

Clifford Algebras and Liquid Crystalline Fermions

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We show that Clifford algebras provide a natural language to describe the physics of liquid crystal defects in 3D. This framework shows that most of these defects have fermionic nature, as the director field profile on a 2D cross section can algebraically be represented by a spinor. Defects in uniaxial, biaxial nematics, and cholesterics are represented by elements belonging to different Clifford algebras, suggesting that there are fundamental distinctions between topological defects in each of these phases. Our theory allows nematic defects to be interpreted as Majorana-like spinors, as defects and antidefects are topologically equivalent, while some cholesteric defects, such as screw dislocations, are better viewed as Weyl-like spinors of well-defined chirality. Defects can be described by a “defect bivector,” an algebraic element that quantifies the roto-translation associated with them. In cholesterics, fermionic defects of different types can combine to yield composite quasiparticles with either fermionic or bosonic nature. Under cylindrical confinement, these quasiparticles provide the way to understand the structure of screw dislocations. In the bulk, they may condensate to form topological phases, such as blue phases or skyrmion lattices. Our results provide a surprising link between liquid crystals, particle physics, and topological quantum matter.

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I. INTRODUCTION

In addition to being of considerable importance in technological applications, liquid crystals provide a fascinating and fertile playground for practical physical applications of abstract ideas from topology and geometry [1,2]. This is particularly true of defects, which are characterized in terms of mathematical groups from homotopy theory [3] and give, in turn, rise to quasiparticles and topological phases, such as skyrmions, hopfions, and blue phases in 3D cholesterics [1,4–12].

Topological theories of liquid crystals rely on specifying an underlying disclination algebra, which associates different algebraic elements to each defect. While this formalism disregards energetic considerations related to liquid crystalline elasticity, it nevertheless proves useful to predict the

result of a combination of—or collision between—defects, as this can be done by multiplying the algebraic elements corresponding to the combining defects. The relevant algebra for defects in biaxial nematics, cholesterics, and smectics is normally taken to be that of quaternions in all cases. Quaternions allow a compact way to describe 3D rotations, and their algebra also provides a powerful mathematical tool to characterize 3D disclination loops in nematics—uniaxial and biaxial alike [13,14].

An outstanding puzzling observation in the topological theory of liquid crystals is that applying algebraic rules to the composition of cholesteric defects and disclinations gives results that are difficult to reconcile with observations from numerical simulations and experiments [15]. The standard theory states that there are three types of cholesteric defects in 3D, which are singular for two fields out of the triad made up by the director field, the helical axis, and the normal to the two. These are typically associated with the three quaternions i , j , and k , as they also constitute the first homotopy group of both biaxial nematics and cholesterics [3]. While some of the algebraic quaternion rules account for physically observed phenomena—for

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example, the formation of an edge dislocation as a composite defect, which corresponds to the quaternion formula $ij = k$ —others predict equivalences that are not observed in practice. For instance, because $i^2 = j^2 = k^2 = -1$, one would be led to predict that different pairs of singularities of the same type should be equivalent in cholesterics, as they are in biaxial nematics. Instead, as discussed in Ref. [15], experiments and numerical simulations show that a χ^{+1} dislocation cannot be smoothly transformed into a λ^{+1} disclination, and the conversion between the two requires the creation of an additional array of disclination loops. As a consequence, we arrive at a paradoxical conclusion, as the existence of three distinct axes in cholesterics likens them to biaxial nematics, whereas the quaternion algebra describing 3D rotations works well for biaxial nematics, but only partially for cholesterics [3,15].

Here, we introduce a different way to study the algebraic properties of 3D liquid crystals, based on Clifford algebras, which are sets of mathematical elements traditionally used in particle physics, for instance, to derive the Dirac equation [16,17]. Clifford algebras are useful in our present context because they provide a general framework to describe any combinations of 3D rotation and translations, which are required to characterize the geometry of disclinations and dislocations in liquid crystals. We shall show that the Clifford algebras needed to describe uniaxial nematics, biaxial nematics, and cholesterics are different, and we shall propose an identification of defects in each case with elements of the algebra. Most notably, quaternions are suitable to describe disclinations in nematics algebraically, but they are not sufficient to characterize cholesteric defects, where translations along the helical axis play a role, besides 3D rotations. The algebra required for cholesterics is a subset of the geometric projective algebra, which can be described by dual quaternions [18]. These have double the degrees of freedom of traditional quaternions, to account for three translational degrees of freedom, in addition to the three rotational ones associated with usual quaternions.

The algebraic description of 3D liquid crystalline defects we propose provides a natural way to resolve the cholesteric puzzle outlined above. The framework we develop additionally uncovers a fundamental link between liquid crystal physics and particle physics, as the defect profiles behave as spinors; hence, these topological excitations can be likened to fermions. More specifically, we will show that different defects in nematics and cholesterics can be identified as Majorana and Weyl spinors. Additionally, Clifford algebras naturally describe 3D defects in cholesterics as a combination of dual quaternions encoding both rotations and translations, in a way that formally mirrors the way magnetic and electric fields are represented by the electromagnetic tensor [19].

Finally, the intrinsic chirality of cholesterics, together with the non-Abelian nature of dual quaternions, allows

composite quasiparticles to be formed in these materials. The resulting quasiparticles can be of either fermionic or bosonic nature and are important to understand the emergent phase behavior in chiral liquid crystals. Specifically, they provide the basis for the structure of edge and screw dislocations, as well as the key to understand the formation of topological phases such as blue phases or skyrmion lattices. In the latter case, the resulting composite quasiparticles have bosonic nature and can accumulate, in a way that is reminiscent of that in which Cooper pairs condense in superconductors.

It is important to highlight that the topological analysis we propose applies to cholesterics in which the length scale of deformations imposed on the natural helicoidal structure is on the order of the cholesteric pitch. Defects such as disclinations and dislocations considered in this paper are examples of such “weakly” distorted cholesterics. In contrast, when the scale of deformations is much larger than the pitch, the elastic properties of cholesterics are akin to those of lamellar liquid crystals with equidistant phase surfaces. Strong departures from the equilibrium pitch and thus continuous deformations used in topological description are not allowed. As a result, large-scale defects such as focal conics in cholesterics are not the subject of our consideration.

II. CLIFFORD ALGEBRA OF UNIAXIAL PLANAR NEMATIC SPINORS

We start by studying the algebra of disclinations in 3D uniaxial nematics, focusing on the case where local defect profiles are planar, as considered in Ref. [20]. As we shall see, the underlying Clifford algebra in this case can be chosen as $\text{Cl}(2, 0)$, with generators

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (1)$$

The algebra is such that the generators anticommute and have square equal to unity [16,17],

$$e_1^2 = e_2^2 = 1, \quad e_1 e_2 + e_2 e_1 = 0. \quad (2)$$

The product of the two generators gives

$$e_{12} = -e_{21} = e_1 e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (3)$$

and $e_{12}^2 = -1$. This Clifford algebra, therefore, contains a scalar (1, the 2×2 identity matrix), two vectors (the generators), and a bivector (e_{12} , which is also a pseudo-scalar). The representation we have chosen is purely real, and the elements of the algebra can be mapped (via a bijection) to all 2×2 real matrices.

Introducing the Pauli matrices σ_x , σ_y , and σ_z , we can identify $e_1 = \sigma_z$, $e_2 = \sigma_x$, and $e_{12} = i\sigma_y$.

An important fact to note is that the local nematic director profile corresponding to a defect—which we shall refer to as the “local defect profile”—can be viewed as a spinor. To see why, we denote by $\hat{e}_x$ and $\hat{e}_y$ the two linearly independent unit vectors in the plane perpendicular to the disclination line which contains the defect under consideration. For a planar defect profile (Fig. 1), the director field can then be expressed as a two-component vector in the $(\hat{e}_x, \hat{e}_y)$ plane as

$$\mathbf{n} = \begin{pmatrix} \cos(s\phi + \phi_0) \\ \sin(s\phi + \phi_0) \end{pmatrix}. \quad (4)$$

The representation in Eq. (4) as a 2×1 column vector is already suggestive of the defect profile being a spinor. To understand this more in depth, let us consider for concreteness a specific reference profile, a $-1/2$ defect with $\phi_0 = 0$, which we shall denote $\mathbf{n}_0$. In $\text{Cl}(2, 0)$, an analogous quantity is the element $\mathbf{N}_0$ [20],

$$\begin{aligned} \mathbf{N}_0 &= \begin{pmatrix} \cos(\phi/2) & \sin(\phi/2) \\ -\sin(\phi/2) & \cos(\phi/2) \end{pmatrix} \\ &= \cos(\phi/2)\mathbb{1} + \sin(\phi/2)e_{12}, \end{aligned} \quad (5)$$

which can be readily recognized as a rotation matrix (with the angle $-\phi/2$). The defect profile corresponding to the $-1/2$ triradius defect can be identified with either of the column vectors in Eq. (5), as these correspond to $s = -1/2$ defect profiles with $\phi_0 = 0$ and $\phi_0 = \pi/2$, respectively. The $+1/2$ defect profile is instead given by Eq. (5) with $\phi \rightarrow -\phi$.

To show that Eq. (5) describes a spinor, one can see that left multiplication by the $\text{SO}(2)$ matrix

$$R(\gamma) = \cos(\gamma/2)\mathbb{1} - \sin(\gamma/2)e_{12} \quad (6)$$

rotates the profile by $\gamma/2$ —i.e., sends $-\phi \rightarrow -\phi + \gamma$ in Eq. (5). (Equivalently, it sends $\phi \rightarrow \phi + \gamma$ for the comet profile.) This corresponds to the physically well-known fact that the defect profile needs to rotate by 4π to return to itself, which is expected of a spinor. Alternatively, we may observe that projection from the right by the operator

$$P = \frac{\mathbb{1} + e_1}{2} \quad (7)$$

gives the algebraic element $\mathbf{N}_0 P$, which still encodes enough information to describe the triradius profile, while being a minimal left ideal, which represents the mathematical definition of a spinor in a Clifford algebra [16].

It is useful for the more complex cases to be discussed later to note that the spinor represented by Eq. (5) involves

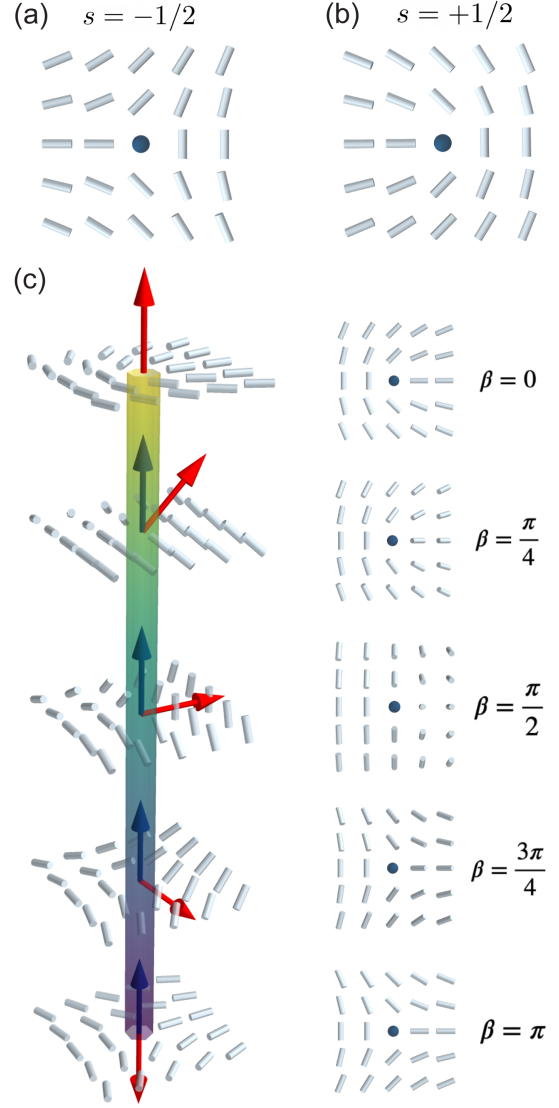


FIG. 1. Planar defect profiles in uniaxial nematics. (a) Triradius profile, with topological charge $s = -1/2$. (b) Comet profile, with topological charge $s = +1/2$. The Clifford algebra $\text{Cl}(2, 0)$ elements representing these profiles are the bivectors e_{12} (a) and $-e_{12} = e_{21}$ (b). Examples of in-plane and out-of-plane director field pattern about a disclination line (c), showing the disclination tangent unit vector $\hat{\mathbf{T}}$ and the defect profile rotation vector $\hat{\mathbf{\Omega}}$, in blue and red, respectively. (c) Defect profiles interpolating from a $-1/2$ (bottom, $\beta = \pi$) to a $+1/2$ (top, $\beta = -\pi$) local profile, where $\cos(\beta) = \hat{\mathbf{T}} \cdot \hat{\mathbf{\Omega}}$. The profiles with $-\pi < \beta < \pi$ are all nonplanar and include twist as well as splay and bend deformations. The Clifford algebra element corresponding to any defect profile in (c) is the defect bivector Δ , given in Eq. (8) in terms of $\hat{\mathbf{\Omega}}$ and the bivectors in $\text{Cl}(3, 0)$.

only even elements in the Clifford algebra—i.e., products of an even number of vectors, either 0 or 2 in this case. The even elements 1 and e_{12} generate a subalgebra of $\text{Cl}(2, 0)$, which we denote, following normal convention, as $\text{Cl}(2, 0)^{[0]}$ —in the case under consideration, this subalgebra is equivalent to the $\text{spin}(2)$ group. It is not a

surprise that spinors reside here, as this is in line with the generic algebraic construction of spinors in terms of bivectors [21,22].

