

Disclinations, Dislocations, and Emanant Flux at Dirac Criticality

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What happens when fermions hop on a lattice with crystalline defects? The answer depends on topological quantum numbers that specify the action of lattice rotations and translations in the low energy theory. One can understand the topological quantum numbers as a twist of continuum gauge fields in terms of crystalline gauge fields. We find that disclinations and dislocations—defects of crystalline symmetries—generally lead to a certain “emanant” quantized magnetic flux in the continuum. To demonstrate these facts, we study in detail tight-binding models whose low-energy descriptions are $(2 + 1)$ D Dirac cones. Our map from lattice to continuum defects explains the crystalline topological response to disclinations and dislocations, and motivates the fermion crystalline equivalence principle used in the classification of crystalline topological phases. When the gap closes, the presence of emanant flux leads to pair creation from the vacuum with the particles and antiparticles swirling around the defect. We compute the associated currents and energy density using the tools of defect conformal field theory. There is a rich set of renormalization group fixed points, depending on how particles scatter from the defect. At half flux, there is a defect conformal manifold leading to a continuum of possible low-energy theories. We present extensive numerical evidence supporting the emanant magnetic flux at lattice defects, and we test our map between lattice and continuum defects in detail. We also point out a no-go result, which implies that a single $(2 + 1)$ D Dirac cone in symmetry class AII is incompatible with a commuting C_M rotational symmetry with $(C_M)^M = +1$.

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I. INTRODUCTION

A fundamental question in the study of quantum matter is to understand the effect of crystalline symmetries, such as lattice translations, rotations, and reflections, in quantum many-body systems. The past two decades have seen significant progress in our understanding of how crystalline symmetry can enrich the set of possible gapped quantum phases of matter (for a partial list of references, see, e.g., Refs. [1–27]). These phases are typically referred to as crystalline topological phases and can be described by a topological quantum field theory (TQFT) in the infrared

limit. The TQFT is characterized by crystalline topological invariants that determine the universal response to defects of the crystalline symmetry, such as lattice disclinations and dislocations.

In this paper, we consider how crystalline symmetry enriches quantum critical points. We develop a general framework for studying crystalline defects at quantum criticality. As an example that is perhaps of most experimental interest, we focus on the case of two-dimensional lattice systems described at low energies by Dirac fermions. Along the way, we also show how our perspective can be used to explain many previously known facts about the gapped (insulating) phases and the fermionic crystalline equivalence principle: Topological phases of fermions with a symmetry group G that contains spatial symmetries can be effectively classified using the formalism for internal symmetries with a symmetry G_{eff} [28] [16,29–31].

First, we explain how, in the presence of an M -fold rotational symmetry, implemented by an operator $C_{M,o}$

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about a fixed point o , each Dirac fermion has an angular-momentum quantum number s_o , in a basis that diagonalizes $C_{M,o}$. If $(C_{M,o})^M = +1$, s_o is quantized to a half-integer modulo M , whereas if $(C_{M,o})^M = (-1)^F$, where F is the fermion number, then s_o is quantized to an integer modulo M . The dependence of s_o on o is related to the lattice momentum of the Dirac point. More generally, we have a group homomorphism $\rho: G_{UV} \rightarrow G_{IR}$ from the lattice symmetry group G_{UV} to the symmetry group of the IR quantum field theory, G_{IR} . The quantum numbers $\{s_o\}$ are part of the data parametrizing this group homomorphism. The constraints on $\{s_o\}$ provide an explanation of the fermionic crystalline equivalence principle for rotational symmetries [16,30,31].

A central question that we address in this paper is how to detect $\{s_o\}$ and the group homomorphism ρ more generally from universal physical observables at the critical point. The key point that enables this is that crystalline symmetry defects, labeled by the microscopic lattice symmetries $g \in G_{UV}$, should be modeled in the IR QFT by $\rho(g) \in G_{IR}$ defects, defined in terms of the symmetries of the infrared theory. In particular, we argue that this correspondence implies that lattice disclinations with disclination angle $2\pi/M$ should be modeled for each Dirac fermion in the IR by a conical defect with angle $2\pi/M$ together with a $U(1)$ holonomy $2\pi s_o/M$ (in the basis of Dirac fermions that diagonalizes the lattice rotation operator). We argue that lattice dislocations are modeled for each Dirac fermion in the IR by a $U(1)$ holonomy set by the momentum of the Dirac point and the Burgers vector (in the basis of Dirac fermions that diagonalizes the lattice translation operator) [32].

Therefore, the continuum Dirac fermions see an effective magnetic flux entirely due to the lattice defects, even if no magnetic flux is applied at the defect in the lattice model. We refer to this effective magnetic flux as an emanant magnetic flux, following terminology introduced in Ref. [33], for reasons we explain later. We demonstrate some consequences of this emanant flux, including particle creation from the vacuum, a universal shift in the dependence of the electric current and ground-state energy density as a function of applied magnetic flux in the presence of the lattice defects (see Fig. 1), and various other interesting observables.

For gapped phases, we use our map from lattice to continuum defects to obtain topological invariants of crystalline systems. Indeed, in addition to the familiar Chern number of insulating bands, there are several topological invariants having to do with crystalline rotations and translations, leading to measurable physical effects, for instance, charge accumulation at a disclination [20,23,25]. We will show how several of the known topological crystalline invariants [20,24] can be understood from well-known facts about continuum Dirac fermions together with the map ρ from the lattice to the continuum.

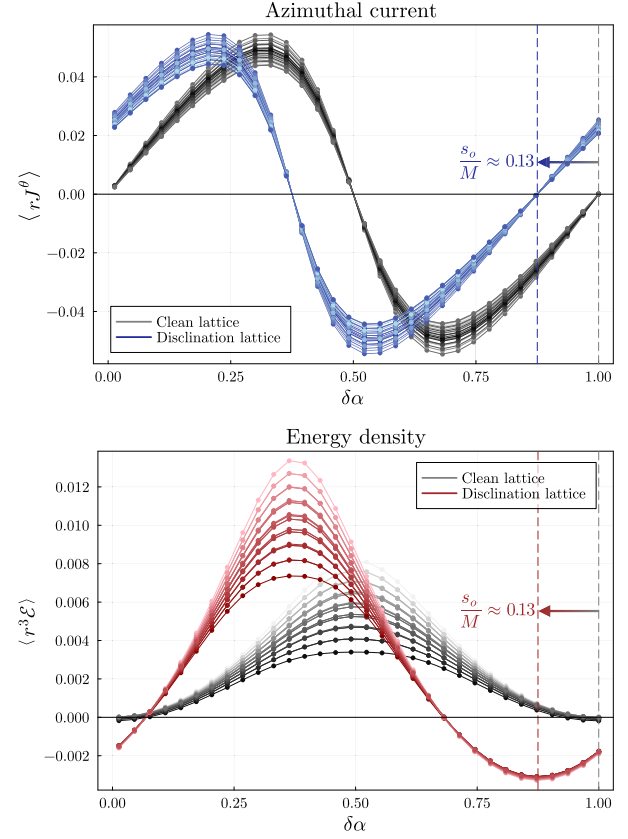


FIG. 1. Observables measured in the QWZ model ground state at criticality on lattices with and without a crystalline defect, as a function of applied flux $\delta\alpha$ inserted at the rotation center. Notice that the defect lattice data are offset along the $\delta\alpha$ axis relative to the clean lattice data due to the emanant flux generated by the crystalline defect. Each curve corresponds to a particular point in the bulk of the lattice, with lighter colors denoting curves closer to the rotation center. The magnitude of the clean lattice current (upper panel) is scaled for ease of comparison with the disclination lattice current. Remarkably, the magnitude of the minima and maxima of the energy density is larger with the disclination as compared to the clean case, which also matches the theoretical prediction in Sec. VI.

At the gapless point, the massless Dirac fermion obeys a conformal symmetry $so(3,2)$. Lattice defects flow to defect conformal theories, preserving the subalgebra $sl(2, \mathbb{R}) \subset so(3,2)$. The continuum theory is therefore in the framework of defect conformal field theory (DCFT). (For recent reviews, see Refs. [34–37].) This description applies to the long-distance limit of dislocations, disclinations, and Aharonov-Bohm fluxes.

We characterize the $sl(2, \mathbb{R})$ -invariant theory at long distances in all of these cases, focusing on its most important consequences. In particular, we study the particle-antiparticle creation due to a localized magnetic field, which leads to a current and nonzero energy density in the vacuum, even away from the defect. This case is to be contrasted with the Aharonov-Bohm effect, where

nonrelativistic particles and antiparticles circulate around the solenoid. Here, particles are created from the vacuum and start circulating. The net particle number vanishes in our case in the Dirac field theory. The particle creation can be understood from the fact that, to adiabatically turn on a localized magnetic field, one has to create an electromotive force that will rip out electron-hole pairs from the vacuum, and those pairs will start circulating around the defect in opposite directions.

An important quantity that needs to be specified is how the particles scatter from the defect. Generically, there are discrete choices that preserve the $sl(2, \mathbb{R})$ symmetry at long distances and correspond to fixed points of the RG. One special choice corresponds to a stable infrared fixed point, and we expect it to agree with lattice models absent special fine-tuning. An exception occurs at half flux, i.e., when the $U(1)$ flux is $\pi \bmod 2\pi$. Then, there is a continuum of choices of infrared stable fixed points with an exactly marginal operator (conformal manifold); see also Ref. [38]. By dialing the exactly marginal defect operator, the azimuthal current varies in a certain range while the energy density remains constant. We remark that such a defect conformal manifold is particular for the Dirac criticality; it is absent, for example, for the free boson model studied in Appendix B. A conceptual way of understanding its appearance is by using the g theorem [39]. It also has observable consequences for the fate of lattice model observables near the π flux: In the absence of additional symmetry pinning the current to zero at π flux, the conformal manifold allows it to flow to nonzero values in the IR. This point is discussed further in Sec. VII.

In this work, we develop the formalism of defect conformal field theory for conical singularities and discuss the absence of the displacement operator for such defects, even though they preserve $sl(2, \mathbb{R})$. Conical defects, unlike symmetry defects (monodromy defects), are therefore very slightly outside the usual framework of DCFT. The results of our DCFT analysis are twofold. First, we obtain the universal functional dependence of the azimuthal current and energy density on total magnetic flux, with or without conical singularities. For a generic lattice model with disclinations, these observables are determined by the quantum number s_o at a sufficiently large system size. By matching results from the DCFT analysis to numerical calculations in critical tight-binding models, we are able to extract the quantum number s_o of each Dirac fermion and verify our mapping between defects on the lattice and the continuum. Second, we find that finite-size corrections are significantly enhanced when the total magnetic flux approaches $\pi \bmod 2\pi$ [40]. This finding presents a fundamental challenge in numerical studies, which we circumvent by measuring observables away from the π flux. We further discuss observable imprints of this effect in critical tight-binding models. We remark that the same

methodology can be applied to extract the crystalline quantum numbers for systems whose UV completions are more complicated.

A. Related work

The emanant magnetic flux is reminiscent of the pseudomagnetic field that arises due to elastic strain in two-dimensional Dirac materials such as graphene [41], except that here it arises from topological defects of the lattice. The emanant magnetic flux induced by lattice disclinations has been noticed in earlier work on large fullerene molecules [42,43], graphitic cones [44–47], and disordered graphene sheets [48]. These works utilized this observation to compute electronic spectra and local density of states (LDOS) from continuum Dirac models. Reference [49] further used these observations, along with the emanant magnetic flux induced by lattice dislocations, to study weak localization in graphene. Furthermore, the perspective that lattice defects can be viewed as fluxes of background gauge fields—and, consequentially, that lattice defects bind fractional charge in gapped phases—has been employed widely in prior works [16,17,19,23–27,50–55]. Our work builds on these results in the following ways:

- (i) We place the understanding of the emanant magnetic flux within a broader context that applies to strongly interacting systems. The emanant magnetic flux arising from disclinations and dislocations can be understood to be a consequence of the UV-IR homomorphism, which also exists for generic interacting systems beyond the single particle limit and other conformal fixed points.
- (ii) As we explain, the fact that there must necessarily be a nonzero emanant magnetic flux at disclinations for certain theories is directly tied to the fact that the Dirac fermions have spin $1/2$ while the lattice $2\pi/M$ rotational symmetry has order M , which is intimately related to the fermion crystalline equivalence principle.
- (iii) Our work focuses on identifying the universal observable consequences of the emanant magnetic flux and the UV-IR homomorphism, whereas the electronic spectra and LDOS are nonuniversal and depend on details of the defect core. In the gapped phases, this finding explains the crystalline topological invariants found in recent work. At the critical points, we identify the space of defect fixed points and RG flows between them for the simplest system of free Dirac cones. Our analysis extends existing results on the consequences of monodromy defects in conformal field theories. A collection of existing results about monodromy defects in conformal theories can be found in Refs. [38,56–62]. Our presentation of the subject will be self-contained and will review some of the pertinent results.

II. UV AND IR SYMMETRIES

Let us consider a UV theory that could be a many-body lattice system or a QFT with relevant deformations. In the limit of long wavelengths, we assume an IR QFT describes the low-energy physics of this UV theory. A precise treatment of the RG flow that connects UV and IR is often too difficult in strongly coupled systems. Instead, in this section, we discuss how symmetries in the UV and IR are matched.

A. Generalities

Suppose G_{UV} denotes the spatial and internal group symmetries of the UV theory while G_{IR} denotes that of the IR QFT. Then, there is a group homomorphism

$$\rho: G_{\text{UV}} \rightarrow G_{\text{IR}}, \quad (1)$$

which determines how symmetries in the UV map to symmetries in the IR. One has to define what the IR QFT is—we will be working at a gapless point or its immediate neighborhood, so only very light modes are kept and G_{IR} is well defined.

A corollary of this construction is that we have a map between symmetry defect configurations. Given a theory with a symmetry G , we can consider deforming the theory to create symmetry defects, labeled by $g \in G$. For example, these can physically correspond to inserting the magnetic flux of a background gauge field or creating lattice disclinations or dislocations. The group homomorphism ρ can be thought of as determining a collection of RG flows, where the UV system with a symmetry defect $g \in G_{\text{UV}}$ is described in the IR QFT with a symmetry defect $\rho(g) \in G_{\text{IR}}$. For completeness, let us define what a symmetry defect is. Given a field theory with a symmetry G , whose elements $g \in G$ label topological operators, we can create a defect by taking the topological operator to be a semi-infinite plane and orienting the boundary in the time direction. Then, in space, we have a topological line that ends at a point. At that point, there is curvature for the background gauge field. The idea of “topological operators” is more difficult to define for lattice quantum many-body systems. However, we can still define symmetry defects by introducing boundary conditions on the lattice where lattice fields are forced to undergo a symmetry transformation as they travel around the defect.

Let us consider the case where the IR QFT is a fixed point of the RG flow and G_{IR} is the symmetry group of the fixed point QFT. In many cases, such as those of interest in this paper, there is a discrete set of possible choices of the group homomorphism ρ . In these cases, ρ can be viewed as a topological invariant of the UV theory since it cannot be changed by local perturbations of the system without encountering a singularity in the ground-state energy that fundamentally alters the nature of the RG flow. If we have

two UV Hamiltonians H_1 and H_2 corresponding to two different homomorphisms ρ_1 and ρ_2 , then H_1 cannot be continuously deformed into H_2 without encountering a singularity in the ground-state energy. In the case where the IR QFT is critical, ρ therefore parametrizes topologically distinct ways that quantum critical points can be enriched with the symmetry G_{UV} . We refer to these as distinct symmetry-enriched quantum critical points [63].

While it is not of direct concern for us in this paper, we note that, in general, ρ is the bottom layer of a more complicated mathematical structure. It is expected that quantum field theories in D space-time dimensions have a generalized symmetry that forms the structure of a higher category. Therefore, in general, we expect that the UV and IR symmetries form the structure of higher categories, $\mathcal{G}_{\text{UV}}$ and $\mathcal{G}_{\text{IR}}$, and we have a higher functor between these higher categories, $\mathcal{G}_{\text{UV}} \rightarrow \mathcal{G}_{\text{IR}}$. These ideas are currently a topic of intense research and have only been fully developed in special cases. For example, $(2+1)\text{D}$ topological phases of matter with symmetry group $G = G_{\text{UV}}$ can be described using G -crossed modular categories, which can be viewed as a 3-functor between 3-categories; for details, see Refs. [64,65]. As another example, there is a fusion 2-category associated with $(1+1)\text{D}$ quantum field theories, which is the fusion 2-category of topological defects of the bulk $(2+1)\text{D}$ topological quantum field theory that hosts the $(1+1)\text{D}$ theory at its boundary.

We now comment on a few basic properties of ρ . When ρ has a nontrivial kernel, some UV internal symmetries become trivial in the low-energy effective theory. Physically, this result can occur when massive degrees of freedom decouple from the spectrum, whose associated symmetries become invisible to the IR QFT. When ρ has a nontrivial cokernel, it could be that the IR QFT is endowed with emergent (accidental) symmetries, where symmetry-breaking operators are irrelevant and suppressed at low energy. The emanant symmetries studied in Ref. [33] provide an example where $\rho(g)$ and g generate distinct groups; such symmetries are exact in the IR QFT even though the group generated by $\rho(g)$ may not be a subgroup of G_{UV} ; the group generated by $\rho(g)$ is said to emanate from the symmetry group generated by g . Such examples are common for lattice translation symmetries.

The primary focus of this paper is on cases where the UV theory is a fermion lattice model and the IR theory consists of N free massless or very light Dirac fermions. We thus have a symmetry group G_{UV}^f that acts faithfully on fermionic operators. Let $\mathbb{Z}_2^f$ be the order-2 group generated by fermion parity $(-1)^F \in \mathbb{Z}_2^f$. The bosonic symmetry group is such that $G_{\text{UV}}^b \equiv G_{\text{UV}}^f / \mathbb{Z}_2^f$. More specifically, we are concerned with lattice models with UV symmetry group

$$G_{\text{UV}}^f = U(1)^f \times (\mathbb{Z}^2 \rtimes \mathbb{Z}_M), \quad (2)$$

which is relevant for describing charged fermions on a lattice. Here, $U(1)^f$ refers to a central extension of the bosonic global symmetry $U(1)$ by $\mathbb{Z}_2^f$. For the spatial symmetries, $\mathbb{Z}_M$ is the M -fold lattice rotational symmetry group, and $\mathbb{Z}^2$ is the lattice translation group. Physically, we have magnetic translations that commute up to a $U(1)$ transformation.

We remark that lattice reflections are beyond the scope of this paper. Defects of reflection symmetry require nonlocal operations in the UV theory, and they require us to study the theory on nonorientable manifolds [66]. The action of the crystalline reflection symmetries in the IR theory also contains several additional subtleties (see, e.g., Refs. [31,67]), which we leave for future work.

B. Dirac fermion quantum numbers

We consider a theory of N massless $(2+1)$ D Dirac fermions as the IR theory. For simplicity, we assume that they all have the same velocity (which we set to 1). The bosonic symmetry group is then

$$G_{\text{IR}}^b = \frac{U(N)}{\mathbb{Z}_2} \times SO(3,2), \quad (3)$$

where $SO(3,2)$ is the conformal group associated with $(2+1)$ D Minkowski space. The fermionic symmetry group is a $\mathbb{Z}_2^f$ central extension:

$$G_{\text{IR}}^f = [U(N)^f \times \text{Spin}(3,2)]/\mathbb{Z}_2, \quad (4)$$

where $U(N)^f$ refers to the group $U(N)$ that acts faithfully on the fermions and the -1 element corresponds to fermion parity. To obtain the bosonic symmetry group, we need to mod out by fermion parity, which leads to the bosonic internal symmetry group $U(N)/\mathbb{Z}_2$. Note that the IR theory may also contain charge conjugation, reflection, and/or time-reversal symmetries, which we ignore in this discussion.

For simplicity, we assume that the $U(1)$ symmetry in the UV acts as a $U(1)$ symmetry in the IR where each Dirac fermion carries charge 1. If the UV model is a free fermion lattice model, this is the only possibility. If it includes interactions, then each IR Dirac fermion, in principle, could carry any odd integer charge [68].

Let us consider a translation operator $T_{\vec{v}}$ acting on a Dirac fermion operator $\Psi = (\Psi_1, \dots, \Psi_N)^T$, where $\vec{x}$ is the spatial coordinate and Ψ_i , $i = 1, \dots, N$, is a single 2-component Dirac fermion. We have

$$T_{\vec{v}}: \Psi(t, \vec{x}) \rightarrow U_{T_{\vec{v}}} \Psi(t, \vec{x} + \vec{v}), \quad (5)$$

where $U_{T_{\vec{v}}} \in U(N)^f$. Lattice dislocations investigated in this paper are created by $|\vec{v}|$ on the order of the lattice spacing, which is taken to zero in the continuum limit. As we will discuss in subsequent sections, the long-distance

physics of dislocations is therefore solely determined by $U_{T_{\vec{v}}}$. For a given lattice model, the N Dirac cones will occur at momenta $\{\vec{k}_\star^{(i)}\}$ in the Brillouin zone. We can thus expect

$$(U_{T_{\vec{v}}})_{ij} = \delta_{ij} e^{i\vec{k}_\star^{(i)} \cdot \vec{v}}. \quad (6)$$

Now, let us consider the action of $2\pi/M$ rotations around a high-symmetry point o in the lattice, implemented by an operator $C_{M,o}$ in the UV theory with $(C_{M,o})^M = 1$. We have

$$C_{M,o}: \Psi(t, \vec{x}) \rightarrow U_{M,o} \Lambda(2\pi/M) \Psi(t, \vec{x}'), \quad (7)$$

where $\vec{x}'$ is obtained from $\vec{x}$ by a $2\pi/M$ rotation about the fixed point o . Here, $\Lambda(2\pi/M)$ is a $2\pi/M$ spatial rotation in the IR field theory, which generates the spatial rotation group $SO(2)^f \subset \text{Spin}(3,2)$, and $U_{M,o} \subset U(N)$ is an internal rotation.

Therefore, we see that $C_{M,o}$ acts in the IR theory as a combination of the field theory spatial rotation together with an internal $U(N)$ rotation, sometimes referred to as a “twisted rotation.” Since the Dirac fermion is a spin-1/2 representation of the space-time symmetry group of the field theory, we have that $(\Lambda(2\pi/M))^M = -1$ on the Dirac fermion. Since $(C_{M,o})^M = 1$, it follows that $(U_{M,o})^M = -1$. If we instead consider a $2\pi/M$ rotation that satisfies $(C_{M,o})^M = (-1)^F$, then we would have $(U_{M,o})^M = 1$. In other words, the $(C_{M,o})^M$ would then have order $2M$ and act in the IR via an internal $U(N)$ operation that has order M .

In general, $C_{M,o}$ does not commute with translations. We can find a basis $\tilde{\Psi}$ where $C_{M,o}$ is diagonal:

$$C_{M,o}: \tilde{\Psi}_i \rightarrow e^{2\pi i s_o^{(i)}/M} \Lambda(2\pi/M) \tilde{\Psi}_i. \quad (8)$$

We refer to the quantum numbers $s_o^{(i)}$ as the orbital angular momenta, which are defined modulo M . From the above discussion, we see that $s_o^{(i)}$ are necessarily half-odd integers in the case we consider, where $(C_{M,o})^M = 1$. If instead we consider rotation symmetry operators where $(C_{M,o})^M = (-1)^F$, then $s_o^{(i)}$ are integers.

Rotations about different fixed points, such as vertex-centered vs plaquette-centered fixed points, are related by translations:

$$C_{M,o+\vec{v}} = T_{\vec{f}(\vec{v})} C_{M,o}, \quad (9)$$

where $\vec{f}(\vec{v}) = (1 - R(2\pi/M))\vec{v}$ and $R(2\pi/M)$ is a rotation by angle $2\pi/M$. If the Dirac fermion Ψ_i occurs at a momentum in the Brillouin zone that is invariant under $2\pi/M$ rotations, then there is a simple relationship between

the orbital angular momentum for rotations about different high-symmetry points:

$$s_{o+\vec{v}}^{(i)} = s_o^{(i)} + \frac{M}{2\pi} \vec{f}(\vec{v}) \cdot \vec{k}_\star^{(i)} \mod M. \quad (10)$$

C. Relation to fermionic crystalline equivalence principle

The above discussion provides an explanation of the correspondence predicted by the fermionic crystalline equivalence principle between spatial rotation symmetries satisfying $(C_M)^M = +1$ and internal symmetries that have order $2M$, as well as that between spatial rotation symmetries satisfying $(C_M)^M = (-1)^F$ and internal symmetries that have order M . Namely, if we want to classify gapped phases of fermions using topological quantum field theory, then the spin-statistics theorem tells us that the fermions in the TQFT should have half-odd integer spin under the continuous rotational symmetry of the IR TQFT. The discussion above then shows us how the C_M operation must come with an additional internal symmetry that has order $2M$ in order to satisfy the relation $(C_M)^M = 1$. Correspondingly, for the case where $(C_M)^M = (-1)^F$, the additional internal symmetry operation must have order M . Classifying topological phases with a C_M rotational symmetry can then be done by classifying ways in which the corresponding internal symmetry acts in the IR theory.

We can deduce that the crystalline equivalence principle could then break down in situations where the IR theory cannot be Lorentz invariant, for instance, for gapped fracton models [69], Fermi liquids, or anisotropic quantum critical points.

Later, we will see that using the twisted action of the lattice translations and lattice rotation on the continuum fermions, we will be able to derive several known topological invariants associated with crystalline symmetries, which is an important facet of the fermion crystalline equivalence principle that we can now illuminate.

D. No-go results

The existence of the group homomorphism ρ can place strong constraints on UV completions for IR theories that are Lorentz invariant and contain fermions. Below, we discuss two examples.

1. Constraints from time-reversal symmetry

Time-reversal symmetry introduces additional constraints on the symmetry actions described above. For example, consider the symmetry group

$$G_{\text{UV}}^f = \frac{U(1)^f \rtimes \mathbb{Z}_4^{T,f}}{\mathbb{Z}_2} \times [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]. \quad (11)$$

Here, $\mathbb{Z}_4^{T,f}$ is an order-4 group generated by time-reversal $\mathcal{T}$ such that $\mathcal{T}^2 = (-1)^F$, and the equivalence relation identifies the fermion parity elements from the $U(1)^f$ and $\mathbb{Z}_4^{T,f}$ factors. This example can be thought of as symmetry class AII in the Altland-Zirnbauer classification, supplemented with the additional wallpaper group symmetry $\mathbb{Z}^2 \rtimes \mathbb{Z}_M$. For such a symmetry group, time-reversal and spatial rotations commute, $\mathcal{T}C_{M,o} = C_{M,o}\mathcal{T}$.

Consider the case of a single Dirac cone. Then, from the above discussion, since $s_o \neq 0$, it follows that $\mathcal{T}$ and $C_{M,o}$ cannot commute. We see that a single Dirac cone is necessarily incompatible with the symmetry of Eq. (11). A similar argument shows that any odd number of Dirac fermions is inconsistent with commuting $\mathcal{T}$ and $C_{M,o}$ symmetries.

If we instead have a spatial rotation symmetry $(C_{M,o})^M = (-1)^F$, then it is possible to have $s_o = 0$, in which case $C_{M,o}$ and $\mathcal{T}$ commute.

Since a single Dirac fermion in class AII occurs at the surface of a $(3+1)$ D topological insulator, we can further conclude that topological insulators in this symmetry class are incompatible with an order- M C_M rotational symmetry that commutes with $\mathcal{T}$. Instead, for C_M to commute with $\mathcal{T}$, such systems must have a “spinful” spatial rotation symmetry, where $(C_M)^M = (-1)^F$. This result gives a more abstract perspective on the necessity of spin-orbit coupling for topological insulators [2,70] since spatial rotations must necessarily be supplemented with internal symmetry operations.

