

Free and Interacting Fermionic Conformal Field Theories on the Fuzzy Sphere

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The fuzzy-sphere regularization is a powerful tool to study conformal field theories (CFT) in three spacetime dimensions. In this paper, we extend its scope to CFTs with local fermionic operators. We realize the free-Majorana-fermion CFT on a setup with one flavor of bosons and one flavor of fermions on the lowest Landau level with a $1/2$ angular momentum mismatch and allow conversion between two bosons and two fermions, and use a relative chemical potential as the tuning parameter. On the phase diagram, we observe two continuous transitions described, respectively, by a free Majorana fermion and a gauged Ising CFT. We numerically confirm the emergent conformal symmetry through the operator spectrum and the two-point correlation function of the local Majorana fermion. We further establish a correspondence between the fuzzy-sphere models and the field-theory Lagrangians, and extend it to an interacting fermionic CFT—the super-Ising theory with emergent superconformal symmetry.

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I. INTRODUCTION

Conformal field theory (CFT) is one of the central topics of modern physics. CFT has provided important insights into various aspects of theoretical physics. In condensed-matter physics, it has produced useful predictions about the critical phenomena [1–3]. Many classical and quantum phase transitions are conjectured to have emergent conformal symmetry in the infrared (IR). In high-energy physics and quantum field theories, CFT also plays important role in understanding string theory [4], duality with gravitational theories on anti-de Sitter (AdS) space [5], and renormalization group flow structure [6]. In two dimensions, the infinite-dimensional conformal algebra has made many theories exactly solvable [7–9]. However, going to the higher dimensions, the CFTs are less well studied due to a much smaller conformal group, although some of the conformal data are determined at high precision by conformal bootstrap [10,11].

Fermionic CFTs, i.e., CFTs that include fermionic local operators, constitute a particularly interesting type of CFTs, including Gross-Neveu-Yukawa theories and certain gauge theories with or without Chern-Simons (CS) term. In some

special cases, these CFTs can have emergent supersymmetry. For example, the theory of a single Majorana fermion χ coupled to a real Yukawa scalar field σ flows to a superconformal field theory (SCFT) with $\mathcal{N} = 1$ in the IR, known as the super-Ising theory [12–16], where σ , χ , and $\epsilon = \sigma^2$ combine into a superconformal multiple and their scaling dimensions are locked together. Experimentally, these fermionic CFTs are easier to detect than their bosonic counterparts, as they produce rich electric signals, to which most experimental probes are naturally sensitive. For example, the frequency dependence of conductivity can be measured through electric transport experiments, and it corresponds to the OPE of the conserved symmetry current $J \times J$ [17,18], including the central charge C_J from its two-point function. An outstanding platform is the Moiré materials [19–21]. Through tuning many available knobs like potential amplitude, they can be used to realize plateau transitions between fractional Chern insulators [22–24]. However, the nature and critical data of these theories remain poorly understood.

The fuzzy-sphere regularization [25] has emerged as a new powerful method to study 3D CFTs. This method involves studying quantum systems on a sphere that is “fuzzy” (noncommutative) due to a magnetic monopole at its center [26]. It offers distinct advantages including the exact preservation of rotation symmetry, the direct observation of emergent conformal symmetry and the efficient extraction of conformal data. In the fuzzy-sphere method, the state-operator correspondence plays an essential role.

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Specifically, there is a one-to-one correspondence between the eigenstates of the critical Hamiltonian on the sphere and the CFT operators, where the energy gaps are proportional to the scaling dimensions. The power of this approach has been first demonstrated in the context of the 3D Ising transition [25], where the presence of emergent conformal symmetry has been convincingly established. Since its proposal, various studies have greatly expanded its horizon, including

- (1) accessing the “conformal data” of the 3D Ising CFT, such as the OPE coefficients [27], correlation functions [28], entropic F function [29], conformal generators [30,31], and cross-cap coefficients [32],
- (2) developing and applying techniques to improve the numerical precision, such as quantum Monte Carlo simulation [33] and conformal perturbation [34],
- (3) realising various 3D CFTs at integer filling, such as the free scalar [35,36] and Wilson-Fisher [37] CFTs, critical gauge theories realized through nonlinear σ model including $SO(5)$ [38,39] and $O(4)$ [40] deconfined criticality and a series of new $Sp(N)$ CFTs [41], Potts model [42], and Yang-Lee criticality [43–45],
- (4) exploring fractional quantum Hall (FQH) transitions, such as the Ising CFT on a FQH background [46] and the confinement transition of the $\nu = 1/2$ bosonic Laughlin state [47], and
- (5) studying the conformal defects (e.g., the pinning-field defect [48] and its g function [49] and cusp [50]), conformal boundaries [51,52], and lower-dimensional CFTs [53].

All the CFTs that have been realized on the fuzzy sphere so far have purely bosonic operator content. Apart from the fuzzy-sphere regularization, a recent exploratory attempt to access the fermionic Gross-Neveu-Yukawa CFT considered Dirac fermions on a conventional sphere, using a basis of spinor spherical harmonics [54]. Although intuitive, this approach suffers from ultraviolet (UV) divergences similar to those in standard QFTs on continuous space, and thus further theoretical and numerical efforts may be required to justify its validity. In contrast, the fuzzy-sphere approach, similar to the highly successful lattice regularization of QFTs and strongly interacting condensed-matter lattice models, is free of UV divergences [35,55]. Consequently, the study of fermionic CFTs would greatly benefit from a fuzzy-sphere realization, especially in light of the success achieved for their bosonic counterparts.

Nevertheless, the fuzzy-sphere realization of fermionic CFTs faces some challenges. Most fuzzy-sphere models are built of microscopic fermions, i.e., the fermions with $U(1)$ electric charge on the lowest Landau level. These microscopic fermions experience the magnetic field generated by the monopole and thus become fuzzy, in the sense that they have strong noncommutativity inherited from the magnetic field [56]. The resolution is that these microscopic fermions

must be gapped, so that the electric charge $U(1)$ is decoupled from the IR critical theory. Gapless “local” CFT operators on the fuzzy sphere tend to involve bilinear expressions in the microscopic fermions, which are charge neutral and bosonic. Moreover, another challenge is that the fermionic field in the CFT should carry half-integer spin (angular momentum), a requirement that microscopic fermions may not automatically satisfy. In this paper, we show that these challenges can be overcome by a fuzzy-sphere model with boson-fermion mixture. Concretely, we consider a setup with both microscopic bosons b and fermions f with the same electric charge and let their angular momenta differ by $1/2$. Their bilinears $b^\dagger f$ or $f^\dagger b$ are neutral under electric charge and have fermionic statistics as well as half-integer spins. Therefore, the bilinears can be used to realize fermionic local operators in a CFTs [57].

The simplest fermionic CFT to target is a massless free Majorana fermion. Its Lagrangian is

$$\mathcal{L}_{\text{Majorana}} = \frac{1}{2} \bar{\chi} (i \not{\partial} - m) \chi, \quad (1)$$

where χ is a two component spinor, $\bar{\chi} = \chi^T \epsilon$, $\epsilon = i\gamma^2$ [58], and $\not{\partial} = \gamma^\mu \partial_\mu$. The free Majorana fermion χ has exact scaling dimension $\Delta_\chi = 1$ and is subject to the equation of motion $\not{\partial} \chi = 0$ [59]. It describes a transition driven by the mass m . Masses with different signs flow, respectively, to a trivially gapped phase and a $\nu_K = 1$ Majorana quantum Hall (MQH) phase [60,61]. The MQH is a gapped phase with no topological order characterized by thermal Hall conductance or chiral central charge $c_- = 1/2$ and a chiral free Majorana fermion mode on the edge. The Kitaev index ν_K appears in the “16-fold way” classification of topological superconductors from spinless fermion [62] and gives the number of chiral edge Majorana fermions and the chiral central charge $c_- = \nu_K/2$ [63]. It is also known as the p -wave topological chiral superconductor (TSC) [64].

In this paper, we study the setup with one flavor of fermions and one flavor of bosons on the fuzzy sphere with the same electric charge but a mismatch $1/2$ in the angular momenta. We set the total filling $\nu = 1$ and allow the conversion between two fermions and two bosons. By tuning a relative chemical potential, we identify a fermionic integer quantum Hall (fIQH) phase at $\nu = 1$ when fermions are energetically favorable, a bosonic Pfaffian (bPf) phase [60,65] at $\nu_b = 1$ with topological order characterized by a $SU(2)_2$ Chern-Simons theory when bosons are energetically favorable, and an intermediate gapped phase corresponding to a $\nu_K = 3$ MQH (i.e., a $\nu_K = 1$ MQH on the background of a $\nu = 1$ fIQH state, or a f -wave TSC). We show theoretically and numerically that the fIQH-MQH transition is described by a free Majorana fermion. In particular, the energy gap, operator spectrum,

and two-point correlation function of the local fermion all agree well with the free-Majorana-fermion CFT.

On the other hand, the MQH-bPf transition is described by an Ising CFT coupled to a $\mathbb{Z}_2$ gauge field. The gauged Ising universality, also known as Ising*, was first studied in the confinement transition of $\mathbb{Z}_2$ topological order [66–72]. While the Ising* criticality was first proposed to describe the transition between a trivially gapped state and a $\mathbb{Z}_2$ topological order, in our setup, confining a $\mathbb{Z}_2$ -gauge field drives the bPf phase to a MQH with trivial topological order. The only difference is a MQH background that does not change the universality. This is numerically confirmed by the agreement of the operators spectrum with the $\mathbb{Z}_2$ -even sector of the Ising CFT. An interesting feature of the gauged Ising CFT is the topological Wilson line defect. We show that a $\mathbb{Z}_2$ -gauge charge as the end point of the Wilson line can be inserted by inserting a 2π monopole flux, and the end-point operators correspond to the $\mathbb{Z}_2$ -odd sector of the Ising CFT.

We observe an analogy between the fuzzy-sphere models and the QFT Lagrangians in the realization of both free Majorana fermion and critical real scalar. In particular, a flavor serves as the reference state from which the CFT ground state can be regarded as a perturbation. The elementary fields that constitute the field theory are identified as hopping from the reference flavor to the excitation flavors. The interaction terms in the Hamiltonian can be identified with the terms in the QFT Lagrangian. We extend this correspondence to a simple interacting fermionic theory—the super-Ising theory with a Majorana fermion coupled to a real scalar through a Yukawa interaction. We provide numerical evidence for the emergence of superconformal symmetry and the agreement with the conformal bootstrap results.

In short, our boson-fermion mixture construction successfully extends the fuzzy-sphere approach to free and interacting fermionic CFTs, opening avenues to explore a broad class of theories, including the Gross-Neveu-Yukawa and quantum Hall transitions relevant to Moiré materials. A key ingredient is to create a half-integer fermionic field through the angular momentum offset of $1/2$ between microscopic fermions and bosons. This idea might be generalizable to define CFTs such that the most relevant operators have nontrivial spin: a “pure-symmetry” CFT such that the simplest local operators are conserved currents of spin-1, or a “pure-energy” CFT such that the simplest local operator is the stress tensor, a property expected from holographic duals to theories of quantum gravity [73–75]. Both would be examples of “self-organized” critical theories stable to any deformation. It may also give a route to engineer spin-1 gauge fields and thus implement general renormalizable Lagrangians of quantum field theories on the fuzzy sphere. It may also be possible to reach effective field theories with higher spin quantum fields. The realization of the gauged Ising CFT

provides new possibilities in studying line defects like the monodromy defect in Ising CFT.

This paper is organized as the following: In Sec. II, we present the setup of fuzzy sphere and our model Hamiltonian. In Sec. III, we present the phase diagram supported by the numerical evidences for the gapped phases. In Sec. IV, we analyze the critical theories that describe the two continuous transitions between the three gapped phases. Then, Secs. V and VI are devoted to the numerical results, respectively, for the free-Majorana-fermion CFT and the gauged Ising CFT. In Sec. VII, we establish the correspondence between fuzzy-sphere models and QFT Lagrangians and discuss the realization of the super-Ising theory in light of this, and present in Sec. VIII the numerical evidence for the emergent superconformal symmetry. Finally, in Sec. IX, we make a brief summary and discuss the implications of our work and future directions.

II. MODEL

In the setup of the fuzzy sphere [76], we consider one flavor of fermions f and one flavor of bosons b with the same electric charge on the sphere under the influence of a magnetic monopole at its center. We let the monopole charge they experience differ by 2π , i.e., we let the fermion experience a $4\pi q$ monopole and the bosons experience a $4\pi(q - \frac{1}{2})$ monopole. As will be elaborated below, the mismatch of the monopole flux reflects the coupling between the electric charge and the spatial curvature, and it also allows the fermionic local operators like $f^\dagger b$ to have half-integer angular momenta. Because of the presence of the monopole, the single-particle eigenstates form highly degenerate quantized Landau levels [77]. The lowest Landau level (LLL) has angular momentum q and degeneracy $N_{mf} = 2q + 1$ for the fermions and angular momentum $q - \frac{1}{2}$ and degeneracy $N_{mb} = 2q$ for the bosons. By setting the single-particle gap the leading energy scale, we project the system onto the LLL and express the fermion and boson operators in terms of the creation and annihilation operators on the LLL

$$\begin{aligned} f^\dagger(\mathbf{r}) &= \frac{1}{R} \sum_{m=-q}^q f_m^\dagger Y_{qm}^{(q)}(\mathbf{r}), \\ b^\dagger(\mathbf{r}) &= \frac{1}{R} \sum_{m=-q+1/2}^{q-1/2} b_m^\dagger Y_{q-1/2,m}^{(q-1/2)}(\mathbf{r}), \end{aligned} \quad (2)$$

where the radius of the sphere is set as $R = N_{mf}^{1/2}$ [25], and the single-particle wave functions are the spin-weighted spherical harmonics [78,79].

The symmetries we preserve are

- (1) A $U(1)_e$ corresponding to the conservation of total electric charge $N_e = N_b + N_f$. This symmetry

decouples from the low energy dynamics at the critical points, but it can still lead to the nontrivial topological effect (cf., Sec. VI). We set $N_e = N_{mf}$ and allow the conversion between bosons and fermions.

- (2) The SU(2) sphere rotation symmetry. The spatial parity is not manifest in our model, which frees us from the constraint of parity anomaly in theories like single free Majorana fermion. The conservation of fermion parity, whose charge is $(-1)^{N_f}$, is a part of this rotation symmetry. Because the boson and fermion orbitals have an angular momentum mismatch of $1/2$, the fermion parity even states have integer spin under SU(2) and the fermion parity odd states have half-integer spin under SU(2). In other words, we are building in spin statistics rather than having it emerge in the relativistic IR CFT.

The Hamiltonian we study includes the simplest symmetry-allowed terms:

- (1) The local electric charge density interaction $H_U = \int d^2\mathbf{r} (n_f(\mathbf{r}) + n_b(\mathbf{r}))^2$, where $n_f(\mathbf{r}) = f^\dagger(\mathbf{r})f(\mathbf{r})$, $n_b(\mathbf{r}) = b^\dagger(\mathbf{r})b(\mathbf{r})$. This term is a natural form of interaction that exists in many other fuzzy-sphere models (e.g., Refs. [25,35,38,47]).

- (2) A local boson-fermion pair conversion H_t which annihilates two fermions and creates two bosons or vice versa. This term breaks the separate conservation of boson and fermion numbers to the total electric charge conservation. The conversion must occur in a pair to preserve the fermion parity.
- (3) A relative chemical potential μN_f , which enables us to energetically favor bosons or fermions.

Altogether,

$$H = H_U + tH_t + \mu N_f \quad (3)$$

$$\begin{aligned} H_U &= \int d^2\mathbf{r} (n_f(\mathbf{r}) + n_b(\mathbf{r}))^2 \\ &= \sum_{\{m_i\}} M_{m_1 m_2 m_3 m_4}^{(b)} b_{m_1}^\dagger b_{m_2}^\dagger b_{m_3} b_{m_4} \\ &\quad + M_{m_1 m_2 m_3 m_4}^{(bf)} b_{m_1}^\dagger f_{m_2}^\dagger f_{m_3} b_{m_4} \\ H_t &= \sum_{\{m_i\}} M_{m_1 m_2 m_3 m_4}^{(t)} b_{m_1}^\dagger b_{m_2}^\dagger f_{m_3} f_{m_4} + \text{H.c.} \end{aligned} \quad (4)$$

The four-fermion contact terms are equivalently expressed in terms of the pseudopotentials [25,77]

$$\begin{aligned} M_{m_1 m_2 m_3 m_4}^{(t)} &= \delta_{m_1+m_2, m_3+m_4} (4q-1) \begin{pmatrix} q-\frac{1}{2} & q-\frac{1}{2} & 2q-1 \\ m_2 & m_1 & -m_1-m_2 \end{pmatrix} \begin{pmatrix} q & q & 2q-1 \\ m_3 & m_4 & -m_3-m_4 \end{pmatrix}, \\ M_{m_1 m_2 m_3 m_4}^{(b)} &= \delta_{m_1+m_2, m_3+m_4} (4q-1) \begin{pmatrix} q-\frac{1}{2} & q-\frac{1}{2} & 2q-1 \\ m_2 & m_1 & -m_1-m_2 \end{pmatrix} \begin{pmatrix} q-\frac{1}{2} & q-\frac{1}{2} & 2q-1 \\ m_3 & m_4 & -m_3-m_4 \end{pmatrix}, \\ M_{m_1 m_2 m_3 m_4}^{(bf)} &= \delta_{m_1+m_2, m_3+m_4} 4q \begin{pmatrix} q & q-\frac{1}{2} & 2q-\frac{1}{2} \\ m_2 & m_1 & -m_1-m_2 \end{pmatrix} \begin{pmatrix} q & q-\frac{1}{2} & 2q-\frac{1}{2} \\ m_3 & m_4 & -m_3-m_4 \end{pmatrix}, \end{aligned}$$

where $\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ is the Wigner's $3j$ symbol. Note that in H_t , the orbital indices of the two bosons are symmetrized and fermions are antisymmetrized. In real space, the pair conversion is equivalent to

$$\begin{aligned} H_t &= \int d^2\mathbf{r} \eta(\mathbf{r}) D_+ \eta(\mathbf{r}) + \text{H.c.}, \\ \eta(\mathbf{r}) &= b^\dagger(\mathbf{r}) f(\mathbf{r}), \\ D_\pm &= D_\theta \pm iD_\phi, \end{aligned} \quad (5)$$

where D_μ is the covariant derivative on the sphere [81]. This pair conversion can be understood as a kinetic term of the Majorana fermion (cf. Sec. VII).