While Eq. (5) describes the triradius profile locally in each point in the plane, a global description of a triradius can be given as a director rotation of $-\pi$, so that we simply associate it with the element $e_{12} = -e_{21}$. This simpler representation is useful for our purposes, as it can be generalized more easily and leads to a simpler algebra for liquid crystalline disclinations. It should also be observed that e_{12} is the (single) generator of the Lie algebra $\mathfrak{so}(2)$, which generates the $SO(2)$ group associated with the previous representation, Eq. (5).

Interpreting e_{12} as the triradius defect profile clarifies a few key properties. First, the “antidefect” of the triradius, which annihilates it to leave a nematic background free of defects, is the comet defect ($s = +1/2$). Analogous considerations to those above suggest that the comet defect should be represented by $e_{21} = -e_{12}$ (as this would generate a π rotation of the director profile). Algebraically, $e_{21}e_{12} = \mathbb{1}$, in line with the standard interpretation of the two defects as “antiparticles” of each other. Second, a spinor is defined up to a field, e_{12} and $e_{21} = -e_{12}$ are algebraically equivalent—i.e., they describe the same spinor—so this representation naturally incorporates the well-known equivalence between $+1/2$ and $-1/2$ defects in 3D, as rotation of the director field out of the plane can map the two defects onto each other [3]. Third, a direct consequence of these two properties is that the uniaxial nematic defect (e_{12}) behaves as a Majorana spinor because it is equivalent to its antiparticle, as discussed in Ref. [20]. In the bivector representation, the square of the Clifford algebra element corresponding to a $\pm 1/2$ defect profile squares to $-\mathbb{1}$ (as $e_{12}^2 = -\mathbb{1}$): As we shall see, this is a signature of the fermionic (spinor) nature of such a defect profile when viewed as a quasiparticle.

III. OUT-OF-PLANE PROFILES AND THE DEFECT BIVECTOR

In the previous section, we considered a planar defect profile, where the director did not get out of the $x - y$ plane (perpendicular to the local tangent of the disclination). We discuss here how the resulting spinor representation of defects in the Clifford algebra—which is simply $\pm e_{21}$ for planar profiles of topological charge $\pm 1/2$ —can be generalized for arbitrary out-of-plane profiles [Fig. 1(c)].

A generic out-of-plane profile may be written as a rotation (of π) around a generic axis (unit vector) with polar angles (α, β) , or $\hat{\Omega} = (\hat{\Omega}_1, \hat{\Omega}_2, \hat{\Omega}_3) = (\sin(\beta) \cos(\alpha), \sin(\beta) \sin(\alpha), \cos(\beta))$. To represent this out-of-plane defect profile, we need a larger algebra than $Cl(2, 0)$, because we need to describe rotations around an arbitrary axis. This can be achieved by using $Cl(3, 0)$, which is, for instance, the

algebra generated by the Pauli matrices, with $e_1 = \sigma_x$, $e_2 = \sigma_y$, and $e_3 = \sigma_z$. The Clifford algebra element corresponding to the defect is then

$$\Delta = \hat{\Omega}_1 e_{32} + \hat{\Omega}_2 e_{13} + \hat{\Omega}_3 e_{21}, \quad (8)$$

where $e_{32} = e_3 e_2$, $e_{13} = e_1 e_3$, and $e_{21} = e_2 e_1$ —note that these provide a representation of the quaternions i, j , and k . We refer to Δ as the local defect bivector. We note that Δ is a proper generalization of the $\pm e_{21}$ element representing planar defects, as it has the two following key analogous properties: (i) It squares to $-\mathbb{1}$, and (ii) its antidefect is $-\Delta$, which is equivalent to itself—as spinors are defined up to a minus sign. Therefore, this defect is still a Majorana-like spinor, as for planar defect profiles in $Cl(2, 0)$.

The vector $\hat{\Omega}$ can be constructed by starting from the disclination tensor [23],

$$D_{ij} = \epsilon_{i\mu\nu} \epsilon_{jlk} \partial_l Q_{\mu\alpha} \partial_k Q_{\nu\alpha}, \quad (9)$$

where $i, j, k, \alpha, \mu, \nu$ are tensor indices and where the Einstein summation convention of repeated indices has been used. The $\mathbf{Q}$ tensor Q_{ij} is a traceless and symmetric tensor that describes the orientational order in the liquid crystal and whose largest eigenvalue determines the average direction of order [24]. For uniaxial liquid crystals in 3D, the $\mathbf{Q}$ tensor is given in terms of the director profile $\mathbf{n}$ as

$$Q_{ij} = q \left(n_i n_j - \frac{\delta_{ij}}{3} \right), \quad (10)$$

where q is the (local) magnitude of order.

Through singular value decomposition, the disclination tensor D_{ij} can be written as

$$D_{ij} = s(\mathbf{r}) \hat{\Omega}_i \hat{T}_j, \quad (11)$$

where $s(\mathbf{r})$ is a positive scalar field that is maximum at the disclination core, equal to the square of the Frobenius norm of D_{ij} , and $\hat{\mathbf{T}}$ is the local tangent to the disclination line. Therefore, from Q_{ij} and D_{ij} , we can reconstruct $\hat{\Omega}$ and hence the defect bivector.

We note that Ref. [25] gave a related spinor representation of out-of-plane director field profiles, in terms of $SU(2)$ Pauli matrices. While the two representations are, in practice, equivalent (as there is a bijection between them), there is a formal difference in them that is important for our treatment. The representation in Ref. [25] is given in terms of the Pauli matrices or, equivalently, the vectors of $Cl(3, 0)$, whereas the one we propose uses the bivectors (e_{32}, e_{13}, e_{21}). In other words, that representation lies in the odd part of that algebra, or $Cl(3, 0)^{[1]}$, rather than in

$\text{Cl}(3,0)^{[0]}$ [26]. The representation in Ref. [25] revealed an interesting mapping between defect profiles and qubits in topological quantum matter. However, the fact that the framework does not use bivectors renders it less amenable to generalizations within our method. We shall now discuss how defects in biaxial nematics and cholesterics can be represented by algebraic spinors in suitable Clifford algebras.

IV. ALGEBRAIC SPINORS FOR 3D BIAxIAL NEMATICS

We first generalize our spinor construction to biaxial nematics. Like uniaxial liquid crystals, orientational ordering in biaxial nematics is also described by the $\mathbf{Q}$ tensor. However, while for uniaxial liquid crystals two eigenvalues of the $\mathbf{Q}$ tensor are degenerate, in a biaxial nematic all three eigenvalues are nondegenerate, and correspondingly, the phase can exhibit order along two different directions.

Defect profiles in biaxial nematics may be of three types [3,27,28]. These are usually called λ , χ , and τ , and we shall follow this convention here. This classification corresponds to a local 2D profile (Fig. 2) or an infinite straight disclination line with translational invariance along it. The symbol λ denotes the primary director field (the eigenvector corresponding to the largest eigenvalue of $\mathbf{Q}$), χ denotes the secondary director field axis (the eigenvector corresponding to the middle eigenvalue), and τ is the remaining perpendicular direction. Defects of type λ have a singularity in the orientation of the secondary director field and in the direction τ : The primary director field (direction λ) is nonsingular. Similarly, defects of type τ are nonsingular for the τ direction and have a singularity in the primary and secondary director fields. Finally, defects of type χ have a singularity in the τ and λ directions.

Sticking to planar defect profiles, a popular algebraic representation in the literature [28] is in terms of quaternions, respectively, $\lambda \leftrightarrow i$, $\tau \leftrightarrow j$, and $\chi \leftrightarrow k$. To obtain a representation in $\text{Cl}(3,0)$, we start from the three generators e_a , e_b , and e_c . These Latin indices refer to the three nonequivalent axes of the biaxial molecules, or orientational order— a is the direction of the long axis or primary director field, b of the middle axis or secondary director field, and c is that of the small axis or the remaining normal direction. We stress that these indices are different from the numerical indices previously used, which in our notation refer to Cartesian coordinates. The defect profiles can then be represented algebraically as follows:

$$\begin{aligned} \lambda &\leftrightarrow e_{cb} = e_c e_b = -e_b e_c, \\ \tau &\leftrightarrow e_{ba} = e_b e_a, \\ \chi &\leftrightarrow e_{ca} = e_c e_a, \end{aligned} \quad (12)$$

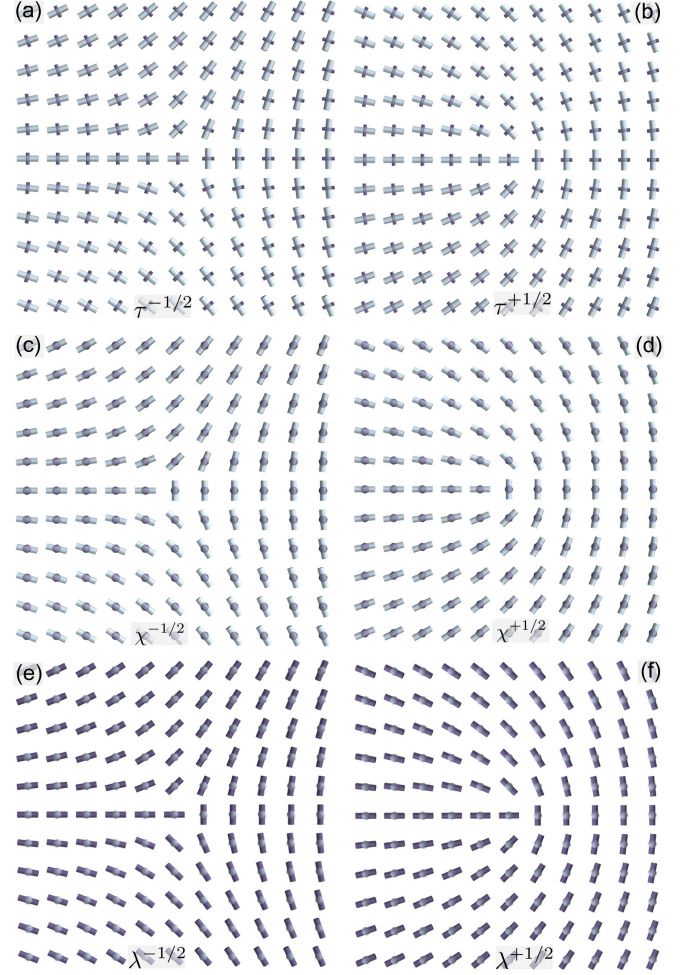


FIG. 2. Examples of two-dimensional τ , χ , and λ defect patterns in biaxial nematics. Defect profiles shown correspond to $\tau^{-1/2}$ (a), $\tau^{+1/2}$ (b), $\chi^{-1/2}$ (c), $\chi^{+1/2}$ (d), $\lambda^{-1/2}$ (e), and $\lambda^{+1/2}$ (f). The Clifford algebra elements corresponding to these defect profiles are given in Eq. (12) and are $\pm e_{ba}$ (a), (b), $\pm e_{ca}$ (c), (d), $\pm e_{cb}$ (e), (f).

where the two indices in the bivectors are those for which the corresponding eigenvector, or unit vector, is singular in the defect profile.

We highlight that Eq. (12) give the same algebra as quaternions, the group defined by their generators is cyclic, and that the three elements in the algebra (λ , τ , and χ) anticommute, as pertains to different components of a fermionic field. Additionally, as in the uniaxial nematic case, the antiparticles corresponding to each component and disclinations can be represented by the same bivectors with the opposite sign. Therefore, each defect is equivalent to its antidefect (again, because spinors are defined up to a sign), and each 2D defect profile can be viewed as a Majorana fermion at rest. Hence, the quasiparticle excitations in biaxial nematics are equivalent to three independent flavors of Majorana spinors.

An important feature of the algebraic representations we have discussed is that five conjugacy classes correspond to

combinations of the three types of disclination. In the formalism of Eq. (12), these combination rules—sometimes referred to as fusion algebra [29]—are the following:

$$\begin{aligned}\lambda\tau &= \chi, & \text{as } e_{ca} &= e_{cb}e_{ba}, \\ \chi\lambda &= \tau, & \text{as } e_{ba} &= e_{ca}e_{cb}, \\ \tau\chi &= \lambda, & \text{as } e_{cb} &= e_{ba}e_{ca}, \\ \lambda^2 &= \tau^2 = \chi^2 = -\mathbb{1}.\end{aligned}\quad (13)$$

It should be noted that two λ (or two τ or two χ) disclinations can also combine to create a defect-free configuration, or $+\mathbb{1}$, which is the fifth conjugacy class [3], because $\pm\lambda$ are equivalent, being spinors. These combination rules have permutation symmetry ($\lambda \rightarrow \tau \rightarrow \chi \rightarrow \lambda$), hence the three disclinations are all equivalent within the biaxial spinor subgroup in Eq. (12). As we shall see, this equivalence is broken in the case of cholesterics, which is one of the reasons why this representation is not suitable to fully describe defects in the latter systems.

