Noninvertible Bosonic Chiral Symmetry on the Lattice

Lukasz Fidkowski,[1] Cenke Xu,[2] and Carolyn Zhang[3]

[1]*Department of Physics, University of Washington, Seattle, Washington 98195, USA*
[2]*Department of Physics, University of California, Santa Barbara, California 93106, USA*
[3]*Department of Physics and Astronomy, and Quantum Matter Institute, University of British Columbia, Vancouver, British Columbia, V6T 1Z1, Canada*

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In this work, we realize the $3+1$D noninvertible $\mathbb{Z}_N$ chiral symmetry generator as an operator in a many-body lattice Hilbert space. A crucial ingredient in our construction is the use of infinite-dimensional U(1) rotor site Hilbert spaces. Specifically, our Hilbert space is that of a U(1) lattice gauge theory coupled to a charge 1 scalar in the Villain formulation, which allows for direct access to monopoles and for a simple definition of a magnetic $\mathbb{Z}_N$ one-form symmetry $Z_m^{(1)}$, at the lattice Hamiltonian level. We construct the generator of the $\mathbb{Z}_N$ chiral symmetry as a locality-preserving unitary operator not in the entire Hilbert space but only in the subspace of $Z_m^{(1)}$-invariant states. Using a lattice-level duality based on gauging $Z_m^{(1)}$, we find a dual description of this subspace, as the subspace of a charge $1/N$ gauge theory invariant under an electric one-form symmetry $Z_e^{(1)}$. We show that, in this dual formulation, the chiral symmetry generator extends unitarily to the entire Hilbert space but has a mixed anomaly with the $Z_e^{(1)}$ symmetry.

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I. INTRODUCTION

Symmetries play an important role in both continuum quantum field theory and in condensed matter many-body quantum lattice systems. Recently, the notion of symmetry has expanded to include higher-form symmetries, whose defects have codimension greater than one [1], higher-group symmetries, which mix defects of different codimensions [2–5], and noninvertible symmetries, whose defects do not form a group under fusion [6–11]. Among the noninvertible symmetries, one particularly interesting example is that of the discrete noninvertible chiral symmetry in certain $3+1$D U(1) gauge theories [12–14]. This is a discrete remnant of the $\mathrm{U}(1)_A$ axial symmetry that is broken by the Adler-Bell-Jackiw (ABJ) anomaly [15,16]. The corresponding symmetry defect hosts a quantum Hall state with a fractional Hall conductivity. Such a state must also have anyons with fractional statistics, and it is the noninvertibility of this $2+1$D state that, in the field theory interpretation, is the origin of the noninvertible nature of the $3+1$D chiral symmetry [14]. However, this intuition does not directly lead to an operator-theoretic understanding of the noninvertible chiral symmetry: how does a noninvertible $2+1$D fractional quantum Hall state living on the symmetry defect lead to a noninvertible operator implementing the action of the symmetry in a many-body lattice Hilbert space?

In this work, we answer this question by explicitly constructing the action of the discrete noninvertible chiral symmetry in a bosonic many-body lattice Hilbert space [17]. In our approach, we first construct the action of an axial $\mathrm{U}(1)_A$ symmetry in the "ungauged" theory, i.e., one with a global on-site $\mathrm{U}(1)_V$ symmetry. This is nontrivial due to the Nielsen-Ninomiya theorem [18,19] and its extension to interacting systems [20]. However, these no-go theorems can be evaded in various ways, for example, by changing the $\mathrm{U}(1)_A$ to an $\mathbb{R}$ symmetry [21] or by relaxing the requirement that $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_V$ commute, resulting in an Onsager algebra [22,23]. In this work, we retain $\mathrm{U}(1)_A$ as a compact symmetry and insist that it commute with $\mathrm{U}(1)_V$ but instead evade the no-go theorems by working with a lattice of infinite-dimensional rotor degrees of freedom [24]. In this setting, the on-site $U(1)_V$ simply shifts the rotors, while the $\mathrm{U}(1)_A$ multiplies by a certain non-on-site rotor-dependent phase, given as a sum of a local charge density $*j_A^{\mathrm{matter}}$. The mixed 't Hooft anomaly that characterizes these symmetries is manifested by the fact that acting with a $2\pi\mathrm{U}(1)_A$ rotation pumps a U(1) bosonic symmetry-protected topological (SPT) state [26], characterized by nonzero Hall conductivity, out to the

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boundary. Indeed, in the case of the standard axial symmetry of a Dirac fermion, a 2π axial rotation pumps out an integer quantum Hall (IQH) state to the boundary; equivalently, a fundamental vortex in the Dirac mass term hosts a $c = 1$ chiral edge mode [20], which is the obstruction to obtaining a gapped $U(1)_A$-symmetric by quantum disordering the Dirac mass order parameter [27] Our lattice situation is different in two ways: we have a bosonic system instead of a fermionic one, and our $U(1)_A$ symmetry has no self-anomaly. The latter property is reflected by the fact that we pump an SPT phase, which has no net chiral central charge, only U(1) Hall conductivity. We provide a field theory realization of this anomaly in Sec. IV A. The ability of a many-body Hilbert space made up of infinite-dimensional rotor Hilbert spaces on each site to evade the no-go theorems of Refs. [18–20] is closely tied to the existence of $2 + 1$D commuting projector Hamiltonians in rotor Hilbert spaces with nonzero Hall conductivity [30], which evade the no-go theorems of Refs. [31,32].

The second step in our approach is to gauge the $U(1)_V$ symmetry by coupling to a $U(1)_V$ gauge field, at the level of the spatial lattice. It is important for us to retain control over the monopoles, so we use the Villain formalism, where the $U(1)_V$ vector potential a is real valued and defined on the edges of our spatial lattice and the integer-valued magnetic variable n is defined on the plaquettes; the real-valued rotors ϕ live on the vertices [33]. The Hilbert space $\mathcal{H}$ consists of all wave functions of ϕ, a, and n that are invariant under gauge invariance [Eq. (6)] and the Villain condition [Eq. (7)]. The latter can be thought of as a gauge invariance condition arising from gauging a $\mathbb{Z}$ subgroup of the $\mathbb{R}$ one-form symmetry implementing shifts of a, with n acting as the $\mathbb{Z}$ two-form gauge field. The matter rotor Hilbert space can be viewed as a subspace $\mathcal{H}_{\text{matter}} \subset \mathcal{H}$ where we have made the gauge field trivial, i.e., set $a = d\lambda + m$, $n = dm$, where λ is real and m is integer valued. We then write down a local charge density $*j_A$ which may be thought of as a "gauging" of $*j_A^{\text{matter}}$, in that it agrees with $*j_A^{\text{matter}}$ on $\mathcal{H}_{\text{matter}}$. Our $*j_A$ does not generate a U(1) symmetry, but it does have the property that

$$\exp\left[2\pi i \int_{M^3} *j_A\right] = \exp\left[2\pi i \int_{M^3} (da \cup a - a \cup n - n \cup a - a \cup_1 dn - 2\phi \cup dn)\right],$$

where M^3 is the spatial manifold and the integral is really a sum over 3-simplices. The lattice-level cup products in this equation are defined for both branched simplicial decompositions (see, e.g., Appendix A in Ref. [34]) and for ordinary cubic lattices [35], so we can work with either kind of lattice. The expression on the right-hand side of the above equation is both gauge invariant and invariant under the Villain condition [36]. It has previously been used to define a $2 + 1$D action for lattice Chern-Simons theory [36,37], but here it is interpreted as an operator in a $3 + 1$D theory. The above equation shows that $*j_A$ generates a family of local unitaries parametrized by $\mathbb{R}$, as opposed to U(1), and also that these unitaries fail to be symmetries for any Hamiltonian that includes an electric field operator. This is the expected anomalous nonconservation of $U(1)_A$ in the presence of a $U(1)_V$ gauge field.

Since we are interested in constructing only a $\mathbb{Z}_N$ subgroup of $U(1)_A$, however, we could attempt to define the generator of this $\mathbb{Z}_N$ group by

$$\exp\left[\frac{2\pi i}{N} \int_{M^3} (*j_A - (da \cup a - a \cup n - n \cup a - a \cup_1 dn - 2\phi \cup dn))\right]. \tag{1}$$

The Nth power of this operator is the identity by construction, and, for a gauge field configuration with no monopoles ($dn = 0$), the only ϕ dependence in the above exponent is in $*j_A$. Thus, this operator is equal to $\exp\{[(2\pi i)/N] \int_{M^3} *j_A\}$ on $\mathcal{H}_{\text{matter}}$; i.e., it has the correct axial action on the matter degrees of freedom. However, the problem is that this is not actually a well-defined operator in $\mathcal{H}$, because the expression in the exponent is not invariant under the Villain constraint [Eq. (7)]. Indeed, this can immediately be seen from the fact that the operator in Eq. (1) is not invariant under integer shifts of ϕ.

This problem is, of course, expected, since we do not expect to define an invertible symmetry. Guided by our expectation of obtaining a noninvertible symmetry, we attempt only to define a $\mathbb{Z}_N$ generator on a certain subspace $\mathcal{H}^m$ of $\mathcal{H}$. Our $\mathbb{Z}_N$ generator then involves a projection onto this subspace, which is the origin of its noninvertible nature. The subspace $\mathcal{H}^m$ is the set of states invariant under the $\mathbb{Z}_N$ magnetic one-form symmetry

$$Z_m^{(1)}(\Sigma) \cdot \Psi(\phi, a, n) = \exp\left[\frac{2\pi i}{N} \int_\Sigma n\right] \Psi(\phi, a, n),$$

where Σ is a homology 2-surface in M^3:

$$\mathcal{H}^m = \{|\psi\rangle \in \mathcal{H} | Z_m^{(1)}(\Sigma) \cdot |\psi\rangle = |\psi\rangle \text{ for all } \Sigma\}.$$

Again, since we are working on a lattice, all integrals are to be interpreted as sums, so that the integral over Σ is really a sum over the 2-simplices making up Σ. Invariance under $\mathbb{Z}_m^{(1)}(\Sigma)$ for all Σ implies that n is an exact 2-cocycle when viewed modulo N, i.e., that there exists an integer-valued 1-cochain α such that $n = d\alpha \bmod N$. We also use the notation $\alpha = d^{-1}n$. $d^{-1}n$ is unique up to shifts by integer

valued 1-cochains which are closed modulo N. We then define

$$U_{1/N}(M^3) = \exp\left[\frac{2\pi i}{N}\int_{M^3}(*j_A - [(da-n)\cup(a-d^{-1}n) - a\cup_1 dn - 2\phi\cup dn])\right]\cdot P_{\mathcal{H}^m}, \tag{2}$$

where $P_{\mathcal{H}^m}$ is the projection onto $\mathcal{H}^m$. This is now a well-defined operator in $\mathcal{H}$ (in particular, note that $dn \in N\mathbb{Z}$ in $\mathcal{H}^m$, so the operator is invariant under integer shifts of ϕ) and is our definition of the noninvertible chiral symmetry operator. Since on configurations with no monopoles ($dn = 0$) the only ϕ dependence is in $*j_A$, and $\mathcal{H}_{\text{matter}}$ is in the image of the projector $P_{\mathcal{H}^m}$, $U_{1/N}(M^3)$ performs the requisite invertible axial rotation on the matter degrees of freedom.

The expression subtracted from $*j_A$ in Eq. (2), together with the factor of $1/N$, may be thought of as a particular way of defining a gauge-invariant fractional Chern-Simons term "$(1/N)ada$" on the spatial lattice. However, the connection between this expression and the topological quantum field theory (TQFT)-based method of curing the ABJ anomaly [14] is not immediately clear. To elucidate this connection, we gauge the $\mathbb{Z}_N$ one-form symmetry $Z_m^{(1)}$ by minimally coupling to a two-form spatial lattice $\mathbb{Z}_N$ gauge field. Similarly to what happens in simpler examples in $1+1$D [38–40] and $3+1$D [8,9,41–44], such gauging gives a locality-preserving isomorphism between $\mathcal{H}^m$ and another Hilbert space, $\mathcal{H}^e \subset \mathcal{H}^N$, where $\mathcal{H}^N$ is the Villain Hilbert space of a charge $1/N$ scalar coupled to a U(1) gauge field and $\mathcal{H}^e$ is the subspace invariant under an electric one-form symmetry $Z_e^{(1)}$ which effectively just measures the fractional part of the charge of a Wilson line. Intuitively, $\mathcal{H}^m \cong \mathcal{H}^e$ because the operator algebras of the two Hilbert spaces match: in addition to the electric and magnetic fields, both contain integer charge Wilson lines and magnetic charge N 't Hooft lines.

In summary, we consider four Hilbert spaces:

(i) $\mathcal{H}$ is a Hilbert space of an integer charge scalar coupled to a U(1) gauge field via the Villain formalism;
(ii) $\mathcal{H}^m \subset \mathcal{H}$ is the subspace of $\mathcal{H}$ symmetric under the magnetic one-form symmetry, which allows for only magnetic charge N monopoles;
(iii) $\mathcal{H}^N$ is a Hilbert space with a different Villain condition, resulting in a period N scalar carrying electric charge $1/N$ coupled to a U(1) gauge field with magnetic charge N monopoles;
(iv) $\mathcal{H}^e \subset \mathcal{H}^N$ is the subspace of $\mathcal{H}^N$ invariant under the electric one-form symmetry, which removes the fractional electric charges.

