

Fractional Quantum Hall States under Density Decoherence

Zijian Wang^{1,2,*}, Ruihua Fan^{1,2,*}, Tianle Wang^{1,2,3}, Samuel J. Garratt^{2,4}, and Ehud Altman^{2,5}

¹*Institute for Advanced Study, Tsinghua University, Beijing, China*

²*Department of Physics, University of California, Berkeley, California 94720, USA*

³*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*

⁴*Department of Physics, Princeton University, New Jersey 08544, USA*

⁵*Materials Sciences Division, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*



(Received 5 December 2025; revised 31 May 2026; accepted 4 August 2026; published 21 September 2026)

Fractional quantum Hall states are promising platforms for topological quantum computation due to their capacity to encode quantum information in topologically degenerate ground states and in the fusion space of non-Abelian anyons. We investigate how the information encoded in two paradigmatic states, the Laughlin and Moore-Read states, is affected by density decoherence—coupling of local charge density to nonthermal noise. We identify a critical filling factor ν_c , above which the quantum information remains fully recoverable for arbitrarily strong decoherence. The $\nu = 1/3$ Laughlin state and $\nu = 1/2$ Moore-Read state both lie within this range. Below ν_c both classes of states undergo a decoherence-induced Berezinskii-Kosterlitz-Thouless (BKT) transition into a critical decohered phase. For Laughlin states, information encoded in the topological ground state manifold degrades continuously with decoherence strength inside this critical phase, vanishing only in the limit of infinite decoherence strength. On the other hand, quantum information encoded in the fusion space of non-Abelian anyons of the Moore-Read states remains fully recoverable under arbitrarily strong decoherence even beyond the BKT transition. These results lend further support to the promise of non-Abelian FQH states as platforms for topological quantum computation. They reveal how the interplay between chiral topological order and global symmetry can give rise to novel mixed-state phases under decoherence, distinguished by their capacity for storing quantum information.

DOI: [10.1103/bw7c-y5dc](https://doi.org/10.1103/bw7c-y5dc)

Subject Areas: Condensed Matter Physics,
Quantum Information,
Strongly Correlated Materials

I. INTRODUCTION

One route towards realizing topological quantum computation is through solid-state systems that host anyonic excitations [1–6]. A paradigmatic example is the fractional quantum Hall (FQH) platform, where charged anyons can be manipulated with electrostatic controls to perform quantum computation [7–11]. There are two major sources of errors that arise in such devices, one is from thermally excited quasiparticles, and the other from decoherence caused by local noise in the device, cosmic rays, and other uncontrolled effects. The thermal errors can be suppressed by cooling the system and safely ignored at experimentally accessible low temperatures, e.g., in graphene quantum Hall systems [12,13]. However, the second type of error is

harder to eliminate and persists throughout the experiment. In this work, we examine the effect of such local decoherence on the encoded quantum information.

There are two ways to encode quantum information in a topologically ordered FQH state [3]. The first is to store the information in the topologically degenerate ground state subspace, which applies to both Abelian and non-Abelian FQH states. The second is to prepare a non-Abelian quantum Hall state and encode information into the fusion space of non-Abelian anyons. In practice, the first route may require using bilayer FQH systems or fractional quantum spin Hall insulators with engineered edge couplings [14–16]. For the purpose of analyzing robustness against decoherence, it is more convenient to consider single-layer FQH states on a torus as a theoretical simplification. We will investigate the robustness of quantum information stored in both approaches.

From a theoretical perspective, our investigation aligns with the broader program of understanding topological order in mixed quantum states [17–34]. Quantifying the capacity for encoding quantum information amounts to characterizing the long-range entanglement in the presence

*These authors contributed equally to this work.

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

of decoherence. We can then view it as a probe of topological order in the mixed-state setting. Most prior work in this direction has focused on fixed-point models with commuting projector Hamiltonians. As we will see below, decohered FQH states are different in several important ways, raising new technical and conceptual challenges. Moreover, decoherence-induced transitions of non-Abelian topological orders remain much less explored than the corresponding transitions in Abelian states [30,31]. In particular, little is known about the fate of non-Abelian anyons under decoherence. This is one of the central questions addressed in this work.

For topological stabilizer codes or the quantum double models, one can always specify a decoding algorithm and discuss its success probability [35–38]. While the non-stabilizer and chiral nature of FQH states complicates an explicit construction of this kind, we can infer the threshold of the optimal decoder by analyzing certain information-theoretic quantities [17]. In particular, we can use the quantum coherent information to quantify the recoverable quantum information in a decohered system [39,40]. A sharp drop in this quantity as the decoherence increases beyond a threshold signals the breakdown of the FQH state as a quantum memory [17].

In realistic platforms such as graphene heterostructures, the decoherence in the bulk conserves the total electron number. After the ground state is prepared, the subsequent dynamics have two components. First, decoherence continually creates small pairs of anyons. Second, these anyons move diffusively under the combined effect of Hamiltonian evolution and decoherence. In this work we isolate the effect of density decoherence, neglecting the diffusion of anyons. We will comment on the effect of more generic decoherence dynamics in Sec. VI.

Decoherence coupling purely to the electron density is described by the following Lindblad equation

$$\frac{d\hat{\rho}}{dt} = \gamma \int d^2z (\hat{n}(z)\hat{\rho}\hat{n}(z) - \frac{1}{2}\{\hat{n}(z)^2, \hat{\rho}\}), \quad (1)$$

where $\hat{\rho}$ is the density matrix, $\hat{n}(z)$ the electron density operator. By dimensional analysis, the decoherence strength γ has the unit of $[\text{time}]^{-1}[\text{length}]^2$, where the length scale can only come from the short-distance regularization. Therefore, it is important to keep a *finite* short-distance cutoff to ensure that the density decoherence has nonvanishing effects [41].

We examine the effect of decoherence in the Laughlin and Moore-Read states, the simplest Abelian and non-Abelian FQH states, respectively [42,43]. The Laughlin state offers the simplest arena for analyzing the robustness of information encoded in degenerate ground states and also helps with the development of tools before addressing the more intricate non-Abelian states. The Moore-Read state is the simplest non-Abelian FQH state that can be used

for quantum computation with anyons. We derive our results by using the well-known model wave functions of these states. They admit well-developed plasma analogies and conformal field theory (CFT) descriptions that allow us to solve the problem analytically [42–45]. One can also employ the Chern-Simons theory description of the quantum Hall states and derive the same results. Therefore, we expect our results to be applicable even beyond the ideal wave functions.

The rest of the paper is organized as follows. In Sec. II we provide a non-technical overview of the main results and motivate them with physical arguments, leaving the technical derivations to subsequent sections. In Sec. III we use the plasma analogy to identify and characterize a decoherence-induced transition in Laughlin states. Then, in Sec. IV we determine how the quantum coherent information, quantifying the recoverable quantum memory, evolves across the transition. To accomplish this, we utilize the field theory description of the Laughlin state in terms of a $1+1$ -dimensional CFT. In Sec. V we use the CFT formulation to identify the decoherence-induced transition in Moore-Read states and determine its effect on information encoded in the fusion space of non-Abelian anyons. We conclude with general remarks on the results and outlook in Sec. VI.

II. OVERVIEW

In this section, we give an overview of the main results, emphasizing the physical intuition. Our starting point is an FQH state that encodes quantum information either in the topological ground state degeneracy or the fusion space of non-Abelian anyons. We assume that this logical qudit is entangled in some way with another qudit R , as it would be if it were taking part in a quantum computation. The amount of information that can be recovered using operations on the system Q , subject to decoherence, can be quantified by the quantum coherent information $I_c(R|Q) = S_Q - S_{RQ}$, where S_Q and S_{RQ} are von Neumann entropies of these systems after the decoherence.

The density matrix of the system affected by the decoherence channel (1) has a simple form in the particle position basis:

$$\rho(z^1, z^2) = \Psi(z^1)\Psi^*(z^2)e^{-\frac{1}{2}V(z^1, z^2)}. \quad (2)$$

Here $z^\alpha \equiv \{z_1^\alpha, \dots, z_N^\alpha\}$ denote electron configurations, $\Psi(z^\alpha)$ is the first quantized wave function of the original pure state, and the decoherence generates V that depends on the electron density $n_i(z)$ as

$$V(z^1, z^2) = \gamma t \int d^2z [n_1(z) - n_2(z)]^2. \quad (3)$$

It is convenient to define a dimensionless measure of the decoherence strength $\mu = \gamma t / 2\pi a^2$, where a is a short-distance cutoff much smaller than the magnetic length.

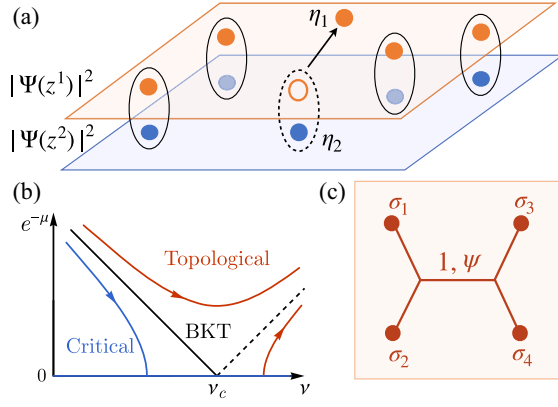


FIG. 1. (a) Schematics for the purity of the decohered state. Two layers are the two copies of states, orange and blue dots are the electrons, black circles represent their tight binding at the strong decoherence limit, dashed circles represent the unbinding events when deviating from the limit. (b) Renormalization-group flow for decohered Laughlin and Moore-Read states. Below a critical filling the system enters a critical phase through a BKT transition. Above it the system remains topological. (c) Encoding a single qubit in the two-dimensional fusion space of four well-separated fundamental quasiholes of the Moore-Read state.

In the strong decoherence limit $\mu \rightarrow +\infty$, the coupling V suppresses off-diagonal elements and thus the density matrix reduces to a classical probability distribution.

The simplest proxy for the behavior of the quantum coherent information is the Rényi-2 version of this quantity. Thus, as a first step, we analyze the purity as a generating function for all other Rényi-2 information quantities. Following Eq. (2) we have

$$\text{tr} \rho^2 = \int \mathcal{D}z^1 \mathcal{D}z^2 |\Psi(z^1)|^2 |\Psi(z^2)|^2 e^{-V(z^1, z^2)}, \quad (4)$$

which can be interpreted as a partition function of *classical* electron fluids residing in two layers as depicted in Fig. 1(a). The intralayer interactions among fluid particles derive from the structure of the initial state $\Psi(z)$ through the modulus squared. At the same time, the decoherence introduces an interlayer attraction (3), which grows with the decoherence strength. Thus, in the pure initial state, electrons in the two layers fluctuate independently, while in the opposite limit $\mu \rightarrow +\infty$ they form tightly bound pairs, depicted as the solid circles in Fig. 1(a). The nature of the transition between the two extremes can be understood by expanding about the strong decoherence limit in the small fugacity $e^{-\mu}$ of unbound interlayer pairs.

A. Decohered Laughlin states

The physics of unlocking the two electron fluids near the strong decoherence limit is particularly intuitive in the Laughlin states with filling factor $\nu = 1/m$. This is addressed in Sec. III. Through Laughlin's plasma analogy,

the wave function gives rise to Coulomb interactions between particles of effective charge $\sqrt{2m}$ within each layer. In the pure state $\mu = 0$, the two plasmas are independently in their screening phase (for $m < 70$), while in the opposite limit $\mu \rightarrow \infty$ they are locked into tightly bound pairs, which screen only the symmetric component of the charge. Because the interaction between unbound (relative) charges is not fully screened in the decohered state, their tendency to proliferate depends on the usual competition between the Coulomb energy cost and entropy gain, leading to the Berezinskii-Kosterlitz-Thouless (BKT) RG flow depicted in Fig. 1(b) [46–48]. Interestingly, the filling factor ν plays the role of the dielectric constant or inverse stiffness. So, increasing the decoherence strength moves the physical system (i.e., the initial condition for the flow) vertically downward on this diagram. This results in a BKT transition at a finite value of μ if $\nu < \nu_c$. Remarkably, if the filling factor $\nu \geq \nu_c$ the screening phase persists to arbitrarily large μ ($\nu_c = 1/4$ in the analysis of the purity). This is suggestive, and we will show that it indeed implies, that a fully recoverable quantum memory persists to arbitrarily strong decoherence.

An important aspect of our analysis is that the fully decohered state falls on a line of critical states, with properties determined by universal aspects of the FQH state. In fact, the expansion around the fully decohered state, which reveals the decoherence-induced transitions, relies on the same universal properties rather than on the detailed structure of the underlying state. Therefore, the main results of this analysis are expected to hold for FQH states in the Laughlin series beyond specific model wave functions.

The critical decohered state, discussed in Sec. III B, is characterized by algebraic decay of Rényi correlations in both the electron and quasihole (anyon) operators, which originate from universal properties of the initial pure state. It is interesting to connect this result with the understanding that decoherence-induced transitions arise from anyon condensation in $2 + 0$ dimensions [18]. In the decohered Laughlin state anyons cannot fully condense because they carry a (fractional) charge of the global charge $U(1)$ symmetry. Instead, they form a critical condensate exhibiting quasi-long-range order.

The critical nature of the fully decohered state also helps us generalize from a theory of the purity to one of Rényi-1 and von Neumann quantities, which have more transparent information-theoretic meanings. This is done in Sec. III C. The key, again, is to develop an expansion in small fugacity about the diagonal, fully decohered state. Divergence of the expansion then signals an instability of the critical diagonal density matrix toward the encoding phase. While we are not able to obtain the exact critical filling factor, we derive rigorous bounds placing it within the range $1/8 \leq \nu_c \leq 1/4$.

So far we have not addressed directly how these transitions affect the quantum information encoded in

the m -dimensional ground state subspace on a torus. To access these m states, which are not easily discerned within the plasma analogy, we reformulate the problem in Sec. IV in terms of an effective field theory, using the well-known correspondence between the FQH states and correlations in $1+1$ -dimensional CFTs. In particular, a basis of m Laughlin states on a torus is given by

$$\Psi_\ell(z) = \left\langle \prod_{i=1}^N e^{im\varphi(z_i)} e^{-\frac{i}{2\pi} \int d^2z \varphi} \right\rangle_\ell, \quad (5)$$

where the N -point correlation is evaluated in the CFT of a chiral boson φ with a twist of $2\pi\ell/m$ in the boundary conditions along $\hat{x}$ and $\ell = 0, \dots, m-1$.

In the expression for the purity (4), the chiral and antichiral modes originating from the two wave functions and their conjugates add up to two nonchiral bosonic modes $\phi_\pm$. These fields can be interpreted as the phases conjugate to the plasmas associated with the symmetric and anti-symmetric (relative) charge between the two layers. The resulting field theory naturally reproduces the decoherence-induced transitions found through the plasma analogy. In the pure state ($\mu = 0$), where the two plasmas are in the screening phase, both fields are locked to zero and the effective theory is gapped. In the opposite limit ($\mu \rightarrow \infty$), corresponding to the fully decohered state, only the symmetric charge density is screened in the plasma analogy, hence ϕ_+ is still gapped, but ϕ_- is critical. Away from the infinite decoherence limit, the expansion of the purity in small fugacity leads to the low-energy effective sine-Gordon theory

$$Z_- = \int \mathcal{D}\phi_- \exp \left(- \int \frac{m}{4\pi} (\nabla\phi_-)^2 - 2e^{-\mu} \cos m\phi_- \right), \quad (6)$$

which exhibits the same BKT transition discussed above [49]. The field theory formulation now allows us to compute the evolution of the coherent information across the transition separating the gapped screening phase, in which the cosine term is relevant, and the critical decohered state, in which it is irrelevant.

To have a non-trivial coherent information, the decoherence channel needs to operate on a Laughlin state on a torus Q , whose ground state manifold is entangled with a reference qudit R . In Sec. IV we show that the purities that enter the Rényi-2 coherent information can be written as sums over the sine-Gordon partition functions (6) with different twisted boundary conditions of the field ϕ_- or of its dual θ_-/m . The Rényi-2 coherent information is then given by

$$I_C^{(2)}(R)Q = \log \frac{1 + \sum_{\ell=1}^{m-1} e^{-\Delta F_{(0,\ell)}}}{1 + \sum_{\ell=1}^{m-1} e^{-\Delta F_{(\ell,0)}}}. \quad (7)$$

Here $\Delta F(\ell, 0)$ and $\Delta F(0, \ell)$ are the relative free energy costs of inserting a twist of $2\pi\ell/m$ in the boundary

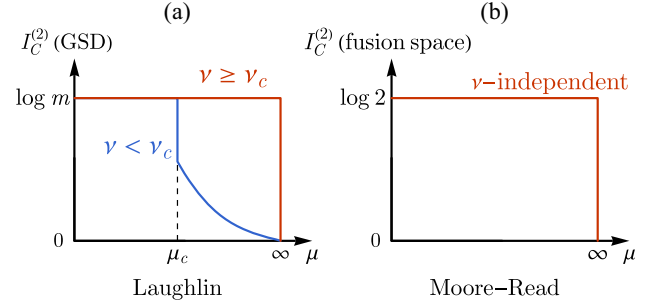


FIG. 2. Rényi-2 quantum coherent information as a function of the decoherence strength. (a) For Laughlin states, the information is encoded in the degenerate ground states. For systems below the critical filling it starts to decrease after a finite time and becomes zero only at the strong decoherence limit. (b) For Moore-Read states, the information is encoded in the fusion space and is infinitely robust against decoherence.

conditions of, respectively, ϕ_- and its dual θ_-/m along the $\hat{x}$ direction of the torus.

In the screening phase ($\mu < \mu_c$) the cosine term dominates and any nonvanishing twist of ϕ_- incurs a free energy cost of order L , while twists in θ_- are free. Thus we have $I_C^{(2)} = \log m$ as expected. For $\mu > \mu_c$ the cosine term is irrelevant and the system flows to the critical phase. This, however, does not imply the vanishing of the coherent information. The contribution of the irrelevant cosine term to the free energy cost of a ϕ_- twist can be calculated perturbatively and is easily seen to be of order one. Thus, the coherent information jumps upon crossing the transition at μ_c from $\log m$ to a smaller nonuniversal positive value. As μ exceeds μ_c , the coherent information decreases monotonically, as shown by the blue curve in Fig. 2(a), approaching zero in the limit $\mu \rightarrow \infty$. This result, a signature of the critical nature of the mixed state above the error threshold, stands in sharp contrast to the vanishing of the coherent information across the error threshold in the toric code [17].

If the filling factor is above the critical filling ν_c , the cosine term is relevant even at the infinite decoherence limit. Consequently, the coherent information stays at $\log m$ for any finite decoherence strength in the thermodynamic limit, as is shown by the red curve in Fig. 2(a). In a finite system of linear size L , however, the cosine term becomes effectively irrelevant when $\mu \sim \mathcal{O}(\log L)$. As a result, the lifetime of the memory diverges logarithmically in the system size.

B. Decohered Moore-Read states

Moore-Read states are the simplest FQH states that host non-Abelian anyons. Though these anyons do not support a fault-tolerant universal gate set, one can store protected quantum information in their fusion space, as is illustrated in Fig. 1(c). Our aim is to understand the decoherence-induced transitions in Moore-Read states and how they

affect the information encoded in the fusion channels of non-Abelian anyons. To this end, we use both an effective field theory description and a plasma analogy [45] of Moore-Read states.

Like the Laughlin states, Moore-Read states can also be represented as a correlation function in a $1 + 1$ -dimensional CFT, which in this case includes two sectors: a chiral Ising (fermion) CFT and a chiral boson CFT that we refer to as the bosonic charge sector of the theory. For the description of the purity (4), the chiral and antichiral modes originating from the wave functions and their conjugates combine into a pair of *nonchiral* Ising CFTs and a pair of *nonchiral* boson CFTs. The purity is then formulated as an insertion of strong perturbations in this CFT.

Our strategy again is to expand about the limit $\mu \rightarrow \infty$ of the fully decohered Moore-Read state. In this limit, the nonchiral boson field ϕ_- from the bosonic charge sector is in a free critical state, while the symmetric boson field ϕ_+ is gapped, exactly as in the fully decohered Laughlin state. The Ising sector, on the other hand, forms a more complex strongly coupled theory. Nonetheless, we can obtain the long-range correlations in this sector, needed for the small fugacity expansion, by utilizing the plasma analogy of the Moore-Read states [45].

Having established the correlations in the fully decohered state we use them to formulate the effective theory through the expansion about this reference point. In the bosonic charge sector, this gives rise to an effective sine-Gordon theory similar to (64), which leads to a decoherence-induced BKT transition for filling factors $\nu < \nu_c$. In the Ising sector, however, the expansion about the fully decohered state inevitably generates relevant perturbations. As a result, for any finite decoherence the Ising sector flows back to a topological phase. This suggests the establishment of a new intrinsically mixed phase of matter, which is critical and simultaneously has quantum information preserved only in the non-Abelian sector. No known ground state of a local Hamiltonian exhibits such a phenomenon.

In Sec. VC we verify the robustness of the quantum memory encoded in the fusion space of (four) non-Abelian anyons by directly calculating the quantum coherent information. As a first step, we represent the basis states in this space as conformal blocks in the chiral Ising $\times$ U(1) CFT. This allows us to entangle the initial state with a reference qubit and represent the quantum coherent information of the decohered system in terms of expectation values in the effective field theory. The result, illustrated in Fig. 2(b), shows that the coherent information remains $\log 2$ for arbitrary (yet finite) decoherence strength, indicating fully recoverable quantum memory independent of the filling factor.

At the same time, quantum information encoded in the bosonic charge sector of the degenerate ground state on a torus undergoes a transition similar to that found in the

Laughlin state if $\nu < \nu_c$. Using the expansion in small fugacity about the von Neumann entropy of the fully decohered state we find that this critical filling for the Moore-Read states is bounded by $1/8 \leq \nu_c \leq 1/3$.

III. DECOHERED LAUGHLIN STATES

In this section, we apply the plasma analogy to study decoherence-induced phase transitions in Laughlin states and show how they depend on the filling factor. We start with analyzing the purity of the density matrix, the Rényi-2 quantity, and then the von Neumann quantities.

A. Plasma analogy and filling-dependent transition

The Laughlin state is an incompressible state in the lowest Landau level at filling $\nu = 1/m$. Its universal properties are captured by the following un-normalized model wave function

$$\Psi(\{z_i\}) = \prod_{i < j}^N (z_i - z_j)^m e^{-\sum_i |z_i|^2 / 4\ell_B^2}. \quad (8)$$

Here, m is an odd integer, N is the number of electrons, $z_i = x_i + iy_i$ is the electron coordinate and ℓ_B is the magnetic length. Notably, all analyses in this work also apply to bosonic Laughlin states, where m is an even integer. To simplify the notation, we often omit the curly bracket $\{\dots\}$ and set $\ell_B = 1$ from now on. The squared norm of Ψ takes the form of a partition function of a single-component two-dimensional plasma

$$\langle \Psi | \Psi \rangle = \int \mathcal{D}z e^{-\Phi_L(z_i, \sqrt{2m})},$$

$$\Phi_L(z_i, Q) = -Q^2 \sum_{i < j} \log |z_i - z_j| + \frac{Q^2}{4m} \sum_i |z_i|^2, \quad (9)$$

where $\mathcal{D}z \equiv \prod_{i=1}^N d^2 z_i / N!$ is the integration measure. In this analogy, each particle carries a charge $Q = \sqrt{2m}$ and moves on a uniform neutralizing background of charge density $-Q/2m\pi$. For physically relevant values of m (e.g., $m < 70$), the plasma is in a screening phase [50], and all long-distance properties of the Laughlin state follow.