Using the disclination bivectors and $\mathbb{1}$, we can also write a set of braiding operators [30] as follows:

$$\begin{aligned}A &= \frac{1}{\sqrt{2}}(\mathbb{1} + \tau), \\ B &= \frac{1}{\sqrt{2}}(\mathbb{1} + \lambda), \\ C &= \frac{1}{\sqrt{2}}(\mathbb{1} + \chi).\end{aligned}\quad (14)$$

These braiding operators satisfy the following identities:

$$ABA = BAB, \quad BCB = CBC, \quad ACA = CAC, \quad (15)$$

and they form a representation of the braiding group as discussed in Ref. [30]. The fundamental reason why biaxial disclinations can be braided is that they are described by a nontrivial (non-Abelian) underlying algebra (that of quaternions).

As done for uniaxial nematics, we can associate bivectors corresponding to 3D rotations to biaxial defect profiles as well. For planar profiles, considered up to now, we can associate a defect pattern with a pair (Δ_1, Δ_2) , where $\Delta_{1,2}$ are in the even subalgebra $\text{Cl}(2, 0)^{[0]}$, and describe whether there is a singularity ($\Delta_{1,2} = \pm e_{21}$) or not ($\Delta_{1,2} = \mathbb{1}$) for the primary and secondary director field, respectively. For instance, the defect bivector pair corresponding to a χ defect would be $(\pm e_{21}, \mathbb{1})$. This can be generalized to out-of-plane defects, in which case $\Delta_{1,2}$ are in $\text{Cl}(3, 0)^{[0]}$ (or, equivalently, quaternions). It should be noted that not all possible combinations of bivectors will be realized in physically occurring defect profiles.

V. CHOLESTERIC ALGEBRA: DUAL QUATERNIONS AND WEYL-LIKE SPINORS

We next turn to the algebraic description of cholesteric defects, or spinors, which constitutes the main result of the current work. The underlying key idea is that $\text{Cl}(3, 0)$ is not large enough, as the symmetries of cholesterics include translations along the helical pitch as well as rotations. We, therefore, shall resort to the three-dimensional projective geometric algebra, $\text{Cl}(3, 0, 1)$, which extends $\text{Cl}(3, 0)$ to include translations.

$\text{Cl}(3, 0, 1)$ has four generators, three of which square to 1, and one of which squares to 0. It has 2^4 elements and contains dual quaternions [19] as the subalgebra of even elements (scalar, bivector, and pseudoscalar), $\text{Cl}(3, 0, 1)^{[0]}$. We denote the four generators of $\text{Cl}(3, 0, 1)$ as e_1, e_2, e_3 , and e_4 , with $e_1^2 = e_2^2 = e_3^2 = 1$, and $e_4^2 = 0$.

The three types of geometric transformation encoded by elements of $\text{Cl}(3, 0, 1)^{[0]}$ are the elements of the Euclidean group $\text{SE}(3)$, namely, rotations, translations, and screw transformations—the latter are rotations followed by a translation along the same direction. A general transformation ζ can be viewed as a rotation of an angle ϕ along an axis $\mathbf{a}$ followed by a translation of a distance d along a unit vector $\mathbf{b}$; hence, it can be described as

$$\zeta = \left[\mathbb{1} + \frac{d}{2} \mathbf{b} \cdot \mathbf{e}_t \right] \left[\cos\left(\frac{\phi}{2}\right) \mathbb{1} + \sin\left(\frac{\phi}{2}\right) \mathbf{a} \cdot \mathbf{e}_r \right], \quad (16)$$

where we have defined translation and rotation vectors $\mathbf{e}_t$ and $\mathbf{e}_r$, respectively, as follows:

$$\begin{aligned}\mathbf{e}_t &= (e_{41}, e_{42}, e_{43}), \\ \mathbf{e}_r &= (e_{32}, e_{13}, e_{21}).\end{aligned}\quad (17)$$

As in previous sections, bivectors are defined as $e_{ij} = e_i e_j = -e_j e_i$ with $i \neq j$. A screw transformation along $\hat{\mathbf{h}}$, $\Sigma_{\hat{\mathbf{h}}}(d, p)$, of size d and pitch p , is given by Eq. (16) with $\mathbf{b} = \mathbf{a} \equiv \hat{\mathbf{h}}$, and $\phi = 2\pi d/p$.

Let us now see how we can match elements of $\text{Cl}(3, 0, 1)^{[0]}$ to cholesteric defects. Consider first the defect in Fig. 3. This defect consists of a rotating wedge disclination, or a screw dislocation, whose Burgers vector is along the disclination tangent, which coincides with the cholesteric helix axis (and is directed along z in Fig. 3). This defect is of type χ : As in biaxial nematics, it is called like this as it is nonsingular in the χ field, which in cholesterics denotes the helical axis. Because it entails a rotation and a translation along the same axis, we can algebraically view χ as a screw transformation along z (or e_3) of size $p/2$,

$$\chi \leftrightarrow \Sigma_z(p/2, p) = \left(\mathbb{1} + \frac{p}{4} e_{43} \right) e_{21}. \quad (18)$$

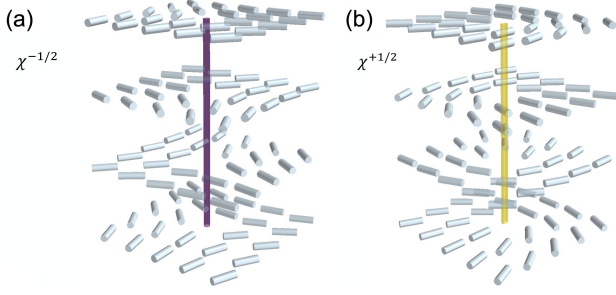


FIG. 3. Examples of screw dislocation patterns, corresponding to a spiraling triradius, or $\chi^{-1/2}$ line (a), and a spiraling comet defect, or $\chi^{+1/2}$ (b). The elements of $\text{Cl}(3, 0, 1)^{[0]}$ corresponding to the defect profiles are $-(\mathbb{1} + [p/4]e_{43})e_{21}$ (a) and $(\mathbb{1} + [p/4]e_{43})e_{21}$ (b); equivalently, the defects are associated with a rotation of π and a half-pitch translation along the helical axis e_3 . Defects are colored according to $\cos(\beta)$ as in Fig. 1, with purple and yellow corresponding to triradius $[\cos(\beta) = -1]$ and comet $[\cos(\beta) = +1]$ profiles, respectively.

The other two types of cholesteric defects, which are called τ and λ , both involve singularities in the helical axis so that the latter axis is not well defined near their core. Examples of such defects are the patterns shown in Fig. 4. The patterns corresponding to defects of type τ can be viewed as a simple rotation along an axis perpendicular to that of the screw (i.e., an axis perpendicular to the plane of the drawing in Fig. 4). Similarly, defects of type λ can be viewed as the same rotation (represented by e_{13} and $-e_{13}$ for π and $-\pi$ rotations, respectively) followed by a screw transformation along the helical axis (a direction in the plane of the drawing in Fig. 4) of size $p/2$. Therefore, these two types of defects can be algebraically represented as follows:

$$\begin{aligned} \tau &\leftrightarrow e_{13}, \\ \lambda &\leftrightarrow -\Sigma_z \left(\frac{p}{2}, p \right) e_{13} = \left(\mathbb{1} + \frac{p}{4} e_{43} \right) e_{32}. \end{aligned} \quad (19)$$

Note that it is also common to refer to the representation of τ in Eq. (19) as $\tau^{+1/2}$, as the corresponding rotation is by $+\pi$. (Correspondingly, $\tau^{-1/2}$ is represented by $-e_{13}$ in this Clifford algebra notation.) In terms of dual quaternions, the representation of the τ , χ , and λ defects in Eq. (19) becomes

$$\begin{aligned} \tau &\leftrightarrow j, \\ \chi &\leftrightarrow \left(1 - \frac{p}{4} \epsilon k \right) k = k + \epsilon \frac{p}{4}, \\ \lambda &\leftrightarrow \left(1 - \frac{p}{4} \epsilon k \right) i = i - \epsilon \frac{p}{4} j, \end{aligned} \quad (20)$$

where we have identified the dual unit ϵ (such that $\epsilon^2 = 0$) with e_{4321} and the quaternions (i, j, k) with the triplet of bivectors (e_{32}, e_{13}, e_{21}) .

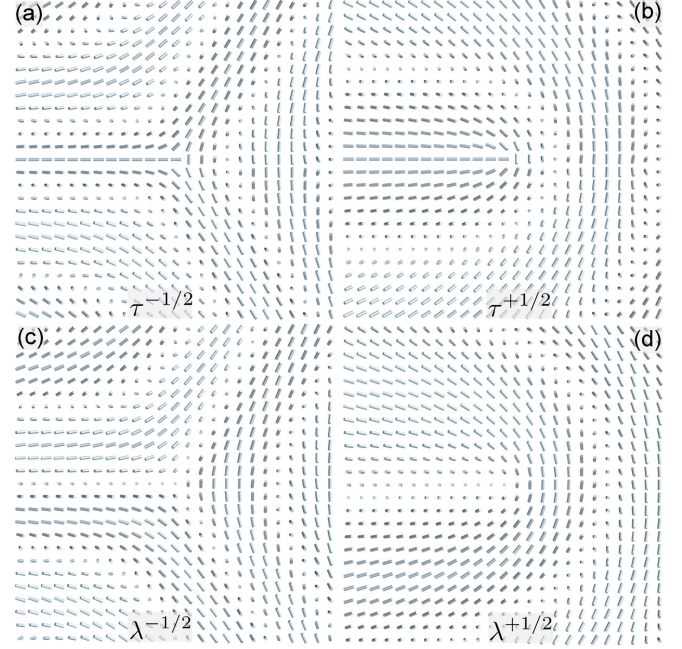


FIG. 4. Examples of two-dimensional τ and λ disclination defect patterns. While τ defects (a),(b) are singular in the director fields, λ defects (c),(d) are not. Defect profiles shown correspond to $\tau^{-1/2}$ (a), $\tau^{+1/2}$ (b), $\lambda^{-1/2}$ (c), and $\lambda^{+1/2}$ (d). In 3D, the disclination line corresponding to these defect profiles is perpendicular to the plane of the drawing (or the $x-y$ plane). The elements of $\text{Cl}(3, 0, 1)^{[0]}$ corresponding to the defect profiles are $-e_{21}$ (a), e_{21} (b), $-(\mathbb{1} + [p/4]e_{42})e_{13}e_{21} = -e_{32} + (p/4)e_{43}$ (c), and $e_{32} - (p/4)e_{43}$ (d).

It is useful to pause to discuss a few key properties of the representation in Eq. (20).

First, the antidefects for τ and λ are $-\tau$ and $-\lambda$. Therefore, these two defect profiles can be viewed as Majorana spinors, as in the nematic case, because spinors are defined up to a minus sign. It can also be seen that both the spinors square to -1 as in the nematic case. The case of the screw dislocation spinor, which corresponds to a screw transformation, is different though. This spinor does *not* square to -1 ; instead, $\chi^2 = -\mathbb{1} - (p/2)e_{43} = -1 + (p/2)\epsilon k$, so that the square of χ entails a translation (of $(p/2)$) as well. As a consequence, generally $-\chi$ defects are not the inverse of χ , and hence they cannot be mathematically viewed as a Majorana spinor. For a pointlike defect, we might use the fact that cholesteric patterns are invariant after translation of half a pitch (which are associated with 2π rotations), to neglect factors of $(p/2)e_{43}$ and hence reestablish a Majorana-like interpretation for χ . However, this symmetry breaks down for extended disclinations (i.e., nonpointlike defects, such as any segments in the lines in Fig. 3). For segments of size l , the relevant algebraic representation of defects as Clifford algebra elements is

$$\begin{aligned}
\chi &\leftrightarrow \Sigma_z(p/2 + l, p) \\
&= \left[\mathbb{1} + \frac{z}{2} e_{43} \right] \left[\cos\left(\frac{\phi}{2}\right) \mathbb{1} + \sin\left(\frac{\phi}{2}\right) e_{21} \right], \\
z &= \frac{p}{2} + l, \quad \phi = \frac{2\pi z}{p}.
\end{aligned} \quad (21)$$

Its inverse is now different from $-\chi$, even if we neglect translations of any multiple of half a pitch: This is because the inherent chirality of cholesterics breaks the symmetry between right- and left-handed local screw disclination profiles, such that the inverse of a $+1/2$ right-handed profile is a right-handed $-1/2$ profile, given explicitly by

$$\left[\mathbb{1} - \frac{z}{2} e_{43} \right] \left[\cos\left(\frac{\phi}{2}\right) \mathbb{1} - \sin\left(\frac{\phi}{2}\right) e_{21} \right], \quad (22)$$

rather than the left-handed $-1/2$ profile, which Eq. (21) transforms onto under a parity transformation. Therefore, (extended or nonpointlike) χ defects are more naturally interpreted as Weyl-like, rather than Majorana-like, spinors, as they have a well-defined chirality or, equivalently, correspond to a specific combination of rotations and translations. These objects are qualitatively similar to Weyl spinors in particle physics, which correspond to a specific combination of rotations and boosts.