Let us consider some limiting cases for μ . At large negative μ , fermions are energetically favorable and the fermion orbitals are fully filled, forming a fermionic integer quantum Hall state with $\nu = 1$. At large positive μ , bosons are energetically favorable, and the boson density

interaction drives a bosonic Pfaffian state [65,82]. The mismatch of number of orbitals and particles $N_e = N_{mb} + 1$ is in agreement with the Wen-Zee shift required by the bPf state [83,84] that arises from the coupling between the electric charge and the spatial curvature. We will discuss this phase in more detail in Sec. III.

III. PHASE DIAGRAM

We study this model numerically through exact diagonalization (ED) using our numerical package FUZZIFIED [85,86]. The maximal system size we reach is $N_{mf} = 13$, and the maximal RAM usage is 24 gigabytes. We first study the phase diagram on the t - μ plane. For finite t , we find that the fermion density drops continuously from 1 to 0 with increasing μ (Fig. 1) [90].

To study the nature of the fIQH-bPf transition, we first fix $t = 0.3$. We calculate the finite-size behavior of the fermionic gap ΔE_f , defined as the energy difference

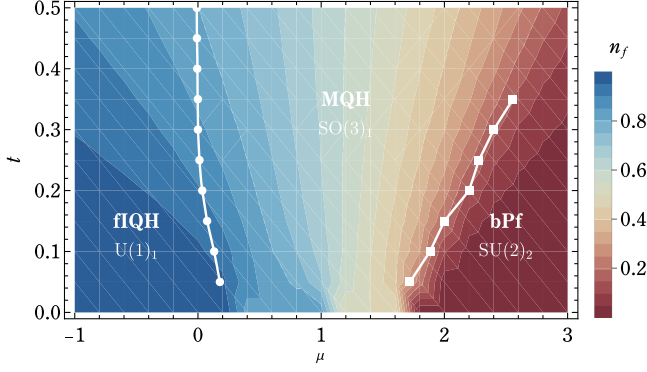


FIG. 1. The phase diagram on the μ - t plane. The color denotes the average fermion density calculated at $N_{mf} = 12$. Selected contours are marked by dashed lines as visual guidance. The circles separate the $\nu = 1$ fermionic integer quantum Hall phase and the $\nu_K = 3$ Majorana quantum Hall phase, and the squares separate MQH and bosonic Pfaffian phase. The approximate phase boundary is determined through the scaling of the energy gap from $N_{mf} = 12$ and 10 data.

between the lowest excited state with angular momentum $l = 1/2$ and the ground state at $l = 0$, and the bosonic gap ΔE_b defined as the energy difference between the two lowest $l = 0$ states. At a Lorentz-invariant critical point, the gap is expected to close as $\Delta E \propto 1/R$. We find two such critical points. At $\mu_{c1} \approx 0$, both bosonic and fermionic gap close; at $\mu_{c2} \approx 2.3$, the bosonic gap closes while the fermionic gap remains (Fig. 2). We keep the data only for even N_{mf} near μ_{c2} because the bPf ground state is only compatible with even N_{mf} ; for odd N_{mf} , the phase and the transition still exist, but the states have an anyon excitation (cf. Sec. VI). Similar scenario holds for other finite t . For each t , we determine the approximate phase boundary by comparing the energy gaps for $N_{mf,1} = 12$ and $N_{mf,2} = 10$ (Fig. 1): We minimize $[\Delta E(N_{mf,1})/\Delta E(N_{mf,2})](N_{mf,1}/N_{mf,2})^{1/2}$, respectively, for the fermion gap and for the boson gap to get μ_{c1} and μ_{c2} . The minima are typically value close to 1.

The existence of two gapless points indicates that there is an intermediate phase whose transitions to the fIQH and bPf phases are both continuous. To determine the nature of this intermediate phase, we calculate the real-space entanglement spectrum with the entanglement cut at the equator [91,92]. For a 3D gapped topological phase, the low-lying part of the ground-state entanglement spectrum for a local bi-partition is generally believed [93] to reproduce the spectral counting of the physical edge CFT of that phase. In the following, we analyze the entanglement spectrum of the three phases, respectively.

A. Fermionic integer quantum Hall phase

The large negative μ belongs to the $\nu = 1$ fermionic integer quantum Hall phase. It can be described by a trivial Chern-Simons topological quantum field theory (TQFT) of $U(1)_1 \cong SO(2)_1$. The 2D chiral CFT on its edge is a free chiral Dirac fermion, or a pair of free chiral Majorana fermion. The two fermions form a $U(1)$ symmetry and carry charge $Q = \pm 1$. The $U(1)$ symmetry is manifest on the fuzzy sphere as the electric charge conservation. By state-operator correspondence, the degeneracy counts in the entanglement spectrum equal the numbers of operators with corresponding $U(1)$ charge Q and angular momentum m in the chiral 2D CFT. We give the detail for the calculation from the character in Appendix A [Eq. (A1)] and label the expected degeneracies on Fig. 3. Practically, Q is half the difference $(Q_{e,A} - Q_{e,B})/2$ between the charge in two hemispheres and m is shifted by $-N_{mf}^2/8$. The counting is consistent with our numerical observation [Fig. 3(a)].

B. Majorana quantum Hall phase

We continue with the analysis of the intermediate phase. We shall show that this phase is a MQH phase with Kitaev index $\nu_K = 3$ [62] (the $\nu = 1$ fIQH can be viewed as $\nu_K = 2$ MQH). The $\nu_K = 3$ MQH is described by a $SO(3)_1$ CS theory. The 2D chiral CFT on its edge is three free chiral Majorana fermions. They form a vector under a

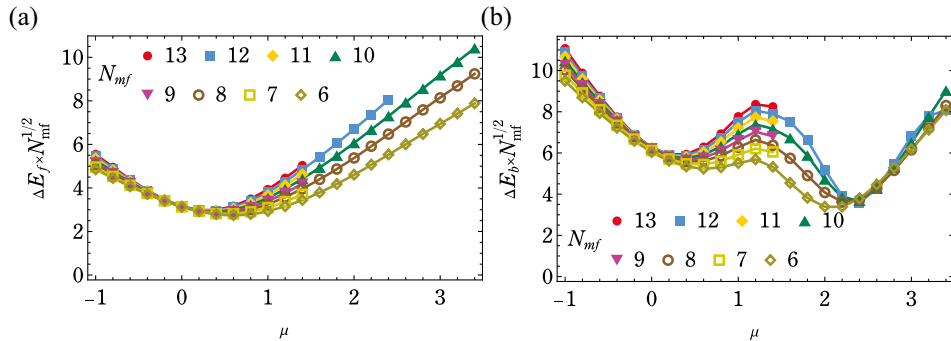


FIG. 2. (a) The rescaled fermionic gap $\Delta E_f N_{mf}^{1/2}$ and (b) the rescaled bosonic gap $\Delta E_b N_{mf}^{1/2}$ as a function of μ . In the calculation, we vary N_{mf} and fix $t = 0.3$. As bPf ground state only exists for even N_{mf} , we only keep the data with $\mu < 1.5$ for odd N_{mf} .

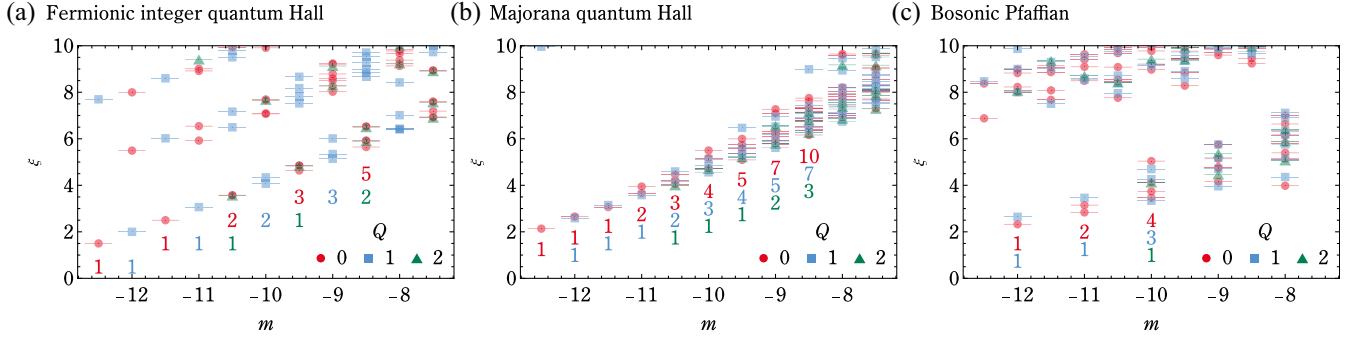


FIG. 3. The real-space entanglement spectrum with the cut at the equator at different phases at different charge sector $Q = (Q_{e,A} - Q_{e,B})/2$: (a) the fermionic integer quantum Hall phase at $\mu = -2$, (b) the $\nu_K = 3$ Majorana quantum Hall phase at $\mu = 1$ and (c) the bosonic Pfaffian phase at $\mu = 5$. Markers are plotted with transparency, so that nearly degenerate levels appear darker. The degeneracies for different charge sectors are labeled beneath. In the calculation we take $N_{mf} = 10$.

SO(3) global symmetry on the edge, of which a U(1) subgroup is manifest on the fuzzy sphere as the charge conservation. In the entanglement spectrum, we observe a well-split lowest chiral edge mode whose degeneracy is consistent with the 2D chiral CFT [Fig. 3(b), Eq. (A2)]. The TQFT has a chiral central charge $c_- = 3/2$. The enhancement from U(1)_e to the SO(3) global symmetry on the edge can be seen from the formation of multiplets. The excited states at $m = 1/2$ and 1 form SO(3)-vectors corresponding to the Majorana fermion operator and the symmetry current. An intuition for its origin is the fluctuating number of fermions in this phase and the pairing pattern given by H_t which antisymmetrizes the orbital index of two fermions.

C. Bosonic Pfaffian

The large positive μ belongs to the bosonic Pfaffian phase. The bosonic Pfaffian state is given by the many-body wave function on a plane with complex coordinates $z, \bar{z}$ [65]

$$\Psi(\{z_i\}) = \text{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i < j} (z_i - z_j), \quad (6)$$

where Pf is the Pfaffian of an even-dimensional square matrix, which also restricts that the bPf state is only compatible with even number of particles, i.e., $N_{mf} \in 2\mathbb{Z}$. It is described by a Chern-Simons theory $\text{SU}(2)_2$. As $\text{SU}(2)$ is a double covering of $\text{SO}(3)$, $\text{SO}(3)_1 \cong \text{SU}(2)_2/\mathbb{Z}_2$, i.e., the $\text{SU}(2)_2$ can be regarded as gauging the $\mathbb{Z}_2$ fermion-parity symmetry from $\text{SO}(3)_1$. Correspondingly, its chiral edge mode is the same as removing the $m \in \mathbb{Z} + \frac{1}{2}$ levels and keeping the $m \in \mathbb{Z}$ part of the chiral edge mode in $\text{SO}(3)_1$ [Fig. 3(c)].

The $\text{SU}(2)_2$ CS theory is a non-Abelian TO with chiral central charge $c_-^{(\text{bPf})} = 3/2$ and three kinds of anyons, viz. $\mathbb{I}$, σ , and ψ . They have statistical angles and fusion rules

$$\begin{aligned} \theta_{\mathbb{I}}^{(\text{bPf})} &= 0, & \theta_{\psi}^{(\text{bPf})} &= \pi, & \theta_{\sigma}^{(\text{bPf})} &= 3\pi/8 \\ \psi \times \psi &= \mathbb{I}, & \psi \times \sigma &= \sigma, & \sigma \times \sigma &= \mathbb{I} + \psi. \end{aligned} \quad (7)$$

Here $\mathbb{I}$ is the vacuum, ψ is a fermionic anyon, and σ is a non-Abelian anyon. The anyons are also referred to as $0, \frac{1}{2}$, and 1. This TO can also be understood as an Ising topological order stacked with an Abelian topological order

$$\frac{\text{Ising} \times \text{U}(1)_4}{\mathbb{Z}_2}.$$

The stacking with $\text{U}(1)_4$ adds 1 to the chiral central charge and adds an additional phase $\pi/4$ to the σ anyon compared with their values $c_-^{(\text{Ising})} = 1/2$ and $\theta_{\sigma}^{(\text{Ising})} = \pi/8$ in the Ising TO. In Kitaev's “16-fold way” classification [62], this phase is described by the gauged MQH with index $\nu_K = 3$.

IV. CRITICAL THEORIES

In this section, we discuss the critical theories that describe the transition between the fIQH, MQH, and bPf phases.

The transition from the fIQH to the MQH phase is described by a single free Majorana fermion. One way to see this is that the chiral central charge c_- increases by $1/2$ across the transition. This transition is captured by the Lagrangian

$$\mathcal{L}_{\text{Majorana}} = \frac{1}{2} \bar{\chi} (i\partial - m) \chi + \frac{5}{4} \text{CS}_g, \quad (8)$$

where χ is a (two-component) Majorana fermion, $\bar{\chi} = \chi^T \epsilon$, $\epsilon = i\gamma^2$, and $\partial = \gamma^\mu \partial_\mu$. The gravitational Chern-Simons term is defined through extending into a 4D manifold X with the original 3D manifold its boundary $M = \partial X$ [94]

$$\int_{M=\partial X} \text{CS}_g = \frac{1}{192\pi} \int_X \text{tr} R \wedge R,$$

where R is the curvature 2-form. A background CS_g level-1 from the background integer quantum Hall state has been included. Adding a positive or negative mass gaps out the Majorana fermion and adds an extra gravitational Chern-Simons response of level $\pm 1/4$.

$$\mathcal{L}_{\text{Majorana}}^> = \frac{3}{2}\text{CS}_g, \quad \mathcal{L}_{\text{Majorana}}^< = \text{CS}_g.$$

They recover the chiral central charge of the fIQH and MQH phases. This transition has another conjectural dual description of $\text{SO}(3)_1$ coupled to a single free scalar [97–95]]. The condensation of the scalar Higgses the gauge theory to $\text{SO}(2)_1$ describing the fIQH phase, and gapping the scalar results in $\text{SO}(3)_1$ corresponding to the MQH phase.

The transition from the MQH to the bPf phase falls into a gauged Ising (Ising*) universality, i.e., Ising type transition coupled to a $\mathbb{Z}_2$ -gauge field, or the confinement transition of a $\mathbb{Z}_2$ -gauge field. One way to understand it is to recognize that bPf can be described by MQH coupled to a $\mathbb{Z}_2$ -gauge field, so its transition to MQH can be realized by Higgsing the $\mathbb{Z}_2$ -gauge field. We can also use the language of CS theory to understand the emergence of gauged Ising transition. The MQH and the bPf phase can be described by $\text{SO}(3)_1 \cong \text{SU}(2)_2/\mathbb{Z}_2$ and $\text{SU}(2)_2$ respectively, and the difference is only a $\mathbb{Z}_2$ -gauge field. So Higgsing the $\mathbb{Z}_2$ center of the $\text{SU}(2)$ gauge group drives a transition from the bPf phase to the MQH phase. In Appendix B, we discuss the Lagrangian of this confinement transition. This transition can be equivalently considered in terms of anyon condensation. In the bPf phase, although the ψ anyon is fermionic, combined with the local fermion f , $f^\dagger\psi$ is bosonic with statistical angle $\theta_{f^\dagger\psi} = 0$. Across the transition, this composite particle $\langle f^\dagger\psi \rangle \neq 0$ is condensed. Because of the braiding of the anyons ψ and σ , σ will also be confined, resulting in the MQH with trivial topological order. Since this transition only changes the neutral sector, the chiral central charge $c_- = 3/2$ remains the same across the transition. This MQH-bPf transition can also be described as the adjoint QCD of $\text{SU}(2)_1$ Chern-Simons theory coupled to a fermion in the adjoint representation of the gauge group [98]]. When the fermion is gapped, it contributes a Chern-Simons level ± 1 depending on the sign of the mass. This drives a transition between $\text{SU}(2)_2$ topological order and a trivial topological order $\text{SU}(2)_0$, which only differs from the MQH phase by an invertible TQFT. This transition is conjectured to be dual to $\text{O}(1)_{-3}$ coupled to a real scalar through the level-rank duality, which is equivalent to the Ising* universality.