Following Eq. (4), the purity admits a similar mapping to the partition function of a classical two-component plasma, from which many properties of the decohered systems follow. Specifically, after time t under the decoherence channel (1), the purity of the Laughlin state is given by

$$\text{tr} \hat{\rho}^2 = \int \mathcal{D}(z^1, z^2) e^{-\Phi_L(z_i^1, \sqrt{2m}) - \Phi_L(z_i^2, \sqrt{2m}) - V}, \quad (10)$$

where $\Phi_L(z_i^1, \sqrt{2m})$ and $\Phi_L(z_i^2, \sqrt{2m})$ are the plasma energy functionals defined in Eq. (9). It is therefore natural to interpret the purity as the partition function of two layers

of plasmas, labeled by 1 and 2, which originate from the bra and ket spaces of the density matrix. The decoherence suppresses off-diagonal elements of the density matrix in the position basis. In the language of plasma, it induces an attractive interaction between the two plasmas, through the potential

$$V = \gamma t \int d^2 z [n_1(z) - n_2(z)]^2. \quad (11)$$

As noted in the overview, V depends on the short-distance regularization. One convenient choice is to smear the density operator in Eq. (1) by a Gaussian

$$\hat{n}(z) \mapsto \frac{1}{\pi a^2} \int d^2 z' e^{-|z-z'|^2/a^2} \hat{n}(z'), \quad (12)$$

which yields

$$V = \mu \sum_{i,j=1}^N e^{-\frac{|z_i^1 - z_j^1|^2}{2a^2}} + e^{-\frac{|z_i^2 - z_j^2|^2}{2a^2}} - 2e^{-\frac{|z_i^1 - z_j^2|^2}{2a^2}}, \quad (13)$$

where $\mu = \gamma t / 2\pi a^2$ is the dimensionless quantity that parametrizes the decoherence strength. The first two terms in Eq. (13) describe a short-range repulsion within each plasma, which is overwhelmed by the Coulomb repulsion and does not qualitatively affect the physics. Below we examine how the last term, representing interlayer attraction, affects the bilayer plasma.

An instructive reference point for understanding the transition is the strong-decoherence limit ($\mu \rightarrow +\infty$). In the plasma analogy, this limit corresponds to having infinitely strong interlayer attractions so that all charges in the two layers form tightly bound pairs. Denoting the density matrix in this limit by ρ_∞ , we have

$$Z_\infty = \text{tr} \rho_\infty^2 = \int \mathcal{D}z e^{-2\Phi_L(z_i, \sqrt{2m})}, \quad \mu \rightarrow +\infty, \quad (14)$$

where $z_i = z_i^1 = z_i^2$ labels the position of each pair [51]. We can reinterpret Z_∞ as the partition function of a single-component plasma formed by the pairs, i.e., $2\Phi_L(z_i, \sqrt{2m}) = \Phi_L(z_i, 2\sqrt{m})$. Each particle carries a charge $Q = 2\sqrt{m}$ and moves on a neutralizing background. For $m < 35$, any test charge that is only coupled to this paired plasma is fully screened, while test charges coupled to one of the two plasmas are partially screened. As a result, the plasma is in a different phase than at $\mu = 0$.

As a first step to characterizing the behavior at finite μ , where unbound pairs may exist, we examine the screening of external test charges in the Z_∞ plasma. Specifically, consider inserting two pairs of test charges: q_1 and q_2 at position η in the two respective layers, and $-q_1$ and $-q_2$ at η' . The coupling to the plasma mediates an effective interaction between the test charges:

$$e^{-U_{\text{eff}}(\eta, \eta')} = \frac{1}{Z_\infty} \int \mathcal{D}z \exp\{-\Phi_L(z_i, \sqrt{2m}; \eta, q_1; \eta', -q_1) - \Phi_L(z_i, \sqrt{2m}; \eta, q_2; \eta', -q_2)\}. \quad (15)$$

Here $\Phi_L(z_i, Q; \eta, q; \eta', q')$ denotes the energy of a charge- Q single-component plasma with two test charges q, q' inserted at η, η' :

$$\begin{aligned} \Phi_L(z_i, Q; \eta, q; \eta', q') &= \Phi_L(z_i, Q) + \frac{Qq}{4m} |\eta|^2 + \frac{Qq'}{4m} |\eta'|^2 \\ &\quad - \sum_i Qq \log |z_i - \eta| + Qq' \log |z_i - \eta'| \\ &\quad - qq' \log |\eta - \eta'|, \end{aligned} \quad (16)$$

where the three lines of additional terms are the interaction energies with the background, mobile particles, and the bare interaction between test charges themselves. We can formally separate the test charges into a pair of “center-of-mass” charges $q_{\text{CM}} = \pm(q_1 + q_2)/\sqrt{2}$ that are coupled only to the charge- $2\sqrt{m}$ paired plasma, and a pair of “relative” charges $q_{\text{rel}} = \pm(q_1 - q_2)/\sqrt{2}$ that interact only with each other. Namely, we rewrite the exponent in Eq. (15) as

$$\begin{aligned} &-\Phi_L(z_i, 2\sqrt{m}; \eta, |q_{\text{CM}}|; \eta', -|q_{\text{CM}}|) \\ &-|q_{\text{rel}}|^2 \log |\eta - \eta'|, \end{aligned} \quad (17)$$

and accordingly

$$U_{\text{eff}}(\eta, \eta') = \Delta F(\eta, \eta') + |q_{\text{rel}}|^2 \log |\eta - \eta'|. \quad (18)$$

The first term is the free energy difference of the charge- $2\sqrt{m}$ plasma with and without the pair of the center-of-mass charges. Since the charge- $2\sqrt{m}$ plasma is in the screening phase for $m < 35$, we expect $\Delta F(\eta, \eta')$ to approach some finite constant when $|\eta - \eta'|$ is much larger than the screening length, and thus

$$U_{\text{eff}}(\eta, \eta') = |q_{\text{rel}}|^2 \log |\eta - \eta'| + \dots, \quad (19)$$

where the ellipsis represents constant corrections. We conclude that the relative charges interact via an unscreened Coulomb interaction.

The last result suggests that at finite decoherence strength μ the bilayer plasma may undergo a BKT transition through unbinding of the relative charges. Following the standard BKT analysis, we estimate the free energy cost f_0 of each individual unpaired particle. To this end, we consider a single unbinding pair, with the two particles separated at a distance of the order of the linear size L . The unbinding introduces a pair of center-of-mass charges that are fully screened by the charge- $2\sqrt{m}$ plasma, and a pair of

relative charges $q_{\text{rel}} = \pm\sqrt{m}$ whose interaction energy is given by Eq. (19). The total energy cost of the unbinding event is then $2\mu + m \log(L/a) + \dots$, where the ellipsis represents the constant terms due to screening. The entropy gain is $\log(NA/a^2)$, where A is the total area, $N \propto A$ the total particle number. Thus for $m < 4$ the free energy cost of unbinding is negative, indicating that the strongly decohered state is unstable towards the unbinding of inter-layer pairs for arbitrarily small deviation from infinite decoherence (i.e., for any non-vanishing fugacity $\propto e^{-\mu}$). We will show below that this also holds for $m = 4$.

For a more systematic treatment of the BKT transition we expand the partition function, representing the purity near the strong decoherence limit, as a sum over the number of unpaired particles. In the dilute limit, the unpaired particles effectively introduce relative charges $\pm\sqrt{m}$ with the interaction (19). We can approximate the partition function by

$$Z = Z_{\infty} Z_{\text{rel}} \\ Z_{\text{rel}} \approx \sum_k \frac{e^{-2\bar{\mu}k}}{(k!)^2} \int \prod_i^{2k} d^2 \xi_i e^{\sum_{i<j} q_i q_j \log |\xi_i - \xi_j|}. \quad (20)$$

Namely, the original bilayer plasma re-organizes into a charge- $2\sqrt{m}$ paired plasma Z_{∞} in its screening phase (for $m < 35$) and a plasma of relative charges Z_{rel} in the grand canonical ensemble, with $q_i = \pm\sqrt{m}$, $\sum_i q_i = 0$, and fugacity $e^{-\bar{\mu}} \propto e^{-\mu}$ [52]. The phase diagram is determined by the standard RG flow of the BKT transition as shown in Fig. 1(b) with $\nu = 1/m$ taking the role of the dimensionless temperature (or inverse stiffness). Increasing the decoherence strength tunes the starting point of the RG flow downward on a vertical line in this phase diagram. Crucially, the RG flow implies that for $m \leq 4$, the relative plasma is always in the screening phase for arbitrarily small fugacity. For $m > 4$ on the other hand, there is a BKT transition at a finite noise strength μ_c , leading to an unscreened phase at $\mu > \mu_c$.

B. Critical phase above the threshold

In this section we examine the critical phase. As explained in [17], only quantities that are nonlinear functions of the density matrix can show the establishment of a new phase beyond the decoherence threshold. Here we characterize the critical phase using the Rényi-2 correlation functions [17–19, 53–55]. In particular the Rényi-2 correlation for electrons is given by

$$C_e^{(2)}(\eta, \eta') = \frac{\text{tr}(c(\eta') \rho c^\dagger(\eta') c(\eta) \rho c^\dagger(\eta))}{\text{tr} \rho^2}. \quad (21)$$

Intuitively, $C_e^{(2)}$ quantifies the distinguishability between the state ρ with a hole at η and another state with a hole at η' .

Below the threshold, we expect $C_e^{(2)}$ to decay exponentially with $|\eta - \eta'|$. Above the threshold, however, we will show that $C_e^{(2)}$ decays algebraically.

We start with the infinite decoherence limit ($\mu = +\infty$), where the numerator of Eq. (21) is given by

$$\text{tr}(c(\eta') \rho c^\dagger(\eta') c(\eta) \rho c^\dagger(\eta)) \\ = \frac{1}{(N-1)!} \int \prod_{i=1}^{N-1} d^2 z_i e^{-\Phi_L(z_i, 2\sqrt{m}; \eta, \sqrt{m}; \eta', \sqrt{m}) - m \log |\eta - \eta'|}.$$

Similar to the analysis of Eq. (15), the Rényi-2 correlation can be rewritten as

$$C_e^{(2)}(\eta, \eta') = \frac{N}{A} e^{-\Delta F'(|\eta - \eta'|) - m \log |\eta - \eta'|}, \quad (22)$$

where N/A is the average particle density, and $\Delta F'$ is the free-energy difference between inserting a pair of test charges with $q = +\sqrt{m}$ at η, η' in a Laughlin plasma with $Q = +2\sqrt{m}$, and that of inserting a single test charge $+2\sqrt{m}$. Because the plasma is in the screening phase (for $m < 35$), $\Delta F'$ is independent of the separation $|\eta - \eta'|$ at large separations, and we have

$$C_e^{(2)}(\eta - \eta') \sim \frac{1}{|\eta - \eta'|^m}, \quad (\mu = \infty). \quad (23)$$

The result for the case $m = 1$ was also observed previously in the context of a fully decohered Chern insulator [56]. For $\mu_c \leq \mu < \infty$ ($m > 4$), the Luttinger parameter is renormalized by the cosine term, so

$$C_e^{(2)}(\eta - \eta') \sim \frac{1}{|\eta - \eta'|^\alpha}, \quad 4 \leq \alpha < m, \quad (24)$$

where $\alpha = 4$ at the critical transition $\mu = \mu_c$.

The power-law correlations in the decohered state are inherited from the power-law correlation of composite bosons in Laughlin states [57, 58]. For the Laughlin state with filling factor $\nu = 1/m$, the composite bosons b are defined by attaching m flux quanta to each electron via a singular gauge transformation:

$$b(z) = c(z) e^{-im \int d^2 z' n(z') \arg(z' - z)}. \quad (25)$$

The pure Laughlin state has off-diagonal quasi-long-range order for such composite bosons,

$$\langle \Psi | b^\dagger(\eta') b(\eta) | \Psi \rangle \sim \frac{1}{|\eta - \eta'|^{m/2}}.$$

For the decohered Laughlin state, the Rényi-2 correlation of composite bosons has a simple plasma analogy for arbitrary dephasing strength:

$$\frac{\text{tr}(b(\eta')\rho b^\dagger(\eta')b(\eta)\rho b^\dagger(\eta))}{\text{tr}\rho^2} = \frac{1}{|\eta - \eta'|^m} \left(\frac{1}{(N-1)!} \right)^2 \frac{\int \prod_{i=1}^{N-1} d^2 z_i^1 d^2 z_i^2 e^{-V - \sum_{\kappa=1,2} \Phi_L(z_i^\kappa, \sqrt{2m}; \eta, \sqrt{m/2}; \eta', \sqrt{m/2})}}{\int \mathcal{D}(z^1, z^2) e^{-V - \sum_{\kappa=1,2} \Phi_L(z_i^\kappa, \sqrt{2m})}}$$

where the test charges at η, η' carry only the “center-of-mass” charges and will be screened at long distance. Therefore, the correlator decays algebraically for any dephasing strength

$$\frac{\text{tr}(b(\eta')\rho b^\dagger(\eta')b(\eta)\rho b^\dagger(\eta))}{\text{tr}\rho^2} \sim \frac{1}{|\eta - \eta'|^m}. \quad (26)$$

The composite boson correlation functions are related to the electron correlations via the nonlocal phase factors implementing the flux attachment. However, at infinite dephasing the phase factors from the two sides of the density matrix exactly cancel because the fully dephased density matrix is diagonal in the local occupation. Therefore, in this limit the Rényi-2 electron correlations are the same as the Rényi-2 composite boson correlations, which leads to Eq. (23). Reference [59] noted a similar effect, where the power-law composite boson correlations can be translated to local observables through measurement and feedback.

We can find the correlation functions of quasiholes in the critical state by replacing the electron operators in Eq. (21) with quasihole operators. Let ρ_η be the decohered state with a single quasihole at η

$$\rho_\eta = e^{\mathcal{L}t} [|\Psi_\eta\rangle\langle\Psi_\eta|],$$

$$\Psi_\eta(\{z_i\}) = \prod_i (z_i - \eta) \prod_{i < j} (z_i - z_j)^m e^{-\sum_i |z_i|^2/4}. \quad (27)$$

Substituting this state into the correlation function we obtain

$$C_{qh}^{(2)}(\eta, \eta') = \frac{\text{tr}\rho_\eta \rho_\eta}{\text{tr}\rho^2} \sim \frac{1}{|\eta - \eta'|^{\alpha/m}}, \quad (28)$$

where α is the exponent appearing in Eq. (24). In contrast to previously studied examples of decoherence-induced transitions in topological states [18], here the anyons do not fully condense even at infinite decoherence. The reason is that they are charged under the global U(1) symmetry, and the density dephasing preserves the original probability distribution in position space.

C. Critical filling for the von Neumann quantities

It is straightforward to generalize the analysis of the previous section to Rényi- n quantities with $n = 2, 3, 4, \dots$. In this case the decoherence-induced transition is described by the statistical mechanics of n layers of coupled plasmas.

All the relative plasma modes exhibit a BKT transition below a critical filling ν_c that depends on the Rényi index:

$$\nu_c(n) = \frac{n-1}{2n}, \quad n = 2, 3, \dots \quad (29)$$

Quantities that have direct information-theoretic meanings, and bear directly on the recoverability of quantum information, require taking the replica limit $n \rightarrow 1$ [39,54,60,61]. However, we show in Appendix B that $\nu_c(n)$ is not analytic at $n = 2$ because taking the limit $n \rightarrow 1$ does not commute with taking the thermodynamic limit. Consequently, the expression (29), derived for integer $n \geq 2$, does not extend to $n < 2$. In Appendix B, we derive the correct expression for non-integer n and prescribe the proper way for taking the $n \rightarrow 1$ limit.

Here, we present a simpler approach, working directly with the Rényi-1 or von Neumann quantities. As in the Rényi-2 case, we take the fully decohered state ρ_∞ as a starting point and show that it is critical also at $n = 1$. Then we assess the stability of the critical state to a small deviation from the infinite decoherence limit. This allows us to bound the critical filling factor within the range $1/8 \leq \nu_c(1) \leq 1/4$.

To establish the critical nature of the fully decohered state ρ_∞ in the $n \rightarrow 1$ limit, it is convenient to calculate the Rényi-1 correlation function for electrons:

$$C_e^{(1)}(\eta, \eta') = \text{tr}(c(\eta')\sqrt{\rho}c^\dagger(\eta')c(\eta)\sqrt{\rho}c^\dagger(\eta)). \quad (30)$$

In the electron occupation basis $|\{z\}\rangle = \prod_{i=1}^N c^\dagger(z_i)|0\rangle$, ρ_∞ is fully diagonal with the matrix element:

$$\langle\{z_i\}|\rho_\infty|\{z'_i\}\rangle = \frac{1}{Z} |\Psi(\{z_i\})|^2 \delta_{\{z_i\}, \{z'_i\}},$$

where Z is a normalization factor. It is then straightforward to calculate $\sqrt{\rho_\infty}$ and the Rényi-1 correlation function, for which we have

$$C_e^{(1)}(\eta, \eta') = \frac{Z^{-1}}{(N-1)!} \int \prod_i d^2 z_i |\Psi(\{z_i, \eta\})\Psi(\{z_i, \eta'\})|, \quad (31)$$

where we introduce the notation

$$\{z_i, \eta\} := (z_1, z_2, \dots, z_{N-1}, z_N = \eta)$$

to denote a set of coordinates with the last one being specified. Using the same steps as the analysis for Eq. (21), we obtain

$$C_e^{(1)}(\eta, \eta') \sim \frac{1}{|\eta - \eta'|^{m/2}}, \quad (\mu = +\infty). \quad (32)$$

We will frequently quote this result in our subsequent analysis of the instability.

As in the Rényi-2 case, we assess the stability of the critical phase by expanding the Rényi-1 or von Neumann quantities in the small deviation from the fully decohered state. The convergence of this expansion determines the stability of the critical phase. The same kind of analysis is valid more generally and we use it also to understand the Moore-Read state under decoherence in Sec. V E. Hence we will keep the following discussion general and specialize to the Laughlin states toward the end of it.

In the fully decohered limit, the density matrix in Eqs. (2) and (3) becomes diagonal in the coarse-grained position basis [62]. Namely, all electrons in the ket and bra spaces are matched into pairs that are bound within the cutoff scale. Going slightly away from this limit, gives finite weight to off-diagonal elements, corresponding to unmatched ket-bra electron pairs. It is natural to organize such off-diagonal contributions by the number of unmatched pairs, with each unmatched pair suppressing the matrix element by a factor of $\sim e^{-\bar{\mu}}$. Here the renormalized value $\mu \rightarrow \bar{\mu}$ absorbs the short-distance effects below the cutoff scale a . Then, to lowest nontrivial order in $e^{-\bar{\mu}}$ we have

$$\begin{aligned} \rho &= \rho_\infty + e^{-\bar{\mu}} \delta\rho + \mathcal{O}(e^{-2\bar{\mu}}), \quad \rho_\infty = \frac{1}{Z} \sum_b |\psi_b|^2 |b\rangle\langle b|, \\ \delta\rho &= \frac{1}{Z} \sum_{(b,c)} \psi_b \psi_c^* |b\rangle\langle c|, \end{aligned} \quad (33)$$

where the ψ_b are the many-electron position basis states corresponding to electron configurations labeled by b . The brackets (b, c) indicate pairs of electron configurations that differ in one particle:

$$\{(b, c) | b = \{z_i, \eta\}, c = \{z_i, \eta'\}, |\eta - \eta'| > a\}.$$

Hence, $\delta\rho$ collects the leading contribution in the off-diagonal coherences.

We now consider the expansion of the von Neumann entropy $S(\rho) = -\text{tr} \rho \log \rho$ to the first nontrivial order in $\delta\rho$. Analyzing other von Neumann quantities yields the same conclusion. To properly expand $S(\rho)$ with respect to $e^{-\bar{\mu}}$, we use the following identity for a positive matrix

$$\log \rho = \int_0^{+\infty} \left(\frac{1}{s+1} - \frac{1}{s+\rho} \right) ds, \quad (34)$$

which leads to

$$\begin{aligned} S(\rho) &\approx S(\rho_\infty) + e^{-2\bar{\mu}} \delta S, \\ \delta S &= - \int_0^{+\infty} \text{tr} [\delta\rho (s + \rho_\infty)^{-1} \delta\rho (s + \rho_\infty)^{-1}] ds. \end{aligned} \quad (35)$$

The fact that ρ_∞ is diagonal in the chosen basis simplifies the calculation, leading to

$$\delta S = - \frac{2}{Z} \sum_{(b,c)} \frac{|\psi_b| |\psi_c|}{|\psi_c|/|\psi_b| - |\psi_b|/|\psi_c|} \log \frac{|\psi_c|}{|\psi_b|}. \quad (36)$$

To derive the upper and lower bound, let us introduce the notation $x = \log(|\psi_c|/|\psi_b|)$. Using the inequality $x/\sinh x \leq 1$ we can find an upper bound on $|\delta S|$

$$(\text{upper bound}): |\delta S| \leq \frac{1}{Z} \sum_{(b,c)} |\psi_b| |\psi_c|. \quad (37)$$

To derive a lower bound, we first use the inequality $x/\sinh x \geq 1/\cosh x$, which gives

$$|\delta S| \geq \frac{2}{Z} \sum_{(b,c)} \frac{|\psi_b| |\psi_c|}{|\psi_c|/|\psi_b| + |\psi_b|/|\psi_c|}.$$

We then apply Jensen's inequality $\overline{x^{-1}} \geq \bar{x}^{-1}$ by interpreting $|\psi_b| |\psi_c|$ as being proportional to the weight of the distribution and $|\psi_b|/|\psi_c| + |\psi_c|/|\psi_b|$ in the denominator as the observable. This leads to the lower bound

$$|\delta S| \geq \frac{2}{Z} \frac{(\sum_{(b,c)} |\psi_b| |\psi_c|)^2}{\sum_{(b,c)} (|\psi_b|^2 + |\psi_c|^2)}.$$

Finally, we note that $\sum_{(b,c)} |\psi_b|^2 + |\psi_c|^2 = 2c_0 N (L/\ell_B)^2 Z$ where c_0 is a nonuniversal constant [63]. We explicitly include ℓ_B in expressions here and below to clarify the units. We have

$$(\text{lower bound}): |\delta S| \geq \frac{\ell_B^2}{c_0 N L^2 Z^2} \left(\sum_{(b,c)} |\psi_b| |\psi_c| \right)^2. \quad (38)$$

Using the formulas (37) and (38), we have

$$\begin{aligned} \frac{\ell_B^2}{c_0 N L^2} \left(\int d^2\eta d^2\eta' C_e^{(1)}(\eta, \eta') \right)^2 \\ \leq |\delta S| \leq \int d^2\eta d^2\eta' C_e^{(1)}(\eta, \eta'), \end{aligned} \quad (39)$$

where $C_e^{(1)}$ is exactly the Rényi-1 correlation function we have encountered in Eq. (31).

The physical instability comes from unbound pairs separated over a macroscopic distance of order the system

size $|\eta - \eta'| \sim O(L)$. To see whether such an instability occurs, we isolate contributions to δS that come from macroscopically separated unpaired particles by defining

$$\delta S_L = \delta S[\delta \rho], \quad (40)$$

where $\delta \rho = \frac{1}{2} \sum_{[b,c]} \psi_b \psi_c^* |b\rangle\langle c|$ includes pairs of configurations described below

$$\{b = \{z_i, \eta\}, c = \{z_i, \eta'\} | |\eta - \eta'| > \alpha L\},$$

with $0 < \alpha < 1$. We determine the critical filling ν_c by examining whether $\delta S_L^{(n)}$ diverges in the thermodynamic limit $L \rightarrow +\infty$.

