

Topological Defect Propagation to Classify Knitted Fabrics

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Knits and crochets are mechanical metamaterials with a long history that can typically be produced from a single yarn. Despite the simplicity of the manufacturing process, they exhibit a wide range of structural configurations with diverse mechanical properties and application potentials. Although there has been recent growing interest in textile-based metamaterials, a rigorous topological characterization of what makes a structure *knittable* has been lacking. Here, we introduce a general criterion based on topological constraints that distinguish knits and crochets from other textile structures. We demonstrate how the introduction of topological defects and their propagation makes this classification practical. Our approach highlights a fundamental link between manufacturing processes and structural fragility. Within this framework, we show how the rationalization of defect propagation unlocks the design of fabrics with controllable damage resistance.

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I. INTRODUCTION

Topology, as embedded structural features robust to external perturbations, plays a central role across many areas of physics, from quantum systems [1–5] to classical and soft matter [6–14]. In soft matter, the interplay between geometry, topology, and mechanics underlies phenomena such as the emergence of unusual rheology of polymers [11,12], frustration [8], mechanical stability of linked ropes [9], localized modes [15–17], solitons [16,17], climbing of

plants [13], and control of the DNA molecules [14]. Topological concepts apply naturally to entangled structures, including textiles [18–21]. Textiles are among the oldest human technologies and continue to evolve as both traditional crafts and advanced manufacturing techniques [22–26]. They include a wide variety of structures produced through processes such as weaving, knitting, crochet, braiding, and nonwoven fabrication, each created through distinct methods of intertwining yarns in space. Fundamentally, most textiles are formed by intertwining yarns through repeated local operations to produce a thin sheet. The resulting structures can be modeled as two-dimensional, planar assemblies of one-dimensional curves that pass over and under one another. This dimensional reduction, from a three-dimensional physical structure to a two-dimensional diagram, referred to in the following as a textile diagram, makes it possible to study the textiles using

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tools from knot theory [18,19,27–31]. Knot theory is a branch of mathematics concerned with the classification and properties of entangled curves in three-dimensional space, often studied through their two-dimensional projections called knot diagrams [32].

Among textiles, knits and crochets form a special category in which a single yarn can be used to create self-supporting fabrics through local loop operations. Loops are created by pulling a yarn through the already entangled portions with a needle or hook. This progressive entanglement of successive portions of the same yarn gives rise to periodic networks of loops. To emphasize this common structure, we refer to knits and crochets collectively as *amimono*, adopting the Japanese term that encompasses both techniques. Yet, knits and crochets have pronounced differences in their physical properties, stemming from their distinct manufacturing methods that define how new loops are made [33,34]. In crochet, a loop may be pulled through an adjacent loop in the direction parallel to the yarn, whereas in knitting, loops generally entangle in a direction transverse to the yarn. Amimono loop-based fabrics exhibit properties that go beyond the behavior of the yarn itself. While the yarns are typically inextensible, the resulting fabrics exhibit remarkable stretchiness. Amimono then possesses the characteristic features of mechanical metamaterials whose macroscopic properties arise not only from material composition, but also from geometry and topology [35]. Linking the local structural features of amimono to their overall mechanical response is an ongoing challenge. Multiple aspects have been investigated, including tensile response [36–38], mechanical stability [39,40], global shape [41–44], and curvature emergence [45–47], reflecting the complex interplay between geometry, topology, and material behavior. An important numerical effort also emerged from the computer graphics community, pioneering numerical tools to model whole fabrics from the yarn level [48–53]. Advanced numerical tools are also leveraged to integrate modern additive fabrication tools, such as mesh-based designs, into manufacturing processes by accounting for topological constraints to generate machine instructions [54–57]. Despite recent advances in modeling, creating textiles with more complex topologies is surprisingly difficult [58]. Topological descriptions, such as those based on network [21,40,56] or knot [18,19] topology, are usually applied to known textile patterns. The possibility of relying on topology alone to classify textile patterns and assess which manufacturing techniques are suited to producing them remains an open question.

In this study, we propose a mathematical framework rooted in knot theory to identify the topological properties that define amimono textiles and assess which ones are considered *knittable*. Starting from a homogeneous and periodic textile diagram, we develop a method to translate the question of knittability into determining whether a

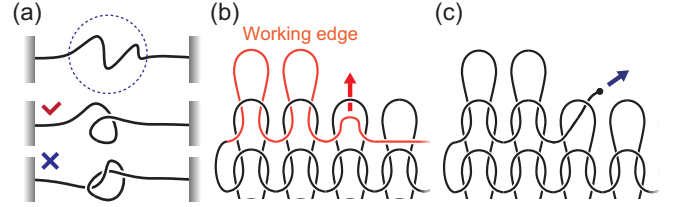


FIG. 1. (a) Configurations that can and cannot be created from the initial state shown on top, using only local manipulations within the dashed circle. (b) Operations that create or remove crossings without moving the yarn ends are permitted in the process of creating a knitting. (c) Operations that move the end points of the yarn are not permitted.

knot or link is trivial. In particular, we evaluate how the introduction of topological defects in a textile diagram, and their subsequent propagation, can disentangle the structure. This classification reveals qualitative differences between amimono and other textiles and further distinguishes knits from crochets based on how these defects behave within the structure. We finally discuss how to leverage this framework to draw parallels with other physical systems, such as marginally stable structures, and to design functional fabrics from a topological perspective.

II. KNITTING PROCESS AND TEXTILE CLASSIFICATION

We model textiles as quasi-two-dimensional structures composed of intertwined one-dimensional yarns. The term yarns will refer to both multiple distinct yarns and multiple segments of a single yarn. This broad definition allows an infinite number of possible patterns, which are mainly determined by the crafting or industrial techniques used to produce them. Knitted and crocheted textiles, commonly referred to as amimono fabrics, result from a very specific process where only local manipulation of the yarn is permitted while its two ends are pinned; see Fig. 1(a). Amimono fabrics essentially consist of sliding loops, which refers to a part of the yarn forming an omega shape, into previous loops [Fig. 1(b)]. With the end pinned, threading the yarn into a previous loop is forbidden [Fig. 1(c)]. Knitting and crocheting techniques can be modeled on a grid where vertically entangled loops form stitches along horizontal yarn paths. The two techniques differ by the order in which the stitches are created. These seemingly restrictive conditions permit the production of amimono fabrics from a single yarn, with one end free at the start, and the other end still deeply buried in a ball or spool of yarn. This remarkable feature of amimono has contributed to its popularity, as knitters need only a ball of yarn and small tools (such as needles or hooks) to manipulate the yarn locally. Since all materials can be easily carried, amimono fabrics can be made anywhere by anyone, outside of a workshop, and without the need for expensive machinery.

Nevertheless, highly sophisticated and innovative materials can be created from these simple processes, including crocheting the hyperbolic plane [59], artificial muscles [60], and actuators [61], among others.

Another consequence of this local fabrication process is the presence of a working edge where loops are added gradually. By working edge, we mean the yarn that creates the highest row of stitches, considering a bottom-to-top construction of the amimono fabric, as illustrated in Fig. 1(b). Local manipulations of the yarn to create the fabric are reversible along this edge, which then constitutes a “weak point” of the fabric. An intact amimono fabric is then a textile that can be untangled only via its working edge, keeping its bulk topologically protected. This topological protection and the sensitivity to defect propagation from the working edge will be at the core of our knittability criteria. However, from a textile pattern alone, distinguishing structures fabricated via an amimono process can be far from obvious (see “real” jersey knit and “fake” knit in the center of Fig. 3). Still, some other classes of textiles, such as chain mail or weaves, can be easily ruled out. The method we develop to assess knittability will focus on patterns that are not easily distinguishable.

Translating the physical characteristics of knits and crochets into mathematical conditions is our first step in assessing if a fabric can be made by one of these two techniques. To distinguish amimono fabrics from other textiles, we first impose the following conditions for the pattern:

- (i) discrete periodicity in two transverse directions,
- (ii) yarns run in parallel directions and each yarn entangles exactly two other yarns,
- (iii) no closed curves are formed by yarns.

The first periodicity condition (i) rules out nonwoven textiles that are made by random arrangements of a multitude of yarns. This condition also means that we restrict ourselves to swatches of fabric with homogeneous patterns, and the knittability criteria will reflect a local property of the structure, independent of its global shape. The second condition (ii) aims to eliminate wovenlike fabrics where threads run in at least two distinct directions, as well as braided-type structures. For example, yarn i in the jersey pattern in Fig. 2 crosses only with the $i - 1$ and $i + 1$ yarns. This condition is satisfied by most common knitted patterns, including standard stockinette and crochet structures. However, certain decorative techniques, such as tuck stitches, that pull a yarn from nonadjacent rows, do not satisfy this condition. The following analysis can be extended to patterns beyond this restriction, but we leave such generalizations outside the scope of this paper. Note that although amimono can be made from a single yarn, we focus here on the bulk properties of the fabric, ignoring its edges. The successive paths of the same yarn are then equivalent to multiple parallel yarns. Condition

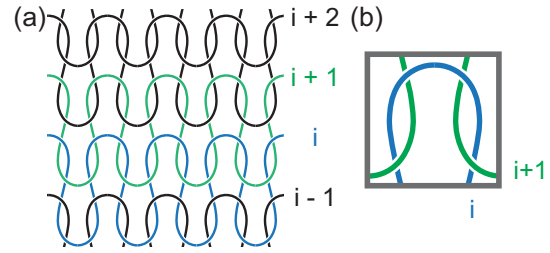


FIG. 2. (a) Textile diagram of a jersey knit with highlighted yarn paths. The yarn i crosses with the $i - 1$ and $i + 1$ yarns. A possible unit cell is shown in (b).

(iii) eliminates chain-mail-like patterns featuring closed yarn, which are readily incompatible with amimono manufacturing methods that use open yarns. In Fig. 3, the three conditions are presented in a Venn diagram and illustrated by matching textile diagrams. The structures of interest in the study are those satisfying the three conditions at the center of the diagram. Satisfying the three conditions still leaves ample room for unknittable diagrams, such as the fake knitting on the center right of Fig. 3, that only slightly differs from the classic jersey knit patterns on its left. To distinguish how these subtle changes in textile diagrams lead to dramatic differences in manufacturability, we leverage tools from knot theory.