2. IR fixed points without internal symmetries

Consider an IR theory containing half-odd-integer spin fermions. The above discussion shows that such a theory can only be compatible with an order- M C_M rotational symmetry in the UV [$(C_M)^M = 1$] if it contains a $\mathbb{Z}_{2M}^f$ internal symmetry in the IR. In other words, QFTs with half-odd-integer spin fermions without a $\mathbb{Z}_{2M}^f$ internal symmetry are fundamentally incompatible with a UV completion containing an M -fold rotational symmetry with $(C_M)^M = 1$. Any rotational symmetry in the UV would instead have to satisfy $(C_M)^M = (-1)^F$.

III. SYMMETRY EIGENVALUES IN FREE FERMION LATTICE MODELS

Here, we review how to determine the UV-IR homomorphism described above for lattice models of free fermions. We also present an example, the QWZ model, which we use to numerically test our ideas throughout the rest of the paper.

A free fermion lattice model is governed by a Hamiltonian

$$H = \sum_{\vec{i}, \vec{j}, a, b} h_{ij}^{ab} c_{i,a}^\dagger c_{j,b}, \quad (12)$$

where $\vec{i}, \vec{j}$ are 2-component vectors denoting lattice sites, a, b denote flavor indices (which could stand for, e.g., orbital, layer, spin), and h_{ij}^{ab} is the first-quantized Hamiltonian. A $2\pi/M$ (magnetic) rotational symmetry about a high-symmetry point o acting on the fermion operators takes the general form

$$C_{M,o} c_{\vec{j},a} C_{M,o}^\dagger = e^{i\lambda_{\vec{j}}^{M,o}} \sum_b V_{M,o}^{a,b} c_{R_{2\pi/M}^o(\vec{j}),b}, \quad (13)$$

where $\lambda_{\vec{j}}^{M,o}$ encodes a possible $U(1)$ phase that is location dependent, $V_{M,o}$ encodes a rotation in the flavor space, and $R_{2\pi/M}^o(\vec{j})$ gives the lattice site obtained by rotating $\vec{j}$ by $2\pi/M$ about the origin o . Note that we consider $(C_M)^M = 1$ by definition, so the magnetic rotation generates a $\mathbb{Z}_M$ symmetry.

Similarly, a (magnetic) translation $T_{\vec{v}}$ by the vector $\vec{v}$ is defined via

$$T_{\vec{v}} c_{\vec{j},a} T_{\vec{v}}^\dagger = e^{i\lambda_{\vec{j}}^{T,\vec{v}}} \sum_b V_{T,\vec{v}}^{a,b} c_{\vec{j}+\vec{v},b}, \quad (14)$$

where $\lambda_{\vec{j}}^{T,\vec{v}}$ encodes a possible position-dependent $U(1)$ transformation and $V_{T,\vec{v}}$ encodes a rotation in the flavor space. These transformations, along with $U(1)$ charge conservation, generate the symmetry group of the lattice model on the infinite two-dimensional Euclidean plane,

$$G_{UV}^f = U(1)^f \times (\mathbb{Z}^2 \rtimes \mathbb{Z}_M), \quad (15)$$

which we discussed extensively in Sec. II. In general, this group must be modified to take into account the non-commutativity of translations when the unit cell contains nonzero total flux, but we will not encounter this complication in this work.

Given a free fermion model with symmetry $C_{M,o}$, which can be described at low energies by N Dirac cones centered at momenta $\vec{k}_\star^{(i)}$, for $i = 1, \dots, N$, we can explicitly determine the UV-IR homomorphism described above by determining the action of the symmetry operators on the single-particle eigenstates at the Dirac crossings. For example, for a critical point with a single Dirac cone, at $\vec{k}_\star$ we have a two-dimensional space of states, with corresponding annihilation operators $\Psi_{\vec{k}_\star} = (\psi_{\vec{k}_\star, \uparrow}, \psi_{\vec{k}_\star, \downarrow})$. We can pick a basis such that

$$C_{M,o} : \Psi_{\vec{k}_\star} \rightarrow e^{i\frac{2\pi}{M}s_o} \Lambda(2\pi/M) \Psi_{\vec{k}_\star}, \quad (16)$$

with $\Lambda(2\pi/M) = e^{i\frac{1}{2M}\sigma^z}$. In other words, the eigenvalues of $C_{M,o}$ are given by $e^{i\frac{2\pi}{M}(s_o \pm 1/2)}$. Therefore, the s_o are

directly related to the crystal angular momentum of the zero energy states at the Dirac crossing. If there are N Dirac cones at a critical point, the procedure is analogous. We can pick a basis for the Dirac cones that diagonalizes $C_{M,o}$, such that $C_{M,o}$ acts by the $2N$ -dimensional matrix $\bigoplus_{i=1}^N e^{i\frac{2\pi}{M}s_o^{(i)}} \Lambda(2\pi/M)$.

Similarly, we can determine the invariants $\vec{k}_\star^{(i)}$ by measuring the eigenvalues of the translation operators along the primitive lattice vectors. For a single Dirac cone, we have

$$T_{\vec{v}} : \Psi_{\vec{k}_\star} \rightarrow e^{i\vec{k}_\star \cdot \vec{v}} \Psi_{\vec{k}_\star} \quad (17)$$

The generalization to multiple fermions is straightforward.

While it is straightforward to determine $s_o^{(i)}$ and $\vec{k}_\star^{(i)}$ numerically given any free fermion lattice model, the primary purpose of this work is to understand its physical consequences and how it can be measured through physical observables.

A. Example: QWZ model

As a prototypical example, we will study the QWZ model [71], which is a straightforward lattice regularization of a continuum Dirac theory. The model consists of two flavors of fermion hopping on a square lattice, which we denote as $\uparrow$ and $\downarrow$. The real-space Hamiltonian is given by

$$H = \frac{1}{2} \sum_{\vec{j}} 2mc_{j_x, j_y}^\dagger \sigma^z c_{j_x, j_y} + c_{j_x+1, j_y}^\dagger [\sigma^z + i\sigma^x] c_{j_x, j_y} + c_{j_x, j_y+1}^\dagger [\sigma^z + i\sigma^y] c_{j_x, j_y} + \text{H.c.}, \quad (18)$$

where the σ^i are Pauli matrices acting in the flavor space. This Hamiltonian is visualized in Fig. 2. It can be expressed compactly in momentum space as

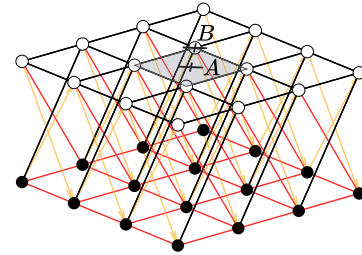


FIG. 2. Real-space QWZ Hamiltonian, where black lines correspond to $h_{i,j} = \frac{1}{2}$, red lines to $h_{i,j} = -\frac{1}{2}$, and orange lines to $h_{i,j} = (i/2)$. White and black circles represent the on-site potentials $+2m$ and $-2m$, respectively. We have visualized the different flavors of fermions as different layers, with the top layer corresponding to the $\uparrow$ flavor. The rotationally invariant unit cell is shaded in gray, with the two inequivalent rotation centers marked with black crosses.

$$H = \sum_{\vec{k}} \left(c_{\vec{k},\uparrow}^\dagger, c_{\vec{k},\downarrow}^\dagger \right) (\vec{d}_{\vec{k}} \cdot \vec{\sigma}) \begin{pmatrix} c_{\vec{k},\uparrow} \\ c_{\vec{k},\downarrow} \end{pmatrix}, \quad (19)$$

where $\vec{\sigma} = (\sigma^x, \sigma^y)^T$ and

$$\vec{d}_{\vec{k}} = \begin{pmatrix} \sin(k_x) \\ \sin(k_y) \\ 2m + \cos(k_x) + \cos(k_y) \end{pmatrix}. \quad (20)$$

Let us denote by $o = A$ and $o = B$ the plaquette and vertex centers, respectively. On a finite lattice with an odd number of sites per side, this model has a magnetic rotation symmetry about its central vertex given by

$$C_{4,B} c_{\vec{i},\uparrow}^\dagger C_{4,B}^\dagger = i c_{R_{\pi/2}^B(\vec{i}),\uparrow}^\dagger, \quad C_{4,B} c_{\vec{i},\downarrow}^\dagger C_{4,B}^\dagger = c_{R_{\pi/2}^B(\vec{i}),\downarrow}^\dagger, \quad (21)$$

where $R_{\pi/2}^B(\vec{i})$ is the lattice site obtained by rotating site $\vec{i}$ about the central vertex by $\pi/2$. Similarly, on a finite lattice with an even number of sites per side, this model has a magnetic rotation symmetry about its central plaquette given by

$$C_{4,A} c_{\vec{i},\uparrow}^\dagger C_{4,A}^\dagger = i c_{R_{\pi/2}^A(\vec{i}),\uparrow}^\dagger, \quad C_{4,A} c_{\vec{i},\downarrow}^\dagger C_{4,A}^\dagger = c_{R_{\pi/2}^A(\vec{i}),\downarrow}^\dagger, \quad (22)$$

where $R_{\pi/2}^A(\vec{i})$ is the lattice site obtained by rotating site $\vec{i}$ about the central plaquette by $\pi/2$. When placed on a square torus, the QWZ model is symmetric under both $C_{4,B}$ about any vertex and $C_{4,A}$ about any plaquette center. In general, the points about which the rotation symmetry is centered will be high-symmetry points of the rotationally invariant unit cell, as seen in Fig. 2. In the QWZ case, the high-symmetry points of the unit cell—namely, its center and corners—correspond to the plaquette centers and vertices of the square lattice. For other models, this need not be the case. All symmetric rotation centers may be located at vertices of the lattice, for example, as we will see for another model in Sec. VII.

The QWZ model also has a translation symmetry generated by a translation by one site in either the x or y direction when placed on a torus:

$$T_{\hat{x}} c_{\vec{i},a} T_{\hat{x}}^\dagger = c_{\vec{i}+\hat{x},a}, \quad T_{\hat{y}} c_{\vec{i},a} T_{\hat{y}}^\dagger = c_{\vec{i}+\hat{y},a}. \quad (23)$$

Finally, the QWZ model is also endowed with an antiunitary particle-hole symmetry P that acts as

$$P c_{\vec{i},a} P^\dagger = c_{\vec{i},\bar{a}}^\dagger, \quad P c_{\vec{i},a}^\dagger P^\dagger = c_{\vec{i},\bar{a}}, \quad (24)$$

where $\bar{a}$ denotes the flavor opposite a . This symmetry induces an effective transformation of the first-quantized Hamiltonian according to

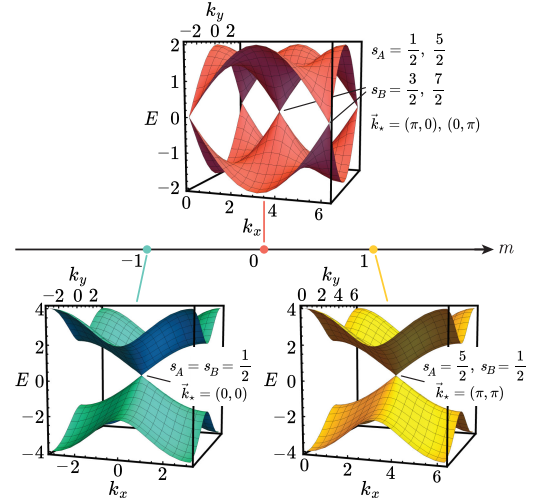


FIG. 3. For each critical point of the QWZ model, we plot the band structure in the first Brillouin zone and label the angular-momentum eigenvalues.

$$P: h_{\vec{i}\vec{j}}^{ab} \mapsto -(h_{\vec{i}\vec{j}}^{\bar{a}\bar{b}})^*. \quad (25)$$

This transformation remains a symmetry of the Hamiltonian on a clean lattice even when π flux is inserted locally at the center of the lattice.

The QWZ model exhibits four distinct gapped phases as the single parameter m is tuned from $-\infty$ to ∞ . The boundaries between these phases lie at $m = -1, 0$, and 1 . The locations of the Dirac cones and values of $s_o^{(i)}$ at each critical point are shown in Fig. 3. At $m = +1$, there is a single Dirac cone at $\vec{k}_* = (\pi, \pi)$, while at $m = -1$, there is a single Dirac cone at $\vec{k}_* = (0, 0)$. At $m = 0$, there are two Dirac cones at $\vec{k}_* = (0, \pi), (\pi, 0)$. The critical points all have different values of $\vec{k}_*$ and s_o and thus realize distinct crystalline symmetry-enriched Dirac critical points. We also show in Appendix A 1 how the various s_o appear analytically in the Dirac Hamiltonian treatment of the QWZ model.

IV. CONTINUUM DEFECTS FROM LATTICE DEFECTS

As we have seen, the QWZ lattice model (along with many other similar constructions) is described in the IR by massless Dirac fermions at various special values of the couplings. To fully determine the map from the lattice to the continuum, we need more information. In particular, we will see how defects on the lattice map to defects in the continuum, which is governed by our homomorphism ρ . Equivalently, we will see that defects on the lattice lead to some nontrivial background gauge fields in the continuum. Therefore, our goal is to use the UV-IR homomorphism to determine the background gauge fields and spin connection of the IR theory in terms of the configuration of lattice

defects in the UV. Before we proceed, we review some basics concerning defects on the lattice.

A. Lattice defects and crystalline gauge fields

For the following discussion, it is not necessary (but it is very convenient) to describe lattice defects in terms of “crystalline gauge fields.” We use quotes because this formalism only makes sense for widely separated defects that can be clearly identified as disclinations and dislocations. (For more complicated lattice geometries, these gauge fields are subject to various constraints, but here we will only consider isolated elementary defects, for which the constraints on these gauge fields are unimportant.) Such crystalline gauge fields are nonetheless very useful for bookkeeping lattice defects. Having outlined this assumption, we drop the quotes below.

Suppose we have a rotation operator $C_{M,o}$ around some high-symmetry point o on the lattice which generates a $\mathbb{Z}_M$ symmetry. Disclinations are defined as $\mathbb{Z}_M$ defects, where we essentially cut out or add a wedge of angle $(2\pi)/M$ and reconnect the lattice via the symmetry action. This approach is described in detail in Refs. [23,25]. We employ a gauge field ω_o whose holonomies are $\int \omega_o \in [(2\pi)/M]\mathbb{Z}$, such that $\int \omega_o = 0$ corresponds to no disclination, and $\int \omega_o = [(2\pi k)/M]$ corresponds to removing k wedges of angle $(2\pi)/M$ for $k > 0$ or adding $|k|$ wedges of angle $(2\pi)/M$ for $k < 0$. Since we cannot remove more than M wedges, we have $k < M$. This finding should be interpreted as a constraint on the gauge field configurations realized by the cutting and gluing procedure we outlined.

Further, suppose we have a pair of translation operators $T_{\vec{v}_i}$ (for $\vec{v}_1$ and $\vec{v}_2$ the primitive vectors of the lattice), which together generate a $\mathbb{Z}^2$ symmetry. Dislocations are defects of this $\mathbb{Z}^2$ symmetry [20]. To record dislocations, we employ a gauge field $\vec{R}$ whose holonomies are valued in $\mathbb{Z}^2$, $[1/(2\pi)] \oint \vec{R} \in \mathbb{Z}^2$. Physically, the holonomy $[1/(2\pi)] \oint \vec{R}$ determines the Burgers vector for the region enclosed by the loop γ ; elementary lattice dislocations correspond to the case where the Burgers vector is an elementary lattice vector. For widely separated dislocations, it makes sense to discuss the gauge field $\vec{R}$.

In general, since rotations and translations do not commute, each defect is labeled by a conjugacy class in $\mathbb{Z}^2 \rtimes \mathbb{Z}_M$. For instance, to describe the effect of disclinations centered at another point o' in terms of ω_o , we need to include $\vec{R}$ holonomies. As a corollary, a vertex-centered disclination and a plaquette-centered antidisclination combine to form a dislocation.

We emphasize that there are considerable subtleties in defining the gauge fields ω_o and $\vec{R}$ from the lattice geometry; we refer to Refs. [25,72] for detailed discussions. As previously mentioned, for the most part, these

subtleties are unimportant here as we only discuss isolated elementary dislocations and disclinations.

B. UV-IR correspondence for symmetry defects

To describe in the infrared a lattice system at or near its gapless points, we consider a $(2+1)$ D theory of N free Dirac fermions, coupled to a background $U(N)$ gauge field A , a spin connection Ω , and mass matrix $\mathbf{M}$:

$$\mathcal{L} = \bar{\Psi} \left(i \not{D} + \not{A} + \frac{1}{2} \not{\Omega}_{\nu\rho} \sigma^{\nu\rho} + \mathbf{M} \right) \Psi. \quad (26)$$

Here, $\Psi = (\Psi_1, \dots, \Psi_N)^T$ consists of N 2-component Dirac fermions. The possible configurations of the parameters A , $\Omega_{\nu\rho}$, and $\mathbf{M}$ depend on the lattice model. In particular, assuming no spontaneous symmetry breaking, they have to be compatible with the lattice symmetries (e.g., $\mathbf{M}$ should commute with $U_{T_{\vec{v}_i}}$ and $U_{M,o}$ introduced in Sec. II B).

We argued above that a $g \in G_{\text{UV}}$ symmetry defect in the UV is modeled in the IR by a $\rho(g) \in G_{\text{IR}}$ symmetry defect. It is therefore convenient to expand the $U(N)$ gauge field A in terms of individual contributions from the UV symmetry defects. In other words, we need a map from the crystalline gauge fields ω_o and $\vec{R}$ to the infrared background fields A , $\Omega_{\nu\rho}$, and $\mathbf{M}$. We restrict ourselves to writing the map at the gapless point $\mathbf{M} = 0$ and discuss the consequences of this approach for nonzero M in the next section.

For simplicity, let us consider the case where $U_{T_{\vec{v}_i}}$ and $U_{M,o}$ are simultaneously diagonal, meaning the Dirac nodes occur at rotation-symmetric lattice momenta (we will relax this condition later).

Our UV theory has some configuration of the $U(1)$ gauge field A_{UV} along with lattice disclinations centered at o and lattice dislocations with Burgers vector $\vec{R}$. We claim that each fermion Ψ_i therefore sees a $U(1)$ gauge field

$$A^{(i)} = A_{\text{UV}} + s_o^{(i)} \omega_o + \frac{1}{2\pi} \vec{k}_\star^{(i)} \cdot \vec{R}, \quad (27)$$

where s_o is the topological quantum number (8) and $\vec{k}_\star$ is the lattice momentum of the Dirac mode. Recall that the holonomy of the ω_o gauge field around any loop γ is quantized as $\oint_\gamma \omega_o \in [(2\pi)/M]\mathbb{Z}$, measuring the disclination angle of the underlying lattice in the interior of γ , and $\oint \vec{R} \in 2\pi\mathbb{Z}^2$ keeps track of the Burgers vector of dislocations.

Furthermore, it is intuitively clear that a lattice disclination centered at o maps in the continuum theory to a geometry with a conical deficit angle of $2\pi k/M$. Thus, we set the spatial part of the spin connection to be

$$(\Omega_{xy})_\mu = \omega_{o,\mu}. \quad (28)$$

(We set the terms involving the time component to zero: $\Omega_{\mu t} = \Omega_{t\mu} = 0$, for $\mu = x, y$.)

Equation (27) shows the map between the gauge field that couples to the Dirac fermions in the continuum description and the gauge field on the lattice, A_{UV} . The appearance of $s_o^{(i)} \omega_o$ means that disclinations on the lattice activate a magnetic field in the continuum description. Similarly, the term $[1/(2\pi)] \vec{k}_\star^{(i)} \cdot \vec{R}$ means that dislocations lead to a magnetic field.

We refer to these magnetic fields in the IR theory that are induced by the UV lattice defects as emanant magnetic fields because the corresponding group element of $U(1)$ arises from the image of translations or rotations under ρ . Even if the $U(1)$ charge conservation in the UV were broken by weak perturbations, the IR would have an exact symmetry generated by $U_{T_{\vec{x}}}, U_{M,o} \subset U(1)$ if the crystalline symmetry is preserved. Thus, the $U(1)$ flux in the IR can be thought of as “emanating” from the translation and rotation symmetry defects in the UV, analogous to the discussion of Ref. [33].

An important part of the mapping between the lattice and continuum that we discussed above is that dislocations with Burgers vectors that are of the lattice scale activate a magnetic field in the continuum (with holonomy $[1/(2\pi)] \vec{k}_\star^{(i)} \cdot \vec{R}$) but carry no geometric consequences otherwise. As we will explain in more detail later, the point is that the geometric response due to such dislocations is irrelevant in the continuum limit. For Burgers vectors that remain finite in the continuum limit, the story may well be very different, and we do not discuss it here (indeed, lattice scale translations map to internal symmetries in the continuum limit, while macroscopic translations map to nontrivial translations in the continuum and hence will have geometric imprints as well).

Finally, if the Dirac fermions do not occur at momenta that are symmetric under $2\pi/M$ rotations, then, as we discussed, the transformation $U_{M,o}$ is not diagonal. In this case, the lattice disclinations induce a nondiagonal $U(N)$ monodromy. A lattice disclination corresponding to the operator $C_{M,o}$ induces a monodromy

$$\mathcal{P} e^{i \oint_{\gamma} A} = e^{i \oint_{\gamma} A_{UV}} U_{M,o}, \quad (29)$$

where we have also included the $U(1)$ gauge field A_{UV} and $\mathcal{P}$ denotes path ordering.

In summary, we have provided a prescription of how to compute the infrared background fields corresponding to certain lattice defects. A lattice disclination should be described in the IR in terms of a conical defect with $U(N)$ monodromy, while a lattice dislocation should be described in the IR purely in terms of a $U(N)$ monodromy. It is important that lattice dislocations and disclinations generically lead to emanant magnetic fields at long distances.

V. CRYSTALLINE INVARIANTS OF GAPPED PHASES

Our perspective of realizing the long-distance limit of lattice defects as standard gauge field configurations in the continuum (27) allows us to study the gapped (insulating) phases. In the continuum, this idea is straightforward and boils down to integrating out the massive fermions at one loop. Once the result is expressed in terms of the original lattice gauge fields, it leads to many predictions, as we will momentarily see. In particular, we show how the crystalline topological invariants predicted in Refs. [20,21,23–25] can be understood in terms of the Dirac fermion quantum numbers.

These results provide another way of deducing the Dirac fermion quantum numbers $s_o^{(i)}$ in terms of the crystalline topological invariants of the nearby gapped phases. They also demonstrate how crystalline symmetry-enriched quantum critical points between different crystalline topological phases can be distinguished and how the crystalline topological invariants can jump due to the appearance of gapless modes. Remarkably, we will see that the results match the results of the symmetry eigenvalue analysis of the preceding sections, as well as the results for the current at the gapless points presented in the subsequent sections.

A. From Dirac fermions to crystalline topological invariants

In this section, we derive expressions for previously studied topological invariants in terms of parameters appearing in an effective Dirac fermion IR description. We begin with the Lagrangian introduced in Sec. IV B, which included massive Dirac fermions coupled to background gauge fields, and integrate out the fermions. The terms in the resultant Lagrangian involve only gauge fields, and the coefficient of each term can be identified with a topological invariant. We always assume that $U_{M,o}$ and $U_{T_{\vec{x}}}$ are simultaneously diagonalized, so the Dirac cones occur at the $2\pi/M$ symmetric momenta. This assumption is important because it allows us to couple each Dirac fermion to a gauge field of the form in Eq. (27), with $s_o^{(i)}$ and $\vec{k}_\star^{(i)}$ simultaneously well defined.

Suppose that we consider a diagonal mass matrix, $\mathbf{M} = \text{diag}(m_1, \dots, m_N)$, so that the effective Lagrangian is given by

$$\mathcal{L} = \sum_{i=1}^N \bar{\Psi}_i (i \not{\partial} + \not{A}^{(i)} + m_i) \Psi_i. \quad (30)$$

Integrating out the fermions then gives an effective Lagrangian [73]

$$\mathcal{L} = \sum_{i=1}^N \frac{\text{sgn}(m_i)}{2} \left(\frac{1}{4\pi} A^{(i)} \wedge dA^{(i)} - \frac{1}{48\pi} \omega_o \wedge d\omega_o \right), \quad (31)$$

where the second term is the contribution from the gravitational Chern-Simons term. Expanding $A^{(i)}$, which is the $U(1)$ connection of the i th Dirac fermion, using Eq. (27), we obtain a response theory of the form

$$\begin{aligned} \mathcal{L} = & \frac{C}{4\pi} A_{UV} \wedge dA_{UV} + \frac{\mathcal{S}_o}{2\pi} A_{UV} \wedge d\omega_o + \frac{\tilde{\mathcal{L}}_o}{4\pi} \omega_o \wedge d\omega_o \\ & + \frac{\vec{\mathcal{P}}_o}{2\pi} \cdot A_{UV} \wedge d\vec{R} + \frac{\vec{\mathcal{Q}}_o}{2\pi} \cdot \omega_o \wedge d\vec{R} + \frac{1}{4\pi} \vec{R} \cdot \Pi \wedge d\vec{R}, \end{aligned} \quad (32)$$

where the coefficients C , $\mathcal{S}_o$, $\tilde{\mathcal{L}}_o$, etc., are dependent on the couplings $s_o^{(i)}$ and $\vec{k}_\star^{(i)}$ and the masses m_i , as presented in Eqs. (33)–(37). The coefficients are quantized topological invariants that characterize the gapped state with crystalline symmetry. In particular, these coefficients determine the response of the system to the introduction of symmetry defects. For example, the second term implies that a disclination (the symmetry defect associated with ω_o) with angle Ω accumulates electric charge (the symmetry charge associated with A_{UV}) equal to $[(\mathcal{S}_o)/(2\pi)]\Omega$. Note also that $X \wedge dY = Y \wedge dX$, in general, so the second term also implies that inserted magnetic flux accumulates angular momentum proportional to $\mathcal{S}_o$. The other terms have similar physical interpretations.

The Chern number of the topological insulator is

$$C = \sum_{i=1}^N \frac{\text{sgn}(m_i)}{2}, \quad (33)$$

which sets the quantized Hall conductivity. The discrete shift is

$$\mathcal{S}_o = \sum_{i=1}^N \frac{\text{sgn}(m_i) s_o^{(i)}}{2}, \quad (34)$$

which gives a fractional quantized contribution to $U(1)$ charge in the vicinity of a lattice disclination and determines the angular momentum of magnetic flux [23], as discussed above. Note that Ref. [23] demonstrated the constraint that $\mathcal{S}_o = (C/2) \bmod 1$. In our current description, we see that this constraint is a consequence of the fact that the orbital angular momenta $s_o^{(i)}$ are half-odd integers. Next, we have

$$\begin{aligned} \tilde{\mathcal{L}}_o &= \ell_o - \frac{c_-}{12} \\ \ell_o &= \sum_{i=1}^N \frac{\text{sgn}(m_i) (s_o^{(i)})^2}{2}, \end{aligned} \quad (35)$$

where $c_- = \sum_{i=1}^N \{[\text{sgn}(m_i)]/2\} = C$ is the chiral central charge. Note that ℓ_o can be determined in a quantum many-body system from ground-state expectation values of partial

rotation operations [24]. Reference [26] demonstrated the constraint $\ell_o = (C/4) \bmod \gcd(M, 2)$, which can also be understood from the above equation and the fact that $s_o^{(i)}$ are half-odd integers.