Numerically, we verify these critical theories by inspecting the energy spectrum at the critical point. A quantum system on a sphere described by a CFT enjoys the state-operator correspondence where the scaling

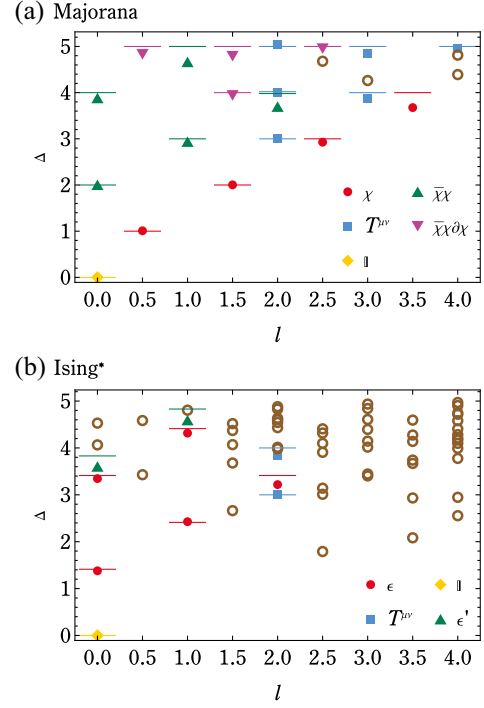


FIG. 4. The rescaled energy spectrum at the conformal points (a) $t = 0.3, \mu = 0.0$ described by the free-Majorana-fermion CFT and (b) $t = 0.1, \mu = 2.25$ described by the Ising* CFT. We identify some multiplets of the lowest primaries in the spectra and the bars denote their expected value in the CFT. The spectra are calibrated by $\Delta_T = 3$. The brown circles denote states not identified. In the calculation, we fix $N_{mf} = 12$.

dimensions of the operators are proportional to the excited energies

$$\Delta_\Phi = \frac{v}{R}(E_\Phi - E_0),$$

where Φ is an arbitrary scaling operator and v is a model-dependent constant that can be determined by calibrating the stress tensor $T^{\mu\nu}$ to $\Delta = 3$. The spectrum of a CFT can be classified into multiplets of primaries and their descendants at integer spacing.

At the fIQH-MQH transition, we observe that most of the operators are close to integer levels and they can be classified into multiplets of the lowest primary operators in the free-Majorana-fermion CFT [Fig. 4(a)], viz. the fermion χ at $\Delta = 1, l = 1/2$, the mass field $\bar{\chi}\chi = \epsilon_{\alpha\beta}\chi^\alpha\chi^\beta$ at $\Delta = 2, l = 0$, the conserved stress tensor $T^{\mu\nu} = (i/2)\bar{\chi}_a\alpha(\gamma^{(\mu})^\alpha{}_\beta\partial^{\nu)}\chi^\beta$ at $\Delta = 3, l = 2$, $\bar{\chi}\chi\partial\chi$ at $\Delta = 4, l = 3/2$, etc. We highlight that spectrum contains $l \in \mathbb{Z} + \frac{1}{2}$ states corresponding to fermionic operators. The free-Majorana-fermion CFT will be analyzed in more detail in Sec. V.

At the MQH-bPf transition, although there are many nonconformal gapped excitations at relatively low energy at

small system size, the low lying states for each integer $l \leq 2$ can be identified as operators in the Ising* CFT [Fig. 4(b)]. The Ising* CFT consists all the $\mathbb{Z}_2$ -even operators in the Ising CFT. We can identify ϵ at $\Delta = 1.412$ and its several descendants, the stress tensor $T^{\mu\nu}$ and ϵ' at $\Delta = 3.830$ in the spectrum. The Ising* CFT will be discussed in more detail in Sec. VI.

V. FREE-MAJORANA-FERMION CFT

In this section, we focus on the free-Majorana-fermion CFT on the fuzzy sphere. In this CFT, the Majorana fermion χ has scaling dimension $\Delta = 1$ and $SU(2)$ spin $l = 1/2$. The spectrum of scaling operator, including the primaries and descendants, can be determined with the help of harmonic oscillators [35]. Here we list the scaling dimension Δ and spin l of the low-lying operators:

$$\begin{aligned} \Delta = 1 & \quad l = 1/2 \\ \Delta = 2 & \quad l = 0, 3/2 \\ \Delta = 3 & \quad l = 1, 2, 5/2 \\ \Delta = 4 & \quad l = 0, 3/2, 2, 2, 3, 7/2 \\ & \vdots \end{aligned}$$

where the replicated numbers denote degenerate operators. To choose an ideal parameter point, we optimize a cost function along the critical line defined as the root mean square of the deviations of the scaling dimension of all operators with $\Delta \leq \Delta_c = 3$ from their expected values [41]. We find that in the region $0.2 < t < 0.4$, the cost function is relatively insensitive to the choice of t , and the critical $\mu_{c1} \approx 0$ up to a precision of 0.01, which may be a hint for a possible enhanced symmetry in the absence of relative chemical potential. We therefore fix $t = 0.3$ and $\mu_c = 0$.

We find that the low lying spectrum agrees well with the theory (Fig. 5). All the states up to $\Delta = 4$ are CFT states, and all the theoretically expected operators appear in the spectrum close to integer scaling dimensions. The discrepancy is within 2.5% for all the $\Delta \leq 3$ operators and most $\Delta = 4$ operators for our maximal accessible size $N_{mf} = 13$. This precision is remarkable with almost no fine-tuning at such small size even compared with similar fuzzy-sphere model (e.g., the free real scalar [35]). With increasing system size, the scaling dimension gradually approaches to expected integer value [Figs. 6(a)–6(d)]. We quantitatively evaluate this through the cost function consisting of all the operators with $\Delta \leq \Delta_c = 2, 3, 4$. We find these cost functions all scales to zero as $Q \sim N_{mf}^{-\alpha/2}$ where $\alpha = 3.4, 3.6, 3.2$, respectively, for $\Delta_c = 2, 3, 4$ [Fig. 6(e)].

Having discussed the operator spectrum, we now move on to the correlation functions of the local operators, of

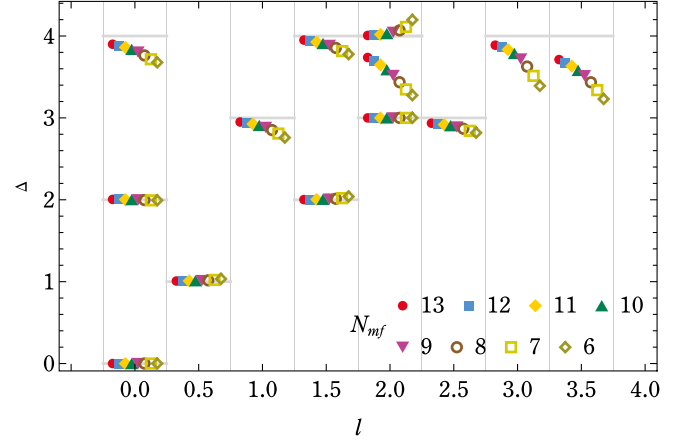


FIG. 5. The spectrum with $\Delta \lesssim 4$ for various systems sizes in the free-Majorana-fermion CFT. The scaling dimensions are calibrated by the stress tensor with $\Delta_T = 3$. The gray bars denote the theoretical value for comparison. In the calculation, we fix $t = 0.3$ and $\mu = 0$. The vertical gridlines with interval 0.5 separates different spins, which are either integers or half-integers.

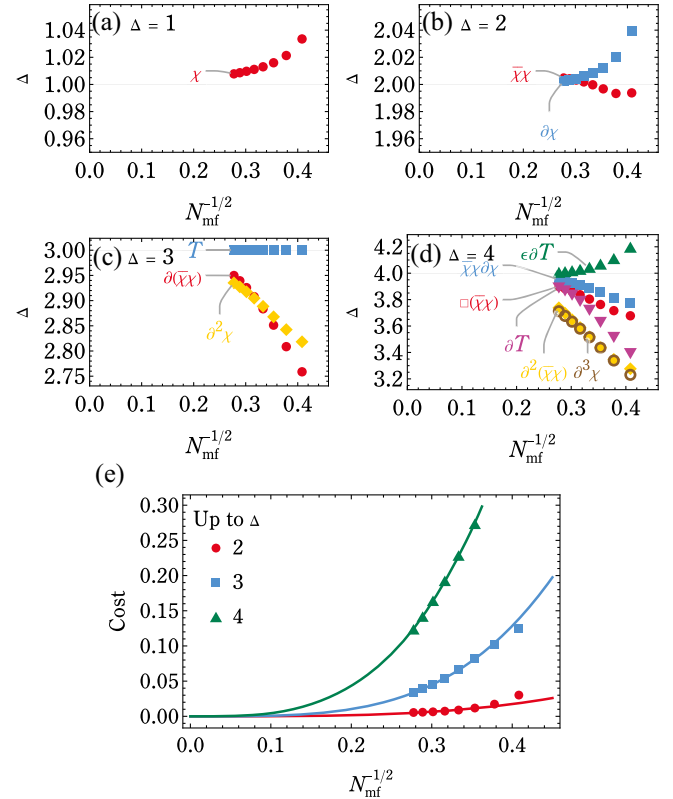


FIG. 6. (a)–(d) The finite-size scaling of scaling dimensions of various operators in the free-Majorana-fermion CFT. The gray lines denote the theoretical value for comparison. (e) The finite-size scaling of cost function consisting of operators with $\Delta \leq \Delta_c = 2, 3, 4$. In the calculation, we fix $t = 0.3$ and $\mu = 0$.

which the most important is the Majorana fermion χ itself. We sketch the process here, and give the details in Appendix C. On the fuzzy sphere, the simplest gapless local fermionic operators are boson-fermion pairs $f^\dagger(\mathbf{r})b(\mathbf{r})$ and $b^\dagger(\mathbf{r})f(\mathbf{r})$. As χ is a two-component spinor, $f^\dagger b$ and $b^\dagger f$, respectively, realizes its two polarizations

$$\begin{aligned}\eta(\mathbf{r}) &= f^\dagger(\mathbf{r})b(\mathbf{r}) = \alpha\chi^{(1/2)}(\mathbf{r}) + \dots, \\ \eta^\dagger(\mathbf{r}) &= b^\dagger(\mathbf{r})f(\mathbf{r}) = \bar{\alpha}\chi^{(-1/2)}(\mathbf{r}) + \dots,\end{aligned}\quad (9)$$

where the two polarizations $\chi^{(s)}(\mathbf{r})$ ($s = \pm 1/2$) of the spinor carry angular momentum s under the $\text{SO}(2)$ rotation along an axis that passes $\mathbf{r}$ (i.e., spin weight), and α is a coefficient that can be determined from the microscopic model. The conformal kinematics requires the two-point correlators to have the form

$$\langle \chi^\alpha(x_1) \bar{\chi}_\beta(x_2) \rangle = \frac{(\gamma_\mu)^\alpha{}_\beta x_{12}^\mu}{(x_{12}^2)^{\Delta_\chi + 1/2}}, \quad (10)$$

where $\Delta_\chi = 1$, $x_{1,2} \in \mathbb{R}^3$, $x_{12} = x_1 - x_2$, $\alpha, \beta = \pm$ are spinor indices, and γ are the Pauli matrices. Taking the polarizations $\chi^{(s)}(\mathbf{r})$ on the sphere, their two-point function

$$\langle 0 | \chi^{\dagger(s_1)}(\mathbf{r}_1) \chi^{(s_2)}(\mathbf{r}_2) | 0 \rangle = \frac{2(-1)^{s_1} \delta_{s_1 s_2}}{R^{2\Delta_\chi} (2 \sin \frac{1}{2} \theta_{12})^{2\Delta_\chi}} \quad (11)$$

has the same form as scalars on the sphere, where θ_{12} is the angular distance between the two points.

We calculate the dimensionless correlation function

$$C_\chi^*(\theta_{12}) = \frac{1}{2i} R^{2\Delta_\chi} \langle 0 | \chi^{\dagger(1/2)}(\mathbf{r}_1) \chi^{(1/2)}(\mathbf{r}_2) | 0 \rangle, \quad (12)$$

whose exact expression is given in Eq. (C25), and plot the results in Fig. 7(a). As the system size increases, the numerical result gradually approaches its theoretical value

$(2 \sin \theta_{12}/2)^{-2\Delta_\chi}$. From this correlation function, the scaling dimension Δ_χ can be extracted in several ways

$$C_\chi^*(\theta_{12})|_{\theta_{12}=\pi} = 2^{-2\Delta_\chi}, \quad (13a)$$

$$\frac{(C_\chi^*)''(\theta_{12})}{C_\chi^*(\theta_{12})} \Big|_{\theta_{12}=\pi} = \frac{\Delta_\chi}{2}, \quad (13b)$$

$$\int_{-1}^1 d \cos \theta_{12} \sin \frac{\theta_{12}}{2} C_\chi^*(\theta_{12}) = \frac{2^{2(1-\Delta_\chi)}}{3-2\Delta_\chi}. \quad (13c)$$

We find the results all approach $\Delta_\chi = 1$ with increasing system size [Fig. 7(b)]. The results from the integral Eq. (13c) has an especially small deviation within 10^{-3} at all system sizes.

VI. GAUGED ISING CFT

In this section, we focus on the gauged Ising (i.e., Ising*) CFT on the fuzzy sphere. Gauging the $\mathbb{Z}_2$ symmetry brings no change to the dynamics. The operator content of the gauged Ising CFT is the $\mathbb{Z}_2$ -even sector in the original Ising CFT, which includes ϵ , $T^{\mu\nu}$, etc. The $\mathbb{Z}_2$ -odd operators (e.g., σ) of the original Ising CFT are no longer local operators in the gauged Ising CFT because they are not gauge invariant.

Numerically, we can indeed identify $\epsilon, \epsilon', T^{\mu\nu}$ and their descendants up to $\Delta < 5$ and $l \leq 2$ in the spectrum [Fig. 8(a)]. In the calculation, we fix $t = 0.1$ with a relatively optimal conformal symmetry, and choose μ for each system size by optimizing a cost function consisting of $\epsilon, \partial^\mu \epsilon, \partial^\mu \partial^\nu \epsilon, \square \epsilon$, and $T^{\mu\nu}$. The optimal μ for each size is listed in Table I (the meaning of the right panel will be explained shortly). We note that although there are non-conformal states due to the lowest gapped excitations in the proximate bosonic Pfaffian phase, the low energy spectrum with $l \leq 2$ show relatively clear conformal signature. At large enough system size, the low energy spectrum should be dominated by CFT states and these gapped excitations

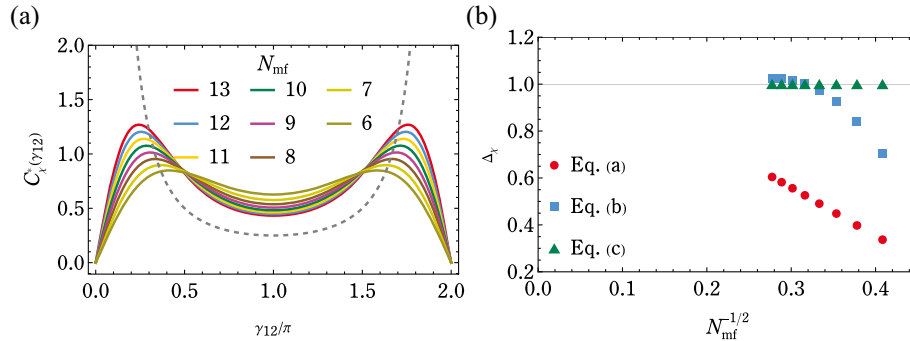


FIG. 7. (a) The dimensionless correlation function of the Majorana fermion χ Eq. (12) at different system sizes. The gray dashed line denotes $(2 \sin \theta_{12}/2)^{-2}$ required by conformal symmetry. (b) The scaling dimension Δ_χ extracted from the correlation function through different ways in Eqs. (13) as a function of system size. In the calculation, we fix $t = 0.3$ and $\mu = 0$.

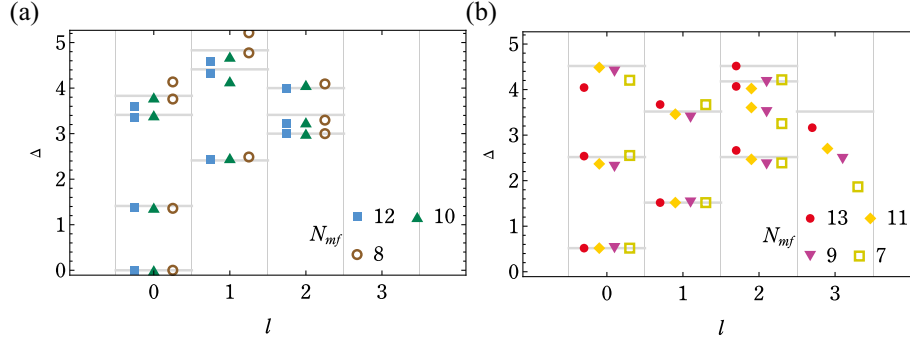


FIG. 8. The spectrum of the identified CFT states with $\Delta < 5$ for various systems sizes. The gray bars denote the theoretical value for comparison. (a) The bulk spectrum at even N_{mf} corresponding to the $\mathbb{Z}_2$ -even part of the Ising CFT spectrum. The scaling dimensions are calibrated by the stress tensor with $\Delta_T = 3$. (b) The Wilson-line end-point spectrum at odd N_{mf} corresponding to the $\mathbb{Z}_2$ -odd part of the Ising CFT spectrum. The scaling dimensions are calibrated by setting $\Delta_\sigma = 0.518$ and $\Delta_{\partial^\mu \sigma} = 1.518$. In the calculation, we fix $t = 0.1$ and choose μ as in Table I.

will move higher, but at the sizes available, these excitations appear at relatively low energy at large l . As the bosonic Pfaffian state is only compatible with even N_{mf} , and the optimal conformal point drifts with system size, we leave the finite-size scaling analysis to future work.