Repeating the derivation of Eq. (39), we can get

$$\begin{aligned} \frac{\ell_B^2}{c_0 N L^2} \left(\int_{|\eta - \eta'| > \alpha L} d^2 \eta d^2 \eta' C_e^{(1)}(\eta, \eta') \right)^2 \\ \leq |\delta S_L| \leq \int_{|\eta - \eta'| > \alpha L} d^2 \eta d^2 \eta' C_e^{(1)}(\eta, \eta'), \end{aligned} \quad (41)$$

Both the upper and lower bounds can be estimated using the result Eq. (32) for the Rényi-1 correlations in the Laughlin state. On one side of the inequality, we have

$$|\delta S_L| \leq C L^{4-m/2}, \quad (42)$$

where C is some constant. Thus, for $m > 8$ the correction is convergent in the IR and there is no unbinding instability. In other words the critical filling $\nu_c \geq 1/8$. The other side of the inequality gives

$$|\delta S_L| \geq C' L^{4-m}, \quad (43)$$

where C' is some other constant. This inequality implies that there must be an instability for $m \leq 4$, therefore the critical filling $\nu_c \leq 1/4$.

IV. DIAGNOSING QUANTUM MEMORY

In this section, we determine how the decoherence-induced phase transition in the Laughlin state, identified using the plasma analogy, impacts the quantum memory encoded in the degenerate ground states on a torus. These degenerate ground states differ only in nonlocal properties and are therefore complicated to represent using the plasma analogy. Instead we turn to an effective field theory description, which also sets the stage for analyzing the Moore-Read state in the next section.

A. Quantum coherent information

The quantum coherent information quantifies the quantum information that is recoverable by an optimal decoder after the action of decoherence. Below we briefly review

the definition of this quantity and discuss its general form in the context of Laughlin states.

Let $|\Psi_\ell\rangle, \ell = 0, \dots, m-1$, denote the m -fold degenerate normalized Laughlin states on the torus. We initialize the system Q into a maximally entangled state with a reference qudit R

$$|\Psi_{RQ}\rangle = \frac{1}{\sqrt{m}} \sum_{\ell=0}^{m-1} |\Psi_\ell\rangle \otimes |\ell\rangle_R.$$

Let ρ_{RQ} denote the state after decoherence, the quantum coherent information is defined as

$$I_c := S(\rho_Q) - S(\rho_{RQ}), \quad (44)$$

where $S(\rho)$ is the von Neumann entropy. The quantum coherent information is maximal $I_c = \log m$ for the initial state $|\Psi_{RQ}\rangle$, and it cannot increase under arbitrary quantum channels. Its value is bounded by $-\log m \leq I_c \leq \log m$. We can perfectly recover the encoded information if and only if $I_c = \log m$. Here, for simplicity, we only consider the Rényi-2 version of the quantum coherent information

$$I_c^{(2)} = \log \frac{\text{tr} \rho_{RQ}^2}{\text{tr} \rho_Q^2}. \quad (45)$$

Besides its computational simplicity, this Rényi-2 version also has an information-theoretic meaning: $I_c^{(2)} > 0$ implies that Q and R are nonseparable, a necessary condition for the decodability [64]. From now on, we focus on the Rényi-2 coherent information and determine how it evolves across the decoherence-induced transition identified in the previous section.

It is convenient to use the doubled space description, where the density matrix is vectorized,

$$\rho = \sum_{a,b} \rho_{ab} |a\rangle\langle b| \rightarrow |\rho\rangle\rangle = \sum_{ab} \rho_{ab} |a\rangle \otimes |b^*\rangle. \quad (46)$$

Here $|b^*\rangle$ denotes the complex conjugate wave function in the density basis. We can view the pure state $|\rho\rangle\rangle$ as a bilayer system, where the two layers correspond to the ket and bra space. According to our definition of the complex conjugation, the density matrix of the undecohered Laughlin state is mapped to a bilayer of Laughlin states with opposite chiralities. In the doubled space, the decoherence channel is mapped to an operator

$$\mathcal{N} = e^{\mathcal{L}t} = e^{-\frac{\eta}{2} \int d^2 z (\hat{n}_1(z) - \hat{n}_2(z))^2}, \quad (47)$$

and we have

$$I_c^{(2)} = \log \frac{\sum_{\ell, \ell'=0}^{m-1} \langle \Psi_\ell \Psi_{\ell'}^* | \mathcal{N}^\dagger \mathcal{N} | \Psi_\ell \Psi_{\ell'}^* \rangle}{\sum_{\ell, \ell'=0}^{m-1} \langle \Psi_{\ell'} \Psi_{\ell'}^* | \mathcal{N}^\dagger \mathcal{N} | \Psi_\ell \Psi_\ell^* \rangle}, \quad (48)$$

where the denominator and numerator have subtle but important differences in the indices. The value of the (Rényi-2) quantum coherent information is independent of the choice of the basis states. Below, we simplify the expression by choosing a particular basis.

We can attribute the m -fold degeneracy of the Laughlin state to its center-of-mass (c.m.) motion [65]. Thus, we can pick our basis states $|\Psi_\ell\rangle$ to have a definite c.m. position along the y direction, or equivalently, a definite c.m. momentum along the x direction. More explicitly, let $t_i(\mathbf{a})$ denote the magnetic translation operator that moves the i th electron by $\mathbf{a}$, and $T(\mathbf{a}) = \prod_i t_i(\mathbf{a})$ the c.m. translation. We can specify the basis states by

$$T_x |\Psi_\ell\rangle = e^{\frac{2\pi i}{m}\ell} |\Psi_\ell\rangle, \quad |\Psi_{\ell+1}\rangle = T_y |\Psi_\ell\rangle, \quad (49)$$

where $T_i = T(\mathbf{e}_i L_i / N_\phi)$, $i = x, y$, moves the c.m. position along the i th direction by L_i/m , and they serve as the logical operators for the code subspace. Note that the effect of the dephasing channel does not change if we apply the magnetic translation T_i to the ket and bra state simultaneously, i.e., $[\mathcal{N}, T_i \otimes T_i^*] = 0$. We then have the following simplified form:

$$I_c^{(2)} = \log \frac{\sum_{\ell=0}^{m-1} \langle \Psi_\ell \Psi_0^* | \mathcal{N}^\dagger \mathcal{N} | \Psi_\ell \Psi_0^* \rangle}{\sum_{\ell=0}^{m-1} \langle \Psi_0 \Psi_0^* | \mathcal{N}^\dagger \mathcal{N} | \Psi_\ell \Psi_\ell^* \rangle}, \quad (50)$$

where we reduce the double summations in Eq. (48) to single ones.

B. Effective field theory of the decohered Laughlin states

Under the plasma analogy, degenerate Laughlin states correspond to configurations with magnetic and electric fluxes inside the torus, which complicates direct statistical analysis. In this section, we reformulate the bilayer plasma discussed in the previous section in terms of an effective field theory, leveraging the established correspondence between Laughlin wave functions and $(1+1)$ -dimensional conformal field theories (CFTs) [43]. This equivalent framework provides a more transparent description of the degenerate ground states, facilitating the analysis of quantum coherent information.

To represent the $\nu = 1/m$ Laughlin wave function, we consider the $U(1)_m$ compact boson CFT

$$S_0 = \frac{m}{8\pi} \int d^2 z (\nabla \phi)^2, \quad \phi \simeq \phi + 2\pi. \quad (51)$$

The bosonic field ϕ can be decomposed into chiral and anti-chiral components as $\phi(z, \bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z})$. The Laughlin

wave function on the torus can be expressed in terms of the correlation function of the chiral vertex operators [43]. Specifically, on an $L_x \times L_y$ torus, we write down an *unnormalized* Laughlin wave function as

$$\Psi_\ell(\{z_i\}) = \text{tr}_{\mathcal{H}_\ell} \left(e^{-L_y H_\varphi} \prod_{i=1}^N e^{i m \varphi(z_i)} e^{-\frac{i}{2\pi} \int d^2 z \varphi} \right), \quad (52)$$

where H_φ is the Hamiltonian of the chiral boson, and $\mathcal{H}_\ell$ denotes the Hilbert space with twisted boundary conditions of the chiral field along the x direction,

$$\varphi(z + L_x) = \varphi(z) + 2\pi \left(k + \frac{\ell}{m} \right), \quad k \in \mathbb{Z}. \quad (53)$$

Importantly, we do not normalize the right-hand side of Eq. (52) by the ℓ -dependent chiral character $\chi_\ell = \text{tr}_{\mathcal{H}_\ell} e^{-L_y H_\varphi}$ so that the resulting wave functions satisfy Eq. (49). Their overlaps take the form

$$\langle \Psi_\ell | \Psi_{\ell'} \rangle = C \delta_{\ell\ell'}, \quad (54)$$

where C is an ℓ -independent positive constant. We therefore adopt the convention (52) in the subsequent analysis. In the following, we use $\langle \cdots \rangle'$ to denote such unnormalized correlation functions and use $\langle \cdots \rangle$ for the normalized expectation values.

Using this representation, we can express the wave function overlaps in the numerator and denominator of Eq. (50) as path integrals of two non chiral fields. More generally, we consider

$$Z(\bar{\ell}_1, \ell_2; \ell_1, \bar{\ell}_2) = \langle \Psi_{\bar{\ell}_1} \Psi_{\ell_2}^* | \mathcal{N}^\dagger \mathcal{N} | \Psi_{\ell_1} \Psi_{\bar{\ell}_2}^* \rangle. \quad (55)$$

We need two chiral fields φ_1, φ_2 for $|\Psi_{\ell_1}\rangle, \langle \Psi_{\ell_2}^*|$, and two antichiral fields $\bar{\varphi}_1, \bar{\varphi}_2$ for $\langle \Psi_{\bar{\ell}_1}|, |\Psi_{\bar{\ell}_2}^*\rangle$, which should obey the following spatial boundary conditions

	$\bar{\varphi}_1$	φ_2	φ_1	$\bar{\varphi}_2$
winding	$\bar{k}_1 - \frac{\bar{\ell}_1}{m}$	$k_2 + \frac{\ell_2}{m}$	$k_1 + \frac{\ell_1}{m}$	$\bar{k}_2 - \frac{\bar{\ell}_2}{m}$

(56)

We can combine them into nonchiral fields $\phi_\alpha = \varphi_\alpha + \bar{\varphi}_\alpha$ and write the overlap as

$$Z(\bar{\ell}_1, \ell_2; \ell_1, \bar{\ell}_2) = \int Dn_{1,2} D\phi_{1,2} e^{-\gamma t \int (n_1 - n_2)^2} \times e^{-\sum_{\alpha=1}^2 S_0[\phi_\alpha] + i m \int (n_\alpha - \bar{n}) \phi_\alpha}, \quad (57)$$

where n_α is the electron density and $\bar{n} = 1/2\pi m$ ensures the correct total electron number.

As a sanity check, we examine Eq. (57) at $\mu = 0$, where n_1, ϕ_1 and n_2, ϕ_2 correspond to the two decoupled

single-component plasmas as described in Eq. (10). The fact that each plasma is screening (for $m < 70$) implies that both ϕ_1 and ϕ_2 are gapped fields. The screening behavior further indicates that magnetic charges are linearly confined. Consequently, threading a magnetic flux through the torus incurs an extensive free energy cost, which leads to the following no-winding condition for the two nonchiral fields:

$$k_\alpha + \bar{k}_\alpha + \frac{\ell_\alpha - \bar{\ell}_\alpha}{m} = 0, \quad \alpha = 1, 2. \quad (58)$$

Thus we recover Eq. (54).

This simple case illustrates the general strategy that we adopt in the subsequent analysis, including the treatment of the Moore-Read state later. Specifically, we start with kinematic analysis with a field theory, e.g., (57), then invoke the screening property from the plasma analogy to determine the low-energy degrees of freedom. With this input, we carry out the remainder of the analysis with the field-theory technique. For example, the confinement of magnetic charge is not immediately evident from the screening property due to its non-local formulation in terms of electric charges. The corresponding no-winding condition emerges naturally as a consequence of the fields being gapped.

We close this section with a remark on the generality of our results. Our analysis so far is fully based on the model wave function of the Laughlin states. The effective field theory approach may have broader applicability. The model wave function can also be obtained at the temporal boundary of the Euclidean-time path integral of $U(1)_m$ Chern-Simons field theory, where the states $|\Psi_\ell\rangle$ are distinguished by the holonomy of the gauge field a around the noncontractible x cycle

$$\oint_{\gamma_x} a = 2\pi \left(k + \frac{\ell}{m} \right), \quad k \in \mathbb{Z}. \quad (59)$$

Deviation from the model wave function can be interpreted as arising from the inclusion of gapped matter fields coupled to the Chern-Simons gauge field. By appealing to the principle of universality, we expect Eq. (57) to retain the dominant contributions relevant for long-wavelength behavior. Accordingly, we regard Eq. (57) as an effective field theory for more general Laughlin states subject to density dephasing.

C. Filling-dependent decodability transition

In the plasma analogy, it is convenient to reorganize the bilayer plasma into a paired plasma and a “relative plasma.” Accordingly, we recombine the field variables as follows:

$$n_\pm = n_1 \pm n_2, \quad \phi_\pm = \frac{1}{2}(\phi_1 \pm \phi_2), \quad \theta_\pm = \theta_1 \pm \theta_2, \quad (60)$$

where $\theta_\alpha = m(\varphi_\alpha - \bar{\varphi}_\alpha)/2$, $\alpha = 1, 2$ are the 2π -periodic nonchiral field dual to ϕ_α . It is necessary to introduce them to keep track of the boundary conditions faithfully. We then recast the partition function as

$$\begin{aligned} Z(\bar{\ell}_1, \ell_2; \ell_1, \bar{\ell}_2) &= \int \mathcal{D}(n_\pm, \phi_\pm) e^{-S_+[\phi_+, n_+] - S_-[\phi_-, n_-]}, \\ S_+[\phi_+, n_+] &= 2S_0[\phi_+] - im \int (n_+ - 2\bar{n})\phi_+, \\ S_-[\phi_-, n_-] &= 2S_0[\phi_-] - \int imn_- \phi_- + \gamma t n_-^2. \end{aligned}$$

Namely, n_+, ϕ_+ correspond to the charge- $2\sqrt{m}$ paired plasma with a uniform background charge, and n_-, ϕ_- correspond to the relative plasma with a decoherence-controlled fugacity.

Despite the formal separation into S_+ and S_- , the fields n_+, ϕ_+ and n_-, ϕ_- are not entirely independent in general. Recall that the paired plasma is in the screening phase (for $m < 35$), thus implying that ϕ_+ remains gapped regardless of the decoherence strength. Consequently, energetically favored configurations of ϕ_+ should obey the no-winding condition, which in turn constrains the spatial boundary conditions for ϕ_- and θ_- :

$$\begin{aligned} \frac{k_1 + \bar{k}_1 + k_2 + \bar{k}_2}{2} + \frac{\ell_1 - \bar{\ell}_1 + \ell_2 - \bar{\ell}_2}{2m} &= 0, \\ \phi_-(z + L_x) &= \phi_-(z) + \left(k_1 + \bar{k}_1 + \frac{\ell_1 - \bar{\ell}_1}{m} \right) 2\pi, \\ \theta_-(z + L_x) &= \theta_-(z) + m \left(k_1 + \bar{k}_2 + \frac{\ell_1 - \bar{\ell}_2}{m} \right) 2\pi. \end{aligned} \quad (61)$$

Furthermore, to account for the quantized charge of the underlying particles, we need to introduce a short-distance cutoff and discretize the space into a lattice so that the density fields n_+ and n_- become integer-valued fields defined on lattice sites. Under this discretization, it becomes clear that n_+ and n_- are not independent either, as their sum must always yield even integers.

Since ϕ_+ remains gapped, the presumed decodability transition should be governed by the dynamics of ϕ_- . To isolate this, we integrate out n_+, ϕ_+ and write

$$\begin{aligned} Z(\bar{\ell}_1, \ell_2; \ell_1, \bar{\ell}_2) &= Z_\infty Z_-(\ell_1 - \bar{\ell}_1, \ell_1 - \bar{\ell}_2), \\ Z_\infty &:= \int \mathcal{D}(n_+, \phi_+) e^{-S_+[\phi_+, n_+]}|_{n_+=0}, \\ Z_-(\ell_1 - \bar{\ell}_1, \ell_1 - \bar{\ell}_2) &:= \int \mathcal{D}(n_-, \phi_-) e^{-S_{\text{eff}}[\phi_-, n_-]}, \end{aligned} \quad (62)$$

where Z_∞ is obtained by integrating out n_+, ϕ_+ at fixed $n_- = 0$; Z_- encodes the dynamics of ϕ_- , which contains a summation over the allowed winding numbers of ϕ_-, θ_- subjected to the no-winding condition (61). We can compare Eq. (62) with Eq. (20) and recognize that Z_∞ corresponds to the paired plasma at the strong decoherence limit, and Z_- to the relative plasma. Equation (20) in the plasma analogy holds only near the strong decoherence limit, while Eq. (62) and the following analysis can be applied more broadly.

To determine the structure of $S_{\text{eff}}[n_-, \phi_-]$, we note that integrating out n_+, ϕ_+ generates terms involving n_- alone, as n_- is the only field locally coupled with them. Moreover, the original action is symmetric under $n_- \rightarrow -n_-$ and $\phi_- \rightarrow -\phi_-$, implying that only even powers of n_- can appear. Performing an expansion in the powers of n_- and its gradients, we have

$$S_{\text{eff}}[n_-, \phi_-] = 2S_0[\phi_-] - \int imn_- \phi_- + \bar{\gamma} t n_-^2 + \dots \quad (63)$$

The coefficient $\bar{\gamma} t$ implicitly includes the renormalization from integrating out n_+, ϕ_+ , and the ellipsis denotes terms involving gradients of n_- or higher powers. Near the strong decoherence limit, n_- only weakly fluctuates around $n_- = 0$ so that it suffices to omit the subleading terms in Eq. (63) and further integrate out n_- . A proper treatment requires us to discretize the space into a lattice with a short-distance cutoff a . This procedure yields a sine-Gordon field theory

$$Z_-(\ell_1 - \bar{\ell}_1, \ell_1 - \bar{\ell}_2) = \int \mathcal{D}\phi_- e^{-S_{\text{eff}}[\phi_-]},$$

$$S_{\text{eff}}[\phi_-] \approx \int \frac{m}{4\pi} (\nabla \phi_-)^2 - 2e^{-\bar{\mu}} \cos m\phi_-, \quad (64)$$

Since the boundary conditions involve the dual field θ_- , it is convenient to recast Eq. (64) with the corresponding Hamiltonian that includes θ_- explicitly:

$$Z_-(\ell_1 - \bar{\ell}_1, \ell_1 - \bar{\ell}_2) \approx \text{tr}_{\ell_1 - \bar{\ell}_1, \ell_1 - \bar{\ell}_2} e^{-L_y H_-},$$

$$H_- = \int_0^{L_x} \frac{(\partial_x \theta_-)^2}{4\pi m} + \frac{m}{4\pi} (\partial_x \phi_-)^2 - 2e^{-\bar{\mu}} \cos m\phi_-. \quad (65)$$

Even though we need approximations that are well controlled only near the strong decoherence, we expect Eqs. (64) and (65) to capture the possible phases and the potential transition away from this limit. For simplicity, we assume $L_x = L_y = L$ in the following analysis. The reader can refer to Appendix C for a discussion on the general cases.

Using Eq. (62), we can write the Rényi-2 quantum coherent information in terms of ϕ_- only:

$$I_c^{(2)} = \log \frac{\sum_{\ell=0}^{m-1} Z_-(0, \ell)}{\sum_{\ell=0}^{m-1} Z_-(\ell, 0)}. \quad (66)$$

It is also convenient to divide both the numerator and denominator by $Z_-(0, 0)$ and write

$$I_c^{(2)} = \log \frac{1 + \sum_{\ell=1}^{m-1} e^{-\Delta F_{(0, \ell)}}}{1 + \sum_{\ell=1}^{m-1} e^{-\Delta F_{(\ell, 0)}}}, \quad (67)$$

where $\Delta F_{(0, \ell)}$ and $\Delta F_{(\ell, 0)}$ are the free energy costs of twisting the boundary conditions of θ_- and ϕ_- . At the maximal dephasing $\mu = \infty$, the sine-Gordon theory is reduced to the compact boson CFT. The duality symmetry $\phi_- \leftrightarrow \theta_-/m$ guarantees that $Z_-(0, \ell) = Z_-(\ell, 0)$, and thus $I_c^{(2)} = 0$. This result is expected because the fully decohered state becomes separable between the system Q and reference R , which cannot contain any positive coherent information. At finite μ , the duality symmetry is explicitly broken, leading to nonzero $I_c^{(2)}$. There are two phases, in which $I_c^{(2)}$ exhibits distinct behavior:

- (1) The topologically ordered encoding phase is established for $\mu < \mu_c$. In this case the cosine interaction is strong enough to gap out ϕ_- and energetically penalize its spatial winding. Accordingly, we have $\Delta F_{(0, \ell)} \approx 0$ while $\Delta F_{(\ell \neq 0, 0)} \sim e^{-\bar{\mu}} L$. Therefore $I_c^{(2)} = \log m$ in the thermodynamic limit.

For $m \leq 4$, the critical decoherence strength μ_c is pushed to infinity in the thermodynamic limit. In this case $I_c^{(2)} = \log m$ for any finite decoherence strength or time. In a finite-size system, on the other hand, $I_c^{(2)}$ will start to deviate from $\log m$ if the free energy cost $\Delta F_{(\ell \neq 0, 0)}$ becomes negligible. This happens when $e^{-\bar{\mu}} L \sim \mathcal{O}(1)$, or equivalently, when $\bar{\mu} \sim \mathcal{O}(\log L)$. Since $\bar{\mu}$ is linear in the decoherence time t , the lifetime of the quantum memory is therefore of order $\mathcal{O}(\log L)$. It is worth pointing out that $\mathcal{O}(\log L)$ time is sufficient for the channel to converge to the maximum dephasing. Therefore, the scaling of the lifetime with L , in this case, saturates the maximum that is achievable under such density dephasing [24].

- (2) The critical phase is established for $\mu_c \leq \mu < \infty$. In this case the cosine term is irrelevant, so we can analyze its contribution perturbatively. Surprisingly, it contributes an additional $\mathcal{O}(1)$ free energy cost for nonzero windings of ϕ_- , i.e., $\Delta F_{(\ell, 0)} > \Delta F_{(0, \ell)}$ ($\ell \neq 0$), which leads to $0 < I_c^{(2)} < \log m$. The calculation is shown in Appendix C.

In Fig. 2(a), we plot the Rényi-2 coherent information as a function of decoherence strength of the Laughlin states (for both $m > 4$ and $m \leq 4$) under dephasing. Remarkably, for $m \leq 4$, the quantum memory is robust for any finite dephasing noise in contrast to that of the toric code under Z dephasing. For $m > 4$, the coherent information remains

finite and decreases continuously after the transition. Operationally, the decoding fidelity of the optimal decoder also decreases continuously, meaning that the critical phase has partial quantum memory. This is in sharp contrast to the toric code under dephasing noise, where the quantum memory is completely destroyed after the transition.

V. DECOHERED MOORE-READ STATES

In this section, we analyze the decohered Moore-Read (MR) states, the simplest non-Abelian quantum Hall states [43]. The most prominent feature of the MR states is the non-Abelian quasihole excitations. Multiple well-separated quasiholes can span a multidimensional fusion space that can be used for storing quantum information and performing topological quantum computation. The main goal of this section is to examine the integrity of this fusion space under decoherence. We again primarily consider Rényi-2 quantities and discuss the von Neumann quantities at the end.