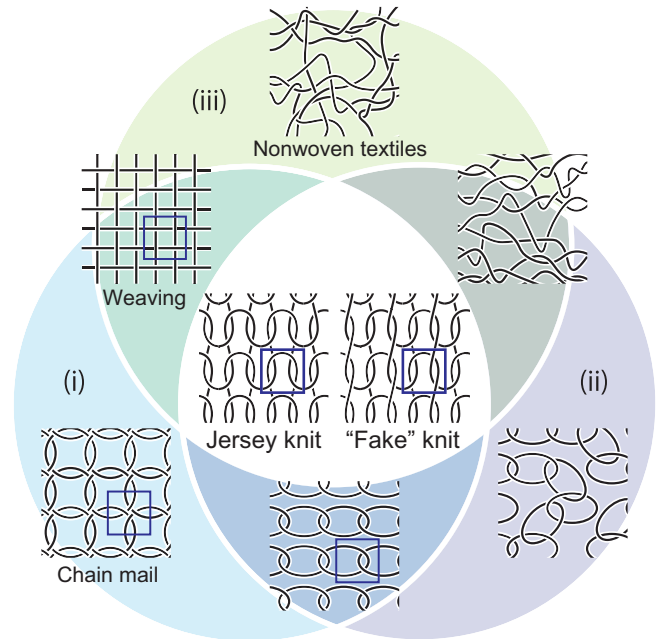


FIG. 3. Venn diagram to classify textiles, built from the three conditions detailed in the text. Each intersecting region features an example of a textile satisfying the corresponding conditions. Our focus is on the center of the Venn diagram, where subtle differences can switch a pattern from knittable (jersey knit, left) and between amimono (such as a jersey knit, center left) and “fake” knitting (center right). For each textile with discrete periodicity, a possible unit cell is depicted by a blue square.

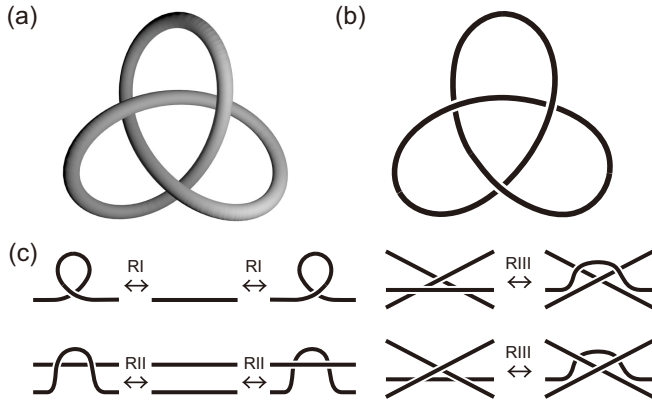


FIG. 4. (a) Illustration of a trefoil knot in 3D space and (b) its corresponding knot diagram. (c) Illustration of the three types of Reidemeister moves, RI, RII, and RIII.

III. TOPOLOGICAL MODEL OF DOUBLY PERIODIC TEXTILES

Some key steps and concepts of this paper will be presented at two levels of comprehension: one in plain language, the other in more rigorous mathematical terms. This section introduces the tools and definitions from knot theory needed to follow the reasoning in plain language. Interested readers may refer to Ref. [32] for an introduction to more advanced mathematical concepts.

Knot theory is the study of closed, entangled curves in space, called links [Fig. 4(a)]. A link consisting of a single curve component is called a knot. Two links are said to be ambient isotopic, or simply equivalent, if one can be transformed into the other by continuous deformations without cutting or gluing the curves. Evaluating their equivalence in three-dimensional space is not trivial. Knot theory reduces the complex 3D equivalence problem to a simpler 2D combinatorial problem through the introduction of link diagrams and isotopy moves. A link diagram is a two-dimensional projection of a link [Fig. 4(b)] in which over and under crossing information is marked at each intersection between two arcs. The continuous deformations of links are then discretized into a set of local transformations on their planar diagrams, called Reidemeister moves, which capture three-dimensional equivalence. There are exactly three types of Reidemeister moves [Fig. 4(c)]: RI, add or remove a single loop; RII, add or remove two crossings; and RIII, slide a strand past a crossing. Two links are equivalent if and only if their diagrams are related by a finite sequence of Reidemeister moves. This statement is known as the Reidemeister theorem. In particular, we call a knot trivial (or an unknot) if its diagram can be transformed, by Reidemeister moves, into a simple circle without any crossings. A trivial link is a set of disjoint unknots, meaning that its corresponding diagram contains no crossings, up to Reidemeister moves.

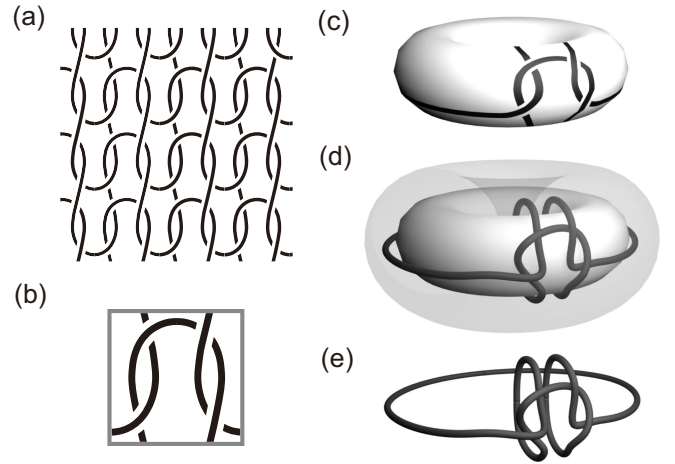


FIG. 5. (a) A 2D textile diagram in the plane. (b) A unit cell extracted from the textile diagram. (c) The same unit cell folded on a torus. (d) A knot generated by lifting the diagram from the torus to a thickened torus. (e) A knot in three-space obtained by forgetting the thickened torus in three-space. The resulting knot is a trefoil knot, which, when projected onto a plane, has a knot diagram equivalent to Fig. 4(b).

In this paper, we model textiles as doubly periodic entanglements of curves embedded in the thickened plane $\mathbb{R}^2 \times I$, where $\mathbb{R}^2$ denotes the Euclidean plane and I the unit interval. As in classical knot theory, we then define textile diagrams as their natural planar projection that extends infinitely in two transverse directions in $\mathbb{R}^2$ with crossing information encoding the over and under intertwining of yarns, as illustrated in Fig. 5(a). Intuitively, a simple cutting-and-stitching process allows one to build a link diagram from a textile diagram. We first extract a parallelogram unit cell from the textile pattern [Fig. 5(b)], then glue its opposite sides together to form a toroidal shape [Fig. 5(c)]. The double periodicity ensures that, when gluing the opposite sides of the unit cell, each cut curve can be joined continuously to another from the opposite side. By interpreting back the diagram on the torus as a 3D curve [Fig. 5(d)], and removing the torus, we obtain a 3D knot [Fig. 5(e)].

More formally, the object resulting from the gluing is a link diagram on the torus T^2 , which we refer to as a torus diagram. A torus diagram is obtained by the quotient of a doubly periodic textile diagram under a periodic lattice (see Refs. [62–64] for more details). In this paper, we use the term unit cell to refer to a flat torus diagram cut along a longitude and a meridian of the torus. A torus diagram can then be lifted in 3D to a set of entangled curves in the thickened torus $T^2 \times I$, with I being the unit interval [Fig. 5(d)]. By viewing the thickened torus in the three-space $\mathbb{R}^3$, one can obtain a link in $\mathbb{R}^3$ by “forgetting” $T^2 \times I$, as illustrated in the example of Fig. 5(e), which can be regarded as a trefoil knot in $\mathbb{R}^3$. However, the resulting link depends on the choice of the unit cell. A rigorous

mathematical investigation, presented in Ref. [62], establishes the equivalence of doubly periodic textile structures in three-space via the torus diagrams associated with their unit cells, independently of the choice of periodic lattice and basis. Defining a suitable unit cell is then a crucial step when applying knot theory to textile diagrams and for classification purposes.

IV. UNIT CELL

Analogously to crystals, modeling textile structures as doubly periodic planar diagrams allows the extraction of unit cells, thereby reducing the complexity of the problem. A unit cell is a portion of the textile diagram that can be tiled infinitely in two transverse directions. More formally, a unit cell is defined as the quotient of the planar textile diagram under an integer point lattice. To the same textile diagram, one may assign infinitely many lattices, which have the type of one of the five 2D Bravais lattices [65]. The corresponding unit cell is then represented as a parallelogram in a lattice, bounded by four edges. The choice of a specific unit cell for a given textile pattern is arbitrary. In what follows, we introduce conditions for rigorously selecting a suitable unit cell for our classification method.

Our knittability assessment methodology will rely on a restricted group of unit cells that satisfy specific conditions, chosen to reflect the amimono process of iteratively creating new loops along a line or column. We define a yarn path as the continuous path of a yarn running through the fabric (see Fig. 6, with one path highlighted in green). The conditions to select a suitable unit cell for our methodology are as follows.

- (I) The unit cell is a parallelogram with one pair of opposite edges parallel to the overall path direction. The other pair is referred to as vertical, and all yarn segments crossing these edges are from the same path.
- (II) The unit cell encompasses exactly two paths of yarn.
- (III) The unit cell satisfying conditions (I) and (II) is chosen to be minimal, meaning that it cannot be divided into smaller unit cells satisfying these conditions.

By isolating two yarn paths, conditions (I) and (II) will be essential for our methodology and the definition of topological defects. It also simplifies the representation of an amimono diagram as a rectangular grid with each row following one yarn path. The fabrication process involves adding a new row on top of the grid, with yarns traveling along rows and loops connecting along columns. These conditions exclude some textile patterns that cannot be reduced to unit cells with only two paths. While extensions of our method to more general cases are possible, in this paper, we focus on patterns that satisfy these conditions for simplicity. Condition (III) on the unit cell minimality is imposed to reduce the number of possible

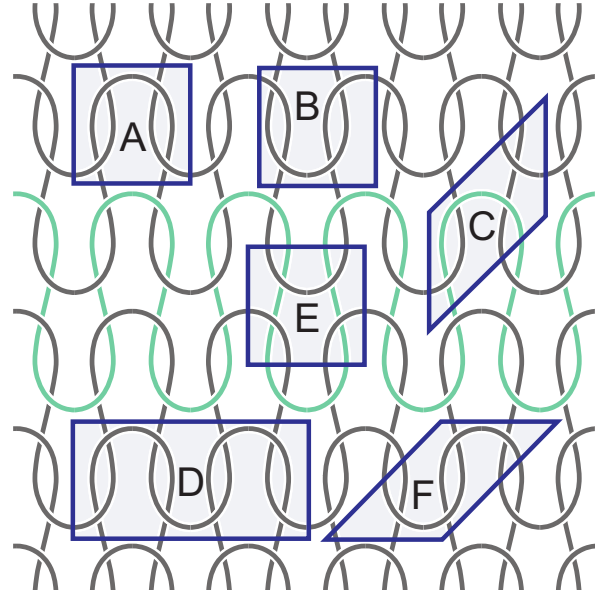


FIG. 6. Cells A–F are examples of possible unit cells for the jersey knit pattern. Only cells A and B satisfy all our conditions for a unit cell. Condition (I) is not respected by cells C and F, while unit cells C and E violate condition (II). Unit cell D is not minimal, hence not satisfying condition (III).

unit cells, the complexity of the topological operations required later to assess knittability, and to enable coherent comparison of different patterns. For a given periodic lattice, a unit cell is defined as a parallelogram whose corners are four lattice points and such that it does not contain any other point of the lattice in its interior. However, this notion of unit depends on the choice of lattice, and one may find a “smaller” unit cell for the same textile diagram if choosing a different lattice. The notion of a minimal unit cell is defined in Ref. [62] as a unit cell that cannot be divided into a smaller one by a change of lattice. Note, however, that area-preserving affine transformations, generated by shearing and translations, as well as Reidemeister moves, preserve minimality. In our case, as the lattice has to respect conditions (I) and (II), it is not necessarily the minimal unit cell of the textile diagram. Examples of unit cells satisfying or not these conditions are illustrated in Fig. 6, while Appendix A shows two unit cells linked by a shear deformation satisfying all conditions.

