26,30,37] belong to the circular orthogonal ensemble (COE) for $\mathcal{T}$ -invariant systems with $\mathcal{T}^2 = 1$. From a technical point of view, it is not clear how the formalism can be extended to systems belonging to the circular unitary ensemble (CUE) in the absence of $\mathcal{T}$ symmetry and the circular symplectic ensemble (CSE) for $\mathcal{T}$ -invariant systems with $\mathcal{T}^2 = -1$.
- (iii) In particular, while the number of diagrams that contribute in the first and second orders in t for the COE and CUE classes is finite, the number of such diagrams is exponentially large in t for the CSE class. Thus, an alternative scheme must be found to evaluate the SFF for the CSE class. We have achieved this goal, which is a key highlight of this paper.

This work calculates the SFF up to two leading orders in t for generic, periodically kicked, interacting, many-body quantum systems with or without $\mathcal{T}$ symmetry, which allows us to investigate all three Dyson circular ensembles [9,10]. A recent study computed the exact SFF of non-interacting fermions with Dyson statistics [40,41]. Our approach involves the RPA to perform the ensemble average, which has been verified to work very well for such systems in the presence of random on-site potentials and long-range interactions [26,30,37]. We have given a justification for RPA for the Hamiltonian with random on-site potentials and random long-range interactions. We show the emergence of universal RMT SFF for Dyson symmetry classes, namely, COE, CUE, and CSE. The RMT predictions for the SFF of these ensembles are

$$K_{\text{COE}}(t) = 2t - t \ln \left(1 + \frac{2t}{\mathcal{N}} \right), \quad (2)$$

$$K_{\text{CUE}}(t) = t, \quad (3)$$

$$K_{\text{CSE}}(t) = 2t - t \ln \left| 1 - \frac{2t}{\mathcal{N}} \right|, \quad (4)$$

for $0 < t < \mathcal{N}$ [9,10,40,42]. The SFF $K_{\text{CSE}}(t)$ differs from the CSE SFF in Refs. [9,42] by a factor of $1/4$ because the CSE SFF in these references is defined with an extra factor of $1/4$. However, we exclude this factor in our derivation for a unified description. Our explicit derivation of SFF is crucial because it explains how (e.g., the mechanism, nonuniversal behavior) and when (timescales) many-body quantum systems acquire a universal RMT form. For the COE class, the leading order in t contribution to the SFF can be interpreted in terms of a return probability $P_t(\underline{n})$ to an initial state $|\underline{n}\rangle$ after t time steps. The diagrams contributing to the second order in t of the SFF resemble the Sieber-Richter pairs of semiclassical periodic orbit theory [12,13]. Most importantly, our results for the SFF not only give the universal SFF at longer times but also the nonuniversal part

of the SFF at short times, which goes beyond the RMT predictions. We particularly notice that the nonuniversal part of the SFF at shorter times comes mainly from the leading order in the t contributions of the SFF. We also determine the system-size scaling of the Thouless timescales t^* beyond which the SFF takes the universal RMT form in chains of spinless or spinful interacting fermions in the presence or absence of a $U(1)$ symmetry. We observe $P_t(\underline{n}) \sim 1$ when $t \ll t^*$ and $P_t(\underline{n}) \sim 1/\mathcal{N}$ when $t \geq t^*$. Although 1D long-range models require more control to probe our predictions, testing them in higher dimensions experimentally would be easier. Our calculation of the SFF is also applicable to higher space dimensions. We specifically notice the validity of the RPA for a shorter range of interactions with increasing coordination numbers that are easily feasible in higher dimensions. This work is highly significant, as it explains the emergence of RMT behavior in generic many-body quantum systems.

The techniques developed in this work share some similarities with previous work on quantum graphs [43–45], as a doubly stochastic matrix arises in both. A doubly stochastic matrix has the property that the largest eigenvalue is 1, and the other eigenvalues have magnitude less than 1. Furthermore, the eigenvector corresponding to the largest eigenvalue is a uniform superposition of all basis states. These properties immediately lead to the decay of some nonuniversal terms over time and to the emergence of universal terms from the largest eigenvalue and its corresponding eigenvector in both studies. However, quantum graphs are models of single-particle systems with a well-defined stochastic or Markov chain analog in the classical limit [45]. Thus, the results from semiclassical periodic orbit theory were directly used within quantum graphs to compute SFF—namely, the diagonal approximation for leading-order SFF and the Sieber-Richter pairing mechanism for second-order corrections. There are no classical limits for many-body quantum systems studied in this paper. Consequently, we encounter diagrams absent in the semiclassical periodic orbit theory, as discussed in Sec. III. In this work, we explain how these diagrams can be analyzed, thereby showing the emergence of universal RMT SFF. Furthermore, these quantum many-body systems within the RPA can potentially be mapped to a supersymmetric nonlinear sigma model using the generalized Hubbard-Stratonovich transformation [46–50] to show the emergence of universal RMT SFF. Nevertheless, our work provides a simple and direct microscopic explanation for the nonuniversal and universal parts of SFF without any field-theory mapping.

The remainder of the paper is organized as follows. In Sec. II, we present the basic models and the RPA used to perform the ensemble average, and we summarize the key technical developments in computing $K(t)$. We summarize the main findings in Sec. III. In Secs. IV–VI, we prove all the main technical steps. In Sec. VII, we derive the SFF up

to two leading orders in time for each class of $\mathcal{T}$ symmetry. In Sec. VIII, we derive the system-size scaling of t^* . In Sec. IX, we extend our $K(t)$ calculation to higher spatial dimensions for many-body systems with an example showing identical system-size scaling of t^* as in lower dimensions. We also discuss the applicability of our present SFF calculation for single-particle systems in higher spatial dimensions. Finally, we present our conclusion in Sec. X. Further technical details are presented in nine appendices.

II. MODELS AND OBSERVABLES OF INTEREST

The main goal of this paper is to provide an explanation for the emergence of RMT behavior in generic, strongly interacting quantum systems. Thus, we study a general class of periodically kicked systems, described by the Hamiltonian

$$\hat{H}(t) = \hat{H}_0 + \tau_p \hat{H}_1 \sum_{n \in \mathbb{Z}} \delta(t - n\tau_p), \quad (5)$$

where $\mathbb{Z}$ is the set of integers, $\delta(t)$ is the Dirac delta function, $\hat{H}_0$ and $\hat{H}_1$ are two noncommuting Hermitian operators, and τ_p is the kicking period. We set $\tau_p = 1$ for the rest of the discussion. We take $\hat{H}_0$ to be $\mathcal{T}$ invariant, and $\hat{H}_1$ with or without $\mathcal{T}$ symmetry. In Table I, we summarize the constraints imposed by these symmetries on relevant quantities related to $\hat{H}_0$ and $\hat{H}_1$. To investigate the emergence of RMT behavior in such systems, we study the SFF, which is defined as the Fourier transform of a connected two-point correlation function of spectral density [19,42]. For periodically driven systems, the spectral density is defined as

$$\rho(\phi) = \frac{2\pi}{\mathcal{N}} \sum_{n=1}^{\mathcal{N}} \delta(\phi - \phi_n), \quad (6)$$

where ϕ_n for $n = 1, \dots, \mathcal{N}$ are eigenphases of the Floquet operator $\hat{U}$. The connected two-point correlation function of $\rho(\phi)$ is defined as

$$C(\theta) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \rho(\phi + \theta/2) \rho(\phi - \theta/2) - 1. \quad (7)$$

We obtain the SFF by taking its Fourier transform:

$$K(t) = \frac{\mathcal{N}^2}{2\pi} \int_0^{2\pi} d\theta C(\theta) e^{-i\theta t} = [\text{tr} \hat{U}^t][\text{tr} \hat{U}^{-t}] - \mathcal{N}^2 \delta_{t,0}. \quad (8)$$

Since $K(t)$ is not self-averaging, we perform additional ensemble averaging denoted by $\langle \dots \rangle$ and explain it later in this section. Thus, the SFF takes the final form

$$K(t) = \langle [\text{tr} \hat{U}^t][\text{tr} \hat{U}^{-t}] \rangle - \mathcal{N}^2 \delta_{t,0}. \quad (9)$$

TABLE I. Constraints imposed by the presence or absence of time-reversal ($\mathcal{T}$) symmetry on the relevant quantities related to $\hat{H}_0$ and $\hat{H}_1$. Previous studies [19,26,30,37] with generic Hamiltonian models only explore the COE class.

	Basis states under time reversal	Phases $\theta_{\underline{n}}$	$\hat{H}_1$ matrix	$\hat{V}(\equiv e^{-i\hat{H}_1})$ matrix
COE	$ \mathcal{T}\underline{n}\rangle = \underline{n}\rangle$	Nondegenerate	Real symmetric	Symmetric unitary
CUE	$ \mathcal{T}\underline{n}\rangle = \underline{n}\rangle$	Nondegenerate	Complex Hermitian	Nonsymmetric unitary
CSE	$\langle \underline{n} \mathcal{T} \underline{n} \rangle = 0$	Doubly degenerate, $\theta_{\mathcal{T}\underline{n}} = \theta_{\underline{n}}$	Complex Hermitian with $\langle \mathcal{T}\underline{n} \hat{H}_1 \mathcal{T}\underline{n}' \rangle = \langle \underline{n} \hat{H}_1 \underline{n}' \rangle^*$, $\langle \underline{n} \hat{H}_1 \mathcal{T}\underline{n}' \rangle = -\langle \mathcal{T}\underline{n} \hat{H}_1 \underline{n}' \rangle^*$, $\langle \mathcal{T}\underline{n} \hat{H}_1 \underline{n}' \rangle = -\langle \underline{n} \hat{H}_1 \mathcal{T}\underline{n}' \rangle^*$	Nonsymmetric unitary with $\langle \mathcal{T}\underline{n} \hat{V} \mathcal{T}\underline{n}' \rangle = \langle \underline{n}' \hat{V} \underline{n} \rangle$, $\langle \underline{n} \hat{V} \mathcal{T}\underline{n}' \rangle = -\langle \underline{n}' \hat{V} \mathcal{T}\underline{n} \rangle$, $\langle \mathcal{T}\underline{n} \hat{V} \underline{n}' \rangle = -\langle \mathcal{T}\underline{n}' \hat{V} \underline{n} \rangle$

For the Hamiltonian in Eq. (5), the operator $\hat{U}$ can be expressed in terms of operators $\hat{H}_0$ and $\hat{H}_1$ as

$$\hat{U} = \mathcal{T} e^{-i \int_0^t dt \hat{H}(t)} = \hat{V} \hat{W}, \quad (10)$$

where $\mathcal{T}$ represents time ordering, $\hat{V} = e^{-i\hat{H}_1}$, and $\hat{W} = e^{-i\hat{H}_0}$. We choose the eigenstates of $\hat{H}_0$, denoted by $|\underline{n}\rangle$, as the basis states to compute the SFF. Thus, the operators $\hat{H}_0$ and $\hat{W}$ act as

$$\hat{H}_0 |\underline{n}\rangle = E_{\underline{n}} |\underline{n}\rangle, \quad (11)$$

$$\hat{W} |\underline{n}\rangle = e^{-i\theta_{\underline{n}}} |\underline{n}\rangle, \quad (12)$$

where the eigenvalues of $\hat{H}_0$ and the eigenphases of $\hat{W}$ are related by $\theta_{\underline{n}} = E_{\underline{n}} \bmod 2\pi$. Inserting identities $\sum_{\underline{n}_\tau} |\underline{n}_\tau\rangle \langle \underline{n}_\tau| = \mathbb{1}_{\mathcal{N}}$ in $\text{tr} \hat{U}^t$ and $\sum_{\underline{n}'_\tau} |\underline{n}'_\tau\rangle \langle \underline{n}'_\tau| = \mathbb{1}_{\mathcal{N}}$ in $\text{tr} \hat{U}^{-t}$ for $\tau = 1, \dots, t$, we obtain

$$K(t) = \sum_{\underline{n}_1, \dots, \underline{n}_t} \sum_{\underline{n}'_1, \dots, \underline{n}'_t} \left\langle e^{-i \sum_{\tau=1}^t (\theta_{\underline{n}_\tau} - \theta_{\underline{n}'_\tau})} \times \prod_{\tau=1}^t V_{\underline{n}_\tau, \underline{n}_{\tau+1}} V_{\underline{n}'_\tau, \underline{n}'_{\tau+1}}^* \right\rangle, \quad (13)$$

where the trace in Eq. (9) forces periodic boundary conditions (PBCs) in time, $t+1 \equiv 1$, and V denotes the matrix representation of $\hat{V}$ in the computational basis with elements $V_{\underline{n}, \underline{n}'} = \langle \underline{n} | \hat{V} | \underline{n}' \rangle$. Next, we apply the RPA to relate the phases $\theta_{\underline{n}_\tau}$ to the phases $\theta_{\underline{n}'_\tau}$. The RPA is based on the assumption that the phases $\theta_{\underline{n}}$ are independent and uniformly distributed over the interval $[0, 2\pi)$ apart from the CSE case, where the Kramers' degeneracy leads to doubly degenerate phases but still uniformly distributed. The consequences of the RPA on the SFF have been numerically tested successfully in the presence of random on-site potentials and long-range interactions in $\hat{H}_0$ for 1D lattices [19,26,30,37]. In Appendix A, we give a justification of RPA in the strong disorder limit for a base Hamiltonian H_0 with random on-site potentials and random

long-range interactions. Later, we discuss how the RPA works better in higher spatial dimensions, even for relatively shorter-range interactions. Within the RPA, we obtain the following by averaging the randomness in $\hat{H}_0$:

$$\langle e^{-i \sum_{\tau=1}^t (\theta_{\underline{n}_\tau} - \theta_{\underline{n}'_\tau})} \rangle = \sum_{\pi} \prod_{\tau=1}^t \delta_{\theta_{\underline{n}'_\tau}, \pi(\theta_{\underline{n}_\tau})}, \quad (14)$$

where π is a permutation over t phases $\{\theta_{\underline{n}_1}, \dots, \theta_{\underline{n}_t}\}$. In general, one or more phases appear multiple times; thus, the collection of phases, $\{\theta_{\underline{n}_1}, \dots, \theta_{\underline{n}_t}\}$, forms a multiset. If there are l distinct phases of multiplicity $p_1, \dots, p_l$, satisfying $\sum_{i=1}^l p_i = t$, the permutations π belong to the group $S_t / (S_{p_1} \times \dots \times S_{p_l})$, where S_x is the symmetric group of degree x for $x = t, p_1, \dots, p_l$. For systems belonging to the COE and CUE classes, all phases are nondegenerate, which implies that the permutation of phases is equivalent to the permutation of states $\underline{n}_\tau$. Thus,

$$\pi(\theta_{\underline{n}_\tau}) = \theta_{\underline{n}_{\pi(\tau)}}. \quad (15)$$

For a system belonging to the CSE class, the phases are doubly degenerate as $\theta_{\underline{n}} = \theta_{\mathcal{T}\underline{n}}$. Thus,

$$\pi(\theta_{\underline{n}_\tau}) = \theta_{\underline{n}_{\pi(\tau)}}, \theta_{\mathcal{T}\underline{n}_{\pi(\tau)}}, \quad (16)$$

where the subscript $\mathcal{T}\underline{n}_{\pi(\tau)}$ represents the time-reversed state $|\mathcal{T}\underline{n}_{\pi(\tau)}\rangle$ of a state $|\underline{n}_{\pi(\tau)}\rangle$. We substitute Eqs. (14) and (15) in Eq. (13) to find the SFF, $K_1(t)$ and $K_0(t)$, respectively, for the COE and CUE classes as

$$K_1(t) = \sum_{\underline{n}_1, \dots, \underline{n}_t} \sum_{\pi} \prod_{\tau=1}^t V_{\underline{n}_\tau, \underline{n}_{\tau+1}} V_{\underline{n}_{\pi(\tau)}, \underline{n}_{\pi(\tau)+1}}^*, \quad (17)$$

$$K_0(t) = \sum_{\underline{n}_1, \dots, \underline{n}_t} \sum_{\pi} \prod_{\tau=1}^t V_{\underline{n}_\tau, \underline{n}_{\tau+1}} V_{\underline{n}_{\pi(\tau)}, \underline{n}_{\pi(\tau)+1}}^*. \quad (18)$$

We stress that while Eqs. (17) and (18) seem to have identical expressions on the right-hand side, the matrix V is

symmetric in Eq. (17) and nonsymmetric in Eq. (18). We substitute Eqs. (14) and (16) in Eq. (13) to obtain the SFF, $K_{-1}(t)$ for the CSE class:

$$K_{-1}(t) = \sum_{\underline{n}_1, \dots, \underline{n}_t} \sum_{\pi} \sum_{\vec{\sigma}} \prod_{\tau=1}^t V_{\underline{n}_\tau, \underline{n}_{\tau+1}} V_{\underline{n}_{\pi(\tau)}, \underline{n}_{\pi(\tau+1)}}^*, \quad (19)$$

where $\vec{\sigma} = (\sigma_1, \dots, \sigma_t)$ and $\sigma_\tau = 0, 1$ for $\tau = 1, \dots, t$ such that $\underline{n}_{\pi(\tau)}^{(0)} = \underline{n}_{\pi(\tau)}$ and $\underline{n}_{\pi(\tau)}^{(1)} = \mathcal{T} \underline{n}_{\pi(\tau)}$. Since the subgroups of permutations π depend on the multiplicity of different phases in the multiset $\{\theta_{\underline{n}_1}, \dots, \theta_{\underline{n}_t}\}$, the summations $\sum_{\underline{n}_1, \dots, \underline{n}_t}$ and $\sum_{\pi}$ in Eqs. (17)–(19) cannot be performed independently. To make further progress, it is necessary to decouple the two summations. We introduce the following procedure to achieve this goal.

Step 1: First, we take $\pi \in S_t$, irrespective of the multiplicity of phases in the multiset $\{\theta_{\underline{n}_1}, \dots, \theta_{\underline{n}_t}\}$. For a multiset containing l distinct phases of multiplicity $p_1, \dots, p_l$, this approach leads to each distinct permutation being considered $\prod_{i=1}^l p_i!$ times. Therefore, all the extra appearances of each distinct permutation must be removed. The following step explains how this result can be achieved.

Step 2: We start by fixing the minimal case of repetition, where a phase appears twice in the multiset, $\theta_{\underline{n}_{\tau_1}} = \theta_{\underline{n}_{\tau_2}}$. Each distinct permutation of such a multiset appears twice. We remove the contributions of such extra appearances by explicitly calculating and then subtracting their contributions as below:

$$K_1(t) = \sum_{\pi \in S_t} \mathcal{X}_\pi - \sum_{\pi \in S_t/S_2} \sum_{\tau_1 < \tau_2} \mathcal{X}_\pi^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}} + \dots, \quad (20)$$

$$K_0(t) = \sum_{\pi \in S_t} \mathcal{Y}_\pi - \sum_{\pi \in S_t/S_2} \sum_{\tau_1 < \tau_2} \mathcal{Y}_\pi^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}} + \dots, \quad (21)$$

$$K_{-1}(t) = \sum_{\pi \in S_t} \sum_{\vec{\sigma}} \mathcal{Z}_{\pi, \vec{\sigma}} - \sum_{\pi \in S_t/S_2} \sum_{\vec{\sigma}} \sum_{\tau_1 < \tau_2} \mathcal{Z}_{\pi, \vec{\sigma}}^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}} - \sum_{\pi \in S_t/S_2} \sum_{\vec{\sigma}} \sum_{\tau_1 < \tau_2} \mathcal{Z}_{\pi, \vec{\sigma}}^{\{\underline{n}_{\tau_1}, \mathcal{T} \underline{n}_{\tau_2}\}} + \dots, \quad (22)$$

where

$$\mathcal{X}_\pi = \sum_{\{\underline{n}\}} \prod_{\tau=1}^t V_{\underline{n}_\tau, \underline{n}_{\tau+1}} V_{\underline{n}_{\pi(\tau)}, \underline{n}_{\pi(\tau+1)}}^*, \quad (23)$$

$$\mathcal{X}_\pi^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}} = \sum_{\{\underline{n}\}} \delta_{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}} \prod_{\tau=1}^t V_{\underline{n}_\tau, \underline{n}_{\tau+1}} V_{\underline{n}_{\pi(\tau)}, \underline{n}_{\pi(\tau+1)}}^*, \quad (24)$$

$$\mathcal{Y}_\pi = \sum_{\{\underline{n}\}} \prod_{\tau=1}^t V_{\underline{n}_\tau, \underline{n}_{\tau+1}} V_{\underline{n}_{\pi(\tau)}, \underline{n}_{\pi(\tau+1)}}^*, \quad (25)$$

$$\mathcal{Y}_\pi^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}} = \sum_{\{\underline{n}\}} \delta_{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}} \prod_{\tau=1}^t V_{\underline{n}_\tau, \underline{n}_{\tau+1}} V_{\underline{n}_{\pi(\tau)}, \underline{n}_{\pi(\tau+1)}}^*, \quad (26)$$

$$\mathcal{Z}_{\pi, \vec{\sigma}} = \sum_{\{\underline{n}\}} \prod_{\tau=1}^t V_{\underline{n}_\tau, \underline{n}_{\tau+1}} V_{\underline{n}_{\pi(\tau)}, \underline{n}_{\pi(\tau+1)}}^*, \quad (27)$$

$$\mathcal{Z}_{\pi, \vec{\sigma}}^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}} = \sum_{\{\underline{n}\}} \delta_{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}} \prod_{\tau=1}^t V_{\underline{n}_\tau, \underline{n}_{\tau+1}} V_{\underline{n}_{\pi(\tau)}, \underline{n}_{\pi(\tau+1)}}^*, \quad (28)$$

$$\mathcal{Z}_{\pi, \vec{\sigma}}^{\{\underline{n}_{\tau_1}, \mathcal{T} \underline{n}_{\tau_2}\}} = \sum_{\{\underline{n}\}} \delta_{\underline{n}_{\tau_1}, \mathcal{T} \underline{n}_{\tau_2}} \prod_{\tau=1}^t V_{\underline{n}_\tau, \underline{n}_{\tau+1}} V_{\underline{n}_{\pi(\tau)}, \underline{n}_{\pi(\tau+1)}}^*, \quad (29)$$

where $\sum_{\{\underline{n}\}} = \sum_{\underline{n}_1, \dots, \underline{n}_t}$. The symbols $\mathcal{X}_\pi$ and $\mathcal{Y}_\pi$ denote, respectively, the total contribution to SFFs $K_1(t)$ and $K_0(t)$ of a permutation $\pi \in S_t$ of all the multisets of states $\{\underline{n}_1, \dots, \underline{n}_t\}$. The symbols $\mathcal{X}_\pi^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}}$ and $\mathcal{Y}_\pi^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}}$ represent, respectively, the correction for a permutation $\pi \in S_t/S_2$ of all multisets $\{\underline{n}_1, \dots, \underline{n}_t\}$, where states at time steps τ_1 and τ_2 are identical. These terms fix the double counting of permutation π of such multisets in $\sum_{\pi \in S_t} \mathcal{X}_\pi$ and $\sum_{\pi \in S_t} \mathcal{Y}_\pi$. However, the overcounting resulting from a higher number of repetitions is still present and requires further correction terms. In this paper, we restrict ourselves to double repetitions only. Similarly, the symbol $\mathcal{Z}_{\pi, \vec{\sigma}}$ denotes a contribution to the SFF, $K_{-1}(t)$, of a permutation $\pi \in S_t$ and a configuration $\vec{\sigma}$ of all the multisets of states $\{\underline{n}_1, \dots, \underline{n}_t\}$. In this case, when a phase repeats two times in $\{\theta_{\underline{n}_1}, \dots, \theta_{\underline{n}_t}\}$, due to the Kramers' degeneracy, either a state appears two times in $\{\underline{n}_1, \dots, \underline{n}_t\}$ or a state and its time-reversed state appear in $\{\underline{n}_1, \dots, \underline{n}_t\}$. Thus, there are two correction terms, denoted by $\mathcal{Z}_{\pi, \vec{\sigma}}^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}}$ and $\mathcal{Z}_{\pi, \vec{\sigma}}^{\{\underline{n}_{\tau_1}, \mathcal{T} \underline{n}_{\tau_2}\}}$.

Calculating the SFF for each case of the Dyson symmetry classes requires a procedure to calculate $\mathcal{X}_\pi$, $\mathcal{X}_\pi^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}}$, $\mathcal{Y}_\pi$, $\mathcal{Y}_\pi^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}}$, $\mathcal{Z}_{\pi, \vec{\sigma}}$, $\mathcal{Z}_{\pi, \vec{\sigma}}^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}}$, and $\mathcal{Z}_{\pi, \vec{\sigma}}^{\{\underline{n}_{\tau_1}, \mathcal{T} \underline{n}_{\tau_2}\}}$ for different permutations π and configurations $\vec{\sigma}$. Based on a diagrammatic representation of π and $\vec{\sigma}$ (e.g., Fig. 1), we present rules to calculate the contribution of each permutation in Appendixes B, E, and G.

We further find that, unlike the COE and CUE cases, the matrix V has a richer structure for the CSE case, as given in Table I. More specifically, the property $\langle \mathcal{T} \underline{n} | \hat{V} | \mathcal{T} \underline{n}' \rangle = \langle \underline{n}' | \hat{V} | \underline{n} \rangle$ immediately leads to the following theorem for such systems:

Theorem 1. For a permutation π and a configuration $\vec{\sigma} = (\sigma_1, \dots, \sigma_t)$, there exists another configuration $\vec{\sigma}' = (1 - \sigma_t, \dots, 1 - \sigma_1)$, such that

$$\mathcal{Z}_{\pi, \vec{\sigma}} = \mathcal{Z}_{\mathcal{R}\pi, \vec{\sigma}'}, \quad (30)$$

where $\mathcal{R}$ is a reflection or anticyclic permutation, which reverses the order of all states in $\{\underline{n}_1, \dots, \underline{n}_t\}$. Therefore,

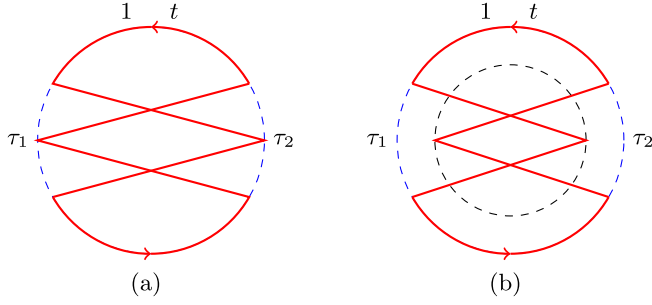


FIG. 1. (a) Diagrammatic representation of a transposition for systems belonging to the COE and CUE classes, where states $\underline{n}_{\tau_1}$ and $\underline{n}_{\tau_2}$ are interchanged. The dashed blue circle connects the states of the initial configuration, $\{\underline{n}_1, \dots, \underline{n}_{\tau_1}, \dots, \underline{n}_{\tau_2}, \dots, \underline{n}_t\}$. The subscripts $(1, \dots, t)$ are in increasing order along the counterclockwise direction on the blue circle. The red curve connects the configuration of states after transposition $\{\underline{n}_1, \dots, \underline{n}_{\tau_2}, \dots, \underline{n}_{\tau_1}, \dots, \underline{n}_t\}$. (b) Diagrammatic representation of a transposition for a system belonging to the CSE class, where states $\underline{n}_{\tau_1}$ and $\underline{n}_{\tau_2}$ are interchanged and $\sigma_{\tau_1} = \sigma_{\tau_2} = 1, \sigma_{\tau} = 0$ for $\tau = 1, \dots, t$ excluding τ_1, τ_2 . The inner black dashed circle connects the time-reversed version of the states on the outer circle.

collectively, permutation $\mathcal{R}\pi$ changes the order of states to $\{\underline{n}_{\pi(t)}, \dots, \underline{n}_{\pi(1)}\}$.

Proof. From Eq. (27), we write

$$\begin{aligned} \mathcal{Z}_{\pi, \vec{\sigma}} &= \sum_{\underline{n}_1, \dots, \underline{n}_t} \prod_{\tau=1}^t V_{\underline{n}_{\tau}, \underline{n}_{\tau+1}} V_{\underline{n}_{\pi(\tau)}, \underline{n}_{\pi(\tau+1)}}^{(\sigma_{\tau})} \\ &= \sum_{\underline{n}_1, \dots, \underline{n}_t} \prod_{\tau=1}^t V_{\underline{n}_{\tau}, \underline{n}_{\tau+1}} V_{\mathcal{T}\underline{n}_{\pi(\tau+1)}, \mathcal{T}\underline{n}_{\pi(\tau)}}^{(\sigma_{\tau})} \\ &= \sum_{\underline{n}_1, \dots, \underline{n}_t} \prod_{\tau=1}^t V_{\underline{n}_{\tau}, \underline{n}_{\tau+1}} V_{\underline{n}_{\pi(\tau+1)}^{(1-\sigma_{\tau+1})}, \underline{n}_{\pi(\tau)}^{(1-\sigma_{\tau})}} \\ &= \mathcal{Z}_{\mathcal{R}\pi, \vec{\sigma}'}. \end{aligned} \quad (31)$$

Following the rules in Appendixes B, E, and G, the leading-order SFF determining the linear ramp can be easily derived for the COE and CUE cases by studying the t cyclic and t anticyclic variants of identity permutation, I ,

$$K_1^{(1)}(t) = \sum_{l=0}^{t-1} (\mathcal{X}_{\mathcal{C}^l I} + \mathcal{X}_{\mathcal{R}\mathcal{C}^l I}), \quad (32)$$

$$K_0^{(1)}(t) = \sum_{l=0}^{t-1} (\mathcal{Y}_{\mathcal{C}^l I} + \mathcal{Y}_{\mathcal{R}\mathcal{C}^l I}), \quad (33)$$

where the superscript (1) represents that the above expressions only contain the leading-order SFF, and $\mathcal{C}^l$ denotes cyclic permutation, whose action is defined as $\underline{n}_{\mathcal{C}^l(\tau)} = \underline{n}_{\tau+l}$ for $\tau = 1, \dots, t$. We explicitly evaluate $K_1^{(1)}(t)$ and $K_0^{(1)}(t)$ in Sec. VII. However, for the CSE case, each permutation π

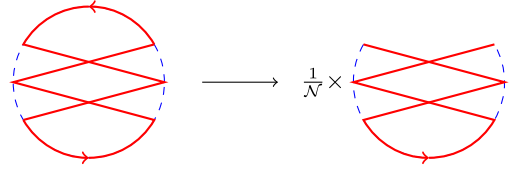


FIG. 2. Reduced diagram (right) obtained by inserting a factor $1/\mathcal{N}$ for a red arc in a diagram (left) representing a transposition.

has further 2^t variants due to configurations $\vec{\sigma}$ in Eq. (22). Therefore, a brute-force calculation based on the rules in Appendix G is practically impossible. Instead, a new approach is required. Our new approach starts with a general analysis of arbitrary diagrams in Sec. IV, which reveals that the contribution of a diagram can be expressed as a sum of three kinds of terms, namely, Type I, Type II, and Type III. The Type I term is $1/\mathcal{N}^\mu$, where μ is a non-negative integer. This term is universal since it is independent of Hamiltonian parameters. Type II and Type III terms are nonuniversal because they depend on Hamiltonian parameters. Furthermore, Type III terms decay exponentially with time, while Type II terms remain finite at longer times.