On the other hand, the $\mathbb{Z}_2$ -odd operators can be made gauge invariant by attaching a Wilson line (of $\mathbb{Z}_2$ -gauge field) to it. In other words, the $\mathbb{Z}_2$ -odd operators become end-point operators of a line defect in the gauged Ising CFT. Although the defect end-point operators arise generically in conformal line defects and have been discussed in various contexts [49,99–101], in the setting of $\mathbb{Z}_2$ - (or any discrete) gauge theory, the Wilson line defect is not only conformal but also topological. In the bPf-MQH transition, we can straightforwardly realize the topological Wilson line on the fuzzy sphere by leveraging the even-odd effect of N_{mf} .

We first note that the bosonic Pfaffian state is sensitive to the parity of particle number, and is only compatible with even N_{mf} . The choice of the odd N_{mf} , on the other hand, is equivalent to the insertion of a gauge charge at the center of the sphere as the end point of a Wilson line defect. The reason is that although the $U(1)_e$ symmetry of electric charge conservation decouples from the dynamics at the critical point, it still has topological contribution. First, odd N_{mf} can be understood as the insertion of a 2π -flux of $U(1)_e$ on top of even N_{mf} , i.e., q is increased by $1/2$. Second, the 2π -flux insertion is bound to a $\mathbb{Z}_2$ -gauge charge. The binding of flux and gauge charge can be seen most evidently from the mutual Chern-Simons $(1/2\pi)adA$

coupling of the electromagnetic field A and the dynamical gauge field a (cf. Appendix B). Here a is a $U(1)$ dynamical gauge field which is Higgsed to $\mathbb{Z}_2$ [102]. Its equation of motion asserts

$$Q_a = \frac{1}{2\pi} \Phi_A,$$

i.e., upon the insertion of 2π -flux of A , a $\mathbb{Z}_2$ gauge charge is also inserted together with an electric charge. Because of the topological nature, the attached Wilson line preserves the $SO(3)$ sphere rotation symmetry, which is different from the usual case of a nontopological conformal line. The defect end-point operators are the $\mathbb{Z}_2$ -odd part of the Ising CFT, and they can be observed from the eigenstates through the state-operator correspondence on the fuzzy sphere, as we will discuss below.

We first confirm check the closure of the bosonic gap for the odd N_{mf} . The bosonic gap ΔE_b closes for the odd N_{mf} at $\mu_{c2} \approx 2.0$, similar to even N_{mf} , verified by overlapping $\Delta E_b \times N_{mf}^{1/2}$ [Fig. 9(a)]. Moreover, the value of $\Delta E_b \times N_{mf}^{1/2}$ for even N_{mf} is approximately 0.7 times that for odd N_{mf} . This is consistent with the expectation from the state-operator correspondence. In the gauge-neutral sector, the lowest two spin-0 operators are vacuum $\mathbb{I}$ and ϵ with a scaling dimension difference 1.412; in the gauge-charged sector, they are σ and $\square\sigma$ with a scaling dimension difference 2.

We then analyze the energy spectrum of the Wilson-line end point. As the gauge charge is locked with N_{mf} , we cannot access the bulk spectrum and Wilson-line end-point spectrum at the same time and use the wave-function overlap scheme [49,103,104] to extract the scaling dimension of end-point operators. Instead, we fix the lowest end-point operator σ at $\Delta = 0.518$, and use the second lowest operator $\partial^\mu \sigma$ at $\Delta = 1.518$ as the calibrator. The lowest operators up to $\Delta < 5$ and $l \leq 3$ can be identified

TABLE I. The optimal conformal point μ_{c2} for the Ising* CFT for each system size. In the calculation, we fix $t = 0.1$.

N_{mf}	12	10	8	13	11	9	7
μ_{c2}	2.254	2.230	2.356	2.290	2.318	2.373	2.396

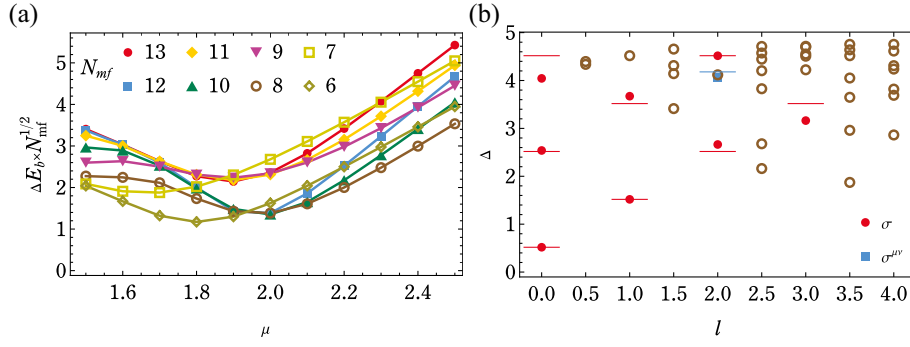


FIG. 9. (a) The rescaled bosonic energy gap $\Delta E_b N_{mf}^{1/2}$ as a function of μ in vicinity of the Ising* transition for both even and odd N_{mf} calculated at $t = 0.1$. The lines with even and odd N_{mf} intersect respectively at $\mu_{c2} \approx 2.0$ except for the smallest system sizes $N_{mf} = 6, 7$. (b) The rescaled energy spectrum at conformal point $t = 0.1, \mu = 2.29$ calculated at $N_{mf} = 13$ described by the Wilson-line end-point operators in the Ising* CFT. We identify some multiplets of the lowest primaries in the spectra and the bars denote their expected value in the CFT. The energy spectrum is rescaled such that $\Delta_\sigma = 0.518$ and $\Delta_{\sigma^{\mu\nu}} = 1.518$.

with $\mathbb{Z}_2$ -odd operators in the Ising CFTs, viz. σ with $\Delta = 0.518$ and its descendants, and $\sigma^{\mu\nu}$ with $\Delta = 4.180$ [Fig. 9(b)]. Like for the gauge-neutral sector, we fix $t = 0.1$ and optimize a cost function consisting $\sigma, \partial^\mu \sigma, \partial^\mu \partial^\nu \sigma, \square \sigma, \square \partial^\mu \sigma$, and $\sigma^{\mu\nu}$ for each system size to find the optimal μ (Table I, right panel). The spectra of identified conformal operators for each system size are plotted in Fig. 8(b). They agree considerably well with the odd sector of Ising CFT.

VII. CORRESPONDENCE WITH QFT LAGRANGIANS

In Ref. [35], an analogy between the fuzzy-sphere models and the field theories have been noted for the free and critical scalars. In this section, we further extend this correspondence to free and interacting Majorana fermions.

For a critical real scalar (Ising), the setup contains two fermionic flavors $f_{0,1}$ that are, respectively, even and odd under the $\mathbb{Z}_2$ symmetry. The f_0 flavor can be identified as a “vacuum” or reference flavor—the CFT ground state can be regarded as a perturbation from the reference state where all the f_0 particles are occupied. As an evidence, the CFT ground state has a substantial overlap with the reference state. As the reference state is a direct product state in the orbital space, the orbital-space entanglement entropy is very small for the CFT ground state. The orbital-space entanglement entropy is measured through a bi-partition between the $m < 0$ orbitals and the $m \geq 0$ orbitals. Correspondingly, the f_1 flavor is identified as an “excited state.” The hopping between the two flavors is identified as the elementary field σ , and the density of “excitations,” or particles in the f_1 flavor, is identified as the mass field $\epsilon = \sigma^2$ up to a constant

$$\sigma \sim n_x = f_1^\dagger f_0 + f_0^\dagger f_1, \quad \epsilon + \text{const} \sim n_1 = f_1^\dagger f_1. \quad (14)$$

Here the “ $\sim$ ” denotes a phenomenological correspondence between a field theory operator and a fuzzy-sphere

operator, or between a term in the field-theory Lagrangian and a term in the fuzzy-sphere Hamiltonian [105]. The correspondence is illustrated in Fig. 10(a). All the terms in the Hamiltonian

$$H = \int d^2 \mathbf{r} [n_e(\mathbf{r})^2 + U n_x(\mathbf{r}) \nabla^2 n_x(\mathbf{r}) - \mu_1 n_1(\mathbf{r})] \quad (15)$$

besides a electric density interaction of $n_e = f_0^\dagger f_0 + f_1^\dagger f_1$ that suppresses the charge fluctuation, have a direct interpretation to the field theory

$$\mathcal{L} = \frac{1}{2} (\partial^\mu \sigma)^2 - \frac{1}{2} m^2 \sigma^2 - \frac{\lambda}{4!} \sigma^4. \quad (16)$$

Similar to the lattice models like the transverse-field Ising model, the Ising-type density-density interaction can be

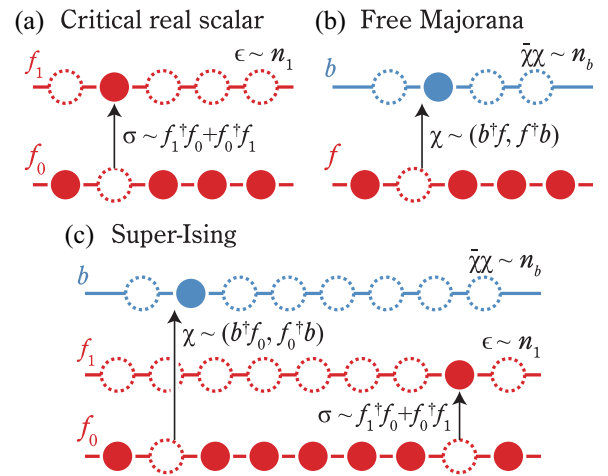


FIG. 10. The illustration of the correspondence between fuzzy-sphere setup and the field-theory operators in (a) the critical real scalar (Ising), (b) the free Majorana fermion, and (c) the super-Ising theory.

interpreted as the ϕ^4 theory at a certain mass m_0 , and the chemical potential fine-tunes the mass term

$$\begin{aligned} Un_x \nabla^2 n_x &\sim \frac{1}{2} (\partial^\mu \sigma)^2 - \frac{1}{2} m_0^2 \sigma^2 - \frac{\lambda}{4!} \sigma^4 \\ \mu_1 n_1 &\sim \frac{1}{2} (m^2 - m_0^2) \sigma^2. \end{aligned} \quad (17)$$

While the correspondence between field-theory Lagrangian and the fuzzy-sphere model is to some extent vague for the critical scalar, it can be seen much more clearly for the free Majorana fermion. The setup now contains one fermionic “reference” flavor f and a bosonic excitation flavor b . The CFT ground state has a substantial overlap $|\langle \text{ref} | 0_{\text{CFT}} \rangle|$ with the reference state where all fermions are occupied [Fig. 11(a)]

$$|\text{ref}\rangle = \prod_{m=-q}^q f_m^\dagger |\emptyset\rangle \quad (18)$$

and its orbital-space entanglement entropy is small [Fig. 11(b)]. The conversion between the two flavors are identified as the elementary spinor field χ , and the density of excitations, or particles in the b flavor, is identified as the mass field $\bar{\chi}\chi$ up to a constant [Fig. 10(b)]

$$\chi \sim (\eta, \eta^\dagger)^T = (f^\dagger b, b^\dagger f)^T, \quad \bar{\chi}\chi + \text{const} \sim n_b = b^\dagger b. \quad (19)$$

The terms in the Hamiltonian can again be interpreted as the kinetic term and the mass term

$$\frac{1}{2} \bar{\chi} i \not{D} \chi \sim t(\eta D_+ \eta + \text{H.c.}), \quad \frac{1}{2} m \bar{\chi} \chi \sim \mu n_b, \quad (20)$$

where $D_+ = D_\theta + iD_\phi$ in addition to an electric density interaction.

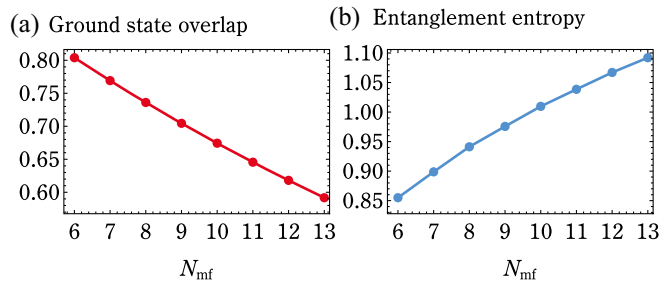


FIG. 11. Properties of the CFT ground state of the free Majorana fermion: (a) The overlap $|\langle \text{ref} | 0_{\text{CFT}} \rangle|$ between the CFT ground state and the reference state (18) where all fermions are occupied; (b) the orbital-space entanglement entropy. The data are measured at $t = 0.3, \mu_{c1} = 0$ and plotted as a function of the system size N_{mf} .

Notably, the pair conversion term H_t bears exactly the same form as the kinetic term in the Hamiltonian of free Majorana fermion upon radial quantization.

$$\begin{aligned} H_t &\propto \int d^2 \mathbf{r} i e^{\mu\nu\rho} \hat{n}_\mu \bar{\chi}_\alpha(\mathbf{r}) (\gamma_\nu)^\alpha_\beta \partial_\rho \chi^\beta(\mathbf{r}) \\ &= \int d^2 \mathbf{r} i \bar{\chi}(\mathbf{r}) (\hat{\mathbf{n}} \times \boldsymbol{\gamma} \cdot \boldsymbol{\partial}) \chi(\mathbf{r}). \end{aligned} \quad (21)$$

As an evidence for this correspondence, the critical point locates at $\mu_c = 0$ to a high precision where the mass term vanishes.

We shall show that these ingredients suffice to construct the interacting theories. The simplest one is a Majorana fermion χ coupled to a real scalar σ through a Yukawa interaction [13,106]

$$\mathcal{L}[\chi, \sigma] = \frac{1}{2} \bar{\chi} i \not{D} \chi + \frac{1}{2} (\partial^\mu \sigma)^2 - \frac{1}{2} m_\sigma^2 \sigma^2 - \frac{\lambda}{4!} \sigma^4 - \frac{g}{2} \sigma \bar{\chi} \chi. \quad (22)$$

The Yukawa interaction locks the Ising $\mathbb{Z}_2$ symmetry to the space-time parity. As a result, this theory has no global symmetry, and σ is odd under space-time parity. When the parity is not manifest, two additional perturbations should also be considered, namely, the fermion mass and the polarization term

$$m_\chi \bar{\chi} \chi + h \sigma.$$

In the infrared, the Lagrangian flows to a $\mathcal{N} = 1$ superconformal field theory known as the super-Ising theory. We will introduce its property in more detail in Sec. VIII.

The corresponding setup on the fuzzy sphere contains two fermionic flavors f_0, f_1 and a bosonic flavor b . We set the bosons and fermions to have a mismatch $q_f - q_b = 1/2$ in the angular momenta. We set the total filling to be 1, i.e., $N_e = N_{mf}$. Similar to the critical real scalar and the free Majorana fermion, we interpret the f_0 flavor as the reference flavor, f_1 as the scalar excitation, and b as the fermion excitation. Hence, the elementary fields χ and σ are realized as the hoppings from the reference fermion flavor f_0 to, respectively, the boson flavor b and the other fermion flavor f_1 [Fig. 10(c)]

$$\eta(\mathbf{r}) = b^\dagger(\mathbf{r}) f_0(\mathbf{r}), \quad \chi(\mathbf{r}) \sim (\eta(\mathbf{r}), \eta^\dagger(\mathbf{r}))^T \quad (23a)$$

$$n_x(\mathbf{r}) = f_1^\dagger(\mathbf{r}) f_0(\mathbf{r}) + f_0^\dagger(\mathbf{r}) f_1(\mathbf{r}), \quad \sigma(\mathbf{r}) \sim n_x(\mathbf{r}). \quad (23b)$$

The fermion and scalar bilinears $\bar{\chi}\chi$ and σ^2 are realized using the density operators

$$n_b = b^\dagger b \sim \bar{\chi}\chi + \text{const}, \quad n_1 = f_1^\dagger f_1 \sim \sigma^2 + \text{const}. \quad (23c)$$

up to a constant. Using these ingredients, we can translate the terms in the Lagrangian to four-particle interactions in

the fuzzy-sphere model. We start with the terms that describe the critical scalar and the free Majorana fermion

$$\frac{1}{2}\bar{\chi}i\partial\chi \sim t(\eta(\mathbf{r})D_+\eta(\mathbf{r}) + \text{H.c.}) \quad (24a)$$

$$\frac{1}{2}(\partial^\mu\sigma)^2 - \frac{1}{2}m_0^2\sigma^2 - \frac{\lambda}{4!}\sigma^4 \sim Un_x(\mathbf{r})\nabla^2n_x(\mathbf{r}). \quad (24b)$$

The Yukawa coupling can be expressed as a contact interaction between n_b and n_x

$$\frac{g}{2}\sigma\bar{\chi}\chi \sim gn_x(\mathbf{r})n_b(\mathbf{r}). \quad (24c)$$

The terms left are the scalar mass, and the two additional terms due to the absence of parity symmetry

$$\frac{1}{2}(m_\sigma^2 - m_0^2)\sigma^2 \sim \mu_1 n_1, \quad \frac{1}{2}m_\chi\bar{\chi}\chi \sim \mu_b n_b, \quad h\sigma \sim hn_x. \quad (24d)$$

Combined with an electric density interaction n_e^2 , where $n_e = f_0^\dagger f_0 + f_1^\dagger f_1 + b^\dagger b$, the total Hamiltonian reads

$$\begin{aligned} H = \int d^2\mathbf{r} [& n_e^2(\mathbf{r}) + t(\eta(\mathbf{r})D_+\eta(\mathbf{r}) + \text{H.c.}) \\ & + Un_x(\mathbf{r})\nabla^2n_x(\mathbf{r}) + gn_x(\mathbf{r})n_b(\mathbf{r}) \\ & - hn_x(\mathbf{r}) - \mu_1 n_1(\mathbf{r}) - \mu_b n_b(\mathbf{r})]. \end{aligned} \quad (25)$$

It contains six parameters t, U, g, h, μ_1 and μ_b . The latter three h, μ_1 , and μ_b control the three relevant perturbations $\sigma, \epsilon = \sigma^2$ and $\sigma' = \bar{\chi}\chi$, so that we can fix the values of t, U, g and fine-tune h, μ_1, μ_b to reach the super-Ising fixed point [107].