Second, the three elements corresponding to λ , τ , and χ , unlike the case of the biaxial nematics, do not provide a closed subgroup. Indeed, it can be seen that $\lambda\tau = \chi$ and $\chi\lambda = \tau$, but $\tau\chi = \lambda - (p/2)e_{42} = \lambda + \epsilon(p/2)j$; hence, the combination entails an extra translation. Again, this is non-negligible for nonpointlike defects. We suggest that the neglect of this feature is the origin of the cholesteric algebra puzzle described in Ref. [15] and reviewed in the Introduction.

Finally, we note that the λ and τ defects are related by left multiplication by a screw transformation of size $(p/2)$ (a translation of $(p/2)$ and a rotation of π). Because of their spinor nature, this operation rotates only the patterns with respect to each other by $(\pi/2)$ and translates them by $(p/4)$, which corresponds to our physical intuition of the relation between the two defects in Fig. 4. (In contrast, a screw transformation Σ on a Clifford vector V would act via a sandwich product, or via $\Sigma V \Sigma^{-1}$.)

While in screw dislocations the disclination tangent $\hat{\mathbf{T}}$ is parallel to the helical axis $\hat{\mathbf{h}}$ (Fig. 3), another relevant geometry is the one in which $\hat{\mathbf{T}}$ is perpendicular to $\hat{\mathbf{h}}$. In this geometry, the relevant χ dislocation is an edge dislocation, associated with a twist disclination, such that the Burgers vector is now perpendicular to the disclination (Fig. 5). Taking $\hat{\mathbf{T}} = e_3$, along the z direction, we can associate χ with a screw along the y direction, τ with a rotation along

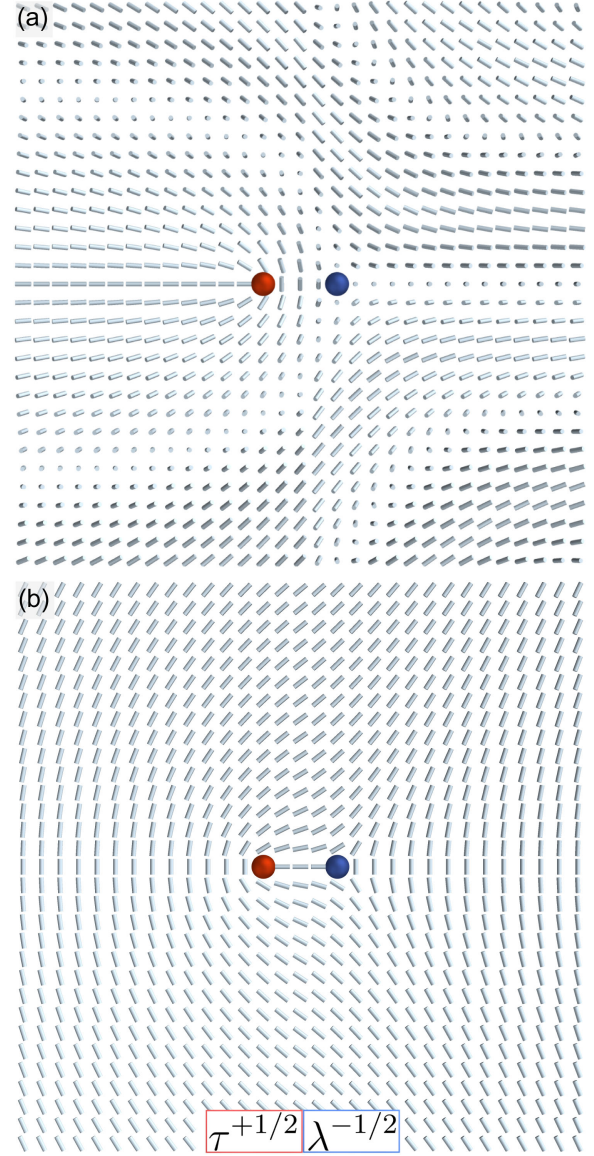


FIG. 5. Example of edge dislocation pattern. (a) Director field profile, showing the splitting of an edge dislocation into τ and λ defects, here a $\tau^{-1/2}$ and a $\lambda^{+1/2}$ pair. (b) Helical direction (χ) pattern, showing singularities at the location of the λ and τ defects and the parallel far field along the vertical direction.

the z direction, and λ again as a rotation along z followed by a screw transformation along a perpendicular direction (here y). Algebraically, this corresponds to

$$\begin{aligned}
\tau &\leftrightarrow e_{21}, \\
\chi &\leftrightarrow \left(\mathbb{1} + \frac{p}{4} e_{42} \right) e_{13} = e_{13} + \frac{p}{4} e_{4321}, \\
\lambda &\leftrightarrow - \left(\mathbb{1} + \frac{p}{4} e_{42} \right) e_{13} e_{21} = e_{23} + \frac{p}{4} e_{43}.
\end{aligned} \quad (23)$$

In terms of dual quaternions, Eq. (23) becomes

$$\begin{aligned}\tau &\leftrightarrow k, \\ \chi &\leftrightarrow \left(1 - \frac{p}{4}\epsilon j\right)j = j + \frac{p}{4}\epsilon, \\ \lambda &\leftrightarrow -\left(1 - \frac{p}{4}\epsilon j\right)jk = -i - \frac{p}{4}\epsilon k.\end{aligned}\quad (24)$$

Similar consideration holds for this representation, in Eqs. (23) and (24), as for the previous one, given by Eq. (20). In this geometry, this representation predicts that $\chi = \tau\lambda$, in line with the well-known fact that a χ edge dislocation can be seen as a composite defect [28,31], or a topological dipole made up, for instance, by a $\tau - \lambda$ pair (in the example in Fig. 5). Here, the algebraic formula accounts for the $(p/4)$ effective size of the edge dislocation—again the finite size renders this structure a Weyl-like spinor algebraically. The explicit pattern in Fig. 5 shows, in addition, that the χ field (the helical axis), as expected, is well defined in the far field and everywhere except near the τ and λ defects. This edge dislocation pattern is an example of composite quasiparticle, where two spinors (τ and λ) combine to form another spinor (χ): The composite quasiparticle is, therefore, fermionic in nature here.

The magnitude of the Burgers vector in the edge dislocation in Fig. 5 is $b = (p/2)$. It is interesting to also consider the case of dislocation with $b = p$, which theory suggests to be a pair of λ defects. These dislocations appear in thicker regions of Grandjean-Cano wedges [31,32] and are naturally found in ferroelectric chiral nematics [32,33].

In the Clifford algebra formalism, a (finite-size) composite of $\lambda^{-1/2}$ - and $\lambda^{+1/2}$ -type elements in Eq. (23) can either equal 1, corresponding to a trivial defect-free pattern, or to $(1 \pm [p/2]e_{42})$, corresponding to a pure translation along the helical axis. Algebraically, which result is selected depends on the order in which the rotations and translations that make up each of the λ elements are composed. For instance, if we choose

$$\begin{aligned}\lambda_1 &\leftrightarrow -\left(1 + \frac{p}{4}e_{42}\right)e_{13}e_{21}, \\ \lambda_2 &\leftrightarrow e_{13}e_{21}\left(1 + \frac{p}{4}e_{42}\right),\end{aligned}\quad (25)$$

we find

$$\lambda_1\lambda_2 \leftrightarrow \left(1 + \frac{p}{2}e_{42}\right), \quad (26)$$

which corresponds to an edge dislocation with $b = p$. (Instead, choosing $\lambda_2 = -\lambda_1$ gives $\lambda_1\lambda_2 = 1$.) In the case of $b = p$ dislocations, the resulting quasiparticles are not fermionic, as Eq. (26) does not square to -1 , but is instead

bosonic in nature. Because of the presence of a nontrivial Burgers vector, the composite can be thought of as a vector boson. Therefore, the Clifford algebra formalism shows there is a fundamental algebraic difference between edge dislocations with $b = (p/2)$ and $b = p$.

VI. DEFECT BIVECTORS IN CHOLESTERICIS

Generally, defects in cholesterics have a rotational part (disclination) and a translational part (dislocation). Therefore, they can be described by the combination of a rotation of π around a unit vector $\mathbf{\Omega}$ followed by a translation along a Burgers vector $\mathbf{b}$ (which has a variable length, corresponding to the extent of the disclination). Algebraically, we can therefore associate a local director profile at a defect with the following linear combination of bivectors:

$$\begin{aligned}\Delta &= [1 + \mathbf{b} \cdot \mathbf{e}_t]\mathbf{\Omega} \cdot \mathbf{e}_r \\ &= \mathbf{\Omega} \cdot \mathbf{e}_r + \epsilon\mathbf{\Omega} \cdot \mathbf{b} + \epsilon(\mathbf{\Omega} \times \mathbf{b}) \cdot \mathbf{e}_r,\end{aligned}\quad (27)$$

which can describe generic cholesteric defect patterns and generalizes the defect bivector introduced for nematics in Eq. (8). We suggest Δ can be used to quantify the local profile both geometrically and topologically. It is interesting to note that Eq. (8) is formally analogous to the representation of the Faraday bivector in the space-time Clifford algebra $\text{Cl}(1, 3)$ [19,34].

The vector $\mathbf{b}$ is, for both screw and edge disclinations, parallel to the helical pitch axis $\hat{\mathbf{h}}$. Like $\hat{\mathbf{\Omega}}$, $\hat{\mathbf{h}}$ can be reconstructed starting from the $\mathbf{Q}$ tensor. Specifically, this can be done by using the chirality tensor,

$$C_{ij} = \epsilon_{j\mu\nu}Q_{\mu\alpha}\partial_i(Q_{\nu\alpha}). \quad (28)$$

In the uniaxial limit, $Q_{\alpha\beta} \propto (n_\alpha n_\beta - \delta_{\alpha\beta}/3)$, this tensor can be written in terms of the director field as $C_{ij} = \epsilon_{j\mu\nu}n_\mu\partial_i n_\nu$, which is the chirality tensor defined in Ref. [15]. The helical axis can be found as the left eigenvector of C_{ij} , which can be identified by singular value decomposition. The trace of C_{ij} can be written as

$$\text{Tr}(C) = -\epsilon_{\alpha\gamma\delta}Q_{\alpha\beta}\partial_\gamma Q_{\delta\beta} = -S_{\text{tw}}, \quad (29)$$

where S_{tw} is the twist parameter defined in Refs. [35–37].

Equation (27) provides a description of a single defect in terms of a $\text{Cl}(3, 0, 1)$ bivector. Similar to what is done in Ref. [25], a pair of stable defects (do not recombine) could be described by an element in $\text{Cl}(3, 0, 1) \times \text{Cl}(3, 0, 1)$, and considering a direct product of n copies of $\text{Cl}(3, 0, 1)$ allows the representation of a set of n nonrecombining defects. For these cases, it would be of interest to analyze the relative positioning of the $\mathbf{\Omega}$ and $\mathbf{b}$ bivectors and characterize how the orientations lock as the defects approach.

VII. MATRIX REPRESENTATION AND CAYLEY FACTORIZATION OF CHOLESTERIC SPINORS

Until now, we have used $\text{Cl}(3,0,1)$ elements without any reference to a specific representation. It is also useful, in practice, to provide a suitable matrix representation both of generic elements of the Clifford algebra and of the even subalgebra (again, in terms of matrices, rather than dual quaternions). Elements in this Clifford algebra can be represented by a subset of 4×4 matrices, with the following generators:

$$\begin{aligned} e_1 &= \begin{bmatrix} 0 & 0 & 0 & +i \\ 0 & 0 & +i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, & e_2 &= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \\ e_3 &= \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & +i \\ +i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, & e_4 &= \begin{bmatrix} +i & 0 & +1 & 0 \\ 0 & +i & 0 & +1 \\ +1 & 0 & -i & 0 \\ 0 & +1 & 0 & -i \end{bmatrix}, \end{aligned} \quad (30)$$

where $e_1^2 = e_2^2 = e_3^2 = 1$, $e_4^2 = 0$, and $e_i e_j = -e_j e_i$ for $i \neq j$.

As cholesteric spinors are combinations of even-grade elements in $\text{Cl}(3,0,1)^{[0]}$, it is useful to have a representation of this even subalgebra, which can be achieved by exploiting Cayley factorization of 4×4 matrices into right and left isoclinic bases [38,39]. A right and a left basis set is provided by the following $A_{x,y,z}$ and $B_{x,y,z}$ matrices, respectively:

$$\begin{aligned} A_x &= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{bmatrix}, & B_x &= \begin{bmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \\ A_y &= \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{bmatrix}, & B_y &= \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \\ A_z &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{bmatrix}, & B_z &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \end{aligned} \quad (31)$$

It can be shown that the matrix algebra of (A_x, A_y, A_z) [or of (B_x, B_y, B_z)] is that of quaternions; hence, the A and B triads can both be mapped to the triad of bivectors

(e_{32}, e_{13}, e_{21}) . Only one set of matrices is needed to represent 3D transformations with sandwich products.