This finding implies that only Type I and Type II terms are significant in determining the SFF at longer times. More specifically, when the contribution of all the permutations is added, Type I terms must add up to give a universal RMT form of SFF, whereas Type II terms must cancel each other out. Type III terms only contribute to the nonuniversal part of SFF and vanish beyond t^* . In addition, Type III terms are insignificant when the goal is to understand the emergence of RMT behavior. Thus, we define reduced diagrams (see, for example, Fig. 2) by eliminating some Type III terms in Sec. V. The reduced diagrams provide further analytical control. We perform a general analysis of the reduced diagrams in Sec. VI. For systems belonging to the COE and CUE classes, we find that, for each reduced diagram without repetition, there exists a diagram with minimal repetition of states that has an identical reduced diagram. In other words, for a given permutation π , there exists a permutation π' such that Type I and Type II contributions of $\mathcal{X}_{\pi}$ and $\mathcal{X}_{\pi'}^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}}$ (or $\mathcal{Y}_{\pi}$ and $\mathcal{Y}_{\pi'}^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}}$) are identical for some appropriate values of τ_1 and τ_2 , as in Fig. 3.

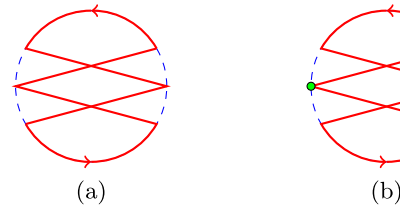


FIG. 3. Reduced diagrams of (a) and (b) are identical. The green circles in diagram (b) indicate that the states at those time steps are identical.

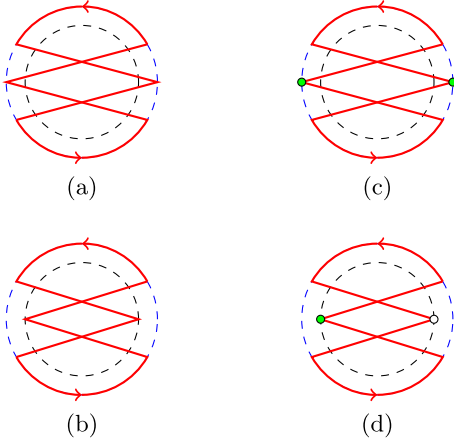


FIG. 4. Diagrams in (a) and (c) have identical reduced diagrams. Diagrams in (b) and (d) have identical reduced diagrams. The green circles in diagram (c) indicate that the states at those time steps are identical. The green and white circles in diagram (d) indicate that the states at those time steps are related by time reversal.

For systems belonging to the CSE class, we find that, for each reduced diagram without repetition, a diagram with minimal repetition of states or a diagram with two states at different time steps related by time reversal has an identical reduced diagram. In other words, for a given permutation π and configuration $\vec{\sigma}$, there exists another permutation π' such that Type I and Type II contributions of either $\mathcal{Z}_{\pi, \vec{\sigma}}$ and $\mathcal{Z}_{\pi', \vec{\sigma}}^{\{n_{r_1}, n_{r_2}\}}$ or $\mathcal{Z}_{\pi, \vec{\sigma}}$ and $\mathcal{Z}_{\pi', \vec{\sigma}}^{\{n_{r_1}, T n_{r_2}\}}$ are identical, as in Fig. 4. Following these findings, we identify the diagrams whose reduced diagrams are not canceled in Eqs. (20)–(22). Such diagrams give the universal RMT form of SFF beyond the leading order. We identify these diagrams contributing to the SFF in second order in t in Sec. VII.

The SFF of physical chaotic systems takes the RMT SFF form only beyond t^* . To find the system-size scaling of t^* , we study strongly interacting fermionic chains with or without $\mathcal{T}$ symmetry in Sec. VIII. To investigate systems belonging to the COE and CUE classes, we consider a chain of spinless fermions described by

$$\hat{H}_0^0 = \sum_{x=1}^L \epsilon_x \hat{n}_x + \sum_{x < y} \frac{U_{xy}}{|x-y|^\alpha} \hat{n}_x \hat{n}_y, \quad (34)$$

$$\hat{H}_1^0 = \sum_{x < y} -J_{xy} \hat{c}_x^\dagger \hat{c}_y + \Delta_{xy} \hat{c}_x^\dagger \hat{c}_y^\dagger + \text{H.c.}, \quad (35)$$

where the superscript 0 represents spinless fermions, and $\hat{n}_x \equiv \hat{c}_x^\dagger \hat{c}_x$, $\hat{c}_x$, and $\hat{c}_x^\dagger$ are, respectively, the fermion occupation number, and annihilation and creation operators at site x . We take on-site energies $\epsilon_1, \dots, \epsilon_L$ as i.i.d. Gaussian random numbers with zero mean and a finite standard deviation of $\Delta\epsilon$. We also take the long-range interaction parameters U_{xy} as i.i.d. Gaussian random numbers with mean U_0 and standard deviation ΔU_0 . With this choice of

disorder, a justification for RPA can be given as presented in Appendix A. The hopping amplitude, denoted by J_{xy} , takes real or imaginary values according to the presence or absence of $\mathcal{T}$ symmetry. The pairing amplitude Δ_{xy} takes only real values.

For systems belonging to the CSE class, we take a chain of interacting spin-1/2 fermions given as

$$\begin{aligned} \hat{H}_0^{1/2} = & \sum_{x=1}^L \epsilon_x (\hat{n}_{x\uparrow} + \hat{n}_{x\downarrow}) + \sum_{x < y} \sum_{\sigma\sigma'} \frac{U_{xy}^{\sigma\sigma'}}{|x-y|^\alpha} \hat{n}_{x\sigma} \hat{n}_{y\sigma'} \\ & + \sum_{x=1}^L U_x \hat{n}_{x\uparrow} \hat{n}_{x\downarrow}, \end{aligned} \quad (36)$$

$$\hat{H}_1^{1/2} = \sum_{x=1}^L \sum_{\sigma\sigma'} J_{\sigma\sigma'} \hat{c}_{x\sigma}^\dagger \hat{c}_{x+1\sigma'} + \Delta_{\sigma\sigma'} \hat{c}_{x\sigma}^\dagger \hat{c}_{x+1\sigma'}^\dagger + \text{H.c.}, \quad (37)$$

where the superscript 1/2 stands for spin-1/2 fermions, $\sigma = \uparrow, \downarrow$ represent the spin states along the z axis, and $\hat{n}_{x\uparrow} \equiv \hat{c}_{x\uparrow}^\dagger \hat{c}_{x\uparrow}$ and $\hat{n}_{x\downarrow} \equiv \hat{c}_{x\downarrow}^\dagger \hat{c}_{x\downarrow}$ are fermion occupation number operators at site x for spin states $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively. The operators $\hat{c}_{x\sigma}$ and $\hat{c}_{x\sigma}^\dagger$ are annihilation and creation operators of fermions at site x with spin σ . The long-range interaction parameters $U_{ij}^{\sigma\sigma'}$ are all chosen as Gaussian random numbers with mean U_0 and standard deviation ΔU_0 to ensure phases are only doubly degenerate due to the Kramers' degeneracy. The particle-particle repulsion U_x at a site x is also chosen as Gaussian random numbers with mean $\bar{U}$ and standard deviation $\Delta \bar{U}$. Additionally, the annihilation operators of spin-1/2 fermions transform under $\mathcal{T}$ as

$$\mathcal{T} \hat{c}_{x\uparrow} \mathcal{T}^{-1} = -\hat{c}_{x\downarrow}, \quad \mathcal{T} \hat{c}_{x\downarrow} \mathcal{T}^{-1} = \hat{c}_{x\uparrow}. \quad (38)$$

Thus, $\hat{H}_0^{1/2}$ and $\hat{H}_1^{1/2}$ are invariant under $\mathcal{T}$ when $U_{ij}^{\uparrow\uparrow} = U_{ij}^{\downarrow\downarrow}$, $U_{ij}^{\uparrow\downarrow} = U_{ij}^{\downarrow\uparrow}$, $J_{\downarrow\downarrow} = J_{\uparrow\uparrow}^*$, $J_{\uparrow\downarrow} = -J_{\downarrow\uparrow}^*$, $\Delta_{\downarrow\downarrow} = \Delta_{\uparrow\uparrow}^*$, and $\Delta_{\uparrow\downarrow} = -\Delta_{\downarrow\uparrow}^*$. We also take PBC on real space for all the models by imposing $\hat{c}_{x+L} \equiv \hat{c}_x$, $\hat{c}_{x+L\sigma} \equiv \hat{c}_{x\sigma}$, and replacing $|x-y|$ by $\min(|x-y|, L-|x-y|)$ in long-range interactions.

III. OUTLINE OF THE RESULTS

We have identified the diagrams determining the universal RMT form of SFF up to two leading orders in time for all three Dyson circular ensembles [9,10]. Different system-size scalings of t^* have been found in recent years for various systems of COE and CUE classes with or without a $U(1)$ symmetry [19,21,25,26,30,37,51,52]. There is hardly any result on the interacting many-body quantum chaos of the CSE class. Here, we provide a unified description of system-size scaling of t^* for all three Dyson

TABLE II. System-size scaling of Thouless time (t^*) for periodically kicked, strongly interacting fermionic chains of length L for the three Dyson symmetry classes. The operators $\hat{N} = \sum_{x=1}^L \hat{n}_x$ and $\hat{N}_t = \sum_{x=1}^L \sum_{\sigma} \hat{n}_{x\sigma}$ denote the total fermion number for spinless and spinful models, respectively, and $\hat{N}_s = \sum_{x=1}^L (-1)^x \hat{n}_x$ is the staggered fermion number for even L . The presence or absence of a $U(1)$ symmetry can be found by their commutation with the chain's Hamiltonian $\hat{H}(t)$. Previous studies [19,26,30] with generic Hamiltonian models found the first three scaling behaviors for the COE class only.

	Parameters (β)	Symmetry	λ_1 in thermodynamic limit	Degeneracy, d_1	Thouless time, t^*
COE	$J \neq 0, \Delta = 0$	$[\hat{H}(t), \hat{N}] = 0$	$1 - c_\beta/L^2$	$\mathcal{O}(L^0)$	$\mathcal{O}(L^2)$
	$J \neq \Delta \neq 0$	$[\hat{H}(t), \hat{N}] \neq 0$	$\mathcal{O}(L^0)$	$\mathcal{O}(L^0)$	$\mathcal{O}(L^0)$
	$ J = \Delta \neq 0$	$[\hat{H}(t), \hat{N}] \neq 0$	$\mathcal{O}(L^0)$	$\mathcal{O}(L^2)$	$\mathcal{O}(\ln L)$
	$J = 0, \Delta \neq 0$	$[\hat{H}(t), \hat{N}_s] = 0$	$1 - c_\beta/L^2$	$\mathcal{O}(L^0)$	$\mathcal{O}(L^2)$
CUE	$J, g \neq 0; \Delta, \Delta' = 0$	$[\hat{H}(t), \hat{N}] = 0$	$1 - c_\beta/L^2$	$\mathcal{O}(L^0)$	$\mathcal{O}(L^2)$
	$J, g \neq 0; \Delta, \Delta' \neq 0$	$[\hat{H}(t), \hat{N}] \neq 0$	$\mathcal{O}(L^0)$	$\mathcal{O}(L^0)$	$\mathcal{O}(L^0)$
	$ J = \Delta , g = \Delta' $	$[\hat{H}(t), \hat{N}] \neq 0$	$\mathcal{O}(L^0)$	$\mathcal{O}(L)$	$\mathcal{O}(\ln L)$
CSE	$J_{\sigma\sigma'} \neq 0; \Delta_{\sigma\sigma'} = 0$	$[\hat{H}(t), \hat{N}_t] = 0$	$1 - c_\beta/L^2$	$\mathcal{O}(L^0)$	$\mathcal{O}(L^2)$
	$J_{\sigma\sigma'} \neq 0; \Delta_{\sigma\sigma'} \neq 0$	$[\hat{H}(t), \hat{N}_t] \neq 0$	$\mathcal{O}(L^0)$	$\mathcal{O}(L^0)$	$\mathcal{O}(L^0)$
	$ J_{\sigma\sigma'} = \Delta_{\sigma\sigma'} $	$[\hat{H}(t), \hat{N}_t] \neq 0$	$\mathcal{O}(L^0)$	$\mathcal{O}(L)$	$\mathcal{O}(\ln L)$

classes in the presence or absence of a $U(1)$ symmetry, which we summarize in Table II.

A. Emergence of universal RMT SFF

1. Leading-order SFF: Linear ramp

The leading-order SFF is determined by the identity permutation and its variants, which can be expressed diagrammatically as in Fig. 5. For systems belonging to the COE class, there are t cyclic and t anticyclic permutations as variants of identity permutation. All these variants have identical contributions due to the symmetric nature of the matrix V , as summarized in Table I, and PBC in time. Therefore, we find the leading-order SFF for the COE class by applying the rules in Appendix B as

$$K_1^{(1)}(t) = 2t \text{tr} \mathcal{M}^t = 2t(1 + \lambda_1^t + \dots + \lambda_{N-1}^t). \quad (39)$$

Here, $\mathcal{M}$ is a doubly stochastic (Markov) matrix given by $\mathcal{M} = V \bullet V^*$, where “ $\bullet$ ” represents the Hadamard product defined as $(A \bullet B)_{i,j} = A_{i,j} B_{i,j}$. The largest eigenvalue of $\mathcal{M}$ is 1, $\lambda_0 = 1$, whereas other eigenvalues $\lambda_1, \dots, \lambda_{N-1}$

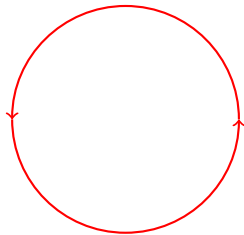


FIG. 5. Diagram contributing to leading-order SFF for systems of the COE and CUE classes.

have magnitude less than 1. The leading order in t contributions to the SFF can be physically interpreted using the Markov matrix in terms of a return probability $P_t(\underline{n}) = \langle \underline{n} | \mathcal{M}^t | \underline{n} \rangle$ to an initial state $|\underline{n}\rangle$ after t time steps. Thus, we can write $K_1^{(1)}(t) = 2t \sum_{\underline{n}} P_t(\underline{n})$.

For systems belonging to the CUE class, the contribution of cyclic variants differs from that of anticyclic variants due to the nonsymmetric nature of the matrix V . We thus obtain the leading-order SFF for the CUE class by applying the rules in Appendix E as

$$K_0^{(1)}(t) = t(\text{tr} \mathcal{M}^t + \text{tr} \tilde{\mathcal{M}}^t) = t(1 + \lambda_1^t + \dots + \lambda_{N-1}^t + \chi_0^t + \dots + \chi_{N-1}^t), \quad (40)$$

where $\tilde{\mathcal{M}}$ is a complex Hermitian matrix defined by $\tilde{\mathcal{M}} = V \bullet V^\dagger$. It can be shown that all eigenvalues of $\tilde{\mathcal{M}}$, $\chi_0, \dots, \chi_{N-1}$, have magnitude less than 1.

For systems belonging to the CSE class, each of the cyclic and anticyclic variants has further 2^t variants corresponding to different configurations $\vec{\sigma} = (\sigma_1, \dots, \sigma_t)$. Analyzing their reduced diagrams, we find that only diagrams with $\vec{\sigma} = (0, \dots, 0)$ [Fig. 6(a)] and $\vec{\sigma} = (1, \dots, 1)$ [Fig. 6(b)] have a nonvanishing contribution at longer times, leading to the SFF for the CSE class at leading order in t as

$$K_{-1}^{(1)}(t) = 2t(\text{tr} \mathcal{M}^t + \text{tr} \tilde{\mathcal{M}}^t) = 2t(1 + \lambda_1^t + \dots + \lambda_{N-1}^t + \chi_0^t + \dots + \chi_{N-1}^t). \quad (41)$$

All the nonuniversal terms decay exponentially with time. Therefore, at long times, we have $K_1^{(1)} \simeq 2t$, $K_0^{(1)} \simeq t$, and $K_{-1}^{(1)}(t) \simeq 2t$, which are identical to the RMT predictions for these classes in Eqs. (2)–(4) up to first order in t . A generic periodically kicked many-body system shows the RMT

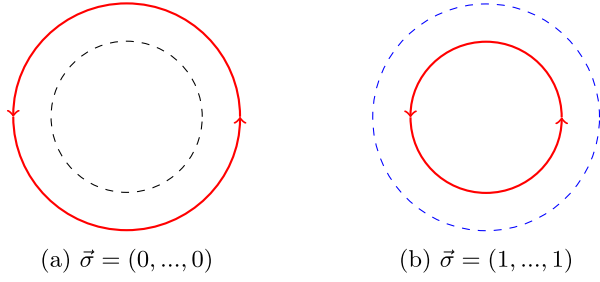


FIG. 6. Diagrams contributing to leading-order SFF for systems of the CSE class.

behavior if the largest eigenvalue of $\mathcal{M}$ is nondegenerate. Since we work in the eigenbasis of $\hat{H}_0$, $\mathcal{M}$ is obtained from V in the eigenbasis of $\hat{H}_0$. When V has a block-diagonal structure on this basis, only then can the largest eigenvalue of $\mathcal{M}$ be degenerate. Therefore, matrices V or $\hat{H}_1$ should not have a block-diagonal structure on the eigenbasis of $\hat{H}_0$ for the RMT behavior to emerge. Further, we show later that the Thouless time should be less than the Heisenberg time or Hilbert space dimension for the RMT behavior.

2. Second-order correction

The second-order term in the RMT SFF is inversely proportional to the Hilbert space dimension $\mathcal{N}$ for the COE and CSE classes, and such a term is absent for the CUE class. From the Type I term for different permutations, we find that the following permutations (see Fig. 7) and their variants contribute to the second-order correction in the SFF. They include (1) single transpositions (T) defined as an interchange of any two states in $\{\underline{n}_1, \dots, \underline{n}_t\}$, (2) subsequence reversal (S) defined as a reversal of the order of states between two time steps, (3) identity permutation with a state repeated twice (R), and (4) subsequence reversal with a state repeated twice (SR). The last permutation reverses the order of states between the two appearances of the repeated state.

Analyzing the reduced diagrams for these permutations, we find that the reduced diagrams of T and R are exactly identical for all three Dyson classes. This finding implies that the Type I and Type II contributions of these permutations exactly cancel each other in Eqs. (20)–(22). Similarly, the reduced diagrams of S are canceled by the reduced diagrams of SR . However, there are two extra SR diagrams with lower red arcs of lengths $\lceil t/2 \rceil - 1$ and $\lceil t/2 \rceil$. Computing all the variants of these diagrams, we obtain the second-order RMT correction to the SFF for all three Dyson classes as

$$K_1^{(2)}(t) = -\frac{2t^2}{\mathcal{N}} + \mathcal{O}(\lambda_1^t), \quad (42)$$

$$K_0^{(2)}(t) = \mathcal{O}(\lambda_1^t, \chi_0^t), \quad (43)$$

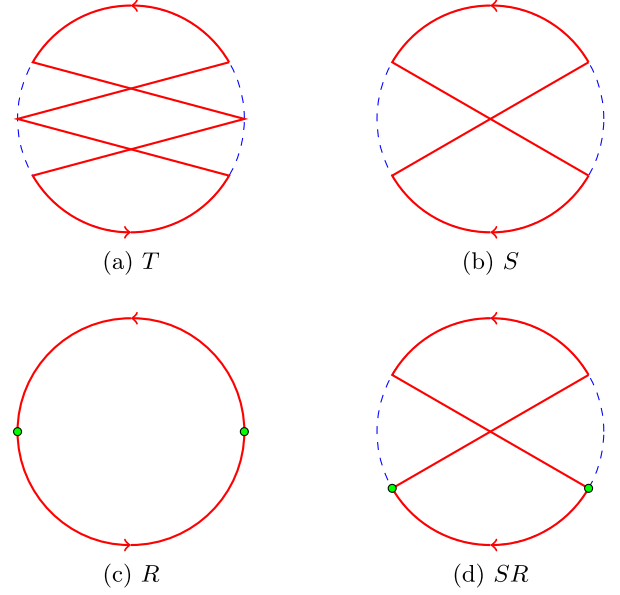


FIG. 7. Diagrams determining the second-order correction in RMT SFF.

$$K_{-1}^{(2)}(t) = \frac{2t^2}{\mathcal{N}} + \mathcal{O}(\lambda_1^t, \chi_0^t). \quad (44)$$

In an alternative diagrammatic representation, the extra SR diagrams resemble the Sieber-Richter pairs of near-miss and self-crossing orbits [12,13]. Nevertheless, there are specific differences in these diagrams, as here we study quantum systems with finite-dimensional local Hilbert spaces compared to semiclassical periodic orbit theory corresponding to continuous variable systems and infinite Hilbert space. We further notice that the nonuniversal terms at second order in t of our calculated SFF do not contribute significantly to the short-time nonuniversal behavior of SFF when we compare the SFF computed within the RPA and that from direct numerics in Fig. 24.

B. Transition from nonuniversal to universal RMT regime: Thouless time

The Thouless time t^* is defined as the timescale beyond which the SFF has a universal RMT form. From Eqs. (39)–(41), we find that it is determined by the second-largest eigenvalue λ_1 of the doubly stochastic matrix $\mathcal{M}$ for all three classes. We find that t^* is related to eigenvalue λ_1 and its degeneracy d_1 through

$$t^* \simeq \frac{\ln(d_1) + 1}{|\ln \lambda_1|}. \quad (45)$$

We determine the system-size scaling of t^* in the presence or absence of a $U(1)$ symmetry by studying the Hamiltonians in Eqs. (34) and (35) for systems with $\mathcal{T}^2 = 1$ and in the absence of $\mathcal{T}$ symmetry. To study the former

case, we choose the hopping and pairing parameters as

$$J_{xy} = \begin{cases} J & \text{if } |x - y| = 1 \\ 0 & \text{otherwise,} \end{cases} \quad (46)$$

$$\Delta_{xy} = \begin{cases} \Delta & \text{if } |x - y| = 1 \\ 0 & \text{otherwise,} \end{cases} \quad (47)$$

where J and Δ are real-valued parameters. We break $\mathcal{T}$ symmetry by adding an imaginary next-nearest-neighbor hopping. Thus, we consider the following parameters to investigate t^* for systems in the absence of $\mathcal{T}$ symmetry.

$$J_{xy} = \begin{cases} J & \text{if } |x - y| = 1 \\ ig & \text{if } |x - y| = 2 \\ 0 & \text{otherwise,} \end{cases} \quad (48)$$

$$\Delta_{xy} = \begin{cases} \Delta & \text{if } |x - y| = 1 \\ \Delta' & \text{if } |x - y| = 2 \\ 0 & \text{otherwise,} \end{cases} \quad (49)$$

where J , g , Δ , and Δ' are real-valued parameters. We explore a chain of strongly interacting spin-1/2 fermions in Eqs. (36) and (37) to determine t^* for systems belonging to the CSE class. We obtain different system-size scalings of t^* for various choices of parameters in all three classes, as summarized in Table II. While the second-largest eigenvalue λ_1 of $\mathcal{M}$ determines when the universal RMT SFF form would emerge for all three classes, the other eigenvalues of $\mathcal{M}$ and $\tilde{\mathcal{M}}$ fix the nonuniversal behavior of the

SFF at short timescales for any physical systems. We have shown a nice comparison between the SFF calculated using our microscopic theory within the RPA and that from direct numerics using Eq. (9) in Fig. 8, both at short and long times for many-body interacting models from all three Dyson classes. Our theory gives the features of the SFF in the universal RMT region and the early nonuniversal region. These two regions are separated by the Thouless time.

IV. GENERAL TREND IN THE CONTRIBUTION OF DIFFERENT DIAGRAMS

Theorem 2. Contributions $\mathcal{X}_\pi$, $\mathcal{X}_\pi^{\{\underline{n}_{r_1}, \underline{n}_{r_2}\}}$, $\mathcal{Y}_\pi$, $\mathcal{Y}_\pi^{\{\underline{n}_{r_1}, \underline{n}_{r_2}\}}$, $\mathcal{Z}_{\pi, \vec{\sigma}}$, $\mathcal{Z}_{\pi, \vec{\sigma}}^{\{\underline{n}_{r_1}, \underline{n}_{r_2}\}}$, and $\mathcal{Z}_{\pi, \vec{\sigma}}^{\{\underline{n}_{r_1}, \mathcal{T}\underline{n}_{r_2}\}}$ can be expressed as a sum of three kinds of terms, namely, Type I, Type II, and Type III. The Type I term is $1/N^\mu$, where μ is a non-negative integer. This term is universal since it is independent of Hamiltonian parameters. Type II and Type III terms are nonuniversal because they depend on Hamiltonian parameters. Furthermore, Type III terms decay exponentially with time, while Type II terms remain finite at longer times.

In the following subsections, we prove Theorem 2 for all three Dyson classes.

A. COE

We first consider a permutation π , whose diagram contains a number r of red arcs of lengths $\nu_1, \dots, \nu_r$ with no repetition of states. Following the rules in Appendix B, the contribution to SFF can be written similarly to Eq. (B2) as

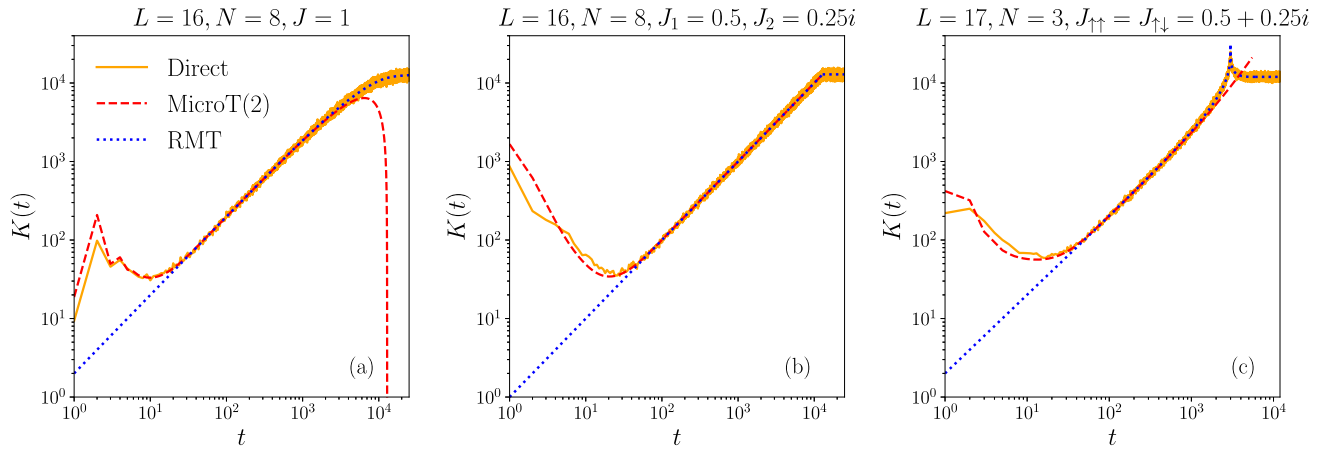


FIG. 8. Comparison of directly calculated SFF, $K(t)$, using Eq. (9) with that obtained using our microscopic theory up to two leading orders in time [MicroT(2)] for periodically kicked, strongly interacting fermionic chains of all three Dyson classes: (a) $\mathcal{T}$ -invariant system with $\mathcal{T}^2 = 1$, (b) system in the absence of $\mathcal{T}$ symmetry, and (c) $\mathcal{T}$ -invariant system with $\mathcal{T}^2 = -1$. For direct simulation, we choose $\langle \epsilon \rangle = 0$, $\Delta \epsilon = 10$, and $\alpha = 1.5$ in all plots. Furthermore, we fix $U_0 = 100\sqrt{2}$ and $\Delta U_0 = 20$ in diagrams (a) and (b), and we choose $U_0 = 40$, $\Delta U_0 = 10$, $\bar{U} = 40$, and $\Delta \bar{U} = 10$ in diagram (c). An averaging over 300–400 realizations of disorder is performed for direct simulation.

$$\mathcal{X}_\pi = \sum_{\{i_q\}} \left(\prod_{q=1}^r \lambda_{i_q}^{\nu_q} \right) \mathcal{Q}_\pi^{i_1 \dots i_r}, \quad (50)$$

where $\sum_{\{i_q\}} = \sum_{i_1, \dots, i_r}$ with $i_q = 0, \dots, \mathcal{N} - 1$, $\forall q$. For a nontrivial permutation, such as the one shown in Fig. 7(a) or 7(b), some of the red line segments do not overlap with the blue circle. Let δ be the total number of such red line segments. Then, we have

$$\sum_q \nu_q = t - \delta. \quad (51)$$

In Eq. (50), the factor $(\prod_{q=1}^r \lambda_{i_q}^{\nu_q})$ determines the behavior of different terms at longer times as $|\lambda_{i_q}| < 1$ when $i_q \neq 0$ and $\lambda_{i_q} = 1$ when $i_q = 0$. Thus, all the terms in Eq. (50) can be divided into three different categories as follows.