VIII. SUPER-ISING SCFT

We have constructed the model to realize the super-Ising theory on the fuzzy sphere. In this section, we present the numerical evidence for the emergence of superconformal symmetry and the agreement with conformal bootstrap.

We first discuss the supersymmetry in the field-theory Lagrangian (22). In the infrared, it flows to a superconformal field theory with super-charge $\mathcal{N} = 1$ known as the super-Ising theory. The supersymmetry is manifest in the Lagrangian at $\lambda = 3g^2$ and $m_\sigma = 0$. The action can be organized in terms of a superfield $\Sigma(x, \theta) = \sigma + \bar{\theta}\chi + \frac{1}{2}\bar{\theta}\theta\epsilon$ where θ and $\bar{\theta}$ are Grassmannian coordinates. The ϵ is an auxiliary field coupled to σ^2 . By the equation of motion $\epsilon = \sigma^2$. The Poincaré supercharge Q^α acts on the fields as

$$\begin{aligned} [Q^\alpha, \sigma] &= \chi^\alpha, \\ \{Q^\alpha, \chi^\beta\} &= -(\gamma^\mu)^{\alpha\beta}\partial_\mu\sigma - \epsilon^{\alpha\beta}\epsilon, \\ [Q^\alpha, \epsilon] &= (\gamma^\mu)^{\alpha\beta}\partial_\mu\chi_\beta. \end{aligned}$$

The superconformal group is homomorphic to a super-Lie group $\text{OSp}(1|4)$ [108,109]. More precisely, the (Lorentzian) conformal transformations acting on fermions give the Lie group $\text{Spin}(3, 2) \cong \text{Sp}(4, \mathbb{R})$ of real symplectic transformations. Besides that, the super-Lie algebra also contains two Grassmannian generators, i.e., the Poincaré supercharge Q^α and the conformal supercharge S^α , which extend the conformal group to $\text{OSp}(1|4, \mathbb{R})$. These generators organize several conformal multiplets into a superconformal multiplet in a similar way as how the conformal symmetry organizes the scaling operators into conformal multiplets [110]. Similar to P^μ and K^μ that act as ladder operators of D increasing and decreasing the scaling dimension Δ by 1, Q^α and S^α act as “half-ladder” operators changing Δ by $\pm 1/2$. The conformal primaries are classified into superconformal primaries that are the lowest-weight state of Q^α , and the superconformal descendants by acting Q^α on the superprimaries. Because of the odd Grassmannian parity of Q^α , each supermultiplet contains at most 4 conformal multiplets. As an example, the Majorana fermion $\chi^\alpha = Q^\alpha\sigma$ and $\epsilon = \bar{Q}_\alpha Q^\alpha\sigma$ are now superconformal descendants of the superconformal primary σ , and their scaling dimensions are not independent $\Delta_\chi = \Delta_\sigma + 1/2$, $\Delta_\epsilon = \Delta_\sigma + 1$. The operator $\bar{Q}Q$ is odd under space-time parity, so σ and ϵ has opposite space-time parity.

The super-Ising SCFT has been studied extensively by perturbative calculation [12,111] and conformal bootstrap [14–16]. The two relevant superprimaries $\sigma, \sigma' = \bar{\chi}\chi$ has been determined to a high precision. They are parity-odd Lorentz scalars and have scaling dimensions $\Delta_\sigma = 0.584$, $\Delta_{\sigma'} = 2.887$ [16]. Besides that, a spin-2 parity-even operator $T^{\mu\nu}$ at $\Delta_{T'} \approx 3.28$ has been observed in the extremal spectrum [16].

To verify that our ansatz (25) realizes the super-Ising SCFT, we carry out an ED calculation up to $N_{mf} = 9$. We fix $t = 1.5$, $U = 0.25$, and $g = 1$ and determine h, μ_1 , and μ_b for each N_{mf} by optimizing a cost function defined as the root mean square of the relative deviations of the scaling dimension of all 14 operators with $\Delta < 3.1$ and $l \leq 2$ (listed in Fig. 12) from the values obtained from the conformal bootstrap [16]. The conformal points for $N_{mf} = 9, 8, 7, 6$ are listed in Table II. The optimal parameters remain relatively stable with the system size. The low-lying spectrum for various N_{mf} are plotted in Fig. 12. For each N_{mf} , the lowest operators remain close to the expected values. The cost functions remain between 2% and 3%.

Going to a slightly higher part of the spectrum (Fig. 13), almost all states at $\Delta \lesssim 5, l \leq 3$ can be classified into

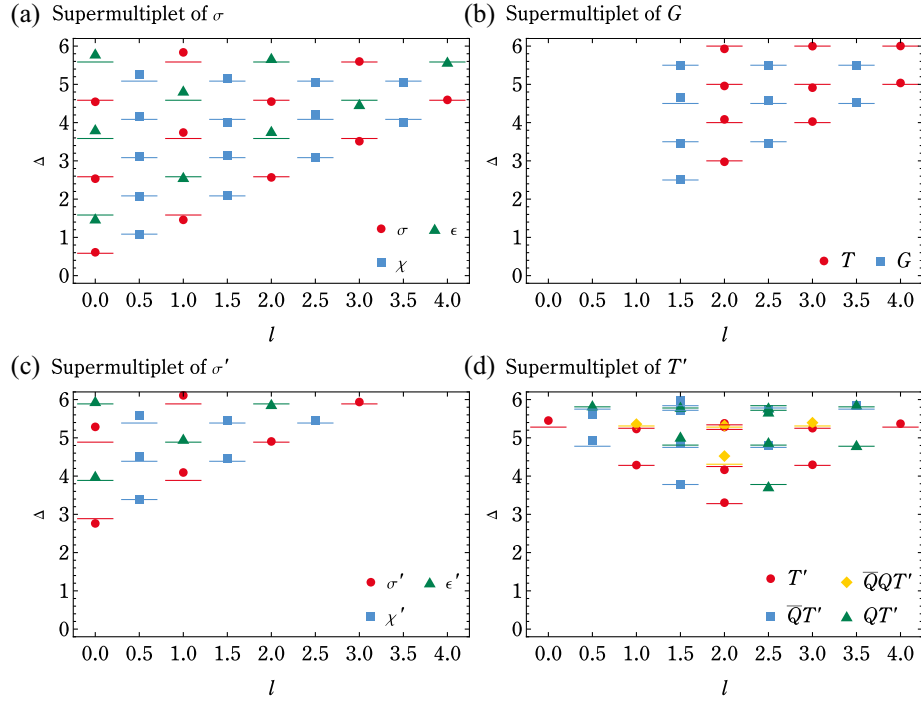


FIG. 14. The superconformal multiplets of the lowest superconformal primaries (a) σ , (b) G , (c) σ' , and (d) T' at $N_{mf} = 9$ and up to $\Delta = 6$ and $l = 4$. The symbols mark the measured scaling dimensions and the bars mark the expected values from a conformal bootstrap. The states in each superconformal multiplet are organized into conformal multiplets. The degenerate states are strongly mixed and are assigned arbitrarily. In the calculation, we fix $t = 1.5$, $U = 0.25$, $g = 1$ and take h, μ_1, μ_b as in Table II.

SCFT on the fuzzy sphere. For the free Majorana fermion, we consider a setup consisting of one bosonic and one fermionic lowest Landau level with a $1/2$ angular momentum mismatch, and allow the pair conversion between them. This microscopic realization of the spin-statistics constraint expected in the IR is an interesting subject for future work. By tuning a relative chemical potential between a fermionic integer quantum Hall phase at high fermion density and a bosonic Pfaffian phase [$SU(2)_2$ Chern-Simons theory] at high boson density, we find an intermediate phase described by a Majorana quantum Hall state with Kitaev index $\nu_K = 3$. The three phases are separated by two continuous phase transitions. Supported by numerical and theoretical evidence, we show that the fIQH-MQH transition is described by a free-Majorana-fermion CFT. Using state-operator correspondence, we find that the energy spectrum agrees well with the operator content of free Majorana fermion up to $\Delta \leq 4$. The local fermion operator is realized as the bilinear $f^\dagger(\mathbf{r})b(\mathbf{r})$ and H.c. on the sphere, whose two-point correlation function matches the expected conformal correlator. On the other hand, the MQH-bPf transition is described by a gauged Ising (Ising*) CFT. Its spectrum agrees with the $\mathbb{Z}_2$ -even sector of the Ising CFT. The transition is only consistent with even N_{mf} . We find that odd N_{mf} is equivalent to the insertion of a $\mathbb{Z}_2$ -gauge charge, and the spectrum

corresponds to the Wilson-line end-point operators of the gauged Ising CFT, i.e., the $\mathbb{Z}_2$ -odd sector of the Ising CFT.

Furthermore, we have established a correspondence between the fuzzy-sphere model Hamiltonians and the field-theory Lagrangians. In this correspondence, a flavor serves as a reference state and the other serves as an excitation flavor. The elementary fields that constitute the field theory are identified as hopping between them. Using this framework, we have translated an interacting fermionic CFT—the super-Ising theory, in which a Majorana fermion χ is coupled to a real scalar σ through a Yukawa interaction—into a fuzzy-sphere model with two fermionic flavors $f_{0,1}$ and one bosonic flavor b . The elementary fields are identified with the hopping operators $\sigma = f_0^\dagger f_1 + f_1^\dagger f_0$ and $\chi = (b^\dagger f_0, f_0^\dagger b)^T$. The three tuning parameters admit direct physical interpretations as the relevant operators σ , $\epsilon = \sigma'$, and $\sigma' = \bar{\chi}\chi$. We have provided numerical evidence for the emergence of superconformal symmetry and the agreement with the conformal bootstrap results. The superconformal multiplet structure with half-integer spacing is clearly observed and most of the low-lying states are identified as SCFT operators.

Our work represents the first fuzzy-sphere realization of CFTs with critical local fermions and constitutes a step towards studying renormalizable Lagrangians of quantum field theories on the fuzzy sphere. Most previous

fuzzy-sphere works were limited to realizing universality classes by constructing Hamiltonian with correct symmetry and anomaly and relying on RG flow to reach the desired conformal fixed point in the IR. The free Majorana fermion, the super-Ising theory, and the free and critical real-scalar models in Ref. [35] offer a natural and versatile playground for studying the connection between fuzzy-sphere constructions and quantum field theories. In particular, our fuzzy-sphere Majorana-fermion model has several notable features that deserve further exploration:

- (1) The critical point lies at $\mu_{c1} = 0$ with high precision over a wide range of t , suggesting the presence of an enhanced symmetry that forbids the chemical-potential perturbation in the free-Majorana-fermion CFT.
- (2) At the critical point $\mu_{c1} = 0$, the Hamiltonian closely resembles the familiar free Majorana fermion QFT: Apart from the charge density interaction term n_e^2 , it takes the same form as the field theory (21).
- (3) Analogous to the density operators, the angular components of the local fermion $\eta_{lm}^{(1/2)} = \int d^2\mathbf{r} \bar{Y}_{lm}^{(1/2)}(\mathbf{r}) f^\dagger(\mathbf{r}) b(\mathbf{r})$ may exhibit an algebraic structure similar to that of the density operators. It would be particularly interesting to determine whether this algebra is a deformation of the fermionic harmonic oscillator, analogous to the free scalar on the fuzzy sphere [35]. Following this line, the ground state could admit a description by a wave-function ansatz resembling a coherent state of η , as in the case of the free scalar.

The small orbital-space entanglement entropy suggests that DMRG calculations could access larger system sizes and provide more definitive answers to these questions. Understanding these features may guide us to generalize the fuzzy-sphere approach to other interacting Lagrangians, e.g., the gauge theories that require the engineering of spin-1 gauge fields.

Realizing renormalizable Lagrangians within the fuzzy-sphere framework will allow for a systematic study of three-dimensional quantum field theories in the deep IR. This approach may address questions such as the IR fate of a given Lagrangian, and the duality between different field-theory Lagrangians [112–114]. For example, the free Majorana fermion that we have studied is conjectured to admit a dual Lagrangian description as the $SO(N)_1$ Chern-Simons gauge theory coupled to a critical scalar in the $SO(N)$ vector representation [95–97]. A fuzzy sphere realization of the latter would allow numerical tests of this duality which go beyond comparisons of the expected phase diagrams, such as those associated with FIQH-MQH transitions.

The correspondence with Lagrangians also provides an intuitive way to design models and understand the physical meaning of the tuning parameters on the fuzzy sphere. By

assigning each Majorana fermion and each real scalar an excitation flavor in addition to a reference flavor, various field theories can be constructed in a similar way to the super-Ising theory, including

- (i) The $O(N_f)$ free Majorana fermion and Gross-Neveu-Yukawa theory [106,115],
- (ii) the $\mathcal{N} = 2$ Wess-Zumino model [116]—or, equivalently, the GNY theory of a Dirac fermion coupled to a complex scalar, and
- (iii) the generalization of Ginzburg-Landau description of a class of nonunitary minimal models to three space-time dimensions [117].

Nevertheless, our current realization of the super-Ising theory still requires the fine-tuning of three relevant operators due to the absence of the parity symmetry or manifest supersymmetry. In the fuzzy-sphere models, parity is usually implemented as the particle-hole symmetry combined with certain flavor transformations. This is not directly applicable for models with microscopic bosons. Finding a way to implement the parity for the boson-fermion-mixed models would greatly simplify targeting the fermionic CFTs. Besides that, several further questions regarding the realization of super-Ising theory deserve attention, e.g., how the correspondence (23) between the CFT and fuzzy-sphere operators holds numerically, whether it is possible to construct the emergent superconformal generators Q^α and S^α in analogy with the conformal generators P^μ and K^μ [30,31], how the reference-state overlap and the small orbital-space entanglement hold for the super-Ising SCFT, etc.

Instead of letting the supersymmetry emerge, one can also try to implement part of the fermionic charges of the superconformal symmetry at the microscopic level. The idea of fuzzy supersphere has indeed been explored over decades [118–120]. The supersphere $S^{2|2}$ is embedded in a superspace with three regular and two Grassmannian coordinates and is equipped with the $OSp(1|2)$ super-rotation symmetry [121,122]. We discuss this construction in more detail in Appendix D. The LLL on the supersphere with a $4\pi q$ supermonopole carries a super-spin- q representation, which branches into two $SU(2)$ multiplets with spin- q and $(q - \frac{1}{2})$ of opposite Grassmannian parity, in direct correspondence with our setup. The local density-density interaction on the LLL also takes a similar form as the Hamiltonian (3) for the free Majorana fermion. However, the embedding of $SU(2)$ rotations in $Sp(4, \mathbb{R})$ conformal transformations does not lift to an embedding of the super-rotation group $OSp(1|2)$ into the $\mathcal{N} = 1$ superconformal group $OSp(1|4)$. An embedding $OSp(1|2) \subset U(1|2) \subset OSp(2|4, \mathbb{R})$ into the $\mathcal{N} = 2$ superconformal group does exist, though. A more serious problem is that the superrotation group $OSp(1|2)$ does not admit any finite-dimensional non-trivial representation with a positive-definite inner product. If one is willing to relax the

requirement of unitarity and allow for an indefinite inner product, this embedding may provide a route to engineer $\mathcal{N} = 2$ SCFTs: as the rotation group is enhanced to a conformal group in the IR, the $\text{OSp}(1|2)$ fermionic super-rotation generators that mix the bosonic and fermionic orbitals must also become part of a larger symmetry supergroup and $\text{OSp}(2|4, \mathbb{R})$ is a natural candidate.