With unit cells and textile patterns satisfying these conditions, a patch of the diagram of our textiles of interest can be simply represented as a rectangular grid. The position of each unit cell is defined by a pair of integers (i, j) , with an arbitrary reference $(0, 0)$ on the bottom left of the grid. A unit cell located directly above the cell at position (i, j) , across a horizontal edge, is assigned the position $(i, j + 1)$. Similarly, a unit cell located directly to the right of the cell at position (i, j) , across a vertical edge, is assigned the position $(i + 1, j)$. For each fixed i , the set

of unit cells with position (i, j) for all j is parallel to what we refer to as the i th path of the yarn.

V. DEFINITION AND PROPAGATION OF TOPOLOGICAL DEFECTS

The conditions we impose on textile diagrams, along with the choice of a suitable unit cell, allow for a simplified view of their underlying structure. A remarkable feature of amimono is the contrast between the topological protection of their bulk and the fragility of their edges. To capture this feature from a topological point of view and exploit it to establish knittability criteria, topological defects in textile diagrams are defined, and their propagation within diagrams is assessed. We generate a topological defect $\tilde{A}$ of a corresponding unit cell A by switching at least one of its over-under crossing information. When placing one or several of these defects $\tilde{A}$ into a grid of unit cells A , defect propagation occurs if at least one unit cell A can be transformed into a defect $\tilde{A}$, using only Reidemeister moves within the diagram.

Among the numerous types of defects that can be produced, we are interested in those that reproduce a free edge within the bulk of the diagram. Within our definition of suitable unit cells, the same yarn path runs in and out of a unit cell via its lateral edges, and it crosses only one other subsequent path. We then coin as *simple defect* a defect that disentangles the two paths passing through a unit cell. In the context of knot theory, a classical method to disentangle a knot is based on unknotting operations, which consist of switching over-under information at selected crossings in a knot diagram [32]. This approach has recently been extended to the study of triply periodic entanglement to disentangle unit cells viewed as link diagrams in the three-torus [66]. However, general disentangling methods are not practical to implement in this setting, since they require exploring a large combinatorial space of crossing changes and Reidemeister moves, leading to rapidly increasing computational cost as the number of crossings grows. Instead, we leverage our simplified unit cell structure by simply considering all the crossings between the i th and $(i+1)$ th paths, and switching them or not to have systematically the $(i+1)$ th path on top, as shown in Fig. 7(a).

A simple defect $\tilde{A}$ is thus created by locally disentangling the two yarn paths crossing the unit cell A . Introducing defects into a textile diagram modifies its local topology. We now assess whether the topological modifications remain localized or propagate to neighboring unit cells. To assess the propagation of a defect, we place one or several defects $\tilde{A}$ in a given pattern within a grid of unit cells A . Then, by evaluating the topological equivalence through Reidemeister moves, it is possible to assess if the original neighboring unit cells A can be transformed into defects $\tilde{A}$ without any net changes to any other unit cells, as

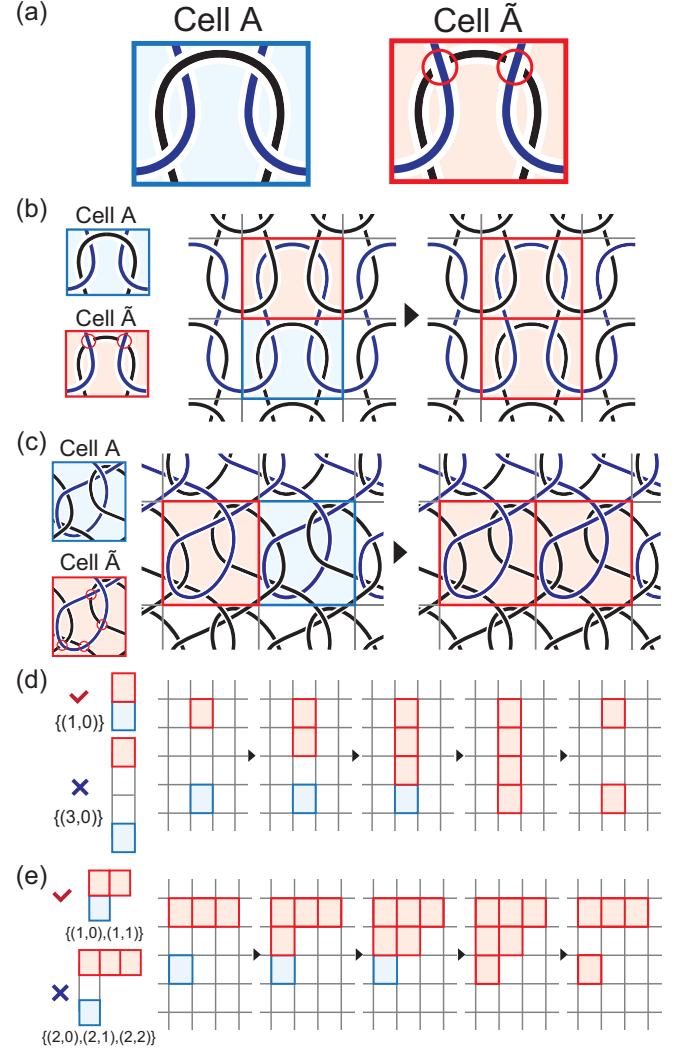


FIG. 7. (a) Illustration of a unit cell and its simple defect counterpart. The crossings that have been switched are circled in red. (b),(c) Propagation of simple defects in a jersey (b) and crochet (c) pattern. The respective propagation sets are $\{(1,0)\}$ and $\{(0,-1)\}$. (d) Illustration of the reduction of the propagation set $\{(3,0)\}$ to $\{(1,0)\}$ by applying several times the latter. As Reidemeister moves are reversible, defects in $(1,0)$ and $(2,0)$ can be reverted back to A to obtain exactly the set $\{(3,0)\}$. (e) Likewise for the propagation sets $\{(2,0), (2,1), (2,2)\}$ and $\{(1,0), (1,1)\}$.

shown in Figs. 7(b) and 7(c). In the example in Fig. 7(b), the cell $\tilde{A}$ can propagate to adjacent cells perpendicular to the direction of the yarn, relative to the original defect. While in the example in Fig. 7(c), the cell $\tilde{A}$ can propagate to an adjacent cell parallel to the direction of the yarn. The sequences of Reidemeister moves that allow the propagation of defects in these two examples are detailed in Appendix B. Thus, the defects do not necessarily propagate along the yarn, but can be propagated in various directions, depending on the topology of the cell A and defect alter ego $\tilde{A}$. While Figs. 7(b) and 7(c) show propagating cases,

numerous topologies do not allow the propagation of their defects.

We now introduce a formalization and classification of the rules of defect propagation. Select first an arbitrary unit cell, denoted A_0 , in the grid representing the textile diagram of interest. Our objective is to evaluate the number and the position of the neighboring unit cells of A_0 which need to be changed into $\tilde{A}$, so that A_0 can be transformed into $\tilde{A}$ by Reidemeister moves. We consider the set of all unit cells in the grid, identified by their relative position from A_0 , $\mathcal{P} = \{(i_1, j_1), (i_2, j_2), \dots\}$. The *propagation set* $\mathcal{S}$ is then defined as a subset of $\mathcal{P}$ satisfying the following propagation conditions.

- (1) If all the unit cells whose positions belong to $\mathcal{S}$ are transformed into defect cells, then there exists a sequence of Reidemeister moves that transform the cell A_0 into a defect $\tilde{A}$, while keeping the other cells unchanged. We then refer to a *propagation pattern* the transformation of A_0 to $\tilde{A}$, or conversely, from $\tilde{A}$ to A_0 .
- (2) Propagation sets that can be reproduced by repeating other propagation patterns are excluded.

The propagation condition (1) sets the requirement for defects to propagate. This condition and resulting propagation pattern are independent of the type of unit cells (whether A or $\tilde{A}$), whose positions do not belong to $\mathcal{S}$. If for the set of unit cell relative positions $\mathcal{P}$ there is no subset $\mathcal{S}$ that satisfies condition (1), then the pattern does not allow defect propagation. The jersey knit pattern [Fig. 7(b)] has a propagation set $\{(1, 0)\}$, so only one defect just above this position is sufficient to transform A_0 into a defect $\tilde{A}$. In contrast, with a crochet-type pattern [Fig. 7(c)], defects propagate with a propagation set $\{(0, -1)\}$; hence, a single defect on the left of A_0 is enough to transform it into $\tilde{A}$ by Reidemeister moves.

Condition (2) reduces the propagation set $\mathcal{S}$ to a minimal set. If a propagation set $\mathcal{S}_1$ can be reproduced by applying multiple times other propagation sets (usually containing closer neighbors), we eliminate $\mathcal{S}_1$. This condition is introduced to reduce the number of propagation sets to a finite number (in most cases one). Note that this condition does not guarantee uniqueness. In general, multiple minimal collections of propagation sets may exist that generate the same propagation. In this work, we do not attempt to characterize the unique representation, as our analysis depends on the existence of propagation, not on the specific choice of generating sets.

Two examples of propagation sets reproduced by minimal sets are provided in Figs. 7(d) and 7(e). In the case of the jersey [Fig. 7(b)], the propagation set $\{(3, 0)\}$ also satisfies condition (1) as defects can propagate on the whole column below a defect. However, it does not satisfy condition (2) as the propagation set $\{(3, 0)\}$ can be obtained by repeating the propagation set $\{(1, 0)\}$, as illustrated in Fig. 7(d). Another example is shown in

Fig. 7(e), where the propagation set $\{(2, 0), (2, 1), (2, 2)\}$ can be reduced to the propagation set $\{(1, 0), (1, 1)\}$ by applying it multiple times.

For a single textile pattern, there are generally multiple ways to choose a unit cell that satisfies conditions (I)–(III). The propagation set introduced here is therefore defined for each unit cell. Because different unit cells may select different yarn segments, the set of crossings that are switched in forming $\tilde{A}$ depends on the chosen unit cell. Consequently, different choices of unit cell may lead to different propagation sets. In what follows, we use the concept of topological defects to define knittability criteria and then apply the propagation set to provide a straightforward method for their implementation. We note that from this stage on, details of the underlying textile diagram, such as the yarn paths and crossings, can be safely ignored and reduced to grids of unit cells and defects, along with their propagation rules on a square grid.