Case 1: $i_q = 0$, $\forall q$. Since $\lambda_0 = 1$, the right-hand side of Eq. (50) gives the term $\mathcal{Q}_\pi^{0 \dots 0}$, which is a function of $\mathcal{M}^{(0)}$, V , and V^* . Since $\mathcal{M}$ is a doubly stochastic matrix, we have $\langle \lambda_0 | \equiv (1/\sqrt{\mathcal{N}})(1 \dots 1)$. Therefore, all the matrix elements of $\mathcal{M}^{(0)} (\equiv |\lambda_0\rangle\langle\lambda_0|)$ are $1/\mathcal{N}$. A study of different permutations reveals that $\mathcal{Q}_\pi^{0 \dots 0}$ can typically be simplified to the form $1/\mathcal{N}^\mu$ (check Appendix H), where μ is a non-negative integer. The absence of Hamiltonian parameters implies that this term is universal. It is the Type I term in Theorem 2. If $\mathcal{Q}_\pi^{0 \dots 0}$ cannot be simplified to a form independent of Hamiltonian parameters, the term is non-universal and considered as a Type II term because it remains finite at longer times.

Case 2: $i_q \neq 0$, $\forall q$. In this case, $|\lambda_{i_q}| < 1$, $\forall q$, so $\prod_{q=1}^r \lambda_{i_q}^{\nu_q}$ can be interpreted as a product of $(t - \delta)$ real numbers of magnitude less than 1. Thus, it decays exponentially with t . Therefore, such terms are Type III.

Case 3: All the remaining terms involve at least one $i_{q'} = 0$ and at least one $i_{q''} \neq 0$. Thus, we can divide the set $\{1, \dots, r\}$ into two subsets $\mathcal{Q}_1 = \{q'_1, \dots, q'_{r_1}\}$ and $\mathcal{Q}_2 = \{q''_1, \dots, q''_{r_2}\}$ with $r_1, r_2 \geq 1$ and $r_1 + r_2 = r$, such that $i_q = 0$, $\forall q \in \mathcal{Q}_1$ and $i_q \neq 0$, $\forall q \in \mathcal{Q}_2$. Therefore,

$$\prod_{q=1}^r \lambda_{i_q}^{\nu_q} = \left(\prod_{q \in \mathcal{Q}_1} \lambda_0^{\nu_q} \right) \left(\prod_{q \in \mathcal{Q}_2} \lambda_{i_q}^{\nu_q} \right) = \prod_{q \in \mathcal{Q}_2} \lambda_{i_q}^{\nu_q}. \quad (52)$$

Since $\sum_{q=1}^r \nu_q = t - \delta$, $\sum_{q=1}^r \nu_q$ diverges with increasing time. We can further write $\sum_{q=1}^r \nu_q = \sum_{q \in \mathcal{Q}_1} \nu_q + \sum_{q \in \mathcal{Q}_2} \nu_q$. Therefore, $\sum_{q \in \mathcal{Q}_2} \nu_q$ exhibits two possible behaviors in the long time limit: (a) It diverges, or (b) it remains finite. Consequently, $\prod_{q=1}^r \lambda_{i_q}^{\nu_q}$ falls exponentially at longer times for case (a), and it remains finite in case (b). We substitute the long-time form of $\prod_{q=1}^r \lambda_{i_q}^{\nu_q}$ in Eq. (50) to find that the terms with case (a) behavior fall exponentially with time, and the terms with case (b) behavior remain

finite. Therefore, the case (a) terms are of Type III, and case (b) terms, as they depend on Hamiltonian parameters, are Type II.

Thus, $\mathcal{X}_\pi$ can be expressed as a sum of Type I, Type II, and Type III terms in accordance with Theorem 2. A similar analysis can be performed to show that Theorem 2 is also valid for $\mathcal{X}_\pi^{\{\mathcal{U}_1, \mathcal{U}_2\}}$.

B. CUE

We now consider a permutation π , whose diagram contains a number r of red arcs with counterclockwise arrows and lengths $\nu_1, \dots, \nu_r$, and a number $\tilde{r}$ of red arcs with clockwise arrows and lengths $\tilde{\nu}_1, \dots, \tilde{\nu}_{\tilde{r}}$. Following the rules in Appendix E, the contribution to SFF can be written similarly to Eq. (E1):

$$\mathcal{Y}_\pi = \sum_{\{i_q\}} \sum_{\{\tilde{i}_q\}} \left(\prod_{q=1}^r \lambda_{i_q}^{\nu_q} \right) \left(\prod_{q=1}^{\tilde{r}} \chi_{\tilde{i}_q}^{\tilde{\nu}_q} \right) \mathcal{Q}_\pi^{i_1 \dots i_r, \tilde{i}_1 \dots \tilde{i}_{\tilde{r}}}, \quad (53)$$

where

$$\sum_{q=1}^r \nu_q + \sum_{q=1}^{\tilde{r}} \tilde{\nu}_q = t - \delta. \quad (54)$$

Since $|\chi_i| < 1$ for $i = 0, \dots, \mathcal{N} - 1$, $|\lambda_i| < 1$ for $i = 1, \dots, \mathcal{N} - 1$, and $\lambda_0 = 1$, all the terms in Eq. (53) can be categorized as follows.

Case 1: $\tilde{r} = 0$ and $i_q = 0$, $\forall q$. The right-hand side of Eq. (53) gives the term $\mathcal{Q}_\pi^{0 \dots 0}$. Like the COE case, typically, $\mathcal{Q}_\pi^{0 \dots 0} = 1/\mathcal{N}^\mu$, where μ is a non-negative integer; thus, it leads to a Type I term. If $\mathcal{Q}_\pi^{0 \dots 0}$ cannot be simplified to a form independent of the Hamiltonian parameters, it is a Type II term.

Case 2: $i_q \neq 0$, $\forall q$ or $r = 0$. In this case, the factor $(\prod_{q=1}^r \lambda_{i_q}^{\nu_q})(\prod_{q=1}^{\tilde{r}} \chi_{\tilde{i}_q}^{\tilde{\nu}_q})$ in Eq. (53) is a product of $(t - \delta)$ real or complex numbers of magnitude less than 1. Therefore, the corresponding terms fall exponentially with time. Such terms are Type III.

Case 3: All the remaining terms have at least one $i_q = 0$, and if $\tilde{r} = 0$, then at least one $i_q \neq 0$. Therefore, the set $\{1, \dots, r\}$ can be divided into two subsets, $\mathcal{Q}_1 = \{q | i_q = 0\}$ and $\mathcal{Q}_2 = \{q | i_q \neq 0\}$. We have

$$\sum_{q=1}^r \nu_q + \sum_{q=1}^{\tilde{r}} \tilde{\nu}_q = \sum_{q \in \mathcal{Q}_1} \nu_q + \sum_{q \in \mathcal{Q}_2} \nu_q + \sum_{q=1}^{\tilde{r}} \tilde{\nu}_q = t - \delta \quad (55)$$

and

$$\begin{aligned} \left(\prod_{q=1}^r \lambda_{i_q}^{\nu_q} \right) \left(\prod_{q=1}^{\tilde{r}} \chi_{\tilde{i}_q}^{\tilde{\nu}_q} \right) &= \left(\prod_{q \in Q_1} \lambda_0^{\nu_q} \prod_{q \in Q_2} \lambda_{i_q}^{\nu_q} \right) \left(\prod_{q=1}^{\tilde{r}} \chi_{\tilde{i}_q}^{\tilde{\nu}_q} \right) \\ &= \left(\prod_{q \in Q_2} \lambda_{i_q}^{\nu_q} \right) \left(\prod_{q=1}^{\tilde{r}} \chi_{\tilde{i}_q}^{\tilde{\nu}_q} \right). \end{aligned} \quad (56)$$

Thus, $(\prod_{q=1}^r \lambda_{i_q}^{\nu_q})(\prod_{q=1}^{\tilde{r}} \chi_{\tilde{i}_q}^{\tilde{\nu}_q})$ can be interpreted as a product of $(\sum_{q \in Q_2} \nu_q + \sum_{q=1}^{\tilde{r}} \tilde{\nu}_q)$ real or complex numbers of magnitude less than 1. Therefore, when $\sum_{q \in Q_2} \nu_q + \sum_{q=1}^{\tilde{r}} \tilde{\nu}_q$ diverges with time, the corresponding terms in Eq. (53) decay exponentially with time. When $\sum_{q \in Q_2} \nu_q + \sum_{q=1}^{\tilde{r}} \tilde{\nu}_q$ remains finite at longer times, the corresponding terms in Eq. (53) also remain finite. Thus, this case includes both Type II and Type III terms. The above analysis can be extended to demonstrate that Theorem 2 is also valid for $\mathcal{Y}_{\pi}^{\{\underline{n}_{r_1}, \underline{n}_{r_2}\}}$.

C. CSE

Here, we take a permutation π and a configuration $\vec{\sigma}$, whose diagram contains a number r_o of red arcs on the outer circle with counterclockwise arrows and lengths $\nu_1, \dots, \nu_{r_o}$, a number $\tilde{r}_o$ of red arcs on the outer circle with clockwise arrows and lengths $\tilde{\nu}_1, \dots, \tilde{\nu}_{\tilde{r}_o}$, a number r_i of red arcs on the inner circle with counterclockwise arrows and lengths $\xi_1, \dots, \xi_{r_i}$, and a number $\tilde{r}_i$ of red arcs on the inner circle with clockwise arrows and lengths $\tilde{\xi}_1, \dots, \tilde{\xi}_{\tilde{r}_i}$. Following the rules in Appendix G, the contribution to SFF can be written following Eq. (G1) as

$$\begin{aligned} \mathcal{Z}_{\pi, \vec{\sigma}} &= \sum_{\{i_q\}} \sum_{\{\tilde{i}_q\}} \sum_{\{j_q\}} \sum_{\{\tilde{j}_q\}} \\ &\times \left(\prod_{q=1}^{r_o} \lambda_{i_q}^{\nu_q} \right) \left(\prod_{q=1}^{\tilde{r}_o} \chi_{\tilde{i}_q}^{\tilde{\nu}_q} \right) \left(\prod_{q=1}^{r_i} \chi_{j_q}^{\xi_q} \right) \left(\prod_{q=1}^{\tilde{r}_i} \lambda_{\tilde{j}_q}^{\tilde{\xi}_q} \right) \\ &\times Q_{\pi}^{i_1 \dots i_{r_o}; \tilde{i}_1 \dots \tilde{i}_{\tilde{r}_o}; j_1 \dots j_{r_i}; \tilde{j}_1 \dots \tilde{j}_{\tilde{r}_i}}, \end{aligned} \quad (57)$$

where the sums of lengths of all the red arcs satisfy

$$\sum_{q=1}^{r_o} \nu_q + \sum_{q=1}^{\tilde{r}_o} \tilde{\nu}_q + \sum_{q=1}^{r_i} \xi_q + \sum_{q=1}^{\tilde{r}_i} \tilde{\xi}_q = t - \delta. \quad (58)$$

Similar to the CUE case in Eq. (53), the different terms on the right-hand side of Eq. (57) can be categorized as follows:

Case 1: $\tilde{r}_o = 0$, $r_i = 0$, and $i_q = 0$ and $\tilde{j}_q = 0$, $\forall q$. The right-hand side of Eq. (57) gives the term $Q_{\pi}^{0 \dots 0; \dots; 0 \dots 0}$. Similarly to the COE and CUE cases, typically $Q_{\pi}^{0 \dots 0; \dots; 0 \dots 0} = 1/\mathcal{N}^{\mu}$, where μ is a non-negative integer. Therefore, this term is Type I. If $Q_{\pi}^{0 \dots 0; \dots; 0 \dots 0}$ cannot be

simplified to a form independent of Hamiltonian parameters, it is a Type II term.

Case 2: $i_q \neq 0$ and $\tilde{j}_q \neq 0$, $\forall q$ or $r_o = \tilde{r}_i = 0$. The factor $(\prod_{q=1}^{r_o} \lambda_{i_q}^{\nu_q})(\prod_{q=1}^{\tilde{r}_o} \chi_{\tilde{i}_q}^{\tilde{\nu}_q})(\prod_{q=1}^{r_i} \chi_{j_q}^{\xi_q})(\prod_{q=1}^{\tilde{r}_i} \lambda_{\tilde{j}_q}^{\tilde{\xi}_q})$ can be interpreted as a product of $(t - \delta)$ real or complex numbers of magnitude less than 1. Therefore, the corresponding terms drop exponentially with time. Consequently, such terms are Type III.

Case 3: All the remaining terms have at least one $i_q = 0$ or at least one $\tilde{j}_q = 0$ and, if $\tilde{r}_o = r_i = 0$, then at least one $i_q \neq 0$ or at least one $\tilde{j}_q \neq 0$. Therefore, the set $\{1, \dots, r_o\}$ can be divided into two subsets, $Q_{o,1} = \{q | i_q = 0\}$ and $Q_{o,2} = \{q | i_q \neq 0\}$. Similarly, the set $\{1, \dots, \tilde{r}_i\}$ can be divided into two subsets, $Q_{i,1} = \{q | \tilde{j}_q = 0\}$ and $Q_{i,2} = \{q | \tilde{j}_q \neq 0\}$. Thus, we have

$$\begin{aligned} &\left(\prod_{q=1}^{r_o} \lambda_{i_q}^{\nu_q} \right) \left(\prod_{q=1}^{\tilde{r}_o} \chi_{\tilde{i}_q}^{\tilde{\nu}_q} \right) \left(\prod_{q=1}^{r_i} \chi_{j_q}^{\xi_q} \right) \left(\prod_{q=1}^{\tilde{r}_i} \lambda_{\tilde{j}_q}^{\tilde{\xi}_q} \right) \\ &= \left(\prod_{q \in Q_{o,1}} \lambda_0^{\nu_q} \prod_{q \in Q_{o,2}} \lambda_{i_q}^{\nu_q} \right) \left(\prod_{q=1}^{\tilde{r}_o} \chi_{\tilde{i}_q}^{\tilde{\nu}_q} \right) \left(\prod_{q=1}^{r_i} \chi_{j_q}^{\xi_q} \right) \\ &\times \left(\prod_{q \in Q_{i,1}} \lambda_0^{\tilde{\xi}_q} \prod_{q \in Q_{i,2}} \lambda_{\tilde{j}_q}^{\tilde{\xi}_q} \right) \\ &= \left(\prod_{q \in Q_{o,2}} \lambda_{i_q}^{\nu_q} \right) \left(\prod_{q=1}^{\tilde{r}_o} \chi_{\tilde{i}_q}^{\tilde{\nu}_q} \right) \left(\prod_{q=1}^{r_i} \chi_{j_q}^{\xi_q} \right) \left(\prod_{q \in Q_{i,2}} \lambda_{\tilde{j}_q}^{\tilde{\xi}_q} \right). \end{aligned} \quad (59)$$

The right-hand side in Eq. (59) can be interpreted as a product of t' real or complex numbers of magnitude less than 1, where

$$t' = \sum_{q \in Q_{o,2}} \nu_q + \sum_{q=1}^{\tilde{r}_o} \tilde{\nu}_q + \sum_{q=1}^{r_i} \xi_q + \sum_{q \in Q_{i,2}} \tilde{\xi}_q. \quad (60)$$

According to Eq. (58), t' can diverge or remain finite at longer times, resulting in Type III and Type II terms, respectively. Thus, $\mathcal{Z}_{\pi, \vec{\sigma}}$ can be expressed as a sum of Type I, Type II, and Type III terms in accordance with Theorem 2. A similar analysis indicates that Theorem 2 is also valid for $\mathcal{Z}_{\pi, \vec{\sigma}}^{\{\underline{n}_{r_1}, \underline{n}_{r_2}\}}$ and $\mathcal{Z}_{\pi, \vec{\sigma}}^{\{\underline{n}_{r_1}, \mathcal{T} \underline{n}_{r_2}\}}$.

The above analysis for the three Dyson classes provides further insight into the contribution of different diagrams. More specifically, the red arcs whose length diverges with increasing time lead to Type III terms, whereas the red arcs whose length remains finite with increasing time lead to Type II terms.

V. REDUCED DIAGRAMS

Computing $\mathcal{X}_{\pi}, \mathcal{X}_{\pi}^{\{\underline{n}_{r_1}, \underline{n}_{r_2}\}}, \mathcal{Y}_{\pi}, \mathcal{Y}_{\pi}^{\{\underline{n}_{r_1}, \underline{n}_{r_2}\}}, \mathcal{Z}_{\pi, \vec{\sigma}}, \mathcal{Z}_{\pi, \vec{\sigma}}^{\{\underline{n}_{r_1}, \underline{n}_{r_2}\}}$, and $\mathcal{Z}_{\pi, \vec{\sigma}}^{\{\underline{n}_{r_1}, \mathcal{T} \underline{n}_{r_2}\}}$ for all permutations using the rules in

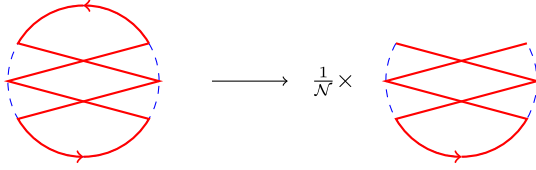


FIG. 9. Diagram and the corresponding reduced diagram obtained by inserting a factor of $1/\mathcal{N}$ for the upper arc. The reduced diagram eliminates some of the Type III terms, which do not contribute to the SFF beyond t^* .

Appendixes B, E, and G is practically impossible. However, Theorem 2 has shown a general pattern in each of them. More specifically, each of them can be expressed as a sum of a universal Type I term, and nonuniversal Type II and Type III terms. Since Type III terms decay exponentially with time, they do not determine the SFF beyond t^* . Therefore, the Type I and Type II terms are only significant for studying the emergence of RMT behavior beyond t^* . With this motivation, we now define a reduced diagram. In the proof of Theorem 2 in Sec. IV, case 2 terms are all Type III for each case of $\mathcal{T}$ symmetry. A reduced diagram eliminates case 2 terms. A precise definition of a reduced diagram is different for the three Dyson symmetry classes. Thus, we present them explicitly for each case.

COE: In Eq. (50), case 2 terms can be eliminated by fixing any one of $i_{q'} = 0$ while allowing $i_q \in \{0, 1, \dots, \mathcal{N} - 1\}$ when $q \neq q'$. This method is diagrammatically equivalent to inserting a factor of $\lambda_0^{\nu_{q'}} \mathcal{M}_{a,b}^{(0)} (= 1/\mathcal{N})$ for a red arc of length $\nu_{q'}$ as in Fig. 9, instead of $\sum_{i_{q'}} \lambda_{i_{q'}}^{\nu_{q'}} \mathcal{M}_{a,b}^{(i_{q'})}$ as mentioned in Appendix B. To ensure that no Type II terms are eliminated, we remove the longest red arc to obtain a reduced diagram.

CUE: In Eq. (53), case 2 terms can be eliminated by fixing any one of $i_{q'} = 0$ while allowing $i_q \in \{0, 1, \dots, \mathcal{N} - 1\}$ when $q \neq q'$. This method is diagrammatically equivalent to inserting a factor of $\lambda_0^{\nu_{q'}} \mathcal{M}_{a,b}^{(0)} (= 1/\mathcal{N})$ for a red arc with a counterclockwise arrow and length $\nu_{q'}$ as in Fig. 10, instead of $\sum_{i_{q'}} \lambda_{i_{q'}}^{\nu_{q'}} \mathcal{M}_{a,b}^{(i_{q'})}$ as given in Appendix E. Similarly to the COE case, we remove the longest red arc with a counterclockwise arrow to obtain a reduced diagram.

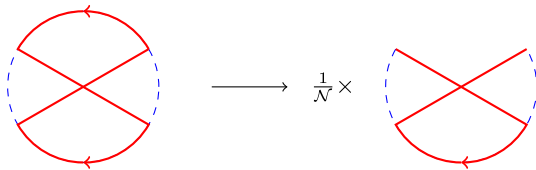


FIG. 10. Diagram and the corresponding reduced diagram obtained by inserting a factor of $1/\mathcal{N}$ for the upper arc with a counterclockwise arrow. The lower arc has a clockwise arrow and cannot be removed to obtain a reduced diagram.

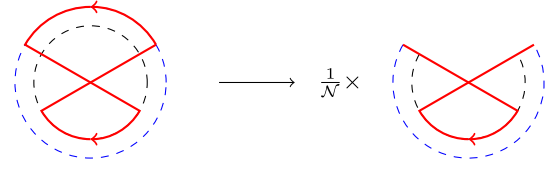


FIG. 11. Diagram and a corresponding reduced diagram obtained by inserting a factor of $1/\mathcal{N}$ for the upper arc with a counterclockwise arrow on the outer circle for the CSE class. The lower arc is present on the inner circle and has a clockwise arrow; therefore, unlike the CUE case in Fig. 10, a reduced diagram can be obtained by removing this arc as well.

CSE: In Eq. (57), case 2 terms can be eliminated by fixing any one of $i_{q'} = 0$ while allowing $i_q \in \{0, 1, \dots, \mathcal{N} - 1\}$ when $q \neq q'$ or fixing any one of $\tilde{j}_{q'} = 0$ while allowing $\tilde{j}_q \in \{0, 1, \dots, \mathcal{N} - 1\}$ when $q \neq q'$. This approach is diagrammatically equivalent to inserting a factor of $\lambda_0^{\nu_{q'}} \mathcal{M}_{a,b}^{(0)} (= 1/\mathcal{N})$ for a red arc with a counterclockwise arrow and length $\nu_{q'}$ on the outer circle or $\lambda_0^{\tilde{\nu}_{q'}} \mathcal{M}_{a,b}^{(0)} (= 1/\mathcal{N})$ for a red arc with a clockwise arrow and length $\tilde{\nu}_{q'}$ on the inner circle as shown in Fig. 11, instead of $\sum_{i_{q'}} \lambda_{i_{q'}}^{\nu_{q'}} \mathcal{M}_{a,b}^{(i_{q'})}$ and $\sum_{\tilde{j}_{q'}} \lambda_{\tilde{j}_{q'}}^{\tilde{\nu}_{q'}} \mathcal{M}_{a,b}^{(\tilde{j}_{q'})}$, respectively, as mentioned in Appendix G. Once again, we choose the longest red arc.

For the CSE case, the basis states related by time reversal are orthogonal to each other, as described in Table I. This property results in the following theorem for the corresponding reduced diagrams.

Theorem 3. For a diagram that contains a red arc with a counterclockwise arrow on the outer circle or a clockwise arrow on the inner circle such that the arc makes a jump to the other circle at one end, the reduced diagram with respect to the same arc has a vanishing contribution. Here, jumps are considered only between the nearest states on the inner and outer circles.

Proof. Figure 12(a) shows a portion of an arbitrary diagram containing a red arc with a counterclockwise arrow on the outer circle. The arc follows the outer circle until state a ; then, instead of connecting the next state b on the same circle, it connects to the time-reversed state $\mathcal{T}b$ on the inner circle. A reduced diagram can be obtained by removing this

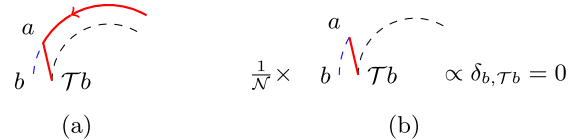


FIG. 12. (a) Portion of an allowed diagram containing a red arc with a counterclockwise arrow on the outer circle and a jump to the inner circle at one end of the arc. (b) Reduced diagram with respect to this arc, which has a vanishing contribution since b and $\mathcal{T}b$ are orthogonal.

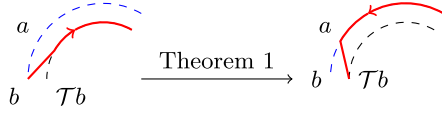


FIG. 13. Portion of an allowed diagram containing a red arc with a clockwise arrow on the inner circle and a jump to the outer circle at one end of the arc. By applying Theorem 1, the red arc can be pushed to the outer circle with the reversed arrow. This new diagram resembles the diagram in Fig. 12(a).

arc and inserting a factor of $1/\mathcal{N}$, as in Fig. 12(b). Applying the rules in Appendix G to the blue and red line segments connected to state a and summing over a implies that the reduced diagram has a vanishing contribution.

In the other case, when the arc is on the inner circle, it can be pushed to the outer circle by applying Theorem 1, as in Fig. 13. Following the arguments presented above, the reduced diagram vanishes once again.

VI. EXACT CANCELLATION OF REDUCED DIAGRAMS

A. COE and CUE

Theorem 4. For each reduced diagram without repetition, there exists a diagram with a minimal repetition of states that has an identical reduced diagram.

Proof. In an arbitrary diagram, e.g., in Fig. 14, the longest red arc starts from a state d following the blue circle up to another state a , and then connects to a state c , other than the immediate next state b on the blue circle. The rest of the red curve is irrelevant for the proof. A reduced diagram can be obtained by removing this arc and inserting a factor of $1/\mathcal{N}$ as given in Fig. 14. We apply the rules in Appendix B or E to the blue and red line segments, respectively, between states a and b and states a and c , and sum over a to find

$$\mathcal{X}_\pi(\text{reduced}) \propto V_{a,b} V_{a,c}^* = \delta_{b,c}. \quad (61)$$

Thus, states b and c are identical in the reduced diagram in Fig. 14. This finding suggests that an original diagram in Fig. 15 with identical states b and c , whose contribution is

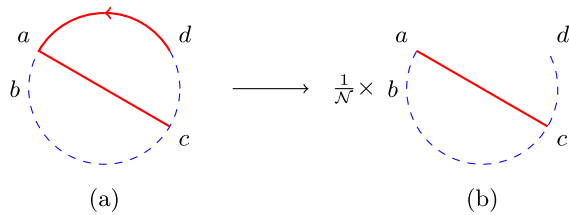


FIG. 14. Diagram with only a portion of the red path emphasizing the general structure at one end of a red arc along with a reduced diagram. A sum over the state a implies that states b and c are identical in the reduced diagram.

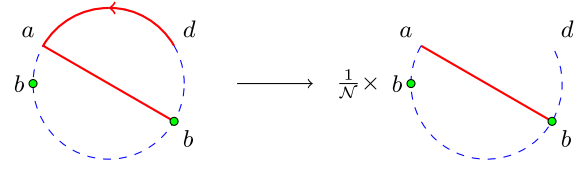


FIG. 15. Diagram with states b and c of the diagram in Fig. 14(a), being identical from the beginning. This diagram also has a reduced diagram identical to that in Fig. 14(b).

denoted by $\mathcal{X}_\pi^{\{n_1, n_2\}}$, also has a similar reduced diagram with respect to the same arc. The diagram on the left of Fig. 15 represents a permutation of t states, two of which are identical.

B. CSE

Theorem 5. For each reduced diagram without repetition, either a diagram with a minimal repetition of states or a diagram with two states at different time steps related by time reversal has an identical reduced diagram.

Proof. In an arbitrary diagram, the longest red arc with a counterclockwise arrow on the outer circle starts from state d following the blue path up to another state a ; it then connects to state c on the outer circle, as in Fig. 16(a), or state Tc on the inner circle, as in Fig. 17(a), other than the immediate next state b on the outer circle. When state c is connected as in Fig. 16(a), an analysis similar to the proof of Theorem 3 implies that there exists an identical diagram with state c the same as state b , as shown in Fig. 16(b). The

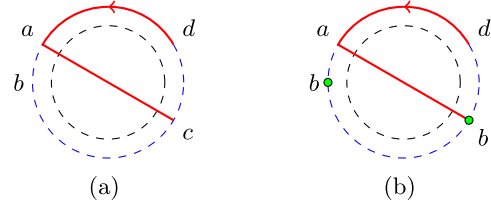


FIG. 16. (a) Portion of a diagram containing a red arc with a counterclockwise arrow on the outer circle. (b) An identical diagram where states b and c are identical from the beginning. The diagrams in (a) and (b) have identical reduced diagrams.

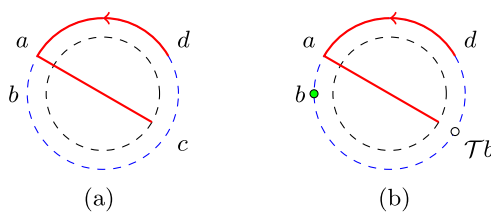


FIG. 17. (a) Portion of a diagram containing a red arc with a counterclockwise arrow on the outer circle. (b) An identical diagram where states b and c are related by time reversal. The diagrams in (a) and (b) have identical reduced diagrams.

two diagrams in Figs. 16(a) and 16(b) have identical reduced diagrams. When state Tc is connected in the diagram in Fig. 17(a), another diagram with an identical reduced diagram has a state and its time-reversed state at two different time steps on the outer circle, as in Fig. 17(b).