The electric polarization is

$$\vec{\mathcal{P}}_o = \frac{1}{2\pi} \sum_{i=1}^N \frac{\text{sgn}(m_i)}{2} \vec{k}_\star^{(i)}, \quad (36)$$

which can be detected in the gapped phase through a variety of methods, including quantized contributions to the electric charge in the vicinity of lattice dislocations and boundaries and the magnetic flux dependence of the momentum of the ground state [25,72]. The o dependence of the rhs is implicit in $\vec{k}_\star^{(i)}$. For example, for free fermion systems, the Bloch functions have an implicit choice of origin in real space, implying that $\vec{k}_\star^{(i)}$ has an implicit choice of real space origin, which is related to the choice o above [74].

Finally, we have

$$\begin{aligned} \vec{\mathcal{Q}}_o &= \frac{1}{2\pi} \sum_{i=1}^N \frac{\text{sgn}(m_i)}{2} s_o^{(i)} \vec{k}_\star^{(i)}, \\ \Pi_{\alpha\beta} &= \frac{1}{4\pi^2} \sum_{i=1}^N \frac{\text{sgn}(m_i)}{2} k_{\star,\alpha}^{(i)} k_{\star,\beta}^{(i)}. \end{aligned} \quad (37)$$

The terms in Eq. (32) above explain the origin of the crystalline topological gauge theory presented in Ref. [20] in terms of the quantum numbers of Dirac fermions.

The crystalline topological gauge theory of Ref. [20] includes two additional terms involving the charge and angular momentum per unit cell. These terms are outside of the Dirac theory and must be included separately from the outset since the Dirac theory does not include information about the charge and angular momentum per unit cell of the underlying lattice system.

The results above should be understood as providing us with the quantum critical theory that describes critical points between gapped phases where the crystalline topological invariants change their values. In particular, as we go through a transition where the mass of N Dirac fermions goes through zero, the changes in the invariants are given by the above formulas, with $\text{sgn}(m_i)$ replaced by the change in the sign of the masses across the transition, $\Delta \text{sgn}(m_i)$. If we know how the crystalline invariants change across a phase transition, we can invert the above equations to obtain a minimal massless Dirac theory with quantum numbers $\vec{k}_\star^{(i)}$, $s_o^{(i)}$ which reproduces the change in crystalline invariants.

The above result yields a simple understanding of some nontrivial properties of the crystalline invariants. The nontrivial relationship between $\mathcal{S}_o$, ℓ_o , and C was already mentioned above. Moreover, one can construct independent invariants of the gapped phase that correspond to

certain combinations of the above coefficients. For example, consider $M = 4$. Across a phase transition where the Dirac masses change sign, we have the independent invariants [26]

$$\begin{aligned}\Delta I_1 &:= \Delta \mathcal{S}_o - \Delta \ell_o - \Delta c_-/4 \in \mathbb{Z}_8, \\ \Delta I_2 &:= \frac{1}{2}(\Delta \ell_o - \Delta c_-/4) \in \mathbb{Z}_2,\end{aligned}\quad (38)$$

which generate a $\mathbb{Z}_8 \times \mathbb{Z}_2$ classification. The fact that these are the appropriate independent invariants and their quantization rules and equivalence relations can be understood from the Dirac theory using the equivalences $s_o^{(i)} \sim s_o^{(i)} + M$ for any given i . Similarly, the equivalence relation $\vec{k}_\star^{(i)} \sim \vec{k}_\star^{(i)} + \vec{K}$, where $\vec{K}$ is a reciprocal lattice vector, gives the equivalence relation for the electric polarization.

Under the assumption that all of the Dirac fermions occur at rotation-symmetric momenta, we can use the origin dependence of $s_o^{(i)}$ in Eq. (10) to determine the o dependence of $\Delta \mathcal{S}_o$. We find

$$\Delta \mathcal{S}_{o+\vec{v}} = \Delta \mathcal{S}_o + M \vec{f}(\vec{v}) \cdot \Delta \vec{\mathcal{P}}_o \pmod{M}. \quad (39)$$

Remarkably, Eq. (39) is consistent with the results of Ref. [25], which were derived using completely different methods. From Ref. [25], we know that Eq. (39) is valid, in general, and does not require the Dirac cones to be centered at rotation-symmetric momenta in the Brillouin zone. Therefore, a more general derivation than that presented above should be possible using the Dirac theory.

A thorough understanding of $\vec{\mathcal{Q}}_o$, Π , and the o dependence of ℓ_o is more complicated and beyond the scope of this paper. Here, we simply note that $\vec{\mathcal{Q}}_o$ and Π combine to give the angular-momentum polarization that was defined in Refs. [20,26].

Note that, using the results of Ref. [25], $\Delta \vec{\mathcal{P}}_o$ is independent of o when the charge per unit cell remains unchanged through the transition, a fact that we will use below, which is indeed the case in these transitions where the Dirac mass changes sign.

In cases where the Dirac fermions do not occur at $2\pi/M$ rotation symmetric momenta, we can use the above equations for $\mathcal{S}_o$, $\vec{\mathcal{P}}_o$, and ℓ_o by considering the response in terms of ω_o and $\vec{R}$ separately. However, the equation for $\vec{\mathcal{Q}}_o$ cannot be used directly.

B. Example: QWZ model

Using the method of partial rotations discussed in Refs. [24,26] and reviewed in Appendix A 2, we measure the topological invariants C , $\mathcal{S}_o$, and ℓ_o in the QWZ model. As we discussed in the previous section, the Dirac theory unambiguously captures changes in the topological

invariants rather than their absolute values. We therefore report the changes of these topological invariants across each of the three critical points in Table I, along with $\Delta \vec{\mathcal{P}}$, which is obtained using Eq. (39). We then deduce the Dirac theory with the minimal number of massless fermions and their corresponding values of $s_o^{(i)}$, $\text{sgn}(m_i)$, and $\vec{k}_\star^{(i)}$, which reproduce the changes in these invariants. We present the results in Table I.

Remarkably, the results agree precisely with those found in Sec. III A using completely different methods, through eigenvalues of rotation and translation operators. In particular, the number of massless fermions at each critical point and topological quantum numbers $s_o^{(i)}$ and $\vec{k}_\star^{(i)}$ match precisely. This correspondence is a nontrivial prediction of the relationship between the UV-IR homomorphism and the field theory analysis discussed above.

VI. DEFECT CONFORMAL FIELD THEORY AT DIRAC CRITICALITY

In the preceding sections, we saw that critical lattice models with crystalline defects map onto continuum theories with conical singularities and with excess magnetic field flux. Our discussion was for gapped theories, and we concentrated on the topological response, extending the Chern number of the insulator by the invariants in Eq. (32).

It is of both theoretical and experimental interest to understand how this excess flux and conical singularities influence observables at the critical point. In order to guide the search for suitable observables, we turn to the framework of defect conformal field theory. The massless bulk fermions preserve the conformal algebra in $2+1$ dimensions $so(3,2)$, which consists of the Lorentz symmetries along with scaling (the fermion field scales like mass) and special conformal transformations. Conical singularities and magnetic flux defects are pointlike in space and, by the general philosophy of the renormalization group, must flow to infrared fixed points, preserving the $so(1,2) \times so(2)$ subalgebra of the conformal algebra, where the $so(2)$ factor corresponds to rotations transverse to the defect while the $so(1,2) \simeq sl(2, \mathbb{R})$ factor corresponds to time translations, scaling symmetry about the location of the defect, and a special conformal transformation. (Naively, one only expects scale invariance at long distances, but the appearance of the full conformal group is common and, in many cases, can be proven to occur. For defects, this case was discussed in Ref. [75].)

For a thorough introduction to the kinematics of these symmetries, see Refs. [34–37]. For our present purposes, it would be most important to identify the fixed points of the defect RG flows and then infer their salient predictions for bulk physics.

We thus identify some relevant observables at the gapless points, understand the scaling of correlation functions, and predict their amplitudes as a function of flux. The results of

TABLE I. Changes in topological invariants and parameters in the Dirac fermion effective Lagrangian for the QWZ model. The values of $s_o^{(i)}$, $k_\star^{(i)}$, and $\text{sgn}(m_i)$ are deduced from matching the changes of the topological invariants to Eqs. (33)–(36). Results agree with those reported in Fig. 3, showing that these independent methods give the same result.

	$s_A^{(i)}$	$s_B^{(i)}$	$\text{sgn}(m_i)$	$\vec{k}_\star^{(i)}$	ΔC	$\Delta \mathcal{S}_A$	$\Delta \mathcal{S}_B$	$\Delta \ell_A$	$\Delta \ell_B$	$\Delta \vec{\mathcal{P}}$
$m = -1$	$\frac{1}{2}$	$\frac{1}{2}$	$+\rightarrow -$	$(0, 0)$	-1	$\frac{7}{2}$	$\frac{7}{2}$	$\frac{15}{4}$	$\frac{15}{4}$	$(0, 0)$
$m = 0$	$\frac{1}{2}$ $\frac{5}{2}$	$\frac{3}{2}$ $\frac{7}{2}$	$-\rightarrow +$ $-\rightarrow +$	$(\pi, 0), (0, \pi)$	2	3	1	$\frac{5}{2}$	$\frac{5}{2}$	$(\frac{1}{2}, \frac{1}{2})$
$m = 1$	$\frac{5}{2}$	$\frac{1}{2}$	$+\rightarrow -$	(π, π)	-1	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{7}{4}$	$\frac{15}{4}$	$(\frac{1}{2}, \frac{1}{2})$

Sec. VI determine what observables we measure in lattice models and help us to interpret the numerical results we find.

Crystalline defects modify the effective metric that emerges in the infrared description of the system (see, e.g., Ref. [76].) We have already discussed the fact that crystalline disclinations lead to conical singularities in space. More generally, one can parametrize a local geometric defect via the curvature

$$R = 4\pi(1 - \beta)\delta^2(x) + \ell(\vec{b} \cdot \partial)\delta^2(x) + O(\ell^2), \quad (40)$$

where ℓ is the ultraviolet scale associated with lattice spacing. The first term of $\delta^2(x)$ corresponds to a conical defect and emerges from lattice disclinations with a deficit angle $2\pi(1 - \beta)$.

The second term, proportional to $\ell\vec{b}$, could arise from lattice dislocations [76]. (Having two opposite disclinations next to each other does not necessarily imply there is a Burgers vector, but it can. Thus, $\vec{b}$ could but does not have to correspond to the Burgers vector on the lattice.) Because of the explicit factor of ℓ at the long-distance limit, the term proportional to $\ell\vec{b}$ is irrelevant and is therefore “screened” in the infrared. Thus, lattice-scale dislocations do not have a geometric imprint at long distances in the continuum description. Therefore, the spatial metric is determined solely by the conical singularity, and our Dirac cones see the continuum effective metric

$$(ds)^2 = -(dt)^2 + (dr)^2 + \beta^2 r^2 (d\theta)^2, \quad (41)$$

where r is the distance to the defect location and $\theta \in \mathbb{S}^1$ is the angular polar coordinate. Remarkably, the metric (41) admits conformal Killing vectors leading to the conformal symmetry algebra $so(1, 2) \times so(2) \subseteq so(3, 2)$, which is precisely the symmetry algebra of infrared fixed points of defects in gapless systems in $2 + 1$ dimensions. Therefore, the conical defect can be viewed as a conformal defect. (There is an important technical difference between the defect conformal field theory of a conical singularity and other, more familiar conformal defects. We will discuss this

difference later.) A simple way to see that the metric (41) admits $so(1, 2) \times so(2)$ conformal Killing vectors is to rewrite

$$(ds)^2 = r^2 \left(\frac{-(dt)^2 + (dr)^2}{r^2} + \beta^2 (d\theta)^2 \right) \sim \frac{-(dt)^2 + (dr)^2}{r^2} + \beta^2 (d\theta)^2, \quad (42)$$

where $\sim$ means Weyl equivalence that is up to a conformal factor. We recognize the factor $[-(dt)^2 + (dr)^2]/r^2$ as AdS_2 (a Minkowski version of the Poincaré disk), and $\beta^2 (d\theta)^2$ is just a circle of radius β . Therefore, the conical defect is conformally equivalent to $\text{AdS}_2 \times \mathbb{S}_\beta^1$, and the $so(1, 2) \times so(2)$ act on the two factors, respectively. For the calculations below, we will analytically continue to the Euclidean signature. The conical singularity is then at the boundary of the Poincaré disk. Of course, these various conformal transformations are allowed only in the massless Dirac theory, which is invariant under the conformal group.

The defect conformal field theory depends on the conical singularity as well as the magnetic fluxes at $r = 0$. In the $\text{AdS}_2 \times \mathbb{S}_\beta^1$ coordinates, the flux at the defect point translates to a holonomy of the gauge field (chemical potential) along $\mathbb{S}_\beta^1$. We will go back and forth between the $\text{AdS}_2 \times \mathbb{S}_\beta^1$ conventions and the original description (41) when convenient.

A. DCFT spectrum

In this section, we focus on a single Dirac cone. We also provide the analysis for the free boson in Appendix B, which has several qualitatively distinct properties that we highlight throughout the discussion.

The low-energy description of the single Dirac cone with the effective conical metric (41) reads

$$S_{\text{Dirac}} = \int_{\text{Cone}} d^3 X \bar{\Psi} \not{\nabla} \Psi. \quad (43)$$

In Eq. (43), $d^3 X$ and $\not{\nabla} = i\not{\partial} + \not{\mathcal{A}} + \frac{1}{2}\not{\phi}_{\nu\rho}\sigma^{\nu\rho}$ are the volume element and Dirac operator on the space (41), respectively.

We also activate a static connection $A_\mu = \alpha \delta_{\mu\theta}$, with $\alpha \in [0, 1)$ corresponding to the Aharonov-Bohm flux at $r = 0$. The explicit form of the Dirac operator is [77]

$$\mathcal{D} = \begin{pmatrix} -i\partial_t & e^{-i\theta} \left(\partial_r + \frac{-i\partial_\theta + \alpha - \frac{\beta}{2}}{r\beta} \right) \\ e^{i\theta} \left(\partial_r + \frac{i\partial_\theta - \alpha - \frac{\beta}{2}}{r\beta} \right) & i\partial_t \end{pmatrix}. \quad (44)$$

Familiar pointlike defects such as the Kondo defect or charge defects can often be dynamically screened; namely, their effect on the infrared is only through irrelevant operators, and the $so(1, 2) \times so(2)$ invariant defect theory is trivial. The conical defect and the monodromy defect (the monodromy defect is the Aharonov-Bohm flux, which leads to an additional phase $e^{i\oint A} = e^{2\pi i\alpha}$ when the Dirac particle Ψ is transported along a contour that encircles the origin) cannot be screened because they are attached to “branch cuts” that go all the way to infinity. Indeed, the monodromy defect can be viewed as the boundary of the $U(1)$ open symmetry surface while the conical defect is the boundary of the open rotations surface. One can say that the conical and monodromy defects are attached to topological surfaces that go all the way to infinity and that is why they cannot be screened.

Next, we need to find the renormalization group fixed points of the defect RG flows. This problem can be approached by first analyzing the near-defect behavior of the field Ψ at $r = 0$. An essential point is that there are different possible boundary conditions at the origin $r = 0$. Without choosing a concrete boundary condition, there is no consistent quantum system to discuss. One has to identify the boundary conditions that respect the defect conformal symmetry and then analyze the flows between different defect fixed points. The analyses of the modes in $\text{AdS}_2 \times \mathbb{S}_\beta^1$ coordinates vs the original coordinates are essentially identical, so we proceed with the $\text{AdS}_2 \times \mathbb{S}_\beta^1$ coordinates.

We can perform a mode expansion over $\mathbb{S}_\beta^1$ and consider the theory as consisting of infinitely many modes in AdS_2 . We denote these modes by $\tilde{\Psi}_s(t, r)$, and they are related to the original fermions in flat space via

$$\Psi(t, r, \theta) = \sum_{s \in \mathbb{Z} + \frac{1}{2}} \frac{e^{i(s - \frac{\sigma}{2})\theta}}{\sqrt{2\pi r}} e^{\frac{i\sigma}{4}\gamma} \tilde{\Psi}_s(t, r). \quad (45)$$

We note that the $e^{-i\frac{\sigma}{2}\theta}$ dependence is necessary to reflect the different spins of the two fermion components, and the $e^{\frac{i\sigma}{4}\gamma}$ factor will be important to diagonalize the action of $\tilde{\Psi}_s(t, r)$ in AdS_2 . The monodromy can also be understood in terms of the phonon $\sigma \in \mathbb{S}^1$ dual to the connection A_μ , such that the local operator $(e^{i\sigma}\Psi) \rightarrow e^{2\pi i\alpha}(e^{i\sigma}\Psi)$ upon $\theta \rightarrow \theta + 2\pi$.

Letting $d^2\tilde{X}$ and $\tilde{\nabla}$ be the volume element and Dirac operator of AdS_2 , we find that the action for our modes $\tilde{\Psi}_s(t, r)$ is

$$S_{\text{Dirac}} = \beta \sum_s \int_{\text{AdS}_2} d^2\tilde{X} \tilde{\Psi}_s \left(\tilde{\nabla} - \frac{s + \alpha}{\beta} \right) \tilde{\Psi}_s. \quad (46)$$

We see that our modes are standard massive fermions on the Poincaré disk, with mass $(s + \alpha)/\beta$. If we now analyze the $r \rightarrow 0$ behavior of the wave functions (which is the boundary of the Poincaré disk), we find two, distinct, consistent possibilities [see Eq. (C15)]:

$$\tilde{\Psi}_s = r^{\Delta_s^\pm} \begin{pmatrix} \psi_s^\uparrow(t) \\ \psi_s^\downarrow(t) \end{pmatrix} + O(r^{\Delta_s^\pm + 1}),$$

where $\Delta_s^\pm = \frac{1}{2} \pm \frac{|s + \alpha|}{\beta}$. (47)

The interpretation of having two distinct possible modes near the defect is crucial to understand. The same issue with boundary conditions was discussed at great length in the context of charged impurities in Refs. [78,79].

As a helpful digression, we note that something similar happens in the nonrelativistic quantum mechanics of a particle with the following Hamiltonian (see, for instance, Refs. [80,81]), $H = -[d^2/(dr^2)] + (h/r^2)$, which allows wave functions that behave for $r \rightarrow 0$ as either $r^{\frac{1}{2} \pm \sqrt{h+1/4}}$. For instance, if $-\frac{1}{4} < h < 0$, then both choices are normalizable and appear physically admissible. The meaning of this ambiguity is well known: The Hamiltonian is not Hermitian and requires an extension, which, in physical terms, means a choice of the $r \rightarrow 0$ boundary condition. Two choices that preserve the scale invariance of the differential operator $-[d^2/(dr^2)] + (h/r^2)$ are to disallow the $r^{\frac{1}{2} + \sqrt{h+1/4}}$ or to disallow the $r^{\frac{1}{2} - \sqrt{h+1/4}}$ mode. The former is a fine-tuned fixed point (unstable fixed point) while the latter is a stable fixed point.

The discussion of the meaning of the two solutions of Eq. (47) is entirely analogous. The modes in Eq. (47) are admissible as long as

$$\Delta_s^\pm \geq 0. \quad (48)$$

This positivity constraint on scaling dimensions is a consequence of unitarity. The solution (+) is known as the standard choice (or standard quantization), and the solution (−) is known as the alternative choice (or alternative quantization) [58,82,83]. Of course, as in the quantum mechanical digression, one can also choose mixed boundary conditions, but those that correspond to fixed points of the defect renormalization group flows are the ones where one of the modes is set to zero. The flow is

implemented by integrating bilinears involving $\psi_s^\uparrow$ and $\psi_s^\downarrow$ on the defect; we will discuss these bilinears later. The scaling dimension of ψ_s is $\Delta_s^\pm$. The $so(2)$ quantum numbers of $\psi_s^\uparrow$ and $\psi_s^\downarrow$ are $s + \alpha + \frac{1}{2}$ and $s + \alpha - \frac{1}{2}$, respectively. A special case worth mentioning is the flat space with no monodromy defect, where $\alpha = 0$, $\beta = 1$, and $\Delta_{\pm\frac{1}{2}}^- = 0$ saturates the bound (48). However, an operator of dimension 0 has to be decoupled from the bulk, and therefore the corresponding modes furnish the Hilbert space of a decoupled qubit on the defect. At $\alpha = 0$, the RG flow to the standard (trivial) fixed point corresponds to gapping the qubit. More generally, we can describe the alternative fixed point for generic values of α by coupling a qubit to the standard fixed point (which is a fermionic analog of the Hubbard-Stratonovich trick). This discussion is perfectly parallel to that of Ref. [84], though the setup is a little different.

As we demonstrate in Appendix C, Eqs. (C15) and (C17), whether we choose standard or alternative quantization is crucial and leads to entirely different physical predictions. Of course, it is more natural to choose the boundary conditions that correspond to stable infrared fixed points of the defect renormalization group flows.

Finally, we comment on the time-reversal symmetry $\mathcal{T}$ that changes the flux $\mathcal{T}:\alpha \rightarrow 1 - \alpha$. Naively, one expects the DCFT to be $\mathcal{T}$ symmetric when $\alpha = 0$ or $\alpha = \frac{1}{2}$. As we will see momentarily, that condition is true for $\alpha = 0$ but is more involved for $\alpha = \frac{1}{2}$. The subtlety can be traced to the bilinear form of the defect spinor $\psi_s = (\psi_s^\uparrow, \psi_s^\downarrow)$, which flips its sign under time reversal $\mathcal{T}:\bar{\psi}_s\psi_s \rightarrow -\bar{\psi}_s\psi_s$.

B. One-point functions and defect parameters

In this subsection, we continue the study of the conical and magnetic flux defect in the free fermion theory (43). At the standard fixed point $\Delta_s = \frac{1}{2} + (|s + \alpha|/\beta)$ for every $s \in \mathbb{Z} + \frac{1}{2}$, we note that all defect deformations are irrelevant for $\alpha \neq \frac{1}{2}$. [Recall that $\alpha \in [0, 1)$ is restricted to the fundamental domain.] Therefore, the standard fixed point is stable in the infrared, and we expect it to match with the lattice theory in the long-distance limit. Corrections are controlled by the cutoff scale of the field theory, which is typically identified with the lattice spacing ℓ . While bulk irrelevant operators lead to corrections suppressed by integer powers of ℓ , those from defect operators such as the defect Dirac masses $\int dt \bar{\psi}_s \psi_s$ scale as $O(\ell^{2|s+\alpha|/\beta})$. Thus, when $|\alpha - \frac{1}{2}| < (\beta/2)$, finite-size defect corrections dominate over those of the bulk. The defect deformation $\int dt \bar{\psi}_{-\frac{1}{2}} \psi_{-\frac{1}{2}}$ approaches marginality as $\alpha \rightarrow \frac{1}{2}$, and it leads to large finite-size corrections.

Note that there are no defect operators that can change the magnetic flux α or the conical angle β since both of these are measurable at infinity. Defect operators, roughly

speaking, can only change the boundary conditions at the defect itself.

The fact that at $\alpha = \frac{1}{2}$ there is a marginal operator in the continuum $\int dt \bar{\psi}_{-\frac{1}{2}} \psi_{-\frac{1}{2}}$ means that the large volume limit and the half-flux limit $\alpha \rightarrow \frac{1}{2}$ do not commute. If we start at $\alpha \neq \frac{1}{2}$ and slowly change α , we will land on a point on the conformal manifold that does not necessarily coincide with the point on which the lattice theory lands.

Quadratic deformations $\int dt \bar{\psi}_s \psi_s$ are analogous to double-trace deformations as in Ref. [85], and they lead to one-loop exact defect RG flows in certain schemes. For $|s + \alpha| < (\beta/2)$, such RG flows connect the alternative fixed point with the IR-stable standard fixed point. As mentioned previously, for $\alpha = \frac{1}{2}$, there is a defect conformal manifold.

The observables that we focus on in the infinite-volume continuum theory are one-point functions, especially the expectation value of the current in the vacuum and the expectation value of energy density. These observables are interesting theoretically and also natural to compare with lattice calculations. The one-point function of the $U(1)$ current $\langle J_\mu \rangle$ at defect fixed points is fixed by $sl(2, \mathbb{R}) \times so(2)$ symmetry and conservation up to a constant C_J [34], such that

$$\langle J_\mu \rangle \equiv \langle i\bar{\Psi}\gamma_\mu\Psi \rangle = \frac{\delta_{\mu\theta}}{2\pi r} C_J. \quad (49)$$

There is no charge density away from the defect, but there can be a current because particles spin around the defect and antiparticles (holes) spin in the opposite direction.

The constant C_J depends on the flux α and the conical singularity β . Similarly, the stress-energy tensor one-point function is fixed by conservation, tracelessness, and the conformal symmetry up to a single coefficient [our metric is as in Eq. (41), $g_{\mu\nu} = \text{diag}(-1, 1, \beta^2 r^2)$],

$$\langle T_{\mu\nu} \rangle \equiv \langle \bar{\Psi}\gamma_\mu\partial_\nu\Psi \rangle = \text{diag}(1, -1, 2\beta^2 r^2) \frac{C_T}{4\pi r^3}. \quad (50)$$

As a function of α and β , C_T is commonly referred to as the defect conformal weight [86,87]. When $\beta = 1$, we also interpret the component $\langle T_{tt} \rangle$ as the local energy density. Note that there are no off-diagonal components in Eq. (50), and there is therefore no angular-momentum density, which aligns with the physical picture that holes and antiholes are rotating around the defect in opposite directions. Finally, we also comment on the charge conjugation $\mathcal{C}:A_\mu \rightarrow -A_\mu$ and time reversal $\mathcal{T}$. The $\mathcal{CT}$ symmetry is preserved by the flux presence, and charged observables vanish identically if it is not spontaneously broken. Indeed, we find that the bulk fermionic condensate $\langle \bar{\Psi}\Psi \rangle = 0$. [Note that, in the free boson model, there is a condensate as shown in Eq. (B10).]

Here, C_J and C_T can be obtained by regularizing certain infinite sums,

$$\begin{aligned}
C_J &= -\beta \sum_{s \in \mathbb{Z} + \frac{1}{2}} \langle \tilde{\Psi}_s \tilde{\Psi}_s \rangle, \\
C_T &= -\frac{1}{\beta} \sum_{s \in \mathbb{Z} + \frac{1}{2}} (s + \alpha) \langle \tilde{\Psi}_s \tilde{\Psi}_s \rangle.
\end{aligned} \quad (51)$$

The detailed computation can be found in Appendix C, Eq. (C18). At the standard fixed point, for $\alpha < \frac{1}{2}$ and $\beta = 1$, we obtain

$$\begin{aligned}
C_J(\alpha) &= \frac{\tan(\pi\alpha)}{8} (1 - 4\alpha^2), \\
C_T(\alpha) &= \frac{\tan(\pi\alpha)}{12} \alpha (1 - 4\alpha^2).
\end{aligned} \quad (52)$$

For $\alpha > \frac{1}{2}$, it follows from the time-reversal symmetry $\mathcal{T}$ that $C_J(\alpha) = -C_J(1 - \alpha)$ and $C_T(\alpha) = C_T(1 - \alpha)$. The $s = -\frac{1}{2}$ alternative fixed point can be reached by fine-tuning the defect Dirac mass $\int dt \tilde{\psi}_{-\frac{1}{2}} \psi_{-\frac{1}{2}}$. From the decomposition (51), we find, for the multicritical fixed point,

$$\begin{aligned}
C'_J &= C_J + \left(\alpha - \frac{1}{2} \right) \tan(\pi\alpha), \\
C'_T &= C_T + \left(\alpha - \frac{1}{2} \right)^2 \tan(\pi\alpha),
\end{aligned} \quad (53)$$

where $\alpha < \frac{1}{2}$ and $\beta = 1$.