A. Review of the Moore-Read wave function

The MR wave function at $\nu = 1/m$ is constructed from the chiral Ising and chiral boson CFT. Let ψ be the chiral fermion field in the Ising sector and φ the chiral boson field with $\langle \varphi(z)\varphi(w) \rangle = -(1/m) \log(z-w)$. The electron corresponds to $\psi e^{im\varphi}$, and the (un-normalized) MR wave function of N electrons on the plane reads

$$\Psi(\{z_i\}) = \left\langle \prod_{i=1}^N \psi(z_i) e^{im\varphi(z_i)} e^{-\frac{i}{2\pi} \int d^2z \varphi} \right\rangle, \quad (68)$$

where we need to assume N is even and $e^{-\frac{i}{2\pi} \int d^2z \varphi}$ provides the neutralizing background charge. For simplicity, we introduce $\langle \cdots \rangle_{\text{bg}}$ as the expectation value with the background charge. If there are nonchiral or multicomponent fields $\{\phi_\alpha\}$, we use the same notation,

$$\langle \cdots \rangle_{\text{bg}} = \langle \cdots e^{-\frac{i}{2\pi} \sum_\alpha \int d^2z \phi_\alpha} \rangle. \quad (69)$$

The reader can infer the correct background charge from the context. The MR state hosts $3m$ types of anyons corresponding to the following primary fields

$$\{e^{i\ell\varphi}, \psi e^{i\ell\varphi}, \sigma e^{i(\ell+\frac{1}{2})\varphi}\}, \quad \ell = 0, 1, \dots, m-1, \quad (70)$$

all of which are local with respect to the operator $\psi e^{im\varphi}$. For instance, $e^{i\varphi}$ creates a Laughlin quasihole carrying charge $1/m$, while $\sigma e^{i\varphi/2}$ corresponds to the fundamental quasihole with charge $1/2m$. Two fundamental quasiholes can fuse into more than one particle

$$\sigma e^{i\varphi/2} \times \sigma e^{i\varphi/2} = e^{i\varphi} + \psi e^{i\varphi}, \quad (71)$$

and thus they are non-Abelian anyons. In the following discussion, “quasihole” always refers to the fundamental quasihole unless specified explicitly.

When there are multiple quasiholes, their wave functions span a degenerate subspace called the fusion space. As the simplest example, having four quasiholes at locations $\eta_1, \eta_2, \eta_3, \eta_4$ yields two basis states

$$\Psi_{\mu, \{\eta_k\}}(\{z_i\}) = \mathcal{F}_\mu(z_i; \eta_k) \times \left\langle \prod_{k=1}^4 e^{i\varphi(\eta_k)/2} \prod_{i=1}^N e^{im\varphi(z_i)} \right\rangle_{\text{bg}}, \quad (72)$$

where $\mu = 1, \psi$ labels the fusion outcome of the first two σ 's, and $\mathcal{F}_1, \mathcal{F}_\psi$ are the two chiral conformal blocks of the following correlation function of nonchiral primaries [66]

$$\left\langle \prod_{k=1}^4 \sigma(\eta_k, \bar{\eta}_k) \prod_{i=1}^N \varepsilon(z_i, \bar{z}_i) \right\rangle = \sum_{\mu=1, \psi} \mathcal{F}_\mu(z_i; \eta_k) \bar{\mathcal{F}}_\mu(\bar{z}_i; \bar{\eta}_k), \quad (73)$$

where $\varepsilon = \bar{\psi}\psi$. When the quasiholes are far apart, the Gram matrix of the two wave functions is [45]

$$\langle \Psi_{\mu, \{\eta_k\}} | \Psi_{\nu, \{\eta_k\}} \rangle = C \delta_{\mu, \nu} + \mathcal{O}(e^{-|\eta_i - \eta_j|/\xi}), \quad (74)$$

where C is a positive constant that is independent of the fusion channel and ξ is a correlation length. Namely, they are two orthogonal states. We can use the fusion space to encode one qubit represented by entangling this state with an external reference qubit

$$|\Psi_{\text{RQ}}\rangle = \frac{1}{\sqrt{2}} (|\Psi_{1, \{\eta_k\}}\rangle \otimes |0_R\rangle + |\Psi_{\psi, \{\eta_k\}}\rangle \otimes |1_R\rangle). \quad (75)$$

As a proxy for the recoverable quantum information we will compute the Rényi-2 quantum coherent information from R to Q of the above state subject to decoherence.

B. Effective field theory near the strong decoherence limit

The CFT construction of the Moore-Read wave functions serves as a convenient starting point for analyzing the effect of decoherence on encoded quantum information. We begin by calculating the purity, which serves as a generating function for various information quantities and can be directly represented using the wave functions (68) in close analogy to the analysis of the decohered Laughlin states.

$$\text{tr}\rho^2 = \int \mathcal{D}(z_i^1, z_i^2) e^{-V(\{z_i^1\}, \{z_i^2\})} \times \left\langle \prod_{\alpha=1}^2 \prod_{i=1}^N \varepsilon_{\alpha}(z_i^{\alpha}, \bar{z}_i^{\alpha}) e^{im\phi_{\alpha}(z_i^{\alpha}, \bar{z}_i^{\alpha})} \right\rangle_{\text{bg}}. \quad (76)$$

This expression of the purity can be viewed as insertions of specific perturbations in a pair of Ising and a pair of U(1) CFTs. Here ε_{α} denotes the energy operator of the Ising CFT α and $\phi_{\alpha} = \varphi_{\alpha} + \bar{\varphi}_{\alpha}$ is the nonchiral boson field of the corresponding U(1) CFT; V represents a local attraction between the two liquids with electron locations given by the complex numbers z_i^{α} . As in the case of Laughlin states, treated above, our strategy for determining the decoherence-induced transitions is to build an effective theory in terms of perturbations from the fully decohered limit.

Before considering the fully decohered state it is worth noting the symmetries intrinsic to the field theory (76), which are present at any decoherence strength. Many of these symmetries originate from the presence of anyons in the MR state. Here we highlight two that are particularly useful for constructing the effective theory. The first symmetry, inherited from the Laughlin quasihole, is a $\mathbb{Z}_m^{\otimes 2}$ symmetry acting on the U(1)² part alone. It shifts the boson fields $\phi_{1,2}$ by

$$(\mathbb{Z}_m)_i: \phi_i \rightarrow \phi_i + \frac{2\pi k_i}{m}, \quad i = 1, 2, \quad (77)$$

where $k_i = 0, 1, \dots, m-1$. The vertex operators $e^{im\phi_1}$ and $e^{im\phi_2}$ are clearly invariant. The extra phase from the background charge can be canceled by adding background vector potential to the corresponding topological defect lines [67–69]. The second symmetry, related to the fundamental quasiholes, is a doubled Kramers-Wannier symmetry that transforms both the Ising and U(1) fields

$$\text{KW}_i: \varepsilon_i \rightarrow -\varepsilon_i, \quad \phi_i \rightarrow \phi_i + \frac{\pi}{m}, \quad i = 1, 2. \quad (79)$$

As with the $\mathbb{Z}_m^{\otimes 2}$ symmetry, the topological defect line implementing the $\text{KW}^{\otimes 2}$ symmetry must be dressed by the background vector potential as well.

In the strong decoherence limit, the particle positions of the two electron fluids become identified, so that the purity reads

$$\text{tr}\rho_{\infty}^2 = \int \mathcal{D}\xi \left\langle \prod_{i=1}^N (\varepsilon_1 \varepsilon_2)(\xi_i, \bar{\xi}_i) e^{2im\phi_+(\xi_i, \bar{\xi}_i)} \right\rangle_{\text{bg}}, \quad (80)$$

where $\phi_{\pm} = (\phi_1 \pm \phi_2)/2$. It is important to note two sets of symmetries, special to the strong decoherence limit, which will be explicitly broken once we deviate from this limit. One is a U(1) symmetry shifting ϕ_{-}

$$\text{U}(1): \phi_{-} \rightarrow \phi_{-} + \alpha, \quad \alpha \in [0, 2\pi), \quad (81)$$

which is manifest from the expression (80). The other is a $\text{U}(1) \times \text{U}(1)$ symmetry acting only on the Ising² part. To see this, recall that the Ising² CFT is equivalent to the $\mathbb{Z}_2$ orbifold of a 2π -periodic compact boson [70]

$$S_{\Phi} = \frac{1}{2\pi} \int d^2z (\nabla\Phi)^2, \quad \Phi \sim \Phi + 2\pi, \quad -\Phi. \quad (82)$$

The latter manifestly has a $\text{U}(1)_m \times \text{U}(1)_w$ symmetry for Φ and its dual Θ

$$\begin{aligned} \text{U}(1)_m: \Phi &\rightarrow \Phi + \alpha_m, & \alpha_m &\in [0, 2\pi), \\ \text{U}(1)_w: \Theta &\rightarrow \Theta + \alpha_w, & \alpha_w &\in [0, 2\pi). \end{aligned} \quad (83)$$

The dictionary of the operator mapping is

$$\begin{aligned} \sigma_1 \sigma_2 &\sim \cos \Phi, & \mu_1 \mu_2 &\sim \sin \Phi, \\ \sigma_1 \mu_2 &\sim \cos \frac{\Theta}{2}, & \mu_1 \sigma_2 &\sim \sin \frac{\Theta}{2}, \\ \varepsilon_1 + \varepsilon_2 &\sim \cos 2\Phi, & \varepsilon_1 - \varepsilon_2 &\sim \cos \Theta, \\ \varepsilon_1 \varepsilon_2 &\sim (\nabla\Phi)^2 - (\nabla\Theta/2)^2. \end{aligned} \quad (84)$$

One can also read out other global symmetry actions from this dictionary. In particular, the aforementioned $\text{KW}^{\otimes 2}$ symmetries act by

$$\begin{aligned} \text{KW}_1: 2\Phi &\leftrightarrow \pi - \Theta, \\ \text{KW}_2: 2\Phi &\leftrightarrow \Theta. \end{aligned} \quad (85)$$

Since the reference theory (80) only contains the insertion of $\varepsilon_1 \varepsilon_2$, the $\text{U}(1)_m \times \text{U}(1)_w$ symmetry is fully preserved.

Similar to our discussion on the purity of decohered Laughlin state, we organize the deviation from the reference point $Z_{\text{MR}_{\infty}} \equiv \text{tr}\rho_{\infty}^2$ by the number of unpaired operators

$$\text{tr}\rho^2 = \sum_n e^{-2\bar{\mu}n} Z_{\text{unpaired}}^{(n)}, \quad Z_{\text{unpaired}}^{(0)} = Z_{\text{MR}_{\infty}}. \quad (86)$$

Near the strong decoherence limit, we can approximate the contribution from unpaired operators as inserting *test* operators to the partition function of the reference theory

$$\frac{Z_{\text{unpaired}}^{(n)}}{Z_{\text{MR}_{\infty}}} \approx \frac{1}{(n!)^2} \int \prod_{\alpha=1}^2 \prod_{i=1}^n d^2 z_i^{\alpha} \left\langle \prod_{\alpha,i} e^{im\phi_{\alpha}} \varepsilon_{\alpha}(z_i^{\alpha}, \bar{z}_i^{\alpha}) \right\rangle_{\text{MR}_{\infty}}, \quad (87)$$

where we introduce the following notation for the correlation function in the reference theory

$$\langle X \rangle_{\text{MR}_\infty} \equiv \frac{1}{Z_{\text{MR}_\infty}} \int \mathcal{D}\xi \left\langle X \prod_{p=1}^N (\varepsilon_1 \varepsilon_2) (\xi_p, \bar{\xi}_p) e^{2im\phi_+(\xi_p, \bar{\xi}_p)} \right\rangle_{\text{bg}}. \quad (88)$$

Equation (87) generalizes the approximation (20) for the decohered Laughlin state in the dilute limit. The inserted test operators $e^{im\phi_a} \varepsilon_a$ preserve the intrinsic symmetries of course, while breaking the symmetries that are special to the fully decohered reference state.

To construct an effective theory we must identify the low-energy modes of the reference theory and their overlap with the test operators. Equivalently, we need to determine whether any of the symmetries are *spontaneously* broken in the infrared. For the $\mathbb{Z}_m^{\otimes 2}$ symmetry and U(1) shift of ϕ_- , the plasma analogy yields

$$\begin{aligned} \lim_{|\eta| \rightarrow \infty} \langle e^{i\phi_+(\eta, \bar{\eta})} e^{-i\phi_+(0,0)} \rangle_{\text{MR}_\infty} &> 0, \\ \langle e^{i\phi_-(\eta, \bar{\eta})} e^{-i\phi_-(0,0)} \rangle_{\text{MR}_\infty} &= |\eta|^{-1/m}. \end{aligned} \quad (89)$$

In the plasma language, inserting $e^{i\phi_+}$ or $e^{i\phi_-}$ amounts to adding two different types of test charge into the plasma. The ϕ_+ charge is completely screened, producing true long-range order, whereas the ϕ_- charge is unscreened and shows only quasi-long-range order. Consequently, we retain only ϕ_- and ignore ϕ_+ in the effective theory, by making the replacement

$$e^{im\phi_1} \rightarrow e^{im\phi_-}, \quad e^{im\phi_2} \rightarrow e^{-im\phi_-}. \quad (90)$$

For the $\text{U}(1)_m$ and $\text{U}(1)_w$ symmetry, note that $\cos \Phi$ translates into test charges in the plasma that are fully screened. Together with the KW symmetry (85), we have

$$\begin{aligned} \lim_{|\eta| \rightarrow \infty} \langle \cos \Phi(\eta, \bar{\eta}) \cos \Phi(0) \rangle_{\text{MR}_\infty} &> 0, \\ \lim_{|\eta| \rightarrow \infty} \left\langle \cos \frac{\Theta(\eta, \bar{\eta})}{2} \cos \frac{\Theta(0)}{2} \right\rangle_{\text{MR}_\infty} &> 0. \end{aligned} \quad (91)$$

Thus both the $\text{U}(1)_m$ and $\text{U}(1)_w$ symmetry are broken spontaneously. On the other hand, Θ creates vortices of Φ (and vice versa), which are linearly confined in the symmetry breaking phase so that their mixed correlators still decay exponentially, e.g.

$$\langle \cos \Phi(\eta, \bar{\eta}) \cos \Theta(0) \rangle_{\text{MR}_\infty} \sim \mathcal{O}(e^{-|\eta|/\xi}). \quad (92)$$

Equations (91) and (92) imply that a Φ condensate and a Θ condensate coexist in the reference state and they make separate contributions to the effective theory [71]. Specifically, we replace ε_1 and ε_2 by constant fields

$$\varepsilon_1 \rightarrow \cos 2\Phi + \cos \Theta, \quad \varepsilon_2 \rightarrow \cos 2\Phi - \cos \Theta, \quad (93)$$

and neglect all correlators that mix Φ and Θ . With these recipes, we have

$$\begin{aligned} \frac{Z_{\text{unpaired}}^{(n)}}{Z_{\text{MR}_\infty}} &\approx \frac{1}{(2n)!} \left\langle \left(2 \cos 2\Phi \int d^2z \cos m\phi_-(z) \right)^{2n} \right\rangle_{\text{MR}_\infty} \\ &+ \frac{1}{(2n)!} \left\langle \left(2 \cos \Theta \int d^2z \cos m\phi_-(z) \right)^{2n} \right\rangle_{\text{MR}_\infty}, \end{aligned} \quad (94)$$

where the two terms account for the contribution from the Φ and Θ condensate, respectively.

Altogether, the effective theory near the strong decoherence limit is

$$\begin{aligned} \text{tr} \rho^2 &\approx \int_0^{2\pi} d\Phi \int D\phi_- e^{-S_{\text{eff}}[\phi_-, \cos 2\Phi]} \\ &+ \int_0^{2\pi} d\Theta \int D\phi_- e^{-S_{\text{eff}}[\phi_-, \cos \Theta]}, \end{aligned} \quad (95)$$

where S_{eff} takes the form of a sine-Gordon theory, e.g.,

$$\begin{aligned} S_{\text{eff}}[\phi_-, \cos 2\Phi] \\ = \int \frac{m}{4\pi} (\nabla \phi_-)^2 - 2e^{-\bar{\mu}} \cos 2\Phi \cos m\phi_-(x). \end{aligned} \quad (96)$$

This closely resembles the effective description of the decohered Laughlin state in Eq. (64), except that the Ising sector of the MR state enters by modifying the coupling of $\cos m\phi_-$ to be proportional to $\cos 2\Phi$ or $\cos \Theta$.

Because Φ and Θ are constant fields, the perturbation retains its scaling dimension $m/2$. For $m < 4$, the relevance of $\cos m\phi_-$ implies no transition at any finite decoherence strength. Minimizing the two relevant perturbations $\cos 2\Phi \cos m\phi_-$ and $\cos \Theta \cos m\phi_-$ pins Φ at four discrete values $0, \pi/2, \pi, 3\pi/2$, and Θ at $0, \pi$. One can verify that Φ and Θ are pinned at the same sets of values for the pristine MR state. Therefore, it suggests that the system remains fully topological at all finite decoherence.

For $m > 4$, a transition occurs at a finite threshold, beyond which ϕ_- becomes a critical degree of freedom. To understand the dynamics of the Φ and Θ field, we can integrate out ϕ_- and have

$$\begin{aligned} \text{tr} \rho^2 &= \int_0^{2\pi} d\Phi e^{-F[\cos 2\Phi]} + \int_0^{2\pi} d\Theta e^{-F[\cos \Theta]}, \\ F[X] &= -\log \int D\phi_- e^{-S_{\text{eff}}[\phi_-, X]}. \end{aligned} \quad (97)$$

For $m > 4$, $\cos m\phi_-$ is irrelevant, so that we can perturbatively expand $F[X]$ in $e^{-\bar{\mu}}$ and have

$$F[X] - F[0] \approx -2e^{-2\bar{\mu}} X^2 L^2 \int d^2x \langle \cos m\phi_-(x) \cos m\phi_-(0) \rangle, \quad (98)$$

where $F[0]$ is the free energy at $e^{-\bar{\mu}} = 0$ and translation invariance simplifies the expression. For $m \geq 4$, the integral converges to a finite positive value that depends on the short-distance cutoff, and we have

$$F[X] - F[0] \approx -\frac{2e^{-2\bar{\mu}} L^2 a^{2-m}}{m-2} X^2, \quad (99)$$

where a is a short-distance cutoff. Although the perturbation is not relevant, it can generate new symmetry-allowed terms that fully determine the dynamics of Φ and Θ . Indeed, Eqs. (97) and (99) are the simplest forms compatible with the $\text{KW} \times \text{KW}$ symmetry. As a result, Φ and Θ are still pinned at the same sets of values for any finite decoherence. It suggests that the Ising sector remains intact despite the fact that the bosonic charge sector is corrupted above the threshold. In what follows, we discuss the precise information-theoretic meaning of this transition.

C. Integrity of the fusion space

In this section, we examine how the decoherence affects the fusion space of multiple fundamental quasiholes. Specifically, we apply decoherence to the state (75) that maximally entangles the fusion space of four quasiholes with a reference qubit, and evaluate the Rényi-2 quantum coherent information

$$I_c^{(2)} = \log \frac{\text{tr} \rho_{\text{RQ}}^2}{\text{tr} \rho_Q^2}.$$

We will show that both the numerator and denominator can be mapped to simple correlation functions in the effective theory (95), enabling a straightforward computation of $I_c^{(2)}$ near the strong decoherence limit.

Using the quasihole wave functions (72) and the relation between conformal blocks and correlation functions of nonchiral primaries (73), we have

$$\begin{aligned} \text{tr} \rho_{\text{RQ}}^2 &= \int \mathcal{D}(z_i^1, z_i^2) e^{-V(\{z_i^1\}, \{z_i^2\})} \\ &\times \left\langle \prod_{k=1}^4 e^{i\phi_+} \sigma_{12}(\eta_k, \bar{\eta}_k) \prod_{\alpha,i} e^{im\phi_a} \varepsilon_\alpha(z_i^\alpha, \bar{z}_i^\alpha) \right\rangle_{\text{bg}}, \end{aligned} \quad (100)$$

where $\sigma_{12} = \sigma_1 \sigma_2$, $\eta_{k=1,\dots,4}$ are locations of the four quasiholes. The only difference between $\text{tr} \rho_{\text{RQ}}^2$ and the purity $\text{tr} \rho^2$ is the insertion of

$$\prod_k e^{i\phi_+} \sigma_{12}(\eta_k, \bar{\eta}_k) \sim \prod_k e^{i\phi_+} \cos \Phi(\eta_k, \bar{\eta}_k).$$

In the effective theory, the above four-point correlator vanishes in the Θ condensate, and we can approximate it by a constant field $\cos^4 \Phi$ in the Φ condensate (96). We have

$$\text{tr} \rho_{\text{RQ}}^2 = \mathcal{N} \langle (\cos \Phi)^4 \rangle_{\text{eff}}, \quad (101)$$

where $\langle \dots \rangle_{\text{eff}}$ stands for the expectation value in the effective theory (96), and $\mathcal{N} = \int \mathcal{D}(\Phi, \phi_-) e^{-S_{\text{eff}}[\phi_-, \cos 2\Phi]}$ is a normalization factor. Analogously, $\text{tr} \rho_Q^2$ corresponds to inserting the following operator (ignoring $e^{i\phi_+}$'s):

$$\begin{aligned} &\frac{1}{2} (\sigma_{12}(\eta_1, \bar{\eta}_1) \sigma_{12}(\eta_2, \bar{\eta}_2) \sigma_{12}(\eta_3, \bar{\eta}_3) \sigma_{12}(\eta_4, \bar{\eta}_4) \\ &+ \mu_{12}(\eta_1, \bar{\eta}_1) \sigma_{12}(\eta_2, \bar{\eta}_2) \sigma_{12}(\eta_3, \bar{\eta}_3) \mu_{12}(\eta_4, \bar{\eta}_4) \\ &+ \sigma_{12}(\eta_1, \bar{\eta}_1) \mu_{12}(\eta_2, \bar{\eta}_2) \sigma_{12}(\eta_3, \bar{\eta}_3) \mu_{12}(\eta_4, \bar{\eta}_4) \\ &+ \mu_{12}(\eta_1, \bar{\eta}_1) \mu_{12}(\eta_2, \bar{\eta}_2) \sigma_{12}(\eta_3, \bar{\eta}_3) \sigma_{12}(\eta_4, \bar{\eta}_4)). \end{aligned}$$

In the effective theory, we get

$$\text{tr} \rho_Q^2 = \frac{\mathcal{N}}{2} \langle (\cos \Phi)^4 \rangle_{\text{eff}} + \frac{3\mathcal{N}}{2} \langle (\cos \Phi \sin \Phi)^2 \rangle_{\text{eff}}, \quad (102)$$

which, similarly, involves only the Φ condensate to evaluate the expectation value. Therefore, the Rényi-2 quantum coherent information is simplified as

$$I_c^{(2)} = \log \frac{2 \langle (\cos \Phi)^4 \rangle_{\text{eff}}}{\langle (\cos \Phi)^4 \rangle_{\text{eff}} + 3 \langle (\cos \Phi \sin \Phi)^2 \rangle_{\text{eff}}} \quad (103)$$

in the effective theory.

We first calculate $I_c^{(2)}$ at the strong decoherence limit as a sanity check of our effective theory. In this case, Φ and ϕ_- are decoupled from each other, which reduces the expectation value to an integral over Φ , e.g.,

$$\langle \cos^4 \Phi \rangle_{\text{eff}} = \frac{1}{2\pi} \int_0^{2\pi} d\Phi \cos^4 \Phi, \quad (e^{-\bar{\mu}} = 0).$$

We thus obtain the expected result

$$I_c^{(2)} = 0, \quad (e^{-\bar{\mu}} = 0). \quad (104)$$

Surprisingly, infinitesimal deviation from the strong decoherence limit restores $I_c^{(2)}$ to its initial value $\log 2$ in the thermodynamic limit. To see this, recall that Φ remains pinned at the same four values $0, \pi/2, \pi, 3\pi/2$ regardless of whether $\cos m\phi_-$ is relevant or not. In all cases, evaluating the expectation values in Eqs. (100) and (102) reduces to a discrete sum. We get

$$I_c^{(2)} = \log 2, \quad (0 < e^{-\bar{\mu}} \leq 1). \quad (105)$$

Strictly speaking, the effective theory and its corollaries are valid near the strong decoherence limit. Since we expect $I_c^{(2)}$ to be monotonic in the decoherence strength, we extend the range of validity for Eq. (105) to include finite decoherence strength. Therefore, although the bosonic charge sector of the topological order is corrupted across the BKT transition, the fusion space of the non-Abelian fundamental quasiholes remains well preserved for any finite dephasing. In a finite-size system, the perturbation becomes negligible if $\bar{\mu} \sim \mathcal{O}(\log L)$. Therefore the lifetime of this quantum memory using the fusion space is also of the order $\mathcal{O}(\log L)$, as in the case of the Laughlin state above the critical filling.