While the matrices in Eq. (31) describe rotations, translations can be represented by $(\epsilon A_x, \epsilon A_y, \epsilon A_z)$ or by $(\epsilon B_x, \epsilon B_y, \epsilon B_z)$, which give alternative basis sets of isoclines.

A general element of $\text{Cl}(3,0,1)^{[0]}$ —or a unit dual quaternion corresponding to a rototranslation—can thus be written in terms of isoclinic rotation matrices as follows (taking, for instance, the $A_{x,y,z}$ basis):

$$\Sigma = \left[\mathbb{1} - \epsilon \frac{d\mathbf{b} \cdot \mathbf{A}}{2} \right] [\cos(\phi) \mathbb{1} + \sin(\phi) \mathbf{a} \cdot \mathbf{A}], \quad (32)$$

where $\mathbf{a}$ and $\mathbf{b}$ are the unit vectors describing the translation and rotation, as in Eq. (16), to which Eq. (32) is equivalent. To see this, we need only to recall that the A basis can be identified with (e_{32}, e_{13}, e_{21}) and that ϵ can be identified with e_{4321} .

An advantage of the formalism described in this section is that the corresponding representation of the even subalgebra $\text{Cl}(3,0,1)^{[0]}$, and hence of cholesteric spinors, is real and constitutes a Lie algebra for e_{32} , e_{13} , and e_{21} . The representation of the translations, while requiring the dual number ϵ , is also a natural generalization of this algebra.

VIII. GEOMETRY OF SCREW DISLOCATIONS UNDER CYLINDRICAL CONFINEMENT

The algebraic representation discussed above provides a powerful way to predict results from combinations of defects. In some cases, the predicted patterns can be observed, in practice, such as for the composite edge dislocation shown in Fig. 5, which spontaneously form in a cholesteric wedge [31]. However, in general, an algebraically plausible pattern need not be realized in practice, if, for instance, it costs too large an amount of elastic free energy.

To explore the thermodynamic stability of cholesteric spinors and defects, in this section we focus on the geometry and structure of screw dislocation patterns, which are comparatively less studied with respect to edge dislocations. Specifically, we analyze configurations found numerically under cylindrical confinement, when initializing the system with a defect pattern corresponding to a screw dislocation, as in Fig. 3—in each cross section perpendicular to the disclination, this structure corresponds to a Weyl-like cholesteric spinor. Confinement is realized by choosing spatially dependent free-energy parameters such that the liquid crystal is in the cholesteric phase inside a cylinder of radius R and in the isotropic phase in the region outside the cylinder (see Appendix A for more details on the functional form of the free energy, the initialization of the system, and the choice of parameters). Our choice of cylindrical confinement is inspired by classical experiments in cholesteric tubes [40–42], although

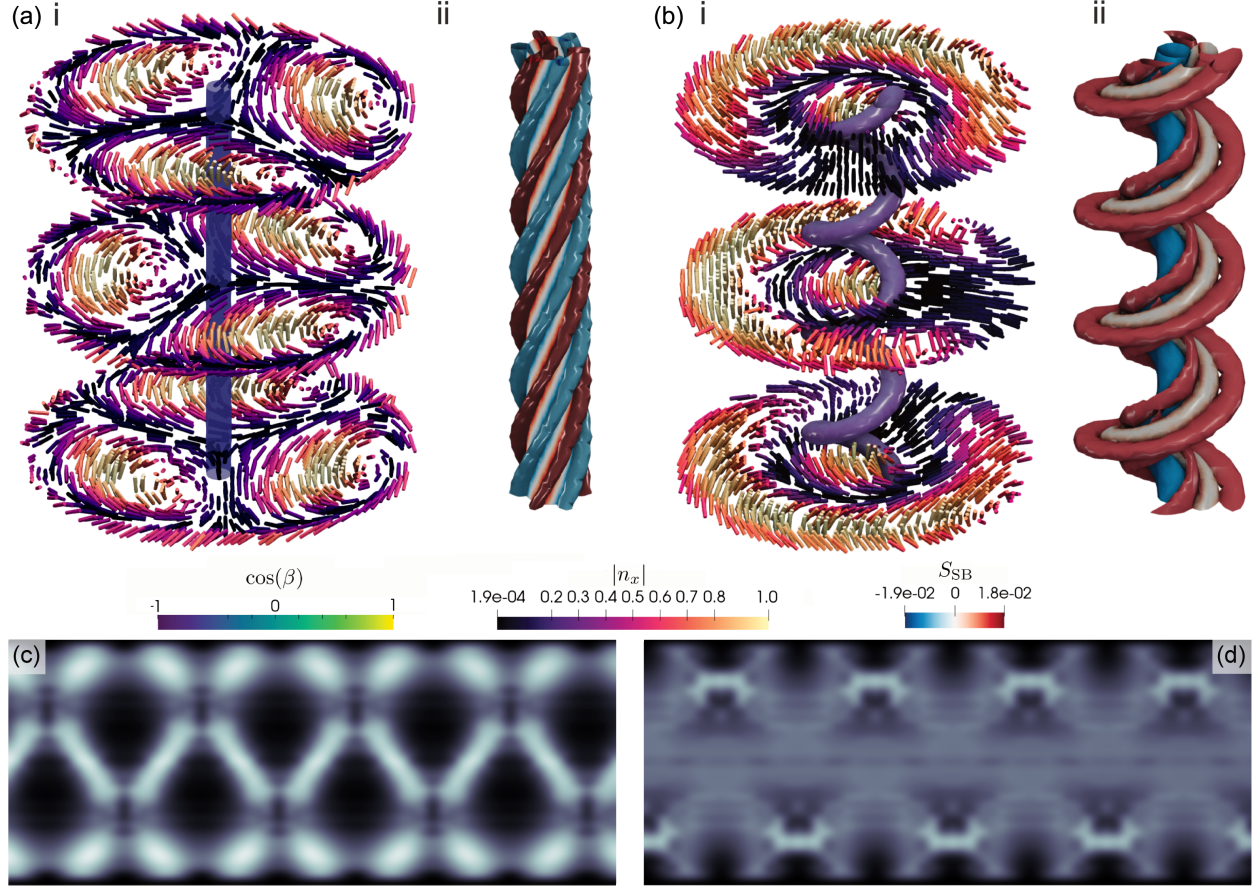


FIG. 6. $\chi^{-1/2}$ (a) and $\chi^{+1/2}$ (b) screw dislocation inside a cholesteric cylinder. Disclinations pierce the cylinder along the central axis. These are visualized as isosurfaces of the highest eigenvalue of the Q tensor [$\lambda^\downarrow(Q) = 0.1$] in (ai) and (bi), and colored accordingly to $\cos(\beta)$. In both cases, the director field $\mathbf{n}$ is visualized as rods on three planes cutting the principal axis of the cylinder and colored accordingly to $|n_x|$. (a ii), (b ii) The isosurfaces of the splay-bend parameter S_{SB} around the disclination lines (blue corresponding to $S_{SB} = -0.01$ and red to $S_{SB} = 0.01$). (c),(d) Simulated polarized optical microscopy images corresponding to the configurations in (a) and (b), respectively (see Appendix B for more details).

in our cases the boundary and initial conditions are distinct and have been chosen to focus on the structure of screw dislocations.

Figure 6 shows the results of simulations initialized with a $\chi^{-1/2}$ [Fig. 6(a)] and with a $\chi^{+1/2}$ [Fig. 6(b)] screw dislocation inside the cholesteric cylinder—in other words, the starting configurations are those in Fig. 3. For the $\chi^{-1/2}$ case, this topology is stable, bar the appearance of double-twist regions (skyrmions) surrounding the central triradius [Fig. 6(a)]. In each of the cylindrical cross sections, the director field pattern in the center retains its Weyl-like character, where the sense of defect profile rotation matches the right-hand thermodynamic chirality of the underlying cholesteric phase. The disclination line has the topology of a twisted ribbon, as can be appreciated by visualizing the splay-bend parameter—defined as

$$S_{SB} = \partial_\alpha \partial_\beta Q_{\alpha\beta}, \quad (33)$$

which displays a “barber-pole” pattern in the director profile close to the defect line [Fig. 6(a ii)].

When starting from a $\chi^{+1/2}$ line, instead, the initially straight disclination curls in 3D to form a right-handed helix, where the cross section is a $-1/2$ profile that does not twist—as can be seen by tracing the red splay filaments in the splay-bend pattern. In other words, the inherent twist in the $\chi^{+1/2}$ line has transformed into 3D writhe. This morphological transition is similar to the conversion between twist and writhe, which can be observed in supercoiled polymers, such as DNA, and that is mathematically described by the linking number theorem [43]. In our structure, a central double-twist region, corresponding to a λ line, accompanies the helical defect, such that the defect itself can be seen as an algebraically allowed $\lambda - \tau$ composite, similar to the case of the edge dislocation in Fig. 5—although here the composition is of the form $\chi^{+1/2} \leftrightarrow \tau^{-1/2} \lambda^{+1/2} \lambda^{+1/2} = \tau^{-1/2} \lambda^{+1}$. The composite nature of the defect can be visualized by tracing the splay-bend

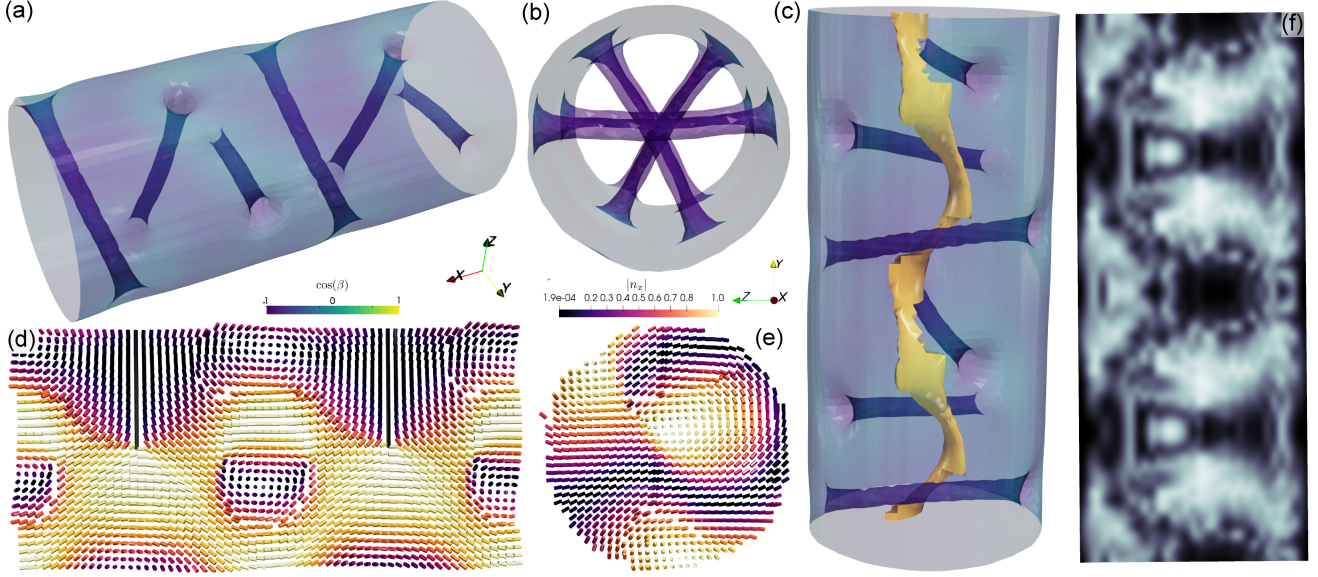


FIG. 7. (a)–(c) Side (a) and top (b) view of a cartwheel disclination pattern. (c) The region where the director field is aligned along the axis $\hat{x}$ of the cylinder (with projection $\mathbf{n} \cdot \hat{x} \geq 0.99$) is also shown, to visualize skyrmion filaments along the cylindrical axis. This cartwheel structure results from an initialization with a straight disclination along the cylinder axis and a spiraling $\tau^{-1/2}$ defect profile on the cross section (see Appendix A for initialization details and parameter list). (d),(e) Cross section along two planes, which are, respectively, parallel (d) and perpendicular (e) to the cylinder axis, of the corresponding steady-state director field pattern. (f) Simulated polarized optical microscopy image.

pattern in 3D [Fig. 6(bii)], as, given the defect geometry, splay, and bend tend to localize close to τ and λ defects, respectively.

To ease comparison of our simulations with future experiments, we also provide simulated polarized optical microscopy images corresponding to the resulting screw dislocation patterns in Figs. 6(c) and 6(d) (see Appendix B and Ref. [44]). When simulating these optical images, we assume that the cholesteric pitch is much larger than the wavelength of probing light, which is in accord with the concept of weakly distorted cholesteric and smooth deformations of the director field.

Figure 7 shows the results obtained when starting from a cholesteric $\tau^{-1/2}$ disclination spiraling along the z direction [as in Fig. 3(a), but with a $\tau^{-1/2}$ pattern as in Fig. 4(a) instead of a triradius pattern at each plane]. *A priori*, one may expect this to lead to the same pattern as for the rotating $\chi^{-1/2}$ disclination, as the patterns in the initial condition are quite similar. Instead, and surprisingly, numerical simulations show that the texture completely changes and transforms into the 3D cartwheel pattern shown in Figs. 7(a) and 7(b). Here, a set of τ disclinations perpendicular to the cylindrical axis is stabilized by a combination of bend deformations and an array of double-twist cylinders (along mutually perpendicular directions). In this case, therefore, a single dislocation has broken up into a whole array of disclinations.