Besides supersymmetric theories, Chern-Simons-matter theories with fermionic content is also an interesting target. For example, the confinement transition of $\nu = 1/3$ Laughlin state which describes its transition with a trivially gapped state, has 2π monopole with spin-3/2 [123] which may be enhanced to the supercurrent in a $\mathcal{N} = 2$ supersymmetry, but the dual supersymmetric theory is expected to be gapped [124]. Another series of examples are the QED with Chern-Simons coupling [22], which describe transitions between Jain states [125]. For example, the $N_f = 3$ QED describes the transition between the states with fillings $\nu = 1/3$ and $\nu = 2/3$, and has 4π monopoles with spin-1/2 and 3/2. The fuzzy sphere is a promising platform to study these theories, as the realization of the proximate topological orders are natural with the fractional quantum Hall setup, e.g., the Laughlin states [126] with filling $\nu = 1/k$ realize the Abelian topological orders captured by Chern-Simons theories $U(1)_{-k}$, the Jain sequence [125] with $\nu = p/(mp + 1)$ realizes the Abelian topological orders with composite fermion descriptions, and the Read-Rezayi sequence with $\nu = p/(mp + 2)$ [127], as a natural extension of the Moore-Read states [65], realizes in general a series of non-Abelian topological orders. Another noticeable direction is to generalize the gauged Ising transition as $SU(2)_1$ coupled to an adjoint fermion to other adjoint QCDs [98], which may describe more exotic phase transitions.

Moving on to the Ising* transition, we explicitly demonstrate the physics of a topological defect of a CFT on the fuzzy sphere. Moreover, the gauging of the $\mathbb{Z}_2$ symmetry offers opportunities to study conformal line defects in the Ising CFT. For example, the monodromy defect in 3d Ising CFT [128] is attached to a half-infinite topological plane across which a $\mathbb{Z}_2$ twist is inserted. After gauging the $\mathbb{Z}_2$ symmetry, it becomes the vortex line of the $\mathbb{Z}_2$ -gauge field which is no longer attached to the topological plane. This means that it can be realized locally at antipodal points on the fuzzy sphere. The vortex line at the Ising* transition is closely related to the σ anyon in the proximate bPf phase: once the bulk is gapped, the vortex line becomes the σ worldline; under radial quantization, this appears as two σ anyons localized at the poles. The attempt to trap and separate anyons [129] in the $\nu = 5/2$ FQH state may give a hint on how to realize the vortex line in the Ising* CFT trap

anyons in the bPf phase. Another interesting related question to explore is the self-dual confinement transition in the $\mathbb{Z}_2$ topological order [130]: While condensing e or m alone in the $\mathbb{Z}_2$ TO drives an Ising* type transition to the trivially gapped phase, the transition condensing the e and m anyon at the same time in a self-dual manner may fall in a different universality. Its nature attracts wide attention, and a few critical exponents have been computed in lattice models through Monte Carlo [131]. This question could be potentially addressed on the fuzzy sphere, as the $\mathbb{Z}_2$ topological order could be produced by coupling the $SU(2)_2$ (i.e., bPf) to its opposite-chirality partner $SU(2)_{-2}$ and condensing certain types of anyons.

Note added. Recently, we became aware of a parallel study [132] that investigates the gauged Majorana CFT (or Majorana*, which is bosonic) in a different setup of quantum Hall bilayer transition. Section VIII on the super-Ising SCFT was added recently. It appears after the first realization of supersymmetry on the fuzzy sphere [133] in the gauged super-Ising universality (or super-Ising*, which is bosonic).

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DATA AVAILABILITY

The data that support the findings of this article are openly available [87–89].

APPENDIX A: EDGE CFT CHARACTER AND ENTANGLEMENT SPECTRUM DEGENERACY COUNT

In this section, we give the detail on calculating the degeneracy count for the entanglement spectrum in Sec. III and Fig. 3 from the character of the 2D chiral CFTs on the edge.

Fermionic integer quantum Hall phase is described by $U(1)_1 \cong SO(2)_1$, the 2D chiral CFT on its edge is a

pair of free chiral Majorana fermion that form a U(1) symmetry and carry charge $Q = \pm 1$. The character of the edge CFT is

$$\begin{aligned}
 \chi^{(\text{fIQH})}(q, y) &= \prod_{Q=\pm 1} q^{1/48} \prod_{k \in \mathbb{Z}+1/2} (1 + y^Q q^k) \\
 &= q^{1/24} (1 + yq^{1/2} + y^{-1}q^{1/2} + y^2q^2 + y^{-2}q^2 + \dots)(1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \dots) \\
 &= q^{1/24} (1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \dots + yq^{1/2} + yq^{3/2} + 2yq^{5/2} + 3yq^{7/2} + 5yq^{9/2} + \dots \\
 &\quad + y^{-1}q^{1/2} + y^{-1}q^{3/2} + 2y^{-1}q^{5/2} + 3y^{-1}q^{7/2} + 5y^{-1}q^{9/2} + \dots \\
 &\quad + y^2q^2 + y^2q^3 + 2y^2q^4 + 3y^2q^5 + 5y^2q^6 + \dots + y^{-2}q^2 + y^{-2}q^3 + 2y^{-2}q^4 + 3y^{-2}q^5 + 5y^{-2}q^6 + \dots + \dots).
 \end{aligned} \tag{A1}$$

Here $y = e^{2\pi iz}$ and z is the fugacity that couples to the U(1). Its chiral central charge is $c_- = 1$. By state-operator correspondence, the coefficient in front of $y^Q q^{m-c_-/24}$ is the degeneracy in the entanglement spectrum with angular momentum m in the z direction (compared with the ground state) and electric charge Q . The factorization shows that the chiral edge modes in different charge

sector have the same degeneracy but have the momentum m shifted.

Majorana quantum Hall with $\nu_K = 3$ is described by $\text{SO}(3)_1$. The 2D chiral CFT on its edge is three free chiral Majorana fermions. They form a vector under a $\text{SO}(3)$ global symmetry on the edge, of which a U(1) subgroup is manifest on the fuzzy sphere. Its character is

$$\begin{aligned}
 \chi^{(\text{MQH})}(q, y) &= \prod_{Q=0, \pm 1} q^{1/48} \prod_{k \in \mathbb{Z}+1/2} (1 + y^Q q^k) \\
 &= q^{1/16} (1 + yq^{1/2} + y^{-1}q^{1/2} + y^2q^2 + y^{-2}q^2 + \dots)(1 + q^{1/2} + q + 2q^{3/2} + 3q^2 + 4q^{5/2} + 5q^3 + 7q^{7/2} + \dots) \\
 &= q^{1/16} (1 + q^{1/2} + q + 2q^{3/2} + 3q^2 + 4q^{5/2} + 5q^3 + 7q^{7/2} + \dots \\
 &\quad + yq^{1/2} + yq + yq^{3/2} + 2yq^2 + 3yq^{5/2} + 4yq^3 + 5yq^{7/2} + \dots \\
 &\quad + y^{-1}q^{1/2} + y^{-1}q + y^{-1}q^{3/2} + 2y^{-1}q^2 + 3y^{-1}q^{5/2} + 4y^{-1}q^3 + \dots \\
 &\quad + y^2q^2 + y^2q^{5/2} + y^2q^3 + 2q^{7/2} + 3y^2q^4 + 4y^2q^{9/2} + 5y^2q^5 + \dots \\
 &\quad + y^{-2}q^2 + y^{-2}q^{5/2} + y^{-2}q^3 + 2q^{7/2} + 3y^{-2}q^4 + 4y^{-2}q^{9/2} + \dots + \dots).
 \end{aligned} \tag{A2}$$

Bosonic Pfaffian is described by $\text{SU}(2)_2$. As $\text{SU}(2)$ is a double covering of $\text{SO}(3)$, $\text{SO}(3)_1 \cong \text{SU}(2)_2/\mathbb{Z}_2$, i.e., the $\text{SU}(2)_2$ can be regarded as gauging the $\mathbb{Z}_2$ fermion-parity symmetry from $\text{SO}(3)_1$. Correspondingly, on its edge

$$\begin{aligned}
 \chi^{(\text{bPf})}(q, y) &= \frac{1}{2} [\chi^{(\text{MQH})}(q, y) \pm (q^{1/2} \rightarrow -q^{1/2})] \\
 &= \left[\chi^{(\text{MQH})}(q, y) \text{ with only } \mathbb{Z} \left(\text{or } \mathbb{Z} + \frac{1}{2} \right) \text{ power of } q \right] \\
 &= q^{1/16} (q^{1/2} + 2q^{3/2} + 4q^{5/2} + 7q^{7/2} + \dots + yq^{1/2} + yq^{3/2} + 3yq^{5/2} + 5yq^{7/2} + \dots \\
 &\quad + y^{-1}q^{1/2} + y^{-1}q^{3/2} + 3y^{-1}q^{5/2} + 5y^{-1}q^{7/2} \dots + y^2q^{5/2} + 2q^{7/2} + 4y^2q^{9/2} + \dots \\
 &\quad + y^{-2}q^{5/2} + 2q^{7/2} + 4y^{-2}q^{9/2} + \dots + \dots) \quad \left(\text{taking } - \text{ and } \mathbb{Z} + \frac{1}{2} \text{ for } N_{mf} \in 4\mathbb{Z} + 2 \right).
 \end{aligned} \tag{A3}$$

The choice of $\pm$ depends on whether the ground state of the chiral CFT on the edge is shifted to an integer or half-integer m , i.e., $+$ for $N_{mf} \in 4\mathbb{Z}$ and $-$ for $N_{mf} \in 4\mathbb{Z} + 2$. Its chiral edge mode is the same as removing the $m \in \mathbb{Z} + \frac{1}{2}$ levels and keeping the $m \in \mathbb{Z}$ part of the chiral edge mode in $\text{SO}(3)_1$.

APPENDIX B: LAGRANGIAN DESCRIPTION OF THE GAUGED ISING TRANSITION

In this section, we construct a Lagrangian that captures the transition between the bPf phase described by $SU(2)_2$ and the MQH phase described by $SU(2)_2/\mathbb{Z}_2 \cong SO(3)_1$. We consider a $U(2)$ gauge field $\alpha + \frac{1}{2}a\mathbb{I}_2$ where α is the traceless part and a is the trace part.

$$S = \int \frac{2}{4\pi} \text{tr} \left(\alpha d\alpha + \frac{2}{3} \alpha^3 \right) + \frac{1}{4\pi} ada + \frac{2}{2\pi} bda + \frac{1}{2\pi} Ada + \frac{1}{4\pi} AdA + d^3 x \left[((\partial_\mu - ia_\mu)\phi)^2 + \frac{m^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 \right]. \quad (\text{B1})$$

We start with the

$$U(2)_{2,1} = \frac{SU(2)_2 \times U(1)_2}{\mathbb{Z}_2}$$

Chern-Simons theory. The trace part a is coupled to the electromagnetic field A . We couple a to another dynamical $U(1)$ gauge field b through a mutual Chern-Simons coupling. This coupling will Higgs a from $U(1)$ to $\mathbb{Z}_2$. This can be seen from

- (1) The equation of motion of b asserts that $da = 0$, and
- (2) The insertion of a Wilson line of b enforces an a -flux $\oint a \in \pi\mathbb{Z}$ around it.

This $\mathbb{Z}_2$ -gauge field is now the center of the $SU(2)$. We then couple the $\mathbb{Z}_2$ -gauge field a to a real scalar ϕ . The phase transition is controlled by the mass of the scalar.

- (1) When $m^2 > m_c^2$, the scalar field is gapped, and the $\mathbb{Z}_2$ -gauge field remains. The resulting theory is $[SU(2)_2 \times \mathbb{Z}_2]/\mathbb{Z}_2 = SU(2)_2$, which describes the topological order of bosonic Pfaffian phase.
- (2) When $m^2 < m_c^2$, the scalar field is condensed, this confines the $\mathbb{Z}_2$ -gauge field. The resulting theory is $SU(2)_2/\mathbb{Z}_2 = SO(3)_1$, which describes the $\nu_K = 3$ MQH.

APPENDIX C: REALIZATION OF LOCAL SPINOR OPERATORS ON THE FUZZY SPHERE

The local Majorana fermion χ is a two-component spinor,

$$\chi(\mathbf{r}) = (\{\chi^M(\mathbf{r})\}_{M=\pm 1/2})^T = \begin{pmatrix} \chi^{-1/2}(\mathbf{r}) \\ \chi^{1/2}(\mathbf{r}) \end{pmatrix}. \quad (\text{C1})$$

Here we denote the two components by an angular momentum index $M = \pm 1/2$, such that acting each component at the origin point will get us an eigenstate

$$\begin{pmatrix} \chi^{-1/2}(0) \\ 0 \end{pmatrix} |\mathbb{I}\rangle = |\chi_{l=1/2, m=-1/2}\rangle, \\ \begin{pmatrix} 0 \\ \chi^{1/2}(0) \end{pmatrix} |\mathbb{I}\rangle = |\chi_{l=1/2, m=1/2}\rangle.$$

At each point on the sphere,

$$\chi(\mathbf{r}) = \chi^{(+1/2)}(\mathbf{r}) \mathbf{e}_{(1/2)} + \chi^{(-1/2)}(\mathbf{r}) \mathbf{e}_{(-1/2)}, \quad (\text{C2})$$

where the two polarizations $\chi^{(s)}(\mathbf{r})$ ($s = \pm 1/2$) of the spinor carry angular momentum s under the rotation along an axis that passes $\mathbf{r}$. Their polarization vectors can be written as

$$e_{(s)}^M = D_{sM}^{1/2}(0, \theta, \phi) = (-1)^{-s} Y^{(-s), 1/2, M}(\mathbf{r}). \quad (\text{C3})$$

Specifically,

$$\mathbf{e}_{(-1/2)} = \begin{pmatrix} u \\ -v \end{pmatrix}, \quad \mathbf{e}_{(+1/2)} = \begin{pmatrix} \bar{v} \\ \bar{u} \end{pmatrix}. \quad (\text{C4})$$

Here $u = e^{-i\phi/2} \cos \theta/2$ and $v = e^{i\phi/2} \sin \theta/2$ compose the Hopf spinor. $(\mathbf{e}_{(-1/2)}(\mathbf{r}), \mathbf{e}_{(1/2)}(\mathbf{r}))$ is an orthonormal basis for spinor fields on the sphere. They satisfy

$$\bar{\mathbf{e}}^{(s)} \mathbf{e}_{(s')} = \delta_{s'}^s, \quad \bar{\mathbf{e}}^{(s)} = (-1)^{-s} \mathbf{e}_{(-s)}^T \mathbf{g} \\ g_{mm'} = \delta_{m+m', 0} (-1)^{-m}. \quad (\text{C5})$$

Altogether, $(\mathbf{e}_{(-1/2)}, \mathbf{e}_{(1/2)})$ composes the Wigner's D -matrix $D^l(\theta, \phi, \psi)$ with $l = 1/2$ and $\psi = 0$, which is a unitary matrix used to rotate spinning objects around the globe.

Here we explain some of the notations. The indices in the parentheses are the spin weight. The expression altogether must carry spin-weight 0. We use a rule that the lower indices lower the angular momentum or spin weight and the upper indices raise them. The indices of spherical harmonics shall also be adapted to this rule. The indices are raised and lowered with a rule such that

$$u^m = (-1)^m u_{-m}, \quad u_m = (-1)^{-m} u^{-m}. \quad (\text{C6})$$

The indices of the spherical harmonics follow the same logic, which makes it a bit different from the standard notation. This notation helps us keep track of all the conjugation relations and minus signs, e.g., the conjugation of spherical harmonics is automatically taken care of

$$\bar{Y}_{(s), l, m} = (-1)^{-s-m} Y^{(-s), l, -m}$$

as well as the orthonormal relations

$$e_m^{(s')} e_m^{(s)} = \delta_{s'}^s.$$

The polarization can be decomposed into spherical components on the basis of spin-weighted spherical harmonics.

$$\begin{aligned}\chi^{(s)}(\mathbf{r}) &= \frac{1}{R^2} \sum_{l=1/2}^{\infty} \sum_{m=-l}^l \chi^{(s),l,m} Y^{(s),l,m}(\mathbf{r}), \\ \chi^{(s),l,m} &= \int d^2\mathbf{r} \bar{Y}_{(s),l,m}(\mathbf{r}) \chi^{(s)}(\mathbf{r}).\end{aligned}\quad (\text{C7})$$

Now we consider the two-point correlation functions of χ . In the standard notation, the conformal kinematics requires

$$\langle \chi^\alpha(x_1) \chi^\beta(x_2) \rangle = \frac{\gamma_\mu^{\alpha\beta} x_{12}^\mu}{(x_{12}^2)^{3/2}}, \quad (\text{C8})$$

where $x_{1,2} \in \mathbb{R}^3$, $x_{12} = x_1 - x_2$, a, b are spinor indices, and γ are the Pauli matrices. It can be equivalently written as

$$\langle \chi^{m_1}(x_1) \chi^{m_2}(x_2) \rangle = \frac{\sqrt{2} C_{1,m}^{\frac{1}{2}m_1, \frac{1}{2}m_2} \hat{x}_{12}^m}{(x_{12}^2)^2}, \quad (\text{C9})$$

where

$$\begin{aligned}\hat{x}^m &= \sqrt{\frac{4\pi}{3}} Y^{1m}(\hat{x}) \\ &= \left(\frac{1}{\sqrt{2}} e^{-i\phi} \sin \theta, \cos \theta, -\frac{1}{\sqrt{2}} e^{i\phi} \sin \theta \right)^T \\ &= C_{10}^{\frac{1}{2}s_1, \frac{1}{2}s_2} C_{\frac{1}{2}m_1, \frac{1}{2}m_2}^{1m} e_{(s_1)}^{m_1}(\hat{x}) e_{(s_2)}^{m_2}(\hat{x}),\end{aligned}\quad (\text{C10})$$

and $C_{lm}^{l_1 m_1; l_2 m_2}$ is the Clebsh-Gordan coefficient. We use this notation such that the contraction of indices are clearer.