D. Corruption of information stored in the Abelian sector

In contrast to the robustness of the fusion space, the filling-dependent transition does affect information stored in the Abelian sector of the MR state. Specifically, consider the MR states on the torus [72]. Similar to the Laughlin states, the degenerate MR wave functions can be constructed from the correlation function of the chiral Ising and compact boson CFT [44]

$$\Psi_{\alpha,\ell}(\{z_i\}) = \left\langle \prod_i^{N_e} \psi(z_i) \right\rangle'_\alpha \left\langle \prod_i^{N_e} e^{im\varphi(z_i)} \right\rangle'_{\text{bg}, \ell - \alpha_x + 1/2}, \quad (106)$$

where $\langle \dots \rangle'$ stands for the unnormalized correlator, $\alpha = (\alpha_x, \alpha_y)$, $\alpha_i = 0, 1/2$ labels the boundary conditions of the Majorana fermion $\psi(z + L_i) = -e^{2i\pi\alpha_i}\psi(z)$, and $\ell - \alpha_x + 1/2$, $\ell = 0, 1, \dots, m-1$, labels different winding number sectors of the chiral boson CFT. When the number of electrons N_e is odd, only $\alpha = (1/2, 1/2)$ gives a nonzero result and the ground-state degeneracy is m . When N_e is even, $\alpha = (0, 0)$, $(0, 1/2)$, and $(1/2, 0)$ all give nonzero results, and the degeneracy is $3m$.

To address the stability of the information in the Abelian sector, we fix the Ising sector label α and use only ℓ for the encoding. In this case the same calculation of the coherent information as in Sec. IV B applies here and will yield the same result. Namely, for systems below the critical filling $\nu = 1/4$, the information encoded with ℓ (characterized by Rényi-2 quantities) will be corrupted across the decoherence-induced transition. The corrupted quantum information exactly corresponds to the c.m. motion of the fractional quantum Hall states. Similar to the Laughlin states on a torus, $\Psi_{\alpha,\ell}(\{z_i\})$ factorizes into a Gaussian part, a holomorphic c.m.-motion part, and a holomorphic relative-motion part [73]

$$\Psi_{\alpha,\ell}(\{z_i\}) = e^{-\sum_i y_i^2/2} F_{\alpha,\ell}(Z) f_\alpha(\{z_i - z_j\}). \quad (107)$$

Importantly, changing ℓ only affects the c.m. part, and the corresponding logical operators are, again, the magnetic translation operators T_x, T_y . Therefore, we can restate the result as the quantum memory of the c.m. motion is corrupted above the threshold.

In principle, we can use both α and ℓ to encode quantum information on the torus. Calculating the coherent information in this general case, especially that from the Ising sector, is beyond the scope of this work. We leave it to future work.

E. Critical filling for the von Neumann quantities

In this section, we extend our discussion to the von Neumann quantities. Using the same approach as in Sec. III C, we analyze the von Neumann entropy to obtain both the upper and lower bounds of the critical filling ν_c .

For the MR state, the instability of the von Neumann entropy near the strong decoherence limit also obeys the general inequality (39). Again, we are interested in δS_L , the contribution of macroscopically separated pairs with $|\eta - \eta'| > \alpha L$ ($0 < \alpha < 1$):

$$\begin{aligned} \frac{1}{2NL^2} \left(\int_{|\eta - \eta'| > \alpha L} d^2\eta d^2\eta' C_e^{(1)}(\eta, \eta') \right)^2 \\ \leq |\delta S_L| \leq \int_{|\eta - \eta'| > \alpha L} d^2\eta d^2\eta' C_e^{(1)}(\eta, \eta'), \end{aligned}$$

where $C_e^{(1)}(\eta, \eta')$ is the Rényi-1 correlator (30) for the fully decohered MR state ρ_∞

$$C_e^{(1)}(\eta, \eta') = \text{tr}(c(\eta') \sqrt{\rho_\infty} c^\dagger(\eta') c(\eta) \sqrt{\rho_\infty} c^\dagger(\eta)).$$

Unlike the decohered Laughlin case, the Rényi-1 correlator itself lacks a direct plasma analogy. Nevertheless, as shown in Appendix E, it admits upper and lower bounds that do map onto a plasma description. For separation $|\eta - \eta'| \gg \xi$ (screening length of the plasma),

$$\frac{c'}{|\eta - \eta'|^{m/2+3/4}} \leq C_e^{(1)}(\eta, \eta') \leq \frac{c}{|\eta - \eta'|^{m/2}}, \quad (108)$$

where c, c' are some $\mathcal{O}(1)$ constants. Integrating these bounds yields

$$C'L^{5/2-m} \leq |\delta S_L| \leq CL^{4-m/2}, \quad (109)$$

for some finite constant C, C' . Therefore, the critical filling for von Neumann quantities satisfies

$$1/8 \leq \nu_c(1) \leq 1/3. \quad (110)$$

The experimentally relevant Moore-Read state at $\nu = 1/2$ lies above this range, and it therefore remains topological at any finite decoherence strength.

VI. DISCUSSION AND OUTLOOK

In this work, we examined the Laughlin and Moore-Read wave functions subject to decoherence that couples to the electron density. We identified a critical filling factor, below which the system can exhibit a BKT transition into a critical phase. Within this phase, the quantum information encoded in the degenerate ground states is partially corrupted, whereas the information stored in the fusion space of non-Abelian anyons (for the Moore-Read states) remains intact.

These results and, in particular, the existence of a critical filling factor, derive from the properties of the fully decohered states. Surprisingly, the fully decohered states are always critical, with a varying scaling dimension, making them unstable to infinitesimal coherence above a critical filling. For this reason, the system exhibits absolute resilience to decoherence above the critical filling. If the fully decohered state were “gapped,” in the sense of having a finite correlation length and Markov length [24,25], then it would have a finite domain of stability and the transition would occur at a finite decoherence strength regardless of filling factor [17–20,23,74].

The fully decohered states are interesting and unusual in other aspects too. Having zero coherent information, rather than the minimum value of $-\log m$ implies that they retain $\log m$ bits of residual information. But although the fully decohered states are classical by virtue of having a diagonal density matrix in the position basis, the retained memory cannot be entirely classical as we explain below.

In Abelian topologically ordered states the retained memory is determined by the subset of anyons that have not been proliferated by the decoherence and that have a trivial braiding with those that have been proliferated [18]. This memory is said to be classical if the residual nontrivial logical (anyon) operators form a commuting algebra. One example is the fully decohered toric code under X or Z errors. If the X strings proliferate due to X errors, the commuting X loops on the two cycles persist as a classical memory.

In the Laughlin state, on the other hand, the loop operators T_x and T_y transporting quasiholes along the two cycles of the torus, form noncommuting logical operators. Because C_4 rotation symmetry is preserved by decoherence, it is impossible to preserve only a commuting subset of nontrivial logical operators and thereby a purely classical memory. All the logical information would be destroyed if the quasiholes fully proliferate because of their nontrivial self-statistics. However, for the density dephasing noise we studied here, the quasiholes are not fully proliferated, and instead establish a critical state, leading to zero (rather than negative) coherent information. This

indicates that the logical information encoded by T_x , T_y is partially preserved. Similar results also hold for more general FQH states under maximal dephasing. A similar analysis also applies to the toric code under $(X + Z)$ noise [29], which leads to a critical state at infinite decoherence. However, this requires a fine-tuned $e - m$ symmetry, and therefore the critical state exists only at a single point and not as an extended critical phase. In Appendix F, we clarify what we mean by nonclassical memory in a classical state, where we prove this is indeed the case for the dephased FQH states, and also provide a more mundane two-qubit example.

The decoherence model analyzed in this work is an idealized limit. In practice, other noise sources, as well as the interplay of the noise with Hamiltonian evolution, are expected to generate diffusion of excited anyons, which is not captured by the simple density dephasing model. Nonetheless, the effective field theories for the Renyi-2 coherent information, developed in Secs. IV and V, generalize to this case as long as we can model the noise by a finite-depth quantum channel. In particular the effect of the noise is still to induce a coupling between the four chiral (temporal) edge modes originating from the ket and bra in the two copies of the density matrix. In the Laughlin state, a general charge-preserving noise can still only lead to the sine-Gordon model

$$S_{\text{eff}} = \int d^2x \frac{K}{4\pi} (\nabla \phi_-)^2 - J \cos m\phi_- + \dots, \quad (111)$$

where the ellipses denote less relevant terms.

A special feature of the density decoherence model defined by Eq. (1) is that it does not alter the charge distribution and, therefore, leaves the bare value of K unchanged, independent of the noise strength ($K = m$). On the other hand, a decoherence channel that also hops charges incoherently, enhances correlations of the phase dual to ϕ_- , i.e., of $e^{i\theta_-}$, and thereby suppresses the bare value of K . So, while the pure density decoherence takes the bare parameters of the effective model on a vertical downward path on the phase diagram in Fig. 1(b) a more generic decoherence model would chart a path that goes both down and to the left (larger values of K correspond to smaller values of the effective filling factor). We conclude that generic noise would still induce the same KT transition as pure density dephasing, but the value of the critical filling factor ν_c above which the system is absolutely stable and even the existence of such a critical value is special to pure density decoherence. This is indeed seen explicitly in subsequent work by some of us [75], where we considered a phenomenological noise model that includes the diffusion of Abelian anyons. We find that in this case there is always a finite decoherence threshold.

The above analysis can be further generalized to include number-nonconserving processes. Such noise could be

induced, for example, by incoherent hopping of electrons in and out of the system. Using the representation of the electron operator as $c \sim e^{im\phi}$, the effective field theory is generalized to

$$S_{\text{eff}} = \int \frac{K}{4\pi} (\nabla\phi_-)^2 - J \cos m\phi_- - g \cos \theta_- . \quad (112)$$

The term $\cos \theta_-$ describing the particle gain and loss breaks the U(1) symmetry and its coefficient g increases with the decoherence strength or time. This form of the effective field theory also implies a finite decoherence threshold. Unlike for the charge-conserving noise, the fully decohered state in this case is not critical, but rather is a “gapped” state dominated by the g term and characterized by vanishing of the coherent information. There are two possible paths through which the system can reach the trivial state with increasing decoherence time or strength: (i) the FQH state first undergoes a BKT transition into the critical phase at one critical noise strength, and then another BKT transition into the trivial phase; (ii) the state undergoes a direct transition from the topological phase to the trivial phase at some threshold. Which scenario occurs depends on the details of the noise model.

The above discussion applies to both the decohered Laughlin states and the U(1) sector of the decohered Moore-Read state. The fate of the non-Abelian sector under generic noise is more complicated and we leave it for future study. As a first step toward a better understanding of non-Abelian topologically ordered states under generic decoherence sources, it is worth considering simpler examples, such as the chiral Ising topological order, not complicated by a global U(1) symmetry or a bosonic charge sector in the CFT. Finally, we note that the quantum coherent information, discussed throughout this work as an intrinsic measure of the transition is not a directly observable quantity. It would be interesting to construct explicit decoders that can witness the intrinsic transition or a proxy of it through the decoding fidelity.

ACKNOWLEDGMENTS

We thank Jing-Yuan Chen, Margarita Davydova, Tarun Grover, Ethan Lake, Shengqi Sang, Chong Wang, Cenke Xu, and Michael Zaletel for helpful discussions. R. F. thanks Yimu Bao and Meng Cheng for insightful discussions that partially inspired the current work. This work was supported in part by the NSF QLCI program through Grant No. OMA-2016245, by a Simons Investigator Award (E. A.) and by the Gordon and Betty Moore Foundation Grant No. GBMF8688 (R. F. and S. G.). Z. W. acknowledges the support from the Tsinghua Visiting Doctoral Students Foundation. T. W. is supported by the U.S. Department of Energy, Office of Science, Office of Basic Energy Sciences, Materials Sciences and

Engineering Division under Contract No. DE-AC02-05-CH11231 (Theory of Materials program KC2301).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX A: CHEAT SHEET ON THE FREE-BOSON AND ISING CFT

In this section, we collect basic facts about the free-boson CFT and the Coulomb-gas formalism for the Ising CFT. We refer the reader to Refs. [70,76] for details.

1. Free-boson CFT on the plane

Consider a nonchiral compact scalar boson ϕ with the following Euclidean action

$$S[\phi] = \frac{g}{4\pi} \int (\nabla\phi)^2 d^2z, \quad \phi \simeq \phi + 2\pi. \quad (A1)$$

The two-point function of ϕ (formally) reads

$$\langle \phi(z_1, \bar{z}_1) \phi(z_2, \bar{z}_2) \rangle = -\frac{\log |z_1 - z_2|}{g}. \quad (A2)$$

Another commonly used convention is to absorb the coupling constant into the radius of the boson

$$S[\phi'] = \frac{1}{8\pi} \int (\nabla\phi')^2 d^2x, \quad \phi' \simeq \phi' + 2\pi R. \quad (A3)$$

The two conventions are related by $g = R^2/2$, $\phi = \phi'/R$. Here, we work with the first one (A1).

The action manifests two U(1) symmetries that are generated by the following two conserved currents

$$J_\mu = \frac{1}{2\pi} \partial_\mu \phi, \quad \tilde{J}_\mu = \frac{1}{2\pi} \epsilon_{\mu\nu} \partial_\nu \phi. \quad (A4)$$

They generate the shift and winding of ϕ , respectively. The local operators, especially the vertex operators, can be organized by charges under the two symmetries.

Specifically, there are two basic types of vertex operators, the electric and magnetic operators. One convenient way of writing them down is to separate the nonchiral field $\phi(z, \bar{z})$ into a chiral (holomorphic) part $\varphi(z)$ and antichiral part $\bar{\varphi}(\bar{z})$

$$\phi(z, \bar{z}) \equiv \varphi(z) + \bar{\varphi}(\bar{z}), \quad (A5)$$

which have the following correlation functions

$$\begin{aligned}\langle\varphi(z_1)\varphi(z_2)\rangle &= -\frac{\log(z_1 - z_2)}{2g}, \\ \langle\bar{\varphi}(\bar{z}_1)\bar{\varphi}(\bar{z}_2)\rangle &= -\frac{\log(\bar{z}_1 - \bar{z}_2)}{2g}.\end{aligned}\quad (\text{A6})$$

The electric and magnetic operators are defined as (we suppress normal ordering in the expression)

$$\begin{aligned}V_e(z, \bar{z}) &\equiv e^{ie(\varphi(z) + \bar{\varphi}(\bar{z}))}, \\ \mathcal{O}_w(z, \bar{z}) &\equiv e^{iwg(\varphi(z) - \bar{\varphi}(\bar{z}))},\end{aligned}\quad (\text{A7})$$

where one often calls $\theta = g(\varphi - \bar{\varphi})$ the dual field. One can also define their composite, the electromagnetic operator

$$\begin{aligned}V_{e,w} &:= V_e \mathcal{O}_w := :e^{i2\sqrt{g}\alpha_{e,w}\varphi + i2\sqrt{g}\tilde{\alpha}_{e,w}\bar{\varphi}}:, \\ \alpha_{e,w} &= \frac{1}{2}\left(\frac{e}{\sqrt{g}} + \sqrt{gw}\right), \quad \tilde{\alpha}_{e,w} = \frac{1}{2}\left(\frac{e}{\sqrt{g}} - \sqrt{gw}\right),\end{aligned}\quad (\text{A8})$$

which has the conformal dimensions

$$h_{e,w} = \alpha_{e,w}^2, \quad \tilde{h}_{e,w} = \tilde{\alpha}_{e,w}^2. \quad (\text{A9})$$

The shift and winding symmetries imply that the correlation function of these operators is nonzero only when they satisfy the two neutrality conditions

$$\sum_k e_k = \sum_a w_a = 0, \quad (\text{A10})$$

and we have

$$\left\langle \prod_k V_{e_k, w_k}(z_k, \bar{z}_k) \right\rangle = \prod_{i < j} (z_i - z_j)^{2\alpha_i \alpha_j} (\bar{z}_i - \bar{z}_j)^{2\tilde{\alpha}_i \tilde{\alpha}_j}. \quad (\text{A11})$$

A charge- e electric operator encircling a charge- w magnetic operator produces a phase $2\pi ew$. More generally, the chiral and antichiral fields $\varphi, \bar{\varphi}$ circling around $V_{e,w}$ are shifted by

$$\varphi \mapsto \varphi + \pi(w + e/g), \quad \bar{\varphi} \mapsto \bar{\varphi} + \pi(w - e/g), \quad (\text{A12})$$

or, equivalently,

$$\phi \mapsto \phi + 2\pi w, \quad \theta \mapsto \theta + 2\pi e. \quad (\text{A13})$$

In a set of mutually local vertex operators, e and w must be integers. It follows that $\theta \simeq \theta + 2\pi$.

2. Coulomb-gas formalism for the Ising CFT

In this section, we briefly review the Coulomb-gas formalism for the Ising CFT. In particular, we show how to represent the thermal operator ε and spin operator σ in this formalism. There are several ways to present this formalism. Here we will not be pedantic and just pick one presentation that can give us the correct answer. We refer interested readers to [76], Chap. 9 and references therein for a more thorough discussion.

We start with the linear dilaton CFT, which has the same action Eq. (A1) but is defined for a noncompact boson and a different energy-momentum tensor. To emphasize the difference, we denote the boson field by X and the chiral energy-momentum tensor reads

$$T(z) = -\frac{1}{2}:\partial X \partial X: + i\sqrt{2}\alpha_0 \partial^2 X, \quad (\text{A14})$$

where α_0 parametrizes the CFT. It follows from the TT OPE that the central charge is $c = 1 - 24\alpha_0^2$. The vertex operators $V_\alpha = e^{i\sqrt{2}\alpha X}$ are the scalar primaries, where the noncompactness of X allows α to be any real number. From the OPE

$$T(z)V_\alpha(0) \sim \frac{\alpha^2 - 2\alpha\alpha_0}{z^2}V_\alpha(0) + \frac{\partial V_\alpha(0)}{z}, \quad (\text{A15})$$

we can read out its conformal dimension $h_\alpha = \alpha^2 - 2\alpha\alpha_0$.

The Coulomb-gas formalism is a prescription to reproduce all the minimal models from the linear dilaton CFT by choosing α_0 to match the central charge and appropriate set of vertex operators to match the operator content. To obtain the Ising CFT, we first set $\alpha_0 = \sqrt{3}/12$ to get the correct central charge $c = 1/2$. Then we select the set of vertex operators with the following α 's so that their conformal dimensions match those of primary fields in the Ising CFT (the choice is not unique):

$$\alpha_{\mathbb{I}} = 0, \quad \alpha_e = \sqrt{3}/2, \quad \alpha_\sigma = \sqrt{3}/4. \quad (\text{A16})$$

However, directly identifying the associated non-chiral vertex operators $V_\alpha(z, \bar{z})$ with the Ising primaries leads to a contradiction. For example, the two-point function of the thermal operator becomes $\langle \varepsilon(z, \bar{z})\varepsilon(0) \rangle = \langle V_{\alpha_e}(z, \bar{z})V_{\alpha_e}(0) \rangle = 0$, where the first equality is the naive identification and the second equality is the charge neutrality condition, i.e., $\langle \prod_i V_{\alpha_i}(z_i, \bar{z}_i) \rangle$ is nonzero if and only if $\sum_i \alpha_i = 0$.

The key idea of the Coulomb-gas formalism is to remedy this issue by introducing two screening operators

$$Q_\pm = \oint dw \oint d\bar{w} e^{i\sqrt{2}\alpha_\pm X(w, \bar{w})}. \quad (\text{A17})$$

They carry nonzero charges $\alpha_{\pm}$ but a zero conformal dimension, i.e., $h_{\alpha_{\pm}} = \alpha_{\pm}^2 - 2\alpha_{\pm}\alpha_0 = 1$. For the Ising CFT, we have $\alpha_+ = 2/\sqrt{3}$, $\alpha_- = -\sqrt{3}/2$. Consequently inserting these screening operators into correlation functions of vertex operators can screen their charges without affecting their conformal properties. Specifically, we define the screened vertex operator

$$V_{\alpha}^{rs}(z, \bar{z}) = Q_-^r Q_+^s V_{\alpha}(z, \bar{z}), \quad (\text{A18})$$

where the integration contours in the screening operators circle the vertex operator. These contours are implicitly radially ordered and can be freely deformed as long as they do not pass other operators. We can instead use appropriate screened vertex operators so that the total charge is neutralized and the result becomes nonzero. We identify the primaries in the Ising CFT with these screened vertex operators, where the screening charges r, s are chosen so that the correlation function obeys the charge neutrality.

For example, the correlation function of an even number of thermal operators ε 's can be written as

$$\begin{aligned} & \langle \varepsilon(z_1, \bar{z}_1) \varepsilon(z_2, \bar{z}_2) \rangle \\ &= \langle V_{\alpha_{\varepsilon}}^{20}(z_1, \bar{z}_1) V_{\alpha_{\varepsilon}}^{00}(z_2, \bar{z}_2) \rangle \\ &= \oint dw_1 d\bar{w}_1 dw_2 d\bar{w}_2 \frac{|w_1 - w_2|^3 |z_1 - z_2|^3}{\prod_{i,j}^2 |z_i - z_j|^3}, \end{aligned} \quad (\text{A19})$$

where the integration contours enclose all z_i 's. Similarly, the σ two-point function is

$$\begin{aligned} & \langle \sigma(\eta_1, \bar{\eta}_1) \sigma(\eta_2, \bar{\eta}_2) \rangle = \langle V_{\alpha_{\sigma}}^{00}(\eta_1, \bar{\eta}_1) V_{\alpha_{\sigma}}^{10}(\eta_2, \bar{\eta}_2) \rangle \\ &= \oint dw d\bar{w} \frac{|w_1 - w_2|^{3/4}}{|w_1 - w|^{3/2} |w_2 - w|^{3/2}}. \end{aligned} \quad (\text{A20})$$

We can also combine and generalize the above results to write down the correlation function of two spin operators and $2N$ thermal operators

$$\begin{aligned} \left\langle \sigma(\eta_1, \bar{\eta}_1) \sigma(\eta_2, \bar{\eta}_2) \prod_{i=1}^{2N} \varepsilon(z_i, \bar{z}_i) \right\rangle &= \oint \prod_{k=1}^{2N+1} dw_k d\bar{w}_k |\eta_1 - \eta_2|^{3/4} \prod_{i < j}^{2N} |z_i - z_j|^3 \\ &\times \prod_{k < l}^{2N+1} |w_k - w_l|^3 \prod_{i=1}^{2N} \prod_{k=1}^{2N+1} |z_i - w_k|^{-3} \prod_{k=1}^{2N+1} |\eta_1 - w_k|^{-3/2} |\eta_2 - w_k|^{-3/2} \\ &\times \prod_{i=1}^{2N} |\eta_1 - z_i|^{3/2} |\eta_2 - z_i|^{3/2}. \end{aligned} \quad (\text{A21})$$

As a final remark, we can apply the so-called Mathur's trick to further replace each pair of the contour integrals in the above expressions by a 2D integral over the complex plane [77]

$$\oint dw \oint d\bar{w} \mapsto \int d^2w. \quad (\text{A22})$$

The resulting expressions are used to derive the plasma analogy for the Moore-Read state [45].