The cases considered up to now pertain to screw dislocations initialized with half-integer defect profile

($\chi^{\pm 1/2}$). It is of interest to consider the case analyzed, for instance, in Refs. [15,45] of a screw dislocation initialized with a defect profile with integer topological charge (a spiral pattern with rotating angle along the dislocation axis; see Appendix A). The results are shown in Fig. 8. The initially straight dislocation breaks up into an array of defect rings, each of which can be seen as a “smoke-ring” quasiparticle, stacked along the dislocation axis (x). This defect pattern is similar to what was observed in Ref. [15] (and consistent with the results obtained for cholesterics under spherical confinement in Ref. [45]). The defect rings are locally $-1/2$ defect profiles, and there is no twisting (so the rings are locally Majorana-like defects). Double-twist regions accompany the rings and wrap around them helically. The simulated polarized microscopy image has a symmetric zigzagging structure, which would be interesting to look for experimentally. Algebraically, the resulting defect ring arrays can be viewed as $[\mathbb{1} + (p/4)e_{41}](-\mathbb{1}) = -[\mathbb{1} + (p/4)e_{41}]$, which is indeed different from $-\mathbb{1}$, which would be expected if $\chi^{\pm 1/2}$ were algebraically represented by a quaternion [15].

IX. BOSONIZATION AND TOPOLOGICAL PHASES OF CHOLESTERIC SPINORS

The striking morphological diversity of the patterns observed for confined screw dislocations resulting from different initial conditions, and discussed in the previous section, is reminiscent of the spontaneous appearance of

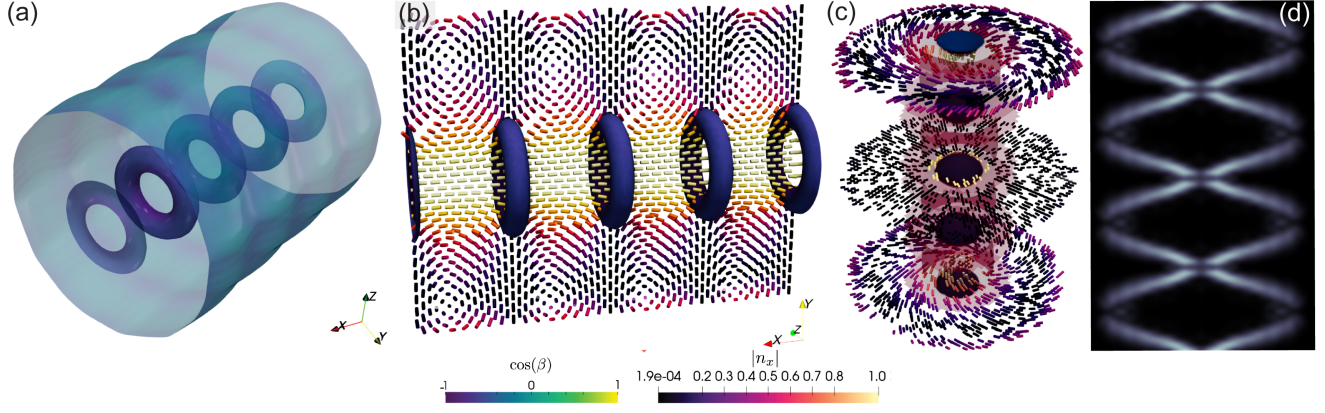


FIG. 8. (a)–(c) Side views of a defect ring array or, equivalently, smoke-ring disclination pattern. (a) The disclination loops are colored with the $\cos \beta$ value, revealing their local $\tau^{-1/2}$ structure. The director profile patterns in (b), (c) show the presence of double-twist regions (b) and the local $+1$ defect profiles perpendicular to the defect ring array (c). (d) Simulated polarized optical microscopy image.

blue phases in 3D bulk cholesterics, where a panoply of different metastable topological phases are found for the same thermodynamic parameters, close to the isotropic-cholesteric transition. As we shall now show, this phenomenon can be thought of as a manifestation of spinor condensation, or bosonization. Here, again, τ and λ defects—or skyrmion filaments—combine to form composite quasiparticles. This time, they have bosonic nature and can, therefore, condense to form topological phases. We highlight here, as well, that the formation of composite defects, which is at the basis of the condensation phenomenon, requires disclinations to entangle, or braid, with one another, which is possible only if the underlying spinor subalgebra is non-Abelian.

Because both λ and τ disclinations have a topological charge, when they combine, they can give rise to structures that are not topologically charged overall, as the half-skyrmion-defect composites which are shown in Fig. 9. Because their effective overall topological charge is 0, these defect-skyrmion complexes behave as composite bosons and can tile the plane to form 2D hexagonal lattices [Figs 9(ai) and 9(aii)]; these two-dimensional blue phases are thermodynamically stable only under an electric field [46,47]. With respect to the case of edge dislocations with $b = p$, these bosonic excitations have scalar, rather than vectorial, character algebraically. These composite quasiparticles are also qualitatively similar to Cooper pairs in superconductors, which are composite bosons made by pairs of fermions [48].

To gain more insight into cholesteric spinor condensation, or bosonization, phenomenon, we numerically study a periodic system with periodic boundary conditions and variable thickness. To visualize skyrmions, we have used the local twist parameter S_{tw} defined in Eq. (29) [35]. Specifically, we have identified skyrmion filaments with regions in space where the absolute value of S_{tw} is above a given threshold. Physically, this is where twist is

maximum, which generalizes the notion of double-twist regions in 2D [Fig. 9(a)].

For thin samples, the hexagonal 2D blue phase is stable, and it stretches into the third dimension to form an array of cylindrical skyrmions separated by straight $-1/2$ disclination lines [Fig. 9(a)]. This structure is also the one found for thin samples in Refs. [7,8].

Activity [49] and geometry [9] were found to morph these ordered lattices into amorphous or quasicrystalline structures. Here, we focus on the effect of sample thickness on the morphology of the disclination line patterns. For sufficiently thick samples, we find that the quasi-2D hexagonal lattice is destabilized and gives way to a fully 3D network of disclination lines, which is interpenetrated by an analogous network of skyrmion filaments [Fig. 9(b)], analogous to blue phases [7,8]. The liquid crystal patterns shown in Fig. 9 (as well as those reported in Refs. [7–9,49]) are all examples of spinor condensation, as there is a finite density of defects in equilibrium, such that the number of defects is extensive in the cross-sectional area of the sample. For the cases considered in this section, all patterns, whether quasi-2D or fully 3D, are locally Majorana-like, as the defect profile along the disclination [away from the disclination junction points in Fig. 9(b)] have locally a triradius structure [$\hat{\Omega} \cdot \hat{T} = \cos(\beta) = -1$] throughout, and there is no twist of the cross-sectional defect profile along the disclination lines. The triradius feature is promoted thermodynamically, and it would be of interest to see whether and how activity changes it. Indeed, in active nematics, extensile, but not contractile, activity promotes twistlike defect profiles over triradii and comets [20,50,51].

More, in general, skyrmions may arise also in different contexts in cholesterics, where they are associated with a skyrmion number [52]. While the skyrmion filaments discussed in Refs. [7,8] are equivalent to our λ disclinations, conventional skyrmions such as those considered in

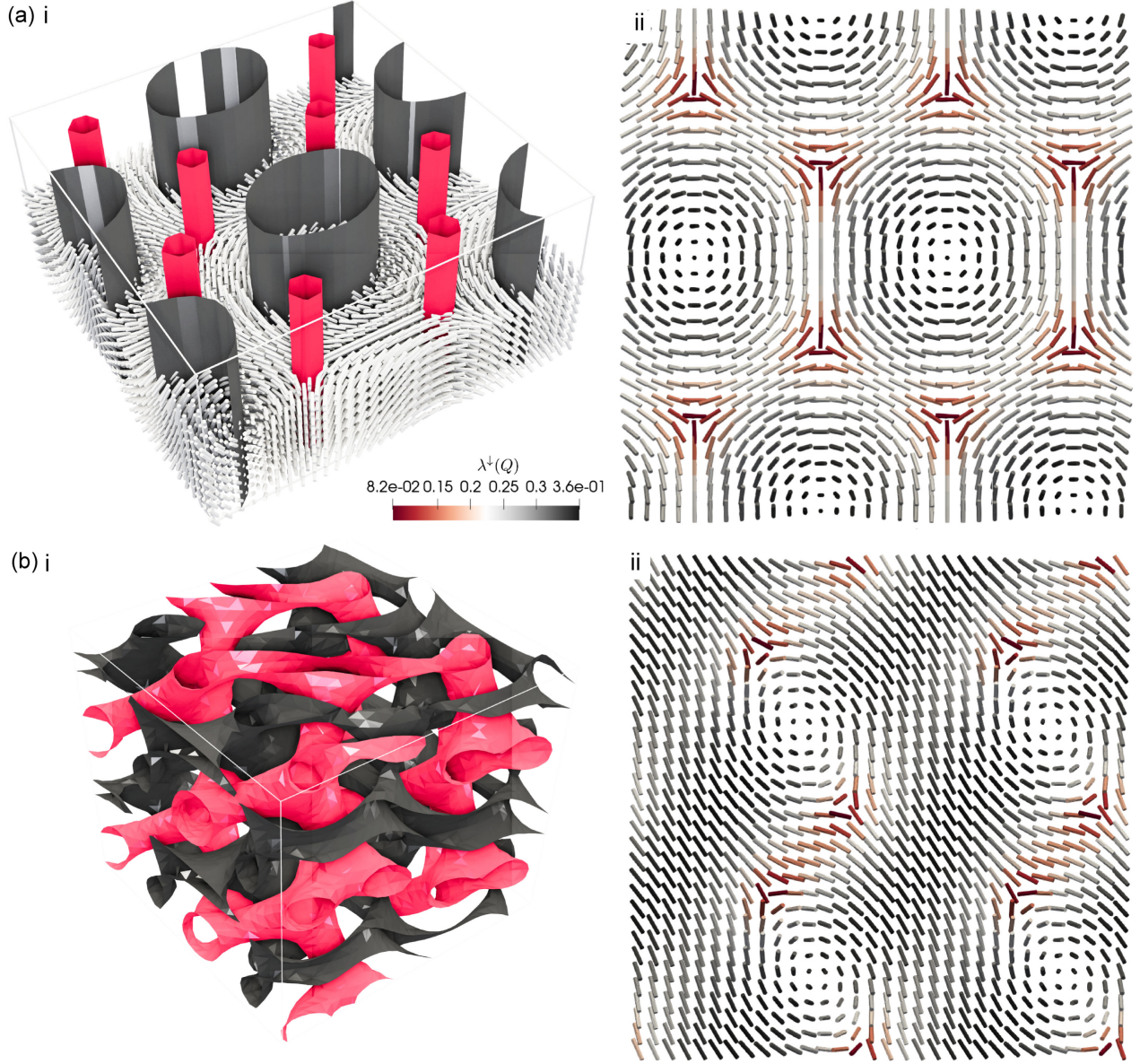


FIG. 9. (a) Quasi-2D phase. (i) Skyrmion filaments (black), visualized as isosurfaces of the twist parameter $S_{tw} = 0.001$, and disclination lines (red) visualized as isosurfaces of the highest eigenvalue of the Q tensor [$\lambda^1(Q) = 0.1$]. (ii) Cut showing the director field colored accordingly to $\lambda^1(Q)$. (b) 3D phase. (i) Skyrmion filaments and disclination lines in 3D. (ii) Cut showing the director field in the midplane. Colors are as in (a).

Ref. [52] are associated with a rotation of the primary director field, which is a multiple of 2π , rather than being equal to $\pm\pi$. Because of this, these quasiparticles are bosonic in our notation, although they are not simple scalar like the defect-half-skyrmion complexes in Fig. 9. Hence, they can be described by elements of the even subalgebra $Cl(3, 0, 1)^{[0]}$, which when squared have a scalar component equal to 1 (rather than -1 for fermionic quasiparticles).

X. DISCUSSION AND CONCLUSIONS

In this work, we have shown that local defect profiles in nematics and cholesterics behave as spinors and as

topological fermionic quasiparticles. For nematics, local 2D defect profiles are naturally interpreted as Majorana-like spinors, as the defects are topologically equivalent to the antidefects that annihilate with them [20]. Instead, cholesteric defects can be either Majorana-like (e.g., straight λ and τ disclinations) or Weyl-like (e.g., extended χ dislocations). In the latter case, the chirality is selected thermodynamically, as the underlying free energy favors one chirality over the other, ultimately due to the microscopic shape of the molecules or interactions between them.