The conformal manifold at $\alpha = \frac{1}{2}$ is due to the $s = -\frac{1}{2}$ AdS₂ mode in Eq. (46) becoming massless. As a consequence, the alternative and the standard quantization are both part of a continuum of possible boundary conditions, all preserving the conformal symmetry. We find that, as we scan over the boundary conditions, the azimuthal current can take values $-[1/(2\pi)] \leq C_J \leq [1/(2\pi)]$ [88] while the conformal weight $C_T = [1/(6\pi)]$ is single valued. Different points of the conformal manifold are reached by the operator $\int dt \tilde{\psi}_{-\frac{1}{2}} \psi_{-\frac{1}{2}}$. We remark that, in general, the defect Dirac mass deformation $\int dt \tilde{\psi}_{-\frac{1}{2}} \psi_{-\frac{1}{2}}$ breaks the time-reversal symmetry $\mathcal{T}$. There is one point on the defect conformal manifold where $C_J = 0$, where $\mathcal{T}$ is preserved. Results regarding the one-point functions C_J and C_T are summarized in Fig. 4. We remark that, in the free boson model where the defect conformal manifold is absent, C_J and C_T in Eq. (B10) as functions of flux are analytic for $\alpha \in [0, 1)$.

An important characteristic of line defects is the g function, which is a monotone of defect RG flows [39, 89, 90] (see also Ref. [91] for earlier, important work). For monodromy defects in flat space ($\beta = 1$), it is easy to prove that, at defect fixed points,

$$\partial_\alpha \log g = 2\pi C_J(\alpha). \quad (54)$$

Therefore, we can compute the g function at fixed points if we only know the current, which, for the standard and

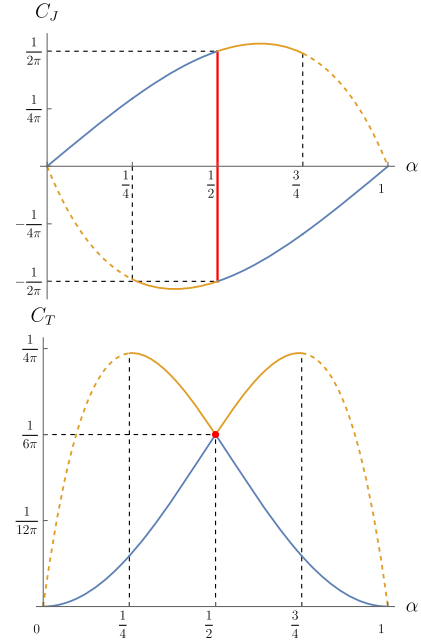


FIG. 4. Azimuthal current C_J and energy density C_T of the defect without conical singularity ($\beta = 1$). Throughout this paper, we use blue curves to denote values at the standard fixed point, while orange curves denote those at the alternative fixed points. The dashed part of the orange curve represents where the defect quartic interaction becomes relevant. The red line and point mark the values at the defect conformal manifold.

alternative fixed points, is given by Eqs. (52) and (53), respectively. From Fig. 4, we see that $g \geq 1$ for $\alpha \in [0, 1)$ of the standard quantization. Finally, for alternative quantization, $g(\alpha = 0) = 2$, which allows for a very simple interpretation of the alternative fixed point in the absence of the defect; namely, it is just a decoupled qubit. Finally, g has to be constant on conformal manifolds, as follows from it being a monotone of the renormalization group. Indeed, it can be explicitly verified that it is a constant on the conformal manifold at $\alpha = \frac{1}{2}$.

C. Flux displacement

An important hallmark of flat space defect conformal field theory is the existence of the displacement operator [92] (see also Refs. [93–95]). The physical reasoning is clear—there must be an operator that moves the location of the defect. Let us consider a local perturbation $\delta X_\perp^i(t)$ to the line defect location, where i 's are the transversal directions. By definition, the displacement operator D^i reads

$$\delta S = \int dt D^i(t) \delta X_\perp^i(t), \quad (55)$$

such that it characterizes the system's response to $\delta X_\perp^i$. From dimensional analysis, D^i is a defect operator of scaling dimension 2, and it clearly has transverse spin 1.

A monodromy defect can be moved around, so we certainly expect that the defect fixed points admit a displacement operator. Indeed, at the standard fixed point with $\beta = 1$, D^i is identified with the lowest-lying spin-1 defect operators $\tilde{\psi}_{-\frac{1}{2}}\psi_{\frac{1}{2}}$ and $\tilde{\psi}_{\frac{1}{2}}\psi_{-\frac{1}{2}}$, whose scaling dimension is exactly as required, $\Delta(\tilde{\psi}_{-\frac{1}{2}}\psi_{\frac{1}{2}}) = \Delta(\tilde{\psi}_{\frac{1}{2}}\psi_{-\frac{1}{2}}) = 2$.

The conical singularity is different. It cannot be moved around by manipulations in the region of the defect. [96] Technically, the difference stems from the fact that the metric, and not the spin connection, is the fundamental variable, while in gauge theories, the gauge connection is fundamental. The metric deformation required to move a conical defect around is non-normalizable, unlike the gauge field deformation. Remarkably, this phenomenon is the continuum manifestation of the well-known fact that disclinations are immobile in crystalline solids (see, e.g., Ref. [97] and references therein).

Indeed, at the standard fixed point with $\beta \neq 1$, we find from Eq. (47) that $\Delta(\tilde{\psi}_{-\frac{1}{2}}\psi_{\frac{1}{2}}) = \Delta(\tilde{\psi}_{\frac{1}{2}}\psi_{-\frac{1}{2}}) = 1 + \frac{1}{\beta}$. Such operators are the lowest-lying ones in the bulk-to-defect OPE of the $U(1)$ current operator. [98] Physically, they describe perturbations to the location of the magnetic flux while keeping the conical singularity unmoved.

D. Current and energy density for conical defects

We now compute the current and energy density in the presence of a conical defect. The details are in Appendix C. For C_J in Eq. (49) at the standard fixed point, we find an integral representation

$$C_J = -\frac{1}{\pi} \left[\int_0^1 \left(\frac{u^{\frac{\alpha}{\beta}} - u^{-\frac{\alpha}{\beta}}}{u^{\frac{1}{2\beta}} - u^{-\frac{1}{2\beta}}} \right) \frac{u^{\frac{v}{2}-1} du}{(1-u)^v} \right]_{v \rightarrow 2}, \quad (56)$$

where the integral $\int du$ is evaluated as the analytical continuation of the index $v \rightarrow 2$. Similarly, for C_T in Eq. (50), we find for the standard fixed point that

$$C_T = \frac{1}{\pi\beta^2} \left[\int_0^1 \left(\frac{(\frac{1}{2} - \alpha)(u^{\frac{2\alpha+2}{2\beta}} + u^{-\frac{2\alpha+1}{2\beta}})}{(u^{\frac{1}{2\beta}} - u^{-\frac{1}{2\beta}})^2} + \frac{(\frac{1}{2} + \alpha)(u^{\frac{2\alpha-1}{2\beta}} + u^{\frac{1-2\alpha}{2\beta}})}{(u^{\frac{1}{2\beta}} - u^{-\frac{1}{2\beta}})^2} \right) \frac{u^{\frac{v}{2}-1} du}{(1-u)^v} \right]_{v \rightarrow 2}. \quad (57)$$

As a consistency check, Eq. (52) is reproduced by taking $\beta = 1$. Denoting the spins for which we impose alternative quantization by $s' \in \mathbb{Z} + \frac{1}{2}$, the generalization of Eq. (53) reads

$$C'_J = C_J + \sum_{s'} \frac{s' + \alpha}{\beta} \cot\left(\pi \frac{|s' + \alpha|}{\beta}\right),$$

$$C'_T = C_T + \sum_{s'} \frac{(s' + \alpha)^2}{\beta^3} \cot\left(\pi \frac{|s' + \alpha|}{\beta}\right). \quad (58)$$

Remember that the $sl(2, \mathbb{R})$ unitarity bound requires that $|s' + \alpha| < (\beta/2)$. Some examples of results for C_J and C_T can be found in Fig. 5.

From Eq. (56), we find $C_J(\alpha = 0) = 0$ (as expected, there is no current in the absence of an Aharonov-Bohm

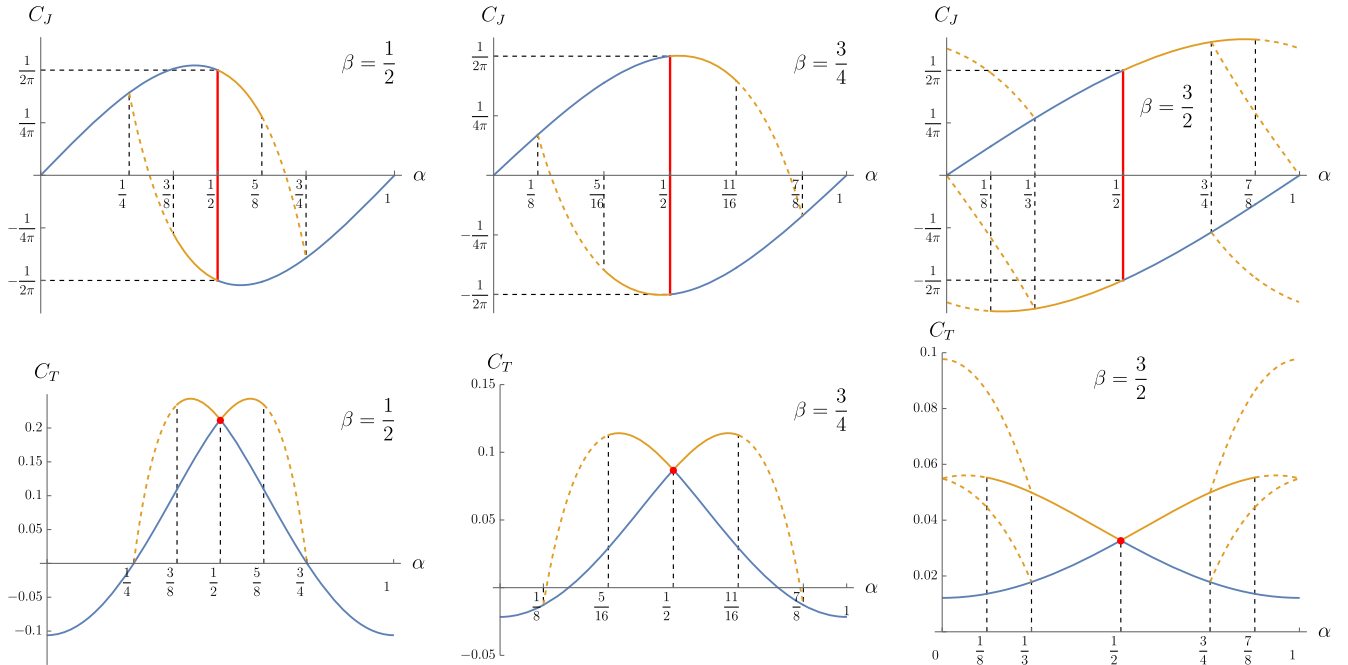


FIG. 5. Azimuthal current C_J and conformal weight C_T of the defects with conical singularity. From left to right: defects with a conical deficit ($\beta = \frac{1}{2}$ and $\beta = \frac{3}{4}$) and defects with a conical excess ($\beta = \frac{3}{2}$).

flux), and, remarkably, $C_J(\alpha = \frac{1}{2})$ ranges from $-[1/(2\pi)]$ to $1/(2\pi)$, regardless of the cone angle $2\pi\beta$.

Time-reversal symmetry imposes $C_J = 0$ at $\alpha = 0$ and $\alpha = \frac{1}{2}$. In this case, we cannot distinguish the half flux and the zero flux using the current only, but we can still measure a difference in the energy density:

$$C_T\left(\alpha = \frac{1}{2}\right) - C_T(\alpha = 0) = \frac{1}{4\pi\beta^2} \left[2 + \int_0^1 \left(\frac{u^{\frac{1}{2\beta}} - 1}{u^{\frac{1}{2\beta}} + 1} \right)^2 \frac{du}{(u-1)^2} \right] > 0, \quad (59)$$

which, among many other observables, distinguishes the time-reversal symmetric defects. [Notably, a similar inequality, Eq. (B11), also holds for the free boson model.]

VII. OBSERVABLES AT CRITICALITY IN LATTICE MODELS

In this section, we discuss how to numerically measure currents and energy densities in lattice models with a disclination or dislocation. We then present numerical results for several free fermion models at criticality, including the QWZ model we have already begun to discuss. The Julia code used to produce these numerical results is publicly available [99].

Our numerical results confirm the following nontrivial theoretical predictions of the defect CFT:

- (i) We find a universal shift in the azimuthal current and energy density as a function of applied flux due to the emanant flux of the disclination and dislocations. This finding provides a direct way to measure the invariants $\{s_o^{(i)}, k_\star^{(i)}\}$ at criticality and confirms the existence of the emanant flux.
- (ii) We find that the absolute values of the maxima and minima of $C_T(\alpha)$ are larger in the presence of the disclination as compared to without the disclination.
- (iii) We find qualitative agreement for the shape of $C_T(\alpha)$ and $C_J(\alpha)$ as a function of the applied flux.
- (iv) The azimuthal current with and without particle-hole symmetry provides evidence for the existence of the conformal manifold at $\alpha = 1/2$.

We have not extracted the scaling exponents $\Delta_s^\pm$, which we leave to future work.

A. Numerical procedure generalities

As we have discussed, we expect a disclination or dislocation to contribute an additional $U(1)$ flux to the continuum fermions proportional to $s_o^{(i)}$ or $\vec{k}_\star^{(i)} \cdot \vec{b}$, respectively. This contribution can be understood as a consequence of the $s_o^{(i)} \omega_o$ and $[1/(2\pi)] \vec{k}_\star^{(i)} \cdot \vec{R}$ terms in the effective Lagrangian as well as the UV-IR homomorphism. The discussion of Sec. VI points to the local energy density $\mathcal{E}_i$

and the azimuthal current J_i^θ as universal observables sensitive to this emanant flux. These quantities can be directly measured in lattice models with or without a crystalline defect.

At critical points where there is only a single Dirac cone, these microscopic observables capture the contribution due to the corresponding massless Dirac fermion. The microscopic current is defined via the continuity equation

$$\frac{\partial n_i}{\partial t} + \vec{\nabla} \cdot \vec{J}_i = 0, \quad (60)$$

where $n_i = \sum_\sigma c_{i,\sigma}^\dagger c_{i,\sigma}$ is the number operator on site $\vec{i}$, $\vec{J}_i$ is the current at site $\vec{i}$, and the divergence is defined on the lattice. The time derivative can be evaluated via Heisenberg evolution of the number operator, and the resulting equation can be solved for the component of the current in the $\vec{r}$ direction:

$$J_i^{\vec{r}} = 2i \sum_{a,b} h_{i,\vec{r}}^{a,b} c_{i,a}^\dagger c_{i+\vec{r},b} - (h_{i,\vec{r}}^{a,b})^* c_{i+\vec{r},a}^\dagger c_{i,b}. \quad (61)$$

To obtain the azimuthal current in which we are interested, we project $J_i^{\vec{r}}$ along the x and y axes to obtain J_i^x and J_i^y . We finally obtain J_i^θ via the linear combination $J_i^\theta = -i_y J_i^x + i_x J_i^y$. The local energy density is defined as

$$\mathcal{E}_i = \sum_{j,a,b} H_{ij}^{ab} = \sum_{j,a,b} h_{ij}^{ab} c_{i,a}^\dagger c_{j,b}. \quad (62)$$

We are interested in the expectation values of these operators in the many-body ground state.

The current as obtained above requires us to define spatial x and y directions. This requirement presents no difficulty on a clean lattice or a lattice with a dislocation. However, for a lattice with a disclination, these directions cannot be defined globally. We can still define them locally and use this local definition to measure the current numerically in a particular patch of the lattice.

In the case where there are multiple massless Dirac fermions at a single critical point (i.e., multiple valleys), the microscopic observables are no longer a good proxy for the one-point functions of the individual Dirac fermions. In some cases, it is possible to explicitly define a valley current and energy density in terms of microscopic creation and annihilation operators, but this can be a subtle process due, in part, to the absence of a valley symmetry in the UV. We will discuss an example where we were able to define valley observables in Sec. VII C 1.

In order to extract the emanant flux due to a lattice defect, we insert flux $\delta\alpha \in [0, 1]$ (in units of 2π) at the defect core and measure the azimuthal current J_i^θ and energy density $\mathcal{E}_i$ at various spatial points in the bulk as a function of $\delta\alpha$. We plot the quantities $r J_i^\theta$ and $r^3 \mathcal{E}_i$, where r is the distance from the defect. Because J_i^θ and $\mathcal{E}_i$ are predicted to

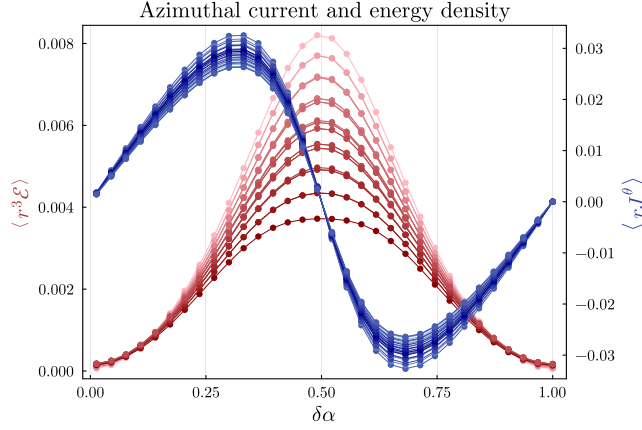


FIG. 6. QWZ model, clean lattice. We plot observables at the $m = -1$ critical point on a lattice without crystalline defects, scaling by r and r^3 , respectively, in order to extract the universal amplitude data. Each curve gives the values of the observables at a particular spatial position in the bulk of the lattice. The shading of the curve corresponds to the distance of the spatial point from the flux insertion center, with lighter colors closer.

scale as $1/r$ and $1/r^3$, respectively, this quantity extracts the universal amplitude of the observables and demonstrates the finite-size deviation from the fixed-point prediction.

In a model without defects, we expect that $J^\theta = 0$ and $\mathcal{E}$ is minimized when $\delta\alpha = 0$. As seen in Fig. 6 for the QWZ model, this occurs when J^θ crosses from negative to positive values for the flux insertion convention we have chosen. To measure the excess flux due to a crystalline defect, we measure the shift in this point away from $\delta\alpha = 0$. The zero crossing in J^θ and minimum in $\mathcal{E}$ will always be at total flux $\alpha = 0$, where

$$\alpha = \delta\alpha + \alpha_{\text{em}} \quad (63)$$

and α_{em} is the emanant flux due to the defect. Thus, if this point is shifted to some nonzero $\delta\alpha$, then the flux due to the defect is $\alpha_{\text{em}} = -\delta\alpha$. In each case, we have calculated the average $\delta\alpha$ at which (1) the two zero crossings in J^θ and (2) the minimum and maximum of $\mathcal{E}$ occur for the plotted data. These measured values are plotted along with the predicted values based on the expected excess flux for each defect.

B. QWZ model

In this section, we continue with the QWZ model example that was discussed in Secs. III A and V B. Because there are two inequivalent choices of the fourfold rotation center in the QWZ model, there are two distinct disclinations that can be inserted into the lattice. These disclination Hamiltonians are constructed using the lattice rotation operators $C_{4,A}$ and $C_{4,B}$ following the procedure discussed in Ref. [23].

The $m = \pm 1$ critical points each have a single Dirac cone, so we can use the microscopic quantities as a proxy for the field theory quantities associated with the Dirac fermion. The $m = 0$ critical point has two Dirac cones and therefore requires a more complicated treatment, so we do not study it here. The measured observables are reported in Fig. 7 for several crystalline defects.

Notice that there is strong agreement between the measured and predicted zero crossing from negative to positive values of J^θ but not always for the zero crossing from positive to negative J^θ . The former corresponds to $\alpha = 0$, while the latter is around $\alpha = \frac{1}{2}$. This phenomenon can be understood in terms of the analysis of Sec. VI. In that section, we showed that there is a conformal manifold of fixed points for $\alpha = \frac{1}{2}$. This finding is precisely the set of field theories corresponding to the situation where $\alpha = \frac{1}{2}$ on the lattice, which has the implication that—unlike at $\alpha = 0$ —the current is *not* forced to be zero at $\alpha = \frac{1}{2}$. In other words, the UV theory at $\alpha = \frac{1}{2}$ does not flow to the IR theory with $J^\theta = 0$ in these cases but to another theory within the conformal manifold at $\alpha = \frac{1}{2}$. This result causes the second zero crossing to be near, but not necessarily at, $\alpha = \frac{1}{2}$. This understanding is quite general and can be applied to any of the models that we study.

In contrast, the measured minima *and* maxima of $\mathcal{E}$ are generally quite close to $\alpha = 0$ and $\frac{1}{2}$. This feature can be understood from the fact that the energy density is invariant throughout the conformal manifold at $\alpha = \frac{1}{2}$, so the conformal manifold is not a source of ambiguity. However, in a finite-size system, numerical results receive corrections from irrelevant operators. When $\alpha \rightarrow \frac{1}{2}$, the dominant corrections are those from the defect fermion bilinear $\int d\tau \bar{\psi}_{-\frac{1}{2}} \psi_{-\frac{1}{2}}$, which is also the deformation that connects the IR-stable standard fixed point with the multicritical alternative one.

Despite the flexibility that is, in principle, afforded by the conformal manifold, the current may be pinned to zero at $\alpha = \frac{1}{2}$ due to additional symmetry. For example, this occurs in the QWZ model due to the presence of the particle-hole symmetry discussed in Sec. III A. This particle-hole symmetry is preserved by the *A*-centered disclination and $\vec{b} = (-1, 0)$ dislocation for $\alpha = 0$ or $\frac{1}{2}$ but not by the *B*-centered disclination. As a result, J^θ is pinned to zero at $\alpha = 0$ and $\frac{1}{2}$. Essentially, the symmetry picks out a particular fixed point in the conformal manifold, namely, that which satisfies $J^\theta = 0$ at $\alpha = \frac{1}{2}$.

We also emphasize that the vertical profiles of the energy-density curves plotted in Fig. 7 are consistent with several predictions from the previous section. In particular, we see that the minimum values are below zero, and the range of energies is larger than on the clean lattice, confirming the theoretical prediction from the defect CFT. While the offset of the energy density along the vertical axis may seem arbitrary because we are free to redefine the zero-point

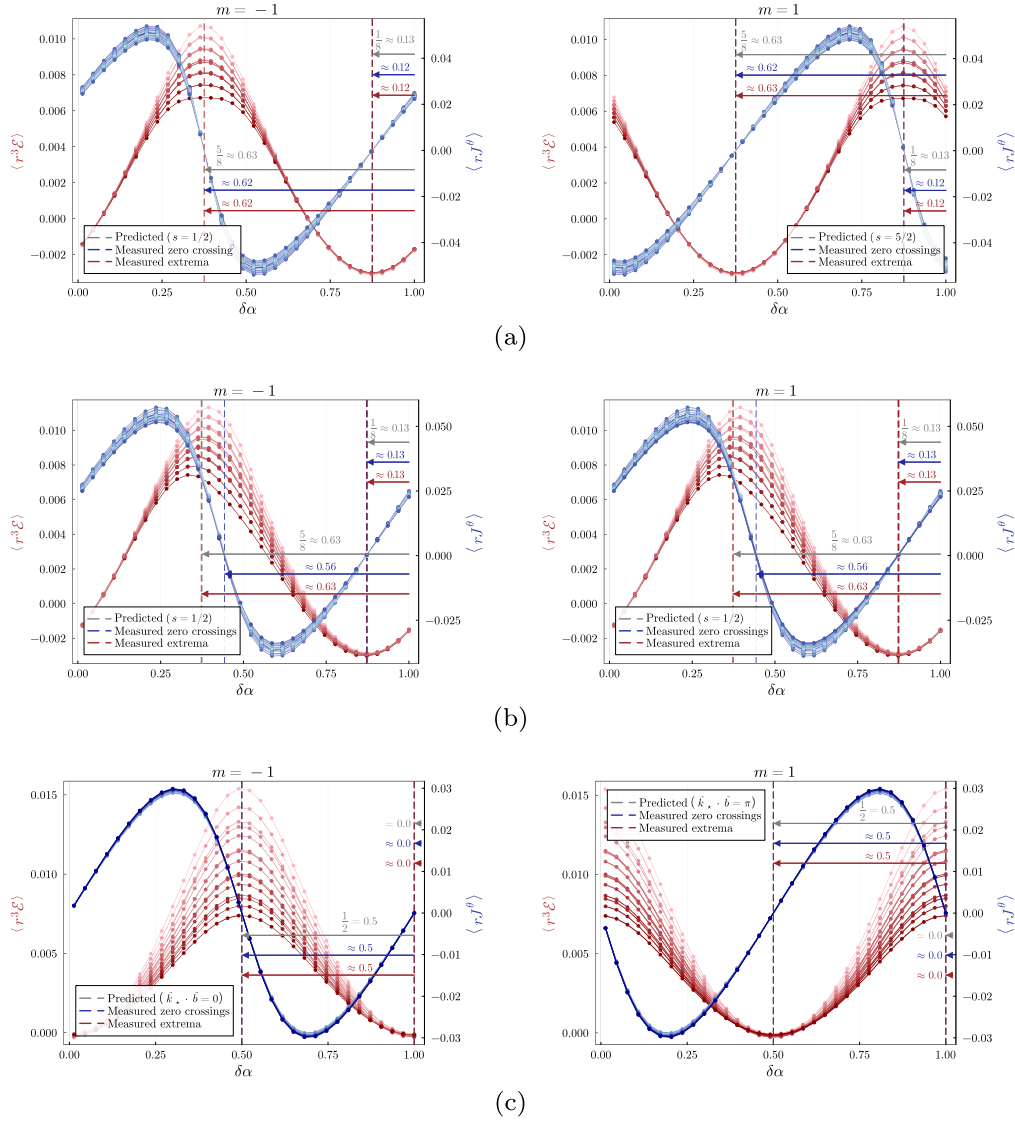


FIG. 7. QWZ model with crystalline defects. Here, we plot the scaled current and energy density in the QWZ model for the $m = \pm 1$ critical points and for several defects. The shading of each curve in the plot indicates the distances from the defect, with lighter curves corresponding to lattice sites closer to the defect. (a) Lattice with $N = 5766$ sites and an A -centered $\beta = 3/4$ disclination. (b) Lattice with $N = 5582$ sites and a B -centered $\beta = 3/4$ disclination. (c) Lattice with $N = 5460$ sites and a $\vec{a}\vec{b} = (-1, 0)$ dislocation.

energy, there is actually a preferred zero point because we are scaling the energy density by r^3 . We find that energy-density curves corresponding to different lattice sites all nearly coalesce at a particular raw value of the energy density. We expect $\langle r^3 \mathcal{E} \rangle$ to be invariant across lattice sites up to finite-size corrections, so we want to preserve this coalescence when we scale by r^3 . We do so by defining $\mathcal{E} = 0$ to be at the point of coalescence, which allows us to talk about positive and negative values of $r^3 \mathcal{E}$ in an absolute sense.