APPENDIX B: CRITICAL FILLING AT GENERAL RÉNYI INDICES AND ITS REPLICHA LIMIT

The goals of this Appendix are twofold. We will derive the Rényi- n critical filling (29) at $n = 2, 3, \dots$ for the decohered Laughlin state. Then we show how to take the replica limit $n \rightarrow 1$ correctly, which complements the direct analysis of von Neumann quantities in Sec. III C.

The basic strategy is to perform a strong-decoherence expansion of the Rényi- n entropy

$$S^{(n)}[\rho] = \frac{1}{1-n} \log \text{tr} \rho^n, \quad (\text{B1})$$

and examine the instability. Specifically, we expand the density matrix as in Eq. (33):

$$\rho = \rho_{\infty} + y \delta \rho + y^2 \delta \rho^{(2)} + \mathcal{O}(y^3),$$

where $y = e^{-\tilde{\mu}}$, ρ_{∞} and $\delta \rho$ are defined in Eq. (33), and $\delta \rho^{(2)}$ denote terms with two particles having different bra/ket positions. Accordingly, the expansion of the Rényi- n entropy with respect to y reads

$$S^{(n)}[\rho] = S^{(n)}[\rho_{\infty}] + \delta S^{(n)}[\delta \rho] + \mathcal{O}(y^3), \quad (\text{B2})$$

where the $\mathcal{O}(y)$ term vanishes and $\delta \rho^{(2)}$ does not enter at this order. We isolate contributions to $\delta S^{(n)}$ that come from far separated unpaired particles by defining

$$\delta S_L^{(n)} = \delta S^{(n)}[\delta \rho'], \quad (\text{B3})$$

where $\delta\rho' = \sum_{[b,c]} \psi_b \psi_c^* |b\rangle\langle c|$ includes pairs of configurations described below

$$\{b = \{z_i, \eta\}, c = \{z_i, \eta'\} | |\eta - \eta'| > \alpha L\},$$

with $0 < \alpha < 1$ and L the linear system size. We determine the critical filling ν_c by examining whether $\delta S_L^{(n)}$ diverges in the thermodynamic limit $L \rightarrow +\infty$.

1. Higher integer Rényi indices

For $n = 2, 3, 4, \dots$, $\text{tr}\rho^n$ has a polynomial expansion in $\delta\rho$, which yields

$$\delta S_L^{(n)} = \frac{ny^2}{2(1-n)} \sum_{k=0}^{n-2} \frac{\text{tr}\rho_\infty^{n-k-2} \delta\rho' \rho_\infty^k \delta\rho'}{\text{tr}\rho_\infty^n}. \quad (\text{B4})$$

The denominator of the summand can be interpreted as a single-component plasma in the screening phase (for $n < 70/m$). Similarly, the numerator is related to the same type of plasma with two test charges. We have

$$\frac{\text{tr}\rho_\infty^{n-k-2} \delta\rho' \rho_\infty^k \delta\rho'}{\text{tr}\rho_\infty^n} = \int_{|\eta-\eta'|>\alpha L} \frac{d^2\eta d^2\eta'}{|\eta-\eta'|^{qq'}} \int \mathcal{D}z e^{-\Phi_L(z_i, \sqrt{2mn}; \eta, q; \eta', q')}, \quad (\text{B5})$$

where $q = \sqrt{[(2m)/n]}(n-k-1)$, $q' = \sqrt{[(2m)/n]}(k+1)$ are the charges carried by the test particles, respectively. Using the screening property, we have

$$\frac{\text{tr}\rho_\infty^{n-k-2} \delta\rho' \rho_\infty^k \delta\rho'}{\text{tr}\rho_\infty^n} \sim \int_{|\eta-\eta'|>\alpha L} \frac{d^2\eta d^2\eta'}{|\eta-\eta'|^{qq'}} \sim L^{4-\frac{2m(k+1)(n-k-1)}{n}}. \quad (\text{B6})$$

In the thermodynamic limit, the summation in (B4) is dominated by the $k=0$ and $k=n-2$ terms. We have

$$\delta S_L^{(n)} \propto L^{4-2m(n-1)/n}. \quad (\text{B7})$$

Equation (B7) diverges with the system size L when the filling factor is above the following critical value

$$\nu_c(n) = \frac{n-1}{2n}, \quad n = 2, 3, 4, \dots \quad (\text{B8})$$

2. Noninteger Rényi indices

Next we turn to the generic case including noninteger n . We will show that the result (B8) no longer holds for $1 < n < 2$. In the general case, we have to use the matrix derivative to expand $\text{tr}\rho^n$ with respect to $\delta\rho$. Given a diagonalizable matrix $A(y)$ and diagonal $A(0)_{ij} = \lambda_i \delta_{ij}$,

the derivative of an arbitrary matrix function $f[A(y)]$ is [78]

$$\begin{aligned} \left. \frac{df(A(y))}{dy} \right|_{y=0} &= \Delta f(\lambda_i, \lambda_j) A'(0)_{ij}, \\ \left. \frac{d^2 f(A(y))}{dy^2} \right|_{y=0} &= 2\Delta^2 f(\lambda_i, \lambda_j, \lambda_k) A'(0)_{ik} A'(0)_{kj} \\ &\quad + \Delta f(\lambda_i, \lambda_j) A''(0)_{ij}, \end{aligned} \quad (\text{B9})$$

where we sum over repeated indices and

$$\begin{aligned} \Delta f(\lambda_1, \lambda_2) &= \frac{f(\lambda_1) - f(\lambda_2)}{\lambda_1 - \lambda_2}, \quad \Delta f(\lambda, \lambda) = f'(\lambda), \\ \Delta^2 f(\lambda_1, \lambda_2, \lambda_3) &= \frac{\Delta f(\lambda_1, \lambda_2) - \Delta f(\lambda_1, \lambda_3)}{\lambda_2 - \lambda_3}, \end{aligned} \quad (\text{B10})$$

are the divided differences. In our case where $f(\rho) = \rho^n$, the expansion around $y=0$ is

$$\begin{aligned} \text{tr}\rho^n &= \text{tr}\rho_\infty^n + \Delta f(\lambda_a, \lambda_a)(y\delta\rho_{aa} + y^2\delta\rho_{aa}^{(2)}) \\ &\quad + y^2\Delta^2 f(\lambda_b, \lambda_b, \lambda_c)\delta\rho_{bc}\delta\rho_{cb}, \end{aligned} \quad (\text{B11})$$

where $\lambda_a = |\psi_a|^2$. The second term vanishes, and the third term can be simplified by symmetrizing b, c . We have

$$\text{tr}\rho^n = \text{tr}\rho_\infty^n + \frac{ny^2}{2} \sum_{(b,c)} \frac{\lambda_b^{n-1} - \lambda_c^{n-1}}{\lambda_b - \lambda_c} \lambda_b \lambda_c, \quad (\text{B12})$$

leading to

$$\delta S_L^{(n)} = \frac{ny^2}{2-2n} \frac{\sum_{[bc]} \lambda_b \lambda_c \frac{\lambda_b^{n-1} - \lambda_c^{n-1}}{\lambda_b - \lambda_c}}{\sum_a \lambda_a^n}. \quad (\text{B13})$$

For $n = 2, 3, 4, \dots$, we can factorize the numerator and reduce it to Eq. (B4). For noninteger n , however, the numerator of Eq. (B13) has to be analyzed by a different method. In particular, it does not have a plasma analogy. Instead, we rewrite the numerator as

$$\frac{\lambda_b \lambda_c \frac{\lambda_b^{n-1} - \lambda_c^{n-1}}{\lambda_b - \lambda_c}}{n-1} = (\lambda_b \lambda_c)^{n/2} \frac{(\lambda_b/\lambda_c)^{\frac{n-1}{2}} - (\lambda_b/\lambda_c)^{-\frac{n-1}{2}}}{(n-1)((\lambda_b/\lambda_c)^{\frac{1}{2}} - (\lambda_b/\lambda_c)^{-\frac{1}{2}})}.$$

Noticing that the second factor monotonically increases with n for $n > 1$, we can derive the following inequality

$$\frac{\lambda_b \lambda_c \frac{\lambda_b^{n-1} - \lambda_c^{n-1}}{\lambda_b - \lambda_c}}{n-1} \leq (\lambda_b \lambda_c)^{n/2}, \quad 1 < n < 2. \quad (\text{B14})$$

We can then apply the plasma analogy to the upper bound and obtain

$$|\delta S_L^{(n)}| \leq cL^{4-mn/2} \Rightarrow \nu_c(n) \geq \frac{n}{8} \quad (n < 2), \quad (\text{B15})$$

where $c > 0$ is some finite constant.

Extending Eq. (B7) to $n < 2$ clearly contradicts the bound derived above. This implies that the critical filling $\nu_c(n)$ is not an analytic function of the Rényi index n on the half-plane $\text{Re}(n) > 1$. The origin of this nonanalyticity is that ν_c is defined only in the thermodynamic limit $L \rightarrow \infty$, i.e., ν_c is extracted from the leading asymptotic behavior of $\delta S_L^{(n)}$ after discarding all subleading (finite-size) terms. If we instead compute the exact finite- L quantity $\delta S_L^{(n)}$, perform the analytic continuation in n at finite L , and only then take $L \rightarrow \infty$, we would be able to obtain the correct ν_c . Indeed, taking the limit $n \rightarrow 1$ in the exact result (B13), we obtain

$$\delta S_L = -\frac{y^2}{2} \frac{\sum_{[bc]} \frac{\lambda_b \lambda_c}{\lambda_b - \lambda_c} \log \frac{\lambda_b}{\lambda_c}}{\sum_a \lambda_a}, \quad (\text{B16})$$

which is consistent with the result (36) from direct perturbation expansion of the von Neumann entropy. In other words, the analytic continuation in n and taking the thermodynamic limit do not commute, so naive extrapolation from integer n could fail.

APPENDIX C: PERTURBATIVE CALCULATION OF QUANTUM COHERENT INFORMATION IN THE CRITICAL PHASE

For Laughlin states below the critical filling, the system enters a critical phase once the decoherence exceeds a threshold. In this Appendix, we examine the Rényi-2 quantum coherent information $I_c^{(2)}$ in this critical phase. Because the perturbation in the corresponding sine-Gordon effective theory is irrelevant, a controlled perturbative expansion applies. We will show that the quantum coherent information remains finite throughout the critical phase and vanishes only at the strong decoherence limit.

Recall that the Rényi-2 quantities can be captured by the following effective Hamiltonian

$$H_- = \int_0^{L_x} \frac{(\partial_x \theta_-)^2}{4\pi m} + \frac{m}{4\pi} (\partial_x \phi_-)^2 - 2e^{-\tilde{\mu}} \cos m\phi_-.$$

The Rényi-2 quantum coherent information reads

$$I_c^{(2)} = \log \frac{\sum_{\ell=0}^{m-1} Z_-(0, \ell)}{\sum_{\ell=0}^{m-1} Z_-(\ell, 0)},$$

where $Z_-(\ell_1, \ell_2) \approx \text{tr}_{\ell_1, \ell_2} e^{-L_y H_-}$ is the partition function of twisted boundary conditions for ϕ_- and θ_- as is specified in Eq. (61).

For notational simplicity, we suppress the subscript “−” on H, ϕ, θ , and define the aspect ratio of the torus:

$$\tau = i\tau_2, \quad \tau_2 = \frac{L_y}{L_x}.$$

We always consider the thermodynamic limit $L_x, L_y \gg \ell_B$.

The cosine term in the effective sine-Gordon field theory is irrelevant, and can be treated as a perturbation. Up to the order $\mathcal{O}(e^{-2\tilde{\mu}})$, the free energy of the twisted sector (ℓ_1, ℓ_2) changes by

$$F_{(\ell_1, \ell_2)} \approx F_{(\ell_1, \ell_2)}^{(0)} - e^{-2\tilde{\mu}} \int d^2 \mathbf{r} d^2 \mathbf{r}' \text{Re} \langle e^{im\phi(\mathbf{r})} e^{-im\phi(\mathbf{r}')} \rangle_{(\ell_1, \ell_2)}, \quad (\text{C1})$$

where $\langle \dots \rangle_{(\ell_1, \ell_2)}$ denotes the expectation value in the free-boson theory with the specified boundary condition.

We separate ϕ and θ into zero-mode and oscillator pieces

$$\begin{aligned} \phi(X) &= \phi_0 + \frac{2\pi w X}{L_x} + \phi_{\text{osc}}(X), \\ \theta(X) &= \theta_0 + \frac{2\pi e X}{L_x} + \theta_{\text{osc}}(X), \end{aligned} \quad (\text{C2})$$

where $w \in \ell_1/m + \mathbb{Z}$, $e/m \in \ell_2/m + \mathbb{Z}$, and $[\phi_0, e] = -[\theta_0, w] = i$. The corresponding zero-mode energy is $H_0(w, e) = \frac{\pi}{L_x} [mw^2 + (e^2/m)]$, so the Boltzmann weight is given by

$$P_{(\ell_1, \ell_2)}^{(w, e)} = \frac{e^{-\pi\tau_2 m(w^2 + (e/m)^2)}}{\sum_{(w', e')} e^{-\pi\tau_2 m(w'^2 + (e'/m)^2)}}. \quad (\text{C3})$$

Accordingly, the two-point function decomposes as (without loss of generality, we assume $Y > 0$):

$$\begin{aligned} &\langle e^{im\phi(X, Y)} e^{-im\phi(0, 0)} \rangle_{(\ell_1, \ell_2)} \\ &= \sum_{(w, e)} P_{(\ell_1, \ell_2)}^{(w, e)} e^{\frac{2\pi i m w Y}{L_x}} \\ &\quad \times K^{+-}(Y) \langle e^{im\phi_{\text{osc}}(X, Y)} e^{-im\phi_{\text{osc}}(0, 0)} \rangle_{\text{osc}}, \end{aligned} \quad (\text{C4})$$

where $K^{+-}(Y)$ is the zero-mode factor:

$$\begin{aligned} K_e^{+-}(Y) &:= \frac{\langle w, e | e^{-(L_y - Y)H_0} e^{+im\phi_0} e^{-YH_0} e^{-im\phi_0} | w, e \rangle}{e^{-L_y H_0(w, e)}} \\ &= \exp[-YH_0(w, e - m) + YH_0(w, e)] \\ &= \exp\left[-\frac{\pi m}{L_x} Y + \frac{2\pi e}{L_x} Y\right]. \end{aligned} \quad (\text{C5})$$

The expectation value $\langle \dots \rangle_{\text{osc}}$ in the “oscillator” part has an exact expression in terms of the elliptic theta function

$$\langle e^{im\phi_{\text{osc}}(X,Y)} e^{-im\phi_{\text{osc}}(0,0)} \rangle_{\text{osc}} = \left(\frac{a}{L_x} \right)^m \left| \frac{\partial_z \theta_1(0|\tau)}{\theta_1(z|\tau)} \right|^m, \quad (\text{C6})$$

where $z \equiv x + iy \equiv (X + iY)/L_x$ and $a \ll L_x, L_y$ is a short-distance cutoff. Plugging these results into Eq. (C1) and writing the integral in a dimensionless form, we have

$$F_{(\ell_1, \ell_2)} \approx F_{(\ell_1, \ell_2)}^{(0)} - \frac{2e^{-2\bar{\mu}} a^m L_x L_y}{L_x^{m-2}} \sum_{(w,e)} P_{(\ell_1, \ell_2)}^{(w,e)} \int_{[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{\tau_2}{2}, \frac{\tau_2}{2}]} d^2 z \Theta\left(|z| - \frac{a}{L_x}\right) \cos(2\pi m w x) e^{-\pi m |y|} \cosh(2\pi e y) \left| \frac{\partial_z \theta_1(0|\tau)}{\theta_1(z|\tau)} \right|^m,$$

where $\Theta(\cdot)$ is the Heaviside step function, imposing the short-distance cutoff. For $m > 4$, the integral is dominated by the ultra-violet divergence:

$$\begin{aligned} & \int_{[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{\tau_2}{2}, \frac{\tau_2}{2}]} d^2 z \Theta\left(|z| - \frac{a}{L_x}\right) \cos(2\pi m w x) e^{-\pi m |y|} \cosh(2\pi e y) \left| \frac{\partial_z \theta_1(0|\tau)}{\theta_1(z|\tau)} \right|^m \\ & \sim \int_{|z| > \frac{a}{L_x}} d^2 z \frac{\cos(2\pi m w x) e^{-\pi m |y|} \cosh(2\pi e y)}{|z|^m} \\ & \sim \int_{|z| > \frac{a}{L_x}} d^2 z \frac{1}{|z|^m} \left[-\frac{1}{2} (2\pi m w x)^2 + \frac{1}{2} (2\pi e y)^2 \right] + \dots \\ & \sim -\frac{2\pi^3 m^2}{m-4} \left(\frac{a}{L_x} \right)^{4-m} \left(w^2 - \frac{e^2}{m^2} \right) + \dots, \end{aligned} \quad (\text{C7})$$

where “...” denotes terms independent of w, e . The free energy of the twisted sector reads

$$F_{(\ell_1, \ell_2)} \approx F_{(\ell_1, \ell_2)}^{(0)} + C + \frac{2\pi^3 a^4 m^2 e^{-2\bar{\mu}} \tau_2}{m-4} \sum_{(w,e)} P_{(\ell_1, \ell_2)}^{(w,e)} \left(w^2 - \frac{e^2}{m^2} \right), \quad (\text{C8})$$

where C collects winding-independent corrections, while the second line gives a finite winding-dependent correction to the free energy. To perturbatively calculate the Rényi-2 quantum coherent information, it is convenient to divide both the numerator and denominator by $Z(0,0)$ and recast the expression in terms of the free-energy difference between different twisted sectors

$$I_c^{(2)} = \log \frac{1 + \sum_{\ell=1}^{m-1} e^{-\Delta F_{(0,\ell)}}}{1 + \sum_{\ell=1}^{m-1} e^{-\Delta F_{(\ell,0)}}}.$$

For free bosons ($\bar{\mu} = +\infty$), we have $\Delta F_{(\ell,0)}^{(0)} = \Delta F_{(0,\ell)}^{(0)} \equiv \Delta F_{\ell}^{(0)}$. Then from Eq. (C8), we can get

$$\begin{aligned} \Delta F_{(\ell,0)} & \approx \Delta F_{\ell}^{(0)} + e^{-2\bar{\mu}} \delta_{\ell}(\tau_2), \\ \Delta F_{(0,\ell)} & \approx \Delta F_{\ell}^{(0)} - e^{-2\bar{\mu}} \delta_{\ell}(\tau_2), \end{aligned} \quad (\text{C9})$$

to order $e^{-2\bar{\mu}}$, where

$$\begin{aligned} \delta_{\ell}(\tau_2) & = \frac{2\pi^3 a^4 m^2 \tau_2}{m-4} \left[\sum_{(w,e)} P_{(\ell,0)}^{(w,e)} \left(w^2 - \frac{e^2}{m^2} \right) \right. \\ & \quad \left. - \sum_{(w,e)} P_{(0,0)}^{(w,e)} \left(w^2 - \frac{e^2}{m^2} \right) \right]. \end{aligned} \quad (\text{C10})$$

For any finite $\bar{\mu}, \tau_2$ and $\ell \neq 0$, we have $\delta_{\ell}(\tau_2) > 0$, which leads to a finite positive coherent information $I_c^{(2)}$. In the anisotropic limit $\tau_2 \rightarrow \infty$, we have $\Delta F_{\ell,0} \propto \tau_2$ for $\ell \neq 0$, so all nontrivial twisted sectors are exponentially suppressed and $I_c^{(2)} \rightarrow 0$. Since the coherent information is invariant under a C_4 rotation $\tau_2 \rightarrow 1/\tau_2$, the same conclusion holds in the opposite limit $\tau_2 \rightarrow 0$.

APPENDIX D: PLASMA ANALOGY OF THE DECOHERED MOORE-READ STATE

Similar to the Laughlin wave function, the Moore-Read (MR) wave function and the quasihole wave functions also have a plasma analogy [45]. In this section, we briefly review the analogy and discuss subtleties of its proper statistical mechanical interpretation that is not explicitly addressed in Ref. [45]. We then apply it to the fully decohered MR state and determine its screening properties via Monte Carlo simulation.

1. Plasma analogy of the Moore-Read state

Recall the definition of the unnormalized MR wave function of N electrons

$$\Psi(\{z_i\}) = \left\langle \prod_{i=1}^N \psi(z_i) e^{im\phi(z_i)} \right\rangle_{\text{bg}}.$$

Using its explicit expression, its squared norm reads

$$\begin{aligned} \langle \Psi | \Psi \rangle &= \int \mathcal{D}z \left\langle \prod_{i=1}^N \varepsilon(z_i, \bar{z}_i) e^{im\phi(z_i)} \right\rangle_{\text{bg}} \\ &= \int \mathcal{D}z \left| \text{Pf} \left(\frac{1}{z_i - z_j} \right) \right|^2 \times e^{-\Phi_L(z_i, \sqrt{2m})}, \end{aligned} \quad (\text{D1})$$

where Φ_L is defined in Eq. (9). As is shown in Ref. [45], we can combine the Coulomb-gas formalism (reviewed in Appendix A 2) of the Ising CFT and the Mathur's trick to rewrite the squared norm of the Pfaffian as a partition function of a two-component Coulomb gas

$$\begin{aligned} \left| \text{Pf} \left(\frac{1}{z_i - z_j} \right) \right|^2 &= \int \prod_{k=1}^N d^2 w_k e^{-\Phi_I(z_i, w_k)}, \\ \Phi_I(z_i, w_k) &= -3 \sum_{i < j} (\log |z_i - z_j| + \log |w_i - w_j|) \\ &\quad + 3 \sum_{i,j} \log |z_i - w_j|. \end{aligned} \quad (\text{D2})$$

The integral on the right-hand side has an ultraviolet divergence as $z_i \rightarrow w_j$ for any z_i and w_j . We will comment on this subtlety at the end of this section. Altogether, we can write the squared norm of the MR wave function as

$$\langle \Psi | \Psi \rangle = \int \mathcal{D}z \prod_{k=1}^N d^2 w_k e^{-\Phi_L(z_i, \sqrt{2m}) - \Phi_I(z_i, w_k)}. \quad (\text{D3})$$

We call Φ_L and Φ_I the energy of the Laughlin and Ising plasma, respectively. The combined plasma has two species of mobile particles: the N electrons are mapped to N particles at $z_1, z_2, \dots, z_N$ carrying both Laughlin and Ising charges $(Q_L, Q_I) = (+\sqrt{2m}, +\sqrt{3})$. There are also N auxiliary particles at $w_1, \dots, w_N$ carrying only negative Ising charges $(Q_L, Q_I) = (0, -\sqrt{3})$. For $m < 70$, the Laughlin plasma and Ising plasma, when not coupled to each other, are both in the screening phase individually. Therefore the combined plasma is in the screening phase as well [45].