VII. EMERGENCE OF UNIVERSAL RMT SFF

A. COE

For the COE class, the leading-order SFF is determined by identity permutation and its t cyclic and t anticyclic variants. All of these variants have an identical contribution due to PBC in time and the symmetric nature of the matrix V . Therefore,

$$K_1^{(1)}(t) = \sum_{l=0}^{t-1} \mathcal{X}_{c^l I} + \mathcal{X}_{\mathcal{R}c^l I} = 2t\mathcal{X}_I, \quad (62)$$

where the subscript I denotes the identity permutation, which is described diagrammatically by overlapping blue and red circles, as shown in Fig. 5. Therefore, by applying the rules in Appendix B, we find

$$\mathcal{X}_I = \text{tr}\mathcal{M}^t, \quad (63)$$

which gives the leading-order SFF:

$$K_1^{(1)}(t) = 2t\text{tr}\mathcal{M}^t = 2t(1 + \lambda_1^t + \cdots + \lambda_{\mathcal{N}-1}^t). \quad (64)$$

Since $|\lambda_i| < 1$ for $i = 1, \dots, \mathcal{N} - 1$, we find, at long times, $K_1^{(1)}(t) \simeq 2t$, which is the leading-order RMT SFF for the COE class in Eq. (2).

As discussed in Sec. III A 2, the second-order correction to the SFF is determined by the permutations whose contributions contain a Type I term of order $1/\mathcal{N}$. In Sec. III A 2, we introduce these permutations contributing to the second-order term of the SFF. Following Eq. (20), the second-order correction can be expressed in terms of the contributions of these permutations, along with their cyclic and anticyclic variants, as follows:

$$K_1^{(2)}(t) = \sum_{l=0}^{t-1} \left[\sum_T (\mathcal{X}_{c^l T} + \mathcal{X}_{\mathcal{R}c^l T}) + \sum_S (\mathcal{X}_{c^l S} + \mathcal{X}_{\mathcal{R}c^l S}) - \sum_R (\mathcal{X}_{c^l I}^{\{n_{\tau_1}, n_{\tau_2}\}} + \mathcal{X}_{\mathcal{R}c^l I}^{\{n_{\tau_1}, n_{\tau_2}\}}) - \sum_{SR} (\mathcal{X}_{c^l S}^{\{n_{\tau_1}, n_{\tau_2}\}} + \mathcal{X}_{\mathcal{R}c^l S}^{\{n_{\tau_1}, n_{\tau_2}\}}) \right]. \quad (65)$$

Since the cyclic and anticyclic variants give identical contributions due to PBC in time and the symmetric nature of the matrix V for systems with $T^2 = 1$, we obtain

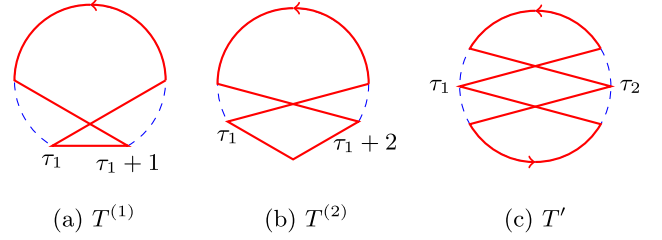


FIG. 18. Transposition (T) diagrams of (a) two nearest-neighbor states at time steps τ_1 and $\tau_1 + 1$, (b) two next-nearest-neighbor states at time steps τ_1 and $\tau_1 + 2$, and (c) two states at time steps τ_1 and τ_2 , where $\tau_2 - \tau_1 > 2$.

$$K_1^{(2)}(t) = 2t \left[\sum_T \mathcal{X}_T + \sum_S \mathcal{X}_S - \sum_R \mathcal{X}_I^{\{n_{\tau_1}, n_{\tau_2}\}} - \sum_{SR} \mathcal{X}_S^{\{n_{\tau_1}, n_{\tau_2}\}} \right]. \quad (66)$$

The right-hand side terms in Eq. (66) can be explicitly calculated using the rules in Appendix B (check Appendix H). However, we proceed with a much more insightful approach based on the reduced diagrams since only Type I and Type II terms can contribute to the SFF at longer times. We begin by analyzing transpositions. We observe that single transpositions can be further classified into three distinct categories based on the different shapes of the diagrams representing them: (a) nearest-neighbor transposition ($T^{(1)}$), where the interchanged states are adjacent on the blue circle, (b) next-nearest-neighbor transposition ($T^{(2)}$), where the interchanged states are separated by two time steps on the blue circle, and (c) all other transpositions (T'). All these transpositions are shown diagrammatically in Fig. 18. From Theorem 4, we find that the reduced diagrams of the diagrams with a state repeated twice in Figs. 19(c) are identical to the

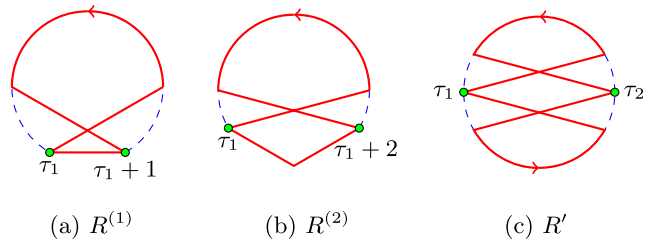


FIG. 19. Diagrams whose reduced diagrams are identical to the reduced diagrams of transpositions in Fig. 18. Reduced diagram of $R^{(1)}$ obtained by removing the upper red arc cancels the reduced diagram of $T^{(1)}$ obtained by removing the same arc. Reduced diagram of $R^{(2)}$ obtained by removing the upper arc cancels the reduced diagram of $T^{(2)}$ obtained by removing the same arc. Finally, reduced diagram of R' obtained by removing the longest arc cancels the reduced diagram of T' obtained by removing the longest arc.

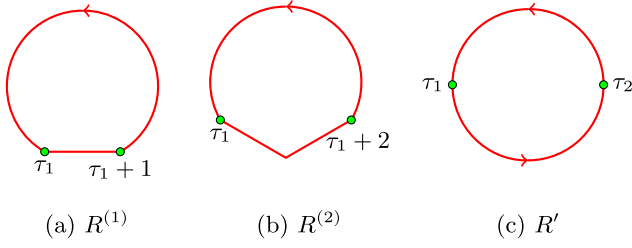


FIG. 20. Diagrams representing an identity permutation with (a) a repeated state at time steps τ_1 and $\tau_1 + 1$, (b) a repeated state at time steps τ_1 and $\tau_1 + 2$, and (c) a repeated state at time steps τ_1 and τ_2 .

reduced diagrams of the diagrams in Figs. 18(c), respectively. Since the diagrams in Figs. 19(c) represent the transposition of a repeated state, they are identical to the identity permutation with a state repeated twice. Therefore, these diagrams are the R diagrams in Fig. 20. Since there are equal numbers of T and R diagrams, as we need to choose two states from t states for both operations, and since the reduced diagrams of T and R diagrams exactly cancel, we obtain

$$\sum_T \mathcal{X}_T - \sum_R \mathcal{X}_R^{\{n_{\tau_1}, n_{\tau_2}\}} = \text{Type III terms.} \quad (67)$$

Because Type III terms decay exponentially in time, we conclude that the T and R diagrams together do not contribute to the SFF in the universal regime. Now, we consider an S diagram with the order of states reversed from τ_1 to τ_2 in Fig. 21(a). When a reduced diagram is obtained by removing the upper arc, Theorem 4 implies that an SR diagram with a repeated state at time steps τ_1 and τ_2 [see Fig. 21(b)] also has an identical reduced diagram after the upper arc is removed. However, when a reduced diagram is obtained by removing the lower arc of the S diagram in Fig. 21(a), a different SR diagram results in an identical reduced diagram. This SR diagram, shown in Fig. 22, has identical states at time steps $\tau_1 - 1$ and $\tau_2 + 1$. Since a reduced diagram is obtained by removing the longest red arc, the upper (lower) arc is removed when its length is greater than or equal to $\lceil t/2 \rceil - 1$. When the upper arc is removed in S and the corresponding SR

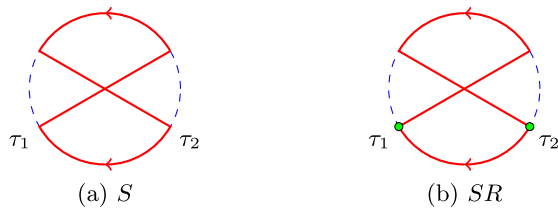


FIG. 21. Diagrams representing (a) a subsequence reversal (S) and (b) a subsequence reversal with a state repeated twice (SR).

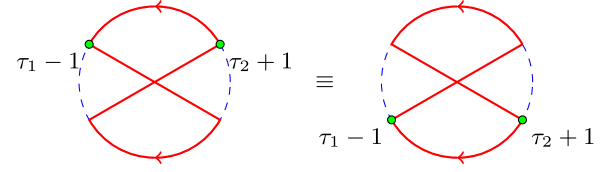


FIG. 22. An SR diagram where the states at the end points, at time steps $\tau_1 - 1$ and $\tau_2 + 1$, of the upper red arc are identical. This diagram has an equivalent representation where a pair of repeated states appear at the end points of the lower red arc.

diagram, its length is identical in both diagrams. However, when the lower arc is removed, its length is $\tau_2 - \tau_1$ in an S diagram and $\tau_2 - \tau_1 + 2$ in the corresponding SR diagram. Therefore, when the length of the lower arc is $\lceil t/2 \rceil - 1$ or $\lceil t/2 \rceil$ in an SR diagram, there is no corresponding S diagram. Thus, there are two extra SR diagrams that determine the second-order correction to SFF. The presence of these extra two SR diagrams can be further understood by noticing that the minimum value of $\tau_2 - \tau_1$ for an S diagram is 3 to avoid overlap with nearest-neighbor and next-nearest-neighbor transpositions, and its maximum value is $t - 5$, as the minimum length of the upper arc is also 3 for the same reason. However, for the SR diagram in Fig. 21(b), the minimum value of $\tau_2 - \tau_1$ is 3 because the repeated states must be at least three time steps away for a subsequence reversal to give a different permutation ($aa \rightarrow aa, aba \rightarrow aba, abca \rightarrow acba$), and its maximum value is $t - 3$ as the minimum length of the upper arc can be 1. Thus, there are two extra SR diagrams compared to S diagrams.

Additionally, instead of representing a repeated state as two separate points, we can denote it by a single point by merging the points. This leads to an alternative diagrammatic representation. In this representation, the two extra SR diagrams can be drawn as in Fig. 23(a). We draw the blue and red curves separately in Fig. 23(b) to find that these curves, respectively, resemble near-miss and self-crossing orbits of the Sieber-Richter pairs [12] of a semi-classical analysis. Applying the rules in Appendix B to the diagram in Fig. 23(a), we obtain

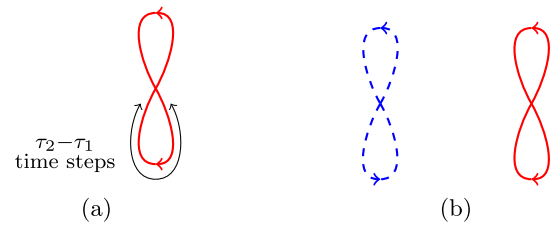


FIG. 23. (a) SR diagram in an alternative diagrammatic representation, where the two points representing a repeated state are merged. In diagram (b), the blue and red curves are separated to show resemblance with the Sieber-Richter pairs.

$$\begin{aligned}
K_1^{(2)}(t) &= -\frac{2t}{2} \sum_{\tau_1=1}^t \sum_{\tau_2=\tau_1+\lceil t/2 \rceil-1}^{\tau_1+\lceil t/2 \rceil} \sum_a \mathcal{M}_{a,a}^{\tau_2-\tau_1} \mathcal{M}_{a,a}^{t-\tau_2+\tau_1} \\
&= -t^2 \sum_a (\mathcal{M}_{a,a}^{\lceil t/2 \rceil-1} \mathcal{M}_{a,a}^{\lfloor t/2 \rfloor+1} + \mathcal{M}_{a,a}^{\lceil t/2 \rceil} \mathcal{M}_{a,a}^{\lfloor t/2 \rfloor}) \\
&= -t^2 \sum_a \sum_{ij} (\lambda_i^{\lceil t/2 \rceil-1} \lambda_j^{\lfloor t/2 \rfloor+1} + \lambda_i^{\lceil t/2 \rceil} \lambda_j^{\lfloor t/2 \rfloor}) \\
&\quad \times \mathcal{M}_{a,a}^{(i)} \mathcal{M}_{a,a}^{(j)}, \tag{68}
\end{aligned}$$

where a represents a repeated state. The factor $2t$ in the first line accounts for the t cyclic and t anticyclic variants, which contribute identically. Furthermore, the extra factor $1/2$ is included to avoid double counting in summing over τ_1, τ_2 since a change in the order of τ_1, τ_2 does not lead to a new permutation when both cyclic and anticyclic variants have been considered (see Appendix H). At long times, we obtain

$$\begin{aligned}
K_1^{(2)}(t) &= -t^2(1+1) \sum_a \mathcal{M}_{a,a}^{(0)} \mathcal{M}_{a,a}^{(0)} + \mathcal{O}(\lambda_1^t) \\
&= -\frac{2t^2}{\mathcal{N}} + \mathcal{O}(\lambda_1^t). \tag{69}
\end{aligned}$$

The first term on the right side of Eq. (69) only survives for $t > t^*$, which matches the second-order correction as predicted by RMT for the COE class. We can further interpret this contribution as $K_1^{(2)}(t) = -2t^2 \sum_{\underline{n}} P_t^{(2)}(\underline{n})$, where $P_t^{(2)}(\underline{n}) \equiv \text{tr} \mathcal{M}^{\lceil t/2 \rceil} \text{tr} \mathcal{M}^{\lfloor t/2 \rfloor}$ is the probability of returning to an initial state $|\underline{n}\rangle$ twice in time t with time intervals $\lceil t/2 \rceil$ and $\lfloor t/2 \rfloor$, respectively. Figure 24 shows a comparison between the SFF computed using Eq. (9) and that obtained using the RPA up to the first- and second-order universal terms along with all the Type III terms (see Appendix H). It shows a good match beyond t^* up to a longer time when the second-order universal term is included. The difference between the SFF computed directly and that obtained using the RPA in the nonuniversal regime of $t < t^*$ in Fig. 24 is due to the RPA and the exclusion of all higher-order contributions beyond second order.

B. CUE

In this case, the contributions of t cyclic variants and t anticyclic variants are different due to the nonsymmetric nature of the matrix V . Therefore, the leading-order SFF is

$$K_0^{(1)}(t) = \sum_{l=0}^{t-1} \mathcal{Y}_{C^l I} + \mathcal{Y}_{RC^l I} = t[\mathcal{Y}_I + \mathcal{Y}_{RI}]. \tag{70}$$

Applying the rules in Appendix E to the cyclic and anticyclic variants of the identity permutation, we obtain

$$\mathcal{Y}_I = \text{tr} \mathcal{M}^t, \tag{71}$$

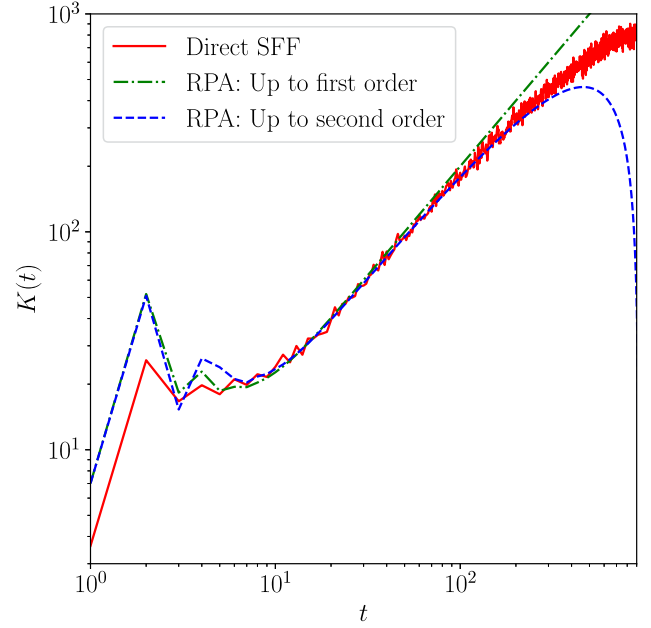


FIG. 24. Comparison of directly calculated SFF, $K(t)$, using Eq. (9) with that obtained using the RPA up to first and second order in time including both the universal and nonuniversal terms for a kicked spinless fermionic chain of the COE class and with particle-number conservation. We use the Hamiltonians in Eqs. (34) and (35) with parameters in Eqs. (46) and (47). We take $L = 12, N = 6, J = 1, \langle \epsilon \rangle = 0, \Delta \epsilon = 10, U_0 = 100\sqrt{2}, \Delta U_0 = 20, \alpha = 1.5$. An averaging over 320 realizations of disorder is performed for the direct simulation of $K(t)$.

$$\mathcal{Y}_{RI} = \text{tr} \tilde{\mathcal{M}}^t. \tag{72}$$

Substituting Eqs. (71) and (72) into Eq. (70), we find

$$\begin{aligned}
K_0^{(1)}(t) &= t(\text{tr} \mathcal{M}^t + \text{tr} \tilde{\mathcal{M}}^t) \\
&= t(1 + \lambda_1^t + \dots + \lambda_{N-1}^t + \chi_0^t + \dots + \chi_{N-1}^t). \tag{73}
\end{aligned}$$

All nonuniversal terms are exponentially decaying with time; therefore, beyond Thouless time, $K_0^{(1)} \simeq t$, which is essentially the RMT prediction, Eq. (3).

The second-order contribution is again determined by the T, R, S , and SR diagrams. However, in this case, the matrix V is not symmetric. Therefore, cyclic and anticyclic variants contribute differently. Consequently, the second-order correction can be expressed as follows:

$$\begin{aligned}
K_0^{(2)}(t) &= t \left[\sum_T (\mathcal{Y}_T + \mathcal{Y}_{RT}) + \sum_S (\mathcal{Y}_S + \mathcal{Y}_{RS}) \right. \\
&\quad - \sum_R (\mathcal{Y}_I^{\{\underline{n}_{r1}, \underline{n}_{r2}\}} + \mathcal{Y}_{RI}^{\{\underline{n}_{r1}, \underline{n}_{r2}\}}) \\
&\quad \left. - \sum_{SR} (\mathcal{Y}_S^{\{\underline{n}_{sr1}, \underline{n}_{sr2}\}} + \mathcal{Y}_{RS}^{\{\underline{n}_{sr1}, \underline{n}_{sr2}\}}) \right]. \tag{74}
\end{aligned}$$

We obtain Type I and Type II terms from a diagram only if it has a reduced diagram, which can be determined from Eq. (53). Thus, the anticyclic variants of the T and R diagrams can only contribute Type III terms, as they contain red arcs with clockwise arrows. Therefore,

$$\sum_T \mathcal{Y}_{\mathcal{RT}} = \text{Type III terms}, \quad (75)$$

$$\sum_R \mathcal{Y}_{\mathcal{RI}}^{\{n_{\tau_1}, n_{\tau_2}\}} = \text{Type III terms}. \quad (76)$$

Furthermore, like the COE case, the cyclic variants of the T and R diagrams cancel out their corresponding reduced diagrams. Therefore,

$$\sum_T \mathcal{Y}_T - \sum_R \mathcal{Y}_I^{\{n_{\tau_1}, n_{\tau_2}\}} = \text{Type III terms}. \quad (77)$$

In addition, all the reduced diagrams of the S diagrams are canceled by the reduced diagrams of the SR diagrams. Once again, there are two extra SR diagrams with length of the lower red arc $\lceil t/2 \rceil - 1$ and $\lceil t/2 \rceil$. Therefore, Eq. (74) gives the following expression, ignoring the Type III terms:

$$K_0^{(2)}(t) = -t \sum_{\text{extra } SR \text{ diagrams}} (\mathcal{Y}_S^{\{n_{\tau_1}, n_{\tau_2}\}} + \mathcal{Y}_{\mathcal{RS}}^{\{n_{\tau_1}, n_{\tau_2}\}}). \quad (78)$$

Since one of the red arcs has a clockwise arrow and the other has a counterclockwise arrow as in Fig. 23(a), following the rules in Appendix E, we find

$$\mathcal{Y}_S^{\{n_{\tau_1}, n_{\tau_2}\}} = \sum_a \tilde{\mathcal{M}}_{a,a}^{\tau_2 - \tau_1} \mathcal{M}_{a,a}^{t - \tau_2 + \tau_1} \quad (79)$$

and

$$\sum_{\text{extra } SR \text{ diagrams}} \equiv \frac{1}{2} \sum_{\tau_1=1}^t \sum_{\tau_2=\tau_1+\lceil t/2 \rceil-1}^{\tau_1+\lceil t/2 \rceil}.$$

Therefore,

$$\begin{aligned} \sum_{\text{extra } SR \text{ diagrams}} \mathcal{Y}_S^{\{n_{\tau_1}, n_{\tau_2}\}} &= \frac{t}{2} \sum_a \sum_{ij} (\chi_i^{\lceil t/2 \rceil - 1} \lambda_j^{\lceil t/2 \rceil + 1} \\ &\quad + \chi_i^{\lceil t/2 \rceil} \lambda_j^{\lceil t/2 \rceil}) \tilde{\mathcal{M}}_{a,a}^{(i)} \mathcal{M}_{a,a}^{(j)} \\ &= \mathcal{O}(\lambda_1^{\lceil t/2 \rceil}, \chi_0^{\lceil t/2 \rceil}). \end{aligned} \quad (80)$$

Similarly,

$$\sum_{\text{extra } SR \text{ diagrams}} \mathcal{Y}_{\mathcal{RS}}^{\{n_{\tau_1}, n_{\tau_2}\}} = \mathcal{O}(\lambda_1^{\lceil t/2 \rceil}, \chi_0^{\lceil t/2 \rceil}). \quad (81)$$

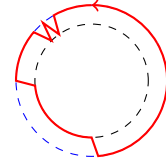


FIG. 25. An arbitrary variant of identity permutation described by $\vec{\sigma}$.

Substituting Eqs. (80) and (81) into Eq. (78), we obtain

$$K_0^{(2)}(t) = \mathcal{O}(\lambda_1^{\lceil t/2 \rceil}, \chi_0^{\lceil t/2 \rceil}). \quad (82)$$

In Appendix I, we show that an exact calculation based on the rules in Appendix E also leads to the same result. Therefore, unlike the COE case, there is no universal term in second order in time, which is identical to the RMT prediction for the CUE class.

C. CSE

Following Eq. (22), the leading-order SFF can be expressed as a sum of contributions of the variants of identity permutation as follows:

$$K_{-1}^{(1)}(t) = \sum_{\vec{\sigma}} \sum_{l=0}^{t-1} (\mathcal{Z}_{\mathcal{C}^l I, \vec{\sigma}} + \mathcal{Z}_{\mathcal{RC}^l I, \vec{\sigma}}). \quad (83)$$

Similar to the CUE class, the matrix V is also nonsymmetric for the CSE class. Furthermore, because of the PBC in time, the t cyclic variants for each $\vec{\sigma}$ give identical contributions. Similarly, the t anticyclic variants for each $\vec{\sigma}$ also give identical contributions but differ from the cyclic variants due to the nonsymmetric nature of the matrix V . Therefore,

$$K_{-1}^{(1)}(t) = t \sum_{\vec{\sigma}} (\mathcal{Z}_{I, \vec{\sigma}} + \mathcal{Z}_{\mathcal{RI}, \vec{\sigma}}). \quad (84)$$

To find which diagrams determine the leading-order SFF, we consider the diagram of an arbitrary variant as shown in Fig. 25. We notice that the diagram contains red arcs on the inner or outer circle connected via jumps between the two circles. Theorem 3 implies that all the reduced diagrams of such variants have vanishing contributions. Therefore, such diagrams do not contribute to the SFF in the ergodic phase. Theorem 3 also implies that Type I contributions can only come from diagrams with no jump to the inner or outer circle, as shown in Figs. 26(a) and 26(b), which correspond to $\vec{\sigma} = (0, \dots, 0)$ and $\vec{\sigma} = (1, \dots, 1)$, respectively. Thus, the leading-order SFF is

$$\begin{aligned} K_{-1}^{(1)}(t) &= t [\mathcal{Z}_{I, (0, \dots, 0)} + \mathcal{Z}_{\mathcal{RI}, (0, \dots, 0)} + \mathcal{Z}_{I, (1, \dots, 1)} \\ &\quad + \mathcal{Z}_{\mathcal{RI}, (1, \dots, 1)}]. \end{aligned} \quad (85)$$

Applying the rules in Appendix G, we obtain

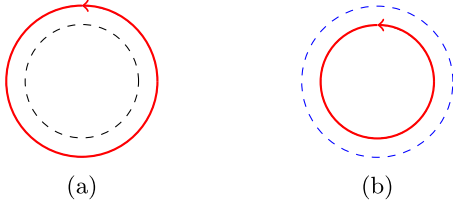


FIG. 26. Variants of identity permutation that determine the leading order SFF of the CSE class, (a) $\vec{\sigma} = (0, \dots, 0)$ and (b) $\vec{\sigma} = (1, \dots, 1)$.

$$\mathcal{Z}_{I,(0,\dots,0)} = \mathcal{Z}_{\mathcal{R}I,(1,\dots,1)} = \text{tr} \mathcal{M}^t, \quad (86)$$

$$\mathcal{Z}_{\mathcal{R}I,(0,\dots,0)} = \mathcal{Z}_{I,(1,\dots,1)} = \text{tr} \tilde{\mathcal{M}}^t. \quad (87)$$

Substituting Eqs. (86) and (87) into Eq. (85), we find

$$\begin{aligned} K_{-1}^{(1)}(t) &= 2t(\text{tr} \mathcal{M}^t + \text{tr} \tilde{\mathcal{M}}^t) \\ &= 2t(1 + \lambda_1^t + \dots + \lambda_{N-1}^t + \chi_0^t + \dots + \chi_{N-1}^t). \end{aligned} \quad (88)$$

Similar to the previous cases, nonuniversal terms decay exponentially. Therefore, beyond the Thouless time, $K_{-1}^{(1)}(t) \simeq 2t$, which is identical to the leading order RMT result, Eq. (4).

The second-order correction, $K_{-1}^{(2)}(t)$, is once again determined by variants of the T , R , S , and SR diagrams. Thus, following Eq. (22), $K_{-1}^{(2)}(t)$ can be expressed as follows:

$$\begin{aligned} K_{-1}^{(2)}(t) &= t \sum_{\vec{\sigma}} \left[\sum_T (\mathcal{Z}_{T,\vec{\sigma}} + \mathcal{Z}_{\mathcal{R}T,\vec{\sigma}}) + \sum_S (\mathcal{Z}_{S,\vec{\sigma}} + \mathcal{Z}_{\mathcal{R}S,\vec{\sigma}}) \right. \\ &\quad - \sum_R \left(\mathcal{Z}_{I,\vec{\sigma}}^{\{n_1, n_2\}} + \mathcal{Z}_{\mathcal{R}I,\vec{\sigma}}^{\{n_1, n_2\}} \right) \\ &\quad - \sum_{SR} \left(\mathcal{Z}_{S,\vec{\sigma}}^{\{n_1, n_2\}} + \mathcal{Z}_{\mathcal{R}S,\vec{\sigma}}^{\{n_1, n_2\}} \right) \\ &\quad - \sum_R \left(\mathcal{Z}_{I,\vec{\sigma}}^{\{n_1, T n_2\}} + \mathcal{Z}_{\mathcal{R}I,\vec{\sigma}}^{\{n_1, T n_2\}} \right) \\ &\quad \left. - \sum_{SR} \left(\mathcal{Z}_{S,\vec{\sigma}}^{\{n_1, T n_2\}} + \mathcal{Z}_{\mathcal{R}S,\vec{\sigma}}^{\{n_1, T n_2\}} \right) \right]. \end{aligned} \quad (89)$$

In Eq. (89), the sum over configurations $\vec{\sigma}$ leads to 2^t different variants for each permutation. We analyze the diagrams of different variants using Theorem 1, Theorem 3, and Theorem 5 to identify the diagrams responsible for the second-order correction. We start by analyzing the T and R diagrams followed by the S and SR diagrams.

1. T and R

An arbitrary variant of a transposition diagram is shown in Fig. 27. A necessary requirement for a variant to have

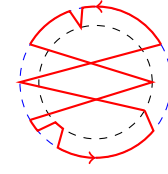


FIG. 27. Diagram representing an arbitrary variant of transposition.

Type I or Type II terms in its contribution is the existence of nonvanishing reduced diagrams. From the definition of reduced diagrams in Sec. V, a diagram must contain either a red arc with a counterclockwise arrow on the outer circle or a red arc with a clockwise arrow on the inner circle. We study the former case in Fig. 27, as the latter case is related to the former by Theorem 1. Theorem 3 implies that the reduced diagrams of such variants give a nonvanishing contribution when the longest red arc does not have a jump to the inner circle at its ends, as shown in Fig. 28(a), where the lower red arc is longest in length. We obtain the reduced diagram in Fig. 28(b) by removing the lower red arc from Fig. 28(a). This reduced diagram has a vanishing contribution due to the time reversal of one of the transposed states. To ensure that the reduced diagram has a nonvanishing contribution, either both of the transposed states must be time reversed or neither should be time reversed. This approach is diagrammatically equivalent to keeping both transposed states either on the outer circle or on the inner

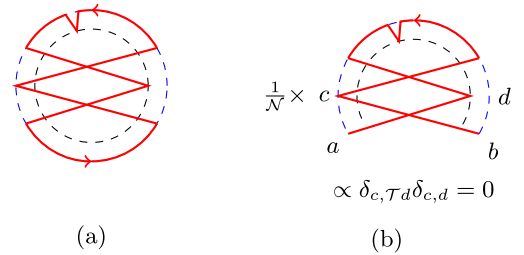


FIG. 28. A variant of a transposition in (a) with a vanishing reduced diagram in (b).