C. π -flux models

We discuss several π -flux models defined on the square lattice, which include nearest-neighbor and next-nearest-neighbor hoppings as well as possibly nonzero on-site

potentials. They are called π -flux models because the hoppings are chosen such that π flux pierces each plaquette of the square lattice. The π -flux models are of the form

$$H = \sum_{\langle \vec{i}, \vec{j} \rangle} e^{iA_{\vec{i}\vec{j}}} c_{\vec{i}}^\dagger c_{\vec{j}} + \frac{t'}{4} \sum_{\langle\langle \vec{i}, \vec{j} \rangle\rangle} e^{iA_{\vec{i}\vec{j}}} c_{\vec{i}}^\dagger c_{\vec{j}} + \Delta \sum_{\vec{i}} \mu_{\vec{i}} c_{\vec{i}}^\dagger c_{\vec{i}}, \quad (64)$$

where single and double brackets indicate sums over nearest and next-nearest neighbors, respectively. Note that $A_{\vec{i}, \vec{j}}$ is a static gauge configuration that inserts π flux through each plaquette, and $\mu_{\vec{i}}$ specifies the configuration of the on-site potential. The three Hamiltonians we study share the same hopping terms and differ only in their

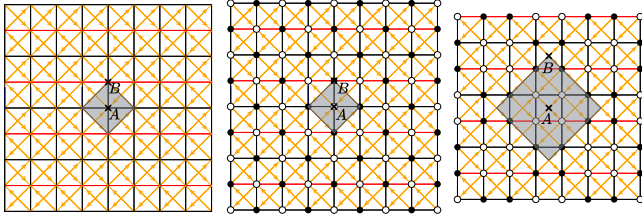


FIG. 8. Real-space π -flux Hamiltonians with no on-site potential and one- and two-site staggered potentials. Black lines correspond to $h_{i,j} = 1$, red lines to $h_{i,j} = -1$, and orange lines to $h_{i,j} = it'/4$. White and black circles represent the on-site potentials $+\Delta$ and $-\Delta$, respectively. The rotation centers are marked with black crosses and labeled.

on-site potentials. The exact Hamiltonians are presented in Fig. 8.

1. No on-site potential

The simplest π -flux model is given by setting $\mu_{\vec{r}} = 0$ everywhere in Eq. (64). This model has a critical point at $t' = 0$ with a pair of Dirac cones. We consider two inequivalent rotation centers, located at the points labeled A and B in Fig. 8. The gauge transformation accompanying each rotation is given by

$$\lambda_{\vec{r}}^{4,A} = \begin{cases} -\pi/2 & i_x, i_y \text{ even} \\ \pi/2 & \text{otherwise,} \end{cases} \quad \lambda_{\vec{r}}^{4,B} = \begin{cases} 0 & i_x, i_y \text{ odd} \\ \pi & \text{otherwise,} \end{cases} \quad (65)$$

where, in each case, we take the origin to be at a site with both coordinates odd. This model also exhibits a translation symmetry along its two primitive lattice vectors:

$\vec{v}_1 = (1, 1)$ and $\vec{v}_2 = (-1, 1)$. The accompanying gauge transformations are given by

$$\lambda_{\vec{r}}^{T_{\vec{v}_1}} = -\lambda_{\vec{r}}^{T_{\vec{v}_2}} = \begin{cases} -\pi/2 & i_x \text{ even} \\ \pi/2 & \text{otherwise.} \end{cases} \quad (66)$$

These transformations must be chosen in such a way that the resulting operators are symmetries and that they form a faithful representation of the spatial symmetry group.

As we discussed in Sec. III, one method to extract topological data at the critical point is to measure the eigenvalues of the symmetry operators. Repeating this analysis in the current model at $t' = 0$, we find

$$s_A^{(i)} = \frac{1}{2}, \quad \frac{3}{2}, \quad s_B^{(i)} = \frac{1}{2}, \quad \frac{7}{2}, \quad \vec{k}_{\star}^{(i)} = (0, 0), \quad (\pi, \pi). \quad (67)$$

Alternatively, we can deduce $s_o^{(i)}$ and $\vec{k}_{\star}^{(i)}$ by first measuring topological invariants in the gapped phases using partial rotations, then constructing the effective Lagrangian Eq. (26) governing the critical point, as discussed in Sec. VA. We report in Table II(a) the changes in each topological invariant across the transition as well as the parameters appearing in the effective Lagrangian that reproduces them. (The absolute values of the measured invariants are given in Appendix D 3.)

Notice that the values of the critical point invariants $s_o^{(i)}$ and $\vec{k}_{\star}^{(i)}$ obtained via the effective Lagrangian exactly match those obtained in Eq. (67) by symmetry eigenvalues. This finding again shows the remarkable agreement between two independent methods of extracting these invariants. It is also interesting to note that the symmetry eigenvalue method involves only measurements at the critical point, while the effective Lagrangian method involves only measurements in the gapped phases; thus, it is possible to obtain $s_o^{(i)}$ and $\vec{k}_{\star}^{(i)}$ in either restricted setting.

TABLE II. Topological invariants in each phase of the three π -flux models discussed in Sec. VII C.

	$s_A^{(i)}$	$s_B^{(i)}$	$\text{sgn}(m_i)$	$\vec{k}_{\star}^{(i)}$	ΔC	$\Delta \mathcal{S}_A$	$\Delta \mathcal{S}_B$	$\Delta \mathcal{L}_A$	$\Delta \mathcal{L}_B$	$\Delta \vec{\mathcal{P}}$
$t' = 0$	$\frac{1}{2}$	$\frac{1}{2}$	$+\rightarrow-$	$(0, 0)$	-2	2	0	$\frac{3}{2}$	$\frac{7}{2}$	$(\frac{1}{2}, \frac{1}{2})$
	$\frac{3}{2}$	$\frac{7}{2}$	$+\rightarrow-$	(π, π)						
(a) No on-site potential.										
	s_A	s_B	$\text{sgn}(m)$	$\vec{k}_{\star}$	ΔC	$\Delta \mathcal{S}_A$	$\Delta \mathcal{S}_B$	$\Delta \mathcal{L}_A$	$\Delta \mathcal{L}_B$	$\Delta \vec{\mathcal{P}}$
$t' = -\Delta$	$\frac{7}{2}$	$\frac{7}{2}$	$+\rightarrow-$	$(0, 0)$	-1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{15}{4}$	$\frac{15}{4}$	$(0, 0)$
$t' = \Delta$	$\frac{5}{2}$	$\frac{1}{2}$	$+\rightarrow-$	(π, π)	-1	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{7}{4}$	$\frac{15}{4}$	$(\frac{1}{2}, \frac{1}{2})$
(b) One-site staggered potential.										
	s_A	s_B	$\text{sgn}(m)$	$\vec{k}_{\star}$	ΔC	$\Delta \mathcal{S}_A$	$\Delta \mathcal{S}_B$	$\Delta \mathcal{L}_A$	$\Delta \mathcal{L}_B$	$\Delta \vec{\mathcal{P}}$
$t' = -\Delta$	$\frac{7}{2}$	$\frac{7}{2}$	$+\rightarrow-$	$(0, 0)$	-1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{15}{4}$	$\frac{15}{4}$	$(0, 0)$
$t' = \Delta$	$\frac{3}{2}$	$\frac{3}{2}$	$+\rightarrow-$	$(0, 0)$	-1	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{7}{4}$	$\frac{7}{4}$	$(0, 0)$
(c) Two-site staggered potential.										

Because this model has two Dirac cones at its critical point, the microscopic current and energy density are no longer a good proxy for the field theory quantities discussed in the previous section. However, we can define a valley current and valley energy density by writing the Hamiltonian in the vicinity of each Dirac point in the Dirac basis, $H = \Psi^\dagger \vec{p} \cdot \vec{\sigma} \Psi$. Using this Ψ , we can express the field theory current and energy-density operators in terms of annihilation operators on the lattice and measure them directly. The results, which are presented in Fig. 17, are consistent with the predicted emanant flux.

2. One-site staggered on-site potential

The next π -flux model we discuss is given by setting $\mu_{\vec{r}} = (-1)^{i_x+i_y}$ in Eq. (64). This model has critical points at $t' = \pm\Delta$, each of which has a single Dirac cone. We consider two inequivalent rotation centers, located at the points labeled *A* and *B* in Fig. 8. Notice that both *A* and *B* are lattice sites in this case. The gauge transformation accompanying each rotation is given by

$$\begin{aligned} \lambda_{\vec{r}}^{4,A} &= \begin{cases} \frac{\pi}{2} & i_x, i_y \text{ even} \\ -\frac{\pi}{2} & \text{otherwise,} \end{cases} \\ \lambda_{\vec{r}}^{4,B} &= \begin{cases} 0 & i_x, i_y \text{ odd} \\ \pi & \text{otherwise,} \end{cases} \end{aligned} \quad (68)$$

where, once again, we take the origin to be located at odd coordinates in each case. This model also exhibits a translation symmetry along the primitive lattice vectors $\vec{v}_1 = (1, 1)$ and $\vec{v}_2 = (-1, 1)$. The accompanying gauge transformation is given by

$$\lambda_{\vec{r}}^{T_{\vec{v}_1}} = -\lambda_{\vec{r}}^{T_{\vec{v}_2}} = \begin{cases} -\frac{\pi}{2} & i_x \text{ odd} \\ \frac{\pi}{2} & \text{otherwise.} \end{cases} \quad (69)$$

The values of the invariants extracted via eigenvalues of the crystalline symmetry operators at the critical points are given by

$$\begin{aligned} t' = -\Delta: s_A &= \frac{7}{2}, \quad s_B = \frac{7}{2}, \quad \vec{k}_\star = (0, 0), \\ t' = \Delta: s_A &= \frac{5}{2}, \quad s_B = \frac{1}{2}, \quad \vec{k}_\star = (\pi, \pi). \end{aligned} \quad (70)$$

The changes in topological invariants across the transition as measured through partial rotations and deduced effective Lagrangian parameters are given in Table II(b). Notice, once again, that the values of s_o and $\vec{k}_\star$ obtained via our two independent methods are identical. We present results for the azimuthal current and local energy density in the presence of lattice defects in Fig. 18. We find that the curves are shifted along the $\delta\alpha$ axis by the predicted emanant flux, namely, $s_o/4$ and $\frac{1}{2\pi}\vec{k}_\star \cdot \vec{b}$.

3. Two-site staggered on-site potential

The final π -flux model we study is given by setting $\mu_{\vec{r}} = 1.68 \times (-1)^{\lfloor \frac{i_x+i_y}{2} \rfloor}$ in Eq. (64), where the scale is chosen so that the critical points remain at $t' = \pm\Delta$. This model exhibits a fourfold magnetic rotation symmetry about two inequivalent rotation centers, located at the points labeled *A* and *B* in Fig. 8. Notice that both *A* and *B* are plaquette centers in this case. The rotation operators about each center are defined with the following gauge transformation:

$$\lambda_{\vec{r}}^{4,A} = \lambda_{\vec{r}}^{4,B} = \begin{cases} 5\pi/4 & i_x, i_y \text{ even} \\ \pi/4 & \text{otherwise,} \end{cases} \quad (71)$$

where we take the site directly to the lower left of each rotation center to have both coordinates even. This model has a translation symmetry along its two primitive lattice vectors: $\vec{v}_1 = (2, 2)$ and $\vec{v}_2 = (-2, 2)$, with a trivial accompanying gauge transformation.

Extracting the topological data through symmetry eigenvalues, we find

$$\begin{aligned} t' = -\Delta: s_A &= s_B = \frac{7}{2}, \quad \vec{k}_\star = (0, 0), \\ t' = \Delta: s_A &= s_B = \frac{3}{2}, \quad \vec{k}_\star = (0, 0). \end{aligned} \quad (72)$$

The values of the topological invariants that are instead measured by performing partial rotations and forming the effective Lagrangian are given in Table II(c). Once again, the values obtained via the two methods coincide. We present results for the azimuthal current and local energy density in the presence of lattice defects in Fig. 19. We find that the curves are shifted along the $\delta\alpha$ axis by the predicted emanant flux, namely, $s_o/4$.

D. Haldane model

In this section, we study the Haldane model [100], the prototypical example of a Chern insulator, with a staggered on-site potential. Unlike the previous models we have discussed, this model has a threefold rotational symmetry. It is given by

$$H = \sum_{\langle \vec{i}, \vec{j} \rangle} c_{\vec{i}}^\dagger c_{\vec{j}} + \frac{it'}{4} \sum_{\langle\langle \vec{i}, \vec{j} \rangle\rangle} \nu_{\vec{i}\vec{j}} c_{\vec{i}}^\dagger c_{\vec{j}} + \Delta \sum_{\vec{i}} \mu_{\vec{i}} c_{\vec{i}}^\dagger c_{\vec{i}}, \quad (73)$$

where $\nu_{\vec{i}\vec{j}}$ assigns signs to the next-nearest-neighbor hoppings such that they are always clockwise and $\mu_{\vec{i}} \approx \pm 1.3$ on the two sublattices of the honeycomb lattice. There are a pair of critical points at $t' = \pm\Delta$, each of which has a single Dirac cone.

This model has a threefold rotation symmetry about the three inequivalent rotation centers labeled *A*, *B*, and *C* in Fig. 9. In each case, the accompanying gauge

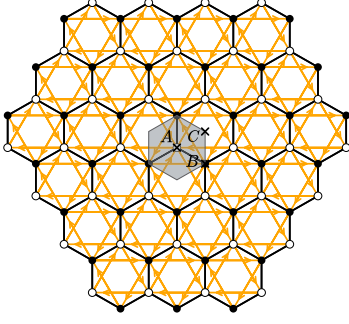


FIG. 9. Real-space Haldane model Hamiltonian where black lines correspond to $h_{i,j} = 1$ and orange lines to $h_{i,j} = i\tau/4$. White and black circles represent the on-site potentials $+\Delta$ and $-\Delta$, respectively. The rotationally invariant unit cell is shaded in gray, and inequivalent rotation centers are marked with black crosses and labeled.

transformation is trivial, so the symmetries are ordinary rotations not magnetic rotations. We also have a translation symmetry by the lattice vectors $\vec{v}_1 = (1, 0)$ and $\vec{v}_2 = [\frac{1}{2}, (\sqrt{3}/2)]$, where the distance between neighboring sites on the same sublattice is 1. Similarly, the accompanying gauge transformation is trivial, so this is an ordinary translation symmetry.

Measuring the eigenvalues of the symmetry operators in the space of zero-energy single-particle states, we find

$$\begin{aligned} t' = -\Delta: s_A = \frac{1}{2}, \quad s_B = \frac{5}{2}, \quad s_C = \frac{3}{2}, \quad \vec{k}_\star = \pi\left(\frac{2}{3}, \frac{4}{3}\right), \\ t' = \Delta: s_A = \frac{5}{2}, \quad s_B = \frac{1}{2}, \quad s_C = \frac{3}{2}, \quad \vec{k}_\star = \pi\left(\frac{4}{3}, \frac{2}{3}\right). \end{aligned} \quad (74)$$

The invariants obtained by instead performing partial rotations in the gapped phases and forming the effective Lagrangians are given in Table III. Notice, once again, that the invariants at each transition exactly match those obtained via rotation eigenvalues. We present results for the azimuthal current and local energy density in the presence of lattice defects in Figs. 10 and 20.

In particular, in Fig. 10, we present results for the same rotation center but with positive and negative disclination angles, i.e., one wedge removed or one wedge added. We find that the curves are shifted along the $\delta\alpha$ axis by equal and opposite values of emanant flux. We also see that the

energy density is shifted above zero for all values of $\delta\alpha$ for $\beta = 4/3$, which is consistent with the theoretical prediction. In the raw $\mathcal{E}$ data for the $\beta = 4/3$ example, we find that there is no point of coalescence of the various curves, as there is in other models. This finding suggests that the energy density never crosses zero, and we define the zero point to be that which minimizes the variance in $\langle r^3 \mathcal{E} \rangle$ across the various curves near their minima. In Fig. 20, we also see agreement with the predicted emanant flux due to either disclinations or dislocations.

VIII. DISCUSSION

We have shown that, on general grounds, the existence of a C_M rotational symmetry where $(C_M)^M = +1$ requires that lattice disclinations induce an azimuthal current when the continuum theory contains Dirac fermions. This requirement is a direct consequence of the emanant flux induced by the lattice disclination, which must be nonzero because the Dirac fermions carry spin 1/2. Similarly, lattice dislocations induce an emanant flux determined by the location of the Dirac cone in the Brillouin zone.

In the case of a single valley (single Dirac cone), the azimuthal current is the total electrical current. In the case where there are multiple valleys (multiple Dirac cones), then each Dirac cone could see a different emanant flux, which leads to a different current for each valley. In particular, in time-reversal symmetric systems with multiple Dirac cones, such as in graphene, the total current vanishes at a disclination, but there is a nonzero valley current. It would be interesting to develop experimental methods to detect this valley current in graphene, especially given that there have been experimental advances in manipulating valley degrees of freedom in two-dimensional materials [101,102].

An interesting generalization of the discussion here is to a situation where the associated flux is that of a dynamical gauge field, in which case the magnetic flux could be partially or completely screened by the induced currents that we have found. We leave a detailed analysis of the case where the gauge field is dynamical to future work. Such an analysis would also be relevant for various Chern-Simons-Dirac quantum critical points enriched with crystalline symmetry, which describes quantum critical points of quantum Hall systems in the presence of a lattice [103–105].

TABLE III. Topological invariants at each critical point of the Haldane model discussed in Sec. VIID.

	s_A	s_B	s_C	$\text{sgn}(m)$	$\vec{k}_\star$	ΔC	$\Delta \mathcal{S}_A$	$\Delta \mathcal{S}_B$	$\Delta \mathcal{S}_C$	$\Delta \mathcal{E}_A$	$\Delta \mathcal{E}_B$	$\Delta \mathcal{E}_C$	$\Delta \vec{\mathcal{P}}$
$t' = -\Delta$	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{3}{2}$	$+\rightarrow-$	$(\frac{2\pi}{3}, \frac{4\pi}{3})$	-1	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{11}{4}$	$\frac{11}{4}$	$\frac{3}{4}$	$(\frac{2}{3}, \frac{1}{3})$
$t' = \Delta$	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$+\rightarrow-$	$(\frac{4\pi}{3}, \frac{2\pi}{3})$	-1	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{11}{4}$	$\frac{11}{4}$	$\frac{3}{4}$	$(\frac{1}{3}, \frac{2}{3})$

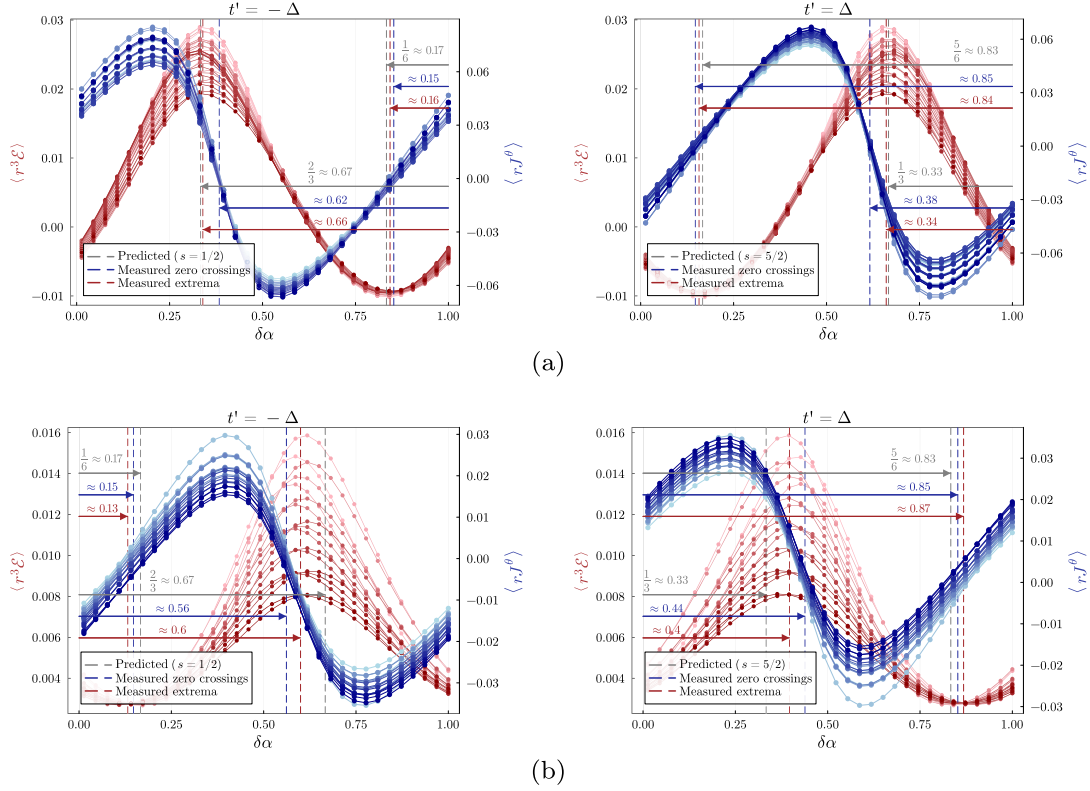


FIG. 10. Haldane model with $\beta = 1 \pm 1/3$ disclinations. Here, we plot the scaled current and energy density at the $t' = \pm\Delta$ critical points. Notice that the emanant flux is of opposite sign for the two different defects, which correspond to $\beta = 1 \pm 1/3$. The shading of each curve in the plot indicates the distances from the defect, with lighter curves corresponding to lattice sites closer to the defect. (a) Lattice with $N = 4761$ sites and A-centered $\beta = 2/3$ disclination. (b) Lattice with $N = 9521$ sites and A-centered $\beta = 4/3$ disclination.

While we have tested a number of theoretical predictions from the defect CFT in our numerics, we have not extracted the defect scaling exponents $\Delta_s^\pm$. This process is particularly challenging as the irrelevant defect operators approach marginality when $\alpha \rightarrow 1/2$, which leads to significant finite-size corrections. It would be interesting to tackle such a difficulty with proper numerical methods and measure $\Delta_s^\pm$. The scaling dimensions provide an alternative way to physically measure the topological quantum numbers of the Dirac fermions without tuning an applied magnetic field.

One remaining question is regarding the total charge in the vicinity of lattice defects at criticality. We find that the charge density vanishes away from the defect. This result is a consequence of conformal invariance. However, there could be localized charges. We leave this question for future work.

In Sec. VA, we assumed that rotations and translations commute and can therefore be simultaneously diagonalized, which is not the case when, for example, there are multiple Dirac cones that map into one another under rotations. We remarked above that this situation can largely be dealt with by choosing the basis that diagonalizes one or the other, depending on which invariant we wish to derive.

However, the invariant $\vec{Q}_o$ is defined via the cross term $\omega_o \wedge d\vec{R}$ and cannot be derived in this manner when rotations and translations do not commute. We expect that the general case can be considered by treating ω_o and $\vec{R}$ as non-Abelian background gauge fields, and we leave this endeavor to future work. We remark that the situation where the magnetic flux per unit cell is nonzero modulo 2π and the two primitive translations do not commute is analogous and should be treated similarly.

We also point out that, in the study of the crystalline topological phases, corners and boundaries exhibit phenomena closely related to the bulk disclinations and dislocations [72,106]. It would be interesting to study the fate of the charges and currents at criticality due to corners and boundaries more systematically.

Interesting open questions were also raised in our DCFT analysis. For monodromy defects in flat space, it seems a natural conjecture that $g \geq 1$ for the IR-stable fixed point, yet we were not able to prove it. On the other hand, we do not know if the g function is an RG monotone in the presence of conical singularities, nor do we know if it is well defined. It is tempting to formally compute the g function from the partition function on hyperbolic space, and the analog to Eq. (54) reads

$$\partial_\alpha \log g = \text{vol}(\mathbb{H}^2) \sum_{s \in \mathbb{Z} + \frac{1}{2}} \langle \tilde{\Psi}_s \tilde{\Psi}_s \rangle = \frac{2\pi}{\beta} C_J(\alpha, \beta). \quad (75)$$

Similarly, we can also obtain $\partial_\beta \log g$. We leave the investigation of these generalized g functions to future work.

We conclude by pointing out that the general discussions of this paper can be applied to quantum critical points more broadly. For example, any CFT appearing as the low-energy limit of a lattice model tuned to a critical point can be enriched by crystalline symmetry. The associated UV-IR homomorphism then determines how disclinations and dislocations induce monodromy and conical defects in the CFT. A particularly relevant example to study in the context of quantum magnetism would be the $O(N)$ fixed point, enriched by crystalline symmetries. Magnetic transitions involving ferromagnets, antiferromagnets, or altermagnets [107] could potentially furnish examples of distinct symmetry-enriched $O(N)$ fixed points. Finally, there are lattice models where the sites support full-fledged gauge theories, such as in Ref. [108] (and references therein). It would be interesting to study crystalline defects in that context.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [99].

APPENDIX A: FURTHER LATTICE MODEL DETAILS

1. QWZ UV-IR homomorphism

In this appendix, we explain in greater detail how to obtain the Hamiltonian in the vicinity of each Dirac point and use it to explicitly derive the UV-IR homomorphism ρ as it acts on spatial rotations.

a. $m = -1$

When $m = -1$, there is a single Dirac point at $\vec{k}_\star := (0, 0)$. We expand around this critical point by

defining $\vec{p}$ via $\vec{k} = \vec{k}_\star + \vec{p}$ and keeping only first-order terms in the components of $\vec{p}$.

The resultant Hamiltonian in the neighborhood of the Dirac point is given by

$$h_{\vec{p}} = \begin{pmatrix} 0 & p_x - ip_y \\ p_x + ip_y & 0 \end{pmatrix} = \vec{p} \cdot \vec{\sigma}. \quad (A1)$$

We therefore have a 2×2 Dirac Hamiltonian:

$$H = \sum_{\vec{p}} \Psi_{\vec{p}}^\dagger (\vec{p} \cdot \vec{\sigma}) \Psi_{\vec{p}}, \quad (A2)$$

where $\Psi_{\vec{p}} = (c_{\vec{p}, \uparrow}, c_{\vec{p}, \downarrow})^T$ and $c_{\vec{p}, \alpha}$ is shorthand for $c_{\vec{k}_\star + \vec{p}, \alpha}$.