Similarly, we can derive a plasma analogy for the quasihole wave function. For our purpose, it suffices to consider the case with two quasiholes

$$\begin{aligned} \Psi_{\{\eta_1, \eta_2\}}(\{z_i\}) &= \left\langle \prod_{k=1}^2 \sigma(\eta_k) e^{i\phi(\eta_k)/2} \prod_{i=1}^N \psi(z_i) e^{im\phi(z_i)} \right\rangle_{\text{bg}}. \end{aligned} \quad (\text{D4})$$

Its squared norm reads

$$\begin{aligned} \langle \Psi_{\{\eta_1, \eta_2\}} | \Psi_{\{\eta_1, \eta_2\}} \rangle &= \int \mathcal{D}z \left\langle \sigma(\eta_1, \bar{\eta}_1) e^{i\phi(\eta_1, \bar{\eta}_1)/2} \sigma(\eta_2, \bar{\eta}_2) e^{i\phi(\eta_2, \bar{\eta}_2)/2} \right. \\ &\quad \times \left. \prod_{i=1}^N \varepsilon(z_i, \bar{z}_i) e^{im\phi(z_i)} \right\rangle_{\text{bg}}. \end{aligned} \quad (\text{D5})$$

We can use the Coulomb-gas representation (A21) and Mathur's trick to rewrite it as

$$\begin{aligned} \langle \Psi_{\{\eta_1, \eta_2\}} | \Psi_{\{\eta_1, \eta_2\}} \rangle &= \int \mathcal{D}z \prod_{k=1}^{N+1} d^2 w_k \\ &\quad \times e^{-\Phi_L(z_i, \sqrt{2m}; \{\eta_1, \eta_2\}, 1/\sqrt{2m}) - \Phi_I(z_i, w_k; \{\eta_1, \eta_2\}, \frac{\sqrt{3}}{2})}, \end{aligned} \quad (\text{D6})$$

where $\Phi_I(z_i, w_k; \{\eta_1, \eta_2\}, \sqrt{3}/2)$ is the energy of the Ising plasma with two test charges $(0, +\sqrt{3}/2)$ at η_1, η_2 , i.e.,

$$\begin{aligned} \Phi_I \left(z_i, w_k; \{\eta_1, \eta_2\}, \frac{\sqrt{3}}{2} \right) &= \Phi_I(z_i, w_k) - \frac{3}{4} \log |\eta_1 - \eta_2| \\ &\quad - \frac{3}{2} \sum_{\nu=1}^2 \left(\sum_{i=1}^N \log |z_i - \eta_\nu| - \sum_{k=1}^{N+1} \log |w_k - \eta_\nu| \right) \end{aligned} \quad (\text{D7})$$

Therefore the squared norm of the two-quasihole wave function $\langle \Psi_{\{\eta_1, \eta_2\}} | \Psi_{\{\eta_1, \eta_2\}} \rangle$ is related to the free energy cost of inserting such two test charges. By the screening property of the combined plasma, the squared norm should asymptote to a finite constant for $|\eta_1 - \eta_2| \rightarrow \infty$.

We close this section with a remark on the ultraviolet divergence in Eq. (D2). To cure the divergence, the most straightforward approach is to impose a short-distance cutoff: only configurations satisfying $|z_i - w_j| > r_0$ for all i, j are included in the integration domain. The advantage of this approach, compared with analytical continuation, is that the regulated integral has a simple stat-mech interpretation: a two-component plasma with a hard-core repulsion. However, it immediately raises a subtlety: Eq. (D2) is no longer a mathematical equality. The left-hand side of Eq. (D2) clearly does not depend on the short-distance cutoff r_0 whereas the regulated integral does. Instead, it can be shown that the regulated integral, despite having complicated dependence on r_0 , has an r_0 -independent part that is equal to the Pfaffian squared norm, in addition to an ultra-violet divergent part. Since the infrared properties of the plasma should be insensitive to

the ultraviolet divergent part, we will directly use the right-hand side of Eq. (D2) with a finite r_0 to deduce properties of the MR wave function.

2. Plasma analogy of the fully decohered Moore-Read state

It is straightforward to generalize the above idea and write down a plasma analogy for the purity of the fully decohered MR state

$$\text{tr} \rho_\infty^2 \equiv Z_{\text{MR}_\infty} = \int \mathcal{D}\xi \prod_{k=1}^N d^2 w_k^1 \prod_{l=1}^N d^2 w_l^2 \times e^{-\Phi_L(\xi_p, 2\sqrt{m}) - \Phi_I(\xi_p, w_k^1) - \Phi_I(\xi_p, w_l^2)}. \quad (\text{D8})$$

Accordingly,

$$\begin{aligned} & \langle \sigma_1 \sigma_2(\eta, \bar{\eta}) \sigma_1 \sigma_2(0) \rangle_{\text{MR}_\infty} \\ &= \frac{1}{Z_{\text{MR}_\infty}} \int \mathcal{D}\xi \prod_{k=1}^{N+1} d^2 w_k^1 \prod_{l=1}^{N+1} d^2 w_l^2 \\ & \times e^{-\Phi_L(\xi_p, 2\sqrt{m}) - \sum_{\alpha=1,2} \Phi_I(\xi_p, w_k^\alpha; \{\eta, 0\}, \frac{\sqrt{3}}{2})}, \end{aligned} \quad (\text{D9})$$

which is related to the free-energy cost of inserting test charges into the Ising plasmas. From the bosonization dictionary (84) and the $\text{KW} \times \text{KW}$ symmetry (85), Eq. (D9) is also proportional to $\langle \cos \Phi(\eta, \bar{\eta}) \cos \Phi(0) \rangle_{\text{MR}_\infty} = \langle \cos \Theta(\eta, \bar{\eta}) \cos \Theta(0) \rangle_{\text{MR}_\infty}$. In the following, we use a generalized KT-type argument as well as Monte-Carlo simulation to show that this plasma is in the screening phase, thus the free-energy costs asymptote to constants at long distances. This result verifies Eq. (91) in the main text, which is further applied to construct the effective theory for the decohered MR state away from the strong decoherence limit.

First, we note that the plasma (D8) is composed of one charge- $2\sqrt{m}$ Laughlin plasma and two charge- $\pm\sqrt{3}$ Ising plasmas. The electrons carry all three types of charges $(Q_L, Q_{I_1}, Q_{I_2}) = (+2\sqrt{m}, +\sqrt{3}, +\sqrt{3})$, and their coordinates are labeled by $\{\xi_i\}$. The two Ising-type plasmas contain additional particles, that carry charges $(0, -\sqrt{3}, 0)$ and $(0, 0, -\sqrt{3})$, with coordinates $\{w_k^1\}$ and $\{w_l^2\}$, respectively. Since all three types of plasmas are in the screening phase when treated individually (for $m < 35$), the combined plasma is expected to be in the screening phase. This can be verified using a KT-type argument. Since in the experimentally relevant region, m is much smaller than 35, the Laughlin plasma always screens. We focus on the screening property of the two Ising plasmas and omit the Q_L component of the charge for simplicity. The Ising plasmas do not screen only when all particles form neutral bound trimers, which consist of one charge- $(+\sqrt{3}, +\sqrt{3})$ particle, one charge- $(-\sqrt{3}, 0)$ particle, and

one charge- $(0, -\sqrt{3})$ particle. Now consider the perturbation that unbinds, without loss of generality, the charge- $(-\sqrt{3}, 0)$ particle from a trimer to an $O(L)$ distance. This unbinding entails a Coulomb energy cost $\Delta E \sim 3 \log L$ and entropy gain $T\Delta S \sim 4 \log L > \Delta E$. Consequently, the combined plasma has an unbinding instability and it is in the screening phase.

We also numerically confirm the screening property via a Monte Carlo simulation of the combined plasma for $m = 2$. We simulate the combined plasma on a periodic $L \times L$ square lattice with lattice constant $a = 1$. On each lattice site, there can be $n_i^{(1)}$ electrons, $n_i^{(2)}$ and $n_i^{(3)}$ the other two particles, where $n_i^{(s)} = 0, 1$ for $s = 1, 2, 3$. They carry three types of charges, labeled by $\alpha = L, I_1, I_2$. The three types of total charges on site i , labeled by $q_i^{(\alpha)}$, are

$$\begin{aligned} q_i^{(L)} &= 2\sqrt{m}n_i^{(1)}, \\ q_i^{(I_1)} &= \sqrt{3}(n_i^{(1)} - n_i^{(2)}), \quad q_i^{(I_2)} = \sqrt{3}(n_i^{(1)} - n_i^{(3)}). \end{aligned} \quad (\text{D10})$$

The Hamiltonian of the combined plasma is

$$\begin{aligned} \Phi &= \sum_{\alpha=L, I_1, I_2} \frac{1}{2} \sum_{i,j} q_i^{(\alpha)} q_j^{(\alpha)} \left((1 - \delta_{ij}) U(\mathbf{r}_i - \mathbf{r}_j) \right. \\ & \quad \left. - \frac{\ln y}{3} (1 - \delta_{\alpha,L}) \delta_{ij} \right), \end{aligned} \quad (\text{D11})$$

which contains a Coulomb interaction $U(\mathbf{r})$ and a finite fugacity $0 < y < 1$ for unpaired Ising charges. The Coulomb potential $U(\mathbf{r})$ is determined by the discretized Poisson's equation

$$\sum_{i=x,y} U(\mathbf{r} + a\hat{e}_i) + U(\mathbf{r} - a\hat{e}_i) - 2U(\mathbf{r}) = -2\pi \left(\delta_{\mathbf{r},0} - \frac{1}{L^2} \right), \quad (\text{D12})$$

The solution is

$$U(\mathbf{r}) = \frac{1}{L^2} \sum_{|\mathbf{k}| \neq 0} \frac{\pi e^{i\mathbf{k} \cdot \mathbf{r}}}{2 - \cos k_x a - \cos k_y a}, \quad (\text{D13})$$

where $k_{x,y} = 2\pi j_{x,y}/L$, $j_{x,y} = 0, 1, \dots, L-1$. In the continuum limit $1 \ll |\mathbf{r}| \ll L$, $U(\mathbf{r}) \approx -\log |\mathbf{r}|$, as expected. The fugacity corresponds to the ultraviolet regularization of the continuum integral expression that we discuss in the previous subsection. We fix the number of particles in the numerical simulation and will neglect the Coulomb potential from the background charge since it merely results in a constant shift of Φ on a periodic lattice.

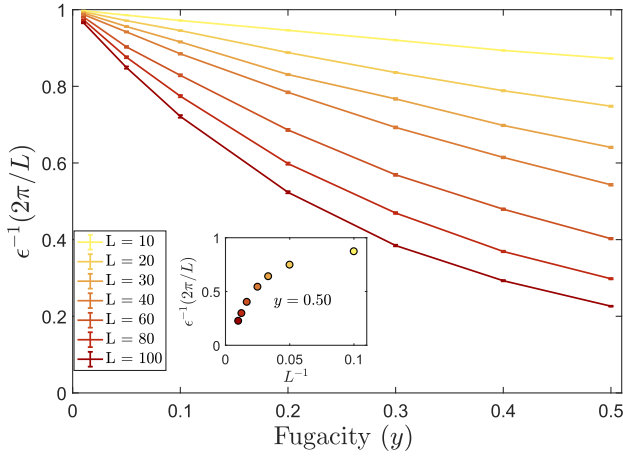


FIG. 3. Inverse dielectric constant of the Ising plasma at $|k| = 2\pi/L$. The error bars are invisible in the plot.

To determine the screening property, we calculate the inverse dielectric constant $\epsilon_0^{-1} = \lim_{|k| \rightarrow 0} \epsilon^{-1}(\mathbf{k})$ of the Ising plasma I_1 . It is finite in an insulating phase and vanishes in a screening phase. Based on the linear response theory, we have [79]

$$\epsilon^{-1}(\mathbf{k}) = 1 - \frac{2\pi}{L^2 T k^2} \langle q_{\mathbf{k}}^{(I_1)} q_{-\mathbf{k}}^{(I_1)} \rangle, \quad (\text{D14})$$

where $q_{\mathbf{k}}^{(I_1)} = \sum_{\text{sites}(x,y)} q_{x,y}^{(I_1)} e^{-ik_x x - ik_y y}$ is the Fourier transformed charge density of plasma I_1 , and $T = 1$ in this problem.

In the simulation, each particle type has a fixed number $N \equiv \sum_i n_i^{(1)} = \sum_i n_i^{(2)} = \sum_i n_i^{(3)} = L^2/10$. Initially, we generate a random configuration for each particle type. At each step, we randomly choose (with probability 1/3) one species of particle, a site (r_x, r_y) that is occupied by that species and another site (r'_x, r'_y) that is not. Then we calculate the energy difference $\Delta\Phi$ of moving the particle from (r_x, r_y) to (r'_x, r'_y) . Then we accept or reject the update according to the standard Metropolis algorithm, that is, we move the particle from (r_x, r_y) to (r'_x, r'_y) with probability $p = \min\{1, e^{-\Delta\Phi}\}$. For each parameter, we take an average over 96 independent trajectories. In each trajectory, we discard the initial $50000 \times 3N$ steps to equilibrate the system. After equilibration, we use $50000 \times 3N$ steps to evaluate the correlation function $\langle q_{\mathbf{k}}^{(I_1)} q_{-\mathbf{k}}^{(I_1)} \rangle$ in Eq. (D14). Finally, for each system size, we average over $\epsilon^{-1}(\mathbf{k})$ for $\mathbf{k} = (2\pi/L, 0)$ and $\mathbf{k} = (0, 2\pi/L)$. In Fig. 3, we show the results for the fugacity ranging from 0.01 to 0.5. Indeed, there is no sign of a phase transition (curves for different L 's should exhibit crossings if there is a BKT transition, as shown in [79]), and the inverse dielectric constant keeps decreasing with the system size, indicating that the plasma remains in the screening phase.

APPENDIX E: RÉNYI-1 CORRELATION OF THE FULLY DECOHERED MOORE-READ STATES

In this Appendix, we show how to use the plasma analogy to bound the long-distance behavior of the Rényi-1 correlation for the fully decohered MR states:

$$C_e^{(1)}(\eta, \eta') = \frac{Z_{\text{MR}\infty}^{-1}}{(N-1)!} \int \prod_i^{N-1} d^2 z_i |\Psi(\{z_i, \eta\}) \Psi(\{z_i, \eta'\})|,$$

where $\Psi(\{z_i, \eta\})$ is the MR wave function with an electron fixed at the location η . Plugging in the MR wave function on the plane, we have

$$\begin{aligned} & |\Psi(\{z_i, \eta\}) \Psi(\{z_i, \eta'\})| \\ &= |\eta - \eta'|^{-m/2} e^{-\Phi_L(z_i, \sqrt{2m}; \{\eta, \eta'\}, \sqrt{m/2})} \\ & \times \left| \text{Pf} \left(\frac{1}{z_i - z_j} \right)_{z_N=\eta} \right| \left| \text{Pf} \left(\frac{1}{z_i - z_j} \right)_{z_N=\eta'} \right|. \end{aligned} \quad (\text{E1})$$

Here, $\Phi_L(z_i, \sqrt{2m}; \{\eta, \eta'\}, \sqrt{m/2})$ is the energy of the Laughlin plasma with two test charges at η and η' , both of which carry charge $\sqrt{m/2}$. The absolute value of the Pfaffian wave function does not have a direct plasma analogy on its own. Below we find its upper and lower bounds that do admit a plasma description, and we apply it to compute the bounds explicitly.

For the upper bound, we use the inequality $2ab \leq a^2 + b^2$ to turn the absolute value into squared norms

$$\begin{aligned} & 2 \left| \text{Pf} \left(\frac{1}{z_i - z_j} \right)_{z_N=\eta} \right| \left| \text{Pf} \left(\frac{1}{z_i - z_j} \right)_{z_N=\eta'} \right| \\ & \leq \left| \text{Pf} \left(\frac{1}{z_i - z_j} \right)_{z_N=\eta} \right|^2 + \left| \text{Pf} \left(\frac{1}{z_i - z_j} \right)_{z_N=\eta'} \right|^2. \end{aligned} \quad (\text{E2})$$

From Eq. (D2), we can interpret each term on the right-hand side as a two-component plasma with a single test charge, e.g.,

$$\left| \text{Pf} \left(\frac{1}{z_i - z_j} \right)_{z_N=\eta} \right|^2 = \int \prod_{k=1}^N d^2 w_k e^{-\Phi_I(\{z_i, \eta\}, \{w_k\})}.$$

As a result, the upper bound on the Rényi-1 correlator has a plasma analogy as

$$\begin{aligned} & \frac{1}{2} \frac{1}{|\eta - \eta'|^{m/2}} \frac{Z_{\text{MR}\infty}^{-1}}{(N-1)!} \\ & \times \int \prod_{i=1}^{N-1} d^2 z_i \prod_{k=1}^N d^2 w_k e^{-\Phi_L(\{z_i\}, \sqrt{2m}; \{\eta, \eta'\}, \sqrt{m/2})} \\ & \times (e^{-\Phi_I(\{z_i, \eta\}, \{w_k\})} + e^{-\Phi_I(\{z_i, \eta'\}, \{w_k\})}). \end{aligned}$$

Using the screening property of the plasma, we have

$$C_e^{(1)}(\eta, \eta') \leq \frac{c}{|\eta - \eta'|^{m/2}}, \quad (|\eta - \eta'| \gg \xi) \quad (\text{E3})$$

for some positive constant c .

For the lower bound, we first use Eq. (D2) to get

$$\begin{aligned} & \left| \text{Pf} \left(\frac{1}{z_i - z_j} \right)_{z_N=\eta} \right| \left| \text{Pf} \left(\frac{1}{z_i - z_j} \right)_{z_N=\eta'} \right| \\ &= \prod_{i < j}^{N-1} |z_i - z_j|^3 \prod_i^{N-1} |z_i - \eta|^{3/2} |z_i - \eta'|^{3/2} \\ & \times \sqrt{\int \tilde{D}w \prod_j^N |w_j - \eta|^{-3} \int \tilde{D}w \prod_k^N |w_k - \eta'|^{-3}}, \end{aligned}$$

where we introduce the short-hand notation

$$\tilde{D}w \equiv \prod_j^N d^2 w_j \prod_{j < k}^N |w_j - w_k|^3 \prod_i^{N-1} \prod_j^N |z_i - w_j|^{-3}.$$

Next, we use the Cauchy-Schwarz inequality:

$$\begin{aligned} & \int \tilde{D}w \prod_j^N |w_j - \eta|^{-3} \int \tilde{D}w \prod_k^N |w_k - \eta'|^{-3} \\ & \geq \left(\int \tilde{D}w \prod_j^N |w_j - \eta|^{-3/2} |w_j - \eta'|^{-3/2} \right)^2, \end{aligned}$$

to obtain a plasma analogy for the lower bound

$$\begin{aligned} & \frac{1}{|\eta - \eta'|^{m/2+3/4}} \frac{Z^{-1}}{(N-1)!} \int \prod_i^{N-1} d^2 z_i \int \prod_j^N d^2 w_j \\ & \times e^{-\Phi_L(\{z_i\}, \sqrt{2m}; \{\eta, \eta'\}, \sqrt{\frac{m}{2}}) - \Phi_I(\{z_i\}, \{w_j\}; \{\eta, \eta'\}, \frac{\sqrt{3}}{2})}, \end{aligned}$$

where

$$\begin{aligned} & \Phi_I(z_i, w_k; \{\eta, \eta'\}, \sqrt{3}/2) \\ &= \Phi_I(z_i, w_k) - \frac{3}{4} \log |\eta - \eta'| \\ & - \frac{3}{2} \sum_i \log |z_i - \eta| + \frac{3}{2} \sum_k \log |w_k - \eta| \\ & - \frac{3}{2} \sum_i \log |z_i - \eta'| + \frac{3}{2} \sum_k \log |w_k - \eta'|, \end{aligned}$$

is the energy of Ising plasma with two test charges of charge $\sqrt{3}/2$ inserted at η, η' . Using the screening property of the combined plasma, we have

$$C_e^{(1)}(\eta, \eta') \geq \frac{c'}{|\eta - \eta'|^{m/2+3/4}}, \quad (|\eta - \eta'| \gg \xi) \quad (\text{E4})$$

for some positive constant c' .

APPENDIX F: NONCLASSICAL MEMORY UNDER MAXIMAL DEPHASING

In Sec. VI, we stated that although the maximally dephased state is diagonal in the density basis, the retained logical information is not purely classical. We clarify what this means in this Appendix. First, clearly all the information must be stored classically, and there is no way to reconstruct an entangled quantum state of the logical qubit and an external reference qubit by acting on the system alone. In this sense, there is no recoverable quantum information. However, it is not a purely classical memory, in the sense that the final classical state retains some residual information about the phase coherence of a given superposition state (for any chosen basis) in the code space. Below, we explain this in more detail.

Suppose that the logical code space $\mathcal{C}$ is initially maximally entangled with a reference system R ,

$$|\Psi_{\text{RQ}}\rangle = \frac{1}{\sqrt{d}} \sum_{\ell=0}^{d-1} |\Psi_{\ell}\rangle \otimes |\ell\rangle_R, \quad d = \dim(\mathcal{C}).$$

After maximal dephasing on Q , the joint state becomes

$$\begin{aligned} \rho_{\text{RQ}}^{(\infty)} &= \int \mathcal{D}z |z\rangle \langle z|_Q \otimes |\chi(z)\rangle \langle \chi(z)|_R, \\ |\chi(z)\rangle_R &= \frac{1}{\sqrt{d}} \sum_{\ell=0}^{d-1} \Psi_{\ell}(z) |\ell\rangle_R, \end{aligned} \quad (\text{F1})$$

where $z = \{z_1, \dots, z_N\}$ denotes an electron configuration. This state is separable across $R|Q$. Therefore, no operation acting on the FQH system alone can reconstruct entanglement between the logical degree of freedom and the reference. In this sense, the maximally dephased state contains no recoverable quantum information.

However, this does not mean a classical memory. Below, we provide a definition of classical memory. Let $\mathcal{C}_L$ be the logical code space at linear system size L , and let $\mathcal{N}_L$ be a noise channel. A purely classical logical memory of the full logical system is associated with a maximal commuting logical algebra

$$\mathcal{A}_{\text{cl}} = \text{span}\{\Pi_i\}_{i=0}^{d-1}, \quad \Pi_i = |\psi_i^L\rangle \langle \psi_i^L|, \quad \Pi_i \Pi_j = \delta_{ij} \Pi_i, \quad (\text{F2})$$

where $\{|\psi_i^L\rangle\}$ is an orthonormal logical basis. The decohered state retains a classical memory associated with $\mathcal{A}_{\text{cl}}$ if, for an arbitrary logical state $|\Psi^L\rangle = \sum_i c_i |\psi_i^L\rangle$, the following two conditions are satisfied. First, the classical

probabilities $|c_i|^2$ in the basis $\{|\psi_i^L\rangle\}$ can be recovered:

$$\lim_{L \rightarrow \infty} \inf_{\mathcal{R}_L} \sup_{|\Psi^L\rangle \in \mathcal{C}_L} \left\| \mathcal{R}_L \circ N_L[|\Psi^L\rangle\langle\Psi^L|] - \sum_{i=0}^{d-1} |c_i|^2 \Pi_i \right\|_1 = 0, \quad (\text{F3})$$

where $\|X\|_1 \equiv \text{tr}\sqrt{X^\dagger X}$ is the trace norm. In particular, all the basis states $|\psi_i^L\rangle$ can be recovered. Second, the information about $\arg c_i$ is lost:

$$\lim_{L \rightarrow \infty} \sup_{\substack{|\Psi^L\rangle, |\Psi'^L\rangle \in \mathcal{C}_L \\ |\langle\psi_i^L|\Psi^L\rangle|^2 = |\langle\psi_i^L|\Psi'^L\rangle|^2, \forall i}} \|N_L[|\Psi^L\rangle\langle\Psi^L|] - N_L[|\Psi'^L\rangle\langle\Psi'^L|]\|_1 = 0. \quad (\text{F4})$$

Namely, there is no residual “quantum coherence.” For example, for toric code under Z dephasing, the quantum memory is degraded to a classical memory above the error threshold $p_c \approx 0.109$, with the classical information encoded in the probability distribution in the eigenbasis of logical Z operators.