We can then pull back $U_{1/N}(M^3)$ defined in Eq. (2) to $\mathcal{H}^e$. Using the explicit form of $*j_A$, we find that

$$U_{1/N}(M^3) = \exp\left[-\frac{2\pi i}{N}\int_{M^3}\lfloor a-d\phi\rceil\cup d\lfloor a-d\phi\rceil\right]\cdot P_{\mathcal{H}^e}, \tag{3}$$

where by abuse of notation we also denote this pullback by $U_{1/N}(M^3)$. Here, $\lfloor x\rceil$ denotes the integer closest to the real number x. As opposed to the action of $U_{1/N}(M^3)$ in $\mathcal{H}^m$, which cannot be extended in a locality-preserving way to all of $\mathcal{H}$, its action in $\mathcal{H}^e$, as defined in Eq. (3), can be extended to all of $\mathcal{H}^N$—in other words, $U_{1/N}(M^3)$ can be written as an invertible operator in $\mathcal{H}^N$ but is necessarily noninvertible in $\mathcal{H}$. In fact, this extension to $\mathcal{H}^N$, given by the formula in Eq. (3) without the projector $P_{\mathcal{H}^e}$, is a constant-depth circuit of local unitaries in $\mathcal{H}^N$. Its key property is that it has a mixed anomaly with the electric $\mathbb{Z}_N$ one-form symmetry $Z_e^{(1)}$. Specifically, $U_{1/N}(M^3)$ is an entangler for a $Z_e^{(1)}$ one-form SPT. The easiest way to see this is by acting with $U_{1/N}(M^3)$ on a particular short-range entangled state in $\mathcal{H}^N$, namely, one where $a-d\phi$ has been "Higgsed" to take integer values, and N-fold monopoles have been condensed, effectively making $a-d\phi$ $\mathbb{Z}_N$ valued. Then, the expression in Eq. (3) is precisely the one-form SPT entangler for a $\mathbb{Z}_N$ one-form degree of freedom written down in Ref. [45]. Another way of diagnosing the mixed anomaly is by localizing the action of $Z_e^{(1)}$ to the boundary of the SPT created by $U_{1/N}(M^3)$ and seeing that these anomalous $Z_e^{(1)}$ line operators are precisely the Wilson lines of the chiral $\mathbb{Z}_N$ anyon TQFT. In particular, the mixed anomaly implies that $U_{1/N}(M^3)$ is a $Z_e^{(1)}$-protected quantum cellular automaton (QCA) on $\mathcal{H}^N$ [46–48]. We note that the one-form SPT produced by $U_{1/N}(M^3)$ acting on a trivial product state in $\mathcal{H}^e$ has a nonchiral boundary theory, and it is only the boundary action of the $\mathbb{Z}_e^{(1)}$ symmetry that is chiral. On the other hand, when acting in $\mathcal{H}^m$, $U_{1/N}(M^3)$ produces a Walker-Wang wave function, which does admit a chiral commuting projector boundary. However, in contrast to the intrinsic QCA that create such wave functions in a tensor product Hilbert space [49,50], $U_{1/N}(M^3)$ is not extendable to the tensor product Hilbert space $\mathcal{H}$.

The interpretation of the noninvertible chiral symmetry as a one-form SPT entangler under one-form gauging is consistent with its field-theory-based construction using one-form gauging [12,13]. For example, in Eq. (2.18) in Ref. [12], the authors construct the action of the non-invertible chiral symmetry of QED by a twisted gauging of its magnetic $\mathbb{Z}_N$ one-form symmetry:

$$\mathcal{L}_{\text{QED}} \to \mathcal{L}_{\text{QED}} + ib\wedge F + iNb\wedge dc + iNkb\wedge b.$$

Although our construction applies to a slightly different field theory, one containing only bosonic charges, the interpretation of this field theory approach is closely related to our lattice-based construction: b is the dynamical two-form gauge field gauging the $Z_m^{(1)}$ symmetry, and the $iNkb \wedge b$ term is the spacetime action for the one-form SPT of the dual $Z_e^{(1)}$ symmetry created by the SPT entangler (here, c is a Lagrange multiplier making b a $\mathbb{Z}_N$ gauge field).

The rest of the paper is structured as follows. In Sec. II, we construct the action of an axial $\mathrm{U}(1)_A$ symmetry in a "matter" Hilbert space consisting of rotors and show that it has the requisite mixed anomaly with the on-site global $\mathrm{U}(1)_V$ symmetry. We also gauge the $\mathrm{U}(1)_V$ symmetry by coupling the rotors to a lattice gauge field in the Villain formulation and construct a gauged (nonconserved) axial charge density $*j_A$. In Sec. III, we write down the expression for our noninvertible chiral $\mathbb{Z}_N$ symmetry in $\mathcal{H}^m \subset \mathcal{H}$, constructed to have the same action on the matter fields as $\exp\{[(2\pi i)/N] \int_{M^3} *j_A\}$. We also find a dual expression for it by gauging the $\mathbb{Z}_N$ magnetic one-form symmetry $Z_m^{(1)}$. We show that the resulting operator, acting in $\mathcal{H}^e \subset \mathcal{H}^N$, is an entangler for a $Z_e^{(1)}$ SPT. In Sec. IV, we connect our results to field theory. We construct a QFT with $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$ global symmetries whose anomaly structure matches that of our lattice rotor model and give a heuristic argument for why this field theory can emerge from the lattice rotor model. Upon gauging the $\mathrm{U}(1)_V$ in the QFT, the $\mathbb{Z}_N$ remnant of $\mathrm{U}(1)_A$ becomes a noninvertible symmetry, and we argue also that this gauged QFT can emerge from our lattice gauge theory model. We conclude in Sec. V with some open directions.

II. $\mathrm{U}(1)_A$ SYMMETRY ON THE LATTICE

In this section, we construct a bosonic lattice Hilbert space of rotors with two U(1) symmetries. The first is $\mathrm{U}(1)_V$, which acts on site by just shifting the rotors. The second is $\mathrm{U}(1)_A$, which is non-on-site but generated by a local charge density $*j_A^{\mathrm{matter}}$. These have a mixed 't Hooft anomaly, captured by the fact that acting with $\mathrm{U}(1)_A$ pumps out nonzero $\mathrm{U}(1)_V$ Hall conductivity. We also gauge the $\mathrm{U}(1)_V$ at the level of the lattice operators, using the Villain formalism, and write down the corresponding gauged local $\mathrm{U}(1)_A$ charge density $*j_A$. We see that the total charge $\int_{M^3} *j_A$ (where M^3 is the spatial manifold) does not commute with the electric field term for the $\mathrm{U}(1)_V$ gauge field, as expected from the ABJ anomaly. Additionally, we see that $*j_A$ generates an $\mathbb{R}$ symmetry instead of U(1).

However, when restricted to the subspace $\mathcal{H}_{\mathrm{matter}} \subset \mathcal{H}$, $*j_A$ agrees with $*j_A^{\mathrm{matter}}$ and, hence, generates a bona fide $\mathrm{U}(1)_A$ symmetry. Our goal in the subsequent section is to construct a noninvertible $\mathbb{Z}_N$ symmetry whose action, when restricted to this matter subspace, matches the $\mathbb{Z}_N$ subgroup of this $\mathrm{U}(1)_A$. Roughly speaking, this means that the noninvertible $\mathbb{Z}_N$ symmetry, defined in the $\mathrm{U}(1)_V$ gauge theory, acts in the correct chiral way on the "matter" degrees of freedom.

A. Matter Hilbert space

We start with a Hilbert space describing the matter degrees of freedom, without any gauge fields:

$$\mathcal{H}_{\mathrm{matter}} = \bigotimes_v \mathcal{H}_v^{\mathrm{rotor}}, \tag{4}$$

where $\mathcal{H}_v^{\mathrm{rotor}} = L^2(S^1)$ and the tensor product is over vertices v of a simplicial decomposition of a 3-manifold M^3. We also assume a branching structure on M^3, which is necessary for defining lattice-level cup products [34,35]. We take the coordinate ϕ_v (the angular coordinate normalized by a factor of $1/(2\pi)$) to be valued in $\mathbb{R}$ and view states in $\mathcal{H}_{\mathrm{matter}}$ as wave functions $\psi(\{\phi_v\})$ invariant under independent integer-valued shifts of ϕ_v. Viewing $\{\phi_v\}$ as a 0-cochain $\phi \in C^0(M^3, \mathbb{R})$, this can be expressed as

$$\psi(\phi + s) = \psi(\phi)$$

for all $s \in C^0(M^3, \mathbb{Z})$. This Hilbert space is equipped with a natural on-site action of a global $\mathrm{U}(1)_V$ symmetry that rotates all the rotors simultaneously.

We now define a non-on-site action of the $\mathrm{U}(1)_A$ symmetry by

$$U_\theta^{\mathrm{matter}}(M^3) = \exp\left[i\theta \int_{M^3} *j_A^{\mathrm{matter}}\right]$$

with

$$*j_A^{\mathrm{matter}} = (d\phi - \lfloor d\phi \rceil) \cup d\lfloor d\phi \rceil.$$

Here, $\lfloor x \rceil$ denotes the integer closest to x, so that this is crucially not a continuous function of the field variables. The interpretation of this expression is as follows. $*j_A^{\mathrm{matter}}$ is an $\mathbb{R}$-valued simplicial three-form, i.e., a collection of $\mathbb{R}$-valued functions for each 3-simplex of M^3. The integral over M^3 is just the oriented sum over all top-dimensional simplices. Since

$$\begin{aligned}&[d(\phi + s) - \lfloor d(\phi + s) \rceil] \cup d\lfloor d(\phi + s) \rceil \\ &= (d\phi - \lfloor d\phi \rceil) \cup d\lfloor d\phi \rceil,\end{aligned}$$

for any $s \in C^1(M^3, \mathbb{Z})$, each such $\mathbb{R}$-valued function can be viewed as a well-defined Hermitian operator in $\mathcal{H}_{\mathrm{matter}}$, diagonal in the ϕ basis. This makes $U_\theta^{\mathrm{matter}}(M^3)$ into a unitary operator for all θ. This definition is motivated by the definition of the $2+1$D bosonic SPT partition function in Ref. [30], but in our setting $*j_A^{\mathrm{matter}}$ is an operator in $3+1$D.

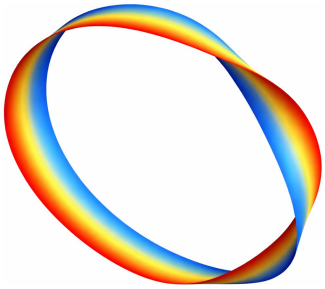


FIG. 1. An example of vortex self-linking. The vortex loop is, in fact, a ribbon, and Eq. (5) gives each linking of the ribbon a phase.

For a closed manifold M^3,

$$U_\theta^{\text{matter}}(M^3) = \exp\left[i\theta \int_{M^3} (d\phi - \lfloor d\phi \rceil) \cup d\lfloor d\phi \rceil\right] = \exp\left[-i\theta \int_{M^3} \lfloor d\phi \rceil \cup d\lfloor d\phi \rceil\right] \tag{5}$$

since $d\phi \cup d\lfloor d\phi \rceil$ is a total derivative. In particular, this means that $U_{2\pi}^{\text{matter}}(M^3) = 1$, so that $U_\theta^{\text{matter}}(M^3)$ does indeed define the action of a U(1) symmetry (as opposed to an $\mathbb{R}$ symmetry) on a closed M^3.

One interpretation of Eq. (5) can be obtained by considering its action on a "superfluid" state, i.e., a single configuration $|\phi\rangle$ consisting of a field ϕ which varies slowly everywhere except at the location of branch cuts, where it jumps by 1. The two-dimensional membrane describing the branch cuts is Poincaré dual to $\lfloor d\phi \rceil$, and its boundary, Poincaré dual to $d\lfloor d\phi \rceil$, describes vortex lines in the superfluid. The exponent in Eq. (5) then computes the integer self-linking number of these vortex lines (see Fig. 1).

This description of the bulk action of $\mathrm{U}(1)_A$ is a $3+1$-dimensional generalization of the corresponding action of $\mathrm{U}(1)_A$ in a $1+1$-dimensional compact boson, where the $\mathrm{U}(1)_A$ counts the "winding number" of the bosonic field around a periodic space—see Ref. [51] for more details.

For our purposes, a more useful property of $\mathrm{U}(1)_A$ is its action on manifolds M^3 with a boundary. We then have

$$U_{2\pi}^{\text{matter}}(M^3) = \exp\left[2\pi i \int_{M^3} (d\phi - \lfloor d\phi \rceil) \cup d\lfloor d\phi \rceil\right] = \exp\left[-2\pi i \int_{M^3} d(d\phi \cup \lfloor d\phi \rceil)\right] = \exp\left[-2\pi i \int_{\partial M^3} d\phi \cup \lfloor d\phi \rceil\right].$$

Thus, the action of $U_{2\pi}^{\text{matter}}(M^3)$ on a trivial insulator $|\phi_0\rangle$, namely, an equal-amplitude superposition of all ϕ configurations, is to pump the two-dimensional state with wave function $\exp\left[-2\pi i \int_{\partial M^3} d\phi \cup \lfloor d\phi \rceil\right]$ out to the boundary of M^3. As shown in Ref. [30], this is the root two-dimensional SPT state of bosons, with Hall conductivity 2 in natural units. This property is a manifestation of the expected mixed anomaly between $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_V$. Similar construction of 3D bosonic SPT wave functions based on nontrivial vortex loop linking and partition function of 2D SPT states was also discussed in Ref. [52].

B. Gauging $\mathrm{U}(1)_V$

We now want to examine the fate of $*j_A^{\text{matter}}$ after $\mathrm{U}(1)_V$ is gauged. Of course, the gauged version of $*j_A^{\text{matter}}$, which we denote by $*j_A$, fails to be conserved, due to the ABJ anomaly. However, we want an expression for this nonconserved $*j_A$ as the first step toward constructing our conserved noninvertible symmetry.

In order to perform the gauging at the level of the lattice, we use the Villain formalism. Thus, in addition to $\phi \in C^0(M^3, \mathbb{R})$, we introduce edge degrees of freedom $a \in C^1(M^3, \mathbb{R})$ [the $\mathrm{U}(1)_V$ vector potential] and Villain variables $n \in C^2(M^3, \mathbb{Z})$. Our gauge theory Hilbert space $\mathcal{H}$ consists of wave functions $\Phi(\phi, a, n)$ invariant under the usual gauge invariance

$$\begin{aligned} &\phi \to \phi + \lambda, \\ &a \to a + d\lambda \end{aligned} \tag{6}$$

for $\lambda \in C^0(M^3, \mathbb{R})$, as well as under the Villain condition

$$\begin{aligned} &a \to a + v, \\ &n \to n + dv \end{aligned} \tag{7}$$

for $v \in C^1(M^3, \mathbb{Z})$.

Before gauging $*j_A^{\text{matter}}$, it is useful to first define a lattice version of the exponentiated Chern-Simons functional $\exp\left[2\pi i \int_{M^3} ada\right]$. Following Ref. [36], we do this by replacing

$$ada \to da \cup (a - d\phi) - (a - d\phi) \cup n - n \cup (a - d\phi) - (a - d\phi) \cup_1 dn, \tag{8}$$

where the higher cup product $\cup_1$ is defined in Ref. [36]. As shown in Ref. [36], the expression on the right-hand side of Eq. (8) changes by an integer under the Villain transformation [Eq. (7)]. It is also manifestly invariant under the usual gauge invariance [Eq. (6)], so that multiplying by $2\pi i$ and taking the exponential yields a well-defined local operator. Furthermore, as shown in Ref. [36], for a closed M^3 we have

$$\exp\left[2\pi i\int_{M^3}(da\cup(a-d\phi)-(a-d\phi)\cup n-n\cup(a-d\phi)-(a-d\phi)\cup_1 dn)\right]$$
$$=\exp\left[2\pi i\int_{M^3}(da\cup a-a\cup n-n\cup a-a\cup_1 dn-2\phi\cup dn)\right],$$

so that, in particular, this expression is independent of the matter field ϕ on configurations with no monopoles ($dn=0$).