We first consider the case where x_2 is at the origin point and x_1 is on the unit sphere. After Weyl transformation, it can be measured through the matrix element $\langle 0 | \chi^{(s)}(\mathbf{r}) | \chi^m \rangle$ on the fuzzy sphere.

$$\langle 0 | \chi^{(s)}(\mathbf{r}) | \chi^m \rangle = \frac{1}{R} e_{m'}^{(s)} \langle \chi^m(\mathbf{r}) \chi^{m'}(0) \rangle = \frac{R^{-1}}{\sqrt{2\pi}} Y^{(s), \frac{1}{2}m}(\mathbf{r}). \quad (\text{C11})$$

Hence, for the spherical components,

$$\langle 0 | \chi^{(s)l'm'} | \chi^m \rangle = \frac{R^{-1}}{\sqrt{2\pi}} \delta_{l', \frac{1}{2}} (-1)^m \delta_{m'+m, 0}. \quad (\text{C12})$$

We then consider the case where both points are on the unit sphere. After the Weyl transformation, it can be measured through $\langle 0 | \chi^{(s_1)}(\mathbf{r}) \chi^{(s_2)}(\mathbf{r}) | 0 \rangle$ on the fuzzy sphere.

$$\begin{aligned}\langle 0 | \chi^{(s_1)}(\mathbf{r}_1) \chi^{(s_2)}(\mathbf{r}_2) | 0 \rangle &= R^{-2} e_{m_1}^{(s_1)}(\mathbf{r}_1) e_{m_2}^{(s_2)}(\mathbf{r}_2) \langle \chi^{m_1}(x_1) \chi^{m_2}(x_2) \rangle \\ &= 2i \delta_{s_1+s_2, 0} x_{12}^{-2} \\ \langle 0 | \chi^{(s_1)}(\mathbf{r}_1) \chi^{(s_2)}(\mathbf{r}_2) | 0 \rangle &= 2(-1)^{1/2-s_1} \delta_{s_2}^{s_1} x_{12}^{-2}.\end{aligned}\quad (\text{C13})$$

Hence, the correlation function of each spin-weighted component behaves exactly like a scalar correlator. In terms of the spherical components,

$$\begin{aligned}\langle 0 | \chi^{(s)}(\mathbf{r}_2) | 0 \rangle &= \sum_{l_1 m_1 l_2 m_2} \langle \chi_{(s)l_1 m_1} \chi^{(s)l_2 m_2} \rangle \bar{Y}_{(s)l_1 m_1}(\mathbf{r}_1) Y^{(s)l_2 m_2} \\ &= \sum_{l_1 m_1 l_2 m_2} \langle \chi_{(s)l_1 0} \chi^{(s)l_1 0} \rangle \delta_{l_1 m_1}^{l_2 m_2} \bar{Y}_{(s)l_1 m_1}(\mathbf{r}_1) Y^{(s)l_2 m_2} \\ &= \sum_l \langle \chi_{(s)l 0} \chi^{(s)l 0} \rangle \sum_m \bar{Y}_{(s)lm}(\mathbf{r}_1) Y^{(s)lm}(\mathbf{r}_2),\end{aligned}\quad (\text{C14})$$

where the second line comes from the rotation symmetry. To perform the summation, we express the spherical Harmonics in terms of the D matrix

$$\begin{aligned}Y^{(s)lm}(\mathbf{r}_2) &= (-1)^s D_{-s,m}^l(0, \theta_2, \phi_2) \\ \bar{Y}_{(s)lm}(\mathbf{r}_1) &= (-1)^{-l} \bar{Y}_{(-s)l,m}(-\mathbf{r}_1) \\ &= (-1)^{-l-s} \bar{D}_{sm}^l(0, \pi - \theta_1, \pi + \phi_1) \\ &= (-1)^{-l-s} D_{ms}^l(-\pi - \phi_1, -\pi + \theta_1, 0)\end{aligned}$$

$$\begin{aligned}\sum_m \bar{Y}_{(s)lm}(\mathbf{r}_1) Y^{(s)lm}(\mathbf{r}_2) &= (-1)^{-l} (D^l(0, \theta_2, \phi_2) D^l(-\pi - \phi_1, -\pi + \theta_1, 0))_{-s,s} \\ &= (-1)^l D_{-s,s}^l(\psi', \pi - \theta_{12}, \phi') \\ &= (-1)^l e^{is(\phi' - \psi')} d_{-s,s}^l(\pi - \theta_{12}) \\ &= e^{is(\phi' - \psi')} d_{-s,s}^l(\theta_{12}) \\ &= e^{is(\phi' - \psi')} \frac{2l+1}{4\pi} \left(\sin \frac{\theta_{12}}{2} \right)^{2s} P_{l-s}^{(2s,0)}(\cos \theta_{12}).\end{aligned}\quad (\text{C15})$$

In the last line, we only calculate for positive s , and negative s follows similarly. There is an overall phase that depends on the relative position of the two points, so we focus on the amplitude. In the third to last line, we have used the superposition of rotations, and θ_{12} is the angular distance between the two points. The $P_n^{(\alpha,0)}$ are the Jacobi polynomials. They are defined on the interval $[-1, 1]$ and have orthonormal relation

$$\int d\Omega \sin^{2s} \theta P_{l-s}^{(2s,0)}(\cos \theta) P_{l'-s}^{(2s,0)}(\cos \theta) = \frac{4\pi}{2l+1} \delta_{ll'}. \quad (\text{C16})$$

The components are

$$C_\chi(\theta_{12}) = \langle 0 | \chi_{(1/2)}(\mathbf{r}_1) \chi^{(1/2)}(\mathbf{r}_2) | 0 \rangle \\ = \sum_l C_{\chi,l} \frac{2l+1}{4\pi} \sin \frac{\theta_{12}}{2} P_{l-1/2}^{(1,0)}(\cos \theta_{12}), \quad (\text{C17})$$

$$C_{\chi,l} = 2\pi \int \sin \theta_{12} d\theta_{12} C_\chi(\theta_{12}) \sin \frac{\theta_{12}}{2} P_{l-1/2}^{(1,0)}(\cos \theta_{12}). \quad (\text{C18})$$

For $C_\chi(\theta_{12}) = (2 \sin \theta_{12}/2)^{-2}$, the decomposed components are

$$C_{\chi,l} = 2\pi.$$

We then briefly discuss how to decompose the local fermionic operators on the fuzzy sphere

$$\eta^{(1/2)}(\mathbf{r}) = f^\dagger(\mathbf{r})b(\mathbf{r}), \quad \eta^{(-1/2)}(\mathbf{r}) = f^\dagger(\mathbf{r})b(\mathbf{r}) \quad (\text{C19})$$

into spherical components:

$$\eta^{(1/2)}(\mathbf{r}) = \eta^{(1/2)lm} Y^{(1/2)lm}(\mathbf{r}) \quad (\text{C20})$$

$$\begin{aligned} \eta^{(1/2)lm} &= \int d^2\mathbf{r} \bar{Y}_{(1/2)lm} \eta^{(1/2)}(\mathbf{r}) \\ &= \int d^2\mathbf{r} \bar{Y}_{(1/2)lm} f^\dagger(\mathbf{r})b(\mathbf{r}) \\ &= \sum_{m_1 m_2} f^{\dagger m_1} b_{m_2} \int d^2\mathbf{r} \bar{Y}_{(1/2)lm} Y^{(q)qm_1} \bar{Y}_{(q-1/2),q-1/2,m_2} \\ &= \sum_{m_1 m_2} f^{\dagger m_1} b_{m_2} \delta_{m_1, m_2+m} (-1)^{-q-m_1} \\ &\quad \times \sqrt{\frac{2q(2q+1)(2l+1)}{4\pi}} \\ &\quad \times \begin{pmatrix} l & q & q-\frac{1}{2} \\ -m & m_1 & -m_2 \end{pmatrix} \begin{pmatrix} l & q & q-\frac{1}{2} \\ 1/2 & -q & q-\frac{1}{2} \end{pmatrix}. \end{aligned} \quad (\text{C21})$$

The $\eta^{(-1/2)}$ follows similarly.

The $\eta^{(\pm 1/2)}$ realizes the two polarizations of χ ,

$$\eta^{(1/2)} = \alpha \chi^{(1/2)}(\mathbf{r}) + \dots \quad (\text{C22})$$

The coefficient α can be settled through Eq. (C12)

$$\alpha = -iR\sqrt{2\pi} \langle 0 | \eta^{(s)\frac{1}{2}, -\frac{1}{2}} | \chi^{\frac{1}{2}} \rangle. \quad (\text{C23})$$

Hence,

$$\langle 0 | \chi_{(1/2)}(\mathbf{r}_1) \chi^{(1/2)}(\mathbf{r}_2) | 0 \rangle = R^{-2} \frac{\langle 0 | \eta_{(1/2)}(\mathbf{r}_1) \eta^{(1/2)}(\mathbf{r}_2) | 0 \rangle}{\frac{1}{2\pi} |\langle 0 | \eta^{(s)\frac{1}{2}, -\frac{1}{2}} | \chi^{\frac{1}{2}} \rangle|^2} \quad (\text{C24})$$

and the dimensionless correlator Eq. (12) is exactly expressed as

$$C_\chi^*(\theta_{12}) = \frac{\langle 0 | \eta_{(1/2)}(\mathbf{r}_1) \eta^{(1/2)}(\mathbf{r}_2) | 0 \rangle}{\frac{1}{\pi} |\langle 0 | \eta^{(s)\frac{1}{2}, -\frac{1}{2}} | \chi^{\frac{1}{2}} \rangle|^2}. \quad (\text{C25})$$

APPENDIX D: THE FUZZY SUPERSPHERE

In this section we explain the idea of the fuzzy supersphere, its connection with our setup for free Majorana fermion, and the embedding of the superrotation symmetry in the $\mathcal{N} = 2$ superconformal group.

1. Supersphere and superrotation

We consider the superspace $E(3|2)$ spanned by the regular coordinates x_μ and Grassmannian coordinates θ_α , where $\mu = 1, 2, 3$ and the spinor index $\alpha = \pm$. The supersphere is defined by the constraint $r^2 = x^\mu x_\mu + \theta_\alpha \epsilon^{\alpha\beta} \theta_\beta$. It is invariant under the superrotation symmetry $\text{OSp}(1|2)$, generated by three bosonic generators J_μ and two fermionic generators J_α [122]

$$J_\mu = L_\mu + \frac{1}{2} \theta_\alpha (\gamma_\mu)^\alpha_\beta \partial^\beta, \\ V_\alpha = \frac{i}{2} \left[\left(1 + \frac{1}{2} \bar{\theta} \theta \right) \epsilon_{\alpha\beta} \partial^\beta - \theta_\beta (\gamma^\mu)^\beta_\alpha L_\mu \right], \quad (\text{D1})$$

where $\partial^\alpha = \partial/\partial\theta_\alpha$, $\bar{\theta}^\alpha = \epsilon^{\alpha\beta} \theta_\beta$ and L_μ is the usual orbital angular momentum $L_\mu = i\epsilon^{\mu\nu\rho} x_\nu \partial_\rho$. They have the following commutation relation [134]:

$$[J_\mu, J_\nu] = i\epsilon_{\mu\nu\rho} J^\rho, \\ [J_\mu, V_\alpha] = \frac{1}{2} J_\beta (\gamma_\mu)^\beta_\alpha, \\ \{V_\alpha, V_\beta\} = \frac{1}{2} J_\mu (\gamma^\mu)_{\alpha\beta}. \quad (\text{D2})$$

These generators are equipped with super-Hermitianity

$$J_\mu^\dagger = J_\mu, \quad J_\alpha^\dagger = \epsilon^{\alpha\beta} J_\beta,$$

a notion enhancing complex conjugation in the superspace to the operator algebra. It has the properties

$$(A^\dagger)^\dagger = (-1)^{\nu(A)} A, \quad (AB)^\dagger = (-1)^{\nu(A)\nu(B)} B^\dagger A^\dagger \quad (\text{D3})$$

for arbitrary operators A, B where $\nu(\bullet) = 0, 1$ is the Grassmannian parity. The super-Hermitian structure of the algebra cannot be realized as the adjoint in any finite-dimensional nontrivial representation with a positive-definite inner product [135]. Nevertheless, the super-Hermitianity requires that the eigenvalues are either real or appear in complex-conjugate pairs, similar to the requirement of $\mathcal{PT}$ symmetry in Yang-Lee singularity. In the following, we relax the requirement of unitarity and

allow for an indefinite inner product in order to construct a quantum-mechanical system on the supersphere.

The nonunitary finite-dimensional representations of $\text{OSp}(1|2)$ are labeled by superspin- j and a Grassmannian degree $\lambda \in \mathbb{Z}_2$ [121,122]. The dimension of the superspin- j representation is $4j + 1$. The components are labeled by angular momenta l and magnetic number m . The quantum numbers take value as

$$j \in \frac{1}{2}\mathbb{Z}, \quad l = j \text{ or } j - \frac{1}{2}, \quad m = -l, -l + 1, \dots, l.$$

Each state $|j\lambda; lm\rangle$ has a Grassmannian number $\lambda + 2(j - l)$, total superspin j , total orbital angular momentum l , and the angular momentum z -direction m . Each representation branches into a two $\text{SU}(2)$ multiplets with spin- j and $(j - \frac{1}{2})$, and their Grassmannian parities are opposite.

2. The fuzzy supersphere

To build a model on the supersphere, we need the basis of lowest Landau level. As on the ordinary sphere, this basis can be constructed from the Hopf spinor [119]. We begin by recalling how the spherical LLL arises from the Hopf spinor of $\mathbb{CP}^1$. The Hopf spinor $(u, v)^T$ defines a $\text{U}(1)$ bundle over $\mathbb{CP}^1$ whose Berry curvature corresponds to the monopole field on S^2 . In Cartesian coordinates it may be written as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{2r(r+x_3)}} \begin{pmatrix} r+x_3 \\ x_1+ix_2 \end{pmatrix}. \quad (\text{D4})$$

The spherical LLL is realized as the space of holomorphic sections of the appropriate power of this bundle, i.e., as homogeneous polynomials in the Hopf spinor components

$$\phi_m(u, v) = C_m u^{q+m} v^{q-m} \quad (m = -q, \dots, q), \quad (\text{D5})$$

where C_m is a normalization constant. In complete analogy, the Hopf spinor of $\mathbb{CP}^{1|1}$ contains two bosonic and one fermionic component

$$\begin{pmatrix} u \\ v \\ \xi \end{pmatrix} = \frac{1}{\sqrt{2r^3(r+x_3)}} \begin{pmatrix} (r+x_3)\left(r - \frac{1}{4(r+x_3)}\bar{\theta}\theta\right) \\ (x_1+ix_2)\left(r + \frac{1}{4(r+x_3)}\bar{\theta}\theta\right) \\ -(r+x_3)\theta_+ - (x_1+ix_2)\theta_- \end{pmatrix}. \quad (\text{D6})$$

It yields the superspherical LLL

$$\begin{aligned} \phi_m(u, v, \xi) &= C_m u^{q+m} v^{q-m} \\ &\quad (m = -q, \dots, q) \\ \psi_m(u, v, \xi) &= C'_m u^{q-\frac{1}{2}+m} v^{q-\frac{1}{2}-m} \xi \\ &\quad \left(m = -q + \frac{1}{2}, \dots, q - \frac{1}{2}\right). \end{aligned} \quad (\text{D7})$$

It carries a super-spin- q representation of the $\text{OSp}(1|2)$ superrotation group. It decomposes into $2q + 1$ Grassmannian-even orbitals and $2q$ Grassmannian-odd orbitals with respective orbital angular momenta q and $q - \frac{1}{2}$. By interpreting the Grassmannian-even orbitals as fermions and the Grassmannian-odd orbitals as bosons, the structure is in complete correspondence with our setup for the Majorana fermion. The mismatch $N_{mf} - N_{mb} = 1$ in the setup for the Majorana fermion is inherited from the $1/2$ angular-momentum difference in the two $\text{SU}(2)$ multiplets. A crucial difference, however, is that implementing the full superrotation symmetry requires a nonunitary (indefinite) realization: as discussed above, the super-Hermitian structure of the algebra does not admit a nontrivial positive-definite finite-dimensional representation.