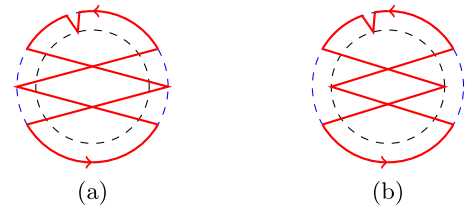


FIG. 29. Variants of a transposition with nonvanishing reduced diagrams. Both the transposed states are on the same circle.

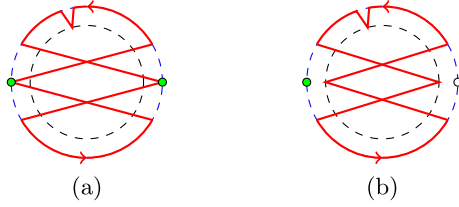


FIG. 30. (a) A transposition diagram with a pair of repeated states whose reduced diagram is the same as that of Fig. 29(a). (b) A transposition diagram containing a state and its time reversed state whose reduced diagram is the same as that of Fig. 29(b).

circle, as shown in Fig. 29. Theorem 5 implies that the diagrams in Fig. 30 containing a state repeated twice or a state and its time-reversed state have identical reduced diagrams to those of the diagrams in Fig. 29. The diagrams in Fig. 30 are equivalent to Fig. 31. We note that Fig. 31(a) is a variant of identity permutation with a state repeated twice, and Fig. 31(b) is a variant of identity permutation with a state and its time-reversed state appearing at two different time steps. We refer to Figs. 31(a) and 31(b) collectively as variants of the R diagrams. Similarly to the COE and CUE cases, the T and R diagrams cancel the reduced diagrams of each other in Eq. (22). Therefore, the T and R diagrams together, once again, do not contribute to the SFF in the universal regime. We can then write following the definition of the reduced diagrams:

$$\sum_{\vec{\sigma}} \left[\sum_T \mathcal{Z}_{T,\vec{\sigma}} - \sum_R \mathcal{Z}_{I,\vec{\sigma}}^{\{n_{r_1}, n_{r_2}\}} - \sum_R \mathcal{Z}_{I,\vec{\sigma}}^{\{n_{r_1}, T n_{r_2}\}} \right] = \text{Type III terms}, \quad (90)$$

$$\sum_{\vec{\sigma}} \left[\sum_T \mathcal{Z}_{RT,\vec{\sigma}} - \sum_R \mathcal{Z}_{RI,\vec{\sigma}}^{\{n_{r_1}, n_{r_2}\}} - \sum_R \mathcal{Z}_{RI,\vec{\sigma}}^{\{n_{r_1}, T n_{r_2}\}} \right] = \text{Type III terms}. \quad (91)$$

2. S and SR

As in the case of transposition, the variants of an S diagram containing a red arc with a counterclockwise arrow on the outer circle or a red arc with a clockwise

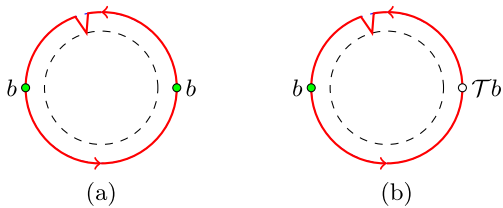


FIG. 31. Variants of the R diagram: (a) diagram whose reduced diagram is the same as that of Fig. 29(a) and (b) diagram whose reduced diagram is the same as that of Fig. 29(b).

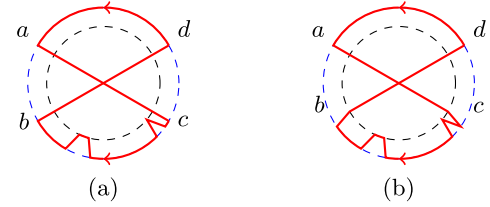


FIG. 32. Variants of the S diagrams.

arrow on the inner circle can contribute Type I and Type II terms. In addition, both the endpoints of the reversed part should lie on the same circle as in Fig. 32. Otherwise, the reduced diagrams have a vanishing contribution similar to a variant of the T diagram in Fig. 28. Applying Theorem 5, the reduced diagram of the SR diagram in Fig. 33(a) obtained by removing the upper red arc cancels out the reduced diagram of the S diagram in Fig. 32(b) obtained by removing the upper red arc. Furthermore, the reduced diagram of the SR diagram in Fig. 33(b) obtained by removing the upper red arc cancels out the reduced diagram of the S diagram in Fig. 32(b) obtained by removing the upper red arc. As for the COE and CUE classes, there are two additional SR diagrams for each case in Fig. 33, where the curve section between b and c has lengths $\lceil t/2 \rceil - 1$ and $\lceil t/2 \rceil$. According to Eq. (57), only diagrams containing red arcs with counterclockwise arrows on the outer circle or red arcs with clockwise arrows on the inner circle can contribute in the ergodic phase. Therefore, all red arcs with clockwise arrows in Fig. 33 must be pushed to the inner circle. Furthermore, in Fig. 33(a), the endpoints of

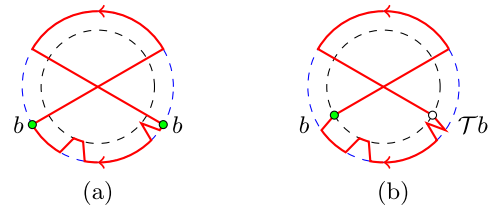


FIG. 33. Variants of the SR diagrams whose reduced diagrams are the same as those of diagrams in Fig. 32.

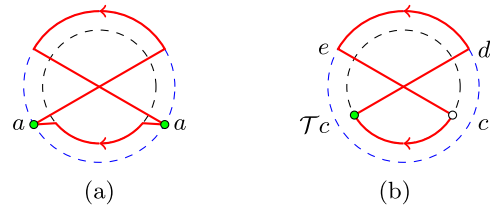


FIG. 34. Variants of the SR diagrams that can contribute to the SFF in the thermal phase according to Eq. (57). The reduced diagram of (a) vanishes. Thus, only the diagram in (b) contributes to the SFF in the thermal phase.

the reversed parts must lie on the outer circle, whereas in Fig. 33(b), they must lie on the inner circle. Thus, only the diagrams in Fig. 34 contribute to the SFF in the ergodic phase. Because of the jumps at the ends of the lower arc in Fig. 34(a), the reduced diagram obtained by removing the lower arc vanishes according to Theorem 3.

Thus, Fig. 34(a) does not contribute to the SFF in the ergodic phase. We evaluate Fig. 34(b) by applying the rules in Appendix G. The symbols $\mathcal{T}c$ and c in Fig. 34(b) represent the states at time steps τ_1 and τ_2 . Therefore, Eq. (89) gives the following expression by ignoring the Type III terms:

$$\begin{aligned} K_{-1}^{(2)}(t) &= -\frac{2t}{2} \sum_{\tau_1=1}^t \sum_{\tau_2=\tau_1+\lceil t/2 \rceil-1}^{\tau_1+\lceil t/2 \rceil} \sum_{ij} \lambda_i^{t-\tau_2+\tau_1-2} \lambda_j^{\tau_2-\tau_1} \mathcal{M}_{d,e}^{(i)} \mathcal{M}_{T,c}^{(j)} V_{e,Tc} V_{c,d} V_{e,Tc}^* V_{-c,d}^* \\ &= -t^2 \sum_{ij} (\lambda_i^{\lfloor t/2 \rfloor-1} \lambda_j^{\lceil t/2 \rceil-1} + \lambda_i^{\lfloor t/2 \rfloor-2} \lambda_j^{\lceil t/2 \rceil}) \mathcal{M}_{d,e}^{(i)} \mathcal{M}_{T,c}^{(j)} V_{e,Tc} V_{c,d} V_{e,Tc}^* V_{-c,d}^* \\ &= -\frac{2t^2}{\mathcal{N}^2} V_{e,Tc} V_{c,d} V_{e,Tc}^* V_{-c,d}^* + \mathcal{O}(\lambda_1^{\lceil t/2 \rceil}), \end{aligned} \quad (92)$$

where the factor of 2 in the numerator of the first line is due to Theorem 1, which says that each diagram has a time-reversed counterpart that contributes identically. By summing over the matrix indices using the unitary property of the matrix V , we find

$$V_{e,Tc} V_{c,d} V_{e,Tc}^* V_{-c,d}^* = -\mathcal{N}. \quad (93)$$

We substitute Eq. (93) in Eq. (92) to obtain

$$K_{-1}^{(2)}(t) = \frac{2t^2}{\mathcal{N}} + \mathcal{O}(\lambda_1^\nu). \quad (94)$$

The last result matches the second-order correction to the SFF predicted by RMT for the CSE class, Eq. (4). Furthermore, the extra SR diagrams for this case do not resemble near-miss and self-crossing orbits of the Sieber-Richter pairs of a semiclassical analysis [53,54]. Unlike the COE class, we could not give a simple probabilistic interpretation of the second-order correction in this case, which probably indicates a fully quantum origin of this contribution.

VIII. THOULESS TIME

By comparing the SFF calculated numerically using Eq. (9) with that from our theory using the RPA in Fig. 8, we notice that the nonuniversal part of the SFF at shorter times mainly comes from leading order in the t contributions of the SFF in Eqs. (64), (73), and (88). Therefore, we can find the Thouless time for systems from all three Dyson classes by analyzing the Type III terms in Eqs. (64), (73), and (88).

A. COE

Since Type III terms are exponentially decaying and $1 \geq |\lambda_1| \geq \dots \geq |\lambda_{\mathcal{N}-1}|$ in Eq. (64), λ_1^t is the Type III term that

vanishes last. Therefore, this term determines t^* . We define t^* as the solution of the equation

$$d_1 \lambda_1^{t^*} \simeq \frac{1}{e}, \quad (95)$$

where d_1 is the degeneracy of eigenvalue λ_1 . We find t^* by solving Eq. (95):

$$t^* \simeq \frac{\ln(d_1) + 1}{|\ln \lambda_1|}. \quad (96)$$

Following Ref. [26], we study a periodically kicked, strongly interacting, spinless fermion chain in Eqs. (34) and (35) with the parameters in Eqs. (46) and (47) to determine t^* in the presence and absence of a $U(1)$ symmetry. When $\Delta = 0$, the Hamiltonians in Eqs. (34) and (35) commute with $\hat{N}$ in Table II. Therefore, the model has $U(1)$ symmetry for $\Delta = 0$. Our numerical study reveals that the matrix $\mathcal{M}$ is $SU(2)$ invariant, which implies the existence of $SU(2)$ multiplets, with descendant states in different particle-number sectors. The eigenvalue λ_1 is independent of the total number of particles. Therefore, the corresponding eigenstate is a descendant state. We can find λ_1 numerically for very large system sizes L for the one-particle sector. In the thermodynamic limit of $L \rightarrow \infty$, we have

$$\lambda_1 \simeq 1 - \frac{c_\beta}{L^2}, \quad (97)$$

where c_β is a constant, which depends on hopping J (see Table II). Substituting Eq. (97) and $d_1 = 2$ (as our numerical study reveals that λ_1 is doubly degenerate) into Eq. (96), we find t^* as [26,30]

$$t^* \propto L^2. \quad (98)$$

We can find λ_1 and d_1 analytically by mapping the matrix $\mathcal{M}$ to a many-body spin-1/2 Hamiltonian in the Trotter regime of small parameters $|J|, |\Delta| \ll 1/\tau_p = 1$ [26,30],

$$\mathcal{M} \simeq (1 - c_1 L) \mathbb{1}_{\mathcal{N}} + \sum_{x=1}^L \sum_{\nu} c_{\nu} s_x^{\nu} s_{x+1}^{\nu}, \quad (99)$$

where $\nu = 1, 2, 3$, $c_1 = (|J|^2 + |\Delta|^2)/2$, $c_2 = c_3 = (|J|^2 - |\Delta|^2)/2$, and s_x^{ν} are the Pauli matrices at site x . In the presence of $U(1)$ symmetry when $\Delta = 0$, $\mathcal{M}$ becomes an XXX -Heisenberg spin-1/2 chain, which gives $\lambda_1 \simeq 1 - 2\pi^2 J^2 / L^2$ for pseudomomenta $k = 2\pi/L$, $2\pi - 2\pi/L$ in the thermodynamic limit. Consequently, $t^* \propto L^2$. When $\Delta \neq 0$, $U(1)$ symmetry is absent, and Eq. (99) becomes an XXZ -Heisenberg chain, which is well known to have a gapped spectrum. Therefore, $\lambda_1 \simeq \mathcal{O}(L^0)$ and $d_1 = 2$ due to reflection symmetry. Thus, $t^* \propto L^0$. Following a similar analysis, the above system-size scalings of t^* were studied in Ref. [26]. However, two other regimes of parameters were missed in Ref. [26]. When $|J| = |\Delta|$, Eq. (99) becomes an Ising Hamiltonian, leading to $\lambda_1 = 1 - 4J^2$ and $d_1 = L(L + 1)/2$. Thus, we have $t^* \propto \ln(L)$, which matches with the system-size scalings of t^* for the local kicking model studied in Kos *et al.* [19]. Furthermore, for $J = 0$ but $\Delta \neq 0$, Eq. (99) is equivalent to an XXZ -Heisenberg Hamiltonian with an anisotropy parameter -1 . For even L , alternating site spins of such a Hamiltonian can be rotated to transform $\mathcal{M}$ back to an XXX -Heisenberg Hamiltonian, which implies $t^* \propto L^2$. In this case, the conserved charge is $\hat{N}_s$ in Table II. Thus, the model is still $U(1)$ symmetric, and we expect $t^* \propto L^2$. The above findings suggest that the various L scalings of λ_1 and its degeneracy d_1 lead to different L scalings of t^* . Here, we provide a unified description of such scalings.

B. CUE

In this case, we conclude from Eq. (73) that t^* is determined by $\max(\lambda_1, \chi_0)$. We next study the Hamiltonians in Eqs. (34) and (35) with the parameters in Eqs. (48) and (49) to determine t^* for a physical model of the CUE class. Our numerical study with different parameters shows that the second-largest eigenvalue λ_1 of $\mathcal{M}$ is larger than the largest eigenvalue χ_0 of $\tilde{\mathcal{M}}$ for longer system sizes, as shown in Figs. 35(a) and 35(b). Thus, λ_1 determines t^* . When $\Delta = \Delta' = 0$, the Hamiltonians in Eqs. (34) and (35) commute with $\hat{N}$ in Table II, indicating a $U(1)$ symmetry of the model. We also numerically find that the matrix $\mathcal{M}$ is $SU(2)$ invariant for arbitrary J and g , which indicates the existence of descendant states. Our numerical study further reveals that λ_1 is independent of the total number of particles, and the corresponding eigenstate is a descendant state. We numerically determine λ_1 for very long system sizes from the one-

particle sector. In the thermodynamic limit of $L \rightarrow \infty$, we find

$$\lambda_1 \simeq 1 - \frac{c_{\beta}}{L^2}, \quad (100)$$

where c_{β} is a constant that depends on hopping J and g . Substituting Eq. (100) and $d_1 = 2$ (as a numerical study shows that λ_1 is doubly degenerate) in Eq. (96), we obtain

$$t^* \propto L^2. \quad (101)$$

We can find an analytical expression for λ_1 by mapping the matrix $\mathcal{M}$ in the Trotter regime ($J, g, \Delta, \Delta' \ll 1/\tau_p = 1$) to an XXZ -Heisenberg spin-1/2 Hamiltonian with nearest-neighbor and next-nearest-neighbor couplings as

$$\mathcal{M} \simeq (1 - (c_1 + c'_1)L) \mathbb{1}_{\mathcal{N}} + \sum_{x=1}^L \sum_{\nu} c_{\nu} s_x^{\nu} s_{x+1}^{\nu} + c'_1 s_x^{\nu} s_{x+2}^{\nu}, \quad (102)$$

where c_{ν} are the same as in the previous subsection, and $c'_1 = (|g|^2 + |\Delta'|^2)/2$ and $c'_2 = c'_3 = (|g|^2 - |\Delta'|^2)/2$. For $\Delta = \Delta' = 0$, Eq. (102) becomes a $SU(2)$ -symmetric XXX -Heisenberg chain with nearest-neighbor and next-nearest-neighbor couplings, which leads to $\lambda_1 \simeq 1 - (2\pi^2 J^2 + 8\pi^2 g^2)/L^2$ in the thermodynamic limit, giving $t^* \propto L^2$ [25]. When $|\Delta| = |J|$ and $|\Delta'| = |g|$, Eq. (102) becomes an Ising Hamiltonian with nearest-neighbor and next-nearest-neighbor couplings, leading to $\lambda_1 = 1 - 4|J|^2 - 4|g|^2$ and $d_1 = L$. Thus, we have $t^* \propto \ln L$, which matches with the system-size scalings of t^* for a quantum circuit model in Chan *et al.* [21]. When $|\Delta| \neq |J|$ or $|\Delta'| \neq |g|$, our numerical study in Fig. 35(b) shows that λ_1 is a constant as L increases, and $d_1 = 1$, which gives $t^* \propto L^0$. Therefore, the system-size scalings of t^* do not change for the CUE class from those within the COE class. We confirm our analytical predictions of $t^* \propto L^2$ in Figs. 36(a) and 36(b) and $t^* \propto L^0$ in Fig. 36(c) by numerically computing the SFF directly using Eq. (9) without the RPA.

C. CSE

In this case also, we conclude from Eq. (88) that t^* is determined by $\max(\lambda_1, \chi_0)$. We study the Hamiltonians in Eqs. (36) and (37) to determine t^* in a physical system. We find numerically that $\lambda_1 > \chi_0$ for longer L and different parameters as shown in Fig. 37. Thus, λ_1 determines t^* in this case also. When $\Delta_{\sigma\sigma'} = 0$, the Hamiltonians in Eqs. (36) and (37) commute with $\hat{N}_i$ in Table II. Therefore, the model has $U(1)$ symmetry. Once again, we find that the matrix $\mathcal{M}$ is $SU(2)$ invariant for arbitrary values of $J_{\sigma\sigma'}$, which indicates the existence of descendant states. Thus, λ_1 is independent of the total

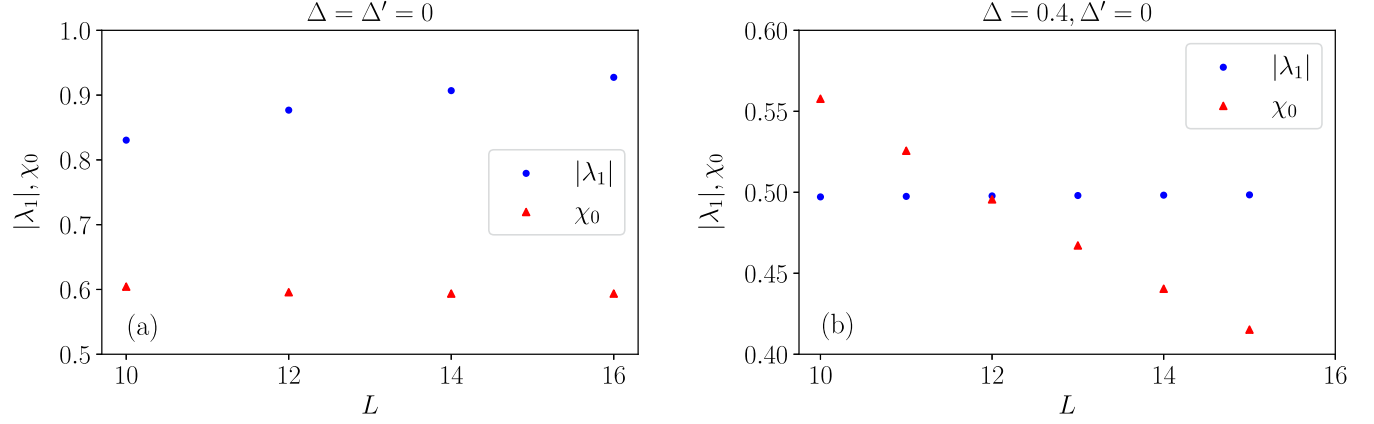


FIG. 35. Second-largest eigenvalue λ_1 of $\mathcal{M}$ and largest eigenvalue χ_0 of $\tilde{\mathcal{M}}$ for different system sizes L for a kicked, strongly interacting, spinless fermionic chain: (a) with ($\Delta = \Delta' = 0$) and (b) without ($\Delta = 0.4, \Delta' = 0$) particle-number conservation. We use the Hamiltonian in Eq. (35) with parameters in Eqs. (48) and (49). We apply the PBC in real space and take half filling for $\Delta = \Delta' = 0$. Here, $J = 0.5$ and $g = 0.25$.

number of particles. We numerically obtain λ_1 for large L 's in the one-particle sector. We find that, in the thermodynamic limit of $L \rightarrow \infty$,

$$\lambda_1 \simeq 1 - \frac{c_\beta}{L^2}, \quad (103)$$

where c_β is a constant, which depends on hopping $J_{\sigma\sigma'}$. We substitute Eq. (103) and $d_1 = 2$ (as our numerical study reveals that λ_1 is doubly degenerate) in Eq. (96) to find

$$t^* \propto L^2. \quad (104)$$

We can analytically determine the eigenvalues of $\mathcal{M}$ by mapping it to a spin-1/2 Hamiltonian in the Trotter regime ($|J_{\sigma\sigma'}|, |\Delta_{\sigma\sigma'}| \ll 1/\tau_p = 1$):

$$\mathcal{M} \simeq \left(1 - L \sum_{\sigma\sigma'} c_1^{\sigma\sigma'}\right) \mathbb{1}_{\mathcal{N}} + \sum_{x=1}^L \sum_{\sigma\sigma'} \sum_{\nu} c_{\nu}^{\sigma\sigma'} s_{x,\sigma}^{\nu} s_{x+1,\sigma'}^{\nu}, \quad (105)$$

where $c_1^{\sigma\sigma'} = (|J_{\sigma\sigma'}|^2 + |\Delta_{\sigma\sigma'}|^2)/2$, $c_2^{\sigma\sigma'} = c_3^{\sigma\sigma'} = (|J_{\sigma\sigma'}|^2 - |\Delta_{\sigma\sigma'}|^2)/2$, $\nu = 1, 2, 3$, and $s_{x,\sigma}^{\nu}$ are the Pauli matrices at site x for each σ . In the presence of $U(1)$ symmetry ($\Delta_{\sigma\sigma'} = 0$), $\mathcal{M}$ is an $SU(2)$ -invariant spin-1/2 ladder

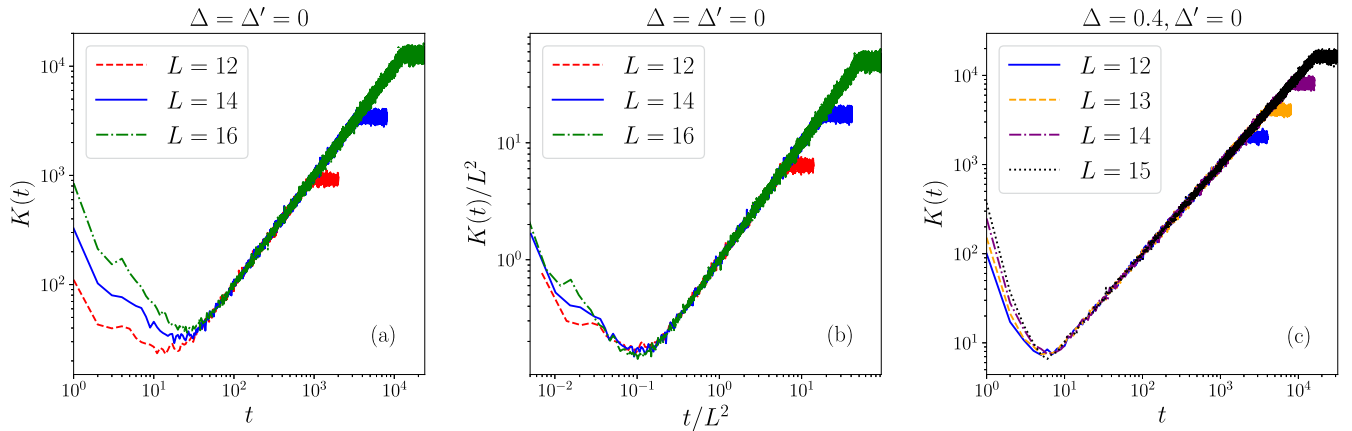


FIG. 36. Spectral form factor $K(t)$ for different system sizes L calculated numerically using Eq. (9) for a kicked, strongly interacting, spinless fermionic chain: (a), (b) with ($\Delta = \Delta' = 0$) and (c) without ($\Delta = 0.4, \Delta' = 0$) particle-number conservation. We use the Hamiltonians in Eqs. (34) and (35) with parameters in Eqs. (48) and (49). Here, $\Delta\epsilon = 10, \alpha = 1.5, J = 0.5, g = 0.25$. For diagrams (a) and (b), we take $U_0 = 80\sqrt{2}, \Delta U_0 = 40$, and for (c) we take $U_0 = 100\sqrt{2}, \Delta U_0 = 20$. We apply the PBC in real space and take half filling $N/L = 1/2$ for $\Delta = \Delta' = 0$. In diagram (b), we show the data collapse in scaled time t/L^2 . An averaging over 250–320 realizations of disorder is performed in each case.

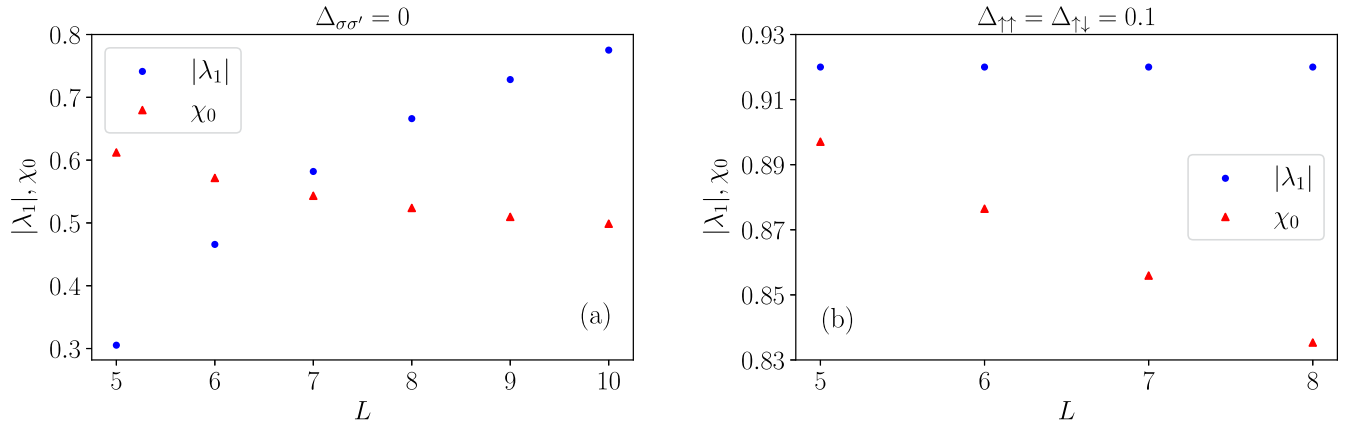


FIG. 37. Second-largest eigenvalue $|\lambda_1|$ of $\mathcal{M}$ and largest eigenvalue χ_0 of $\tilde{\mathcal{M}}$ for different system sizes L for a kicked spinful fermionic chain: (a) with ($\Delta_{\sigma\sigma'} = 0$) and (b) without ($\Delta_{\uparrow\uparrow} = \Delta_{\downarrow\downarrow} = 0.1$) particle-number conservation. We use the Hamiltonian in Eq. (37) and apply the PBC in real space. In diagram (a), we take $J_{\uparrow\uparrow} = 0.5 + 0.25i$, $J_{\downarrow\downarrow} = 0.5 + 0.25i$ and a fixed total number of particles ($N_t = 5$), and in (b), we take $J_{\uparrow\uparrow} = 0.1 + 0.1i$, $J_{\downarrow\downarrow} = 0.1 + 0.05i$.

Hamiltonian. The doubly degenerate second-largest eigenvalue calculated from the one-excitation sector reads as $\lambda_1 \simeq 1 - 4\pi^2(|J_{\uparrow\uparrow}|^2 + |J_{\downarrow\downarrow}|^2)/L^2$ in the limit of $L \rightarrow \infty$. Thus, $t^* \propto L^2$. We confirm our analytical predictions of $t^* \propto L^2$ in Fig. 38 for the systems of CSE class by numerically computing the SFF directly using Eq. (9) without the RPA. When $|J_{\sigma\sigma'}| = |\Delta_{\sigma\sigma'}|$, $\mathcal{M}$ becomes a spin ladder with an Ising-type interaction, which gives $\lambda_1 = 1 - 4|J_{\uparrow\uparrow}|^2 - 4|J_{\downarrow\downarrow}|^2$ with a degeneracy $d_1 \simeq L$. Therefore, $t^* \propto \ln(L)$. When $|J_{\sigma\sigma'}| \neq |\Delta_{\sigma\sigma'}| \neq 0$, a numerical study of the spectra of the matrix $\mathcal{M}$ reveals that λ_1 remains constant as L increases, as shown in Fig. 37(b), and $d_1 = 1$. Therefore, $t^* \propto L^0$. The above L scalings of t^* for strongly interacting systems of the CSE class have not been explored earlier.