VI. KNITTABILITY CRITERIA

We propose two topological constructions that define criteria for knittability and classify amimono. These constructions and ensuing criteria rest on the remarkable property that amimono can be untangled from its working edge only. Here, the working edge [Fig. 1(b)] is characterized in a unit cell as a strand that is distinct from and not tangled with the i th path. For both introduced criteria, we start with a doubly periodic textile diagram generated by a unit cell A . Next, we consider a finite subset G_c of this diagram, consisting of l rows and m columns of unit cells A arranged in an (l, m) grid. In principle, the knittability criteria are expected to be independent of the choice of l and m for a sufficiently large value. In practice, we find that small values (e.g., $l \geq 3$ and $m \geq 2$) are sufficient to obtain stable results. Smaller patches may lead to inconclusive cases if they do not cover the extent of the propagation pattern. We then replace n consecutive rows of G_c by n lines of defects $\tilde{A}$ and we denote by G_o^n this new diagram. If the defects disentangle the threads (no more crossing points up to Reidemeister moves), rows of defects introduce a working edge in G_o^n . Further, by definition of G_c and G_o^n , we can identify the opposite sides of the rectangular grid and obtain a torus diagram. The knittability criteria we now introduce are based on the comparison of the topological properties of the links in three-space that can be created from the grids G_c and G_o^n . In a word, a pattern is knittable if it produces trivial links when folded from the subset with a working edge G_o^n , while the links folded from the original diagram G_c remain topologically protected. The topological processes corresponding to the two criteria are illustrated in Fig. 8.

Knitting criterion I. We construct a link in the three-space $\mathbb{R}^3$ from a grid G_c , made of unit cells A , following the idea illustrated in Fig. 5. First, since G_c forms a torus diagram by gluing its opposite sides, we obtain a link L_c in

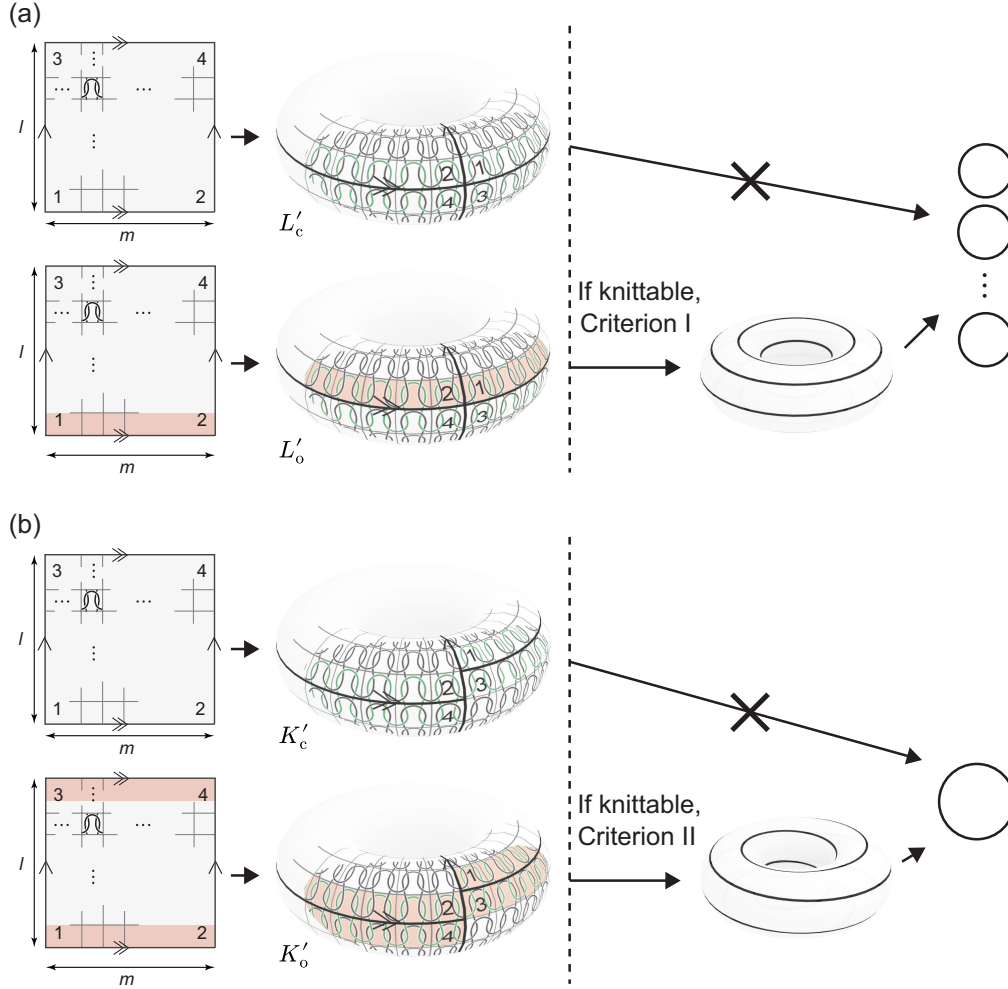


FIG. 8. Illustrations of the two knitting criteria. (a) Criterion I: an $m \times l$ patch G_c is glued into a torus to form a link L_c . We introduce a row of defects (in red) in G_c and fold the resulting grid G_o into the link L_o . If the 3D version L'_c is not a trivial link but L'_o simplifies to a set of disjoint loops, the criterion I is satisfied. (b) Criterion II: here, G_o is defined by transforming two consecutive rows into defects (in red), and then generating the knots K_c and K_o by folding G_c and G_o onto a torus, but shifting the lateral sides by one row. If the 3D knot K'_c is not trivial but K'_o simplifies to a simple loop, criterion II is satisfied.

the thickened torus $T^2 \times I$ (Fig. 8), which leads to a link L'_c in $\mathbb{R}^3$. We then apply the same process to a grid with a line of defects G_o^1 . This generates a link L_o in the thickened torus $T^2 \times I$ [Fig. 8(a)], and the corresponding link L'_o in $\mathbb{R}^3$. The knitting criterion I is defined as follows.

A textile pattern generated by the unit cell A is defined as knittable according to criterion I if, using simple defects to generate G_o^1 , the generated link L'_o is a trivial link in $\mathbb{R}^3$ while L'_c is not.

Knitting criterion II. Akin to the case of criterion I, we consider the torus diagram resulting from the gluing of the opposite sides of the grids G_c and G_o^2 . In particular, the gluing of the left-right sides of the grids forms a meridian of the torus. We now twist the torus along this meridian, clockwise or counterclockwise, such that each row of the grid is glued to its adjacent upper row on one side of the meridian, and to its adjacent bottom row on the other side of

the meridian, as illustrated in Fig. 8(b). This shifts the recombination along the columns of the grids by one row so that the cell $(i, l-1)$ is connected to the cell $(i-1, 0)$. The resulting objects are the knots K_c and K_o in the thickened torus, resulting from the grids G_c and G_o^2 , respectively, after twisting the torus diagrams. These knots are made of a single curve following a helicoid trajectory winding around the torus. Then, by mapping K_c and K_o in the three-space as done in criterion I, we obtain the knots K'_c and K'_o in $\mathbb{R}^3$. The knitting criterion II is defined as follows.

A textile pattern generated by the unit cell A is defined as knittable according to criterion II if there exists a defect $\tilde{A}$ such that the corresponding knot K'_o is trivial and K'_c is not.

For a given textile pattern knittable according to criterion I, the use of simple defects along a row effectively introduces a free boundary into the pattern by disentangling two consecutive paths. This property follows from the

simple defect definition, which ensures that subsequent paths are disentangled, and from the gluing that connects each path to itself. In contrast, criterion II reconnects all the paths into one by twisting the torus along a meridian. As a consequence, self-entanglements of a path now also play a role, and simple defects may not be sufficient to create a free edge. Hence, criterion II is weaker by constraining the knittability to the existence of a suitable defect. For all the examples we considered, a suitable one is the *self-disentangling defect* (defined in detail in Appendix C), which, on top of disentangling consecutive paths like a simple defect, also modifies self-crossings. Another difference lies in that for a given textile diagram, any suitable unit cell satisfying the three conditions stated in Sec. IV will generate the same link L'_0 , even if the propagation pattern of the defect varies. It means that criterion I is independent of the choice of the unit cell, unlike criterion II for which different unit cells may generate different knots K'_0 , some potentially not reducible to a trivial knot. Unlike L'_0 , K'_0 is unit-cell-dependent. Different unit cells may lead to different knots K'_0 because the choice of unit cell determines how over-under crossing information is assigned to the cell or to its neighbors. This changes which crossings are switched when constructing the defect $\tilde{A}$, and therefore affects the resulting knot K'_0 .

For a given textile pattern, these criteria provide a topology-based definition of knittability. Whether a knittable textile satisfies one or two of the criteria also provides a classification method. If a pattern satisfies neither criterion, it is not knittable, within the constraints we imposed on the pattern (Fig. 3). If a textile satisfies the knitting criterion I, we include it in the *knits* category, but if it satisfies only the knitting criterion II, we classify it in the *crochet* category. The unit-cell dependence of criterion II reflects the physical nature of crochet, where the initial loop can be placed in multiple ways. Note that the topological modeling of crochet based on the twist operation of the torus, as presented in this work, is an original mathematical construction. In the next section, we will discuss the implementation of these criteria and the rationale behind the classification.

VII. CRITERIA IMPLEMENTATION

We have reduced the determination of knittability to estimating whether pairs of knots and links are trivial or not. In knot theory, the classification and the unknotting problems are long-standing goals, which motivated the developments of computational tools [67–71]. However, due to the high computational cost of the algorithms (which typically grows exponentially with the number of crossings), it is challenging to find a systematic solution to this problem. A simple heuristic method is, therefore, a reasonable and practical alternative. Here, the concept of defect propagation introduced earlier (Fig. 7) comes into play.

For a given textile diagram, we aim at characterizing whether K'_0 and L'_0 are trivial. Recall from the previous section that these knots and links are intuitively obtained by

folding on the torus the grid G^n_0 into which n rows of unit cells A have been replaced by their defect equivalent $\tilde{A}$, aiming at disentangling the yarns along an effective working edge. We established that these defects can propagate in various directions (Fig. 7); thus, the initial row of defects may invade the whole torus depending on its propagation set. For L'_0 and the knittability criterion I, the row of defect loops back on itself, so it invades the whole torus if propagation has only components along a column, perpendicular to the yarn direction, as in Fig. 7(b). In term of propagation sets, the first coordinate of each position must be nonzero (typically ± 1), with all the same sign, such as $\{(1, -1), (1, 0), (1, 1)\}$ or $\{(-1, -1), (-1, 0)\}$. A propagation along the yarn direction [along a row in the grid, as for Fig. 7(c)] keeps the defect on the same row. For K'_0 and the knittability criterion II, the rows of defects loop back on the row below. In that case, defect propagation along the yarn direction also invades the whole torus by spiraling along the shifted rows. This time, the propagation sets featuring zero or different components in the first coordinate of each position also invade the whole torus.

As defect propagation does not alter the topology of the transformed diagram, depending on the propagation set of the defects, we can establish that K'_0 or L'_0 is equivalent to a link made from a grid containing only defects $\tilde{A}$. The role of the introduced defects is to disentangle the yarns; thus a link made only of defect cells will be greatly simplified, and in most cases, trivial.