We now write down a local Hermitian operator $*j_A$ in $\mathcal{H}$ which is to be viewed as the gauged version of $*j_A^{\text{matter}}$. $*j_A$ performs multiplication by a function of ϕ, a, and n; this function is also denoted $*j_A$. We define

$$*j_A=*j_A^{(\mathrm{I})}+*j_A^{(\mathrm{II})}, \tag{9}$$

where

$$*j_A^{(\mathrm{I})}=da\cup(a-d\phi)-(a-d\phi)\cup n-n\cup(a-d\phi)-(a-d\phi)\cup_1 dn \tag{10}$$

and

$$*j_A^{(\mathrm{II})}=-\lfloor a-d\phi\rceil\cup d\lfloor a-d\phi\rceil+\lfloor a-d\phi\rceil\cup n+n\cup\lfloor a-d\phi\rceil$$
$$+\lfloor a-d\phi\rceil\cup_1 dn-d[(a-d\phi)\cup\lfloor a-d\phi\rceil]. \tag{11}$$

Note that we included a total derivative in the definition of $*j_A^{(\mathrm{II})}$ in order to make $*j_A$, and not just its integral, invariant under the Villain condition [Eq. (7)]. Note also that $*j_A^{(\mathrm{I})}$ and $*j_A^{(\mathrm{II})}$ are not individually well-defined operators, because they are not individually invariant under Eq. (7). Our operator $*j_A$ satisfies, for a closed M^3,

$$\exp\left[2\pi i\int_{M^3}*j_A\right]=\exp\left[2\pi i\int_{M^3}(da\cup a-a\cup n-n\cup a-a\cup_1 dn-2\phi\cup dn)\right]. \tag{12}$$

This may be thought of as the lattice-level analog of the continuum equation $\exp[2\pi i\int_{M^3}(*j_A-ada)]=1$, which follows in the continuum from $d*j_A=da\wedge da$. Some motivation for our definition of $*j_A$ is given in Appendix A.

Because $*j_A^{(\mathrm{II})}$ is an integer, the condition in Eq. (12) is satisfied. Furthermore, $*j_A^{(\mathrm{I})}(\phi,0,0)=0$ and $*j_A^{(\mathrm{II})}(\phi,0,0)=d\phi\cup d\lfloor d\phi\rceil-\lfloor d\phi\rceil\cup d\lfloor d\phi\rceil$, so that $*j_A(\phi,0,0)=*j_A^{\text{matter}}(\phi)$. Because of Villain and gauge invariance, we have more generally that $*j_A(\phi+\lambda,d\lambda+v,dv)=*j_A^{\text{matter}}(\phi)$, for $\lambda\in C^0(M^3,\mathbb{R})$ and $v\in C^1(M^3,\mathbb{Z})$. Wave functions supported on these configurations define the subspace $\mathcal{H}_{\text{matter}}\subset\mathcal{H}$, so we have that, on $\mathcal{H}_{\text{matter}}$ and closed M^3,

$$P_{\mathcal{H}_{\text{matter}}}\left(\exp\left[\frac{2\pi i}{N}\int_{M^3}*j_A\right]\right)P_{\mathcal{H}_{\text{matter}}}$$
$$=P_{\mathcal{H}_{\text{matter}}}\left(\exp\left[-\frac{2\pi i}{N}\int_{M^3}\lfloor a-d\phi\rceil\cup d\lfloor a-d\phi\rceil\right]\right)P_{\mathcal{H}_{\text{matter}}}. \tag{13}$$

III. NONINVERTIBLE CHIRAL $\mathbb{Z}_N$ SYMMETRY

Let us now write down a noninvertible chiral $\mathbb{Z}_N$ symmetry operator $U_{1/N}(M^3)$ that reduces to Eq. (13) on flat gauge field configurations. The noninvertible nature of $U_{1/N}(M^3)$ comes from a projection $P_{\mathcal{H}^m}$ onto the subspace $\mathcal{H}^m$ of states invariant under the $\mathbb{Z}_N$ magnetic one-form symmetry $Z_m^{(1)}$. Specifically, for any homology 2-surface $\Sigma\in M^3$, we define

$$Z_m^{(1)}(\Sigma)\cdot\Psi(\phi,a,n)=\exp\left[\frac{2\pi i}{N}\int_\Sigma n\right]\Psi(\phi,a,n)$$

and define

$$\mathcal{H}^m=\left\{|\psi\rangle\in\mathcal{H}|Z_m^{(1)}(\Sigma)\cdot|\psi\rangle=|\psi\rangle\text{ for all }\Sigma\right\}.$$

We note that, while $\mathcal{H}^m$ does not contain configurations with single magnetic monopoles, it does allow N-fold monopoles. The constraint that $\exp\{[(2\pi i)/N]\int_\Sigma n\}=1$ for all Σ means that $n \bmod N$ is an exact cocycle in $C^2(M^3,\mathbb{Z}_N)$, which, in turn, implies that $n=dv+Nn'$, where $v\in C^1(M^3,\mathbb{Z})$ and $n'\in C^2(M^3,\mathbb{Z})$. Let us define $d^{-1}n\equiv v$. The ambiguity in this definition is $d^{-1}n\to d^{-1}n+\alpha$, where α is any cochain in $C^1(M^3,\mathbb{Z})$ that satisfies $d\alpha\in C^2(M^3,N\mathbb{Z})$. We then

define, in $\mathcal{H}^m \subset \mathcal{H}$,

$$U_{1/N}(M^3)\cdot\Psi(\phi,a,n) = \exp\left[-\frac{2\pi i}{N}\int_{M^3}(\lfloor a-d\phi\rceil - d^{-1}n)\cup(d\lfloor a-d\phi\rceil - n)\right]\cdot\Psi(\phi,a,n). \tag{14}$$

To see that this operator $U_{1/N}(M^3)$ is well defined, we first note that it is invariant under $d^{-1}n \to d^{-1}n+\alpha$, with α as above. Also, it is manifestly gauge invariant [Eq. (6)] and invariant under the Villain transformation [Eq. (7)]. Additionally, it reduces to $\exp[(2\pi i)/N]\int_{M^3} *j_A\}$, as seen from Eq. (13), on $\mathcal{H}_{\text{matter}}$.

Recall that the fusion rules for the symmetry defects include [12,13]

$$D_{1/N}\times D_{1/N}^\dagger = C, \qquad D_{1/N}\times D_{1/N} = A^{N,2}D_{2/N} \quad (\text{odd } N), \tag{15}$$

where C is a condensation operator and $A^{N,2}$ is a minimal Abelian TQFT with anyons a^n with spin $e^{2\pi i n^2/N}$ [42]. The lattice noninvertible operators above satisfy the first fusion rule after identifying the projector with C as in Ref. [43], because the unitary part commutes with the projector onto $\mathcal{H}^m$. The second fusion rule is more subtle. The $A^{N,2}$ factor comes from, in the field theory description, each noninvertible operator carrying its own dynamical gauge fields, leading to a decoupled theory $A^{N,2}$ theory when two are fused together. In our system, the noninvertible projectors all project onto the same subspace, so we instead get $D_{1/N}\times D_{1/N} = D_{2/N}$. We believe that the discrepancy is related to the difference between "composition" and "stacking," where the former describes more naturally global symmetry operators on a Hilbert space while the latter describes symmetry defects. We believe that a local implementation of our noninvertible symmetries, i.e., by an matrix-product operator (see Sec. V), which can introduce extra virtual space at the defects, would likely be able to match the fusion rules of Eq. (15) including the extra $A^{N,2}$.

In Eq. (14), the locality properties of $U_{1/N}(M^3)$ are not entirely manifest due to the presence of the $d^{-1}n$ term. In the following, we give a slightly different derivation of Eq. (14), which makes its locality properties more manifest, and make the connection to field theory more clear.

A. Gauging magnetic $\mathbb{Z}_N$ one-form symmetry

We gauge the $Z_m^{(1)}$ one-form symmetry by coupling to a pure-gauge two-form dynamical $\mathbb{Z}_N$ gauge field. Just as the Kramers-Wannier duality in $1+1$D can be thought of as resulting from gauging the $\mathbb{Z}_2$ zero-form symmetry, gauging $Z_m^{(1)}$ in our $3+1$D Hilbert space $\mathcal{H}$ results in a dual electric one-form symmetry $Z_e^{(1)}$ acting in $\mathcal{H}^N$, the Villain lattice gauge theory of a charge $1/N$ scalar. Defining $\mathcal{H}^e$ as the subspace of $\mathcal{H}^N$ invariant under $Z_e^{(1)}$, we then have that the symmetric subspaces $\mathcal{H}^m$ and $\mathcal{H}^e$ are isomorphic. While the $U_{1/N}(M^3)$ constructed in Eq. (14) does not extend as a locality-preserving unitary operator to all of $\mathcal{H}$ [53]—i.e., it is noninvertible on $\mathcal{H}$—we see that, when viewed as an operator on $\mathcal{H}^e$, it does extend as a local quantum circuit to all of $\mathcal{H}^N$, albeit one with a mixed anomaly with $Z_e^{(1)}$.

We define $\mathcal{H}^N$ to consist of wave functions $\Psi(\phi,a,n)$, with $\phi\in C^0(M^3,\mathbb{R})$, $a\in C^1(M^3,\mathbb{R})$, and $n\in C^2(M^3,N\mathbb{Z})$, invariant under the usual gauge invariance $\phi\to\phi+\lambda$, $a\to a+d\lambda$ for $\lambda\in C^0(M^3,\mathbb{R})$, as well as the Villain condition rescaled by a factor of N;

$$a\to a+Nv, \qquad n\to n+Ndv \tag{16}$$

for $v\in C^1(M^3,\mathbb{Z})$. $\mathcal{H}^N$ is simply the Villain lattice gauge theory Hilbert space of a charge $1/N$ scalar.

We then define the action of an "electric" $\mathbb{Z}_N$ one-form symmetry $Z_e^{(1)}$ in $\mathcal{H}^N$ as follows. Let $v\in C^1(M^3,\mathbb{Z})$ be a cochain that satisfies $dv\in C^2(M^3,N\mathbb{Z})$. Such a v defines a cocycle in $C^1(M^3,\mathbb{Z}_N)$ under a reduction of coefficients mod N, which by abuse of notation we also denote v. We then define $Z_e^{(1)}(v)$ in terms of its action on wave functions as follows:

$$Z_e^{(1)}(v):\Psi(\phi,a,n)\to\Psi(\phi,a-v,n-dv). \tag{17}$$

Let $\mathcal{H}^e$ be the $Z_e^{(1)}$-invariant subspace of $\mathcal{H}^N$:

$$\mathcal{H}^e = \Big\{|\psi\rangle\in\mathcal{H}^N \,|\, Z_e^{(1)}(v)\cdot|\psi\rangle = |\psi\rangle \text{ for all } v\in C^1(M^3,\mathbb{Z}) \text{ such that } dv\in N\mathbb{Z}\Big\}.$$

Physically, $\mathcal{H}^e$ is a Hilbert space with integer electric charge Wilson lines and N times integer magnetic charge 't Hooft lines. In particular, combining the usual gauge invariance [Eq. (6)] with integer-valued λ together with $Z_e^{(1)}(d\lambda)$ shows that all states in $\mathcal{H}^e$ are invariant under $\phi_v\to\phi_v+1$.

To see how $\mathcal{H}^e\subset\mathcal{H}^N$ is the result of gauging the $Z_m^{(1)}$ one-form symmetry of $\mathcal{H}^m$ by minimally coupling to a two-form dynamical $\mathbb{Z}_N$ gauge field at the level of the Hamiltonian lattice, see the derivation in Appendix B. Intuitively, we expect that, under such $\mathbb{Z}_N$ gauging, the $\mathbb{Z}_N$ degrees of freedom should shift from living on the faces to living on the edges (faces of the dual lattice), in the same way that in $1+1$D Kramers-Wannier duality the $\mathbb{Z}_2$ degrees of freedom shift from the original to the dual lattice. This is indeed what happens when we go from $\mathcal{H}^m$ to $\mathcal{H}^e$: the Villain variable n, defined on faces, goes from

being integer valued to being N times integer valued, while the spacing between the allowed quantized electric field values on edges goes from 1 in $\mathcal{H}^m$ to $1/N$ in $\mathcal{H}^e$. This N-fold increase in the allowed values for the electric field can be viewed as the introduction of a $\mathbb{Z}_N$ dual two-form gauge field. For more details, see Appendix B.

It is worth emphasizing that the unitary duality isomorphism

$$\rho\colon \mathcal{H}^m \longrightarrow \cong \mathcal{H}^e \tag{18}$$

resulting from this one-form gauging is a map between two different Hilbert spaces. The analogous map in the $1+1$D $\mathbb{Z}_2$ setting would be

$$X_i \to X_{i-1/2}X_{i+1/2},$$
$$Z_iZ_{i+1} \to Z_{i+1/2}.$$

This is the map that results from coupling a system of $\mathbb{Z}_2$ matter spins (living on integer sites) to a $\mathbb{Z}_2$ gauge field (living on integer plus one-half sites), imposing the Gauss law, and reading off the wave function in a gauge where the matter spins all point up. Its domain is the Hilbert space of matter spin wave functions invariant under the spin flip $\prod_i X_i$ symmetry, and its range is the space of gauge field wave functions invariant under $\prod_i Z_{i+1/2}$. The latter is just the holonomy of the $\mathbb{Z}_2$ gauge field, which is conserved because the $\mathbb{Z}_2$ gauge field has no dynamics. To make contact with the usual Kramers-Wannier transformation, one composes with a Hadamard transformation ($X \leftrightarrow Z$) and a half-integer spatial shift, which maps $\prod_i Z_{i+1/2}$ back to $\prod_i X_i$ and, hence, results in a map from the original Hilbert space to itself. In our $3+1$D situation, we are not performing this second step; in fact, it is important for us that our duality isomorphism ρ is a map between two different presentations of the symmetric Hilbert space.