Using the magnetic rotation operators defined in Eqs. (21) and (22), we can derive the transformation properties of the $\vec{k}$ -space creation and annihilation operators. In particular, we find

$$\begin{aligned} C_{4,B} c_{\vec{k}, \uparrow} C_{4,B}^{-1} &= i c_{R_{\pi/2}^B(\vec{k}), \uparrow}, \\ C_{4,B} c_{\vec{k}, \downarrow} C_{4,B}^{-1} &= c_{R_{\pi/2}^B(\vec{k}), \downarrow}, \\ C_{4,A} c_{\vec{k}, \uparrow} C_{4,A}^{-1} &= i e^{-ik_x} c_{R_{\pi/2}^B(\vec{k}), \uparrow}, \\ C_{4,A} c_{\vec{k}, \downarrow} C_{4,A}^{-1} &= e^{-ik_x} c_{R_{\pi/2}^B(\vec{k}), \downarrow}. \end{aligned} \quad (A3)$$

Notice that $R_{\pi/2}^B(\vec{k}_\star + \vec{p}) = \vec{k}_\star + R_{\pi/2}^B(\vec{p})$ and $e^{-ik_x} = 1$, so $C_{4,A}$ and $C_{4,B}$ have the same action on the spinor at the Dirac point:

$$\begin{pmatrix} c_{\vec{k}_\star, \uparrow} \\ c_{\vec{k}_\star, \downarrow} \end{pmatrix} \xrightarrow{C_{4,A/B}} \begin{pmatrix} i c_{\vec{k}_\star, \uparrow} \\ c_{\vec{k}_\star, \downarrow} \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_{\vec{k}_\star, \uparrow} \\ c_{\vec{k}_\star, \downarrow} \end{pmatrix}. \quad (A4)$$

In other words, we have

$$\Psi_{\vec{p}} \xrightarrow{C_{4,A/B}} \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \Psi_{R_{\pi/2}^{A,B}(\vec{p})} = e^{i\frac{\pi}{4}\sigma_z} \Lambda(\pi/2) \Psi_{R_{\pi/2}^{A,B}(\vec{p})}, \quad (A5)$$

where $\Lambda(\pi/2) = e^{i\frac{\pi}{4}\sigma_z}$ is the spinor transformation that accompanies a spatial $\pi/2$ transformation for a spin-1/2 particle. Matching to the form of Eq. (16), we see that $s_v = s_p = \frac{1}{2}$, which matches the value of s for the massless fermion at the transition that was determined in Sec. VB by measuring topological invariants and computing the effective Lagrangian.

b. $m = 0$

When $m = 0$, there is a pair of Dirac points, one at $\vec{k}_\star^x := (\pi, 0)$ and one at $\vec{k}_\star^y := (0, \pi)$. Once again, we let $\vec{k} = \vec{k}_\star^{x/y} + \vec{p}$ and expand to first order in the components of $\vec{p}$. We find an effective Hamiltonian in the vicinity of each Dirac point:

$$\begin{aligned} h_{\vec{p}}^x &= \begin{pmatrix} 0 & -p_x - ip_y \\ -p_x + ip_y & 0 \end{pmatrix}, \\ h_{\vec{p}}^y &= \begin{pmatrix} 0 & p_x + ip_y \\ p_x - ip_y & 0 \end{pmatrix}. \end{aligned} \quad (\text{A6})$$

Rotating the basis of $h_{\vec{p}}^x$ by $-Y$ and $h_{\vec{p}}^y$ by X , we find

$$\tilde{h}_{\vec{p}}^x = \tilde{h}_{\vec{p}}^y = \begin{pmatrix} 0 & p_x - ip_y \\ p_x + ip_y & 0 \end{pmatrix}. \quad (\text{A7})$$

We can combine the Hamiltonian in each valley into a single 4×4 Hamiltonian:

$$H = \sum_{\vec{p}} \Psi_{\vec{p}}^\dagger [(\vec{p} \cdot \vec{\sigma}) \otimes I] \Psi_{\vec{p}}, \quad (\text{A8})$$

where $\Psi_{\vec{p}} = (ic_{\vec{p},\downarrow,x} - ic_{\vec{p},\uparrow,x}c_{\vec{p},\downarrow,y}c_{\vec{p},\uparrow,y})^T$ and $c_{\vec{p},\alpha,x}$ is shorthand for $c_{\vec{k}_\star + \vec{p},\alpha}$ and similarly for $c_{\vec{p},\alpha,y}$.

Although Eq. (A8) is in the form of a Dirac Hamiltonian, we see that it is still not in the basis that diagonalizes the spinor action of the microscopic $C_{4,B}$. Notice first that $R_{\pi/2}^B(\vec{k}_\star + \vec{p}) = \vec{k}_\star + R_{\pi/2}^B(\vec{p})$ and $R_{\pi/2}^B(\vec{k}_\star + \vec{p}) = \vec{k}_\star + R_{\pi/2}^B(\vec{p})$, so using Eq. (A3), we have

$$\Psi_{\vec{p}} \xrightarrow{C_{4,B}} \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \Psi_{R_{\pi/2}^B(\vec{p})}. \quad (\text{A9})$$

We can recover a diagonal spinor transformation by rotating the basis by

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -i & 0 & 1 & 0 \\ 0 & -i & 0 & 1 \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix}. \quad (\text{A10})$$

In this new basis, we have

$$H = \sum_{\vec{p}} \tilde{\Psi}_{\vec{p}}^\dagger [(\vec{p} \cdot \vec{\sigma}) \otimes I] \tilde{\Psi}_{\vec{p}}, \quad (\text{A11})$$

where

$$\tilde{\Psi}_{\vec{p}} = \frac{i}{\sqrt{2}} \begin{pmatrix} c_{\vec{p},\downarrow,x} + c_{\vec{p},\downarrow,y} \\ -c_{\vec{p},\uparrow,x} + c_{\vec{p},\uparrow,y} \\ -c_{\vec{p},\downarrow,x} + c_{\vec{p},\downarrow,y} \\ c_{\vec{p},\uparrow,x} + c_{\vec{p},\uparrow,y} \end{pmatrix}. \quad (\text{A12})$$

Under the microscopic $C_{4,B}$, we now have

$$\tilde{\Psi}_{\vec{p}} \xrightarrow{C_{4,B}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \tilde{\Psi}_{R_{\pi/2}^B(\vec{p})}. \quad (\text{A13})$$

We see that there is a different transformation matrix for each Dirac fermion:

$$\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = e^{i\frac{7\pi}{4}} \Lambda(\pi/2), \quad \begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix} = e^{i\frac{3\pi}{4}} \Lambda(\pi/2). \quad (\text{A14})$$

The corresponding invariants are therefore $s = \frac{7}{2}$ and $s = \frac{3}{2}$.

Let us now work out the homomorphism under the plaquette-centered rotation. Applying Eq. (A3), we find

$$\Psi_{\vec{p}=\vec{0}} \xrightarrow{C_{4,A}} \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \Psi_{\vec{p}=\vec{0}}. \quad (\text{A15})$$

Once again, we see that we are not in the basis that diagonalizes the spinor action of $C_{4,A}$. We can recover a diagonal spinor transformation by rotating the basis by

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \quad (\text{A16})$$

In this new basis, we have

$$H = \sum_{\vec{p}} \tilde{\Psi}_{\vec{p}}^\dagger [(\vec{p} \cdot \vec{\sigma}) \otimes I] \tilde{\Psi}_{\vec{p}}, \quad (\text{A17})$$

where

$$\tilde{\Psi}_{\vec{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} ic_{\vec{p},\downarrow,x} + c_{\vec{p},\downarrow,y} \\ -ic_{\vec{p},\uparrow,x} + c_{\vec{p},\uparrow,y} \\ -ic_{\vec{p},\downarrow,x} + c_{\vec{p},\downarrow,y} \\ ic_{\vec{p},\uparrow,x} + c_{\vec{p},\uparrow,y} \end{pmatrix}. \quad (\text{A18})$$

Under the microscopic $C_{4,A}$, we now have

$$\tilde{\Psi}_{\vec{p}=\vec{0}} \xrightarrow{C_{4,A}} \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tilde{\Psi}_{\vec{p}=\vec{0}}. \quad (\text{A19})$$

We see that there is a different transformation matrix for each Dirac fermion:

$$\begin{pmatrix} -i & 0 \\ 0 & -1 \end{pmatrix} = e^{i\frac{5\pi}{4}}\Lambda(\pi/2), \quad \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} = e^{i\frac{\pi}{4}}\Lambda(\pi/2). \quad (\text{A20})$$

The corresponding invariants are therefore $s = \frac{5}{2}$ and $s = \frac{1}{2}$.

c. $m = 1$

When $m = 1$, there is a single Dirac point at $\vec{k}_\star := (\pi, \pi)$. Expanding around this Dirac point in small momentum, we find

$$h_{\vec{p}} = \begin{pmatrix} 0 & -p_x + ip_y \\ -p_x - ip_y & 0 \end{pmatrix}. \quad (\text{A21})$$

Rotating the basis by Z , we have

$$\tilde{h}_{\vec{p}} = \begin{pmatrix} 0 & p_x - ip_y \\ p_x + ip_y & 0 \end{pmatrix} = \vec{p} \cdot \vec{\sigma}. \quad (\text{A22})$$

We therefore have a 2×2 Dirac Hamiltonian:

$$H = \sum_{\vec{p}} (c_{\vec{p},\uparrow}^\dagger - c_{\vec{p},\downarrow}^\dagger)(\vec{p} \cdot \vec{\sigma}) \begin{pmatrix} c_{\vec{p},\uparrow} \\ -c_{\vec{p},\downarrow} \end{pmatrix}, \quad (\text{A23})$$

where $c_{\vec{p},\alpha}$ is shorthand for $c_{\vec{k}_\star + \vec{p},\alpha}$.

In this case, $R_{\pi/2}^B(\vec{k}_\star + \vec{p}) = \vec{k}_\star + R_{\pi/2}^B(\vec{p})$ and $e^{-ik_\star^x} = -1$, so using Eq. (A3), we have

$$\begin{aligned} \Psi_{\vec{p}} &\xrightarrow{C_{4,B}} \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \Psi_{R_{\pi/2}^B(\vec{p})}, \\ \Psi_{\vec{p}} &\xrightarrow{C_{4,A}} \begin{pmatrix} -i & 0 \\ 0 & -1 \end{pmatrix} \Psi_{R_{\pi/2}^B(\vec{p})}. \end{aligned} \quad (\text{A24})$$

The transformation matrices are therefore given by

$$\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} = e^{i\frac{\pi}{4}}\Lambda(\pi/2), \quad \begin{pmatrix} -i & 0 \\ 0 & -1 \end{pmatrix} = e^{i\frac{5\pi}{4}}\Lambda(\pi/2). \quad (\text{A25})$$

The corresponding invariants are therefore $s_v = \frac{1}{2}$ and $s_p = \frac{5}{2}$.

2. Topological invariants from partial rotations

In this section, we review the method of partial rotations introduced in Refs. [24,26]. Suppose we have a tight binding model with an M -fold magnetic rotation symmetry about the high-symmetry point o , which is realized by

$C_{M,o}$. We can define an operator

$$C_{M,o}^- := e^{-i\frac{\pi}{M_o}\hat{N}} C_{M,o}, \quad (\text{A26})$$

where N is the fermion number operator $\hat{N} = \sum_{\vec{r}} c_{\vec{r}}^\dagger c_{\vec{r}}$. Because $C_{M,o}$ is a symmetry, $C_{M,o}^-$ will also be a symmetry. However, it satisfies $(C_{M,o}^-)^M = (-1)^F$, in contrast to $(C_{M,o})^M = 1$, where $(-1)^F$ is fermion parity. We may then define two families of operators

$$C_{M,o,\chi} := e^{i\chi\frac{2\pi}{M}\hat{N}} C_{M,o}, \quad C_{M,o,\chi}^- := e^{i\chi\frac{2\pi}{M}\hat{N}} C_{M,o}^-, \quad (\text{A27})$$

where $\chi = 0, \dots, M-1$. These will also be symmetries.

Next, we fix a rotation-invariant region of lattice sites D , for a lattice on either open or periodic boundary conditions. In order to extract topological data, we measure expectation values in the many-body ground state $|\Psi\rangle$ of rotation operators restricted to the region D , which are called partial rotations. Letting $|_D$ denote the restriction of an operator to sites in D , we find

$$\langle \Psi | C_{M,o,\chi}^{(-)} |_D | \Psi \rangle = e^{\frac{2\pi i}{M} l_{D,o,\chi}^{(-)} - \gamma_{D,o,\chi}^{(-)}}, \quad (\text{A28})$$

where $\gamma_{D,o,\chi}^{(-)} \propto \partial D$ sets the real-valued amplitude and $l_{D,o,\chi}^{(-)}$ encodes the topological data. In particular, $l_{D,o,\chi}^{(-)}$ satisfies

$$l_{D,o,\chi}^{(-)} = \frac{C\chi^2}{2} + \mathcal{S}_o^{(-)}\chi + K_o^{(-)} \pmod{M}. \quad (\text{A29})$$

For a given set of $l_{D,o,\chi}^{(-)}$, we find C , $\mathcal{S}_o^{(-)}$, and $K_o^{(-)}$ by fitting to the measured values of $l_{D,o,\chi}^{(-)}$, keeping in mind the quantization of each invariant. Once C , $\mathcal{S}_o^{(-)}$, and $K_o^{(-)}$ are obtained, we define the real-space invariants

$$\begin{aligned} \Theta_o &:= \begin{cases} K_o \pmod{\frac{M}{2}}, M \text{ even} \\ K_o \pmod{M}, M \text{ odd}, \end{cases} \\ \Theta_o^- &:= \begin{cases} K_o^- \pmod{M}, M \text{ even} \\ K_o^- \pmod{\frac{M}{2}}, M_o \text{ odd}. \end{cases} \end{aligned} \quad (\text{A30})$$

We can finally calculate ℓ_o via

$$\ell_o = \begin{cases} \frac{11}{12}C + 2\Theta_o \pmod{3}, & M = 3 \\ \frac{5}{4}C + 2\Theta_o \pmod{4}, & M = 4. \end{cases} \quad (\text{A31})$$

Following this procedure, we find the values of C , $\mathcal{S}_o$, and ℓ_o reported in the main text.

APPENDIX B: FREE BOSON MODEL

As a supplement to the Dirac fermion model (43) in the main text, we discuss the conical monodromy defect in a

$(2+1)d$ free boson model. The scalar field $\Phi_b \in \mathbb{C}$ is subject to a $U(1)$ 0-form symmetry, and we couple it to $A_\mu = \delta_{\mu\theta}\alpha \in [0, 1)$, which implements the monodromy. We consider the bulk action in the conical space (41),

$$S_{\text{boson}} = \int_{\text{Cone}} d^3X |\nabla_\mu \Phi_b|^2, \quad (\text{B1})$$

where d^3X is the covariant integral measure. Let $\sigma \in \mathbb{S}^1$ be the phonon dual to the connection A_μ . We find the local

gauge-invariant field operator $\Phi = e^{i\sigma}\Phi_b$. An angular mode expansion to Φ reads

$$\Phi(t, r, \theta) = \frac{1}{\sqrt{2\pi r}} \sum_{n \in \mathbb{Z}} e^{i(n+\alpha)\theta} \tilde{\Phi}_n(t, r). \quad (\text{B2})$$

We now apply the same trick we used in the main text (46) that maps the cone (41) to $\text{AdS}_2 \times \mathbb{S}_\beta^1$. The action of angular Kaluza-Klein modes on AdS_2 is such that

$$S_{\text{boson}} = \beta \sum_{n \in \mathbb{Z}} \int_{\text{AdS}_2} d^2\tilde{X} \tilde{\Phi}_n^* \left[-\tilde{\square}^2 - \frac{1}{4} + \frac{(n+\alpha)^2}{\beta^2} \right] \tilde{\Phi}_n. \quad (\text{B3})$$

DCFT data regarding Eq. (B1) can be obtained by studying the asymptotic scaling of AdS_2 propagators, Eqs. (C5) and (C7). We note the expansion

$$\tilde{\Phi}_n = r^{\Delta_n^\pm} \phi_n(t) + O(r^{\Delta_n^\pm+1}), \quad \text{where } \Delta_n^\pm = \frac{1}{2} \pm \frac{|n+\alpha|}{\beta} > 0. \quad (\text{B4})$$

Equation (B4) is also known as the bulk-to-defect OPE of the standard (+) and alternative (−) quantization. The defect primary ϕ_n is of the $SL(2, \mathbb{R})$ conformal dimension $\Delta_n^\pm$ and $SO(2)$ transversal rotation spin $n+\alpha$. Remarkably, the defect bilinear deformation $\int dt |\phi_0|^2$ when $\alpha=0$ is marginal at the tree level. It is similar to the fermion sliding operator $\int dt \bar{\psi}_{-\frac{1}{2}} \psi_{-\frac{1}{2}}$ at $\alpha=\frac{1}{2}$, except that the fermion bilinear deformation is exactly marginal while $\int dt |\phi_0|^2$ is marginally irrelevant. Such a fact can be checked either from a conformal perturbation calculation or directly from the logarithmic divergence in the singular solution Eq. (C7).

Now, let us consider the bulk one-point functions of the $U(1)$ current $J_\mu = i(\Phi \partial_\mu \Phi^* - \Phi^* \partial_\mu \Phi)$ and stress-energy

tensor $T_{\mu\nu} = \partial_\mu \Phi^* \partial_\nu \Phi$. Note that $\langle J_\mu \rangle$ and $\langle T_{\mu\nu} \rangle$ of the free massless boson are, respectively, of the tensor structure in Eqs. (49) and (50), which is fixed by the residual conformal symmetry. We find

$$\begin{aligned} C_J &= 2 \sum_{n \in \mathbb{Z}} (n+\alpha) \langle |\tilde{\Phi}_n|^2 \rangle, \\ C_T &= \sum_{n \in \mathbb{Z}} \left(\frac{n+\alpha}{\beta} \right)^2 \langle |\tilde{\Phi}_n|^2 \rangle. \end{aligned} \quad (\text{B5})$$

More specifically, Eq. (B5) in the standard quantization admits an integral representation similar to Eqs. (56) and (57). For the azimuthal current density, we find that

$$C_J = \frac{1}{\pi\beta} \left[\int_0^1 \frac{(1-\alpha)(u^{\frac{\alpha}{\beta}} - u^{-\frac{\alpha}{\beta}}) + \alpha(u^{\frac{\alpha-1}{\beta}} - u^{\frac{1-\alpha}{\beta}})}{(u^{\frac{1}{2\beta}} - u^{-\frac{1}{2\beta}})^2 (1-u)^{v-1}} u^{\frac{v-3}{2}} du \right]_{v \rightarrow 2}. \quad (\text{B6})$$

On the other hand, the energy density is such that

$$C_T = \frac{1}{2\pi\beta^3} \left\{ \int_0^1 [(2\alpha^2 - 2\alpha - 1)(u^{\frac{\alpha}{\beta} - \frac{1}{2\beta}} + u^{-\frac{\alpha}{\beta} + \frac{1}{2\beta}}) - (1-\alpha)^2(u^{\frac{\alpha}{\beta} + \frac{1}{2\beta}} + u^{-\frac{\alpha}{\beta} - \frac{1}{2\beta}}) - \alpha^2(u^{\frac{\alpha}{\beta} - \frac{3}{2\beta}} + u^{-\frac{\alpha}{\beta} + \frac{3}{2\beta}})] \frac{u^{\frac{v-3}{2}} du}{(u^{\frac{1}{2\beta}} - u^{-\frac{1}{2\beta}})^3 (1-u)^{v-1}} \right\}_{v \rightarrow 2}. \quad (\text{B7})$$

In addition to the azimuthal current and energy density, the boson field develops a nonzero particle density profile

$$\langle |\Phi|^2 \rangle = \frac{1}{2\pi r} \sum_{n \in \mathbb{Z}} \langle |\tilde{\Phi}_n|^2 \rangle \equiv \frac{C_{|\Phi|^2}}{2\pi r}, \quad \text{where } C_{|\Phi|^2} = -\frac{1}{2\pi\beta} \left(\int_0^1 \frac{u^{-\frac{\alpha}{\beta} + \frac{1}{2\beta}} + u^{\frac{\alpha}{\beta} - \frac{1}{2\beta}}}{u^{\frac{1}{2\beta}} - u^{-\frac{1}{2\beta}}} \frac{u^{\frac{v-3}{2}} du}{(1-u)^{v-1}} \right)_{v \rightarrow 2}. \quad (\text{B8})$$

At alternative fixed points, we also find a relation similar to Eq. (58), such that

$$\begin{aligned} C'_{|\Phi|^2} &= C_{|\Phi|^2} + \frac{1}{2\beta} \sum_{n'} \tan\left(\pi \frac{|n' + \alpha|}{\beta}\right), \\ C'_J &= C_J + \sum_{n'} \frac{n' + \alpha}{\beta} \tan\left(\pi \frac{|n' + \alpha|}{\beta}\right), \\ C'_T &= C_T + \frac{1}{2\beta} \sum_{n'} \frac{(n' + \alpha)^2}{\beta^2} \tan\left(\pi \frac{|n' + \alpha|}{\beta}\right). \end{aligned} \quad (\text{B9})$$

When $\beta = 1$ and there is no conical singularity, we find for $\alpha \in [0, 1)$ that

$$\begin{aligned} C_{|\Phi|^2} &= -\frac{1}{4}(1 - 2\alpha) \tan(\pi\alpha), \\ C_J &= \frac{1}{8}(1 - 2\alpha)^2 \tan(\pi\alpha), \\ C_T &= \frac{1}{12}(1 - \alpha)\alpha(1 - 2\alpha) \tan(\pi\alpha), \end{aligned} \quad (\text{B10})$$

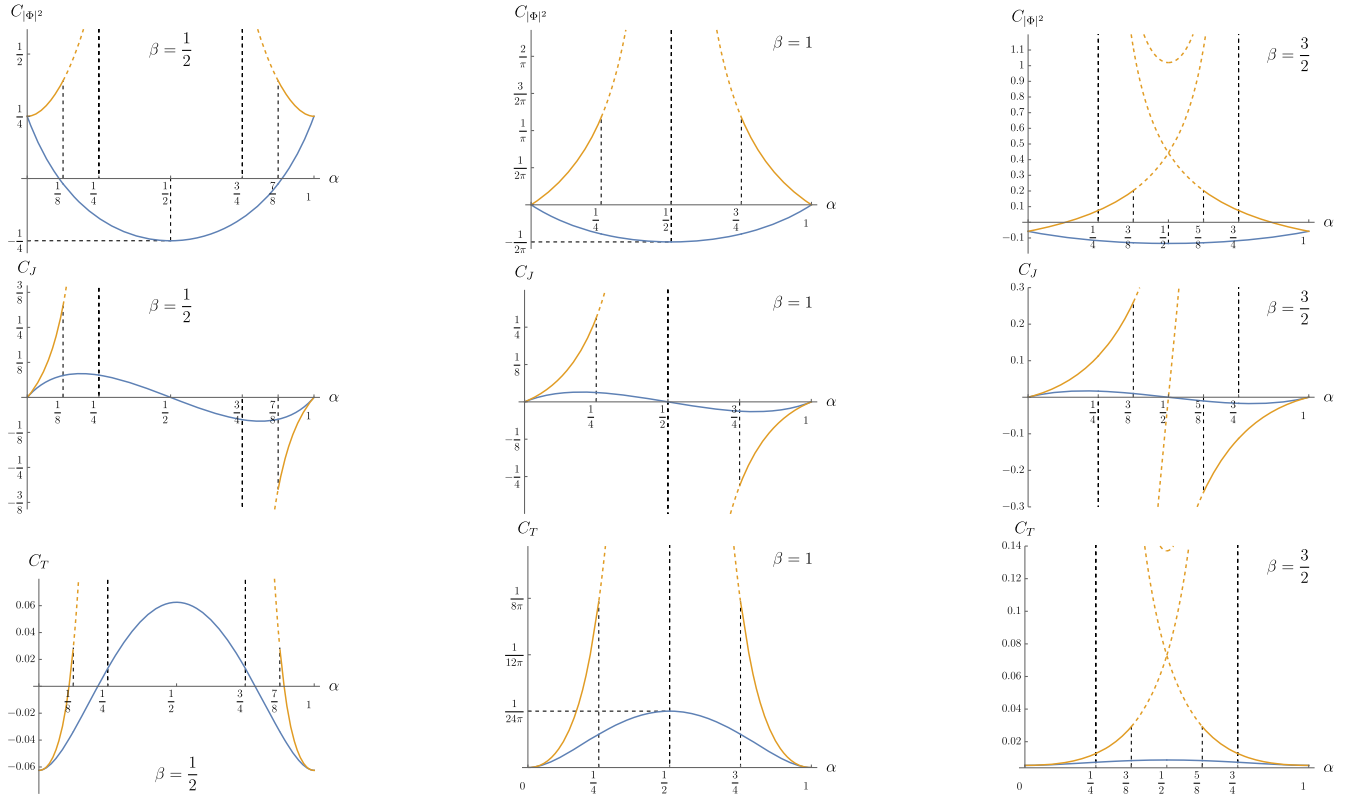


FIG. 11. Particle density $C_{|\Phi|^2}$, azimuthal current C_J , and conformal weight C_T of the free massless boson defect. From left to right: defect with a conical deficit ($\beta = \frac{1}{2}$), defect in flat space ($\beta = 1$), and defect with a conical excess ($\beta = \frac{3}{2}$).

in agreement with Refs. [38,58]. A gallery of $C_{|\phi|^2}$, C_J , and C_T can be found in Fig. 11. We remark on two distinctions between the free boson and the free fermion defect: First, free boson observables C_J and C_T as a function of α are differentiable at $\alpha = \frac{1}{2}$. There exists no defect conformal manifold at π flux, and the current vanishes, $C_J(\alpha = \frac{1}{2}) = 0$, at the standard fixed point. Second, the bulk energy density of the free boson at π flux is lower than that at the 0 flux. We find the energy-density difference between time-reversal invariant DCFTs where $C_J = 0$, such that

$$C_T\left(\alpha = \frac{1}{2}\right) - C_T(\alpha = 0) = -\frac{1}{2\pi\beta^2} \int_0^1 \frac{u^{\frac{1}{2\beta}} - u^{-\frac{1}{2\beta}}}{(u^{\frac{1}{2\beta}} + u^{-\frac{1}{2\beta}})^3} \frac{du}{1-u^2} > 0, \quad (\text{B11})$$

which is similar to Eq. (59).

APPENDIX C: AdS PROPAGATORS

In this appendix, we discuss the analytical properties of free field propagators in AdS_d . Let the coordinate system be $x = (r, \vec{x})$, where r is the distance to the AdS_d boundary and $\vec{x}$ are transversal directions. The metric reads

$$(ds)^2 = \frac{(dr)^2 + (d\vec{x})^2}{r^2}, \quad (\text{C1})$$

and the connected isometry group is $SO(d, 1)$. We note that the geodesic distance between two points $x_{1,2} \in \text{AdS}_d$ is fixed by the isometry to depend solely on the variable

$$\xi \equiv \frac{2r_1 r_2}{r_1^2 + r_2^2 + (\vec{x}_{12})^2}, \quad (\text{C2})$$

with $\vec{x}_{12} \equiv \vec{x}_1 - \vec{x}_2$, which we will use intensively in the following.