IX. HIGHER DIMENSIONS

Previous studies using RPA have explored the emergence of a universal RMT form of the SFF, mainly in 1D physical models of strongly interacting fermions, bosons, qubits, and their mixtures [19,26,30,37]. Nevertheless, as we mentioned in Sec. I, it would be easier to experimentally probe our many results in higher dimensions with less control. Furthermore, higher-dimensional models are abundant in nature, contrary to strictly 1D models, which are more of a simplification. Some examples below show that the above formalism to derive the SFF analytically works in higher dimensions. Particularly, Eqs. (20)–(22) are equally valid for higher-dimensional single-particle or many-body systems.

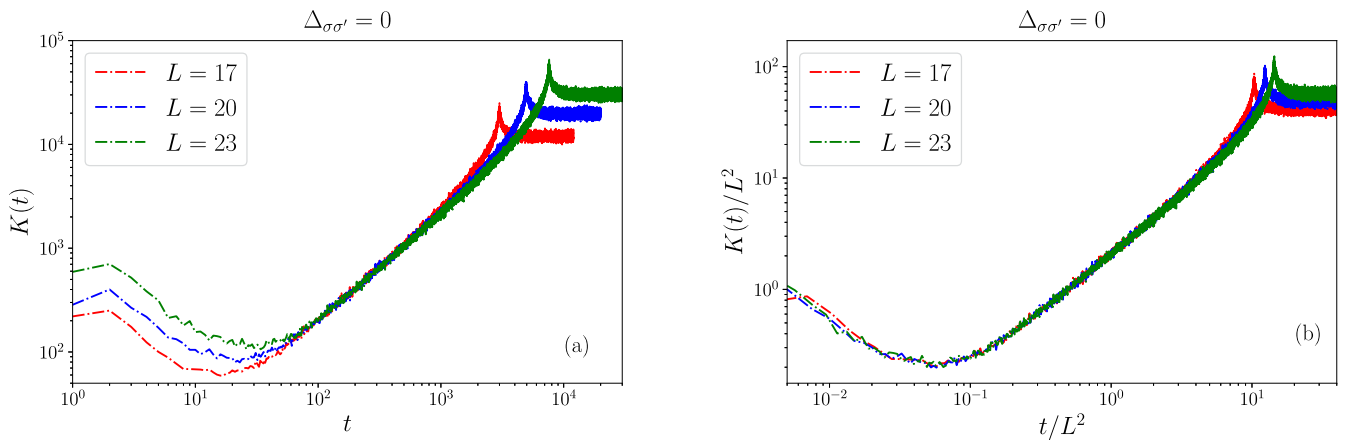


FIG. 38. Spectral form factor $K(t)$ for different system sizes L calculated numerically using Eq. (9) for a kicked, strongly interacting, spinful fermionic chain belonging to the CSE class and with particle-number conservation. We use the Hamiltonians in Eqs. (36) and (37). Here, $\Delta\epsilon = 10$, $U_0 = 40$, $\Delta U_0 = 10$, $\alpha = 1.3$, $\tilde{U} = 40$, $\Delta\tilde{U} = 10$, $J_{\uparrow\uparrow} = 0.5 + 0.25i$, $J_{\downarrow\downarrow} = 0.5 + 0.25i$. We apply the PBC in real space and take a fixed total number of particles ($N_t = 3$). In diagram (b), we show the data collapse in scaled time t/L^2 . An averaging over 400 realizations of disorder is performed in each case.

A. Single-particle systems

We consider the following Hamiltonian of a single spinless fermion on a square lattice in d spatial dimensions for the COE class. The length of the lattice in each direction is L , and we take the PBC

$$\hat{H}(t) = \hat{H}_0 + \hat{H}_1 \sum_{n \in \mathbb{Z}} \delta(t - n), \quad (106)$$

where

$$\hat{H}_0 = \sum_{\vec{x}} \epsilon_{\vec{x}} \hat{n}_{\vec{x}}, \quad (107)$$

$$\hat{H}_1 = \sum_{\langle \vec{x}, \vec{y} \rangle} -J \hat{c}_{\vec{x}}^\dagger \hat{c}_{\vec{y}} + \text{H.c.}, \quad (108)$$

where $\vec{x} \equiv (x_1, \dots, x_d)$ is the position of a lattice site and $x_i = 1, \dots, L$, $\forall i$. The on-site potentials $\epsilon_{\vec{x}}$ are independent Gaussian random variables with zero mean and a finite standard deviation of $\Delta\epsilon$. Here, $\langle \vec{x}, \vec{y} \rangle$ represents nearest-neighbor sites $\vec{x}$ and $\vec{y}$. Following Eq. (13), the SFF of this system can be written in the eigenbasis of $\hat{H}_0$ as

$$K(t) = \sum_{\underline{n}_1, \dots, \underline{n}_t} \sum_{\underline{n}'_1, \dots, \underline{n}'_t} \left\langle e^{-i \sum_{\tau=1}^t \theta_{\underline{n}_\tau} - \theta_{\underline{n}'_\tau}} \right\rangle \times \prod_{\tau=1}^t V_{\underline{n}_\tau, \underline{n}_{\tau+1}} V_{\underline{n}'_\tau, \underline{n}'_{\tau+1}}^*, \quad (109)$$

where $\theta_{\underline{n}_\tau}$'s are eigenvalues of $\hat{H}_0$, which can be expressed in terms of $\epsilon_{\vec{x}}$ and the fermion occupation number $n_{\vec{x}, \tau}$ as follows:

$$\theta_{\underline{n}_\tau} = \sum_{\vec{x}} \epsilon_{\vec{x}} n_{\vec{x}, \tau}. \quad (110)$$

Therefore,

$$\left\langle e^{-i \sum_{\tau=1}^t \theta_{\underline{n}_\tau} - \theta_{\underline{n}'_\tau}} \right\rangle = \left\langle \exp \left\{ -i \sum_{\vec{x}} \epsilon_{\vec{x}} \sum_{\tau=1}^t (n_{\vec{x}, \tau} - n'_{\vec{x}, \tau}) \right\} \right\rangle. \quad (111)$$

Since there is only one particle on the lattice, we have $|\underline{n}_\tau\rangle = |\vec{x}(\tau)\rangle$ and $|\underline{n}'_\tau\rangle = |\vec{x}'(\tau)\rangle$. Therefore,

$$n_{\vec{x}, \tau} = \begin{cases} 1 & \text{if } \vec{x} = \vec{x}(\tau) \\ 0 & \text{otherwise,} \end{cases} \quad (112)$$

and

$$n'_{\vec{x}, \tau} = \begin{cases} 1 & \text{if } \vec{x} = \vec{x}'(\tau) \\ 0 & \text{otherwise,} \end{cases} \quad (113)$$

$$\left\langle e^{-i \sum_{\tau=1}^t \theta_{\underline{n}_\tau} - \theta_{\underline{n}'_\tau}} \right\rangle = \left\langle \exp \left\{ -i \sum_{\tau=1}^t (\epsilon_{\vec{x}(\tau)} - \epsilon_{\vec{x}'(\tau)}) \right\} \right\rangle. \quad (114)$$

Performing the disorder average over the on-site potentials, we find

$$\left\langle \exp \left\{ -i \sum_{\tau=1}^t (\epsilon_{\vec{x}(\tau)} - \epsilon_{\vec{x}'(\tau)}) \right\} \right\rangle = \delta_{\sum_{\tau=1}^t \epsilon_{\vec{x}(\tau)}, \sum_{\tau=1}^t \epsilon_{\vec{x}'(\tau)}} + \mathcal{O}(e^{-\Delta\epsilon^2}). \quad (115)$$

Therefore, in the limit of strong disorder ($\Delta\epsilon \gg 1$),

$$\left\langle e^{-i \sum_{\tau=1}^t \theta_{\underline{n}_\tau} - \theta_{\underline{n}'_\tau}} \right\rangle \simeq \delta_{\{\vec{x}(1), \dots, \vec{x}(t)\}, \{\vec{x}'(1), \dots, \vec{x}'(t)\}}, \quad (116)$$

where $\delta_{\{\vec{x}(1), \dots, \vec{x}(t)\}, \{\vec{x}'(1), \dots, \vec{x}'(t)\}}$ is 1 when $\{\vec{x}'(1), \dots, \vec{x}'(t)\}$ is a permutation of $\{\vec{x}(1), \dots, \vec{x}(t)\}$ and zero otherwise. This finding is equivalent to the conclusion obtained by applying the RPA. Again, the cyclic and anticyclic variants of identity permutation give the leading-order SFF:

$$K^{(1)}(t) = 2t \text{tr} \mathcal{M}^t = 2t(1 + \lambda_1^t + \dots). \quad (117)$$

For such single-particle systems, in the Trotter regime of small parameters ($|J| \ll 1$), we find that the second-largest eigenvalue of $\mathcal{M}$ is $\lambda_1 = 1 - 2J^2 \sum_{i=1}^d (1 - \cos(2\pi/L)) \approx 1 - 4\pi^2 J^2 d/L^2$. The Thouless time is related to λ_1 by Eq. (45). Therefore, we find $t^*(L) \propto L^2$ for any dimension. Nevertheless, the Hilbert space dimension is L^d ; therefore, such single-particle systems are chaotic only in two or higher spatial dimensions.

B. Many-particle systems

We take a two-dimensional (2D) version of the system of interacting spinless fermions for the COE class given in Eqs. (34) and (35):

$$\hat{H}_0^0 = \sum_{\vec{x}} \epsilon_{\vec{x}} \hat{n}_{\vec{x}} + \frac{1}{2} \sum_{\vec{x} \neq \vec{y}} \frac{U_{xy}}{|\vec{x} - \vec{y}|^\alpha} \hat{n}_{\vec{x}} \hat{n}_{\vec{y}}, \quad (118)$$

$$\hat{H}_1^0 = \sum_{\langle \vec{x}, \vec{y} \rangle} -J \hat{c}_{\vec{x}}^\dagger \hat{c}_{\vec{y}} + \Delta \hat{c}_{\vec{x}}^\dagger \hat{c}_{\vec{y}}^\dagger + \text{H.c.}, \quad (119)$$

where $\vec{x} \equiv (x_1, x_2)$ is the position of a site on the 2D square lattice with $1 \leq x_1 \leq L_1, 1 \leq x_2 \leq L_2$, and $\langle \vec{x}, \vec{y} \rangle$ denotes nearest-neighbor sites. We use the PBC in both directions of the lattice. Here, J and Δ represent the hopping and pairing amplitudes between nearest-neighbor sites in both directions. We choose them to be real for the COE class. Furthermore, α determines whether particle-particle interaction is long-range or short-range. We notice that the RPA works better for 2D models than 1D ones, even for larger α values (e.g., $\alpha > 1.5$), which is due to the higher number of

neighbors (coordination numbers) at a fixed distance for 2D rather than 1D, which improves the assumption of the RPA.

We can again write the leading-order SFF for the 2D model using Eq. (64). The Markov matrix $\mathcal{M}$ can now be mapped to a 2D XXZ-Heisenberg Hamiltonian in the Trotter regime of small parameters ($|J|, |\Delta| \ll 1$):

$$\mathcal{M} = (1 - c_1 L_1 L_2) \mathbb{1}_{\mathcal{N}} + \sum_{(\vec{x}, \vec{y})} \sum_{\nu} c_{\nu} s_{\vec{x}}^{\nu} s_{\vec{y}}^{\nu} + \mathcal{O}(J^4, \Delta^4), \quad (120)$$

where $c_1 = (|J|^2 + |\Delta|^2)/2$, $c_2 = c_3 = (|J|^2 - |\Delta|^2)/2$ and $s_{\vec{x}}^{\nu}$, $\nu = 1, 2, 3$ are Pauli matrices at site $\vec{x}$. In the absence of pairing ($\Delta = 0$), we have an $SU(2)$ -invariant mapped Hamiltonian in Eq. (120), whose second-largest eigenvalue λ_1 is independent of the total number of particles, $\hat{N} = \sum_{\vec{x}} \hat{n}_{\vec{x}}$. Thus, λ_1 can be calculated analytically in the single-particle sector. We find $\lambda_1 \approx 1 - 2\pi^2 J^2 / L^2$ in the thermodynamic limit, where $L = \max(L_1, L_2)$. This finding leads to $t^* \propto L^2$, which is similar to the 1D case. For $\Delta = J$, the mapped Hamiltonian in Eq. (120) becomes a 2D Ising Hamiltonian:

$$\mathcal{M} = (1 - J^2 L_1 L_2) \mathbb{1}_{\mathcal{N}} + J^2 \sum_{(\vec{x}, \vec{y})} s_{\vec{x}}^1 s_{\vec{y}}^1 + \mathcal{O}(J^4, \Delta^4). \quad (121)$$

The second-largest eigenvalue of $\mathcal{M}$ for this case is $1 - 8J^2$ with a degeneracy of $L_1 L_2$. Therefore, we then have $t^* \propto \ln(L_1 L_2)$, indicating a logarithmic system-size dependence, which is again similar to the 1D case. While we demonstrate that our formalism works for 2D systems of the COE class, the above argument can be generalized for all three Dyson classes in both 2D and 3D systems.

X. CONCLUSION

This work extends many recent efforts to analytically calculate the spectral statistics in periodically kicked, interacting, many-body quantum systems in chaotic regimes to all three of Dyson's circular ensembles. To the best of our knowledge, this work is the first study of such systems with a CSE class. For this purpose, we have developed an innovative scheme to include contributions from many diagrams of different permutations of basis states. We derived SFFs up to two leading orders in time for generic periodically kicked systems supporting the RPA. We showed that leading-order SFF is determined by identity permutations similar to the diagonal approximation in the semiclassical proof [11]. In contrast, the diagrams leading to second-order corrections for the COE class are similar to the Sieber-Richter pairs [13], even in pure quantum systems without classical limits. We provided general rules to calculate the contribution of different diagrams and discovered reduced diagrams that

contain information on the contribution to SFF in the ergodic phase. Reduced diagrams allow more analytical control and can play a significant role in deriving the complete SFF in the ergodic phase. Our study also revealed an underlying stochastic mechanism determining the emergence of universal RMT behavior. More specifically, the general properties of the doubly stochastic matrix $\mathcal{M}$ determine the universal RMT form of the SFF in all three Dyson symmetry classes. Furthermore, we studied chains of spinless and spinful fermions to derive system-size scaling of the Thouless time using the second-largest eigenvalue of $\mathcal{M}$, which can be obtained numerically for large system sizes and analytically calculated in the Trotter regime due to the $SU(2)$ symmetry of $\mathcal{M}$ for all three classes in the presence of $U(1)$ symmetry. Without $U(1)$ symmetry, a numerical study reveals that the second-largest eigenvalue of $\mathcal{M}$ is independent of system size. If the eigenvalue has degeneracy $\mathcal{O}(L^0)$, then t^* is also $\mathcal{O}(L^0)$, whereas if the degeneracy is $\mathcal{O}(L^5)$, then t^* is $\mathcal{O}(\ln(L))$. Finally, we showed that similar system-size scalings of the Thouless time can also be found in higher dimensions.

It is apt here to make some further remarks on the technical similarities of our method with previous works on quantum graphs. For both the quantum graphs and our method in this paper, a doubly stochastic matrix (denoted $\mathcal{M}$ in our paper) arises in the derivation of the SFF since the SFF can, in some cases, be expressed as a product of propagation amplitudes along pairs of overlapping closed paths. There are no further similarities between the quantum graphs and our method, either through direct derivations in Appendixes H and I for the COE and CUE classes or through the reduced diagrams for all three Dyson classes. The diagrams encountered in the study of many-body quantum systems are very different from those experienced in the study of quantum graphs.

In recent years, many proposals have been made using cold atoms [36] and quantum simulators [35] to detect spectral statistics in various many-body quantum chaotic systems [55]. There have also been recent efforts to measure the SFF and various correlations in different many-body quantum chaotic systems using quantum processors [56] or computers [57]. Thus, we hope our findings, particularly system-size scaling of t^* for the emergence of RMT SFF form with different unitary and antiunitary symmetries (Table II), can be tested in the coming years using these methods. The techniques developed in this article can be applied to derive other physically relevant quantities like correlation functions and entanglement entropies in periodically kicked, interacting, many-body quantum systems supporting the RPA [23]. It would be interesting to check if the formalism can be extended to account for other symmetries leading to the tenfold classification of Altland and Zirnbauer [58]. Some of the

challenges in this direction in the near future are improving the RPA beyond the present form and deriving spectral statistics in autonomous systems.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [59].

APPENDIX A: JUSTIFICATION OF RANDOM PHASE APPROXIMATION

Here, we provide a justification for the RPA for a base Hamiltonian with random on-site potentials and random long-range interactions. Specifically, we show the emergence of RPA in the limit of a large disorder. Following Eqs. (34) and (35), we take the following Hamiltonian, which describes multiple spinless interacting fermions on a 1D lattice of length L :

$$\hat{H}(t) = \hat{H}_0^0 + \hat{H}_1^0 \sum_{n \in \mathbb{Z}} \delta(t - n). \quad (\text{A1})$$

The SFF of this system can be expressed as Eq. (13), where $\theta_{\underline{n}_\tau}$ now takes the form

$$\theta_{\underline{n}_\tau} = \sum_{x=1}^L \sum_{\tau=1}^t \epsilon_x n_{x,\tau} + \sum_{x < y} \frac{U_{xy}}{|x-y|^\alpha} \sum_{\tau=1}^t n_{x,\tau} n_{y,\tau}, \quad (\text{A2})$$

where $n_{x,\tau}$ is the occupation of the fermion at site x in the many-body Fock state $\underline{n}_\tau$. Performing the disorder averaging over ϵ_x 's and U_{xy} 's, we obtain

$$\begin{aligned} \langle e^{-i \sum_{\tau=1}^t \theta_{\underline{n}_\tau} - \theta_{\underline{n}'_\tau}} \rangle &= \int_{-\infty}^{\infty} \left(\prod_{x=1}^L \frac{d\epsilon_x}{\sqrt{2\pi\Delta\epsilon^2}} \exp\left\{-\frac{\epsilon_x^2}{2\Delta\epsilon^2}\right\} \right) \left(\prod_{x < y} \frac{dU_{xy}}{\sqrt{2\pi\Delta U_0^2}} \exp\left\{-\frac{(U_{xy} - U_0)^2}{2\Delta U_0^2}\right\} \right) \\ &\quad \times \exp\left\{-i \sum_{x=1}^L \epsilon_x \sum_{\tau=1}^t (n_{x,\tau} - n'_{x,\tau})\right\} \exp\left\{-i \sum_{x < y} \frac{U_{xy}}{|x-y|^\alpha} \sum_{\tau=1}^t (n_{x,\tau} n_{y,\tau} - n'_{x,\tau} n'_{y,\tau})\right\} \\ &= \exp\left\{-\frac{\Delta\epsilon^2}{2} \sum_{x=1}^L \left(\sum_{\tau=1}^t n_{x,\tau} - \sum_{\tau=1}^t n'_{x,\tau}\right)^2\right\} \exp\left\{-\sum_{x < y} \frac{\Delta U_0^2}{2|x-y|^{2\alpha}} \left(\sum_{\tau=1}^t n_{x,\tau} n_{y,\tau} - \sum_{\tau=1}^t n'_{x,\tau} n'_{y,\tau}\right)^2\right\} \\ &\quad \times \exp\left\{-i \frac{U_0}{|x-y|^\alpha} \sum_{\tau=1}^t (n_{x,\tau} n_{y,\tau} - n'_{x,\tau} n'_{y,\tau})\right\}. \end{aligned} \quad (\text{A3})$$

In the strong disorder limit of $\Delta\epsilon \gg 1$, $\Delta U_0 \gg 1$, we obtain

$$\langle e^{-i \sum_{\tau=1}^t \theta_{\underline{n}_\tau} - \theta_{\underline{n}'_\tau}} \rangle \simeq \left(\prod_{x=1}^L \delta_{\sum_{\tau=1}^t n_{x,\tau}, \sum_{\tau=1}^t n'_{x,\tau}} \right) \left(\prod_{x < y} \delta_{\sum_{\tau=1}^t n_{x,\tau} n_{y,\tau}, \sum_{\tau=1}^t n'_{x,\tau} n'_{y,\tau}} \right), \quad (\text{A4})$$

which vanishes unless the following two conditions are met:

$$C_1: \sum_{\tau=1}^t n'_{x,\tau} = \sum_{\tau=1}^t n_{x,\tau}, \quad (\text{A5})$$

$$C_2: \sum_{\tau=1}^t n'_{x,\tau} n'_{y,\tau} = \sum_{\tau=1}^t n_{x,\tau} n_{y,\tau}. \quad (\text{A6})$$

Since $n_{x,\tau}$ and $n'_{x,\tau}$ take values 0 or 1, the condition C_1 can be written as

$$\sum_{\tau=1}^t n'_{x,\tau} n'_{x,\tau} = \sum_{\tau=1}^t n_{x,\tau} n_{x,\tau}. \quad (\text{A7})$$

Therefore, conditions C_1 and C_2 together can be expressed as

$$\sum_{\tau=1}^t n'_{x,\tau} n'_{y,\tau} = \sum_{\tau=1}^t n_{x,\tau} n_{y,\tau}, \quad \forall x, y. \quad (\text{A8})$$

To understand the implications of Eq. (A8), we express many-body Fock states as L component column vectors,

$$|\underline{n}_\tau\rangle \equiv p_\tau = (n_{1,\tau}, \dots, n_{L,\tau})^T, \quad (\text{A9})$$

$$|\underline{n}'_\tau\rangle \equiv p'_\tau = (n'_{1,\tau}, \dots, n'_{L,\tau})^T. \quad (\text{A10})$$

Next, we define $L \times L$ matrices

$$w_\tau = p_\tau p_\tau^T, \quad (\text{A11})$$

$$w'_\tau = p'_\tau p'^T_\tau, \quad (\text{A12})$$

where $\tau = 1, \dots, t$. The elements of w_τ and w'_τ are $(w_\tau)_{xy} = n_{x,\tau} n_{y,\tau}$ and $(w'_\tau)_{xy} = n'_{x,\tau} n'_{y,\tau}$, respectively. Now, Eq. (A8) can be expressed as a matrix equation,

$$\sum_{\tau=1}^t w'_\tau = \sum_{\tau=1}^t w_\tau. \quad (\text{A13})$$

For a given multiset $\{w_1, \dots, w_t\}$, a multiset $\{w'_1, \dots, w'_t\}$ satisfying Eq. (A13) can be obtained by permuting elements of all matrices $w_1, \dots, w_t$ labeled with the same indices,

$$(w'_\tau)_{xy} = (w_{\pi_{xy}(\tau)})_{xy}, \quad (\text{A14})$$

where π_{xy} is a permutation. In general, π_{xy} can be different for different indices xy . However, w'_τ must be a rank 1 matrix according to Eq. (A12). Furthermore, the elements of the vectors p_τ (p'_τ) are 0 or 1. Thus, nonzero rows and columns of w_τ (w'_τ) are identical. Therefore, permutations π_{xy} for different xy cannot be independent. Permutations of the corresponding elements of the matrices $w_1, \dots, w_t$ should be such that each of the resulting matrices w'_τ has identical nonzero rows and columns. For an arbitrary multiset of matrices $\{w_1, \dots, w_t\}$, this requirement is satisfied if $\pi_{xy} = \pi_{x'y'} = \pi$, which is equivalent to permuting the matrices $w_1, \dots, w_t$. Thus,

$$w'_\tau = w_{\pi(\tau)}, \quad (\text{A15})$$

$$\Rightarrow p'_\tau = p_{\pi(\tau)}. \quad (\text{A16})$$

Therefore, we find that states $\underline{n}'_1, \dots, \underline{n}'_t$ satisfying Eq. (A8) can mostly be obtained from states $\underline{n}_1, \dots, \underline{n}_t$ by permuting them, $\underline{n}'_\tau = \underline{n}_{\pi(\tau)}$. This finding is exactly the consequence of the RPA in Eq. (14), which we have justified here. The RPA can also be justified by comparing directly calculated SFF, $K(t)$, using Eq. (9) for a base Hamiltonian in the strong disorder limit with that obtained using the random phase

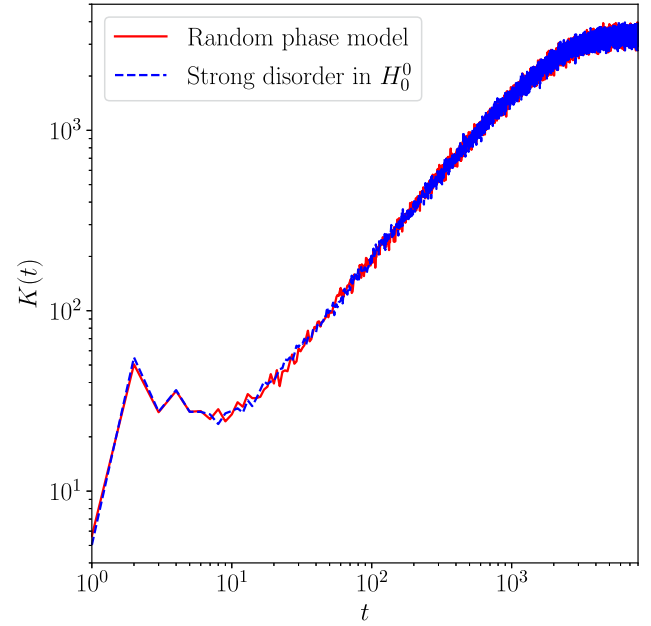


FIG. 39. Comparison of directly calculated SFF, $K(t)$, using Eq. (9) for a base Hamiltonian in Eq. (34) in the strong disorder limit with that obtained using the random phase model, where we replace H_0^0 by a diagonal random matrix, where each element is an independent random number distributed uniformly over the interval $[0, 2\pi)$. The kicking Hamiltonian for the plots is described by Eq. (35) with parameters in Eqs. (46) and (47). We take $L = 14$, $N = 7$, $J = 1$, and $\Delta = 0$. For strong disorder, we take $\Delta\epsilon = 10$, $U_0 = 80\sqrt{2}$, $\Delta U_0 = 20$, and $\alpha = 1.5$. An averaging over 320 realizations of disorder is performed in each case.

model, where we replace the base Hamiltonian by a diagonal random matrix, where each element is an independent random number distributed uniformly over the interval $[0, 2\pi)$. This comparison is depicted in Fig. 39 for the COE class, which shows an excellent match between the two plots. The random phase modeling is precisely the main assumption of the RPA.

The above analysis also explains the role of long-range interactions. Condition C_2 is valid for all pairs of x and y . It imposes permutations on all elements of the matrices $w_1, \dots, w_t$. If only nearest-neighbor interactions were present, then, with the open boundary condition, only the diagonal and leading off-diagonal elements of the matrices $w_1, \dots, w_t$ would be constrained for permutations. With no constraint on other elements of the matrices $w_1, \dots, w_t$ and $w'_1, \dots, w'_t$, there can be states $\underline{n}'_1, \dots, \underline{n}'_t$, which are not just permutations of states $\underline{n}_1, \dots, \underline{n}_t$, while still satisfying conditions C_1 and C_2 (only the cases with $|x - y| = 1$). The above analysis also reveals that the RPA may not be accurate in the thermodynamic limit, which can be understood from Eq. (A3). When x and y are macroscopically separated, we do not obtain the condition C_2 for such far-separated positions even for a large but finite

$$\begin{aligned}
\mathcal{X}_\pi &= \sum_{\underline{n}_1, \dots, \underline{n}_t} \left(\prod_{\tau=2}^{\tau_1-1} \mathcal{M}_{\underline{n}_\tau, \underline{n}_{\tau+1}} \right) \left(\prod_{\tau=\tau_1+1}^{\tau_2-1} \mathcal{M}_{\underline{n}_\tau, \underline{n}_{\tau+1}} \right) \left(\prod_{\tau=\tau_2+1}^{t-1} \mathcal{M}_{\underline{n}_\tau, \underline{n}_{\tau+1}} \right) \mathcal{M}_{\underline{n}_t, \underline{n}_1} \\
&\quad \times V_{\underline{n}_1, \underline{n}_2} V_{\underline{n}_1, \underline{n}_{\tau_1+1}} V_{\underline{n}_2, \underline{n}_{\tau_2+1}} V_{\underline{n}_1, \underline{n}_{\tau_2}}^* V_{\underline{n}_{\tau_1+1}, \underline{n}_2}^* V_{\underline{n}_{\tau_1}, \underline{n}_{\tau_2+1}}^* \\
&= \sum_{\substack{\underline{n}_1, \underline{n}_2 \\ \underline{n}_{\tau_1}, \underline{n}_{\tau_1+1} \\ \underline{n}_{\tau_2}, \underline{n}_{\tau_2+1}}} (\mathcal{M}^{\tau_1-2})_{\underline{n}_2, \underline{n}_{\tau_1}} (\mathcal{M}^{\tau_2-\tau_1-1})_{\underline{n}_{\tau_1+1}, \underline{n}_{\tau_2}} (\mathcal{M}^{t-\tau_2})_{\underline{n}_{\tau_2+1}, \underline{n}_1} \\
&\quad \times V_{\underline{n}_1, \underline{n}_2} V_{\underline{n}_{\tau_1}, \underline{n}_{\tau_1+1}} V_{\underline{n}_{\tau_2}, \underline{n}_{\tau_2+1}} V_{\underline{n}_1, \underline{n}_{\tau_2}}^* V_{\underline{n}_{\tau_1+1}, \underline{n}_2}^* V_{\underline{n}_{\tau_1}, \underline{n}_{\tau_2+1}}^*.
\end{aligned} \tag{C2}$$

We insert the eigendecomposition of $\mathcal{M}^n = \sum_i \lambda_i^n \mathcal{M}^{(i)}$, where $\mathcal{M}^{(i)} = |\lambda_i\rangle\langle\lambda_i|$, to find

$$\begin{aligned}
\mathcal{X}_\pi &= \sum_{\substack{\underline{n}_1, \underline{n}_2 \\ \underline{n}_{\tau_1}, \underline{n}_{\tau_1+1} \\ \underline{n}_{\tau_2}, \underline{n}_{\tau_2+1}}} \left(\sum_i \lambda_i^{\tau_1-2} \mathcal{M}_{\underline{n}_2, \underline{n}_{\tau_1}}^{(i)} \right) \left(\sum_j \lambda_j^{\tau_2-\tau_1-1} \mathcal{M}_{\underline{n}_{\tau_1+1}, \underline{n}_{\tau_2}}^{(j)} \right) \left(\sum_k \lambda_k^{t-\tau_2} \mathcal{M}_{\underline{n}_{\tau_2+1}, \underline{n}_1}^{(k)} \right) \\
&\quad \times V_{\underline{n}_1, \underline{n}_2} V_{\underline{n}_{\tau_1}, \underline{n}_{\tau_1+1}} V_{\underline{n}_{\tau_2}, \underline{n}_{\tau_2+1}} V_{\underline{n}_1, \underline{n}_{\tau_2}}^* V_{\underline{n}_{\tau_1+1}, \underline{n}_2}^* V_{\underline{n}_{\tau_1}, \underline{n}_{\tau_2+1}}^*.
\end{aligned} \tag{C3}$$

The right-hand side of Eq. (C3) naturally leads to the rules outlined in Appendix B.