At the level of wave functions, our unitary duality isomorphism ρ can be described by

$$\rho(\Psi)(\phi, a, n) = c\Psi(\phi, a, n),$$

where

$$c = \frac{N^{(|\text{faces}|-|\text{edges}|)/2}}{\sqrt{|\{\beta|\partial\beta = 0\}|}}$$

and $\partial = *d*$ is the Hodge dual of the coboundary operator. This is derived in Appendix B 4 [see Eq. (B12)]. In other words, given a wave function Ψ in $\mathcal{H}^m$, up to a normalization constant the corresponding wave function $\rho(\Psi) \in \mathcal{H}^e$ is simply given by restricting Ψ to configurations where $n \in C^2(M^3, N\mathbb{Z})$. Note that, due to invariance under the Villain condition, this restriction never annihilates the state, ensuring that the isomorphism is indeed unitary. To describe ρ^{-1}, suppose $\Psi' \in \mathcal{H}^e$, and suppose that $n = dv + Nn'$, for $v \in C^1(M^3, \mathbb{Z})$, $n' \in C^2(M^3, \mathbb{Z})$. Then,

$$\rho^{-1}(\Psi')(\phi, a, n) = c^{-1}\Psi'(\phi, a - v, Nn'). \tag{19}$$

The fact that the above expression is independent of v follows from the $\mathbb{Z}_e^{(1)}$ invariance of Ψ'.

The isomorphism ρ between $\mathcal{H}^m$ and $\mathcal{H}^e$ can also be analyzed at the level of its action on local operators. To find the local operator algebras of these two Hilbert spaces, first let us consider the local operator algebra $\mathcal{A}(\mathcal{H})$. This is generated by the electric field operator $e_l = [1/(2\pi i)] \cdot [d/(da_l)]$ for every edge l, the magnetic field operator $b_f = (da - n)_f$ for every face f, the open Wilson line operator $W_l = \exp[2\pi i(a - d\phi)_l]$ for every edge l, and the open 't Hooft line operator $T_m\colon \Psi(\phi, a, n) \to \Psi(\phi, a, n + m)$ for any $n' \in C^2(M^3, \mathbb{Z})$:

$$\mathcal{A}(\mathcal{H}) = \langle \{e_l\}_l, \{b_f\}_f, \{W_l\}_l, \{T_m\}_{m \in C^2(M^3,\mathbb{Z})} \rangle.$$

Now, wave functions in $\mathcal{H}^m$ are supported on configurations $n = dv + Nn'$, $v \in C^1(M^3, \mathbb{Z})$, and $n' \in C^2(M^3, \mathbb{Z})$, so the operator algebra is generated by the same operators, except the 't Hooft lines can shift n only in multiples of N:

$$\mathcal{A}(\mathcal{H}^m) = \langle \{e_l\}_l, \{b_f\}_f, \{W_l\}_l, \{T_m\}_{m \in C^2(M^3,N\mathbb{Z})} \rangle.$$

By the same argument, $\mathcal{H}^N$ has Wilson lines that transport fractional $1/N$ charge, but $\mathcal{H}^e \subset \mathcal{H}^N$ has only integer Wilson line operators:

$$\mathcal{A}(\mathcal{H}^e) = \langle \{e_l\}_l, \{b_f\}_f, \{W_l\}_l, \{T_m\}_{m \in C^2(M^3,N\mathbb{Z})} \rangle.$$

Now, the isomorphism from $\mathcal{A}(\mathcal{H}^m)$ to $\mathcal{A}(\mathcal{H}^e)$ induced by ρ is simply the one that looks like the identity in the above presentation. Indeed, this is clear from the wave function formula for ρ [Eq. (18)].

B. Mixed anomaly between $U_{1/N}(M^3)$ and $Z_e^{(1)}$

The noninvertible chiral $\mathbb{Z}_N$ symmetry $U_{1/N}(M^3)$ is defined in Eq. (14). It is a unitary operator within $\mathcal{H}^m \subset \mathcal{H}$. Conjugating by the isomorphism ρ in Eq. (18), one obtains the following operator, which by abuse of notation we also refer to as $U_{1/N}(M^3)$:

$$U_{1/N}(M^3) = \exp\left[-\frac{2\pi i}{N}\int_{M^3} \lfloor a - d\phi \rceil \cup d\lfloor a - d\phi \rceil\right]. \tag{20}$$

Note that $U_{1/N}(M^3)$ is manifestly a constant-depth quantum circuit of local unitaries in $\mathcal{H}^N$, commutes as an operator with $Z_e^{(1)}$, and, hence, defines a unitary operator in $\mathcal{H}^e$.

The nontrivial property of $U_{\frac{1}{N}}(M^3)$ is that it has a mixed anomaly with $Z_e^{(1)}$. One way to diagnose this mixed anomaly is as follows. First, define $|\psi_{\text{Higgs}}\rangle \in \mathcal{H}^N$ to be an equal-amplitude superposition of all configurations $\{\phi, a, n\}$ for which $r \equiv a - d\phi \in C^1(M^3, \mathbb{Z})$. Since this constraint pins the covariant derivative of ϕ, it a Higgsed state. Then, $U_{1/N}(M^3)|\psi_{\text{Higgs}}\rangle$ is a state whose amplitude depends only on $r = a - d\phi$ (in particular, not on n) and is $\exp\{-[(2\pi i)/N]\int_{M^3} r \cup dr\}$. This is the wave function of a nontrivial one-form SPT of $Z_e^{(1)}$ [45]. This means that, although $U_{1/N}(M^3)$ is a constant-depth circuit, its gates cannot all be invariant under $Z_e^{(1)}$, so that $U_{1/N}(M^3)$ is a symmetry-protected QCA like all (symmetric) SPT entanglers. This is equivalent to the statement that there is a mixed anomaly between $U_{1/N}(M^3)$ and $Z_e^{(1)}$. Another way to see this anomaly is to follow the idea of Else and Nayak [54] to construct the $2+1$D "boundary" action of $Z_e^{(1)}$ at the boundary of the SPT state created by $U_{1/N}(M^3)$ and read off the anomaly from the nontrivial commutation (braiding) relations of the resulting $Z_e^{(1)}$ string operators. We give more details on this approach in Appendix C.

IV. FIELD THEORIES WITH THE NONINVERTIBLE CHIRAL SYMMETRY

The main goal of this work was to construct a noninvertible $\mathbb{Z}_N$ chiral symmetry as an operator acting in a many-body lattice gauge theory Hilbert space [Eq. (14)]. One can then consider lattice Hamiltonians that commute with this noninvertible symmetry. It is desirable to construct at least one such Hamiltonian that flows, at low energies, to some familiar field theory with a noninvertible chiral $\mathbb{Z}_N$ symmetry. In this section, we first construct such a "target" field theory and discuss some of its key properties and then argue that it can emerge from our microscopic setup.

A. Target field theory

The simplest field theory discussed in the context of the noninvertible chiral symmetry is $3+1$D QED, with $N_f = 1$ flavor of Dirac fermion:

$$\mathcal{L}_{\text{QED}}(\Psi, \bar{\Psi}, A) = \frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}(\partial_\mu - iA_\mu)\gamma^\mu\Psi.$$

However, one impediment to emerging QED out of a microscopic lattice Hilbert space is that the axial current

$$j_{\text{QED}_A}^{\mu} = \bar{\Psi}\gamma_5\gamma^\mu\Psi$$

has, in addition to the standard $\mathrm{U}(1)_V$ piece, a gravitational contribution to its divergence $\mathcal{D}_\mu j_{\text{QED}_A}^{\mu} \propto \mathrm{Tr}R \wedge R$ [55], where R is the curvature of the Levi-Civita connection. This turns out to be an obstacle for simple lattice constructions [20], so it is useful to cancel out this gravitational contribution. For QED, the simplest way to do this is to introduce an additional Dirac fermion Ψ', neutral under the $\mathrm{U}(1)_V$, resulting in the action

$$\mathcal{L}_{\text{QED}'}(\Psi, \bar{\Psi}, \Psi', \bar{\Psi}', A) = \frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}(\partial_\mu - iA_\mu)\gamma^\mu\Psi + i\bar{\Psi}'\partial_\mu\gamma^\mu\Psi'. \tag{21}$$

Because Ψ' still couples to the metric, the gravitational contribution to the anomalous divergence of

$$j_{\text{QED}'_A}^{\mu} = \bar{\Psi}\gamma_5\gamma_\mu\Psi - \bar{\Psi}'\gamma_5\gamma_\mu\Psi'$$

vanishes. Then, the gravitational contribution to the divergence of $j_{\text{QED}'_A}^{\mu}$ is 0, whereas the $\mathrm{U}(1)_V$ contribution remains nonzero.

One important feature of $j_{\text{QED}'_A}^{\mu}$ is the following. Let us work in the "ungauged" theory, where $\mathrm{U}(1)_V$ is a global symmetry. Then, we just have a $\mathrm{U}(1)_V$-charged Dirac fermion Ψ together with a neutral one Ψ'. Let $B \subset \mathbb{R}^3$ be a three-dimensional spatial ball, and define $Q_A^{(B)} = \int_B j_{\text{QED}'_A}^{0}$ to be the operator that measures the axial charge in B. Also, let $|\Psi_0\rangle$ be the short-range entangled state obtained by gapping out both Ψ and Ψ'. Then, $\exp(2\pi i Q_A^{(B)})|\Psi_0\rangle$ looks like $|\Psi_0\rangle$ away from the boundary of B, [56] whereas near the boundary it has an IQH state for the Ψ fermions (electric Hall conductance e^2/h and chiral central charge 1) stacked with an IQH state for the Ψ' fermions (electric Hall conductance 0 and chiral central charge 1). This is precisely the root $2+1$D fermionic SPT state for U(1) symmetry, which carries a Hall conductivity of e^2/h. Thus, the current $j_{\text{QED}'_A}^{\mu}$ generates a unitary that, at rotation angle 2π, "pumps out" the root U(1) SPT state of fermions. This is the key property that encodes the mixed 't Hooft anomaly between $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$.

In this work, we target a field theory that has a bosonic analog of this property: namely, the axial current generates a unitary that pumps out the root $2+1$D U(1) SPT state of bosons, with Hall conductivity $(2e^2)/h$ [57]. To find the target field theory with this property, we must simply find a bosonic version of Eq. (21), i.e., a theory that can emerge out of an underlying bosonic Hilbert space, as opposed to a $\mathbb{Z}_2$-graded one.

Since we want a theory that pumps out the root U(1) SPT state of bosons, let us first discuss a parton construction for such $2+1$D bosonic SPT. Let us consider fermionic partons ψ_1, ψ_2 and f_1, f_2. We assign ψ_α physical charge 1 and f_α physical charge 0. ψ_α and f_α are also both SU(2) fundamental doublets of a SU(2) gauge group. All physical objects should be gauge-invariant bosons with integer charges. One gauge-invariant boson is

$$b \sim \psi_1 f_1^\dagger + \psi_2 f_2^\dagger.$$

As a "mean field" state, we put ψ_α both in a $C = 1$ Chern band and f_α both in a $C = -1$ Chern band. We claim that this parton state, combined with the SU(2) gauge field, gives us the desired boson SPT state, for the following reasons.

(i) At the mean field level, the $1+1$D boundary of the $2+1$D state constructed above is a nonchiral conformal field theory (CFT) $\mathrm{U}(2)_1 \times \overline{\mathrm{U}(2)}_1$, with nonchiral central charge $c = 2$. The boundary CFT decomposes as

$$\mathrm{U}(2)_1 = (\mathrm{SU}(2)_1 \times \mathrm{U}(1)_2)/\mathbb{Z}_2,$$

with the same decomposition for $\overline{\mathrm{U}(2)}_1$. The SU(2) gauge field erases the $\mathrm{SU}(2)_1 \times \overline{\mathrm{SU}(2)}_1$ part of the boundary CFT, leaving only a nonchiral $c = 1$ CFT $\mathrm{U}(1)_2 \times \overline{\mathrm{U}(1)}_2$.

(ii) Since ψ_α both carry charge 1, the total Hall conductivity is $\sigma_{xy} = 2$, consistent with our expectation for a bosonic U(1) SPT state.

(iii) There is no net Chern-Simons term in the $2+1$D for the SU(2) gauge field, as the Chern-Simons terms generated from ψ_α and f_α cancel out. Therefore, the SU(2) gauge field can safely confine.

Now we are ready to construct our target $(3+1)$-dimensional theory. Let us consider QCD with an SU(2) gauge field and $N_f = 2$ flavors of Dirac fermions, one flavor being the SU(2) fundamental ψ_α and having charge 1 under a global $\mathrm{U}(1)_V$ symmetry and the other being another SU(2) fundamental f_α having charge 0 under the global $\mathrm{U}(1)_V$ symmetry [note that α is the SU(2) color index]. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr} G_{\mu\nu} G^{\mu\nu} + \bar{\psi} i\gamma^\mu D_\mu \psi + \bar{f} i\gamma^\mu D_\mu f, \quad (22)$$

where the $G_{\mu\nu}$ is the SU(2) field strength and D_μ is the SU(2) covariant derivative. We now define the axial rotation to be the usual axial symmetry for ψ combined with the opposite one for f. That is, the Weyl fermions ψ_L and f_R carry axial charge $+1$, and ψ_R and f_L carry axial charge -1. We claim that the 2π axial rotation then generates the desired 2D SPT state at the boundary, for the following reasons.

(i) If we couple the theory in Eq. (22) to a dynamical $\mathrm{U}(1)_V$ gauge field A_μ, the 2π axial rotation generates a $\Theta = 4\pi$ term for A_μ in the $(3+1)$D bulk, which is consistent with the response of $\sigma_{xy} = [(2q^2)/h]$ for the root bosonic U(1) SPT at the boundary.

(ii) The 2π axial rotation does not generate a net Θ term for the SU(2) gauge field.

(iii) The 2π axial rotation does not generate a net gravitational Θ term.

We now give a heuristic argument that the field theory in Eq. (22) can emerge from our microscopic rotor model while preserving $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_V$ and that its $\mathrm{U}(1)_V$-gauged version can emerge from the corresponding lattice field theory while preserving the noninvertible $\mathbb{Z}_N$ symmetry.

B. Global $\mathrm{U}(1)_V$ symmetry

Let us now give a heuristic argument that the field theory in Eq. (22) can emerge from our microscopic rotor model with global $\mathrm{U}(1)_V$ symmetry. This argument rests on one crucial assumption: that we can realize two conjugate copies of $\mathcal{L}$ with a microscopic Hamiltonian invariant under on-site actions of both the diagonal $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_V$.