We assess the triviality of the knot obtained from a single defect cell $\tilde{A}$. If this knot is not trivial, by periodicity, any grid constructed from copies of $\tilde{A}$ is also nontrivial [72]. Therefore, even if defects propagate and cover the torus, the resulting link L'_0 cannot be trivial, and criterion I is not satisfied. If no defect can yield a trivial knot, then criterion II is also not verified. In the case that the knot yielded by the defect is trivial, the propagation sets provide a way to show the existence of a sequence of Reidemeister moves that drives the number of crossings in the diagram of K'_0 or L'_0 to 0. Evidence that K'_0 or L'_0 is trivial is then given solely by the propagation set of the defects. By finding a unit cell and a corresponding defect that can propagate over the diagram, we can guarantee that a given textile is knittable. The problem of knittability is then shifted from characterizing a complex link to the heuristic determination of propagation sets. Now, we have to provide evidence that the knots and links without defects, K'_c and L'_c , are, in contrast, nontrivial. If K'_c and L'_c were trivial, it means that the adjacent paths would not entangle upon constructing the fabric, and this absence of entanglement is already reflected at the level of the unit cell. In such nonentangled textiles, each unit cell could be reduced, by Reidemeister moves within the cell, to a configuration equivalent to its simple defect $\tilde{A}$. If a unit cell A is equivalent to its simple defect $\tilde{A}$ by Reidemeister moves within the unit cell, then the textile is not knittable as it would form a fully disentangled structure.

	Original	Defect	Propagation	Knittability		Original	Defect	Propagation	Knittability
(a)			 $\{(1,0)\}$	I, II	(i)			 $\{(1,0)\}$	I, II
(b)			-	×	(j)			 $\{(0,-1)\}$	II
(c)			 $\{(1,0)\}$	I, II	(k)			-	×
(d)			-	×	(l)			 $\{(0,-1)\}$	II
(e)			 $\{(1,0)\}$	I, II	(m)			 $\{(1,0),(1,-1)\}$	I, II
(f)			 $\{(0,-1)\}$	II	(n)			-	×
(g)			-	×	(o)			 $\{(1,-1)\}$ $\{(1,0)\}$	I, II
(h)			-	×	(p)			-	×

FIG. 9. Application of the knittability criteria to a selection of textile patterns. For each example (a)–(p), a suitable unit cell, the corresponding simple defect [or self-disentangling defect for (j) and (l), marked by *], the propagation set, and the criteria satisfaction are featured. As they satisfy at least one criterion, the patterns (a),(c),(e),(f),(i),(j),(l),(m) are considered knittable.

To summarize, the implementation of the knittability criteria is as follows.

- From a given textile diagram, extract a suitable unit cell A that satisfies all the conditions presented in Sec. IV.
- Determine a simple defect $\tilde{A}$ from A by disentangling consecutive paths. If A and $\tilde{A}$ are topologically equivalent, the pattern is unknittable. In addition, if the knot generated by folding $\tilde{A}$ on a torus is nontrivial, the pattern is unknittable according to criterion I (see Appendix D for an example).
- We assess the propagation set of the simple defect. If defects propagate along a column (perpendicular to the yarn direction), then both K'_0 and L'_0 are equivalent to

trivial links and both criteria are satisfied. If the defects propagate along a row (the direction of the yarn), only the knittability criterion II is satisfied. In both cases, the pattern is deemed knittable.

- If a propagation set for the simple defect cannot be found, the pattern does not verify criterion I. We then repeat steps (b) and (c) with another kind of defect, such as the self-disentangling defect (defect that also removes self-entanglement within a path; see Appendix C). If no defect can be found to propagate, criterion II is also not verified, and the pattern is deemed unknittable under our definition.

We implemented this approach over a wide variety of patterns (Fig. 9) and successfully distinguished known

knittable patterns from unknittable ones. For knittable patterns, satisfaction of one or two of the knittability criteria enables a distinction between two subcategories. Patterns whose defect propagation set has only vertical components satisfy both knittability criteria, and we classify them as *knits*. Those whose propagation set has a horizontal component satisfy only the knittability criterion II and are coined as *crochet*. The set of knits comprises all the patterns classically associated with weft knitting, while the crochet categories encompass patterns used in crochet and warp knitting. The fabrication processes associated with these two subcategories differ substantially, so, beyond knittability, details of the fabrication techniques are encoded in the patterns' topological properties. For instance, the patterns of Figs. 9(a) and 9(f) represent, respectively, a standard jersey knitted and crocheted unit cell. The knit [Fig. 9(a)] propagates vertically following a propagation set $\{(1, 0)\}$; hence, an entire line of defects in G_c is necessary to propagate the defects everywhere on the torus. This defect line reflects the presence of a free working edge during knit fabrication. In contrast, the crocheted unit in Fig. 9(f) propagates laterally according to the set $\{(-1, 0)\}$, then remarkably, introducing only a single simple defect in G_c is sufficient to untangle completely K'_o . The structural fragility to a single defect reflects that crocheting progresses loop by loop, with the previous loop topologically secured, reducing the working edge to a single loop. Considering simple defects only is sufficient for most of the knittable patterns, except for Figs. 9(j) and 9(l), which necessitates self-disentangling to verify criterion II. For all patterns deemed unknittable, no propagating defect was found.

This method is efficient for characterizing knittability when, by a succession of Reidemeister moves, we identify the propagation sets of a defect. However, finding a propagation set is not a trivial task, and we cannot prove its absence. Therefore, unknittability cannot be strictly proven with the method described. We now aim to confirm our results, obtained from propagation, with knot theory standard tools. Since our knittability criteria are evaluated on the triviality of K'_o and L'_o , an ideal tool would detect triviality unambiguously. In knot theory, a topological invariant is any quantity assigned to a link that remains unchanged under equivalence. In particular, all trivial links with the same number of components necessarily share identical values of any given invariant. However, the converse fails in general: No known invariant is both necessary and sufficient for triviality, so that some non-trivial links may nonetheless share the same invariant values as trivial links. The only widely studied exception is Khovanov homology, a tool that assigns algebraic structures to a knot in such a way that only the trivial knot produces the simplest possible result [71]. While Khovanov homology can detect the unknot, its high computational cost for moderately complex knots and its

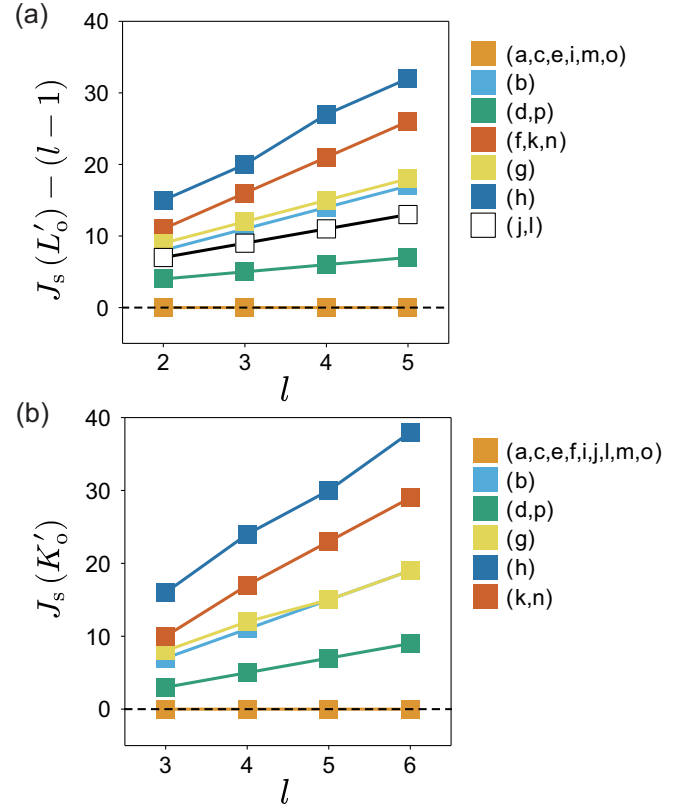


FIG. 10. (a) The quantity $J_s(L'_o) - (l - 1)$, with $J_s(L'_o)$ the span of the Jones polynomial of the links L'_o , is plotted against l , the number of rows in the grid. All patterns shown in Fig. 9 are assessed, and those with $J_s(L'_o) - (l - 1) = 0$ are knittable according to criterion I. (b) For the same patterns, $J_s(K'_o)$ is also plotted against l . Patterns with $J_s(K'_o) = 0$ satisfy criterion II. These computations provide a quantitative validation of the knittability criteria estimated via defect propagation.

inability to detect trivial links motivate the search for a more direct, systematic approach. We therefore adopt a brute-force strategy of computing simpler invariants for each pattern's associated links K'_o and L'_o .

Rather than relying on completeness, one can exploit an invariant's ability to measure complexity. We use the Jones polynomial, which associates to a given link L a polynomial $V_L(t)$ [73]. The Jones polynomial is widely used in knot theory due to its computability and its ability to distinguish between different knot and link types. Its span J_s , which is defined as the difference between the highest and lowest polynomial degrees, provides a single scalar summarizing entanglement complexity. It takes especially simple values on trivial links: For the unknot, $J_s = 0$, and for a trivial link of l components, $J_s = l - 1$. Moreover, computing J_s remains feasible for unit cells with a moderate number of crossings. We therefore calculate the span of $V_{K'_o}(t)$ and $V_{L'_o}(t)$ using the Knotfolio library [74], applied to textile diagrams with two columns ($m = 2$) and a varying number of rows l (Fig. 10). Our results mirror those from the propagation method: Knittable patterns

under the knittability criterion I yield $J_s(L'_0) = l - 1$, and those under the knittability criterion II yield $J_s(K'_0) = 0$ for all l . In contrast, unknittable patterns exhibit higher and increasingly diverse spans as l grows, reflecting the greater topological complexity of their associated links. Combined with the propagation approach, this span-based method thus offers a practical conjecture for the knittability or unknittability of textile patterns whose unit cells have manageable crossing numbers. We empirically confirmed that triviality of L'_c and L'_o are independent of l and m for $m \geq 2$ and $l \geq 2$. However, a more rigorous approach that imposes conditions on (l, m) based on the propagation set is left for future work.

Akin to reducing a many-body problem to a one-body problem, we leveraged periodicity and local propagation sets to reduce the complexity of knots and links. It enables a straightforward way to determine whether a textile pattern is knittable. We validated the approach by directly comparing the defect-propagation criteria with the estimates of a knot invariant. To keep the definition of the criteria as simple as possible, we restrict our discussion to a representative set of knitted patterns. This restriction is not fundamental, and the same criterion could be applied to broader classes of patterns by using the developed methodology. In the next section, we further discuss physical analogies and potential applications of the relation between topology, knittability, and defect propagation.

VIII. DISCUSSION

A. Amimono as marginally constrained textiles

The criteria we implemented to define amimono fabrics highlight a remarkable property: Knits and crochets are marginally constrained textiles. Knitted fabrics are stable structures, but the introduction of some defects completely destroys the structure. One can draw an analogy with other marginally constrained systems whose stability is assessed by constraint counting. Counting the number of degrees of freedom (d.o.f.) and constraints has been instrumental in identifying floppy modes of metamaterials, such as frames [75] or origami [17], but also to locate rigidity transitions in many-body systems [76,77]. With f the number of degrees of d.o.f., c the number of constraints, we define their difference by $\nu = f - c$. Overconstrained “rigid” systems have $\nu < 0$, while underconstrained “floppy” systems have $\nu > 0$. The isostatic point at $\nu = 0$ marks the rigidity transition.