We now show that the fully dephased FQH states fail the second condition, Eq. (F4). For any logical basis $\{|\psi_i^L\rangle\}$, define the corresponding many-body wave functions

$$\psi_i^L(z) = \langle z | \psi_i^L \rangle. \quad (\text{F5})$$

Then, for any logical state $|\Psi^L\rangle = \sum_i c_i |\psi_i^L\rangle$ under maximal dephasing, we get

$$\mathcal{N}_L^\infty[|\Psi^L\rangle\langle\Psi^L|] = \int \mathcal{D}z \left(\sum_{i,j} c_i c_j^* \psi_i^L(z) \psi_j^{L*}(z) \right) |z\rangle\langle z|. \quad (\text{F6})$$

Then the second condition (F4) requires

$$P_{ij}(L) \equiv \int \mathcal{D}z |\psi_i^L(z)| \cdot |\psi_j^L(z)| \rightarrow 0 \quad (L \rightarrow \infty), \quad \forall i \neq j. \quad (\text{F7})$$

However, we show below that for $\nu = 1/m$ Laughlin states on a torus with fixed modular parameter $\tau = iL_y/L_x$, $P_{ij}(L)$ does not vanish in the thermodynamic limit. An arbitrary state ψ_i^L ($L \equiv \sqrt{L_x L_y}$) in the ground-state manifold can be written as

$$\psi_i^L(\{z_a\}) = \Psi_{\text{rel}}^L(\{\xi_a, \bar{\xi}_a\}) \exp \left[-\frac{(\text{Im } Z)^2}{2N} \right] F_{i,\text{c.m.}} \left(\frac{Z}{L_x} \right), \quad (\text{F8})$$

where

$$Z = \sum_{a=1}^N z_a, \quad \xi_a = z_a - \frac{Z}{N}, \quad (\text{F9})$$

are the c.m. and relative coordinates, respectively. The relative part Ψ_{rel}^L is common to all degenerate ground states, and the c.m. wave function $F_{i,\text{c.m.}}$ can be chosen from an m -dimensional space $\mathcal{V}_{m,\tau}$ spanned by the generalized Jacobi ϑ functions:

$$\mathcal{V}_{m,\tau} = \text{span} \left\{ \vartheta \left[\begin{matrix} \alpha + \ell/m \\ \beta \end{matrix} \right] (m\zeta | m\tau) \right\}_{\ell=0}^{m-1}, \quad (\text{F10})$$

where α, β are determined by the magnetic boundary conditions. After integrating over the relative coordinates, the overlap P_{ij} reduces to

$$P_{ij}(L) = \int_{\mathbb{T}_\tau} d^2\zeta \exp \left[-\frac{2\pi m}{\text{Im } \tau} (\text{Im } \zeta)^2 \right] \times |F_{i,\text{c.m.}}^L(\zeta)| \cdot |F_{j,\text{c.m.}}^L(\zeta)|, \quad (\text{F11})$$

where $\zeta = Z/L_x$. Here we used $L_x L_y = 2\pi m N \ell_B^2$ and imposed the normalization $\int_{\mathbb{T}_\tau} d^2\zeta \exp \{ -[(2\pi m)/(\text{Im } \tau)] (\text{Im } \zeta)^2 \} [F_{i,\text{c.m.}}^L(\zeta)]^* F_{j,\text{c.m.}}^L(\zeta) = \delta_{ij}$. Consequently, at fixed m, τ , $P_{ij}(L)$ is independent of L . Since the integrand is non-negative, and the generalized ϑ functions have isolated zeros, we have

$$\lim_{L \rightarrow \infty} \inf_{i \neq j} P_{ij}(L) > 0. \quad (\text{F12})$$

This leads to failure of the condition (F4) of a classical memory.

A similar argument applies to a generic FQH state at fractional filling $\nu = n/m$, where n and m are coprime. The many-body magnetic translations T_x and T_y generate m -dimensional c.m. multiplets [65]. If the total ground-state degeneracy exceeds m , the ground-state manifold may contain several such multiplets, distinguished by a relative or neutral-sector label. Within any fixed multiplet $\mathcal{C}_{\text{c.m.}}$, all states share a common relative-coordinate wave function and differ only in their center-of-mass dependence, so that an arbitrary state in $\mathcal{C}_{\text{c.m.}}$ retains the factorized form of Eq. (F8). See also the related discussion in Sec. V D. The above argument for Laughlin states therefore applies separately to each such c.m. logical subspace. We conclude that maximal density dephasing does not erase all relative-phase information encoded in these c.m. logical sectors. More precisely, the resulting classical electron-configuration distribution cannot depend solely on the probabilities in any single logical basis: it necessarily retains distinguishable records of superpositions that differ only by their relative phases. The residual information is therefore not equivalent to a purely classical memory.

The assumption of fixed τ is essential. For Laughlin states in the thin-torus limit, $L_x \gg \ell_B \gg L_y$, a charge density-wave order occurs [80,81], and thus under maximal dephasing a classical memory is realized in this particular case.

That a fully dephased state can have nonclassical memory might appear mysterious and related to the local indistinguishability of logical states. In the following, we provide a more mundane two-qubit example, showing that such a phenomenon is actually quite common.

We consider a two-qubit code with the following stabilizer

$$(\cos \theta Z_1 + \sin \theta X_1)(\cos \theta Z_2 + \sin \theta X_2). \quad (\text{F13})$$

The code space is two-dimensional, and a generic pure state in the code space takes the form (up to a global phase factor):

$$|\Psi_{\kappa,\alpha}\rangle = \kappa(c|0\rangle + s|1\rangle)^{\otimes_2} + \sqrt{1-\kappa^2}e^{i\alpha}(s|0\rangle - c|1\rangle)^{\otimes_2},$$

with $c \equiv \cos \frac{\theta}{2}$, $s \equiv \sin \frac{\theta}{2}$, $0 \leq \kappa \leq 1$, $0 \leq \alpha < 2\pi$.

$$(\text{F14})$$

Then we add maximal Z dephasing to both qubits, which leads to

$$\begin{aligned} \rho(\kappa, \alpha) &= \mathcal{N}^\infty[|\Psi_{\kappa,\alpha}\rangle\langle\Psi_{\kappa,\alpha}|] \\ &= |\kappa c^2 + \sqrt{1-\kappa^2}e^{i\alpha}s^2|^2|00\rangle\langle 00| \\ &\quad + |\kappa s^2 + \sqrt{1-\kappa^2}e^{i\alpha}c^2|^2|11\rangle\langle 11| \\ &\quad + c^2s^2|\kappa - \sqrt{1-\kappa^2}e^{i\alpha}|^2(|01\rangle\langle 01| + |10\rangle\langle 10|). \end{aligned} \quad (\text{F15})$$

For generic θ , the final state $\rho(\kappa, \alpha)$ lies in a $2d$ region $\mathcal{K}$ in the $4d$ space generated by the basis vectors $|00\rangle\langle 00|, |01\rangle\langle 01|, |10\rangle\langle 10|, |11\rangle\langle 11|$.

On the other hand, a classical memory is encoded in the orthogonal basis $|\psi_0\rangle, |\psi_1\rangle$, the condition (F4) is replaced by

$$\mathcal{N}^\infty[|\psi_0\rangle\langle\psi_1|] = 0. \quad (\text{F16})$$

This means the final state belongs to the following one-parameter family:

$$\rho = |c_0|^2 \mathcal{N}^\infty[|\psi_0\rangle\langle\psi_0|] + (1 - |c_0|^2) \mathcal{N}^\infty[|\psi_1\rangle\langle\psi_1|]. \quad (\text{F17})$$

However, this is only true in the special cases $\theta = n\pi/2, n = 0, 1, 2, 3$, where $\mathcal{K}$ collapses to a 1D line. Therefore, for other values of θ , this two-qubit code realizes a nonclassical memory under maximal Z dephasing.

- [1] A. Kitaev, *Unpaired Majorana fermions in quantum wires*, *Phys. Usp.* **44**, 131 (2001).
- [2] A. Kitaev, *Anyons in an exactly solved model and beyond*, *Ann. Phys. (Amsterdam)* **321**, 2 (2006).
- [3] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, *Non-Abelian anyons and topological quantum computation*, *Rev. Mod. Phys.* **80**, 1083 (2008).
- [4] J. Alicea, *New directions in the pursuit of Majorana fermions in solid state systems*, *Rep. Prog. Phys.* **75**, 076501 (2012).
- [5] T. Karzig *et al.*, *Scalable designs for quasiparticle-poisoning-protected topological quantum computation with Majorana zero modes*, *Phys. Rev. B* **95**, 235305 (2017).
- [6] K. Klocke, Y. Liu, G. B. Halász, and J. Alicea, *Spin-liquid-based topological qubits*, [arXiv:2411.08093](https://arxiv.org/abs/2411.08093).
- [7] R. E. Prange, S. M. Girvin *et al.*, *The Quantum Hall Effect* (Springer US, New York, 1989).
- [8] B. I. Halperin and J. K. Jain, *Fractional Quantum Hall Effects: New Developments* (World Scientific, Singapore, 2020).
- [9] S. Das Sarma, M. Freedman, and C. Nayak, *Topologically protected qubits from a possible Non-Abelian fractional quantum Hall state*, *Phys. Rev. Lett.* **94**, 166802 (2005).
- [10] M. Freedman, C. Nayak, and K. Walker, *Towards universal topological quantum computation in the $\nu = (5)/(2)$ fractional quantum Hall state*, *Phys. Rev. B* **73**, 245307 (2006).
- [11] S. Bravyi, *Universal quantum computation with the $\nu = 5/2$ fractional quantum Hall state*, *Phys. Rev. A* **73**, 042313 (2006).
- [12] A. Assouline, T. Wang, H. Zhou, L. A. Cohen, F. Yang, R. Zhang, T. Taniguchi, K. Watanabe, Roger S. K. Mong, M. P. Zaletel, and A. F. Young, *Energy gap of the Even-Denominator fractional quantum Hall state in bilayer graphene*, *Phys. Rev. Lett.* **132**, 046603 (2024).
- [13] Y. Hu, Y.-C. Tsui, M. He, U. Kamber, T. Wang, A. S. Mohammadi, K. Watanabe, T. Taniguchi, Z. Papic, M. P. Zaletel, and A. Yazdani, *High-Resolution tunneling spectroscopy of fractional quantum Hall states*, *Nat. Phys.* **21**, 716 (2025).
- [14] N. H. Lindner, E. Berg, G. Refael, and A. Stern, *Fractionalizing Majorana fermions: Non-Abelian statistics on the edges of Abelian quantum Hall states*, *Phys. Rev. X* **2**, 041002 (2012).
- [15] M. Barkeshli, *Charge $2e/3$ superconductivity and topological degeneracies without localized zero modes in bilayer fractional quantum Hall states*, *Phys. Rev. Lett.* **117**, 096803 (2016).
- [16] R. Wen and A. C. Potter, *Cheshire qudits from fractional quantum spin Hall states in twisted MoTe₂*, [arXiv preprint arXiv:2407.03401](https://arxiv.org/abs/2407.03401) (2024).
- [17] R. Fan, Y. Bao, E. Altman, and A. Vishwanath, *Diagnostics of mixed-state topological order and breakdown of quantum memory*, *PRX Quantum* **5**, 020343 (2024).
- [18] Y. Bao, R. Fan, A. Vishwanath, and E. Altman, *Mixed-state topological order and the errorfield double formulation of decoherence-induced transitions*, *Phys. Rev. Lett.* **136**, 220402 (2026).
- [19] J. Y. Lee, C.-M. Jian, and C. Xu, *Quantum criticality under decoherence or weak measurement*, *PRX Quantum* **4**, 030317 (2023).

- [20] Z. Wang, Z. Wu, and Z. Wang, *Intrinsic mixed-state topological order*, *PRX Quantum* **6**, 010314 (2025).
- [21] R. Sohal and A. Prem, *Noisy approach to intrinsically mixed-state topological order*, *PRX Quantum* **6**, 010313 (2025).
- [22] T. D. Ellison and M. Cheng, *Toward a classification of mixed-state topological orders in two dimensions*, *PRX Quantum* **6**, 010315 (2025).
- [23] S. Sang, Y. Zou, and T. H. Hsieh, *Mixed-state quantum phases: Renormalization and quantum error correction*, *Phys. Rev. X* **14**, 031044 (2024).
- [24] S. Sang and T. H. Hsieh, *Stability of mixed-state quantum phases via finite Markov length*, *Phys. Rev. Lett.* **134**, 070403 (2025).
- [25] S. Sang, L. A. Lessa, R. S. Mong, T. Grover, C. Wang, and T. H. Hsieh, *Mixed-state phases from local reversibility*, *arXiv:2507.02292* [Phys. Rev. X (to be published)].
- [26] T.-H. Yang, B. Shi, and J. Y. Lee, *Topological mixed states: Axiomatic approaches and phases of matter*, *arXiv:2506.04221* [Phys. Rev. X (to be published)].
- [27] C. Zhang, Y. Xu, J.-H. Zhang, C. Xu, Z. Bi, and Z.-X. Luo, *Strong-to-weak spontaneous breaking of 1-form symmetry and intrinsically mixed topological order*, *Phys. Rev. B* **111**, 115137 (2025).
- [28] L. A. Lessa, S. Sang, T.-C. Lu, T. H. Hsieh, and C. Wang, *Higher-form anomaly and long-range entanglement of mixed states*, *arXiv:2503.12792*.
- [29] Y.-H. Chen and T. Grover, *Unconventional topological mixed-state transition and critical phase induced by self-dual coherent errors*, *arXiv:2403.06553*.
- [30] P. Sala, J. Alicea, and R. Verresen, *Decoherence and wavefunction deformation of D_4 non-Abelian topological order*, *Phys. Rev. X* **15**, 031002 (2025).
- [31] P. Sala and R. Verresen, *Stability and loop models from decohering non-Abelian topological order*, *Phys. Rev. Lett.* **134**, 250403 (2025).
- [32] T. Yoshida, K. Kudo, H. Katsura, and Y. Hatsugai, *Fate of fractional quantum Hall states in open quantum systems: Characterization of correlated topological states for the full Liouvillian*, *Phys. Rev. Res.* **2**, 033428 (2020).
- [33] Z. Li, D. Lee, and B. Yoshida, *How much entanglement is needed for topological codes and mixed states with anomalous symmetry?*, *Phys. Rev. X* **15**, 021090 (2025).
- [34] Y. Tang and Y. Bao, *Phases of Floquet code under local decoherence*, *Phys. Rev. A* **112**, 062437 (2025).
- [35] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, *Topological quantum memory*, *J. Math. Phys. (N.Y.)* **43**, 4452 (2002).
- [36] D. Gottesman, *An introduction to quantum error correction and fault-tolerant quantum computation*, *Proc. Symp. Appl. Math.* **68**, 13 (2010).
- [37] B. M. Terhal, *Quantum error correction for quantum memories*, *Rev. Mod. Phys.* **87**, 307 (2015).
- [38] K. Fujii, *Quantum computation with topological codes: From qubit to topological fault-tolerance*, *arXiv:1504.01444*.
- [39] B. Schumacher and M. A. Nielsen, *Quantum data processing and error correction*, *Phys. Rev. A* **54**, 2629 (1996).
- [40] B. Schumacher and M. D. Westmoreland, *Approximate quantum error correction*, *Quantum Inf. Process.* **1**, 5 (2002).
- [41] It is natural to expect that the universal physics is insensitive to the precise form of the regularization. One specific and possibly physical choice is to use the lowest-Landau-Level projected density operator in Eq. (1) and the short-distance length scale is the magnetic length. We will comment on its effect at the end of the manuscript.
- [42] R. B. Laughlin, *Anomalous quantum Hall effect: An incompressible quantum fluid with fractionally charged excitations*, *Phys. Rev. Lett.* **50**, 1395 (1983).
- [43] G. Moore and N. Read, *Nonabelions in the fractional quantum Hall effect*, *Nucl. Phys.* **B360**, 362 (1991).
- [44] N. Read, *Non-Abelian adiabatic statistics and hall viscosity in quantum Hall states and $p_x + ip_y$ paired superfluids*, *Phys. Rev. B* **79**, 045308 (2009).
- [45] P. Bonderson, V. Gurarie, and C. Nayak, *Plasma analogy and non-Abelian statistics for Ising-type quantum Hall states*, *Phys. Rev. B* **83**, 075303 (2011).
- [46] V. Berezinskii, *Destruction of long-range order in one-dimensional and two-dimensional systems possessing a continuous symmetry group. II. Quantum systems*, *Sov. Phys. JETP* **34**, 610 (1972).
- [47] J. M. Kosterlitz and D. J. Thouless, *Ordering, metastability and phase transitions in two-dimensional systems*, *J. Phys. C* **6**, 1181 (1973).
- [48] J. M. Kosterlitz, *The critical properties of the two-dimensional xy model*, *J. Phys. C* **7**, 1046 (1974).
- [49] J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, *Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model*, *Phys. Rev. B* **16**, 1217 (1977).
- [50] J. M. Caillol, D. Levesque, J. J. Weis, and J. P. Hansen, *A Monte Carlo study of the classical two-dimensional one-component plasma*, *J. Stat. Phys.* **28**, 325 (1982).
- [51] Equation (14) implicitly depends on the short-distance cutoff a . First, it holds only when the distances between these pairs are larger than a . Second, we omit a prefactor $(\kappa a^2)^N$ that accounts for the residual relative motion of each pair. Namely, the two particles in each pair can still move relative to each other within a disk of radius a . And κ is a nonuniversal geometric factor.
- [52] The fugacity comes from the unbinding energy as well as the entropic effect analyzed above, i.e., $e^{-\beta} \sim e^{-\mu} \times \sqrt{N/Aa^2}$. We also omitted an $O(1)$ prefactor that depends on the details of the screening due to the charge- $2\sqrt{m}$ plasma and the short-distance cutoff.
- [53] R. Ma, J.-H. Zhang, Z. Bi, M. Cheng, and C. Wang, *Topological phases with average symmetries: The decohered, the disordered, and the intrinsic*, *Phys. Rev. X* **15**, 021062 (2025).
- [54] L. A. Lessa, R. Ma, J.-H. Zhang, Z. Bi, M. Cheng, and C. Wang, *Strong-to-weak spontaneous symmetry breaking in mixed quantum states*, *PRX Quantum* **6**, 010344 (2025).
- [55] P. Sala, S. Gopalakrishnan, M. Oshikawa, and Y. You, *Spontaneous strong symmetry breaking in open systems: Purification perspective*, *Phys. Rev. B* **110**, 155150 (2024).
- [56] A. Sarma, Y. Bao, N. Myerson-Jain, T. Kiely, and C. Xu, *Effective conformal field theory generated from pure and dephased Chern insulator*, *Phys. Rev. B* **112**, 085130 (2025).

- [57] S. M. Girvin and A. H. MacDonald, *Off-diagonal long-range order, oblique confinement, and the fractional quantum Hall effect*, *Phys. Rev. Lett.* **58**, 1252 (1987).
- [58] S. C. Zhang, T. H. Hansson, and S. Kivelson, *Effective-Field-theory model for the fractional quantum Hall effect*, *Phys. Rev. Lett.* **62**, 82 (1989).
- [59] T.-C. Lu, Z. Zhang, S. Vijay, and T. H. Hsieh, *Mixed-state long-range order and criticality from measurement and feedback*, *PRX Quantum* **4**, 030318 (2023).
- [60] Z. Liu, L. Chen, Y. Zhang, S. Zhou, and P. Zhang, *Diagnosing strong-to-weak symmetry breaking via wightman correlators*, [arXiv:2410.09327](#).
- [61] Z. Weinstein, *Efficient detection of strong-to-weak spontaneous symmetry breaking via the Rényi-1 correlator*, *Phys. Rev. Lett.* **134**, 150405 (2025).
- [62] Coarse grained because the noise inevitably has a short-range cutoff $a \ll \ell_B$.
- [63] To understand this equality, we first note $\sum_{(b,c)} |\psi_b|^2 + |\psi_c|^2 = 2 \sum_{(b,c)} |\psi_b|^2$. We then compute $\sum_{(b,c)} |\psi_b|^2$ by counting electron configurations c that differ from a given configuration b by exactly one electron position. There are N choices for which electron is moved, and the allowed area available to that electron is $c_0 L^2$ with a nonuniversal factor $c_0 < 1$.
- [64] Any separable state across R and Q can be written as a convex sum $\rho_{QR} = \sum_k p_k |Q_k\rangle\langle Q_k| \otimes |R_k\rangle\langle R_k|$, $p_k > 0$, where $|Q_k\rangle$ and $|R_k\rangle$ are sets of generically nonorthogonal but normalized pure states on Q and R . It is then easy to see that $\text{tr} \rho_{QR}^2 \leq \text{tr} \rho_Q^2$ and $I_c^{(2)}$ cannot be positive.
- [65] F. D. M. Haldane, *Many-particle translational symmetries of two-dimensional electrons at rational Landau-level filling*, *Phys. Rev. Lett.* **55**, 2095 (1985).
- [66] A. Belavin, A. Polyakov, and A. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, *Nucl. Phys.* **B241**, 333 (1984).
- [67] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, *Generalized global symmetries*, *J. High Energy Phys.* **02** (2015) 172.
- [68] C.-M. Chang, Y.-H. Lin, S.-H. Shao, Y. Wang, and X. Yin, *Topological defect lines and renormalization group flows in two dimensions*, *J. High Energy Phys.* **01** (2019) 026.
- [69] Specifically, the symmetry is generated by the following line operators
- where C is a loop and $\mathbf{A}$ is the vector potential for the background magnetic field. They are related to the worldline of charge- e/m Laughlin quasiholes. The partition function $Z_{\text{MR},\infty}$ with $U_i(C)$ inserted is invariant under continuous deformation of C , i.e., the line operators (78) are topological, ensuring the symmetry.
- [70] P. H. Ginsparg, *Applied conformal field theory*, in *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena* (1988), [arXiv:hep-th/9108028](#).
- [71] The $U(1)_m$ and $U(1)_w$ symmetries have a mixed anomaly. Therefore, the coexistence of Φ and Θ condensates is also a coexistence of the ordered and disordered phase for each of the $U(1)$ symmetries, which typically happen at first order transitions. The current situation is even more special as the Mermin-Wagner theorem forbids continuous symmetry breaking in usual 2D theories. Our theory, not described by a local Lagrangian, can evade this no-go constraint.
- [72] N. Read and D. Green, *Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect*, *Phys. Rev. B* **61**, 10267 (2000).
- [73] S. B. Chung and M. Stone, *Explicit monodromy of Moore-Read wavefunctions on a torus*, *J. Phys. A* **40**, 4923 (2007).
- [74] K. Su, Z. Yang, and C.-M. Jian, *Tapestry of dualities in decohered quantum error correction codes*, *Phys. Rev. B* **110**, 085158 (2024).
- [75] V. Temkin, Z. Weinstein, R. Fan, D. Podolsky, and E. Altman, *Charge-informed quantum error correction*, [arXiv:2512.22119](#).
- [76] P. Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer Science & Business Media, New York, 2012).
- [77] S. D. Mathur, *Quantum Kac-Moody symmetry in integrable field theories*, *Nucl. Phys.* **B369**, 433 (1992).
- [78] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis* (Cambridge University Press, Cambridge, England, 1994).
- [79] J.-R. Lee and S. Teitel, *Phase transitions in classical two-dimensional lattice Coulomb gases*, *Phys. Rev. B* **46**, 3247 (1992).
- [80] R. Tao and D. J. Thouless, *Fractional quantization of Hall conductance*, *Phys. Rev. B* **28**, 1142(R) (1983).
- [81] E. J. Bergholtz and A. Karlhede, *Quantum Hall system in Tao-Thouless limit*, *Phys. Rev. B* **77**, 155308 (2008).

$$U_i(C) = \exp\left(i \frac{k_i}{m} \oint_C (\nabla \theta_i + e \mathbf{A}) \cdot d\ell\right), \quad (78)$$