Following Refs. [58,82,83], let us first discuss the free complex boson ϕ of the action

$$\tilde{S}_{\text{boson}} = \int_{\text{AdS}_d} d^d \tilde{X} \phi^* \left[-\square - \frac{(d-1)^2}{4} + m^2 \right] \phi. \quad (\text{C3})$$

The Green's function $\tilde{G}_\phi(x_1, x_2) \equiv \langle \phi^*(x_1) \phi(x_2) \rangle$ is subject to the second-order differential equation

$$\left[-\square_1 - \frac{(d-1)^2}{4} + m^2 \right] \tilde{G}_\phi(x_1, x_2) = \delta_{\text{AdS}_d}(x_1 - x_2), \quad (\text{C4})$$

which generally admits two linearly independent solutions. Noticing that the isometry fixes $\tilde{G}_\phi(x_1, x_2) = \tilde{G}_\phi(\xi)$, we find that, when $m \neq 0$,

$$\tilde{G}_\phi(\xi) = \frac{\Gamma(\Delta)(\xi/2)^\Delta}{2\pi^{\frac{d-1}{2}}\Gamma(\Delta + \frac{3-d}{2})} {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta + \frac{3-d}{2}; \xi^2\right), \quad (\text{C5})$$

where $\Delta = [(d-1)/2] \pm |m|$ marks the asymptotic scaling as one of the correlation points approaches the AdS_d boundary at $r = 0$:

$$\tilde{G}_\phi(\xi \ll 1) = \frac{\Gamma(\Delta)(\xi/2)^\Delta}{2\pi^{\frac{d-1}{2}}\Gamma(\Delta + \frac{3-d}{2})} + O(\xi^{\Delta+2}). \quad (\text{C6})$$

When $m = 0$, one solution follows Eqs. (C5) and (C6) with $\Delta = [(d-1)/2]$, while the other one reads

$$\tilde{G}_\phi^{\text{log}}(\xi) = \frac{\xi^{\frac{d-1}{2}}(1-\xi^2)^{1-\frac{d}{2}}}{4\pi^{\frac{d-2}{2}}\sin(\frac{\pi d}{2})\Gamma(2-\frac{d}{2})} {}_2F_1\left(\frac{3-d}{4}, \frac{5-d}{4}; 2-\frac{d}{2}; 1-\xi^2\right). \quad (\text{C7})$$

We remark that the solution (C7) yields logarithmic divergence when one point is close to the AdS_d boundary:

$$\tilde{G}_\phi^{\text{log}}(\xi \ll 1) = \frac{(\xi/2\pi)^{\frac{d-1}{2}} \log(\xi)}{(d-1)\sin(\frac{\pi d}{2})\Gamma(\frac{1-d}{2})} + O((\log \xi)^0, \xi^{\frac{d+1}{2}}). \quad (\text{C8})$$

In the main text, as we calculate the one-point functions, we also need the propagator (C5) at the coincident point:

$$\tilde{G}_\phi(\xi = 1) = \frac{\Gamma(1-\frac{d}{2})\Gamma(\Delta)}{(4\pi)^{d/2}\Gamma(\Delta+2-d)} = \frac{1}{2\pi^{\frac{d-1}{2}}\Gamma(\frac{3-d}{2})} \int_0^1 \frac{u^{\Delta-1}}{(1-u)^{d-1}} du, \quad (\text{C9})$$

which follows Gauss's summation theorem. Particularly, the coincident point is singular when Δ approaches zero:

$$\tilde{G}_\phi(\xi = 1, |\Delta| \ll 1) = \frac{1}{2\pi^{\frac{d-1}{2}}\Gamma(\frac{3-d}{2})\Delta} + O(\Delta^0). \quad (\text{C10})$$

Now, let us follow Refs. [58,109–111] and discuss the free Dirac fermion

$$\tilde{S}_{\text{Dirac}} = \int_{\text{AdS}_d} d^d \tilde{X} \bar{\psi} (\tilde{\gamma} \cdot \tilde{\nabla} - m) \psi. \quad (\text{C11})$$

The spinor Green's function $\langle \bar{\psi}(x_1) \psi(x_2) \rangle = \tilde{G}_\psi(x_1, x_2)$, similar to the boson case, is subject to the Dirac equation

$$(\tilde{\gamma} \cdot \tilde{\nabla}_1 - m) \tilde{G}_\psi(x_1, x_2) = \delta_{\text{AdS}_d}(x_1 - x_2). \quad (\text{C12})$$

Explicitly, the Dirac operator on AdS_d reads

$$\tilde{\gamma} \cdot \tilde{\nabla} = r \tilde{\gamma}^\mu \partial_\mu - \frac{d-1}{2} \tilde{\gamma}_r, \quad (\text{C13})$$

where $\tilde{\gamma}_\mu$ are the flat-space Clifford matrices. Equation (C12) is a first-order differential equation of two variables $\psi = (\psi_u, \psi_d)$, which also admits two solutions distinguished by their asymptotic scaling dimension $\Delta = [(d-1)/2] \pm |m|$. When $m \neq 0$, we find that

$$\begin{aligned} \tilde{G}_\psi(x_1, x_2) = & \frac{\Gamma(\frac{d}{2} \pm |m|)}{2\pi^{\frac{d-1}{2}}\Gamma(\frac{1}{2} \pm |m|)} \left(\frac{\xi}{2-2\xi} \right)^{\pm|m|} \times \left[\frac{\tilde{\gamma} \cdot \vec{x}_{12} + \tilde{\gamma}_r(r_1 - r_2)}{\sqrt{r_1 r_2}} \left(\frac{\xi}{2-2\xi} \right)^{\frac{d}{2}} {}_2F_1 \left(\pm|m|, \frac{d}{2} \pm |m|; 1 \pm 2|m|; \frac{2\xi}{\xi-1} \right) \right. \\ & \left. \pm \text{sgn}(m) \frac{\tilde{\gamma}_r \tilde{\gamma} \cdot \vec{x}_{12} - (r_1 + r_2)}{\sqrt{r_1 r_2}} \left(\frac{\xi}{2+2\xi} \right)^{\frac{d}{2}} {}_2F_1 \left(\pm|m|, \frac{2-d}{2} \pm |m|; 1 \pm 2|m|; \frac{2\xi}{\xi-1} \right) \right]. \end{aligned} \quad (\text{C14})$$

Let one of the correlation points x_1 approach the AdS_d boundary. We obtain an asymptotic expansion to Eq. (C14) as

$$\tilde{G}_\psi(r_1 \ll r_2, |\vec{x}_{12}|) = (1 \pm \tilde{\gamma}_r \text{sgn}(m)) \frac{(\tilde{\gamma} \cdot \vec{x}_{12} - \tilde{\gamma}_r r_2) \Gamma(\frac{d}{2} \pm |m|)}{2\pi^{\frac{d-1}{2}}\Gamma(\frac{1}{2} \pm |m|)} \frac{(r_1 r_2)^{\frac{d-1}{2} \pm |m|}}{[(\vec{x}_{12})^2 + r_2^2]^{\frac{d}{2} \pm |m|}} + O(r_1^{\frac{d+1}{2} \pm |m|}). \quad (\text{C15})$$

On the other hand, when $m = 0$, the two solutions read

$$\tilde{G}_\psi(x_1, x_2) = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \left[\frac{\tilde{\gamma} \cdot \vec{x}_{12} + \tilde{\gamma}_r(r_1 - r_2)}{\sqrt{r_1 r_2}} \left(\frac{\xi}{2-2\xi} \right)^{\frac{d}{2}} \pm \frac{\tilde{\gamma}_r \tilde{\gamma} \cdot \vec{x}_{12} - (r_1 + r_2)}{\sqrt{r_1 r_2}} \left(\frac{\xi}{2+2\xi} \right)^{\frac{d}{2}} \right]. \quad (\text{C16})$$

One important distinction between the bosonic case in Eqs. (C7) and (C8) is that such fermionic massless solutions are degenerate in their asymptotic scaling close to the boundary:

$$\tilde{G}_\psi(r_1 \ll r_2, |\vec{x}_{12}|) = (1 \pm \tilde{\gamma}_r) (\tilde{\gamma} \cdot \vec{x}_{12} - \tilde{\gamma}_r r_2) \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \frac{(r_1 r_2)^{\frac{d-1}{2}}}{[(\vec{x}_{12})^2 + r_2^2]^{\frac{d}{2}}} + O(r_1^{\frac{d+1}{2}}). \quad (\text{C17})$$

Finally, we note the coincident point functions

$$\tilde{G}_\psi(x_1 = x_2) = \mp \frac{\text{sgn}(m) \Gamma(1 - \frac{d}{2}) \Gamma(\frac{d}{2} \pm |m|)}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{2-d}{2} \pm |m|)} = \mp \frac{\text{sgn}(m) \Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}} \Gamma(1-d)} \int_0^1 \frac{u^{\pm|m| + \frac{d}{2} - 1}}{(1-u)^d} du. \quad (\text{C18})$$

For the $m = 0$ case, one can generally choose a linear combination of the solutions in Eq. (C16), such that $-\Gamma(d/2)/(4\pi)^{\frac{d}{2}} \leq \tilde{G}_\psi(x_1 = x_2) \leq \Gamma(d/2)/(4\pi)^{\frac{d}{2}}$.

APPENDIX D: FURTHER NUMERICAL DETAILS

In this appendix, we present additional technical details about how the numerical results presented throughout the paper were obtained. We also present

numerical results for several models introduced in the main text.

1. Defect Hamiltonians

In this section, we present explicit representative Hamiltonians for each defect studied in this work. While the Hamiltonians we show here include a relatively small number of lattice sites for ease of visualization, the actual Hamiltonians used in numerics are qualitatively the same but on larger lattices. The Hamiltonians are presented in Figs. 13–16 with an accompanying legend in Fig. 12.

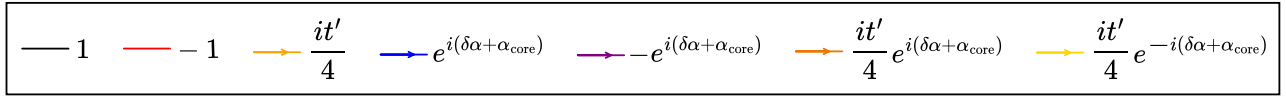
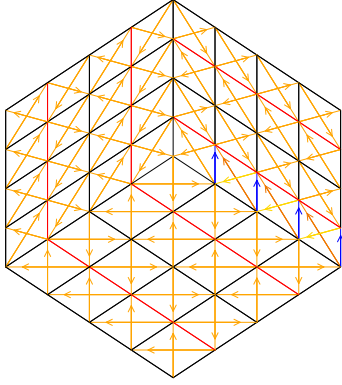
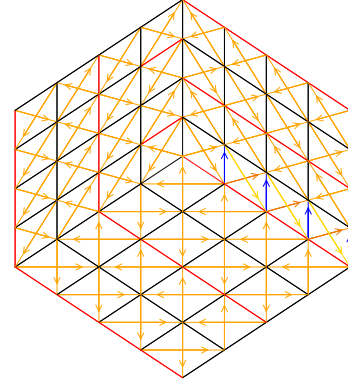


FIG. 12. Legend defining the hoppings used in the following figures. Here, α_{core} denotes a potentially nonzero flux at the defect core that depends on the definition of the symmetry operator used to create the defect, which varies from model to model.

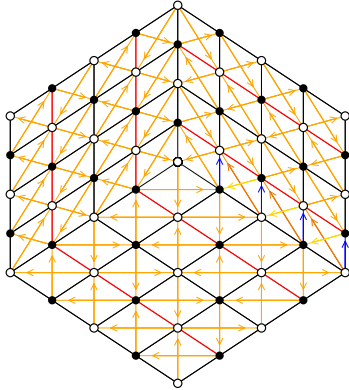


(a) A -centered disclination, $\beta = 3/4$

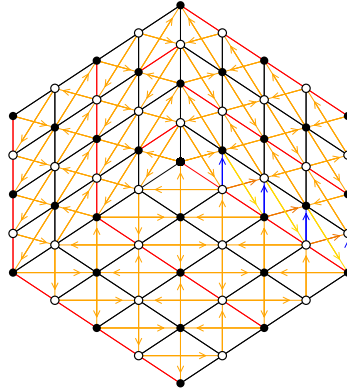


(b) B -centered disclination, $\beta = 3/4$

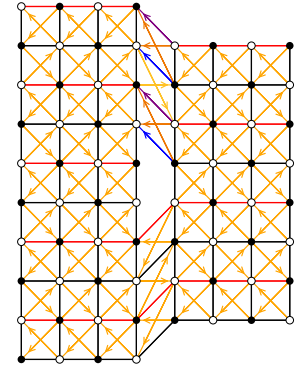
FIG. 13. No on-site potential π -flux model.



(a) A -centered disclination, $\beta = 3/4$



(b) B -centered disclination, $\beta = 3/4$



(c) $\vec{b} = (1, 1)$ dislocation

FIG. 14. One-site staggered potential π -flux model.

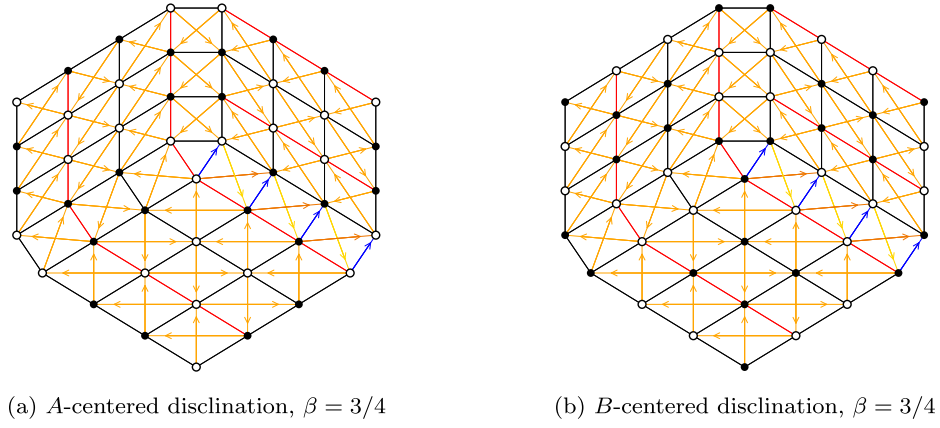
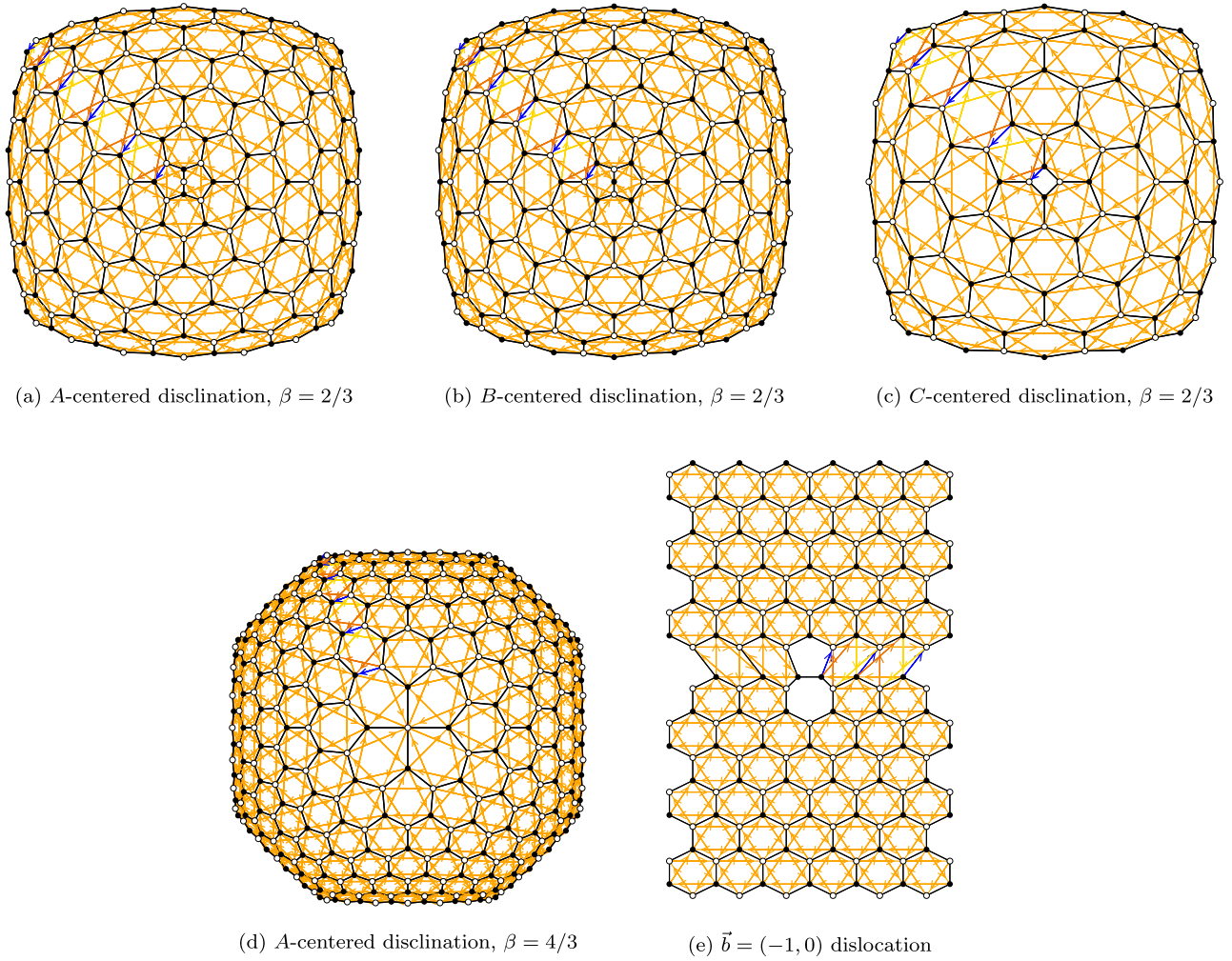
FIG. 15. Two-site staggered potential π -flux model.

FIG. 16. Haldane model.

2. Additional numerical results

In this section, we present additional numerical results for the azimuthal current and energy density for models which were introduced in the main text.

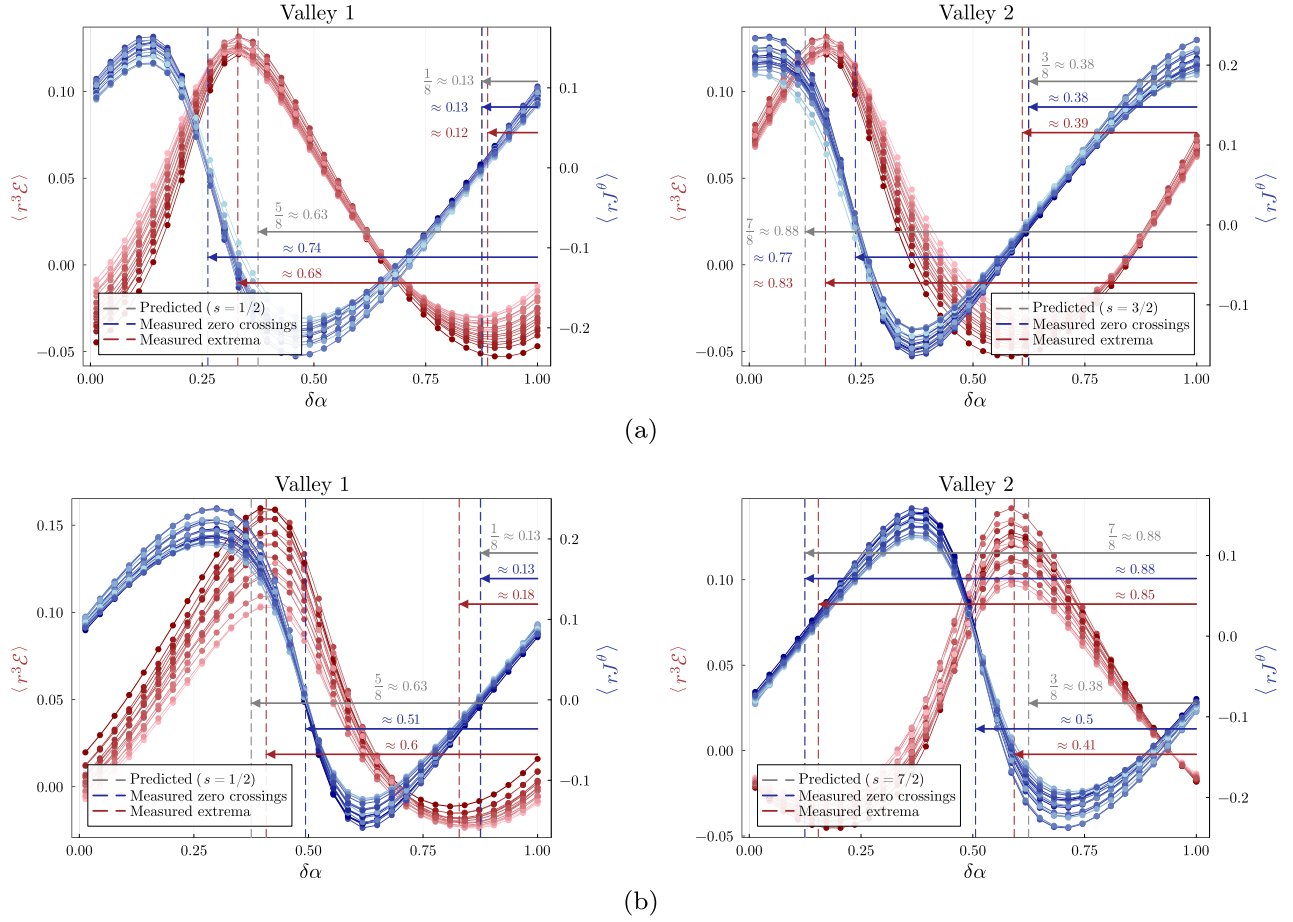


FIG. 17. No on-site potential π -flux model with crystalline defects. (a) Lattice with $N = 4761$ sites and A-centered $\beta = 2/3$ disclination. (b) Lattice with $N = 9521$ sites and A-centered $\beta = 4/3$ disclination. (b) Lattice with $N = 5419$ sites and B-centered $\beta = 3/4$ disclination.

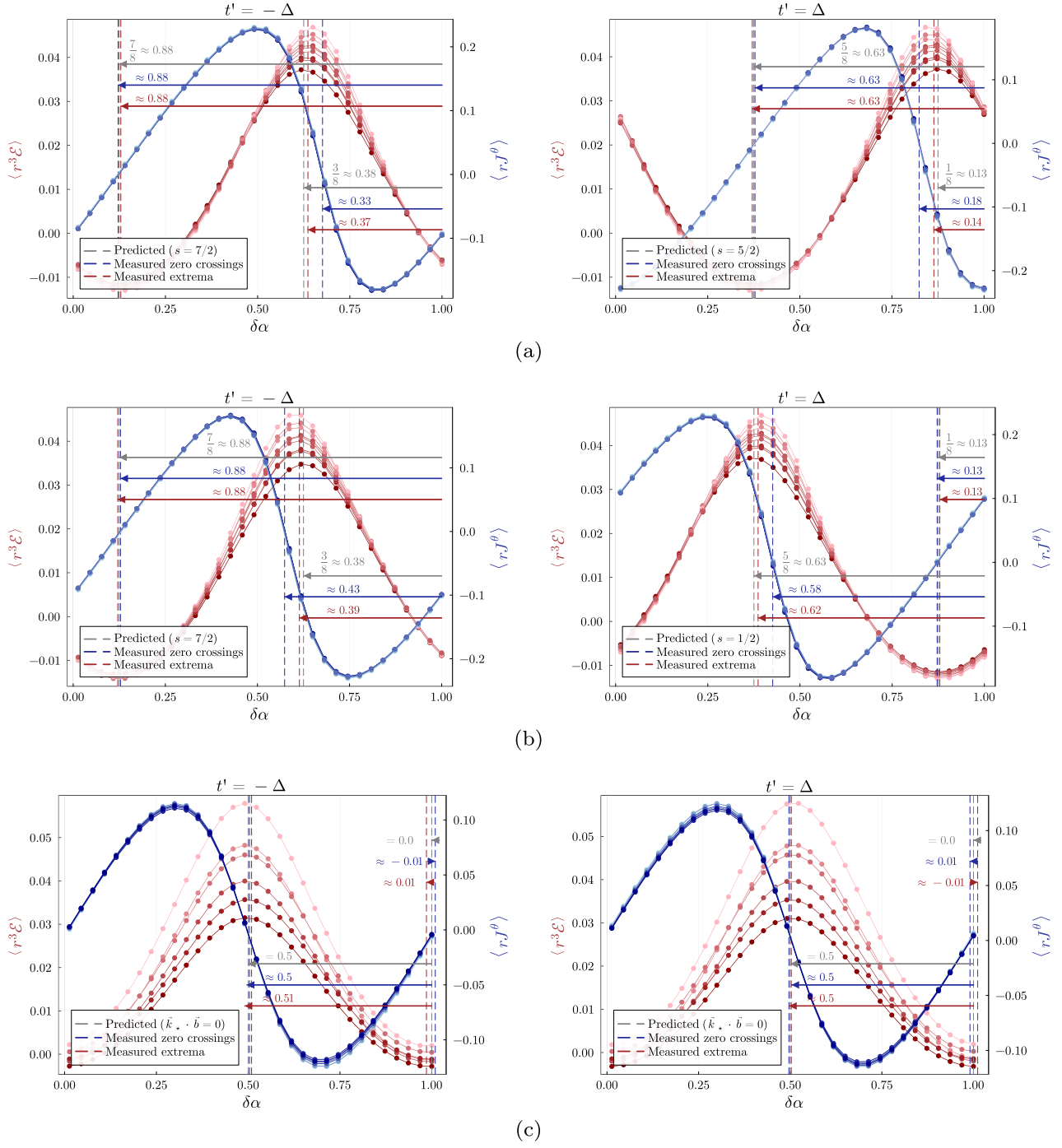


FIG. 18. One-site staggered potential π -flux model with crystalline defects. (a) Lattice with $N = 5419$ sites and A -centered $\beta = 3/4$ disclination. (b) Lattice with $N = 4921$ sites and B -centered $\beta = 3/4$ disclination. (c) Lattice with $N = 6440$ sites and $\vec{b} = (1, 1)$ dislocation.

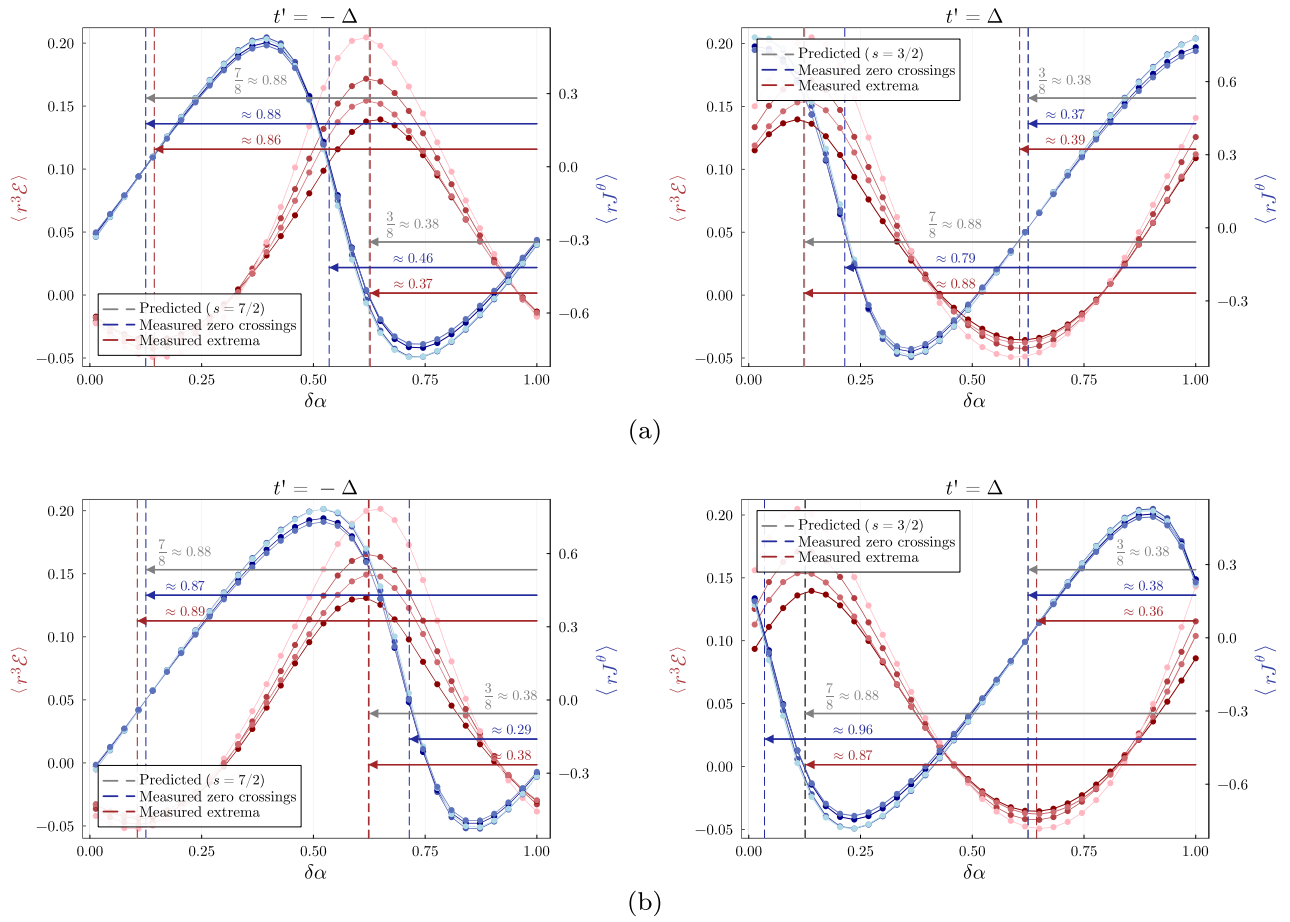


FIG. 19. Two-site staggered potential π -flux model with crystalline defects. (a) A-centered disclination lattice with $N = 4801$ sites. (b) B-centered disclination lattice with $N = 4801$ sites.