By analogy, we define a d.o.f. density for a patch of textile as $\nu = -\tilde{N}/N$, where $\tilde{N}$ is the minimal number of defects required to disentangle the patch, and $N = l \times m$ is the number of unit cells. Since ν is defined for a finite patch, it may depend on the choice of l and m . Its asymptotic behavior is governed by how $\tilde{N}$ scales with system size. From the viewpoint of knot theory, ν is related to the unknotting number. As the number of cells increases,

the number of crossings increases as well. In this case, the increase in the number of crossings is due to the increase in the grid size $l \times m$, not due to Reidemeister moves. Thus, since the number of unknotting operations is fixed within a unit cell, increasing l and m necessarily increases the number of crossings to be switched, whereas introducing defects decreases it. The absolute value of the ratio $|\nu|$ can thus be interpreted as the average of the unknotting number per unit cell.

We now estimate the asymptotic behavior of ν for unknittable and knittable textiles. For unknittable textiles, defects do not propagate, and disentangling requires modifying each unit cell independently, so that $\tilde{N} = N$, yielding $\nu = -1$. In contrast, in amimono, defects propagate along rows or columns. In knitting, defects typically propagate perpendicular to the yarn direction; hence $\tilde{N} \propto m$ and $\nu \propto l^{-1}$. Whereas in crochet, defects propagate along the yarn; hence $\tilde{N} \propto l$ and $\nu \propto m^{-1}$. In both cases, $\tilde{N}$ scales with one linear dimension of the system, rather than with the total number of cells. Therefore, in the limit of large system size with a fixed aspect ratio l/m , d.o.f. will systematically converge toward the isostatic limit:

$$\nu = -\frac{\tilde{N}}{N} \rightarrow 0. \quad (1)$$

This asymptotic behavior characterizes amimono as marginally constrained textiles.

The marginal stability of amimono has direct practical implications. As a method of identification, it suffices to remove or cut a yarn in a textile and observe the propagation of the damage to assess its method of fabrication. If the damage easily propagates, the textile has probably been manufactured using local manipulations of the yarn, as in amimono. When such a destructive method is not adapted, as for textiles excavated from archaeological sites [24,78] or synthesized at the molecular scale [79,80], we can extract the topological structure and virtually introduce defects to infer their propagation.

B. Defect approach to functional textiles

Another direct implication is the rationalization of a very well-known and troublesome feature of knits, laddering [Fig. 11(a)]. When a loop is missed during manufacturing or if the yarn is damaged and locally cut, a “ladder” rapidly forms in a knitted textile under tension. Laddering is a direct visualization of topological defect propagation in amimono, with a breakdown of the structure over a long range, even if initiated by a local perturbation. A remarkable property of laddering is that the nature of the propagated damage is topological and not material. Apart from the initial cut, the thread remains intact. As loops get freed by the propagation of the topological defect, the tension in the fabric straightens the freed loops, making the ladder apparent without any further damage or even

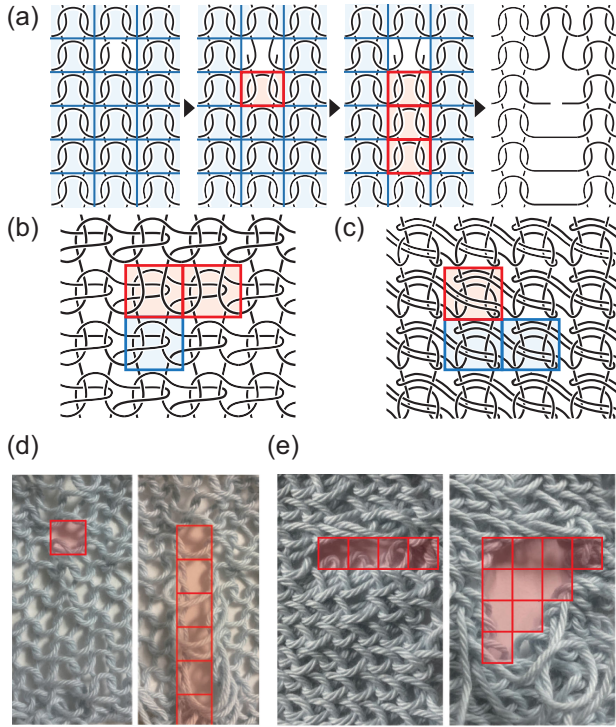


FIG. 11. (a) Defect propagation from a cut yarn, leading to the laddering effect of knitted fabrics. (b) Robust knit, designed such that two defects on the upper row are needed to propagate one defect on the row below. (c) Fragile knit, one defect propagates into two defects on the row below. (d) Laddering of the widely used jersey knit pattern. (e) Realization of the robust knit, where four defects were initially placed on one row. When propagating to a subsequent row, the number of defects is reduced by one, limiting the topological fracture to 4 rows and 10 defects in total.

minimal deformation of the threads. The direction of propagation is also completely determined by topology, and not by the direction of the yarn. Laddering can then be classified as a topological fracture, in contrast with classic fractures induced by the irreversible breaking of materials. Since the propagation of the topological fracture is driven by local and reversible deformation of the yarn, it is completely reversible, except for the initial damage. All knitters have likely experienced a slipping loop that propagates via laddering, but we also know that this does not mean the knit is ruined: By carefully looping back the ladder, we apply a sequence of Reidemeister moves that heals the topological fracture.

The criteria established to determine whether a textile can be knitted or not provide a topological basis to design functional textiles. In particular, one can use the analysis of defect propagation to bring specific properties and functions to fabrics. For example, we propose a geometry that mitigates laddering. The two patterns in Figs. 11(b) and 11(c) are both knittable textiles with radically different laddering resistance embedded in their structures. The textile in Fig. 11(b) restrains the propagation of laddering

via its defect propagation sets. The propagation, expressed as $\mathcal{S} = \{(1, 0), (1, 1)\}$, is such that from n adjoint defects along a yarn, the yarn below will have $n - 1$ defects. This structure forces the number of defects to shrink as they propagate, limiting the extent of laddering. In contrast, the textile in Fig. 11(c) is designed to be extremely sensitive to defects. This time, the propagation set given by $\mathcal{S} = \{(1, -1)\}$ and $\{(1, 0)\}$ features two directions of propagation. The structure of the resulting fabric is then entirely destroyed with a single defect.

The possible operations in amimono closely resemble the dynamics of kinetically constrained models (KCMs) [81]. As discussed in Sec. V, whether a given cell in an amimono structure can be transformed between the states A and $\tilde{A}$ depends on the configuration of neighboring cells. This constraint is analogous to a defining feature of KCMs, in which the possibility of spin flipping at a site is determined by the local spin configuration. This analogy provides a direct correspondence between the propagation sets characterizing defect propagation in amimono and the dynamical constraints in KCMs. For instance, the propagation set $\{(1, 0)\}$ studied in this work corresponds directly to the East model [82,83], a well-known example within the KCM class. This analogy suggests that tools developed for KCMs may be applicable to amimono, particularly for analyzing the asymmetry between knitting and unknitting operations and the stability of randomly generated knit structures.

IX. CONCLUSION

Textiles come in many topologies, reflecting the various crafting techniques developed across human history. In this work, we established a framework based on knot theory to create a direct link between the topology of a textile and its fabrication method. Knits and crocheted works are produced by manipulating yarn with fixed ends, a feature encoded in the fabric's topology that we leverage to assess knittability. We start by considering doubly periodic textile patterns and rule out obvious nonknittable patterns. The resulting textile patterns can be described as a rectangular grid of repeating unit cells, with yarns running along the rows. We then introduce the concept of topological defects in textile diagrams, which serve to locally disentangle the yarns within a unit cell. Depending on the textile topology, topological defects may propagate to neighboring cells by transforming them into defects without cutting the yarns.

Subsequently, knittability criteria are established by considering a swatch of unit cells, introducing defects, folding the swatch into a torus, and finally assessing the triviality of the resulting knot or link. The crafting constraints of amimonoes, with only local manipulation of loops, make their structure sensitive to topological defects, which, in a sense, locally invert the crafting process. Hence, if defects can propagate, they will cover the entire torus and disentangle the yarns. A textile is then knittable if its defect-

infused diagram folded onto a torus results in a trivial knot or link. Furthermore, we introduce two criteria to distinguish whether defects propagate along columns or rows by adding a slight twist to the torus. The two criteria enable refined classification between *knit* and *crochet* textiles, which share the local manipulation feature but differ in the loop entanglement method.

This work opens both fundamental questions and practical applications. The assessment of defect propagation is currently performed heuristically, but using geometric or combinatorial methods to automatically establish propagation patterns would enable handling arbitrarily complex textiles with large unit cells. Additionally, the pattern of defects infused into the swatch is chosen such that standard amimono diagrams are deemed knittable. Yet an interesting direction is to determine the minimal set of defects. We utilized the propagation set only to judge knittability based on the criteria, but it also enables finer classification and the extraction of deep topological features of textiles. We also demonstrated that the breakdown of a textile is tunable by designing the defect propagation pattern, paving the road for failure-aware functional fabrics. Ultimately, the framework we built reveals surprising analogies between textiles and other physical systems, including marginally constrained structures and kinetically constrained models. Exploring these relations further may help us understand how the topology influences the physics and mechanics of entangled materials with, beyond textiles, applications in soft robotics, polymer chemistry, and biological macromolecules.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: EQUIVALENCE OF UNIT CELLS

For the textile diagram of warp knit shown in Fig. 12, we highlight two possible unit cells, A and B, that both satisfy the three conditions. They are linked by an area-preserving shear transformation (full Dehn twist along a longitude of

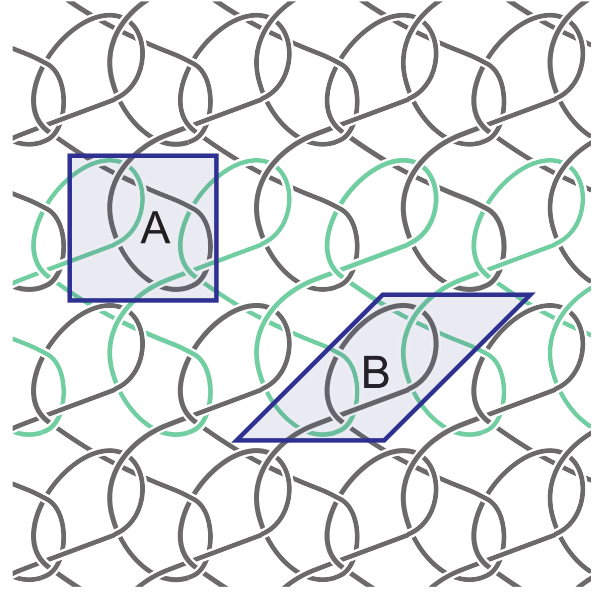


FIG. 12. Both unit cells A and B satisfy the conditions (I)–(III) for a unit cell, but they generate different knot K'_0 when assessing knittability criterion II.

the torus). The knot generated in the criterion II, K'_0 , will depend on whether unit cell A or B is chosen.