APPENDIX D: PROOF THAT THE SPECTRAL RADIUS OF $\tilde{\mathcal{M}}$ IS LESS THAN 1

Geršgorin Circle Theorem. Let M be a complex $\mathcal{N} \times \mathcal{N}$ matrix, with entries $m_{i,j}$. For each row index $i \in \{1, \dots, \mathcal{N}\}$, define

$$R_i = \sum_{j \neq i} |m_{i,j}| \tag{D1}$$

as the sum of the absolute values of the nondiagonal entries in the i th row. Let $D(m_{i,i}, R_i) \subseteq \mathbb{C}$ be a closed disc centered at $m_{i,i}$ with radius R_i . Such a disc is called a Geršgorin disc. The theorem states that every eigenvalue of M lies within at least one of the Geršgorin discs $D(m_{i,i}, R_i)$ [60,61].

We use this theorem to show that the eigenvalues of the $\mathcal{N} \times \mathcal{N}$ matrix $\tilde{\mathcal{M}}$ have magnitudes less than 1. The matrix elements of $\tilde{\mathcal{M}}$ are

$$\tilde{\mathcal{M}}_{a,b} = V_{a,b} V_{b,a}^*, \tag{D2}$$

where $a, b = 1, \dots, \mathcal{N}$. Since V is a unitary matrix, all the columns of V form a complete set of orthonormal basis states. The same is true for all the rows as well. However, the row states are generally different from the column states for a nonsymmetric V . Next, we consider the sum of the absolute values of the elements of $\tilde{\mathcal{M}}$ along the a th row:

$$\sum_b |\tilde{\mathcal{M}}_{a,b}| = \sum_b |V_{a,b}| |V_{b,a}|. \tag{D3}$$

The right-hand side of the above expression can be interpreted as the Euclidean inner product between two real non-negative vectors normalized to unity, which are

$$\begin{pmatrix} |V_{a,1}| \\ |V_{a,2}| \\ \vdots \\ |V_{a,\mathcal{N}}| \end{pmatrix}, \quad \begin{pmatrix} |V_{1,a}| \\ |V_{2,a}| \\ \vdots \\ |V_{\mathcal{N},a}| \end{pmatrix}.$$

If the two vectors are distinct for each a , their inner product is always less than 1, i.e.,

$$\sum_b |V_{a,b}| |V_{b,a}| < 1 \Rightarrow \sum_b |\tilde{\mathcal{M}}_{a,b}| < 1. \tag{D4}$$

Thus, the radius of the Geršgorin disc, $R_a = \sum_{b \neq a} |\tilde{\mathcal{M}}_{a,b}| < 1 - |\tilde{\mathcal{M}}_{a,a}|$. The center of the disc is $\tilde{\mathcal{M}}_{a,a}$, and the farthest point of the a th Geršgorin disc from the origin of the complex plane has a radial distance from the origin given by $R_a + |\tilde{\mathcal{M}}_{a,a}| < 1$. Therefore, all the eigenvalues will have magnitude less than 1. If the two vectors are identical for some a , then the largest eigenvalue can be 1. Our numerical study of the spectra of $\tilde{\mathcal{M}}$ for Hamiltonian $\hat{H}_1$ in Eq. (36) with arbitrary complex hopping parameters reveals that the largest eigenvalue of $\tilde{\mathcal{M}}$ is 1 when $\hat{H}_1$ has a nonconventional time-reversal symmetry [42].

APPENDIX E: RULES TO CALCULATE $\mathcal{Y}_\pi$ AND $\mathcal{Y}_\pi^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}}$ FOR THE CUE CLASS

Here, we list the rules for finding $\mathcal{Y}_\pi$ and $\mathcal{Y}_\pi^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}}$ for the CUE class.

- (1) First, draw a diagram representing a permutation π (e.g., a subsequence reversal, as shown in Fig. 40).
- (2) Represent a state repeated twice by green circles at the corresponding time steps [e.g., Fig. 3(b)].
- (3) For a red arc of length n time steps with a counterclockwise arrow, insert a factor as

$$\begin{array}{c} n \text{ time steps} \\ \curvearrowleft \\ b \quad a \end{array} \equiv (\mathcal{M}^n)_{a,b} = \sum_{i=0}^{N-1} \lambda_i^n \mathcal{M}_{a,b}^{(i)}. \quad \text{The}$$

doubly stochastic matrix $\mathcal{M}$ is nonsymmetric due to the nonsymmetric nature of the matrix V . Thus, $\mathcal{M}^{(i)} = |\lambda_i\rangle_{RL} \langle \lambda_i|$, where $|\lambda_i\rangle_R$ and $|\lambda_i\rangle_L$ are the right and left eigenvectors of $\mathcal{M}$ with eigenvalue λ_i .

- (4) For a red arc of length n time steps with a clockwise arrow, insert a factor as

$$\begin{array}{c} n \text{ time steps} \\ \curvearrowright \\ b \quad a \end{array} \equiv (\tilde{\mathcal{M}}^n)_{a,b} = \sum_{i=0}^{N-1} \chi_i^n \tilde{\mathcal{M}}_{a,b}^{(i)}.$$

The matrix $\tilde{\mathcal{M}}$ is a complex Hermitian with eigenvalues, $\chi_0, \chi_1, \dots, \chi_{N-1}$, satisfying $1 > |\chi_0| \geq |\chi_1| \geq \dots \geq |\chi_{N-1}|$ (following the Geršgorin circle theorem [60,61] as we explain in Appendix D), and

$\tilde{\mathcal{M}}^{(i)} = |\chi_i\rangle \langle \chi_i|$, where $|\chi_i\rangle$ is an eigenvector of $\tilde{\mathcal{M}}$ corresponding to eigenvalue χ_i .

- (5) $a \dashrightarrow b \equiv V_{a,b}$, $a \dashleftarrow b \equiv V_{b,a}$.
- (6) $a \xrightarrow{\text{red}} b \equiv V_{a,b}^*$, $a \xleftarrow{\text{red}} b \equiv V_{b,a}^*$.
- (7) Take the sum over all the matrix indices, which are repeated twice or more. We give a derivation of the above rules in Appendix F. We can apply the above rules to evaluate the contribution $\mathcal{Y}_S$ of a subsequence reversal described in Fig. 40 as

$$\mathcal{Y}_S = \sum_{i,j=0}^{N-1} \lambda_i^{\nu_1} \chi_j^{\nu_2} Q_S^{i,j}, \quad (\text{E1})$$

where $Q_S^{i,j} = \mathcal{M}_{d,a}^{(i)} \tilde{\mathcal{M}}_{b,c}^{(j)} V_{a,b} V_{c,d} V_{a,c}^* V_{b,d}^*$, and ν_1, ν_2 are the same as in Appendix B. As mentioned in the rules above, the contributions of red arcs with counterclockwise and clockwise arrows are different for the CUE case. Therefore, we separate their indices by a semicolon, as in $Q_S^{i,j}$.

APPENDIX F: DERIVATION OF THE RULES TO EVALUATE $\mathcal{Y}_\pi$ AND $\mathcal{Y}_\pi^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}}$

We again consider the permutation diagram in Fig. 41. According to Eq. (25), the contribution to the SFF is

$$\begin{aligned} \mathcal{Y}_\pi = & \sum_{\underline{n}_1, \dots, \underline{n}_t} V_{\underline{n}_1, \underline{n}_2} \left(\prod_{\tau=2}^{\tau_1-1} V_{\underline{n}_\tau, \underline{n}_{\tau+1}} \right) V_{\underline{n}_{\tau_1}, \underline{n}_{\tau_1+1}} \left(\prod_{\tau=\tau_1+1}^{\tau_2-1} V_{\underline{n}_\tau, \underline{n}_{\tau+1}} \right) V_{\underline{n}_{\tau_2}, \underline{n}_{\tau_2+1}} \left(\prod_{\tau=\tau_2+1}^{t-1} V_{\underline{n}_\tau, \underline{n}_{\tau+1}} \right) V_{\underline{n}_t, \underline{n}_1} \\ & \times V_{\underline{n}_1, \underline{n}_{\tau_2}}^* \left(\prod_{\tau=\tau_1+1}^{\tau_2-1} V_{\underline{n}_{\tau+1}, \underline{n}_\tau}^* \right) V_{\underline{n}_{\tau_1+1}, \underline{n}_2}^* \left(\prod_{\tau=2}^{\tau_1-1} V_{\underline{n}_\tau, \underline{n}_{\tau+1}}^* \right) V_{\underline{n}_{\tau_1}, \underline{n}_{\tau_2+1}}^* \left(\prod_{\tau=\tau_2+1}^{t-1} V_{\underline{n}_\tau, \underline{n}_{\tau+1}}^* \right) V_{\underline{n}_t, \underline{n}_1}^*. \end{aligned} \quad (\text{F1})$$

Since the matrix V is not symmetric for the CUE case, we define $\tilde{\mathcal{M}}_{\underline{n}, \underline{n}'} = V_{\underline{n}, \underline{n}'} V_{\underline{n}', \underline{n}}^*$ and $\mathcal{M}_{\underline{n}, \underline{n}'} = V_{\underline{n}, \underline{n}'} V_{\underline{n}', \underline{n}}^*$. Thus, the above expression for $\mathcal{Y}_\pi$ reduces to

$$\begin{aligned} \mathcal{Y}_\pi = & \sum_{\underline{n}_1, \dots, \underline{n}_t} \left(\prod_{\tau=2}^{\tau_1-1} \mathcal{M}_{\underline{n}_\tau, \underline{n}_{\tau+1}} \right) \left(\prod_{\tau=\tau_1+1}^{\tau_2-1} \tilde{\mathcal{M}}_{\underline{n}_\tau, \underline{n}_{\tau+1}} \right) \left(\prod_{\tau=\tau_2+1}^{t-1} \mathcal{M}_{\underline{n}_\tau, \underline{n}_{\tau+1}} \right) \mathcal{M}_{\underline{n}_t, \underline{n}_1} \\ & \times V_{\underline{n}_1, \underline{n}_2} V_{\underline{n}_{\tau_1}, \underline{n}_{\tau_1+1}} V_{\underline{n}_{\tau_2}, \underline{n}_{\tau_2+1}} V_{\underline{n}_1, \underline{n}_{\tau_2}}^* V_{\underline{n}_{\tau_1+1}, \underline{n}_2}^* V_{\underline{n}_{\tau_1}, \underline{n}_{\tau_2+1}}^* \\ = & \sum_{\substack{\underline{n}_1, \underline{n}_2 \\ \underline{n}_{\tau_1}, \underline{n}_{\tau_1+1} \\ \underline{n}_{\tau_2}, \underline{n}_{\tau_2+1}}} (\mathcal{M}^{\tau_1-2})_{\underline{n}_2, \underline{n}_{\tau_1}} (\tilde{\mathcal{M}}^{\tau_2-\tau_1-1})_{\underline{n}_{\tau_1+1}, \underline{n}_{\tau_2}} (\mathcal{M}^{t-\tau_2})_{\underline{n}_{\tau_2+1}, \underline{n}_1} \\ & \times V_{\underline{n}_1, \underline{n}_2} V_{\underline{n}_{\tau_1}, \underline{n}_{\tau_1+1}} V_{\underline{n}_{\tau_2}, \underline{n}_{\tau_2+1}} V_{\underline{n}_1, \underline{n}_{\tau_2}}^* V_{\underline{n}_{\tau_1+1}, \underline{n}_2}^* V_{\underline{n}_{\tau_1}, \underline{n}_{\tau_2+1}}^* \\ = & \sum_{\substack{\underline{n}_1, \underline{n}_2 \\ \underline{n}_{\tau_1}, \underline{n}_{\tau_1+1} \\ \underline{n}_{\tau_2}, \underline{n}_{\tau_2+1}}} \left(\sum_i \lambda_i^{\tau_1-2} \mathcal{M}_{\underline{n}_2, \underline{n}_{\tau_1}}^{(i)} \right) \left(\sum_j \chi_j^{\tau_2-\tau_1-1} \tilde{\mathcal{M}}_{\underline{n}_{\tau_1+1}, \underline{n}_{\tau_2}}^{(j)} \right) \left(\sum_k \lambda_k^{t-\tau_2} \mathcal{M}_{\underline{n}_{\tau_2+1}, \underline{n}_1}^{(k)} \right) \\ & \times V_{\underline{n}_1, \underline{n}_2} V_{\underline{n}_{\tau_1}, \underline{n}_{\tau_1+1}} V_{\underline{n}_{\tau_2}, \underline{n}_{\tau_2+1}} V_{\underline{n}_1, \underline{n}_{\tau_2}}^* V_{\underline{n}_{\tau_1+1}, \underline{n}_2}^* V_{\underline{n}_{\tau_1}, \underline{n}_{\tau_2+1}}^*, \end{aligned} \quad (\text{F2})$$

where we use the eigendecomposition of $\mathcal{M}^n = \sum_i \lambda_i^n \mathcal{M}^{(i)}$ and $\tilde{\mathcal{M}}^n = \sum_i \chi_i^n \tilde{\mathcal{M}}^{(i)}$ in the last line, and $\mathcal{M}^{(i)} = |\lambda_i\rangle_{RL} \langle \lambda_i|$ and $\tilde{\mathcal{M}}^{(i)} = |\chi_i\rangle \langle \chi_i|$. The right-hand side in Eq. (F2) naturally leads to the rules in Appendix E.

APPENDIX G: RULES TO CALCULATE

$\mathcal{Z}_{\pi, \vec{\sigma}}, \mathcal{Z}_{\pi, \vec{\sigma}}^{\{n_{\tau_1}, n_{\tau_2}\}}$ AND $\mathcal{Z}_{\pi, \vec{\sigma}}^{\{n_{\tau_1}, T_{n_{\tau_2}}\}}$ FOR THE CSE CLASS

We give the rules for evaluating $\mathcal{Z}_{\pi, \vec{\sigma}}, \mathcal{Z}_{\pi, \vec{\sigma}}^{\{n_{r_1}, n_{r_2}\}}$, and $\mathcal{Z}_{\pi, \vec{\sigma}}^{\{n_{r_1}, \mathcal{T}_{n_{r_2}}\}}$ for the CSE class.

- (1) Draw a dashed blue circle representing the initial states $\{\underline{n}_1, \dots, \underline{n}_l\}$ and a concentric dashed black circle of smaller radius denoting the time-reversed states $\{\mathcal{T}\underline{n}_1, \dots, \mathcal{T}\underline{n}_l\}$. Draw the red curve representing a permutation π and a configuration $\vec{\sigma}$ (e.g., Fig. 42).
- (2) Draw green circles for a state repeated twice at the corresponding time steps [e.g., Fig. 4(c)]. Draw green and white circles for two states related by $\mathcal{T}$ at the corresponding time steps [e.g., Fig. 4(d)].
- (3) For a red arc of length n time steps with a counterclockwise arrow on the outer circle, insert a factor as

$$\overset{n \text{ time steps}}{\underset{b}{\curvearrowright}} \overset{a}{\curvearrowright} \equiv (\mathcal{M}^n)_{a,b} = \sum_{i=0}^{\mathcal{N}-1} \lambda_i^n \mathcal{M}_{a,b}^{(i)} \quad \text{The}$$
 doubly stochastic matrix $\mathcal{M}$ is nonsymmetric due to the nonsymmetric nature of the matrix V ; thus, $\mathcal{M}^{(i)} = |\lambda_i\rangle_{RL} \langle \lambda_i|$, where $|\lambda_i\rangle_R$ and ${}_L \langle \lambda_i|$ are, respectively, the right and left eigenvectors of $\mathcal{M}$ corresponding to eigenvalue λ_i .

- (4) For a red arc of length n time steps with a clockwise arrow on the outer circle, insert a factor as

$$\begin{array}{c} n \text{ time steps} \\ \xrightarrow{\text{red arc}} \\ b \quad \quad \quad a \\ \text{dashed arc} \end{array} \equiv \left(\tilde{\mathcal{M}}^n \right)_{a,b} = \sum_{i=0}^{N-1} \chi_i^n \tilde{\mathcal{M}}_{a,b}^{(i)}.$$

The matrix $\tilde{\mathcal{M}}$ is complex Hermitian with eigenvalues $\chi_0, \dots, \chi_{N-1}$ satisfying $1 > |\chi_0| \geq |\chi_1| \geq \dots \geq |\chi_{N-1}|$ (again following the Geršgorin circle theorem as discussed in Appendix D), and $\tilde{\mathcal{M}}^{(i)} = |\chi_i\rangle\langle\chi_i|$, where $|\chi_i\rangle$ is an eigenvector of $\tilde{\mathcal{M}}$ corresponding to eigenvalue χ_i .

- (5) For a red arc with a clockwise arrow on the inner circle, insert a factor

as  $\equiv (\mathcal{M}^n)_{a,b} = \sum_{i=0}^{N-1} \lambda_i^n \mathcal{M}_{a,b}^{(i)}$

- (6) For a red arc with a counterclockwise arrow on the inner circle, insert a factor

as  $\equiv (\tilde{\mathcal{M}}^n)_{a,b} = \sum_{i=0}^{\mathcal{N}-1} \chi_i^n \tilde{\mathcal{M}}_{a,b}^{(i)}$.

- (7) $a \overset{\text{blue}}{\dashrightarrow} b \equiv V_{a,b}, \quad a \overset{\text{blue}}{\dashleftarrow} b \equiv V_{b,a}.$
- (8) $a \overset{\text{red}}{\rightarrow} b \equiv V_{a,b}^*, \quad a \overset{\text{red}}{\leftarrow} b \equiv V_{b,a}^*.$
- (9) Take the sum over all the matrix indices, which are

repeated twice or more. Since the matrix V is nonsymmetric in the computational basis, the rules here are identical to those in Appendix E for systems belonging to the CUE class, except the rules (v) and (vi), which follow naturally from Theorem 1. Following the above rules, the contribution $\mathcal{Z}_{T\bar{a}}$ of a

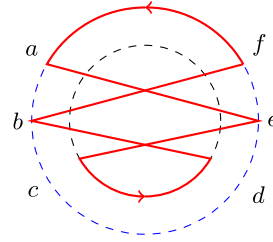


FIG. 42. Diagram representing a transposition of states at time steps τ_1 and τ_2 with $\vec{\sigma} = (\sigma_1, \dots, \sigma_t)$ such that $\sigma_\tau = 1$ for $\tau \in [\tau_1 + 1, \tau_2 - 1]$ and $\sigma_\tau = 0$ for $\tau \notin [\tau_1 + 1, \tau_2 - 1]$. The labels at the vertices denote states on the blue circle at those time steps, $a \equiv \underline{n}_{\tau_1-1}, b \equiv \underline{n}_{\tau_1}, c \equiv \underline{n}_{\tau_1+1}, d \equiv \underline{n}_{\tau_2-1}, e \equiv \underline{n}_{\tau_2}, f \equiv \underline{n}_{\tau_2+1}$. The labels b and e represent the transposed states.

transposition with configuration $\vec{\sigma}$, as shown in Fig. 42, can be expressed as

$$\mathcal{Z}_{T,\vec{\sigma}} = \sum_{i,j=0}^{N-1} \lambda_i^{\nu_1} \chi_j^{\nu_2} Q_T^{i;j}, \quad (\text{G1})$$

where

$$Q_T^{i,j} = \mathcal{M}_{f,a}^{(i)} \tilde{\mathcal{M}}_{c,d}^{(j)} V_{a,b} V_{b,c} V_{d,e} V_{e,f} \times V_{ae}^* V_{e\mathcal{T}c}^* V_{\mathcal{T}d b}^* V_{b,f}^*, \quad (\text{G2})$$

and ν_1, ν_2 are the same as in Appendix B. In general, i_1, i_2, i_3 , and i_4 in the superscript of $Q_\pi^{i_1 i_2 i_3 i_4}$ represent, respectively, the indices coming from red arcs with a counterclockwise arrow on the outer circle, the indices coming from red arcs with a clockwise arrow on the outer circle, the indices coming from red arcs with a counterclockwise arrow on the inner circle, and the indices coming from red arcs with a clockwise arrow on the inner circle. Since Fig. 42 does not contain arcs with a clockwise arrow on the inner or outer circle, $Q_T^{i i j}$ contains empty slots, separated by semicolons, for such arcs.

APPENDIX H: DERIVATION OF THE SECOND-ORDER CORRECTION FOR THE COE CLASS USING ONLY THE RULES IN APPENDIX B

1. Transposition (T)

A single transposition leads to three kinds of diagrams as shown in Fig. 43: (a) a transposition between nearest-neighbor states ($T^{(1)}$), (b) a transposition between next-nearest-neighbor states ($T^{(2)}$), and (c) a transposition of all other pairs of states (T'). Following the rules in Appendix B, we can find the contribution of the transpositions as

$$\mathcal{X}_{T'} = \left(\sum_i \lambda_i^{\tau_2 - \tau_1 - 2} \mathcal{M}_{c,d}^{(i)} \right) \left(\sum_j \lambda_j^{t - \tau_2 + \tau_1 - 2} \mathcal{M}_{f,a}^{(j)} \right) \times V_{a,b} V_{b,c} V_{d,e} V_{e,f} V_{a,e}^* V_{e,c}^* V_{d,b}^* V_{b,f}^*, \quad (\text{H1})$$

$$\mathcal{X}_{T^{(1)}} = \left(\sum_i \lambda_i^{t-3} \mathcal{M}_{d,a}^{(i)} \right) V_{a,b} \mathcal{M}_{b,c} V_{c,d} V_{a,c}^* V_{b,d}^*, \quad (\text{H2})$$

$$\mathcal{X}_{T^{(2)}} = \left(\sum_i \lambda_i^{t-4} \mathcal{M}_{e,a}^{(i)} \right) V_{a,b} (\mathcal{M}^2)_{b,d} V_{d,e} V_{a,d}^* V_{b,e}^*. \quad (\text{H3})$$

We separately calculate the contribution of different single transpositions and their cyclic and anticyclic variants by summing over allowed values of τ_1, τ_2 and multiplying by a factor of $2t$, respectively. For each case of transposition, $1 \leq \tau_1 \leq t$. Since each arc has a minimum size of 1 and we have the PBC in time, $\tau_1 + 3 \leq \tau_2 \leq t + \tau_1 - 3$ for T' . Therefore,

$$\bar{\mathcal{X}}_{T'} = (2t) \frac{1}{2} \sum_{\tau_1=1}^t \sum_{\tau_2=\tau_1+3}^{t+\tau_1-3} \sum_{ij} \lambda_i^{\tau_2 - \tau_1 - 2} \lambda_j^{t - \tau_2 + \tau_1 - 2} Q_{T'}^{ij}, \quad (\text{H4})$$

$$\bar{\mathcal{X}}_{T^{(1)}} = (2t) \sum_{\tau_1=1}^t \sum_i \lambda_i^{t-3} Q_{T^{(1)}}^i, \quad (\text{H5})$$

$$\bar{\mathcal{X}}_{T^{(2)}} = (2t) \sum_{\tau_1=1}^t \sum_i \lambda_i^{t-4} Q_{T^{(2)}}^i, \quad (\text{H6})$$

where

$$Q_{T'}^{ij} = \mathcal{M}_{c,d}^{(i)} \mathcal{M}_{d,a}^{(j)} V_{a,b} V_{b,c} V_{d,e} V_{e,f} V_{a,e}^* V_{e,c}^* V_{d,b}^* V_{b,f}^*, \quad (\text{H7})$$

$$Q_{T^{(1)}}^i = \mathcal{M}_{d,a}^{(i)} V_{a,b} \mathcal{M}_{b,c} V_{c,d} V_{a,c}^* V_{b,d}^*, \quad (\text{H8})$$

$$Q_{T^{(2)}}^i = \mathcal{M}_{e,a}^{(i)} V_{a,b} (\mathcal{M}^2)_{b,d} V_{d,e} V_{a,d}^* V_{b,e}^*. \quad (\text{H9})$$

A factor of $1/2$ is inserted in Eq. (H4) to avoid double counting, as changing the order of τ_1, τ_2 does not lead to a new transposition. We need the following relations for our next step:

$$\mathcal{M}_{a,b}^{(0)} = \frac{1}{\mathcal{N}}, \quad (\text{H10})$$

$$\mathcal{M}^{(i)} \mathcal{M}^{(j)} = \mathcal{M}^{(i)} \delta_{ij}, \quad (\text{H11})$$

$$\text{tr} \mathcal{M}^{(i)} = 1. \quad (\text{H12})$$

Applying these relations along with the unitary property of the matrix V , we obtain

$$Q_{T'}^{00} = \frac{1}{\mathcal{N}}, \quad (\text{H13})$$

$$Q_{T'}^{i0} = Q_{T'}^{0i} = \frac{\lambda_i^2}{\mathcal{N}}, \quad i = 1, 2, \dots, \mathcal{N} - 1 \quad (\text{H14})$$

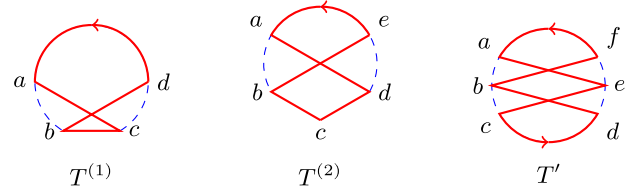


FIG. 43. State b at time τ_1 and state c at time $\tau_1 + 1$ interchanged in transposition $T^{(1)}$; state b at time τ_1 and state d at time $\tau_1 + 2$ interchanged in transposition $T^{(2)}$; and state b at time τ_1 and state e at time τ_2 interchanged in transposition T' .

$$Q_{T^{(1)}}^{(0)} = \frac{1}{\mathcal{N}} \sum_i \lambda_i, \quad (\text{H15})$$

$$Q_{T^{(2)}}^{(0)} = \frac{1}{\mathcal{N}} \sum_i \lambda_i^2. \quad (\text{H16})$$

We next substitute Eqs. (H13) and (H14) in Eq. (H4), Eq. (H15) in Eq. (H5), and Eq. (H16) in Eq. (H6) to find

$$\bar{\mathcal{X}}_{T'} = \frac{t^2(t-5)}{\mathcal{N}} + \sum_{i \neq 0} \frac{2t^2(\lambda_i^3 - \lambda_i^{t-2})}{\mathcal{N}(1 - \lambda_i)} + t^2 \sum_{\substack{i \neq 0 \\ j \neq 0}} \frac{\lambda_i^{t-4} \lambda_j - \lambda_i \lambda_j^{t-4}}{\lambda_i - \lambda_j} Q_{T'}^{ij}, \quad (\text{H17})$$

$$\bar{\mathcal{X}}_{T^{(1)}} = \frac{2t^2}{\mathcal{N}} + \frac{2t^2}{\mathcal{N}} \sum_{i \neq 0} \lambda_i + 2t^2 \sum_{i \neq 0} \lambda_i^{t-3} Q_{T^{(1)}}^i, \quad (\text{H18})$$

$$\bar{\mathcal{X}}_{T^{(2)}} = \frac{2t^2}{\mathcal{N}} + \frac{2t^2}{\mathcal{N}} \sum_{i \neq 0} \lambda_i^2 + 2t^2 \sum_{i \neq 0} \lambda_i^{t-4} Q_{T^{(2)}}^i. \quad (\text{H19})$$

We find the total contribution of all the single transpositions (T) by adding the contributions in Eqs. (H17)–(H19):

$$\begin{aligned} \bar{\mathcal{X}}_T &= \frac{t^2(t-1)}{\mathcal{N}} + \frac{2t^2}{\mathcal{N}} \sum_{i \neq 0} \frac{\lambda_i}{1 - \lambda_i} \\ &+ t^2 \sum_{i \neq 0} \left(-\frac{2\lambda_i^{t-2}}{\mathcal{N}(1 - \lambda_i)} + 2\lambda_i^{t-3} Q_{T^{(1)}}^i + 2\lambda_i^{t-4} Q_{T^{(2)}}^i \right) \\ &+ t^2 \sum_{\substack{i \neq 0 \\ j \neq 0}} \frac{\lambda_i^{t-4} \lambda_j - \lambda_i \lambda_j^{t-4}}{\lambda_i - \lambda_j} Q_{T'}^{ij}. \end{aligned} \quad (\text{H20})$$

2. Subsequence reversal (S)

A subsequence reversal permutation reverses the order of some consecutive states. Let us consider a configuration of states on the blue circle for $t = 9$ as $\{\underline{n}_1, \underline{n}_2, \underline{n}_3, \underline{n}_4, \underline{n}_5, \underline{n}_6, \underline{n}_7, \underline{n}_8, \underline{n}_9\}$. The configuration $\{\underline{n}_1, \underline{n}_2, \underline{n}_6, \underline{n}_5, \underline{n}_4, \underline{n}_3, \underline{n}_7, \underline{n}_8, \underline{n}_9\}$ can be obtained by performing a subsequence reversal from time steps 3 to 6. A subsequence reversal is shown diagrammatically in Fig. 44.