The physical setup for a fuzzy supersphere involves a supermonopole that exerts a connection on the supersphere [119,120]

$$\begin{aligned} A_\mu &= -\frac{q}{r(r+x_3)} \epsilon_{\mu\nu 3} x^\nu \left[1 + \frac{2r+x_3}{4r^2(r+x_3)} \bar{\theta}\theta \right] \\ A_\alpha &= \frac{iq}{r^3} x_\mu \theta_\beta (\gamma^\mu)^\beta{}_\alpha. \end{aligned} \quad (\text{D8})$$

The single-particle Hamiltonian becomes

$$H_0 = \frac{1}{2mr^2} (D^\mu D_\mu + D_\alpha \epsilon^{\alpha\beta} D_\beta) \quad (\text{D9})$$

where the covariant operators

$$D_\mu = \partial_\mu + iA_\mu, \quad D_\alpha = \partial_\alpha - iA_\alpha.$$

The single-particle states are the monopole superspherical harmonics $Y_{jlm}^{(q)}$

$$\begin{aligned} j-s \in \mathbb{Z} \quad Y_{jlm}^{(q)} &= \frac{1 - \frac{1}{2}j\bar{\theta}\theta}{\sqrt{2j+1}} Y_{jlm}^{(q)} \\ Y_{j,j-\frac{1}{2},m}^{(q)} &= i \left[\sqrt{\frac{j+m+\frac{1}{2}}{2j+1}} \theta_- Y_{j,m+\frac{1}{2}}^{(q)} - \sqrt{\frac{j-m+\frac{1}{2}}{2j+1}} \theta_+ Y_{j,m-\frac{1}{2}}^{(q)} \right] \\ j-s \in \mathbb{Z} + \frac{1}{2} \quad Y_{j,j-\frac{1}{2},m}^{(q)} &= \frac{1 - \frac{1}{2}(j+\frac{1}{2})\bar{\theta}\theta}{\sqrt{2j}} Y_{j-\frac{1}{2},m}^{(q)} \\ Y_{jlm}^{(q)} &= i \left[\sqrt{\frac{j-m}{2j}} \theta_- Y_{j-\frac{1}{2},m+\frac{1}{2}}^{(q)} + \sqrt{\frac{j+m}{2j}} \theta_+ Y_{j-\frac{1}{2},m-\frac{1}{2}}^{(q)} \right]. \end{aligned} \quad (\text{D10})$$

Here j is the total super-spin index, l is the total orbital angular momentum index, and the indices take value as

$$j = q, q + \frac{1}{2}, q + 1, \dots$$

$$l = j, j - \frac{1}{2}, \quad m = -l, -l + 1, \dots, l.$$

and the lowest Landau level wave functions are

$$\phi_m = Y_{qqm}^{(q)}(x_\mu, \theta_\mu), \quad \psi_m = Y_{q, q-\frac{1}{2}, m}^{(q)}(x_\mu, \theta_\mu).$$

We consider the superfield operator on the supersphere

$$\Upsilon(x_\mu, \theta_\alpha) = \Psi_0(x_\mu) + \bar{\theta}^\alpha \Phi_\alpha(x_\mu) + \frac{1}{2} \bar{\theta} \theta \Psi_1(x_\mu), \quad (\text{D11})$$

where $\Psi_{0,1}$ are fermionic fields and Φ_α are bosonic fields. We can then project it onto the lowest Landau level (D7)

$$\Upsilon(x_\mu, \theta_\alpha) = \sum_m f_m^\dagger Y_{qqm}^{(q)}(x_\mu, \theta_\mu) + \sum_m b_m^\dagger Y_{q, q-\frac{1}{2}, m}^{(q)}(x_\mu, \theta_\mu). \quad (\text{D12})$$

Here $b_m^\dagger$ and $f_m^\dagger$ are the creation operators on the lowest Landau level.

Like on the fuzzy sphere, we can express the interaction Hamiltonian in terms of $f_m^{(\dagger)}$ and $b_m^{(\dagger)}$ on the fuzzy supersphere. The interaction theories may be built as transitions between gapped states like the super-Haldane-Laughlin states [118,136–139] analogous to the Laughlin states on the regular LLL. A convenient approach to build the interaction Hamiltonian is through the density operator

$$n(x_\mu, \theta_\alpha) = \Upsilon^\dagger(x_\mu, \theta_\alpha) \Upsilon(x_\mu, \theta_\alpha).$$

As an example, the local density-density interaction of a single LLL reads

$$\begin{aligned} & \int d^2x_\mu d^2\theta_\alpha n(x_\mu, \theta_\alpha)^2 \\ &= \sum_{\{m_i\}} [\alpha^{(b)} M_{m_1 m_2 m_3 m_4}^{(b)} b_{m_1}^\dagger b_{m_2}^\dagger b_{m_3} b_{m_4} \\ &+ \alpha^{(bf)} M_{m_1 m_2 m_3 m_4}^{(bf)} b_{m_1}^\dagger f_{m_2}^\dagger f_{m_3} b_{m_4} \\ &+ \alpha^{(t)} M_{m_1 m_2 m_3 m_4}^{(t)} (b_{m_1}^\dagger b_{m_2}^\dagger f_{m_3} f_{m_4} - f_{m_4}^\dagger f_{m_3}^\dagger b_{m_2} b_{m_1})], \end{aligned} \quad (\text{D13})$$

where the coefficients $\alpha^{(b)}$, $\alpha^{(bf)}$, and $\alpha^{(t)}$ are constrained by the superrotation. It has similar form with the Majorana fermion Hamiltonian (3) with $\mu = 0$, except that the super-Hermiticity (D3) requires $b_{m_1}^\dagger b_{m_2}^\dagger f_{m_3} f_{m_4}$ and $f_{m_4}^\dagger f_{m_3}^\dagger b_{m_2} b_{m_1}$ to have opposite coefficients (instead of the same under Hermiticity). This interaction alone produces the super-Haldane-Laughlin state.

3. Embedding into the superconformal algebra

We need to embed the super-rotation group $\text{OSp}(1|2)$ into the superconformal algebra. The superconformal generators in $d = 3$ generates a super-Lie group $\text{OSp}(\mathcal{N}|4)$ with bosonic dimension $10 + \frac{1}{2}\mathcal{N}(\mathcal{N} - 1)$ and fermionic dimension $4\mathcal{N}$ [108,109]. The bosonic generators are $P_\mu, M_{\mu\nu}, D, K_\mu, A_{ab}$ and fermionic generators are Q^{aa}, S^{aa} where $a = 1, \dots, \mathcal{N}$ is the supercharge index. We first consider the case of $\mathcal{N} = 1$. Although $\text{OSp}(1|2)$ is indeed a sub-group of $\text{OSp}(1|4)$, the embedding is in general impossible due to the incompatibility of the commutators. Besides the super-rotation, one also needs to embed the time translation $-iH$ that commutes with every other generator. The commutation with rotation $[H, M^{\mu\nu}] = 0$ constrains that $-iH = D$, but no combination of the fermionic generators Q and S commutes with D . Hence, there exists no consistent embedding of superrotation $\text{OSp}(1|2)$ and $-iH$ into the superconformal group $\text{OSp}(1|4)$.

We then consider a larger superconformal group $\text{OSp}(2|4)$ for $\mathcal{N} = 2$. Besides the conformal generators, the generators include the fermionic $Q_\alpha, \bar{Q}_\alpha, S_\alpha, \bar{S}_\alpha$ and an additional bosonic $U(1)_R$ generator A . We construct the embedding

$$J_\mu = \epsilon_{\mu\nu\rho} M^{\nu\rho}$$

$$V_\alpha = \frac{1}{2} (Q_\alpha + i\bar{S}_\alpha). \quad (\text{D14})$$

One can check explicitly that they satisfy the superrotation algebra $\mathfrak{osp}(1|2)$ (D2). Here we list the relevant nonzero commutators

$$\begin{aligned} [M_{\mu\nu}, Q_\alpha] &= i \frac{1}{2} (\gamma_{[\mu} \gamma_{\nu]})_{\alpha\beta} Q^\beta \\ [M_{\mu\nu}, \bar{S}_\alpha] &= i \frac{1}{2} (\gamma_{[\mu} \gamma_{\nu]})_{\alpha\beta} \bar{S}^\beta \\ \{Q_\alpha, \bar{S}_\beta\} &= -i (\gamma^{[\mu} \gamma^{\nu]})_{\alpha\beta} M_{\mu\nu} + 2i\epsilon_{\alpha\beta} (D - A). \end{aligned} \quad (\text{D15})$$

The time translation generated by $-iH$ cannot be identified as the dilatation D anymore, as it does not commute with the fermionic generators

$$[D, V_\alpha] = \frac{i}{4} (Q_\alpha + i\bar{S}_\alpha),$$

but V_α commutes with the combination of D and the generator of the $U(1)_R$ symmetry A , which we can now define as Hamiltonian.

$$-iH = D + \frac{1}{2}A, \quad [H, V_\alpha] = 0. \quad (\text{D16})$$

Its eigenvalue E would be a combination of the scaling dimension Δ and the $U(1)_R$ charge r , where we denote

$$E = \Delta + \frac{1}{2}r$$

as the energy. In this way, we have embedded all the manifest symmetries into the superconformal group.

We comment on the rest of the superconformal generators, which include

$$U_\alpha = \frac{1}{2}(Q_\alpha - i\bar{S}_\alpha), \quad \bar{Q}, \quad S, \quad \text{and} \quad -i\bar{H} = D - \frac{1}{2}A.$$

These generators have commutation relations with the Hamiltonian

$$[H, U^\alpha] = 0, \quad [H, \bar{Q}^\alpha] = \bar{Q}^\alpha, \quad [H, S^\alpha] = -S^\alpha. \quad (\text{D17})$$

Therefore, U does not change the energy, $\bar{Q}$ increases the energy by 1 and S decreases the energy by 1.

4. Superconformal multiplet structure

We then consider how the energy levels organize themselves into multiplets in our setup. The conformal multiplets combine into superconformal multiplets [110]. Each multiplet is marked by (l, Δ, r) of its primary, containing the Lorentz spin l , scaling dimension Δ and $U(1)_R$ charge r . To be more general, we list the superconformal descendants in a long multiplet.

$$\begin{array}{ccccc}
 \boxed{E} & & (l, \Delta, r) & \xrightarrow{Q} & (l \pm \frac{1}{2}, \Delta + \frac{1}{2}, r - 1) & \xrightarrow{Q} & (l, \Delta + 1, r - 2) \\
 & & \downarrow \bar{Q} & & \downarrow \bar{Q} & & \downarrow \bar{Q} \\
 \boxed{E+1} & & (l \pm \frac{1}{2}, \Delta + \frac{1}{2}, r + 1) & \xrightarrow{Q} & (l \oplus l \oplus l \pm 1, \Delta + 1, r) & \xrightarrow{Q} & (l \pm \frac{1}{2}, \Delta + \frac{3}{2}, r - 1) \\
 & & \downarrow \bar{Q} & & \downarrow \bar{Q} & & \downarrow \bar{Q} \\
 \boxed{E+2} & & (l, \Delta + 1, r + 2) & \xrightarrow{Q} & (l \pm \frac{1}{2}, \Delta + \frac{3}{2}, r + 1) & \xrightarrow{Q} & (l, \Delta + 2, r)
 \end{array} \quad (\text{D18})$$

Here we organize the table such that each line have the same eigenvalue under H , and for the primary $E = \Delta + \frac{1}{2}r$. After considering the super-rotation symmetry, two multiplets whose conformal primaries have the same energy E and spin $l = j$ and $l = j - \frac{1}{2}$ combine into one, which we denote as “bi-multiplet.” Each bi-multiplet is marked by $[j, E]$ containing the super-spin j and energy E . Subsequently, considering the first line of Diagram (D18), they combine into two bi-multiplets

$$\left\{ \begin{array}{c} (l, \Delta, r) \\ (l \pm \frac{1}{2}, \Delta + \frac{1}{2}, r - 1) \\ (l, \Delta + 1, r - 2) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} [l, E] \\ [l - \frac{1}{2}, E] \end{array} \right\}, \quad (\text{D19})$$

where $E = \Delta + \frac{1}{2}$. These two bi-multiplets are connected by the spinor generator U . To show the combination explicitly, we look at the simplest example of a scalar superconformal primary Φ . The conformal multiplets of $\Phi, Q\Phi, Q^2\Phi$ will be reorganized into two bi-multiplets $[\frac{1}{2}]$ and $[0]$, where $Q^2 = Q^\alpha \epsilon_{\alpha\beta} Q^\beta$

$$\begin{aligned}
 \left[\frac{1}{2}, E\right] &= (Q\Phi) \oplus [Q^2\Phi - 2i(\Delta + r)\Phi] \\
 [0, E] &= [Q^2\Phi + 2i(\Delta + r)\Phi].
 \end{aligned} \quad (\text{D20})$$

The superconformal primary does not belong to any of the two bi-multiplets. Instead, the content of the bi-multiplets are linear combinations of the primary and its superconformal descendants.

Therefore, the 16 conformal multiplets in the superconformal multiplet can be reorganized into 8 bi-multiplets.

$$\begin{array}{ccc}
 \boxed{E} & & [j, E] \xleftarrow{U} [j + \frac{1}{2}, E] \\
 & & \downarrow \bar{Q} \qquad \qquad \downarrow \bar{Q} \\
 \boxed{E+1} & & [j \pm \frac{1}{2}, E + 1] \xleftarrow{U} [j \oplus j + 1, E + 1] \\
 & & \downarrow \bar{Q} \qquad \qquad \downarrow \bar{Q} \\
 \boxed{E+2} & & [j, E + 2] \xleftarrow{U} [j + \frac{1}{2}, E + 2]
 \end{array} \quad (\text{D21})$$

Notably, one hallmark of this superconformal structure is that each level is twofold degenerate. These superconformal towers, together with the conformal descendants, determine the integer-level structure in superconformal symmetry.

Special attention should be paid to the short supercurrent superconformal multiplets, which consist of all the conserved currents of the superconformal symmetry, including

- (1) Stress tensor $T^{\mu\nu}$ with quantum numbers $l = 2$, $\Delta = 3$, $r = 0$, which integrates into the conformal charges P_μ , D , K_μ , and $M_{\mu\nu}$;
- (2) Supersymmetry current $G^{\mu\alpha}$ with quantum numbers $l = \frac{3}{2}$, $\Delta = \frac{5}{2}$, $r = 1$ which integrates into the supercharges Q^α , $\bar{Q}^\alpha$, and its conjugate $\bar{G}^{\mu\alpha}$ with $l = \frac{3}{2}$, $\Delta = \frac{5}{2}$, $r = -1$;
- (3) The conserved $U(1)_R$ current with quantum numbers $l = 1$, $\Delta = 2$, $r = 0$.

These four conserved currents are connected by the super-charges Q and $\bar{Q}$, and they combine into two bi-multiplets on the supersphere. The J_R^μ and $\bar{G}^{\mu\alpha}$ combine into a bi-multiplet with $j = \frac{3}{2}$ and $E = 2$; the $G^{\mu\alpha}$ and $T^{\mu\nu}$ combine into a bi-multiplet with $j = 2$ and $E = 3$.

$$\begin{array}{ccc}
 (l, \Delta, r) & [j = \frac{3}{2}, E = 2] & (l, \Delta, r) \\
 J_R^\mu(1, 2, 0) & \xrightarrow{Q} & \bar{G}^{\mu\alpha}(\frac{3}{2}, \frac{5}{2}, -1) \\
 \downarrow \bar{Q} & & \downarrow \bar{Q} \\
 G^{\mu\alpha}(\frac{3}{2}, \frac{5}{2}, 1) & \xrightarrow{Q} & T^{\mu\nu}(2, 3, 0) \\
 & [j = 2, E = 3] &
 \end{array} \quad (D22)$$

APPENDIX E: THE HIGHER SPECTRUM OF THE SUPER-ISING SCFT

In the ED calculation, we calculate the 120 lowest states respectively in the $L^z = 0$ and $1/2$ sectors. These cover the spectrum up to $\Delta \approx 6$. We plot all these states in Fig. 15. All the states up to $l \leq 2$, $\Delta \leq 5.5$, and most states with $l \leq 3$ can be identified as superconformal states.

The identified multiplets from bootstrap result [16] below $\Delta = 6$ include [140]

- (1) The elementary fields σ with $l = 0$, $\Delta_\sigma = 0.5844435(83)$ and parity odd,
- (2) The supercurrent G with $l = 3/2$ and $\Delta_G = 2.5$,
- (3) σ' corresponding to the fermion mass $\bar{\chi}\chi$ with $l = 0$, $\Delta_{\sigma'} = 2.8869(25)$ and parity odd,
- (4) $T'^{\mu\nu}$ in the twisted trajectory of $[\sigma\sigma]$ identified through the extremal spectrum with $l = 2$, $\Delta_{T'} = 3.28(1)$ and parity even,
- (5) ϵ'' identified through the extremal spectrum with $l = 2$, $\Delta_{\epsilon''} = 4.38(1)$ and parity even,
- (6) $T_-^{\mu\nu}$ in the twisted trajectory of $[\sigma\epsilon]$ identified through the extremal spectrum with $l = 2$, $\Delta_{T_-} = 4.58(1)$ and parity odd,
- (7) $[\sigma\chi]_{l=7/2}$ in the twisted trajectory of $[\sigma\chi]$ with $l = 7/2$ and $\Delta_{[\sigma\chi]_{l=7/2}} \approx 4.570$, and
- (8) $[\sigma\sigma]_{l=4}$ in the twisted trajectory of $[\sigma\sigma]$ with $l = 4$, $\Delta_{[\sigma\sigma]_{l=4}} \approx 5.257$.
- (9) $G'^{\mu\alpha}$ at $l = 3/2$ and $\Delta \approx 5.28$ that is observed in the extremal spectrum (5.78 in the $Q^+ \mathcal{F}^{3/2}$ sector) by the conformal bootstrap, but is subject to large numerical uncertainty.



FIG. 15. The spectrum of super-Ising field theory up to $\Delta \approx 6$ and $l = 4$ at $N_{mf} = 9$. The symbols marks the measured scaling dimensions and the bars mark the expected values from conformal bootstrap. Each identified state is labeled by the measured and expected (in parentheses in subscript) scaling dimensions. In the calculation, we fix $t = 1.5$, $U = 0.25$, $g = 1$ and take h, μ_1, μ_b as in Table II.

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