APPENDIX B: REIDEMEISTER MOVES SEQUENCES FOR DEFECT PROPAGATION

For the pattern examples shown in Figs. 7(b) and 7(c), we demonstrate the sequences of Reidemeister moves that constitute their respective propagation pattern (Fig. 14).

APPENDIX C: SELF-DISENTANGLING DEFECT

By connecting consecutive paths with criterion II, simple defects may not be enough to create a free edge, and disentanglement has to also be carried out not only between paths, but also within a path. To do so, we introduce the self-disentangling defect, generated by the following process. We consider a textile diagram that contains at least two

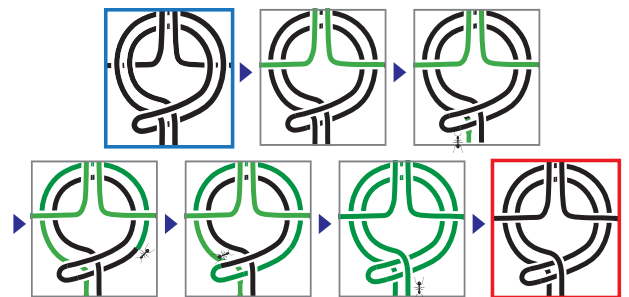


FIG. 13. Detailed steps for the generation of a self-disentangling defect. First, the two paths are disentangled as for a simple defect, then the self-crossing path is disentangled by switching the topology depending on the arclength information.

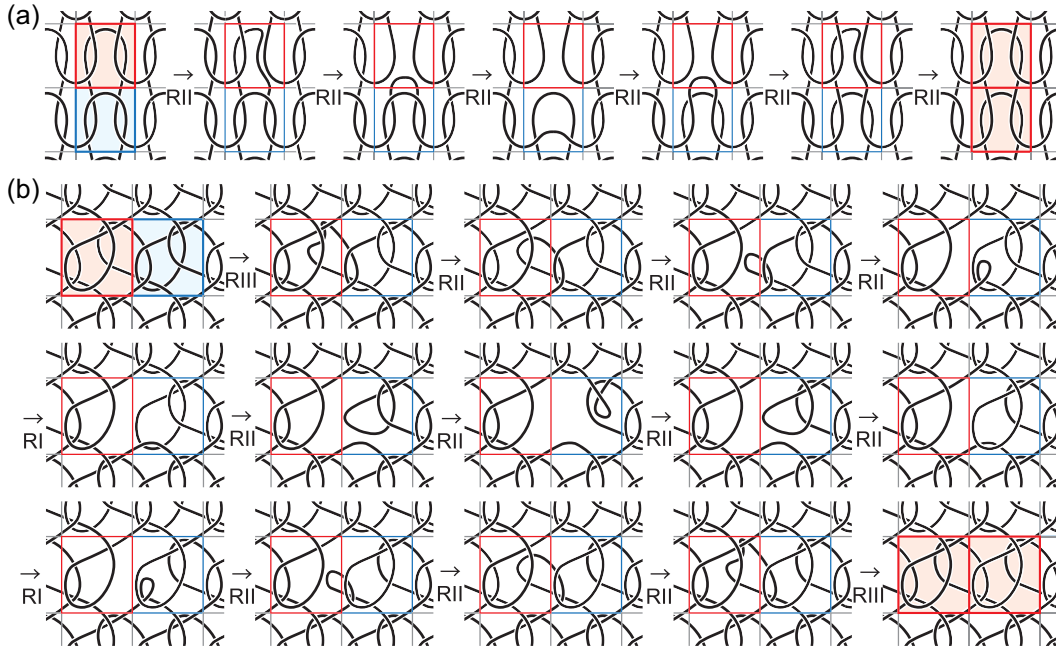


FIG. 14. (a),(b) The sequences of Reidemeister moves to propagate the defects in examples in Figs. 7(b) and 7(c), respectively.

full yarn paths (see Fig. 2 for instance), and we associate to each yarn path a trajectory parametrized by an arclength $t_i \in \mathbb{R}$, with i the index of the yarn path and an arbitrary reference point $t_i = 0$. Each crossing in the diagram is characterized by the indices (c_1, c_2) of the two yarns crossing, the respective arclengths of the crossing along the yarns (t_{c_1}, t_{c_2}) , and the topology of the crossing, which determines whether c_1 is above or below c_2 . A suitable unit cell of this diagram respecting the conditions set earlier (Fig. 6) features only two yarn paths that we denote i and $i + 1$.

To derive the self-disentangling defect of this unit cell, we consider all the crossing points that appear within the unit cell, such that $c_1, c_2 \in \{i, i + 1\}$. The disentangling process starts by following the arclength gradient along the trajectory of the yarn path i , starting from the point with the lowest arclength. Along this path, whenever a crossing is encountered, we update the over-under crossing information based on the crossing's properties. Without loss of generality, we can set $c_1 = i$, and we have $c_2 \in \{i, i + 1\}$ for all the crossings along i . If $c_2 = i + 1 \neq c_1$, two different paths cross, and we flip the topology of the crossing to have systematically $c_2 = i + 1$ above. In the case of a self-crossing $c_2 = i$, c_2 is placed on top only if $t_{c_1} < t_{c_2}$. In other words, the latter portion, with greater arc length, is always on top. Once the highest arclength point is reached for i , the same process is applied to path $i + 1$. With this protocol, the i th path is systematically disentangled from the $(i + 1)$ th path by placing it below at each crossing, as in the case of simple defects. Knots potentially formed by self-crossing are also removed by switching crossings only once. This

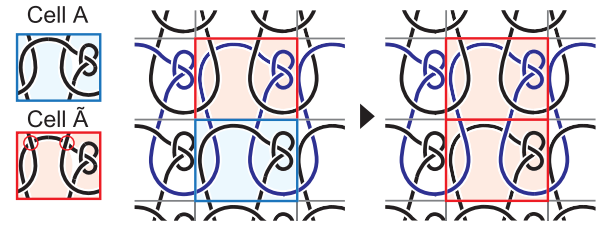


FIG. 15. A unit cell and corresponding simple defect for a jersey knit with a trefoil knot along one of the paths. The defect propagates perpendicular to the yarn as in standard jersey, but the textile is not knittable, as folding the simple defect does not yield a trivial knot.

process is detailed for a unit cell with self-crossing in Fig. 13.

APPENDIX D: TEXTILE WITH A SELF-KNOT

Figure 15 shows a unit cell and its corresponding simple defect of a jersey knit with self-knotting. Simple defects propagate following $\{(1, 0)\}$ as for standard jersey knit. However, the knot obtained by folding the defect on the torus yields a nontrivial trefoil knot. So even if the defect propagates, this textile is not knittable.

- [1] J.E. Avron, R. Seiler, and B. Simon, *Homotopy and quantization in condensed matter physics*, *Phys. Rev. Lett.* **51**, 51 (1983).

- [2] M. Kohmoto, *Topological invariant and the quantization of the Hall conductance*, *Ann. Phys. (N.Y.)* **160**, 343 (1985).
- [3] C. L. Kane and E. J. Mele, Z_2 topological order and the quantum spin Hall effect, *Phys. Rev. Lett.* **95**, 146802 (2005).
- [4] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, *Topological phases of non-Hermitian systems*, *Phys. Rev. X* **8**, 031079 (2018).
- [5] C. F. Barenghi, *Knots and unknots in superfluid turbulence*, *Milan J. Math.* **75**, 177 (2007).
- [6] N. D. Mermin, *The topological theory of defects in ordered media*, *Rev. Mod. Phys.* **51**, 591 (1979).
- [7] R. D. Kamien, *The geometry of soft materials: A primer*, *Rev. Mod. Phys.* **74**, 953 (2002).
- [8] G. M. Grason, *Perspective: Geometrically frustrated assemblies*, *J. Chem. Phys.* **145** (2016).
- [9] V. P. Patil, J. D. Sandt, M. Kolle, and J. Dunkel, *Topological mechanics of knots and tangles*, *Science* **367**, 71 (2020).
- [10] L. Tubiana, G. P. Alexander, A. Barbensi, D. Buck, J. H. Cartwright, M. Chwastyk, M. Cieplak, I. Coluzza, S. Čopar, D. J. Craik *et al.*, *Topology in soft and biological matter*, *Phys. Rep.* **1075**, 1 (2024).
- [11] M. Kapnistos, M. Lang, D. Vlassopoulos, W. Pyckhout-Hintzen, D. Richter, D. Cho, T. Chang, and M. Rubinstein, *Unexpected power-law stress relaxation of entangled ring polymers*, *Nat. Mater.* **7**, 997 (2008).
- [12] D. Parisi, S. Costanzo, Y. Jeong, J. Ahn, T. Chang, D. Vlassopoulos, J. D. Halverson, K. Kremer, T. Ge, M. Rubinstein *et al.*, *Nonlinear shear rheology of entangled polymer rings*, *Macromolecules* **54**, 2811 (2021).
- [13] S. Isnard and W. K. Silk, *Moving with climbing plants from Charles Darwin's time into the 21st century*, *Am. J. Bot.* **96**, 1205 (2009).
- [14] K. Shimokawa, K. Ishihara, I. Grainge, D. J. Sherratt, and M. Vazquez, *FtsK-dependent XerCD-dif recombination unlinks replication catenanes in a stepwise manner*, *Proc. Natl. Acad. Sci. U.S.A.* **110**, 20906 (2013).
- [15] C. L. Kane and T. C. Lubensky, *Topological boundary modes in isostatic lattices*, *Nat. Phys.* **10**, 39 (2014).
- [16] X. Mao and T. C. Lubensky, *Maxwell lattices and topological mechanics*, *Annu. Rev. Condens. Matter Phys.* **9**, 413 (2018).
- [17] B. G. G. Chen, B. Liu, A. A. Evans, J. Paulose, I. Cohen, V. Vitelli, and C. D. Santangelo, *Topological mechanics of origami and kirigami*, *Phys. Rev. Lett.* **116**, 135501 (2016).
- [18] S. Grishanov, V. Meshkov, and A. Omelchenko, *A topological study of textile structures. Part I: An introduction to topological methods*, *Textile research Journal : publication of Textile Research Institute, Inc and the Textile Foundation* **79**, 702 (2009).
- [19] S. Grishanov, V. Meshkov, and A. Omelchenko, *A topological study of textile structures. Part II: Topological invariants in application to textile structures*, *Textile research Journal : publication of Textile Research Institute, Inc and the Textile Foundation* **79**, 822 (2009).
- [20] Y. Kyosev, *Topology-Based Modeling of Textile Structures and Their Joint Assemblies* (Springer, New York, 2019).
- [21] L. Kapllani, C. Amanatides, G. Dion, V. Shapiro, and D. E. Breen, *Topoknit: A process-oriented representation for modeling the topology of yarns in weft-knitted textiles*, *Graph. Models* **118**, 101114 (2021).
- [22] I. Emery, *The Primary Structures of Fabrics: An Illustrated Classification* (Whitney Library of Design, Washington, 1995).
- [23] A. Seiler-Baldinger, *Textiles: A Classification of Techniques* (Smithsonian Institution Press, Washington, 1994).
- [24] I. Good, *Archaeological textiles: A review of current research*, *Annu. Rev. Anthropol.* **30**, 209 (2001).
- [25] S. Maity, S. Rana, P. Pandit, and K. Singha, *Advanced Knitting Technology* (Woodhead Publishing, Cambridge, 2021).
- [26] M. Friedman, *On Joachim Jungius' Texturæ Contemplatio* (Springer, New York, 2023).
- [27] S. Mahmoudi, *On the classification of periodic weaves and universal cover of links in thickened surfaces*, *Commun. Korean Math. Soc.* **39**, 997 (2024).
- [28] S. G. Markande and E. Matsumoto, *Knotty knits are tangles in tori*, in *Proceedings of Bridges 2020: Mathematics, Art, Music, Architecture, Education, Culture* (2020), pp. 103–112, <http://archive.bridgesmathart.org/2020/bridges2020-103.html>.
- [29] M. Fukuda, M. Kotani, and S. Mahmoudi, *Classification of doubly periodic untwisted (p, q) -weaves by their crossing number and matrices*, *J. Knot Theory Ramif.* **32**, 2350032 (2023).
- [30] S. Mahmoudi, R. Rimerman, R. D. Kamien, and G. Dion, *Modelling knitting using knit mosaics* (to be published).
- [31] M. Kuzbary, S. G. Markande, E. A. Matsumoto, and S. Pritchard, *A study of 2-periodic weft-knitted textiles using the theory of knots and links*, [arXiv:2510.08384](https://arxiv.org/abs/2510.08384).
- [32] C. Adams, *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots* (American Mathematical Society, Providence, RI, 2004).
- [33] C. Karp, *Defining crochet*, *Text. Hist.* **49**, 208 (2018).
- [34] J. L. Storck, D. Gerber, L. Steenbock, and Y. Kyosev, *Topology based modelling of crochet structures*, *J. Ind. Text.* **52** (2022).
- [35] K. Bertoldi, V. Vitelli, J. Christensen, and M. Van Hecke, *Flexible mechanical metamaterials*, *Nat. Rev. Mater.* **2**, 1 (2017).
- [36] D. Liu, D. Christe, B. Shakibajahromi, C. Knittel, N. Castaneda, D. Breen, G. Dion, and A. Kontsos, *On the role of material architecture in the mechanical behavior of knitted textiles*, *Int. J. Solids Struct.* **109**, 101 (2017).
- [37] S. Poincloux, M. Adda-Bedia, and F. Lechenault, *Geometry and elasticity of a knitted fabric*, *Phys. Rev. X* **8**, 021075 (2018).
- [38] K. Singal, M. S. Dimitriyev, S. E. Gonzalez, A. P. Cachine, S. Quinn, and E. A. Matsumoto, *Programming mechanics in knitted materials, stitch by stitch*, *Nat. Commun.* **15**, 2622 (2024).
- [39] J. Crassous, S. Poincloux, and A. Steinberger, *Metastability of a periodic network of threads: Shapes of a knitted fabric*, *Phys. Rev. Lett.* **133**, 248201 (2024).
- [40] L. K. Maharaj, C. Amanatides, G. Dion, V. Shapiro, and D. E. Breen, *Topology, integrity, and stability analysis of weft-knitted textiles*, *Text. Res. J.* **95**, 2377 (2025).
- [41] S.-M. Belcastro, *Every topological surface can be knit: A proof*, *J. Math. Arts* **3**, 67 (2009).