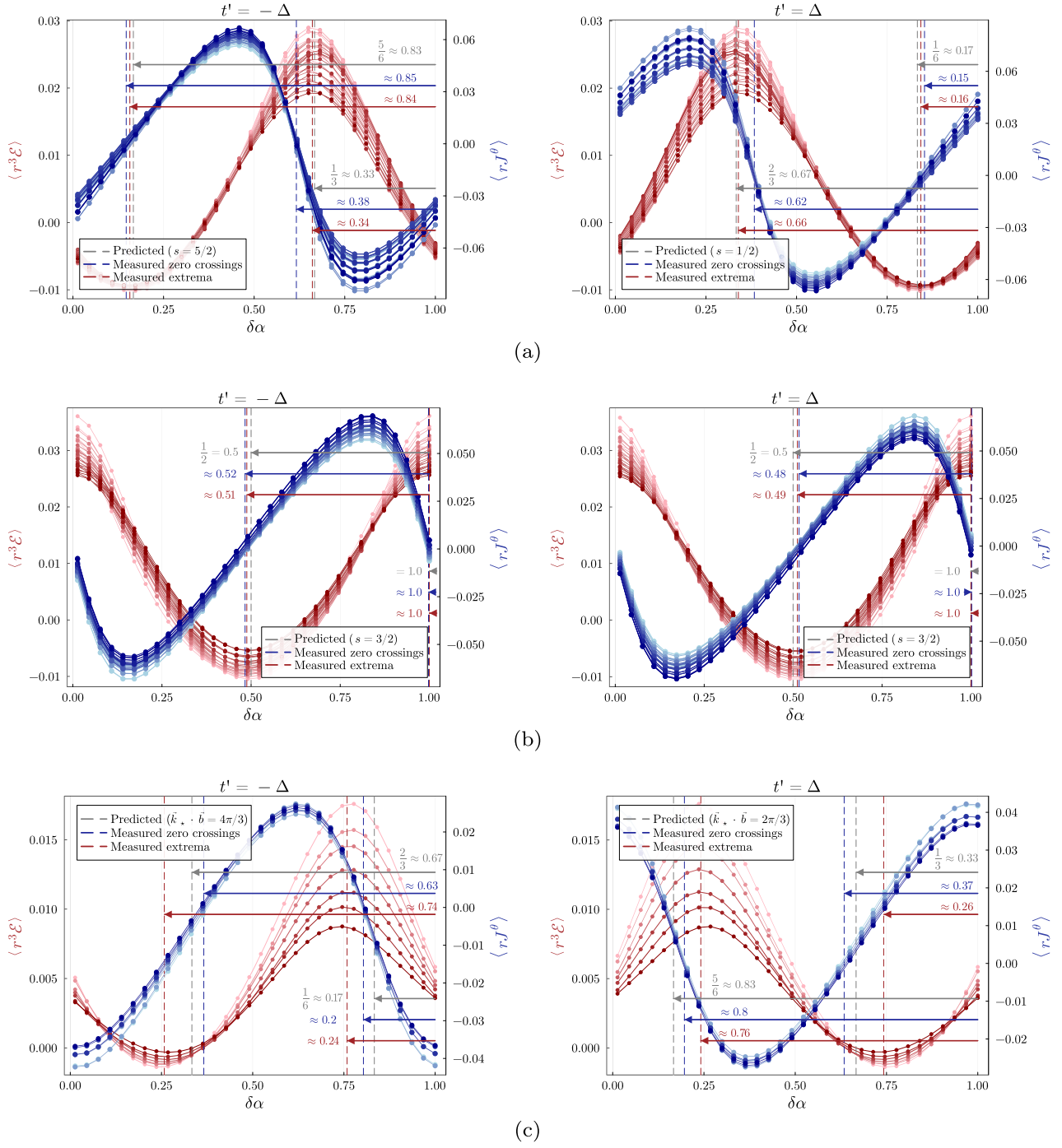


FIG. 20. Haldane model with crystalline defects. (a) Lattice with $N = 4761$ sites and a B -centered $\beta = 2/3$ disclination. (b) Lattice with $N = 4356$ sites and a C -centered $\beta = 2/3$ disclination. (c) Lattice with $N = 9408$ sites and a $\vec{a}\vec{b} = (-1, 0)$ dislocation.

3. Topological invariants

In this section, we present the absolute values of the topological invariants measured through partial rotations in Tables IV–VI.

TABLE IV. Topological invariants in each phase of the QWZ model discussed in Secs. III A, V B, and VII B.

	C	$\mathcal{S}_A$	$\mathcal{S}_B$	ℓ_A	ℓ_B
$m < -1$	0	0	1	0	0
$-1 < m < 0$	-1	$\frac{7}{2}$	$\frac{1}{2}$	$\frac{15}{4}$	$\frac{15}{4}$
$0 < m < 1$	1	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{9}{4}$
$m > 1$	0	0	1	0	2

TABLE V. Topological invariants in each phase of the three π -flux models discussed in Sec. VII C.

	C	$\mathcal{S}_A$	$\mathcal{S}_B$	ℓ_A	ℓ_B
$t' < 0$	1	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{9}{4}$	$\frac{1}{4}$
$t' > 0$	-1	$\frac{7}{2}$	$\frac{1}{2}$	$\frac{15}{4}$	$\frac{15}{4}$
(a) No on-site potential.					
	C	$\mathcal{S}_A$	$\mathcal{S}_B$	ℓ_A	ℓ_B
$t' < -\Delta$	1	$\frac{7}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
$-\Delta < t' < \Delta$	0	0	1	0	0
$t' > \Delta$	-1	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{7}{4}$	$\frac{15}{4}$
(b) One-site staggered potential.					
	C	$\mathcal{S}_A$	$\mathcal{S}_B$	ℓ_A	ℓ_B
$t' < -\Delta$	1	$\frac{7}{2}$	$\frac{7}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
$-\Delta < t' < \Delta$	0	0	0	0	0
$t' > \Delta$	-1	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{7}{4}$	$\frac{7}{4}$
(c) Two-site staggered potential.					

TABLE VI. Topological invariants in each phase of the Haldane model discussed in Sec. VII D.

	C	$\mathcal{S}_A$	$\mathcal{S}_B$	$\mathcal{S}_C$	ℓ_A	ℓ_B	ℓ_C
$t' < -\Delta$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{9}{4}$
$-\Delta < t' < \Delta$	0	0	1	0	0	0	0
$t' > \Delta$	-1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{11}{4}$	$\frac{11}{4}$	$\frac{3}{4}$

- [1] X.-G. Wen, *Quantum orders and symmetric spin liquids*, *Phys. Rev. B* **65**, 165113 (2002).
[2] M. Z. Hasan and C. L. Kane, *Colloquium: Topological insulators*, *Rev. Mod. Phys.* **82**, 3045 (2010).
[3] M. Barkeshli and X.-L. Qi, *Topological nematic states and non-Abelian lattice dislocations*, *Phys. Rev. X* **2**, 031013 (2012).

- [4] A. M. Essin and M. Hermele, *Classifying fractionalization: Symmetry classification of gapped $\mathbb{Z}_2$ spin liquids in two dimensions*, *Phys. Rev. B* **87**, 104406 (2013).
[5] W. A. Benalcazar, J. C. Y. Teo, and T. L. Hughes, *Classification of two-dimensional topological crystalline superconductors and Majorana bound states at disclinations*, *Phys. Rev. B* **89**, 224503 (2014).
[6] Y. Ando and L. Fu, *Topological crystalline insulators and topological superconductors: From concepts to materials*, *Annu. Rev. Condens. Matter Phys.* **6**, 361 (2015).
[7] H. Watanabe, H. C. Po, A. Vishwanath, and M. Zaletel, *Filling constraints for spin-orbit coupled insulators in symmetric and nonsymmetric crystals*, *Proc. Natl. Acad. Sci. U.S.A.* **112**, 14551 (2015).
[8] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, *Classification of topological quantum matter with symmetries*, *Rev. Mod. Phys.* **88**, 035005 (2016).
[9] H. C. Po, A. Vishwanath, and H. Watanabe, *Symmetry-based indicators of band topology in the 230 space groups*, *Nat. Commun.* **8**, 50 (2017).
[10] H. Song, S.-J. Huang, L. Fu, and M. Hermele, *Topological phases protected by point group symmetry*, *Phys. Rev. X* **7**, 011020 (2017).
[11] S.-J. Huang, H. Song, Y.-P. Huang, and M. Hermele, *Building crystalline topological phases from lower-dimensional states*, *Phys. Rev. B* **96**, 205106 (2017).
[12] K. Shiozaki, H. Shapourian, and S. Ryu, *Many-body topological invariants in fermionic symmetry-protected topological phases: Cases of point group symmetries*, *Phys. Rev. B* **95**, 205139 (2017).
[13] J. Kruthoff, J. de Boer, J. van Wezel, C. L. Kane, and R.-J. Slagter, *Topological classification of crystalline insulators through band structure combinatorics*, *Phys. Rev. X* **7**, 041069 (2017).
[14] B. Bradlyn, L. Elcoro, and e. a. Jennifer Cano, *Topological quantum chemistry*, *Nature (London)* **547**, 298 (2017).
[15] F. Schindler, A. M. Cook, M. G. Vergniory, Z. Wang, S. S. P. Parkin, B. A. Bernevig, and T. Neupert, *Higher-order topological insulators*, *Sci. Adv.* **4**, eaat0346 (2018).
[16] R. Thorngren and D. V. Else, *Gauging spatial symmetries and the classification of topological crystalline phases*, *Phys. Rev. X* **8**, 011040 (2018).
[17] S. Liu, A. Vishwanath, and E. Khalaf, *Shift insulators: Rotation-protected two-dimensional topological crystalline insulators*, *Phys. Rev. X* **9**, 031003 (2019).
[18] Z. Song, C. Fang, and Y. Qi, *Real-space recipes for general topological crystalline states*, *Nat. Commun.* **11**, 4197 (2020).
[19] T. Li, P. Zhu, W. A. Benalcazar, and T. L. Hughes, *Fractional disclination charge in two-dimensional C_n -symmetric topological crystalline insulators*, *Phys. Rev. B* **101**, 115115 (2020).
[20] N. Manjunath and M. Barkeshli, *Crystalline gauge fields and quantized discrete geometric response for Abelian topological phases with lattice symmetry*, *Phys. Rev. Res.* **3**, 013040 (2021).
[21] N. Manjunath and M. Barkeshli, *Classification of fractional quantum Hall states with spatial symmetries*, *arXiv: 2012.11603*.

- [22] J. Cano and B. Bradlyn, *Band representations and topological quantum chemistry*, *Annu. Rev. Condens. Matter Phys.* **12**, 225 (2021).
- [23] Y. Zhang, N. Manjunath, G. Nambiar, and M. Barkeshli, *Fractional disclination charge and discrete shift in the Hofstadter butterfly*, *Phys. Rev. Lett.* **129**, 275301 (2022).
- [24] Y. Zhang, N. Manjunath, R. Kobayashi, and M. Barkeshli, *Complete crystalline topological invariants from partial rotations in $(2+1)$ D invertible fermionic states and Hofstadter's butterfly*, *Phys. Rev. Lett.* **131**, 176501 (2023).
- [25] Y. Zhang, N. Manjunath, G. Nambiar, and M. Barkeshli, *Quantized charge polarization as a many-body invariant in $(2+1)$ d crystalline topological states and Hofstadter butterflies*, *Phys. Rev. X* **13**, 031005 (2023).
- [26] N. Manjunath, V. Calvera, and M. Barkeshli, *Characterization and classification of interacting $(2+1)$ -dimensional topological crystalline insulators with orientation-preserving wallpaper groups*, *Phys. Rev. B* **109**, 035168 (2024).
- [27] R. Kobayashi, Y. Zhang, N. Manjunath, and M. Barkeshli, *Crystalline invariants of fractional Chern insulators*, *Phys. Rev. B* **112**, 035147 (2025).
- [28] In particular, M -fold spatial rotation symmetries satisfying $(C_M)^M = +1$ should be described using internal symmetries that have order $2M$ while spatial rotation symmetries satisfying $(C_M)^M = (-1)^F$ should be described in terms of internal symmetries that have order M .
- [29] A. Debray, *Invertible phases for mixed spatial symmetries and the fermionic crystalline equivalence principle*, [arXiv:2102.02941](https://arxiv.org/abs/2102.02941).
- [30] J.-H. Zhang, S. Yang, Y. Qi, and Z.-C. Gu, *Real-space construction of crystalline topological superconductors and insulators in 2D interacting fermionic systems*, *Phys. Rev. Res.* **4**, 033081 (2022).
- [31] N. Manjunath, V. Calvera, and M. Barkeshli, *Nonperturbative constraints from symmetry and chirality on Majorana zero modes and defect quantum numbers in $(2+1)$ dimensions*, *Phys. Rev. B* **107**, 165126 (2023).
- [32] Since one can always apply magnetic flux at the core of the lattice defect, it is important that the Hamiltonian in the presence of the lattice defect is determined, up to local operators at the core, by the lattice rotation and translation operators of interest.
- [33] M. Cheng and N. Seiberg, *Lieb-Schultz-Mattis, Luttinger, and 't Hooft-anomaly matching in lattice systems*, *SciPost Phys.* **15**, 051 (2023).
- [34] M. Billò, V. Gonçalves, E. Lauria, and M. Meineri, *Defects in conformal field theory*, *J. High Energy Phys.* **04** (2016) 091.
- [35] N. Andrei *et al.*, *Boundary and defect CFT: Open problems and applications*, *J. Phys. A* **53**, 453002 (2020).
- [36] A. Söderberg Rousu, *Defects, renormalization and conformal field theory*, Ph.D. thesis, Acta Universitatis Upsaliensis, 2023.
- [37] A. Chalabi, *Exact methods for defects in conformal field theory*, Ph.D. thesis, University of Southampton, 2023.
- [38] L. Bianchi, A. Chalabi, V. Procházka, B. Robinson, and J. Sisti, *Monodromy defects in free field theories*, *J. High Energy Phys.* **08** (2021) 013.
- [39] G. Cuomo, Z. Komargodski, and A. Raviv-Moshe, *Renormalization group flows on line defects*, *Phys. Rev. Lett.* **128**, 021603 (2022).
- [40] For example, consider the total magnetic flux of $(1+\varepsilon)\pi \approx \pi$ accumulated at the defect of conical angle $2\pi\beta$. In a lattice model of system size L^2 , the leading finite-size correction scales as $L^{-2\varepsilon/\beta}$, which approaches $O(1)$ in the limit $\varepsilon \rightarrow 0$. We refer to Sec. VI for detailed discussions.
- [41] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, *The electronic properties of graphene*, *Rev. Mod. Phys.* **81**, 109 (2009).
- [42] J. González, F. Guinea, and M. A. H. Vozmediano, *Continuum approximation to fullerene molecules*, *Phys. Rev. Lett.* **69**, 172 (1992).
- [43] J. Gonzalez, F. Guinea, and M. A. Vozmediano, *The electronic spectrum of fullerenes from the Dirac equation*, *Nucl. Phys.* **B406**, 771 (1993).
- [44] A. Krishnan, E. Dujardin, M. Treacy, J. Hugdahl, S. Lynum, and T. Ebbesen, *Graphitic cones and the nucleation of curved carbon surfaces*, *Nature (London)* **388**, 451 (1997).
- [45] P. E. Lammert and V. H. Crespi, *Topological phases in graphitic cones*, *Phys. Rev. Lett.* **85**, 5190 (2000).
- [46] P. E. Lammert and V. H. Crespi, *Graphene cones: Classification by fictitious flux and electronic properties*, *Phys. Rev. B* **69**, 035406 (2004).
- [47] Y. A. Sitenko and N. Vlasii, *Electronic properties of graphene with a topological defect*, *Nucl. Phys.* **B787**, 241 (2007).
- [48] J. González, F. Guinea, and M. Vozmediano, *Electron-electron interactions in graphene sheets*, *Phys. Rev. B* **63**, 134421 (2001).
- [49] A. Morpurgo and F. Guinea, *Intervalley scattering, long-range disorder, and effective time-reversal symmetry breaking in graphene*, *Phys. Rev. Lett.* **97**, 196804 (2006).
- [50] B. Han, H. Wang, and P. Ye, *Generalized Wen-Zee terms*, *Phys. Rev. B* **99**, 205120 (2019).
- [51] J. May-Mann and T. L. Hughes, *Crystalline responses for rotation-invariant higher-order topological insulators*, *Phys. Rev. B* **106**, L241113 (2022).
- [52] N. Manjunath, A. Prem, and Y.-M. Lu, *Rotational symmetry protected edge and corner states in Abelian topological phases*, *Phys. Rev. B* **107**, 195130 (2023).
- [53] M. R. Hirsbrunner, A. D. Gray, and T. L. Hughes, *Crystalline electromagnetic responses of higher-order topological semimetals*, *Phys. Rev. B* **109**, 075169 (2024).
- [54] M. R. Hirsbrunner, O. Dubinkin, F. J. Burnell, and T. L. Hughes, *Anomalous crystalline-electromagnetic responses in semimetals*, *Phys. Rev. X* **14**, 041060 (2024).
- [55] A. Jahin, Y.-M. Lu, and Y. Wang, *Many-body higher-order topological invariant for C_n -symmetric insulators*, *Phys. Rev. B* **109**, 205123 (2024).
- [56] D. Gaiotto, D. Mazac, and M. F. Paulos, *Bootstrapping the 3D Ising twist defect*, *J. High Energy Phys.* **03** (2014) 100.
- [57] A. Söderberg, *Anomalous dimensions in the WF $O(N)$ model with a monodromy line defect*, *J. High Energy Phys.* **03** (2018) 058.

- [58] S. Giombi, E. Helfenberger, Z. Ji, and H. Khanchandani, *Monodromy defects from hyperbolic space*, *J. High Energy Phys.* **02** (2022) 041.
- [59] A. Gimenez-Grau and P. Liendo, *Bootstrapping monodromy defects in the Wess-Zumino model*, *J. High Energy Phys.* **05** (2022) 185.
- [60] J. S. Dowker, *Entanglement entropy and C_T for monodromy defects of fields on odd-dimensional spheres*, [arXiv:2201.13358](https://arxiv.org/abs/2201.13358).
- [61] J. S. Dowker, *On the Green function for an Aharonov-Bohm flux tube*, [arXiv:2205.08477](https://arxiv.org/abs/2205.08477).
- [62] A. Söderberg Rousu, *The $O(N)$ -flavoured replica twist defect*, *J. High Energy Phys.* **07** (2023) 022.
- [63] More generally, deformation classes of ρ should be viewed as invariants of the UV theory.
- [64] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, *Symmetry fractionalization, defects, and gauging of topological phases*, *Phys. Rev. B* **100**, 115147 (2019).
- [65] C. Jones, D. Penneys, and D. Reutter, *A 3-categorical perspective on G -crossed braided categories*, *J. Lond. Math. Soc.* **107**, 333 (2023).
- [66] M. Barkeshli, P. Bonderson, M. Cheng, C.-M. Jian, and K. Walker, *Reflection and time reversal symmetry enriched topological phases of matter: Path integrals, non-orientable manifolds, and anomalies*, *Commun. Math. Phys.* **374**, 1021 (2019).
- [67] S. D. Pace, G. Delfino, H. T. Lam, and O. M. Aksoy, *Gauging modulated symmetries: Kramers-Wannier dualities and non-invertible reflections*, *SciPost Phys.* **18**, 021 (2025).
- [68] Note that, for concreteness, we are discussing models with no dynamical gauge fields in the IR description.
- [69] R. M. Nandkishore and M. Hermele, *Fractons*, *Annu. Rev. Condens. Matter Phys.* **10**, 295 (2019).
- [70] X.-L. Qi and S.-C. Zhang, *Topological insulators and superconductors*, *Rev. Mod. Phys.* **83**, 1057 (2011).
- [71] X.-L. Qi, Y.-S. Wu, and S.-C. Zhang, *Topological quantization of the spin Hall effect in two-dimensional paramagnetic semiconductors*, *Phys. Rev. B* **74**, 085308 (2006).
- [72] Y. Zhang and M. Barkeshli, *Electric polarization and discrete shift from boundary and corner charge in crystalline Chern insulators*, *Phys. Rev. B* **111**, 075168 (2025).
- [73] As we discuss later, this Lagrangian should be understood as determining changes in the couplings relative to a reference. Strictly speaking, Eq. (31) should have suitable counterterms, which specify the reference, in order to make this term well defined, in general.
- [74] Y. Zhang and M. Barkeshli, *Electric polarization in Chern insulators: Unifying many-body and single-particle approaches*, *Phys. Rev. B* **112**, 115124 (2025).
- [75] Y. Nakayama, *Is boundary conformal in CFT?*, *Phys. Rev. D* **87**, 046005 (2013).
- [76] R. Kupferman, M. Moshe, and J. P. Solomon, *Metric description of singular defects in isotropic materials*, *Arch. Ration. Mech. Anal.* **216**, 1009 (2015).
- [77] Our convention for gamma matrices is $\gamma_t = i\sigma_z$, $\gamma_r = \cos(\theta)\sigma_x + \sin(\theta)\sigma_y$, and $\gamma_\theta = \beta r(-\sin(\theta)\sigma_x + \cos(\theta)\sigma_y)$.
- [78] O. Aharony, G. Cuomo, Z. Komargodski, M. Mezei, and A. Raviv-Moshe, *Phases of Wilson lines in conformal field theories*, *Phys. Rev. Lett.* **130**, 151601 (2023).
- [79] O. Aharony, G. Cuomo, Z. Komargodski, M. Mezei, and A. Raviv-Moshe, *Phases of Wilson lines: Conformality and screening*, *J. High Energy Phys.* **12** (2023) 183.
- [80] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (Elsevier, New York, 2013), Vol. 3.
- [81] D. B. Kaplan, J.-W. Lee, D. T. Son, and M. A. Stephanov, *Conformality lost*, *Phys. Rev. D* **80**, 125005 (2009).
- [82] C. J. Burges, D. Z. Freedman, S. Davis, and G. Gibbons, *Supersymmetry in anti-de Sitter space*, *Ann. Phys. (N.Y.)* **167**, 285 (1986).
- [83] E. D'Hoker and D. Z. Freedman, *Supersymmetric gauge theories and the AdS/CFT correspondence*, in *Strings, Branes and Extra Dimensions: TASI 2001* (World Scientific, Singapore, 2004), pp. 3–159.
- [84] I. Nagar, A. Sever, and D.-I. Zhong, *Planar RG flows on line defects*, *J. High Energy Phys.* **06** (2024) 110.
- [85] S. S. Gubser and I. R. Klebanov, *A universal result on central charges in the presence of double-trace deformations*, *Nucl. Phys.* **B656**, 23 (2003).
- [86] A. Kapustin, *Wilson-'t Hooft operators in four-dimensional gauge theories and s -duality*, *Phys. Rev. D* **74**, 025005 (2006).
- [87] L.-Y. Hung, R. C. Myers, and M. Smolkin, *Twist operators in higher dimensions*, *J. High Energy Phys.* **10** (2014) 178.
- [88] The Zamolodchikov distance on the conformal manifold is $\arcsin(2\pi C_J)$. We thank A. Sharon for a helpful discussion.
- [89] H. Casini, I. Salazar Landea, and G. Torroba, *Entropic g theorem in general spacetime dimensions*, *Phys. Rev. Lett.* **130**, 111603 (2023).
- [90] H. Casini, I. Salazar Landea, and G. Torroba, *Irreversibility, QNEC, and defects*, *J. High Energy Phys.* **07** (2023) 004.
- [91] N. Kobayashi, T. Nishioka, Y. Sato, and K. Watanabe, *Towards a C -theorem in defect CFT*, *J. High Energy Phys.* **01** (2019) 039.
- [92] D. M. McAvity and H. Osborn, *Energy momentum tensor in conformal field theories near a boundary*, *Nucl. Phys.* **B406**, 655 (1993).
- [93] D. M. McAvity and H. Osborn, *Conformal field theories near a boundary in general dimensions*, *Nucl. Phys.* **B455**, 522 (1995).
- [94] P. Liendo, L. Rastelli, and B. C. van Rees, *The bootstrap program for boundary CFT_d*, *J. High Energy Phys.* **07** (2013) 113.
- [95] K. Jensen and A. O'Bannon, *Constraint on defect and boundary renormalization group flows*, *Phys. Rev. Lett.* **116**, 091601 (2016).
- [96] We thank J. Maldacena for a discussion of this topic.
- [97] A. Gromov and L. Radzihovsky, *Colloquium: Fracton matter*, *Rev. Mod. Phys.* **96**, 011001 (2024).
- [98] In special cases where $\beta \in \mathbb{Z}$ is an integer, there exist defect operators with spin 1 and scaling dimension 2. However, it is only when $\beta = 1$ that they are the lowest-lying spin-1 operators.
- [99] C. Fechin, *Crystalline Dirac Criticality*, github.com/fechin/crystalline-dirac-criticality (2025), 10.5281/zenodo.17980048.

- [100] F. D. M. Haldane, *Model for a quantum Hall effect without Landau levels: Condensed-matter realization of the “parity anomaly”*, *Phys. Rev. Lett.* **61**, 2015 (1988).
- [101] J. R. Schaibley, H. Yu, G. Clark, P. Rivera, J. S. Ross, K. L. Seyler, W. Yao, and X. Xu, *Valleytronics in 2D materials*, *Nat. Rev. Mater.* **1**, 1 (2016).
- [102] S. A. Vitale, D. Nezich, J. O. Varghese, P. Kim, N. Gedik, P. Jarillo-Herrero, D. Xiao, and M. Rothschild, *Valleytronics: opportunities, challenges, and paths forward*, *Small* **14**, 1801483 (2018).
- [103] W. Chen, M. P. A. Fisher, and Y.-S. Wu, *Mott transition in an anyon gas*, *Phys. Rev. B* **48**, 13749 (1993).
- [104] M. Barkeshli and J. McGreevy, *Continuous transition between fractional quantum Hall and superfluid states*, *Phys. Rev. B* **89**, 235116 (2014).
- [105] T. Grover and A. Vishwanath, *Quantum phase transition between integer quantum Hall states of bosons*, *Phys. Rev. B* **87**, 045129 (2013).
- [106] W. A. Benalcazar, T. Li, and T. L. Hughes, *Quantization of fractional corner charge in C_n -symmetric higher-order topological crystalline insulators*, *Phys. Rev. B* **99**, 245151 (2019).
- [107] L. Šmejkal, J. Sinova, and T. Jungwirth, *Emerging research landscape of altermagnetism*, *Phys. Rev. X* **12**, 040501 (2022).
- [108] S. S. Razamat, *Quivers and fractons*, *Phys. Rev. Lett.* **127**, 141603 (2021).
- [109] W. Mück, *Spinor parallel propagator and Green function in maximally symmetric spaces*, *J. Phys. A* **33**, 3021 (2000).
- [110] S. Giombi, E. Helfenberger, and H. Khanchandani, *Fermions in AdS and Gross-Neveu BCFT*, *J. High Energy Phys.* **07** (2022) 018.
- [111] S. Giombi, E. Helfenberger, and H. Khanchandani, *Line defects in fermionic CFTs*, *J. High Energy Phys.* **08** (2023) 224.