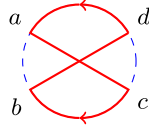


FIG. 44. Diagram representing subsequence reversal, where the order of states from time step τ_1 to τ_2 is reversed and the states at time steps τ_1 and τ_2 are denoted by b and c , respectively.

We apply the rules in Appendix B to find

$$\mathcal{X}_S = \sum_{ij} \lambda_i^{\tau_2 - \tau_1} \lambda_j^{t - \tau_2 + \tau_1 - 2} Q_S^{ij}, \quad (\text{H21})$$

where

$$Q_S^{ij} = \mathcal{M}_{b,c}^{(i)} \mathcal{M}_{d,a}^{(j)} V_{a,b} V_{c,d} V_{a,c}^* V_{b,d}^*. \quad (\text{H22})$$

To determine the contribution of all the S diagrams, we sum over all the allowed values of τ_1 and τ_2 . Naturally, $1 \leq \tau_1 \leq t$; however, we observe that the S diagrams resemble those of nearest-neighbor and next-nearest-neighbor transpositions. Therefore, the allowed values of τ_2 should be such that the S diagrams do not overlap with nearest-neighbor and next-nearest-neighbor transposition diagrams, which leads to $\tau_1 + 3 \leq \tau_2 \leq t + \tau_1 - 5$. Additionally, we observe that changing the order of τ_1 , τ_2 leads to a different subsequence reversal, as shown in Fig. 45(b). However, the new subsequence reversal is just an anticyclic variant, shown in Fig. 45(d), of a subsequence reversal from $\tau_1 + 1$ to $\tau_2 - 1$, shown in Fig. 45(c). Since we sum over all allowed values of τ_1 , τ_2 and explicitly include contributions of cyclic and anticyclic variants by including a factor of $2t$, we must include a factor of $1/2$ while summing the contribution of all the S diagrams:

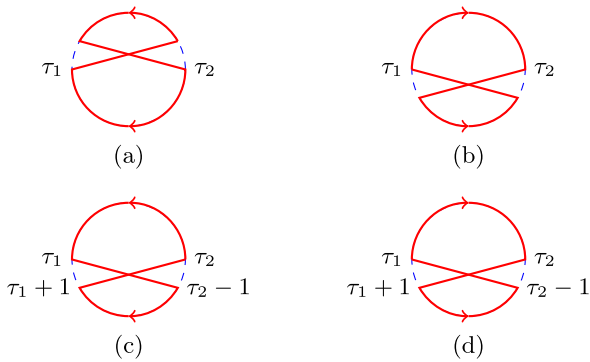


FIG. 45. (a) An S diagram where the order of states from time step τ_1 to τ_2 is reversed. (b) An S diagram where order of states from time step τ_2 to $t + \tau_1 (\equiv \tau_1)$ is reversed. (c) Another S diagram in which the order of states from time step $\tau_1 + 1$ to $\tau_2 - 1$ is reversed. Its anticyclic variant is shown in (d), which matches exactly the diagram in (b). Therefore, changing the order of τ_1 and τ_2 does not lead to extra permutation when cyclic and anticyclic variants are taken care of.

$$\begin{aligned} \bar{\mathcal{X}}_S &= (2t) \frac{1}{2} \sum_{\tau_1=1}^t \sum_{\tau_2=\tau_1+3}^{t+\tau_1-5} \mathcal{X}_S \\ &= t^2 \sum_{ij} \frac{\lambda_i^{t-4} \lambda_j^3 - \lambda_i^3 \lambda_j^{t-4}}{\lambda_i - \lambda_j} Q_S^{ij} \\ &= \frac{t^2(t-7)}{\mathcal{N}} + 2t^2 \sum_{i \neq 0} \frac{\lambda_i^3}{\mathcal{N}(1-\lambda_i)} \\ &\quad - 2t^2 \sum_{i \neq 0} \frac{\lambda_i^{t-3}}{\mathcal{N}(1-\lambda_i)} + t^2 \sum_{\substack{i \neq 0 \\ j \neq 0}} \frac{\lambda_i^{t-4} \lambda_j^3 - \lambda_i^3 \lambda_j^{t-4}}{\lambda_i - \lambda_j} Q_S^{ij}, \end{aligned} \quad (\text{H23})$$

where we have used $Q_S^{00} = 1/\mathcal{N}$, $Q_S^{i0} = Q_S^{0i} = 1/\mathcal{N}$.

3. Identity permutation with repetition (R)

The contribution of the R diagram in Fig. 46 can be found using the rules in Appendix B. It is

$$\begin{aligned} \mathcal{X}_I^{\{\underline{n}_1, \underline{n}_2\}} &= \left(\sum_i \lambda_i^{\tau_2 - \tau_1} \mathcal{M}_{a,a}^{(i)} \right) \left(\sum_j \lambda_j^{\tau_2 - \tau_1} \mathcal{M}_{a,a}^{(j)} \right) \\ &= \sum_{ij} \lambda_i^{\tau_2 - \tau_1} \lambda_j^{t - \tau_2 + \tau_1} Q_R^{ij}, \end{aligned} \quad (\text{H24})$$

where

$$Q_R^{ij} = \mathcal{M}_{a,a}^{(i)} \mathcal{M}_{a,a}^{(j)}. \quad (\text{H25})$$

For this case, all possible combinations of τ_1 , τ_2 are allowed. Thus, $1 \leq \tau_1 \leq t$ and $\tau_1 + 1 \leq \tau_2 \leq t + \tau_1 - 1$. We also include a factor of $1/2$ to avoid double counting of pairs of τ_1 , τ_2 :

$$\begin{aligned} \bar{\mathcal{X}}_I^{\{\underline{n}_1, \underline{n}_2\}} &= (2t) \frac{1}{2} \sum_{\tau_1=1}^t \sum_{\tau_2=\tau_1+1}^{t+\tau_1-1} \mathcal{X}_I^{\{\underline{n}_1, \underline{n}_2\}} \\ &= (2t) \frac{1}{2} \sum_{\tau_1=1}^t \sum_{\tau_2=\tau_1+1}^{t+\tau_1-1} \sum_{ij} \lambda_i^{\tau_2 - \tau_1} \lambda_j^{t - \tau_2 + \tau_1} Q_R^{ij} \\ &= t^2 \sum_{ij} \frac{\lambda_i^t \lambda_j - \lambda_i \lambda_j^t}{\lambda_i - \lambda_j} Q_R^{ij}. \end{aligned} \quad (\text{H26})$$

We have $Q_R^{00} = 1/\mathcal{N}$ and $Q_R^{0i} = Q_R^{i0} = 1/\mathcal{N}$ for $i = 1, \dots, \mathcal{N} - 1$. Thus,

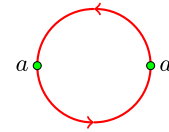


FIG. 46. Diagram representing identity permutation with repetition, where the states at time steps τ_1 and τ_2 are the same and referred to as a .

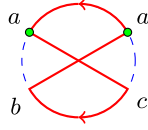


FIG. 47. Diagram representing subsequence reversal with repetition, where the order of states from time steps τ_1 to τ_2 is reversed and the states at time steps τ_1 and τ_2 are the same and denoted by a .

$$\begin{aligned} \bar{\chi}_I^{\{\underline{n}, \underline{n}\}} &= \frac{t^2(t-1)}{\mathcal{N}} + 2t^2 \sum_{i \neq 0} \frac{\lambda_i}{\mathcal{N}(1-\lambda_i)} \\ &\quad - 2t^2 \sum_{i \neq 0} \frac{\lambda_i^t}{\mathcal{N}(1-\lambda_i)} + t^2 \sum_{\substack{i \neq 0 \\ j \neq 0}} \frac{\lambda_i^t \lambda_j - \lambda_i \lambda_j^t}{\lambda_i - \lambda_j} Q_{SR}^{ij}. \end{aligned} \quad (\text{H27})$$

4. Subsequence reversal with repetition (SR)

The contribution of the SR diagram in Fig. 47 can be calculated using the rules in Appendix B. We obtain

$$\chi_S^{\{\underline{n}_1, \underline{n}_2\}} = \sum_{ij} \lambda_i^{\tau_2 - \tau_1 - 2} \lambda_j^{t - \tau_2 + \tau_1} Q_{SR}^{ij}, \quad (\text{H28})$$

where

$$Q_{SR}^{ij} = \sum_{a,b,c} \mathcal{M}_{c,b}^{(i)} \mathcal{M}_{a,a}^{(j)} V_{a,b} V_{c,a} V_{a,c}^* V_{b,a}^*. \quad (\text{H29})$$

The allowed values of τ_1 and τ_2 are $1 \leq \tau_1 \leq t, \tau_1 + 3 \leq$

$\tau_2 \leq t + \tau_1 - 3$. Therefore, the total contribution is

$$\begin{aligned} \bar{\chi}_S^{\{\underline{n}, \underline{n}\}} &= (2t) \frac{1}{2} \sum_{\tau_1=1}^t \sum_{\tau_2=\tau_1+3}^{t+\tau_1-3} \chi_S^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}} \\ &= (2t) \frac{1}{2} \sum_{\tau_1=1}^t \sum_{\tau_2=\tau_1+3}^{t+\tau_1-3} \sum_{ij} \lambda_i^{\tau_2 - \tau_1 - 2} \lambda_j^{t - \tau_2 + \tau_1} Q_{SR}^{ij} \\ &= t^2 \sum_{ij} \frac{\lambda_i^{t-4} \lambda_j^3 - \lambda_i \lambda_j^{t-2}}{\lambda_i - \lambda_j} Q_{SR}^{ij}. \end{aligned} \quad (\text{H30})$$

Using $Q_{SR}^{00} = 1/\mathcal{N}$, $Q_{SR}^{0i} = Q_{SR}^{i0} = 1/\mathcal{N}$, we find

$$\begin{aligned} \bar{\chi}_S^{\{\underline{n}, \underline{n}\}} &= \frac{t^2(t-5)}{\mathcal{N}} + 2t^2 \sum_{i \neq 0} \frac{\lambda_i^3}{\mathcal{N}(1-\lambda_i)} \\ &\quad - 2t^2 \sum_{i \neq 0} \frac{\lambda_i^{t-2}}{\mathcal{N}(1-\lambda_i)} \\ &\quad + t^2 \sum_{\substack{i \neq 0 \\ j \neq 0}} \frac{\lambda_i^{t-4} \lambda_j^3 - \lambda_i \lambda_j^{t-2}}{\lambda_i - \lambda_j} Q_{SR}^{ij}. \end{aligned} \quad (\text{H31})$$

The second-order correction to the SFF is

$$K_1^{(2)}(t) = \bar{\chi}_T + \bar{\chi}_S - \bar{\chi}_I^{\{\underline{n}, \underline{n}\}} - \bar{\chi}_S^{\{\underline{n}, \underline{n}\}}. \quad (\text{H32})$$

Substituting Eqs. (H20), (H23), (H27), and (H31) in Eq. (H32), we obtain

$$\begin{aligned} K_1^{(2)}(t) &= -\frac{2t^2}{\mathcal{N}} + 2t^2 \sum_{i \neq 0} \left(\lambda_i^{t-3} Q_{T(1)}^i + \lambda_i^{t-4} Q_{T(2)}^i - \frac{\lambda_i^{t-3}}{\mathcal{N}(1-\lambda_i)} + \frac{\lambda_i^t}{\mathcal{N}(1-\lambda_i)} \right) \\ &\quad + t^2 \sum_{\substack{i \neq 0 \\ j \neq 0}} \left(\frac{\lambda_i^{t-4} \lambda_j - \lambda_i \lambda_j^{t-4}}{\lambda_i - \lambda_j} Q_{T'}^{ij} + \frac{\lambda_i^{t-4} \lambda_j^3 - \lambda_i \lambda_j^{t-4}}{\lambda_i - \lambda_j} Q_S^{ij} - \frac{\lambda_i^t \lambda_j - \lambda_i \lambda_j^t}{\lambda_i - \lambda_j} Q_R^{ij} - \frac{\lambda_i^{t-4} \lambda_j^3 - \lambda_i \lambda_j^{t-2}}{\lambda_i - \lambda_j} Q_{SR}^{ij} \right) \\ &= -\frac{2t^2}{\mathcal{N}} + \mathcal{O}(\lambda_1^t). \end{aligned} \quad (\text{H33})$$

We include the contribution of all the Type III terms in Eq. (H33), which we ignore in our calculation with the reduced diagrams in Eq. (69).

APPENDIX I: DERIVATION OF THE SECOND-ORDER CORRECTION FOR THE CUE CLASS USING ONLY THE RULES IN APPENDIX E

1. Transposition

Following the rules in Appendix E, the calculation for the cyclic variants of transposition is carried out in a similar

way to the COE case.

$$\begin{aligned} \sum_{l=0}^{t-1} \sum_T \mathcal{Y}_{C^l T} &= \frac{t^2(t-1)}{2\mathcal{N}} + \frac{t^2}{\mathcal{N}} \sum_{i \neq 0} \frac{\lambda_i - \lambda_i^{t-2}}{1 - \lambda_i} \\ &\quad + t^2 \sum_{i \neq 0} (\lambda_i^{t-3} Q_{T(1)}^i + \lambda_i^{t-4} Q_{T(2)}^i) \\ &\quad + \frac{t^2}{2} \sum_{\substack{i \neq 0 \\ j \neq 0}} \frac{\lambda_i^{t-4} \lambda_j - \lambda_i \lambda_j^{t-4}}{\lambda_i - \lambda_j} Q_{T'}^{ij}. \end{aligned} \quad (\text{I1})$$

The contribution from the anticyclic variants is

$$\tilde{Q}_{T'}^{ij} = \tilde{\mathcal{M}}_{f,a}^{(i)} \tilde{\mathcal{M}}_{c,d}^{(j)} V_{a,b} V_{b,c} V_{d,e} V_{e,f} V_{e,a}^* V_{c,e}^* V_{b,d}^* V_{f,b}^*, \quad (I3)$$

$$\begin{aligned} \sum_{l=0}^{t-1} \sum_T \mathcal{Y}_{\mathcal{RC}^l T} &= \frac{t^2}{2} \sum_{ij} \frac{\chi_i^{t-4} \chi_j - \chi_i \chi_j^{t-4}}{\chi_i - \chi_j} \tilde{Q}_{T'}^{ij} \\ &+ t^2 \sum_i \chi_i^{t-3} \tilde{Q}_{T^{(1)}}^i + t^2 \sum_i \chi_i^{t-4} \tilde{Q}_{T^{(2)}}^i, \end{aligned} \quad (I2)$$

$$\tilde{Q}_{T^{(1)}}^i = \tilde{\mathcal{M}}_{d,a}^{(i)} V_{a,b} V_{b,c} V_{c,d} V_{c,a}^* V_{b,c}^* V_{d,b}^*, \quad (I4)$$

$$\tilde{Q}_{T^{(2)}}^i = \tilde{\mathcal{M}}_{e,a}^{(i)} V_{a,b} V_{b,c} V_{c,d} V_{d,e} V_{d,a}^* V_{c,d}^* V_{b,c}^* V_{e,b}^*. \quad (I5)$$

where

The total contribution from all single transposition diagrams is then

$$\begin{aligned} \bar{\mathcal{Y}}_T &= \sum_{l=0}^{t-1} \sum_T \mathcal{Y}_{\mathcal{C}^l T} + \sum_{l=0}^{t-1} \sum_T \mathcal{Y}_{\mathcal{RC}^l T} \\ &= \frac{t^2(t-1)}{2\mathcal{N}} + \frac{t^2}{\mathcal{N}} \sum_{i \neq 0} \frac{\lambda_i}{1-\lambda_i} + t^2 \sum_{i \neq 0} \left(-\frac{\lambda_i^{t-2}}{\mathcal{N}(1-\lambda_i)} + \lambda_i^{t-3} Q_{T^{(1)}}^i + \lambda_i^{t-4} Q_{T^{(2)}}^i \right) + \frac{t^2}{2} \sum_{\substack{i \neq 0 \\ j \neq 0}} \frac{\lambda_i^{t-4} \lambda_j - \lambda_i \lambda_j^{t-4}}{\lambda_i - \lambda_j} Q_{T'}^{ij} \\ &+ \frac{t^2}{2} \sum_{ij} \frac{\chi_i^{t-4} \chi_j - \chi_i \chi_j^{t-4}}{\chi_i - \chi_j} \tilde{Q}_{T'}^{ij} + t^2 \sum_i \chi_i^{t-3} \tilde{Q}_{T^{(1)}}^i + t^2 \sum_i \chi_i^{t-4} \tilde{Q}_{T^{(2)}}^i. \end{aligned} \quad (I6)$$

2. Subsequence reversal

The contribution of the subsequence reversal diagrams can be found again following the rules in Appendix E as

$$\begin{aligned} \bar{\mathcal{Y}}_S &= \frac{t}{2} \sum_{\tau_1=1}^t \sum_{\tau_2=\tau_1+3}^{t+\tau_1-5} \sum_{ij} \chi_i^{\tau_2-\tau_1} \lambda_j^{t-\tau_2+\tau_1-2} Q_S^{ij} + \frac{t}{2} \sum_{\tau_1=1}^t \sum_{\tau_2=\tau_1+3}^{t+\tau_1-5} \sum_{ij} \lambda_i^{\tau_2-\tau_1} \chi_j^{t-\tau_2+\tau_1-2} \tilde{Q}_S^{ij} \\ &= \frac{t^2}{2} \sum_{ij} \frac{\chi_i^{t-4} \lambda_j^3 - \chi_i^3 \lambda_j^{t-4}}{\chi_i - \lambda_j} Q_S^{ij} + \frac{t^2}{2} \sum_{ij} \frac{\lambda_i^{t-4} \chi_j^3 - \lambda_i^3 \chi_j^{t-4}}{\lambda_i - \chi_j} \tilde{Q}_S^{ij}, \end{aligned} \quad (I7)$$

where

$$Q_S^{ij} = \tilde{\mathcal{M}}_{b,c}^{(i)} \mathcal{M}_{d,a}^{(j)} V_{a,b} V_{c,d} V_{a,c}^* V_{b,d}^*, \quad (I8)$$

$$\tilde{Q}_S^{ij} = \mathcal{M}_{b,c}^{(i)} \tilde{\mathcal{M}}_{d,a}^{(j)} V_{a,b} V_{c,d} V_{c,a}^* V_{d,b}^*. \quad (I9)$$

It can be shown that $Q_{SR}^{i0} = \tilde{Q}_{SR}^{0i} = \text{tr} \tilde{\mathcal{M}}^{(i)} / \mathcal{N} = 1/\mathcal{N}$. Thus, we obtain

$$\bar{\mathcal{Y}}_S = \frac{t^2}{\mathcal{N}} \sum_i \frac{\chi_i^{t-2} - \chi_i^3}{\chi_i - 1} + \frac{t^2}{2} \sum_i \sum_{j \neq 0} \frac{\chi_i^{t-4} \lambda_j^3 - \chi_i^3 \lambda_j^{t-4}}{\chi_i - \lambda_j} Q_S^{ij} + \frac{t^2}{2} \sum_{i \neq 0} \sum_j \frac{\lambda_i^{t-4} \chi_j^3 - \lambda_i^3 \chi_j^{t-4}}{\lambda_i - \chi_j} \tilde{Q}_S^{ij}. \quad (I10)$$

3. Identity permutation with repetition

The calculation of the cyclic variants of repetitions is also similar to the COE case,

$$\begin{aligned} \sum_{l=0}^{t-1} \sum_R \mathcal{Y}_{\mathcal{C}^l I}^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}} &= \frac{t^2(t-1)}{2\mathcal{N}} + \frac{t^2}{\mathcal{N}} \sum_{i \neq 0} \frac{\lambda_i - \lambda_i^t}{1-\lambda_i} \\ &+ \frac{t^2}{2} \sum_{\substack{i \neq 0 \\ j \neq 0}} \frac{\lambda_i^t \lambda_j - \lambda_i \lambda_j^t}{\lambda_i - \lambda_j} Q_R^{ij}. \end{aligned} \quad (I11)$$

Following the rules in Appendix E, the total contribution of all the anticyclic variants takes the following form:

$$\sum_{l=0}^{t-1} \sum_R \mathcal{Y}_{\mathcal{RC}^l I}^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}} = \frac{t^2}{2} \sum_{ij} \frac{\chi_i^t \chi_j - \chi_i \chi_j^t}{\chi_i - \chi_j} \tilde{Q}_R^{ij}, \quad (I12)$$

where

$$\tilde{Q}_R^{ij} = \sum_a \tilde{\mathcal{M}}_{a,a}^{(i)} \tilde{\mathcal{M}}_{a,a}^{(j)}. \quad (I13)$$

Thus, we obtain the total contribution from the cyclic and anticyclic variants as

$$\begin{aligned}\bar{\mathcal{Y}}_R^{\{\underline{n}, \underline{n}\}} &= \sum_{l=0}^{t-1} \sum_R \mathcal{Y}_{C^l I}^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}} + \sum_{l=0}^{t-1} \sum_R \mathcal{Y}_{RC^l I}^{\{\underline{n}_{\tau_1}, \underline{n}_{\tau_2}\}} \\ &= \frac{t^2(t-1)}{2\mathcal{N}} + \frac{t^2}{\mathcal{N}} \sum_{i \neq 0} \frac{\lambda_i}{1-\lambda_i} - \frac{t^2}{\mathcal{N}} \sum_{i \neq 0} \frac{\lambda_i^t}{1-\lambda_i} \\ &\quad + \frac{t^2}{2} \sum_{\substack{i \neq 0 \\ j \neq 0}} \frac{\lambda_i^t \lambda_j - \lambda_i \lambda_j^t}{\lambda_i - \lambda_j} Q_R^{ij} + \frac{t^2}{2} \sum_{ij} \frac{\chi_i^t \chi_j - \chi_i \chi_j^t}{\chi_i - \chi_j} \tilde{Q}_R^{ij}.\end{aligned}\quad (I14)$$

$$\begin{aligned}Q_{SR}^{ij} &= \tilde{\mathcal{M}}_{b,c}^{(i)} \mathcal{M}_{a,a}^{(j)} V_{a,b} V_{c,a} V_{a,c}^* V_{b,a}^* \\ &= (\tilde{\mathcal{M}} \tilde{\mathcal{M}}^{(i)} \tilde{\mathcal{M}})_{a,a} \mathcal{M}_{a,a}^{(j)} = \chi_i^2 \tilde{\mathcal{M}}_{a,a}^{(i)} \mathcal{M}_{a,a}^{(j)},\end{aligned}\quad (I16)$$

$$\tilde{Q}_{SR}^{ij} = \mathcal{M}_{b,c}^{(i)} \tilde{\mathcal{M}}_{a,a}^{(j)} V_{a,b} V_{c,a} V_{c,a}^* V_{a,b}^* = \lambda_i^2 \mathcal{M}_{a,a}^{(i)} \tilde{\mathcal{M}}_{a,a}^{(j)}.\quad (I17)$$

4. Subsequence reversal with repetition

Following the rules in Appendix E, we find the contribution from the subsequence reversal with a repetition diagram as

$$\begin{aligned}\bar{\mathcal{Y}}_S^{\{\underline{n}, \underline{n}\}} &= \frac{t}{2} \sum_{\tau_1=1}^t \sum_{\tau_2=\tau_1+3}^{t+\tau_1-3} \sum_{ij} \chi_i^{\tau_2-\tau_1-2} \lambda_j^{t-\tau_2+\tau_1} Q_{SR}^{ij} \\ &\quad + \frac{t}{2} \sum_{\tau_1=1}^t \sum_{\tau_2=\tau_1+3}^{t+\tau_1-3} \sum_{ij} \lambda_i^{\tau_2-\tau_1-2} \chi_j^{t-\tau_2+\tau_1} \tilde{Q}_{SR}^{ij} \\ &= \frac{t^2}{2} \sum_{ij} \frac{\chi_i^{t-4} \lambda_j^3 - \chi_i \lambda_j^{t-2}}{\chi_i - \lambda_j} Q_{SR}^{ij} \\ &\quad + \frac{t^2}{2} \sum_{ij} \frac{\lambda_i^{t-4} \chi_j^3 - \lambda_i \chi_j^{t-2}}{\lambda_i - \chi_j} \tilde{Q}_{SR}^{ij},\end{aligned}\quad (I15)$$

Thus, we can find

$$Q_{SR}^{i0} = \frac{\chi_i^2}{\mathcal{N}} \text{tr} \tilde{\mathcal{M}}^{(i)} = \frac{\chi_i^2}{\mathcal{N}},\quad (I18)$$

$$\tilde{Q}_{SR}^{0i} = \frac{1}{\mathcal{N}} \text{tr} \tilde{\mathcal{M}}^{(i)} = \frac{1}{\mathcal{N}}.\quad (I19)$$

We substitute Eqs. (I18) and (I19) in Eq. (I15) to find

$$\bar{\mathcal{Y}}_S^{\{\underline{n}, \underline{n}\}} = \frac{t^2}{\mathcal{N}} \sum_i \frac{\chi_i^3}{1-\chi_i} - \frac{t^2}{\mathcal{N}} \sum_i \frac{\chi_i^{t-2}}{1-\chi_i} + \frac{t^2}{2} \sum_i \sum_{j \neq 0} \frac{\chi_i^{t-4} \lambda_j^3 - \chi_i \lambda_j^{t-2}}{\chi_i - \lambda_j} Q_{SR}^{ij} + \frac{t^2}{2} \sum_{i \neq 0} \sum_j \frac{\lambda_i^{t-4} \chi_j^3 - \lambda_i \chi_j^{t-2}}{\lambda_i - \chi_j} \tilde{Q}_{SR}^{ij}.\quad (I20)$$

Therefore, we obtain the total second-order correction to the SFF for systems belonging to the CUE class as

$$\begin{aligned}K_0^{(2)}(t) &= \bar{\mathcal{Y}}_T + \bar{\mathcal{Y}}_S - \bar{\mathcal{Y}}_R^{\{\underline{n}, \underline{n}\}} - \bar{\mathcal{Y}}_S^{\{\underline{n}, \underline{n}\}} \\ &= t^2 \sum_{i \neq 0} \left(-\frac{\lambda_i^{t-2}}{\mathcal{N}(1-\lambda_i)} + \lambda_i^{t-3} Q_{T(1)}^i + \lambda_i^{t-4} Q_{T(2)}^i \right) + \frac{t^2}{2} \sum_{\substack{i \neq 0 \\ j \neq 0}} \frac{\lambda_i^{t-4} \lambda_j - \lambda_i \lambda_j^{t-4}}{\lambda_i - \lambda_j} Q_{T'}^{ij} + t^2 \sum_{ij} \frac{\lambda_i^{t-4} \chi_j - \lambda_i \chi_j^{t-4}}{\chi_i - \chi_j} \tilde{Q}_{T'}^{ij} \\ &\quad + t^2 \sum_i \chi_i^{t-3} \tilde{Q}_{T(1)}^i + t^2 \sum_i \chi_i^{t-4} \tilde{Q}_{T(2)}^i + \frac{t^2}{2} \sum_i \sum_{j \neq 0} \frac{\chi_i^{t-4} \lambda_j^3 - \chi_i^3 \lambda_j^{t-4}}{\chi_i - \lambda_j} Q_S^{ij} + \frac{t^2}{2} \sum_{ij} \frac{\lambda_i^{t-4} \chi_j^3 - \lambda_i^3 \chi_j^{t-4}}{\lambda_i - \chi_j} \tilde{Q}_S^{ij} \\ &\quad + \frac{t^2}{\mathcal{N}} \sum_{i \neq 0} \frac{\lambda_i^t}{1-\lambda_i} - \frac{t^2}{2} \sum_{\substack{i \neq 0 \\ j \neq 0}} \frac{\lambda_i^t \lambda_j - \lambda_i \lambda_j^t}{\lambda_i - \lambda_j} Q_R^{ij} - \frac{t^2}{2} \sum_{ij} \frac{\chi_i^t \chi_j - \chi_i \chi_j^t}{\chi_i - \chi_j} \tilde{Q}_R^{ij} + \frac{t^2}{\mathcal{N}} \sum_i \frac{\chi_i^{t-2}}{1-\chi_i} \\ &\quad - \frac{t^2}{2} \sum_{ij} \frac{\chi_i^{t-4} \lambda_j^3 - \chi_i \lambda_j^{t-2}}{\chi_i - \lambda_j} Q_{SR}^{ij} - \frac{t^2}{2} \sum_{ij} \frac{\lambda_i^{t-4} \chi_j^3 - \lambda_i \chi_j^{t-2}}{\lambda_i - \chi_j} \tilde{Q}_{SR}^{ij} \\ &= \mathcal{O}(\lambda_1^t, \chi_0^t).\end{aligned}\quad (I21)$$

Equation (I21) shows the absence of a universal term at second order in time, which explains the emergence of the RMT SFF for the CUE class. Equation (I21) includes the contributions of all Type III terms, which are ignored in our calculation with the reduced diagrams in Eq. (82).

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