The Hilbert space $\mathcal{H}_{\mathrm{matter}}$ consists of rotor degrees of freedom on vertices, as defined in Eq. (4). The $\mathrm{U}(1)_V$ is just the on-site rotation of the rotors, whereas the $\mathrm{U}(1)_A$ is generated by

$$*j_A^{\mathrm{matter}} = (d\phi - \lfloor d\phi \rceil) \cup d\lfloor d\phi \rceil.$$

Our goal is to find a microscopic Hamiltonian on this Hilbert space, together with ancillas described below, which commutes with $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$ and whose low-energy dynamics is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr} G_{\mu\nu} G^{\mu\nu} + \bar{\psi} i\gamma^\mu D_\mu \psi + \bar{f} i\gamma^\mu D_\mu f, \quad (23)$$

such that ψ carries charge 1 under the $\mathrm{U}(1)_V$, f is neutral, and the $\mathrm{U}(1)_A$ acts according to

$$j_A^\mu = \bar{\psi}\gamma_5\gamma^\mu\psi - \bar{f}\gamma_5\gamma^\mu f.$$

Here, $G_{\mu\nu}$ is the SU(2) field strength, and ψ and f are both fundamentals under the SU(2). We show that we can do this if we are allowed to introduce ancilla degrees of freedom into the Hilbert space, which transform in an on-site way under both $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$.

To proceed, we first argue that one can realize two conjugate copies $\mathcal{L}$ and $\mathcal{L}'$ in an ancilla microscopic Hilbert space $\mathcal{H}_{\mathcal{L}+\mathcal{L}'}$ where the diagonal $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$ both act on site. By conjugate copy $\mathcal{L}'$, we mean the same Lagrangian as $\mathcal{L}$:

$$\mathcal{L}' = -\frac{1}{4}\mathrm{Tr} G'_{\mu\nu} G'^{\mu\nu} + \overline{\psi'} i\gamma^\mu D_\mu \psi' + \overline{f'} i\gamma^\mu D_\mu f', \quad (24)$$

but with ψ' having charge -1 under the $\mathrm{U}(1)_V$ and f' being neutral and the $\mathrm{U}(1)_A$ acting according to

$$j'^\mu_A = -\overline{\psi'}\gamma_5\gamma^\mu\psi + \overline{f'}\gamma_5\gamma^\mu f'.$$

This is plausible, because with two conjugate copies the diagonal $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$ symmetries have no net 't Hooft anomalies, so there is no anomaly obstruction to this happening [58]. In our theory of interest, there are also SU(2) gauge fields, and this kind of rearrangement would result in chiral, albeit nonanomalous, SU(2) gauge theories. Although we cannot prove that this is possible, we still assume that a microscopic Hamiltonian $H_{\mathcal{L}+\mathcal{L}'}$ acting in $\mathcal{H}_{\mathcal{L}+\mathcal{L}'}$ and commuting with an on-site action of $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$ can realize the low-energy field theory $\mathcal{L}+\mathcal{L}'$.

The next step is to work in the Hilbert space $\mathcal{H}_{\mathcal{L}+\mathcal{L}'} \otimes \mathcal{H}_{\mathrm{matter}}$, starting with the Hamiltonian $H_{\mathcal{L}+\mathcal{L}'}$ (extending it to act trivially in $\mathcal{H}_{\mathrm{matter}}$). For specificity, we assume that, possibly after some coarse graining, the microscopic spin degrees of freedom of $\mathcal{H}_{\mathcal{L}+\mathcal{L}'}$ live on the same vertices v as the matter degrees of freedom. Our goal is then to deform this Hamiltonian by coupling the matter and ancilla degrees of freedom, thereby gapping out all of the matter and $\mathcal{L}'$, while preserving the combined $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$. The only low-energy degrees of freedom then left are those of $\mathcal{L}$, giving us our desired realization of $\mathcal{L}$. The procedure is sketched below:

$$\mathcal{H}_{\mathcal{L}+\mathcal{L}'} \times \mathcal{H}_{\mathrm{matter}} \Rightarrow \mathcal{H}_{\mathcal{L}} \times (\mathcal{H}_{\mathcal{L}'} \times \mathcal{H}_{\mathrm{matter}})_{\mathrm{Gap}}.$$

The simplest gapped state is at expense of breaking the $\mathrm{U}(1)_A$, by introducing Dirac mass terms for $\mathcal{L}'$, and the trivial rotor Hamiltonian that puts all the rotors in their 0 angular momentum state; let us call such a $\mathrm{U}(1)_V$-conserving but $\mathrm{U}(1)_A$-breaking perturbation V:

$$V = \int_{\mathbb{R}^3} m_0(\psi'^{\dagger}\gamma^0\psi' + f'^{\dagger}\gamma^0 f') - \sum_v \frac{d^2}{d{\phi_v}^2},$$

where m_0 is fixed. We then consider $H_{\mathcal{L}+\mathcal{L}'} + \lambda V$, with λ small enough so that the effects of the perturbation λV can be analyzed within field theory. Thus, the Dirac mass terms gap out $\mathcal{L}'$ and the trivial rotor Hamiltonian gaps out the matter degrees of freedom.

Of course, the problem is that $H_{\mathcal{L}+\mathcal{L}'} + \lambda V$ explicitly breaks the $\mathrm{U}(1)_A$ symmetry. To fix this, we couple V to a dynamical order parameter χ whose quantum fluctuations restore the $\mathrm{U}(1)_A$ symmetry. Specifically, we let $\chi \in C^0(M^3, \mathbb{R})$ be another set of rotors, representing the would-be $\mathrm{U}(1)_A$ order parameter. To make χ compact, we introduce a gauge field $\beta \in C^1(M^3, \mathbb{Z})$ and demand invariance under $\chi \to \chi + s, \beta \to \beta + ds,\ s \in C^1(M^3, \mathbb{Z})$. For now, we view χ and β as classical background fields, though soon we give them dynamics.

Consider now a χ configuration without vortices, i.e., with $d\beta = 0$. On $\mathbb{R}^3$ we can represent this as the gauge equivalence class of $\beta = 0$. Consider then the unitary operator $W[\chi]$ generating a spatially nonuniform $\mathrm{U}(1)_A$ rotation parametrized by χ:

$$W[\chi] = \exp\left[2\pi i \int_{\mathbb{R}^3} \chi \cdot (*j_A + *j'_A + *j_A^{\mathrm{matter}})\right]$$

and let

$$V[\chi] = W[\chi]^{\dagger} \cdot V \cdot W[\chi]. \tag{25}$$

Clearly, $V[\chi]$ is $\mathrm{U}(1)_V$ invariant. For a sufficiently slowly spatially varying χ, we have explicitly

$$\begin{aligned} V[\chi] = &\int_{\mathbb{R}^3} m_0\Big(\cos(2\pi\chi)\psi'^{\dagger}\gamma^0\psi' + \sin(2\pi\chi)\psi'^{\dagger}\gamma^0\gamma^5\psi'\Big) \\ &+ \int_{\mathbb{R}^3} m_0\Big(\cos(2\pi\chi)f'^{\dagger}\gamma^0 f' + \sin(2\pi\chi)f'^{\dagger}\gamma^0\gamma^5 f'\Big) \\ &- \sum_v \exp\left(-2\pi i \int_{\mathbb{R}^3} \chi(x) \cdot *j_A^{\mathrm{matter}}(x)\right) \cdot \frac{d^2}{d{\phi_v}^2} \cdot \\ &\cdot \exp\left(2\pi i \int_{\mathbb{R}^3} \chi(x) \cdot *j_A^{\mathrm{matter}}(x)\right). \end{aligned} \tag{26}$$

For such a slowly spatially varying configuration χ, the Hamiltonian $H_{\mathcal{L}+\mathcal{L}'} + \lambda V[\chi]$ is gapped, because it is a sum of two decoupled pieces, each of which is individually gapped: (i) in the ancilla Hilbert space, it is just a slowly spatially varying Dirac mass term, which can be analyzed within the field theory, where it is clearly gapped, and (ii) in the matter sector, it is the unitary conjugate of an operator that clearly gaps the matter degrees of freedom, as seen in Eq. (26).

Now consider a configuration χ which is still slowly varying in space but may have some vortices. To represent the vortices we need nonzero β; specifically, the vortex locations are along $d\beta \neq 0$. In this case, we can still construct the portion of the operator in Eq. (26) in the bulk of the system, away from the vortex locations, since there we can locally make a gauge transformation to $\beta = 0$. Let us denote this perturbation (which is 0 at the vortex cores) by $V^{\mathrm{bulk}}[\chi, \beta]$. Then, with respect to the Hamiltonian $H_{\mathcal{L}+\mathcal{L}'} + \lambda V^{\mathrm{bulk}}[\chi, \beta]$, the vortices host gapless modes in both the ancilla degrees of freedom and the matter degrees of freedom. This is because in both cases such vortices behave as edges of conjugate copies of $2+1$D bosonic SPTs of $\mathrm{U}(1)_V$, arising from the fact that domain walls between different values of χ have nonzero U(1) Hall conductivity proportional to the difference in χ. By design, this Hall conductivity cancels between the field theory and matter degrees of freedom, so that these gapless chiral modes can be gapped against each other with a local $\mathrm{U}(1)_V$-invariant term $V^{\mathrm{vortex}}[\beta]$. Implicit in this expression is that the gapping term can be made to depend only on the vortex locations and not on the field χ. Letting $V[\chi, \beta] = V^{\mathrm{bulk}}[\chi, \beta] + V^{\mathrm{vortex}}[\beta]$, we have that $H_{\mathcal{L}+\mathcal{L}'} + V[\chi, \beta]$ is fully gapped and $\mathrm{U}(1)_V$ invariant for all configurations χ, β.

For a spatially constant χ_0, $W[\chi_0]$ just generates a global $U(1)_A$ rotation by χ_0. From Eq. (25) and the definition of $V[\chi,\beta]$, we have $W[\chi_0]^\dagger V[\chi,\beta]W[\chi_0] = V[\chi+\chi_0,\beta]$. This means that if we now make χ and β dynamical and assign a $U(1)_A$ charge of -1 to χ, then the Hamiltonian $H_{\mathcal{L}+\mathcal{L}'} + V[\chi,\beta]$ is $U(1)_A$ invariant. Formally, to make χ, β dynamical we have to introduce another Hilbert space $\mathcal{H}_\chi$, so that the total Hilbert space is $\mathcal{H}_\chi \otimes \mathcal{H}_{\mathcal{L}+\mathcal{L}'} \otimes \mathcal{H}_{\text{matter}}$, with $\mathcal{H}_\chi \otimes \mathcal{H}_{\mathcal{L}+\mathcal{L}'}$ acting as ancillas.

Even though the Hamiltonian $H_{\mathcal{L}+\mathcal{L}'} + V[\chi,\beta]$ is now $U(1)_A$ invariant, χ and β are still background fields with no dynamics. In order to fully restore $U(1)_A$, we have to make these fields fluctuate and ensure that the resulting Hamiltonian does not spontaneously break the $U(1)_A$. Consider

$$H = H_{\mathcal{L}+\mathcal{L}'} + \lambda V[\chi,\beta] + H_{\chi,\beta}, \tag{27}$$

where $H_{\chi,\beta}$ is a term that acts only on the χ, β degrees of freedom. To fix the form of $H_{\chi,\beta}$, we first consider just the two terms $H_{\mathcal{L}+\mathcal{L}'} + \lambda V[\chi,\beta]$ and integrate out the degrees of freedom in $\mathcal{H}_{\mathcal{L}+\mathcal{L}'}$ and $\mathcal{H}^{\text{matter}}$ to get an effective potential for χ, β. Even though there are still gapless $\mathcal{L}$ degrees of freedom, they do not couple to χ and β, so this effective potential is short-ranged. We then choose $H_{\chi,\beta}$ to exactly cancel out this effective potential and also include a small kinetic term for χ and β. The effective long-distance description of the χ, β degrees of freedom is then as an XY model in the quantum-disordered phase. This quantum-disordering restores the $U(1)_A$ symmetry and gives us our desired realization of a single copy of $\mathcal{L}$, in the Hilbert space $\mathcal{H}_\chi \otimes \mathcal{H}_{\mathcal{L}+\mathcal{L}'} \otimes \mathcal{H}_{\text{matter}}$.

C. Gauged $U(1)_V$

Let us now gauge the $U(1)_V$ symmetry by weakly coupling the Lagrangian L in Eq. (23) to a dynamical $U(1)_V$ gauge field. In this field theory, the $U(1)_A$ is no longer a symmetry, but a $\mathbb{Z}_N$ remnant remains as a noninvertible symmetry, as is the case in QED and other standard examples of noninvertible symmetries [14]. We work on $M^3 = \mathbb{R}^3$. Since $\mathbb{R}^3$ has no nontrivial 2-cycles and the field theory contains no monopoles, the action of the noninvertible symmetry on this particular Hilbert space is actually unitary.

We argue that this theory can emerge from an appropriate Hamiltonian on a slightly modified version of our Villain Hilbert space $\mathcal{H}$, which results from gauging $U(1)_V$ not just on $\mathcal{H}_{\text{matter}}$, but on $\mathcal{H}_\chi \otimes \mathcal{H}_{\mathcal{L}+\mathcal{L}'} \otimes \mathcal{H}_{\text{matter}}$. Since both $U(1)_V$ and $U(1)_A$ act in an on-site fashion in the ancilla Hilbert space $\mathcal{H}_\chi \otimes \mathcal{H}_{\mathcal{L}+\mathcal{L}'}$, this is just a trivial modification of the Villain Hilbert space obtained from gauging $U(1)_V$ in just $\mathcal{H}_{\text{matter}}$. The end result is a Hilbert space very similar to that of our Villain Hilbert space $\mathcal{H}$ but with additional ancilla superspin degrees of freedom living on vertices and transforming in some well-defined on-site representations of $U(1)_V$ and $U(1)_A$.

The first step in our construction is to take zero coupling, i.e., no dynamics for the gauge field a, so that we can view a as a static spatial background gauge field. Also, we include an energetic penalty for monopole configurations, and the nontrivial terms in the Hamiltonian are all sandwiched between projectors onto zero monopoles and commute with such projectors. We then gauge the Hamiltonian H in Eq. (27) by following the minimal coupling prescription, resulting in H^{gauged}. The $U(1)_A$ symmetry of the ungauged theory turns into an $\mathbb{R}$ symmetry for this gauged Hamiltonian. Note that we are still working at zero coupling, i.e., with no fluctuations for a; a nonzero electric field term would fail to commute with this $\mathbb{R}$ symmetry, a reflection of the ABJ anomaly.

H^{gauged} is also invariant under the noninvertible symmetry $U_{1/N}(\mathbb{R}^3)$. This is because the difference between $U_{1/N}(\mathbb{R}^3)$ and an ordinary $U(1)_A$ rotation by $[(2\pi)/N]$, $\exp\{-[(2\pi i)/N]Q_5\}U_{1/N}(\mathbb{R}^3)$, is entirely independent of the charged (matter) fields, i.e., depends only on the gauge field configuration. Since H^{gauged} does not contain any electric field terms, it commutes with this operator and, hence, with $U_{1/N}(\mathbb{R}^3)$.