These topological quasiparticles and their “scattering properties” after collisions are described by Clifford algebras, and we have seen that the algebras that describe

different types of liquid crystal defects—uniaxial nematics, biaxial, and cholesterics—are different. We started by showing that the two-dimensional algebra $\text{Cl}(2, 0)$ and, in particular, its even subalgebra, which can be identified with the complex numbers, is sufficient to describe planar defects in uniaxial nematics. Instead, $\text{Cl}(3, 0)$ or the quaternion algebra is required to algebraically represent out-of-plane uniaxial defects. Biaxial defects can be described by two copies of $\text{Cl}(2, 0)$ or $\text{Cl}(3, 0)$, for planar or out-of-plane configurations, respectively, where the two copies of the Clifford algebra correspond to the primary and secondary director field of the biaxial sample. Cholesterics require an even larger algebra than $\text{Cl}(3, 0)$, which accounts for both rotations and translations. This is $\text{Cl}(3, 0, 1)$, the even subalgebra of which is equivalent to that of dual quaternions. The difference in the underlying algebra shows why the multiplication rules of quaternions apply to the composition of biaxial nematic defects but not always to that of cholesteric ones, as discussed in Ref. [15]: This is because simple quaternions do not include translations along the helical axis as a physical transformation to consider.

The dual quaternion algebra allows us to understand how λ and τ defects can combine to form edge dislocations or helical screw χ dislocations under cylindrical confinement. Algebraically, these dislocations are composite defects with a fermionic nature, and accordingly, the director field features a π (rather than 2π) rotation when we consider a circular path encircling the defect. Composites can also have a bosonic nature. An example is edge dislocations with the Burgers vector equal to the pitch, which arise, for instance, in ferroelectric chiral nematics. Another example is provided by topological phases emerging in 3D bulk cholesteric samples, such as blue phases or other skyrmion lattices, which are made up of unit cells that are topologically uncharged. These unit cells behave as composite quasiparticles with bosonic nature and hence can condense to form space-filling lattices of disclinations. This phenomenon is reminiscent of the condensation of Cooper pairs in superconductors [48].

The algebraic framework we have described also provides a natural way to describe cholesteric defects as a combination of disclinations and dislocations, as the six bivectors in $\text{Cl}(3, 0, 1)^{[0]}$ account for the three rotational and three translational degrees of freedom identifying the Volterra process associated with the defect. This defect bivector constitutes a quantity akin to the electromagnetic tensor in quantum electrodynamics, and we hope it may provide a useful way to quantify the behavior of cholesteric disclinations in 3D in the future. It would also be of interest to combine this algebraic description of cholesteric defects with that provided by contact topology in Ref. [53]. Another avenue to explore in the future is whether the

Clifford algebra formalism can be exploited to hold additional geometric information. For instance, any two nematic local defect profiles can be connected by a rototranslation, which can be encoded by an element of $\text{Cl}(3, 0, 1)$. This description would allow us to capture defect alignment [25] and analyze how this changes, for instance, during recombination.

As discussed in Ref. [20], a 2D local profile behaves like a quasiparticle at rest. Here, we have considered either 2D systems, where this analogy holds, or 3D systems where disclination lines can be viewed as extended quasiparticles or quasistrings. In 3D, a closer qualitative equivalent of a localized quasiparticle would be a loop, and it would be interesting to use our Clifford algebra framework to identify the nature of these types of quasiparticles in cholesterics and biaxial nematics.

In conclusion, the theory we have outlined provides an unexpected connection between liquid crystal physics, particle physics, and quantum condensed matter. It may provide novel ways to realize quantum-mechanical behavior at the classical scale, facilitating the study—by experiments or computer simulations—of the properties of and interactions between quasiparticles with different spinor natures. Looking ahead, activity provides an additional intriguing parameter to consider, as it constitutes a way to make these quasiparticles mobile [20,54], endowing them with nontrivial dispersion relations.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: NUMERICAL SIMULATIONS

To model cholesteric liquid crystals in bulk and under cylindrical confinement, we considered as dynamical fields the nematic Q tensor $Q(\mathbf{r}, t)$, whose principle eigenvector $\mathbf{n}$ —the so-called “director field”—defines the local direction of alignment of the liquid crystal (LC) and the incompressible velocity field $\mathbf{v}(\mathbf{r}, t)$. The equilibrium properties of the system are described by the free energy

$$\mathcal{F} = \int (f^{\text{bulk}} + f^{\text{el}}) d\mathbf{r}, \quad (\text{A1})$$

where

$$f^{\text{bulk}} = A_0 \left[\frac{1}{2} \left(1 - \frac{\chi}{3} \right) \text{Tr} \mathbf{Q}^2 - \frac{\chi}{3} \text{Tr} \mathbf{Q}^3 + \frac{\chi}{4} (\text{Tr} \mathbf{Q}^2)^2 \right], \quad (\text{A2})$$

$$f^{\text{el}} = f^{\text{SB}} + f^{\text{tw}} = \frac{L}{2} [(\nabla \cdot \mathbf{Q})^2 + (\nabla \times \mathbf{Q} + 2q_0 \mathbf{Q})^2]. \quad (\text{A3})$$

In Eq. (A2), the bulk constant $A_0 > 0$, while χ is a temperaturelike parameter that drives the isotropic-nematic transition that occurs for $\chi > \chi_{\text{cr}} = 2.7$ [24]. In the case of cylindrical confinement, we set $\chi > \chi_{\text{cr}}$ inside the cylinder and $\chi < \chi_{\text{cr}}$ otherwise. The elastic free-energy density in Eq. (A3), proportional to the elastic constant L , captures the energy cost of elastic deformations in the single elastic constant approximation [24]. It has been split into a splay-bend contribution f^{SB} and a twist contribution f^{tw} , which stabilizes helical structures when the chiral wave number q_0 is nonzero. For $q_0 > 0$, the equilibrium configuration in unconfined geometries features a right-handed helix with pitch $p_0 = 2\pi/q_0$.

The following set of coupled partial differential equations governs the dynamics of the fields,

$$D_t \mathbf{Q} = \mathbf{S}(\mathbf{W}, \mathbf{Q}) + \gamma^{-1} \mathbf{H}, \quad (\text{A4})$$

$$\rho D_t \mathbf{v} = \nabla \cdot (\boldsymbol{\sigma}^{\text{hydro}} + \boldsymbol{\sigma}^{\text{LC}}), \quad (\text{A5})$$

where the differential operator $D_t = \partial_t + \mathbf{v} \cdot \nabla$ is the material derivative. Equation (A4) is the Beris-Edwards equation ruling the dynamics of the $\mathbf{Q}$ tensor. The operator $\mathbf{S}(\mathbf{W}, \mathbf{Q})$ denotes the corotational derivative and defines the dynamical response of the LC to straining and shearing. Its explicit expression depends on both the velocity gradient $\mathbf{W} = \nabla \mathbf{v}$ and the $\mathbf{Q}$ -tensor configuration [see Eq. (A8) for the explicit expression]. The coefficient γ is the rotational viscosity measuring the importance of advection relative to relaxation, and the molecular field $\mathbf{H} = -(\delta \mathcal{F} / \delta \mathbf{Q}) + (\mathbf{I}/3) \text{Tr}(\delta \mathcal{F} / \delta \mathbf{Q})$. Finally, Eq. (A5) is the Navier-Stokes equation for the incompressible velocity field ($\nabla \cdot \mathbf{v} = 0$) with constant density ρ . Here, the stress tensor has been divided in (i) a hydrodynamic contribution $\boldsymbol{\sigma}^{\text{hydro}} = -P\mathbf{I} + \eta \nabla \mathbf{v}$ accounting for the hydrodynamic pressure P , ensuring incompressibility, and viscous effects, proportional to the viscosity η and (ii) a LC contribution $\boldsymbol{\sigma}^{\text{LC}}$ accounting for elastic and flow-aligning effects. The explicit expression of the corotational derivative $\mathbf{S}(\mathbf{W}, \mathbf{Q})$ appearing in Eq. (A4) is given by

$$\mathbf{S}(\mathbf{W}, \mathbf{Q}) = (\xi \mathbf{D} + \boldsymbol{\Omega})(\mathbf{Q} + \mathbf{I}/3) \quad (\text{A6})$$

$$+ (\xi \mathbf{D} - \boldsymbol{\Omega})(\mathbf{Q} + \mathbf{I}/3) \quad (\text{A7})$$

$$- 2\xi(\mathbf{Q} + \mathbf{I}/3) \text{Tr}(\mathbf{Q}\mathbf{W}). \quad (\text{A8})$$

Here, $\mathbf{D} = (\mathbf{W} + \mathbf{W}^T)/2$ and $\boldsymbol{\Omega} = (\mathbf{W} - \mathbf{W}^T)/2$ are the symmetric and antisymmetric part of the velocity gradient tensor $W_{\alpha\beta} = \partial_\beta v_\alpha$, respectively. The flow alignment parameter ξ determines the aspect ratio of the LC molecules and the dynamical response of the LC to an imposed shear flow. Here, we choose $\xi = 0.7$ to consider flow-aligning rodlike molecules.

The explicit expression of the LC contribution is

$$\begin{aligned} \sigma_{\alpha\beta}^{\text{LC}} = & -\xi H_{\alpha\gamma} \left(Q_{\gamma\beta} + \frac{1}{3} \delta_{\gamma\beta} \right) - \xi \left(Q_{\alpha\gamma} + \frac{1}{3} \delta_{\alpha\gamma} \right) H_{\gamma\beta} \\ & + 2\xi \left(Q_{\alpha\beta} + \frac{1}{3} \delta_{\alpha\beta} \right) Q_{\gamma\mu} H_{\gamma\mu} + Q_{\alpha\gamma} H_{\gamma\beta} - H_{\alpha\gamma} Q_{\gamma\beta} \\ & - \partial_\alpha Q_{\gamma\mu} \frac{\partial f}{\partial (\partial_\beta Q_{\gamma\mu})}. \end{aligned} \quad (\text{A9})$$

We integrated the dynamics of the hydrodynamic fields in Eq. (A5) in a cubic grid of size $\mathcal{L} = 32, 64, 128$ using a predictor-corrector hybrid lattice Boltzmann approach [56,57]. This consists of solving Eq. (A4) with a finite-difference algorithm implementing the first-order upwind scheme and fourth-order accurate stencils for space derivatives, as well as the Navier-Stokes equation through a predictor-corrector lattice Boltzmann scheme on a $D3Q15$ lattice. For technical details on the lattice Boltzmann method, see Refs. [57,58].

In all simulations, the velocity field was initialized to 0, while the $\mathbf{Q}$ tensor was initially set as described in the two following sections. These sections also contain the full parameter sets used to obtain the configurations in Figs. 6–8.

1. Cholesteric liquid crystals under cylindrical confinement

The results presented in Figs. 6 and 7 were obtained by confining a cholesteric liquid crystal inside a cylinder (with no, or free, anchoring at the boundaries). This has been achieved by imposing $\chi = 3$ (corresponding to the cholesteric phase) inside a cylinder of radius $R = 15$, and $\chi = 2$ (corresponding to the isotropic phase) otherwise. Other parameters were set as follows: $q_0 = \pi/16$, $A_0 = 0.02$, $L = 0.01$, $\xi = 0.7$, $\gamma = 0.33775$, and the viscosity $\eta = 5/3$. The simulation box was a cube with size $L_x = L_y = L_z = 64$ and periodic boundary conditions. The cylindrical axis was taken along the x axis.

For Fig. 6(a), we initialized the director field as

$$\begin{aligned} n_x &= 0, \\ n_y &= \cos(-\phi/2 + q_0 x), \\ n_z &= \sin(-\phi/2 + q_0 x), \end{aligned} \quad (\text{A10})$$

while for Fig. 6(b),

$$\begin{aligned} n_x &=, 0 \\ n_y &= \cos(\phi/2 + q_0x), \\ n_z &= \sin(\phi/2 + q_0x). \end{aligned} \quad (\text{A11})$$

For Fig. 7, we set

$$\begin{aligned} n_x &= \sin(q_0\mathbf{r} \cdot \hat{\mathbf{h}}), \\ n_y &= \cos(-\phi/2 + q_0x) \cos(q_0\mathbf{r} \cdot \hat{\mathbf{h}}), \\ n_z &= \sin(-\phi/2 + q_0x) \cos(q_0\mathbf{r} \cdot \hat{\mathbf{h}}), \end{aligned} \quad (\text{A12})$$

where $\mathbf{r} = (x, y, z)$ and $\hat{\mathbf{h}} = (\hat{h}_x, \hat{h}_y, \hat{h}_z)$, with

$$\begin{aligned} \hat{h}_x &= 0, \\ \hat{h}_y &= -\sin(\phi/2), \\ \hat{h}_z &= \cos(\phi/2). \end{aligned} \quad (\text{A13})$$

Finally, to initialize the χ^{+1} screw dislocation resulting in the defect ring array formation in Fig. 8, we set

$$\begin{aligned} n_x &= 0, \\ n_y &= \cos(\phi + q_0x), \\ n_z &= \sin(\phi + q_0x). \end{aligned} \quad (\text{A14})$$

2. 3D cholesterics

For Fig. 9, we performed simulations for 3D bulk cholesterics, with periodic boundary conditions. Parameters were set as follows: $\chi = 3$, $q_0 = \pi/16$, $A_0 = 0.005$, $L = 0.01$, $\xi = 0.7$, $\gamma = 0.33775$, and the viscosity $\eta = 5/3$. The simulation box was a parallelepiped with sizes $L_x = L_y = 32$, $L_z = 16$ in Fig. 9(a) and a cube with sizes $L_x = L_y = L_z = 32$ in Fig. 9(b).