- [42] C. Yuksel, J. M. Kaldor, D. L. James, and S. Marschner, *Stitch meshes for modeling knitted clothing with yarn-level detail*, *ACM Trans. Graph.* **31**, 1 (2012).
- [43] M. Chas, *Crochet topology*, *AMS Feature Column*, May (2018).
- [44] C. E. Knittel, M. Tanis, A. L. Stoltzfus, T. Castle, R. D. Kamien, and G. Dion, *Modelling textile structures using bicontinuous surfaces*, *J. Math. Arts* **14**, 331 (2020).
- [45] A. Kurbak and O. Ekmen, *Basic studies for modeling complex weft knitted fabric structures Part I: A geometrical model for widthwise curlings of plain knitted fabrics*, *Text. Res. J.* **78**, 198 (2008).
- [46] K. Tajiri, R. Murakami, S. Kobayashi, R. Tarumi, and T. G. Sano, *Curling morphology of knitted fabrics: Structure and mechanics*, *Extreme Mech. Lett.* **76**, 102300 (2025).
- [47] L. Niu, G. Dion, and R. D. Kamien, *Geometric modeling of knitted fabrics*, *Proc. Natl. Acad. Sci. U.S.A.* **122**, e2416536122 (2025).
- [48] M. Meißner and B. Eberhardt, *The art of knitted fabrics, realistic & physically based modelling of knitted patterns*, in *Computer Graphics Forum* (Wiley Online Library, 1998), Vol. 17, pp. 355–362.
- [49] M. Bergou, M. Wardetzky, S. Robinson, B. Audoly, and E. Grinspun, *Discrete elastic rods*, in *ACM Trans. Graph.* **27**, 1 (2008).
- [50] J. M. Kaldor, D. L. James, and S. Marschner, *Simulating knitted cloth at the yarn level*, *ACM Trans. Graph.* **27**, 1 (2008).
- [51] G. Cirio, J. Lopez-Moreno, and M. A. Otaduy, *Yarn-level cloth simulation with sliding persistent contacts*, *IEEE Trans. Visualization Comput. Graph.* **23**, 1152 (2016).
- [52] G. Sperl, R. M. Sánchez-Banderas, M. Li, C. Wojtan, and M. A. Otaduy, *Estimation of yarn-level simulation models for production fabrics*, *ACM Trans. Graph.* **41**, 1 (2022).
- [53] X. Ding, V. Sanchez, K. Bertoldi, and C. H. Rycroft, *Unravelling the mechanics of knitted fabrics through hierarchical geometric representation*, *Proc. R. Soc. A* **480**, 20230753 (2024).
- [54] J. McCann, L. Albaugh, V. Narayanan, A. Grow, W. Matusik, J. Mankoff, and J. Hodgins, *A compiler for 3D machine knitting*, *ACM Trans. Graph.* **35**, 1 (2016).
- [55] V. Narayanan, L. Albaugh, J. Hodgins, S. Coros, and J. McCann, *Automatic machine knitting of 3D meshes*, *ACM Trans. Graph.* **37**, 1 (2018).
- [56] K. Wu, H. Swan, and C. Yuksel, *Knittable stitch meshes*, *ACM Trans. Graph.* **38**, 1 (2019).
- [57] V. Narayanan, K. Wu, C. Yuksel, and J. McCann, *Visual knitting machine programming*, *ACM Trans. Graph.* **38**, 1 (2019).
- [58] R. Rutt, *A History of Hand Knitting* (Batsford Books, London, 1987).
- [59] D. Taimina, *Crocheting Adventures with Hyperbolic Planes: Tactile Mathematics, Art and Craft for All to Explore* (CRC Press, Boca Raton, 2018).
- [60] A. Maziz, A. Concas, A. Khaldi, J. Stålhand, N.-K. Persson, and E. W. Jager, *Knitting and weaving artificial muscles*, *Sci. Adv.* **3**, e1600327 (2017).
- [61] J. Abel, J. Luntz, and D. Brei, *Hierarchical architecture of active knits*, *Smart Mater. Struct.* **22**, 125001 (2013).
- [62] I. Diamantis, S. Lambropoulou, and S. Mahmoudi, *Equivalence of doubly periodic tangles*, *Mathematics* **14**, 1071 (2026).
- [63] A. Kawauchi, *Complexities of a knitting pattern*, *React. Funct. Polym.* **131**, 230 (2018).
- [64] M. Bright and V. Kurlin, *Encoding and topological computation on textile structures*, *Comput. Graph.* **90**, 51 (2020).
- [65] D. Schattschneider, *The plane symmetry groups: Their recognition and notation*, *Am. Math. Mon.* **85**, 439 (1978).
- [66] T. Andriamanalina, S. Mahmoudi, and M. E. Evans, *The untangling number of 3-periodic tangles*, *arXiv:2504.01747*.
- [67] W. Haken, *Theorie der normalflächen*, *Acta Math.* **105**, 245 (1961).
- [68] J. Hass, J. C. Lagarias, and N. Pippenger, *The computational complexity of knot and link problems*, *J. Assoc. Comput. Mach.* **46**, 185 (1999).
- [69] M. Khovanov, *A categorification of the Jones polynomial*, *Duke Math. J.* **101**, 359 (2000).
- [70] D. Bar-Natan, *Khovanov's homology for tangles and cobordisms*, *Geom. Topol.* **9**, 1443 (2005).
- [71] P. B. Kronheimer and T. S. Mrowka, *Khovanov homology is an unknot-detector*, *Publ. Math. IHÉS* **113**, 97 (2011).
- [72] Y. Kotorii, S. Mahmoudi, E. A. Matsumoto, and K. Yoshida, *On the isotopies of tangles in periodic 3-manifolds using finite covers*, *arXiv:2505.20940*.
- [73] V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, *Bull. Am. Math. Soc.* **12**, 103 (1985).
- [74] K. Miller, *Knotfolio* (2019), <https://knotfolio.io/>.
- [75] J. Paulose, B. G.-g. Chen, and V. Vitelli, *Topological modes bound to dislocations in mechanical metamaterials*, *Nat. Phys.* **11**, 153 (2015).
- [76] C. S. O'Hern, L. E. Silbert, A. J. Liu, and S. R. Nagel, *Jamming at zero temperature and zero applied stress: The epitome of disorder*, *Phys. Rev. E* **68**, 011306 (2003).
- [77] L. Yan and D. Bi, *Multicellular rosettes drive fluid-solid transition in epithelial tissues*, *Phys. Rev. X* **9**, 011029 (2019).
- [78] H. Obata and Y.-j. Lee, *Nets hidden in pottery: Resurrected fishing nets in the Jomon period, Japan*, *J. Archaeol. Sci.* **179**, 106231 (2025).
- [79] R. S. Forgan, J.-P. Sauvage, and J. F. Stoddart, *Chemical topology: Complex molecular knots, links, and entanglements*, *Chem. Rev.* **111**, 5434 (2011).
- [80] D. P. August, R. A. Dryfe, S. J. Haigh, P. R. Kent, D. A. Leigh, J.-F. Lemonnier, Z. Li, C. A. Muryn, L. I. Palmer, Y. Song et al., *Self-assembly of a layered two-dimensional molecularly woven fabric*, *Nature (London)* **588**, 429 (2020).
- [81] J. P. Garrahan, P. Sollich, and C. Toninelli, *Kinetically constrained models*, *arXiv:1009.6113*.
- [82] J. Jäckle and S. Eisinger, *A hierarchically constrained kinetic Ising model*, *Z. Phys. B* **84**, 115 (1991).
- [83] P. Sollich and M. R. Evans, *Glassy time-scale divergence and anomalous coarsening in a kinetically constrained spin chain*, *Phys. Rev. Lett.* **83**, 3238 (1999).