However, if we are concerned with only retaining invariance under $U_{1/N}(\mathbb{R}^3)$, we can now introduce some fluctuations of a. We use the projected electric field operator $P_{\mathcal{H}^m} \cdot [d/(da_e)] \cdot P_{\mathcal{H}^m}$, which induces fluctuations of the vector potential within the subspace $\mathcal{H}^m$. This term does not commute with $U_{1/N}(\mathbb{R}^3)$, but if we conjugate it by $U_{1/N}(\mathbb{R}^3)^k$ and sum over $k = 0, 1, \ldots, N-1$, we get a term that trivially commutes with $U_{1/N}(\mathbb{R}^3)$. Since the eigenvalues of $U_{1/N}(\mathbb{R}^3)$ are discrete, this electric field term delocalizes the vector potential a over the configuration space of possible a's corresponding to a particular eigenvalue of $U_{1/N}(\mathbb{R}^3)$ (which also depends on ϕ). If we also add a magnetic field term, then at low energies this simply flows to the theory $\mathcal{L}$, weakly coupled to a $U(1)_V$ gauge field, as desired.

V. CONCLUSIONS

In this work, we first considered a $3+1$D tensor product Hilbert space of rotors, with an on-site action of a $U(1)_V$ symmetry that rotates the rotors. We constructed a non-on-site constant-depth circuit action of an axial $U(1)_A$ symmetry on this Hilbert space, which had the form of $\exp[i\theta \int_{M^3} *j_A^{\text{matter}}]$, where $*j_A^{\text{matter}}$ is a collection of mutually commuting operators that define an axial charge density. The defining property of this $U(1)_A$ symmetry was a mixed anomaly with $U(1)_V$, which manifests itself on the lattice through the fact that performing an axial rotation in a region pumps Hall conductivity out to the boundary of the region.

We then gauged this $\mathrm{U}(1)_V$ symmetry at the level of the Hamiltonian lattice and obtained a gauged axial current, $*j_A$. This gauged axial current did not generate a U(1) symmetry but rather an $\mathbb{R}$ symmetry: $\exp[i\theta \int_{M^3} *j_A]$, $\theta \in \mathbb{R}$. Also, it was not conserved, because it failed to commute with the electric field operator. However, focusing on just the subspace $\mathcal{H}^m \subset \mathcal{H}$ of states invariant under a $\mathbb{Z}_N$ one-form magnetic symmetry $Z_m^{(1)}$, we were able to find a $\mathbb{Z}_N$ group of locality preserving unitary operators whose action on the charged matter degrees of freedom matched that of $\exp\{[(2\pi ik)/N]\theta \int_{M^3} *j_A\}$, $k \in \{0, 1, \ldots, N-1\}$. The generator of this group $U_{1/N}(M^3)$ involved a projection onto $\mathcal{H}^m$ and was our definition of the noninvertible chiral symmetry.

We were able to shed more light on the nature of $U_{1/N}(M^3)$ by gauging $Z_m^{(1)}$ to obtain a dual Hilbert space $\mathcal{H}^e \cong \mathcal{H}^m$. $\mathcal{H}^e$ was the subspace of $\mathcal{H}^N$, the Hilbert space of a charge $1/N$ scalar coupled to a $\mathrm{U}(1)_V$ gauge field. Specifically, it was the subspace invariant under a $\mathbb{Z}_N$ electric one-form symmetry $Z_e^{(1)}$. $U_{1/N}(M^3)$, viewed as an operator on $\mathcal{H}^e$, extended to a constant-depth circuit of local unitaries on $\mathcal{H}^N$ with a mixed anomaly with $Z_e^{(1)}$; in particular, it entangled an SPT of $Z_e^{(1)}$.

There are several avenues for further investigation. One is to address gravitational anomalies within this framework. For example, one may ask whether there is a circuit which pumps out a $2+1$D phase with nonzero thermal Hall response, if one allows for infinite-dimensional site Hilbert spaces. This hinges on whether it is possible to construct a commuting projector Hamiltonian for a $2+1$D state with nonzero thermal Hall conductivity, again with infinite-dimensional sites. Perhaps an intermediate step is asking if there is a circuit that pumps out $2+1$D phases with zero thermal Hall response but no gappable boundary [59]. Additionally, within the context of the present work, one can attempt to understand the full $SL(2, \mathbb{Z}_N)$ structure [12] that should be generated by the one-form gauging operator ρ (which plays the role of S) and $U_{1/N}(M^3)$ acting in $\mathcal{H}^e$ (which plays the role of T). In particular, while $U_{1/N}(M^3)$ acts as the Tsui-Wen SPT entangler in $\mathcal{H}^e$ [45], its action in $\mathcal{H}^m$, which should be viewed as the conjugate of T by S, is a Walker-Wang entangler for the chiral $\mathbb{Z}_N$ anyon theory. This means that it also shifts the gravitational theta angle, as can be seen from the nonzero chiral central charge of the chiral $\mathbb{Z}_N$ anyon theory. It would be interesting to relate this to the projective action of $SL(2, \mathbb{Z}_N)$ discussed in Ref. [12]. Additionally, it would be interesting to relate this structure to the $\mathbb{Z}_N$ QCA found in Ref. [49], which are *invertible* operators that produce the Walker-Wang ground state. There has also been related work on noninvertible time-reversal symmetry [60]; the lattice implementation of such operators appears similar to the noninvertible axial rotations described in this work.

Another direction to explore is a fermionic version of this story. This is natural since the simplest formulation of the ABJ anomaly involves a $\mathrm{U}(1)_V$ gauge theory with fermionic charges, namely, QED. The corresponding ungauged theory is just a Dirac fermion, as opposed to our more complicated bosonic theory in Eq. (22) [see Ref. [61] for a construction using the Villain formalism of an exactly solvable microscopic Hamiltonian symmetric under $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$ symmetries]. The resulting symmetry actions are equivalent up to finite depth unitaries, though the Villain formalism allows one to avoid discontinuous functions in the rotor variables and allows one to construct a symmetric exactly solvable Hamiltonian that spontaneously breaks the $\mathrm{U}(1)_V$. In order to cancel the gravitational contribution to the $\mathrm{U}(1)_V$ anomaly, we could consider a charged Dirac fermion stacked together with a neutral one, with opposite actions of $\mathrm{U}(1)_A$. This has the advantage of being a free fermion theory, and it is, therefore, trivial to realize this theory stacked with its conjugate, with the diagonal $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$ symmetries acting on site, in a microscopic lattice Hamiltonian. The existence of such a realization in the bosonic case was a crucial assumption in our emergeability argument in Sec. IV; in the fermionic case, we do not need to make such an assumption. To build the microscopic fermionic Hilbert space with an action of $\mathrm{U}(1)_A$, we need to introduce lattice fermion degrees of freedom and write down a generating time-independent charge that pumps out an integer quantum Hall state of charged fermions stacked with an opposite Chern number integer quantum Hall state of neutral fermions. For some progress in this direction, see Ref. [51].

Another avenue of investigation that is broadly motivated by this work is to compare the classification of QCA in Hilbert spaces built out finite-dimensional sites versus infinite-dimensional rotors. The operator $*j_A^{\text{matter}}$ constructed in this work can be thought of as a generator of a loop in in the space of 3D QCA and, hence, gives rise to a 2D U(1)-protected QCA at the boundary. This 2D U(1)-protected QCA just entangles the bosonic U(1)-protected SPT state and is allowed to exist only in this rotor setting, since it is known that, with finite-dimensional site Hilbert spaces, it is not possible to have a commuting projector parent Hamiltonian for a system with nonzero U(1) Hall conductivity [31,32].

It would also be interesting to make a more direct connection between our lattice methods and the corresponding structures in field theory. For example, in Sec. II, we constructed the operator $*j_A = *j_A^{(\mathrm{I})} + *j_A^{(\mathrm{II})}$. $*j_A$ was a well-defined operator in $\mathcal{H}$, but the individual pieces $*j_A^{(\mathrm{I})}$ and $*j_A^{(\mathrm{II})}$ were not. Nevertheless, $\exp\{[(2\pi i)/N] \int_{M^3} *j_A^{(\mathrm{I})}\}$ and $\exp\{[(2\pi i)/N] \int_{M^3} *j_A^{(\mathrm{II})}\}$ were individually well defined in $\mathcal{H}^e \subset \mathcal{H}^N$, for closed M^3. The latter was just our noninvertible symmetry operator $U_{1/N}(M^3)$, which performs a chiral rotation on the matter fields. The former

does not depend on the matter field ϕ (for gauge field configurations with no monopoles) and can be viewed as performing a shift of the theta angle. Both of these are nontrivial QCA in $\mathcal{H}^e$ (and, therefore, also $\mathcal{H}^m$), whereas their product is a trivial QCA, since $*j_A$ are well-defined local operators. This comports with the field theory idea that a chiral rotation can be "undone" by a shift of theta angle; from our lattice perspective, both of these operations are nontrivial QCA, but taken together they form a trivial QCA, i.e., a circuit.

Finally, our work has only dealt with spacelike codimension-one defects for the noninvertible symmetry, implementing a global action of the symmetry in space. In the future, it would be interesting to study defects that are extended in time and codimension one in space. To achieve this, one might try to formulate the projection operator in our definition of the noninvertible symmetry in a more local way, e.g., via matrix product operator. We leave this and the remaining directions as avenues for future work.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: MOTIVATION FOR THE FORM OF $*j_A$

In this appendix, we provide some more motivation for our definition of $*j_A$ in Eq. (9). This motivation is not strictly necessary for establishing the results in our paper, but it is instructive to elucidate the way that $*j_A$ was obtained. First, we note that we do not expect the gauging of $*j_A^{\text{matter}}$ to be uniquely determined, so some amount of ambiguity in our expression is unavoidable. However, we can attempt to quantify this ambiguity. First, we make the obvious observation that any gauging $*j_A$, being a well-defined operator on $\mathcal{H}$, must be invariant under Eqs. (6) and (7). In order to be a gauging of $*j_A^{\text{matter}}$, it must also agree with $*j_A^{\text{matter}}$ on the configurations $\{\phi, a, n\} = \{\phi, 0, 0\}$. Furthermore, we expect any gauging $*j_A$ to commute, as an operator, with all operators that depend only on a and n (but not ϕ). This is just the statement that if we turn on a background $\mathrm{U}(1)_V$ gauge field configuration, $*j_A$ does not change that configuration.

We now recall that the standard way to gauge a symmetry in the Hamiltonian lattice formalism is via minimal coupling. This procedure is typically applied to a symmetric Hamiltonian but can equally well be applied to a symmetric current such as $*j_A^{\text{matter}}$. One specific minimal coupling prescription is to first project onto the flat gauge field configurations, so that terms which act in the presence of gauge fluxes are simply zeroed out. The rest of the Hamiltonian is then uniquely determined by gauge invariance. Adding in a sufficiently large negative multiple of the identity to the original Hamiltonian terms then ensures that the gauged ground state lies in the zero flux sector [62]. While this sort of minimal coupling prescription works well for discrete symmetries, it is not appropriate for our setting, which involves probing nonflat gauge field configurations.

To make further progress, we note that, as shown in the main text, the defining feature of the loop of finite-depth circuits defined by $\exp(2\pi i\alpha \int_{M^3} *j_A^{\text{matter}})$ is that they pump $\mathrm{U}(1)_V$ SPTs (with nonzero Hall conductivity) out to the boundary of M^3. This can be viewed as a quantized invariant that partitions the set of such loops of finite-depth circuits into discrete classes. We expect that, for the gauged version, i.e., in the presence of a nontrivial $\mathrm{U}(1)_V$ gauge field, the pumping of such $\mathrm{U}(1)_V$ SPTs is reflected in an effective action with a quantized Chern-Simons coefficient corresponding to the quantized Hall conductivity. That is, we expect that $\exp(2\pi i \int_{M^3} *j_A)$ be of the form $\exp(2\pi i \int_{M^3} S_{\text{eff}}(a, n))$, with $S_{\text{eff}}(a, n)$ containing a lattice version of the quantized Chern-Simons term. This picture corresponds to viewing one of the three spatial directions as the "time" along which the circuit pumps the SPT.

We now note that the lattice version of minimal coupling described above will not lead to $*j_A$ with this property, simply because it acts nontrivially only in the presence of zero gauge flux. For example, for a $\mathrm{U}(1)_V$ gauge field configuration with fluxes confined to some set of flux tubes, such a minimal coupling prescription for $*j_A$ would lead to a circuit that pumps $\mathrm{U}(1)_V$ SPTs out to the boundary of such flux tubes but acts as 0 inside the flux tubes. In particular, $\exp(2\pi i \int_{M^3} *j_A)$ would then not be proportional to the identity on the matter degrees of freedom.

Thus, we seek to construct a $*j_A$ with the additional property that $\exp(2\pi i \int_{M^3} *j_A)$ is the exponential of a lattice version of the Chern-Simons action. This is Eq. (12) in the main text:

$$\exp\left[2\pi i \int_{M^3} *j_A\right] = \exp\left[2\pi i k \int_{M^3} (da \cup a - a \cup n - n \cup a - a \cup_1 dn - 2\phi \cup dn)\right]. \tag{A1}$$

We note that the above lattice Chern-Simons term is indeed independent of ϕ for gauge field configurations with no monopoles $dn = 0$. If monopoles do exist, they acquire a

charge under such SPT pumping (the $-2\phi \cup dn$ term), as expected since the SPTs have nonzero Hall conductivity. The specific expression $*j_A$ given in Eq. (9) is then constructed by taking a logarithm of the right-hand side of the above [which results in $*j_A^{(\mathrm{I})}$, given in Eq. (10)] and modifying it by integers [$*j_A^{(\mathrm{II})}$, given in Eq. (11)] to make it invariant under the Villain constraint.

In general, we expect any gauging of $*j_A^{\mathrm{matter}}$ satisfying the property that $\exp(2\pi i \int_{M^3} *j_A)$ depends only on the gauge field to satisfy

$$\exp\left(2\pi i \int_{M^3} *j_A\right) = \exp\left(2\pi i \int_{M^3} S_{\mathrm{eff}}(a,n)\right),$$

where $S_{\mathrm{eff}}(a,n)$ contains the lattice Chern-Simons term above, plus possible additional local gauge-invariant terms. We conjecture that such $\exp(2\pi i \int_{M^3} S_{\mathrm{eff}}(a,n))$, in fact, define nontrivial QCA in the presence of U(1) electric and magnetic one-form symmetries, with the nontrivial QCA nature reflecting the fact that the Chern-Simons term is not the integral of a local gauge-invariant quantity. We leave the investigation of this conjecture to future work.