APPENDIX B: SIMULATED OPTICAL MICROSCOPY IMAGES

The polarized optical microscopy images presented in Figs. 6 and 7 have been obtained following the procedure presented in Ref. [44], using the accompanying PYTHON package LCPOM, deposited in GitHub [59].

To generate single-wavelength (gray scale) micrographs from our simulation data, we begin by obtaining the tensor order parameter for each point in a regular grid. From the $\mathbf{Q}$ tensor, the pipeline extracts the local director and scalar order parameter S . It then computes the ordinary refractive indices at the chosen wavelength (500 nm for all the cases considered, as no appreciable variation in the resulting pattern was observed at other wavelengths). Next, the sample is discretized along the optical axis into a series

of thin layers, each containing uniform orientation and birefringence. To capture how linearly polarized light propagates through these layers, the Jones matrix method is used [60], multiplying the Jones matrices of all layers in succession. This approach determines how the phase and polarization state are changed within each layer, ultimately yielding, for every pixel, a final transmitted intensity measured between ideal crossed polarizers. The resulting two-dimensional intensity distribution thus corresponds directly to a gray-scale polarized optical micrograph, facilitating comparison with experiments.

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- [1] D. C. Wright and N. D. Mermin, *Crystalline liquids: The blue phases*, *Rev. Mod. Phys.* **61**, 385 (1989).
 - [2] Bryan Gin-ge Chen, P. J. Ackerman, G. P. Alexander, R. D. Kamien, and I. I. Smalyukh, *Generating the Hopf fibration experimentally in nematic liquid crystals*, *Phys. Rev. Lett.* **110**, 237801 (2013).
 - [3] N. D. Mermin, *The topological theory of defects in ordered media*, *Rev. Mod. Phys.* **51**, 591 (1979).
 - [4] O. Henrich, K. Stratford, M. E. Cates, and D. Marenduzzo, *Structure of blue phase III of cholesteric liquid crystals*, *Phys. Rev. Lett.* **106**, 107801 (2011).
 - [5] P. J. Ackerman and I. I. Smalyukh, *Diversity of knot solitons in liquid crystals manifested by linking of preimages in torons and hopfions*, *Phys. Rev. X* **7**, 011006 (2017).
 - [6] J.-S. Wu and I. I. Smalyukh, *Hopfions, heliknotons, skyrmions, torons and both Abelian and non-Abelian vortices in chiral liquid crystals*, *Liq. Cryst. Rev.* **10**, 34 (2022).
 - [7] J. Pišljarić, S. Ghosh, S. Turlapati, N. V. S. Rao, M. Škarabot, A. Mertelj, A. Petelin, A. Nych, M. Marincic, A. Pusovnik, M. Ravnik, and I. Mušević, *Blue phase III: Topological fluid of skyrmions*, *Phys. Rev. X* **12**, 011003 (2022).
 - [8] J. Pišljarić, S. Ghosh, S. Turlapati, N. V. S. Rao, M. Škarabot, A. Mertelj, A. Petelin, A. Pusovnik, M. Ravnik, and I. Mušević, *Skyrmions in blue phases of chiral liquid crystals*, *Liq. Cryst.* **50**, 1406 (2023).
 - [9] L. N. Carenza, G. Gonnella, D. Marenduzzo, G. Negro, and E. Orlandini, *Cholesteric shells: Two-dimensional blue fog and finite quasicrystals*, *Phys. Rev. Lett.* **128**, 027801 (2022).
 - [10] G. Negro, L. N. Carenza, G. Gonnella, D. Marenduzzo, and E. Orlandini, *Topological phases and curvature-driven pattern formation in cholesteric shells*, *Soft Matter* **19**, 1987 (2023).
 - [11] G. P. Alexander and J. M. Yeomans, *Stabilizing the blue phases*, *Phys. Rev. E* **74**, 061706 (2006).
 - [12] G. P. Alexander and D. Marenduzzo, *Cubic blue phases in electric fields*, *Europhys. Lett.* **81**, 66004 (2008).
 - [13] S. Čopar and S. Žumer, *Quaternions and hybrid nematic disclinations*, *Proc. R. Soc. A* **469**, 20130204 (2013).
 - [14] S. Čopar, *Topology and geometry of nematic braids*, *Phys. Rep.* **538**, 1 (2014).
 - [15] D. A. Beller, T. Machon, S. Čopar, D. M. Sussman, G. P. Alexander, R. D. Kamien, and R. A. Mosna, *Geometry of the cholesteric phase*, *Phys. Rev. X* **4**, 031050 (2014).

- [16] P. Lounesto, *Clifford Algebras and Spinors* (Cambridge University Press, Cambridge, England, 1997).
- [17] P. Renaud, *Clifford Algebras: Lecture Notes on Applications in Physics* (2020), available online at <https://hal.science/hal-03015551/document>.
- [18] M. A. Clifford, *Preliminary sketch of biquaternions*, *Proc. London Math. Soc.* **s1-4**, 381 (1871).
- [19] C. Doran and A. Lasenby, *Geometric Algebra for Physicists* (Cambridge University Press, Cambridge, England, 2003).
- [20] L. C. Head *et al.*, *Majorana quasiparticles and topological phases in 3d active nematics*, *Proc. Natl. Acad. Sci. U.S.A.* **121**, e2405304121 (2024).
- [21] M. R. Francis and A. Kosowsky, *The construction of spinors in geometric algebra*, *Ann. Phys. (Amsterdam)* **317**, 383 (2005).
- [22] D. Hestenes, *Real spinor fields*, *J. Math. Phys. (N.Y.)* **8**, 798 (1967).
- [23] C. D. Schimming and J. Viñals, *Singularity identification for the characterization of topology, geometry, and motion of nematic disclination lines*, *Soft Matter* **18**, 2234 (2022).
- [24] P. de Gennes and J. Prost, *The Physics of Liquid Crystals*, International Series of Monographs on Physics (Clarendon Press, New York, 1993).
- [25] Ž. Kos and J. Dunkel, *Nematic bits and universal logic gates*, *Sci. Adv.* **8**, eabp8371 (2022).
- [26] Note that the odd part of the algebra, $Cl(3, 0)^{[1]}$, is not a subalgebra.
- [27] M. V. Kurik and O. Lavrentovich, *Defects in liquid crystals: Homotopy theory and experimental studies*, *Sov. Phys. Usp.* **31**, 196 (1988).
- [28] O. D. Lavrentovich and M. Kleman, *Cholesteric liquid crystals: Defects and topology*, in *Chirality in Liquid Crystals*, edited by H.-S. Kitzerow and C. Bahr (Springer, New York, 2001), p. 115.
- [29] A. Kitaev, *Anyons in an exactly solved model and beyond*, *Ann. Phys. (Amsterdam)* **321**, 2 (2006).
- [30] L. H. Kauffman and S. J. Lomonaco Jr., *Braiding with Majorana fermions*, in *Quantum Information and Computation IX* (2016), Vol. 9873, p. 98730E, [10.1117/12.2228510](https://doi.org/10.1117/12.2228510).
- [31] I. I. Smalyukh and O. D. Lavrentovich, *Three-dimensional director structures of defects in Grandjean-Cano wedges of cholesteric liquid crystals studied by fluorescence confocal polarizing microscopy*, *Phys. Rev. E* **66**, 051703 (2002).
- [32] K. Thapa, O. S. Iadlovskaya, B. Basnet, H. Wang, A. Paul, J. T. Gleeson, and O. D. Lavrentovich, *Confinement and magnetic-field effect on chiral ferroelectric nematic liquid crystals in Grandjean-Cano wedge cells*, *Phys. Rev. E* **109**, 054702 (2024).
- [33] X. Zhao, J. Zhou, J. Li, J. Kougo, Z. Wan, M. Huang, and S. Aya, *Spontaneous helielectric nematic liquid crystals: Electric analog to helimagnets*, *Proc. Natl. Acad. Sci. U.S.A.* **118**, e2111101118 (2021).
- [34] D. Hestenes, *Spacetime physics with geometric algebra*, *Am. J. Phys.* **71**, 691 (2003).
- [35] A. R. Fialho, N. R. Bernardino, N. M. Silvestre, and M. M. Telo da Gama, *Effect of curvature on cholesteric liquid crystals in toroidal geometries*, *Phys. Rev. E* **95**, 012702 (2017).
- [36] S. Čopar, T. Porenta, and S. Žumer, *Visualisation methods for complex nematic fields*, *Liq. Cryst.* **40**, 1759 (2013).
- [37] A. Kilian and A. Sonnet, *On the analysis of twisted director configurations*, *Z. Naturforsch.* **50**, 991 (1995).
- [38] A. Perez-Gracia and F. Thomas, *On Cayley's factorization of 4D rotations and applications*, *Adv. Appl. Clifford Algebras* **27**, 523 (2017).
- [39] M. Hunt, G. Mullineux, R. J. Cripps, and B. Cross, *Characterizing isoclinic matrices and the Cayley factorization*, *Proc. Inst. Mech. Eng., Part C* **230**, 3267 (2016).
- [40] H.-S. Kitzerow, B. Liu, F. Xu, and P. P. Crooker, *Effect of chirality on liquid crystals in capillary tubes with parallel and perpendicular anchoring*, *Phys. Rev. E* **54**, 568 (1996).
- [41] F. Lequeux and M. Kléman, *Helicoidal instability in cholesteric capillary tubes*, *J. Phys.* **49**, 845 (1988).
- [42] Y. Bouligand, *Recherches sur les textures des états mésomorphes. 6—Dislocations coins et signification des cloisons de Grandjean-Cano dans les cholestériques*, *J. Phys.* **35**, 959 (1974).
- [43] A. Bates and A. Maxwell, *DNA Topology*, Oxford Bioscience (Oxford University Press, New York, 2005).
- [44] C. Chen, V. Palacio-Betancur, S. Norouzi, P. F. Zubietarico, N. Chang, M. Sadati, S. J. Rowan, and J. J. de Pablo, *LCPO: Precise reconstruction of polarized optical microscopy images of liquid crystals*, *Chem. Mater.* **36**, 3081 (2024).
- [45] D. Seč, T. Porenta, M. Ravnik, and S. Žumer, *Geometrical frustration of chiral ordering in cholesteric droplets*, *Soft Matter* **8**, 11982 (2012).
- [46] R. M. Hornreich, M. Kugler, and S. Shtrikman, *Possibility of a field-induced hexagonal blue phase in cholesteric liquid crystals*, *Phys. Rev. Lett.* **54**, 2099 (1985).
- [47] O. Henrich, D. Marenduzzo, K. Stratford, and M. E. Cates, *Thermodynamics of blue phases in electric fields*, *Phys. Rev. E* **81**, 031706 (2010).
- [48] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Theory of superconductivity*, *Phys. Rev.* **108**, 1175 (1957).
- [49] L. Metselaar, A. Doostmohammadi, and J. M. Yeomans, *Topological states in chiral active matter: Dynamic blue phases and active half-skyrmions*, *J. Chem. Phys.* **150**, 064909 (2019).
- [50] J. Binysh, Ž. Kos, S. Čopar, M. Ravnik, and G. P. Alexander, *Three-dimensional active defect loops*, *Phys. Rev. Lett.* **124**, 088001 (2020).
- [51] G. Negro, L. C. Head, L. N. Carenza, T. N. Shendruk, D. Marenduzzo, G. Gonnella, and A. Tiribocchi, *Topology controls flow patterns in active double emulsions*, *Nat. Commun.* **16**, 1412 (2025).
- [52] C. D. Schimming and J. Viñals, *A tensor density measure of topological charge in three-dimensional nematic phases*, *Proc. R. Soc. A* **480**, 20230564 (2024).
- [53] J. Pollard and G. P. Alexander, *Contact topology and the classification of disclination lines in cholesteric liquid crystals*, *Phys. Rev. Lett.* **130**, 228102 (2023).
- [54] L. C. Head, C. Doré, R. R. Keogh, L. Bonn, G. Negro, D. Marenduzzo, A. Doostmohammadi, K. Thijssen, T. López-León, and T. N. Shendruk, *Spontaneous self-constraint in active nematic flows*, *Nat. Phys.* **20**, 492 (2024).
- [55] <https://www.archer2.ac.uk>.

- [56] S. Succi, *The Lattice Boltzmann Equation: For Complex States of Flowing Matter* (Oxford University Press, Oxford, 2018).
- [57] L. N. Carenza, G. Gonnella, A. Lamura, G. Negro, and A. Tiribocchi, *Lattice Boltzmann methods and active fluids*, *Eur. Phys. J. E* **42**, 81 (2019).
- [58] G. Negro, G. Gonnella, A. Lamura, S. Busuioc, and V. Sofonea, *Growth regimes in three-dimensional phase separation of liquid-vapor systems*, *Phys. Rev. E* **109**, 015305 (2024).
- [59] <https://github.com/depablogroup/lc-pom>.
- [60] P. W. Ellis, E. Pairam, and A. Fernández-Nieves, *Simulating optical polarizing microscopy textures using Jones calculus: a review exemplified with nematic liquid crystal tori*, *J. Phys. D* **52**, 213001 (2019).