APPENDIX B: DETAILS ON GAUGING $\mathbb{Z}_N$ ONE-FORM SYMMETRIES

1. Definition of inner product in $\mathcal{H}$

Because of the Villain condition [Eq. (7)], wave functions $\Psi(\phi,a,n)$ in $\mathcal{H}$ appear to be non-normalizable if one uses the naive norm, obtained from integrating $|\Psi(\phi,a,n)|^2$ over all ϕ and a and summing over n. However, the Villain condition should clearly just be treated as a redundancy when defining the inner product. One way to handle this is to use the Villain condition to move the variables $(a-d\phi)_e$ into the range [0, 1) for all edges e. Each orbit under the Villain condition has precisely one gauge choice where all of the $(a-d\phi)_e$ are in this range. This results in an inner product

$$\langle\Psi|\Psi'\rangle = \int_0^1 \mathcal{D}(a-d\phi) \sum_n \overline{\Psi(\phi,a,n)} \cdot \Psi'(\phi,a,n), \quad \text{(B1)}$$

where

$$\int_0^1 \mathcal{D}(a-d\phi) \equiv \prod_{\text{edges } e} \int_0^1 d(a-d\phi)_e.$$

It is also useful to have an expression for this product in a Fourier-transformed "plane-wave" basis on the configuration space ϕ,a,n. Let $\varepsilon \in C^1(M^3,\mathbb{R}), \vartheta \in C^2(M^3,\mathbb{R}/\mathbb{Z})$, and consider the state $|e,\vartheta\rangle$, defined by the wave function

$$\exp\left[2\pi i \int_{M^3} (-\partial\varepsilon\cdot\phi + \varepsilon\cdot a + \vartheta\cdot n)\right],$$

where we use the notation $a\cdot b \equiv a\cup *b$. The Villain condition in Eq. (7) then turns into the condition $\varepsilon + \partial\vartheta \in C^1(M^3,\mathbb{Z})$, where $\partial = *d*$ and $*$ is the Hodge dual. To understand this, note that integer translations of a and n, applied to the plane-wave state above, yield phase factors of $\exp(2\pi i\varepsilon)$ and $\exp(2\pi i\vartheta)$, respectively. Then the Villain condition in Eq. (7) turns into the requirement that, for all $v \in C^1(M^3,\mathbb{Z})$,

$$\int_{M^3} (v\cdot\varepsilon + dv\cdot\vartheta) \in \mathbb{Z}. \quad \text{(B2)}$$

We have that $\int dv\cdot\vartheta = \int v\cdot\partial\vartheta$, so Eq. (B2) turns into the requirement that $\varepsilon+\partial\vartheta \in C^1(M^3,\mathbb{Z})$. This is just the statement of charge quantization in the Villain formalism.

A general state in our Hilbert space can be written as

$$|\Psi\rangle = \int_0^1 \mathcal{D}\vartheta \sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})| \\ \varepsilon+\partial\vartheta\in C^1(M^3,\mathbb{Z})}} \Psi_{\varepsilon,\vartheta}|\varepsilon,\vartheta\rangle,$$

where $\int_0^1 \mathcal{D}\vartheta$ is shorthand for $\prod_f \int_0^1 d\vartheta_f$ and the product is over all faces. The sum on ε is due to the constraint that $\varepsilon+\partial\vartheta$ be integer valued; for a given configuration ϑ, the allowed values of ε are quantized up to $-\partial\vartheta$. We could have written the above as an integral over both ϑ and ε, but then we would have had to include delta functions imposing these constraints inside the integral. The wave function of the state $|\Psi\rangle$ is

$$\Psi(\phi,a,n) = \int_0^1 \mathcal{D}\vartheta \sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})| \\ \varepsilon+\partial\vartheta\in C^1(M^3,\mathbb{Z})}} \Psi_{\varepsilon,\vartheta} \cdot \exp\left[2\pi i \int_{M^3} (-\partial\varepsilon\cdot\phi + \varepsilon\cdot a + \vartheta\cdot n)\right]. \quad \text{(B3)}$$

Conversely, given $\Psi(\phi,a,n)$ invariant under the usual gauge invariance and the Villain condition, we can recover $\Psi_{\varepsilon,\vartheta}$ by taking the inverse Fourier transform:

$$\Psi_{\varepsilon,\vartheta} = \int_0^1 \mathcal{D}(a-d\phi) \sum_{n\in C^2(M^3,\mathbb{Z})} \times \exp\left[-2\pi i \int_{M^3} (\varepsilon\cdot(a-d\phi) + \vartheta\cdot n)\right] \Psi(\phi,a,n).$$

Here, $\int_0^1 \mathcal{D}(a-d\phi)$ is shorthand for $\prod_e \int_0^1 d(a-d\phi)_e$ where the product is over all edges e.

From taking the mod squared of the above equation and Fourier transforming, we obtain that

$$\int_0^1 \mathcal{D}\vartheta \sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta\in C^1(M^3,\mathbb{Z})}} |\Psi_{\varepsilon,\vartheta}|^2 = \int_0^1 \mathcal{D}(a-d\phi)\sum_n |\Psi(\phi,a,n)|^2.$$

Therefore, an alternative formula for the inner product in Eq. (B1) is

$$\langle\Psi|\Psi'\rangle \equiv \int_0^1 \mathcal{D}\vartheta \sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta\in C^1(M^3,\mathbb{Z})}} \overline{\Psi_{\varepsilon,\vartheta}}\,\Psi'_{\varepsilon,\vartheta}.$$

Let us also discuss the Hilbert space $\mathcal{H}^N$ of the gauge theory of a charge $1/N$ scalar. Here, n is $N\mathbb{Z}$ valued, and the invariance is only under the restricted Villain transformation [Eq. (16)]. By the same argument as for $\mathcal{H}$, we can take account of this restricted Villain transformation and define the inner product as

$$\langle\Psi^N|\Psi'^N\rangle_N = \int_0^N \mathcal{D}(a-d\phi) \sum_{n\in C^2(M^3,N\mathbb{Z})} \overline{\Psi^N(\phi,a,n)} \cdot \Psi'^N(\phi,a,n). \tag{B4}$$

The corresponding dual representation has $\varepsilon\in C^1\times(M^3,\mathbb{R})$, $\vartheta\in C^2\{M^3,\mathbb{R}/[(1/N)\mathbb{Z}]\}$, and the modified Villain condition [Eq. (16)] turns into $\varepsilon+\partial\vartheta\in C^1[M^3,(1/N)\mathbb{Z}]$. A general state in this Hilbert space is

$$|\Psi^N\rangle = \int_0^{1/N} \mathcal{D}\vartheta \sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta\in C^1(M^3,\frac{1}{N}\mathbb{Z})}} \Psi^N_{\varepsilon,\vartheta}|\varepsilon,\vartheta\rangle_N,$$

where $\int_0^{1/N}\mathcal{D}\vartheta$ is shorthand for $\prod_f\int_0^{1/N}d\vartheta_f$. Here, the state $|\varepsilon,\vartheta\rangle_N$ is given by the wave function

$$\exp\left[2\pi i\int_{M^3}(-\partial\varepsilon\cdot\phi+\varepsilon\cdot a+\vartheta\cdot n)\right]$$

with $n\in C^2(M^3,N\mathbb{Z})$. The wave function of the state $|\Psi^N\rangle$ is

$$\Psi^N(\phi,a,n) = \int_0^{1/N}\mathcal{D}\vartheta \sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta\in C^1(M^3,\frac{1}{N}\mathbb{Z})}} \Psi^N_{\varepsilon,\vartheta} \cdot\exp\left[2\pi i\int_{M^3}(-\partial\varepsilon\cdot\phi+\varepsilon\cdot a+\vartheta\cdot n)\right], \tag{B5}$$

where again $n\in C^2(M^3,N\mathbb{Z})$. Conversely, given $\Psi^N(\phi,a,n)$ invariant under the usual gauge invariance and the modified Villain condition in Eq. (16), we can recover $\Psi^N_{\varepsilon,\vartheta}$ by

$$\Psi^N_{\varepsilon,\vartheta} = N^{|\text{faces}|-|\text{edges}|}\int_0^N \mathcal{D}(a-d\phi) \sum_{n\in C^2(M^3,N\mathbb{Z})} \times\exp\left[-2\pi i\int_{M^3}(\varepsilon\cdot(a-d\phi)+\vartheta\cdot n)\right]\Psi^N(\phi,a,n),$$

where the extra factor out front comes from changing the quantization for the fields n and a. Here, $\int_0^N\mathcal{D}(a-d\phi)$ is shorthand for $\prod_e\int_0^N d(a-d\phi)_e$ where the product is over all edges e.

By taking the mod squared of both sides of the above equation and Fourier transforming, we obtain

$$\int_0^{1/N}\mathcal{D}\vartheta \sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta\in C^1(M^3,\frac{1}{N}\mathbb{Z})}} |\Psi^N_{\varepsilon,\vartheta}|^2 = N^{|\text{faces}|-|\text{edges}|}\int_0^N \mathcal{D}(a-d\phi)\sum_{n\in C^2(M^3,N\mathbb{Z})}|\Psi(\phi,a,n)|^2.$$

Therefore, the inner product in Eq. (B4) can alternatively be written as

$$\langle\Psi^N|\Psi'^N\rangle_N \equiv N^{|\text{edges}|-|\text{faces}|}\int_0^{1/N}\mathcal{D}\vartheta \sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta\in C^1(M^3,\frac{1}{N}\mathbb{Z})}} \overline{\Psi^N_{\varepsilon,\vartheta}}\,\Psi'^N_{\varepsilon,\vartheta}. \tag{B6}$$

2. Structure of $\mathcal{H}^e\subset\mathcal{H}^N$

Let us now take a state $|\Psi\rangle\in\mathcal{H}^e\subset\mathcal{H}^N$:

$$|\Psi^N\rangle = \int_0^{1/N}\mathcal{D}\vartheta \sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta\in C^1(M^3,\frac{1}{N}\mathbb{Z})}} \Psi^N_{\varepsilon,\vartheta}|\varepsilon,\vartheta\rangle_N.$$

We already know from the rescaled Villain constraint [Eq. (16)] that $\varepsilon+\partial\vartheta\in C^1[M^3,(1/N)\mathbb{Z}]$. From the definition of $Z_e^{(1)}$ in Eq. (17), we find that invariance under $Z_e^{(1)}$ places additional constraints on which of the coefficients $\Psi_{\varepsilon,\vartheta}$ can be nonzero: namely, $\int_{M^3}v\cdot(\varepsilon+\partial\vartheta)$ must be an integer for any $v\in C^1(M^3,\mathbb{Z})$ satisfying $dv\in C^2(M^3,N\mathbb{Z})$. Taking $v=ds$ for $s\in C^0(M^3,\mathbb{Z})$ gives $\int_{M^3}s\cdot\partial[N(\varepsilon+\partial\vartheta)]\in N\mathbb{Z}$ for all such s, so that $\partial[N(\varepsilon+\partial\vartheta)]\in C^0(M^3,N\mathbb{Z})$. This means that $*N(\varepsilon+\partial\vartheta)$ is a $\mathbb{Z}_N$-valued 2-cocycle, where $\mathbb{Z}_N=\{0,1,\ldots,N-1\}$. Furthermore, given any closed $\mathbb{Z}_N$ 2-chain $\Sigma\in M^3$, let $v=\text{PD}(\Sigma)\in C^1(M^3,\mathbb{Z}_N)$ be its Poincaré dual. Then, $dv=0$ mod N, so

$$\int_\Sigma *[N(\varepsilon+\partial\vartheta)] = N\int_{M^3}v\cdot(\varepsilon+\partial\vartheta)\in N\mathbb{Z}.$$

Since this is true for all closed Σ, $*N(\varepsilon+\partial\vartheta)$ is exact, implying that $N(\varepsilon+\partial\vartheta)=\partial\beta$ for some $\beta\in C^2(M^3,\mathbb{Z}_N)$. Thus, while the constraint on a general state in $\mathcal{H}^N$ is that $\varepsilon+\partial\vartheta$ be valued in $(1/N)\mathbb{Z}$, for a state in $\mathcal{H}^e\subset\mathcal{H}^N$ we have the additional constraint that $\varepsilon+\partial\vartheta=(1/N)\partial\beta$ mod 1, for some integer-valued β. We see below that $\partial\beta$ can be interpreted as a pure-gauge dual two-form gauge field obtained from gauging $\mathbb{Z}_m^{(1)}$ symmetry in $\mathcal{H}^m$.

3. Gauging $\mathbb{Z}_m^{(1)}$ by minimal coupling

The action of the $Z_m^{(1)}$ symmetry in the Fourier transformed basis for $\mathcal{H}$ is

$$Z_m^{(1)}(\Sigma)|\varepsilon,\vartheta\rangle=|\varepsilon,\vartheta+\frac{1}{N}\cdot *\mathrm{PD}(\Sigma)\rangle, \tag{B7}$$

where $\mathrm{PD}(\Sigma)$ is the Poincaré dual of Σ. Let us now gauge $Z_m^{(1)}$ by coupling to a pure gauge $\mathbb{Z}_N$ two-form gauge field. Since the $Z_m^{(1)}$ one-form symmetry acts on the dual edges, its gauge field lives on dual plaquettes, i.e., ordinary edges. Its being pure gauge means that it is equal to $\partial\beta=*d*\beta$, where $\beta\in C^2(M^3,\mathbb{Z}_N)$; in the formulas below, we treat β as being valued in the integers $0,\ldots,N-1$. The wave functions of the $Z_m^{(1)}$-gauged theory now include dependence on the $Z_m^{(1)}$ gauge field $\partial\beta$:

$$|\Psi_g\rangle=\sum_{\partial\beta\in C^1(M^3,\mathbb{Z}_N)}\int_0^1\mathcal{D}\vartheta\sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta-\frac{1}{N}\partial\beta\in C^1(M^3,\mathbb{Z})}}\Psi_{\varepsilon,\vartheta,\partial\beta}|\varepsilon,\vartheta,\partial\beta\rangle. \tag{B8}$$

Let us explain this formula. $|\Psi_g\rangle$ is a state in the Hilbert space of $\mathcal{H}$ tensored with that of a $\mathbb{Z}_N$ degree of freedom living in $C^1(M^3,\mathbb{Z}_N)$. The first sum above is over all dual-exact such 1-cochains, i.e., all such cochains of the form $\partial\beta$ for some β. The inner product on this Hilbert space is inherited from $\mathcal{H}$ and the natural inner product on the $\mathbb{Z}_N$ degrees of freedom. We explain the sum on ε momentarily.

$|\Psi_g\rangle$ must be invariant under the $Z_m^{(1)}$-Gauss law:

$$\Psi_{\varepsilon,\vartheta+(1/N)\cdot\beta',\partial\beta+\partial\beta'}=\Psi_{\varepsilon,\vartheta,\partial\beta} \tag{B9}$$

for any cochain $\beta'\in C^2(M^3,\mathbb{Z}_N)$. The Villain constraint in Eq. (B2) also needs to be conjugated by the $Z_m^{(1)}$-Gauss law, leading to the sum over ε in Eq. (B8) being restricted to

$$\varepsilon+\partial\vartheta-\frac{1}{N}\partial\beta\in C^1(M^3,\mathbb{Z}). \tag{B10}$$

Note that this means that $\partial\beta$ is completely determined, as a $\mathbb{Z}_N$-valued cochain, by the configuration of ε, ϑ.

We now see that the following map is a unitary isomorphism between the space of pure-gauge wave functions in the $Z_m^{(1)}$-gauged theory and $\mathcal{H}^e$:

$$\sum_{\partial\beta\in C^1(M^3,\mathbb{Z}_N)}\int_0^1\mathcal{D}\vartheta\sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta-\frac{1}{N}\partial\beta\in C^1(M^3,\mathbb{Z})}}\Psi_{\varepsilon,\vartheta,\partial\beta}|\varepsilon,\vartheta,\partial\beta\rangle\to N^{(|\mathrm{faces}|-|\mathrm{edges}|)/2}\int_0^{1/N}\mathcal{D}\vartheta\sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta=\frac{1}{N}\partial\beta\ \mathrm{mod}\ 1\\ \text{for some }\beta\in C^2(M^3,\mathbb{Z}_N)}}\Psi_{\varepsilon,\vartheta,N(e+\partial\vartheta)}|\varepsilon,\vartheta\rangle_N \tag{B11}$$

with inverse given by

$$\int_0^{1/N}\mathcal{D}\vartheta\sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta=\frac{1}{N}\partial\beta\ \mathrm{mod}\ 1\\ \text{for some}\beta\in C^2(M^3,\mathbb{Z}_N)}}\Psi^N_{\varepsilon,\vartheta}|\varepsilon,\vartheta\rangle_N\to N^{(-|\mathrm{faces}|+|\mathrm{edges}|)/2}\int_0^1\mathcal{D}\vartheta\sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta=\frac{1}{N}\partial\beta\ \mathrm{mod}\ 1\\ \text{for some}\beta\in C^2(M^3,\mathbb{Z}_N)}}\Psi^N_{\varepsilon,\vartheta}|\varepsilon,\vartheta,N(\varepsilon+\partial\vartheta)\rangle.$$

Note that the wave function on the right-hand side of Eq. (B11) is invariant under $1/N$ shifts of ϑ, due to the $\mathbb{Z}_m^{(1)}$ Gauss law [Eq. (B9)].

4. Isomorphism $\mathcal{H}^m\to\mathcal{H}^e$ from gauging $\mathbb{Z}_m^{(1)}$

Now, given any state

$$\int_0^1\mathcal{D}\vartheta\sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta\in C^1(M^3,\mathbb{Z})}}\Psi_{\varepsilon,\vartheta}|\varepsilon,\vartheta\rangle\in\mathcal{H}^m,$$

we can view it as a state coupled to the trivial background gauge field configuration $\partial\beta=0$. We can then make it gauge invariant with respect to the $\mathbb{Z}_m^{(1)}$ Gauss law [Eq. (B9)] by allowing the gauge field to fluctuate over all pure gauge configurations. We accomplish this by mapping it to the state:

$$\frac{1}{\sqrt{|\{\partial\beta\}|}}\sum_{\partial\beta}\int_0^1\mathcal{D}\vartheta\sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta-\frac{1}{N}\partial\beta\in C^1(M^3,\mathbb{Z})}}\Psi_{\varepsilon,\vartheta-(1/N)\beta}|\varepsilon,\vartheta,\partial\beta\rangle.$$

This map is unitary. Note that $\Psi_{\varepsilon,\vartheta-(1/N)\beta}$ depends only on $\partial\beta$ by virtue of $Z_m^{(1)}$ invariance [Eq. (B7)], so the sum above is really over exact cochains $d\beta$. Composing with the map in Eq. (B11), we get a unitary isomorphism, denoted ρ in the main text, from $\mathcal{H}^m$ to $\mathcal{H}^e$:

$$\int_0^1 \mathcal{D}\vartheta \sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta\in C^1(M^3,\mathbb{Z})}} \Psi_{\varepsilon,\vartheta}|\varepsilon,\vartheta\rangle \to \frac{N^{(|\text{faces}|-|\text{edges}|)/2}}{\sqrt{|\{\partial\beta\}|}} \int_0^{\frac{1}{N}} \mathcal{D}\vartheta \sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta=\frac{1}{N}\partial\beta \bmod 1\\ \text{for some}\beta\in C^2(M^3,\mathbb{Z}_N)}} \Psi_{\varepsilon,\vartheta-(1/N)\beta}|\varepsilon,\vartheta\rangle_N.$$

The map ρ is especially simple in the original (ϕ,a,n) basis. Applying Eq. (B5) to the right-hand side of the above (but without yet including the overall constant $N^{|\text{faces}|-|\text{edges}|/2}/\sqrt{|\{\partial\beta\}|}$), we have, for $n\in C^2(M^3,N\mathbb{Z})$,

$$\begin{aligned}
&\int_0^{1/N} \mathcal{D}\vartheta \sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\vartheta=\frac{1}{N}\partial\beta \bmod 1\\ \text{for some}\beta\in C^2(M^3,\mathbb{Z}_N)}} \Psi_{\varepsilon,\vartheta-(1/N)\beta} \exp\left(2\pi i\int_{M^3}(\varepsilon\cdot(a-d\phi)+\vartheta\cdot n)\right)\\
&=\frac{1}{|\{\beta|\partial\beta=0\}|}\int_0^1 \mathcal{D}\tilde{\vartheta} \sum_{\substack{\varepsilon\in C^1(M^3,\mathbb{R})|\\ \varepsilon+\partial\tilde{\vartheta}\in C^1(M^3,\mathbb{Z})}} \psi_{\varepsilon,\tilde{\vartheta}} \exp\left(2\pi i\int_{M^3}(\varepsilon\cdot(a-d\phi)+\vartheta\cdot n)\right)\\
&=\frac{1}{|\{\beta|\partial\beta=0\}|}\Psi(\phi,a,n).
\end{aligned}$$

Thus,

$$\rho(\Psi)(\phi,a,n)=\frac{N^{(|\text{faces}|-|\text{edges}|)/2}}{|\{\beta|\partial\beta=0\}|\cdot\sqrt{|\{\partial\beta\}|}}\Psi(\phi,a,n) \tag{B12}$$

$$=\frac{N^{(|\text{faces}|-|\text{edges}|)/2}}{\sqrt{|\{\beta|\partial\beta=0\}|}}\Psi(\phi,a,n). \tag{B13}$$

This is Eq. (18) in the main text. One can explicitly check that the map in Eq. (B12) is unitary by using the original definitions of the inner product in the (ϕ,a,n) basis [Eqs. (B1) and (B4)]. This check involves showing that a combination of the above prefactor and other constants is equal to 1 using the fact that the Euler characteristic of an odd-dimensional manifold vanishes, and Hodge duality, which relates the first cohomology and second homology groups of M^3 with $\mathbb{Z}_N$ coefficients.

5. Isomorphism $\mathcal{H}^e\to\mathcal{H}^m$ from gauging $\mathbb{Z}_e^{(1)}$

For completeness, we also show that gauging $Z_e^{(1)}$ (with a pure gauge field) recovers $\mathcal{H}^m$ and that the resulting isomorphism $\mathcal{H}^e\to\mathcal{H}^m$ is inverse to that constructed in Eq. (18). Gauging $Z_e^{(1)}$ requires the introduction of $\mathbb{Z}_N$-valued two-form degrees of freedom in $\mathcal{H}^N$. We do this simply by extending n from being $N\mathbb{Z}$ valued to being $\mathbb{Z}$ valued. Then, n mod N is the $\mathbb{Z}_N$ two-form gauge field for $Z_e^{(1)}$. The second step is to impose a $Z_e^{(1)}$-Gauss law on each edge e, which just amounts to the ordinary Villain condition [Eq. (7)]. The explicit map at the level of wave functions is as follows. Given $\Psi(\phi,a,n)\in\mathcal{H}^e$, we view it as a wave function in which n is allowed to take on all integer values (and we set its value equal to 0 for any configuration involving $n\neq 0$ mod N). Then, we symmetrize it with respect to the $Z_e^{(1)}$-Gauss law. This symmetrization amounts to the following: given a configuration $n=dv+Nn'$, we define

$$\Psi_{\text{sym}}(\phi,a,dv+Nn')\propto\cdot\Psi(\phi,a-v,Nn'). \tag{B14}$$

The normalization factor must be $\sqrt{|\{\beta|\partial\beta=0\}|}\times N^{-(|\text{faces}|-|\text{edges}|)/2}$ to ensure unitarity. This is Eq. (19) in the main text.

APPENDIX C: MIXED ANOMALY BETWEEN $U_{1/N}(M^3)$ AND $Z_e^{(1)}$

One way to derive the mixed anomaly is to explicitly truncate $U_{1/N}(M^3)$ to an open three-dimensional subset $R\subset M^3$ and then conjugate the action of $Z_e^{(1)}$ by this truncated operator. The result should look like $Z_e^{(1)}$ everywhere except near the two-dimensional boundary ∂R. Physically, R can be thought of as a finite three-dimensional region where we create the one-form SPT, and its two-dimensional boundary ∂R is where the one-form symmetry forms an anomalous representation. In the terminology of Else and Nayak [54] (see also Ref. [63] for a general extension to higher-form symmetries), the restriction of the one-form symmetry $Z_e^{(1)}$ to ∂R is an

"effective boundary action" of the symmetry. The symmetry operators must be supported on codimension-one surfaces in ∂R, i.e., strings. We expect the mixed anomaly between $U_{1/N}(M^3)$ and $Z_e^{(1)}$ to be encoded in this boundary action. Specifically, we expect that the string operators implementing $Z_e^{(1)}$ in ∂R do not commute but rather follow the commutation relations of the string operators of the chiral $\mathbb{Z}_N$ anyon theory.

To explicitly see this, we assume (just for the present argument) that we are on a cubic lattice $\mathbb{Z}^3$, with vertices having integer x, y, z coordinates. We also take odd N, for simplicity. The formalism of cup products and higher cup products extends to hypercubic lattices [35], so everything we have done so far can be ported over to this setting. Let R be the set of points (x, y, z) with $z \leq 0$. Then, ∂R is the set of points $(x, y, 0)$, i.e., a two-dimensional square lattice $\mathbb{Z}^2$. For concreteness, let us first consider the one-form action $Z_e^{(1)}(v)$ on $\mathbb{Z}^3$ with v being 0 everywhere except on edges connecting $(0, y, z)$ and $(1, y, z)$, for all y and z, where it is equal to 1. Then, $Z_e^{(1)}(v)$ simply measures the fractional charge of Wilson lines extended along the x direction.

Truncating $Z_e^{(1)}(v)$ to R, we obtain an operator $Z_e^{(1)}(v)_{\text{trunc}}$ which acts as follows. Let v' be 0 everywhere except on edges connecting $(0, y, z)$ and $(1, y, z)$ for $z \leq 0$, where it is 1. Then,

$$Z_e^{(1)}(v)_{\text{trunc}} \colon \Psi(\phi, a, n) \to \Psi(\phi, a - v', n).$$

Note that there is no shift of $n \to n - dv'$, even though $dv' \neq 0$; indeed, there cannot be such a shift, as n must remain valued in $N\mathbb{Z}$. However, this is a valid truncation, in the sense that it acts the same way as $Z_e^{(1)}$ in the bulk of R and as the identity away from R.

We now have

$$\begin{aligned} &U_{1/N}(M^3) Z_e^{(1)}(v)_{\text{trunc}} U_{1/N}(M^3)^\dagger \colon \Psi(\phi, a, n) \\ &\quad \to \exp\left[\frac{2\pi i}{N} \int_{M^3} (dv' \cup \lfloor a - d\phi \rceil + \lfloor a - d\phi \rceil \cup dv' - v' \cup dv')\right] \Psi(\phi, a - v', n) \\ &\quad \propto \exp\left[\frac{2\pi i}{N} \int_{M^3} (dv' \cup \lfloor a - d\phi \rceil + \lfloor a - d\phi \rceil \cup dv')\right] \Psi(\phi, a - v', n). \end{aligned} \tag{C1}$$

We can express this operator as the composition of $Z_e^{(1)}(v)$ in the bulk of R ($z < 0$), together with an operator that acts nontrivially on the slab $0 \leq z \leq 1$. The nontrivial action involves the phase

$$\exp\left[\frac{2\pi i}{N} \int_{M^3} (dv' \cup \lfloor a - d\phi \rceil + \lfloor a - d\phi \rceil \cup dv')\right]$$

which, because $dv' \neq 0$ only on 2-cells with z coordinates between 0 and 1, is localized to the double-layer slab $0 \leq z \leq 1$.

The form of Eq. (C1) is reminiscent of string operators in string net models. Indeed, acting in the slab $0 \leq z \leq 1$, this operator just shifts the value of a by 1 on edges $[(0, y, 0), (1, y, 0)]$ and multiplies by a phase proportional to $\lfloor a - d\phi \rceil$ on edges $[(0, y, 0), (0, y + 1, 0)]$ and $[(1, y, 1), (1, y + 1, 1)]$. If we then define an analogous operator extended in the x direction (i.e., exchange x and y in the definition of v), then these two operators fail to commute precisely on edges of the form $[(0, 0, 0), (1, 0, 0)]$ and $[(0, 0, 0), (0, 1, 0)]$. Furthermore, these two contributions are both equal to $\exp[(2\pi i)/N]$, so that the overall group commutator of these two operators is $\exp[(4\pi i)/N]$. This is equal to the braiding of the generating anyon of the $\mathbb{Z}_N$ anyon theory with itself (recall that we are working with odd N here, for simplicity). One can also perform an exchange by using a T-junction process [64], which involves an anticommutation on only a single edge, leading to an exchange phase of $\exp[(2\pi i)/N]$.

Since the remaining anyons are just powers of the generating anyon, their braiding and exchange phases are reproduced as well. We, thus, conclude that the anomalous one-form symmetry action on the two-dimensional boundary of R gives precisely the lines of the $\mathbb{Z}_N$ anyon theory, as expected.

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