

Středa Formula for Floquet Systems: Topological Invariants and Quantized Anomalies from Cesàro Summation

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The Středa formula establishes a fundamental physical connection between the topological invariants characterizing the bulk of topological matter and the presence of gapless edge modes. In this work, we extend the Středa formula to periodically driven systems, providing a rigorous framework to elucidate the unconventional bulk-boundary correspondence of Floquet systems while offering a physically grounded link between Floquet winding numbers and tractable response functions. Using the Sambe representation of periodically driven systems, we analyze the response of the unbounded Floquet density of states to a magnetic perturbation. This Floquet-Středa response is regularized through Cesàro summation, yielding a well-defined, quantized result within spectral gaps. The response features two physically distinct contributions: a quantized charge flow between edge and bulk and an anomalous energy flow between the system and the drive, offering new insight into the nature of anomalous edge states. This fundamental result rigorously connects Floquet winding numbers to the orbital magnetization density of Floquet states and holds broadly, from clean to disordered and inhomogeneous systems. This is further supported by providing a real-space formulation of the Floquet-Středa response, which introduces a local topological marker suited for periodically driven settings. In translationally invariant systems, the framework yields a remarkably simple expression for Floquet winding numbers involving geometric properties of Floquet-Bloch bands. A concrete experimental protocol is proposed to extract the Floquet-Středa response via particle-density measurements in systems coupled to engineered baths. Finally, by expressing the topological invariants through the magnetic response of the Floquet density of states, this approach opens a promising route toward the topological characterization of interacting driven phases.

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I. INTRODUCTION

Topology has led to an exceptionally fruitful cross-fertilization between different areas of research, establishing a bridge between concepts of mathematics, quantum field theory, and condensed matter physics. Recent developments in topological quantum matter have not only led to a deeper understanding of topological phenomena in

thermodynamic equilibrium [1–6], but also initiated the ambitious task of generalizing these concepts to the realm of out-of-equilibrium settings [7,8].

Subjecting a system to a time-periodic modulation, also known as Floquet engineering, has emerged as one of the most promising research fronts to address this challenge [9–17]. Formally, periodically driven systems can be described by a time-independent Hamiltonian, at the expense of working with an infinite-dimensional space [18,19]. The spectrum of this unbounded Hamiltonian operator defines the *quasienergies* of the system, an infinite set of eigenvalues that are repeated periodically according to the driving frequency, and can, thus, be organized within Floquet-Brillouin zones. In a lattice setting, the quasienergies are classified into Floquet-Bloch bands, whose

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geometric and topological properties can be precisely tailored by adjusting the driving protocol, potentially enabling the on-demand realization of topological phases [9–11,20–23]. Signatures of both Floquet physics and driven-induced topology have been successfully detected in a variety of experimental platforms, ranging from solid state [24–30], cold atoms in optical lattices [31–42], and acoustic settings [43–45] to photonics [46–56].

The topological invariants that are commonly used to characterize bulk bands (e.g., Chern numbers [57]) have proven to be insufficient to capture the topological properties of Floquet systems [9,23]. In particular, the bulk-boundary correspondence, which establishes the edge-state structure in systems with boundaries [1,2], has to be treated with care: In general, the net number of edge modes that are located within a quasienergy gap is dictated by a topological invariant that takes the time-dependent nature of the Hamiltonian into account [10,11,15,16,23,58–61]. The periodicity of the Floquet spectrum with respect to the driving frequency is central to this peculiarity, as it can result in drive-induced boundary modes that are located at the edges of the Floquet-Brillouin zone; see Fig. 1(c). These exotic edge modes, which have no analog in static systems, have been coined *anomalous edge states* [23]. The bulk-boundary correspondence of two-dimensional Floquet systems, which accurately dictates the total number of edge modes (both regular and anomalous), has been established through the introduction of a spatiotemporal winding number built from the full time-evolution operator [15,23,59,60]. The mathematical complexity of this topological index, combined with the difficulty of preparing a steady-state thermal-like occupation of the Floquet bands [62–66], has hindered the identification of practical methods to probe this invariant and its physical consequences in generic settings. A tomography scheme, requiring the tracking of band-touching points and their corresponding topological charges in momentum space [67], was recently implemented in an optical-lattice setup [39] to deduce these intriguing topological numbers. Nonetheless, the measurement of a robustly quantized observable that is directly dictated by the spatiotemporal winding number of Ref. [23] has remained elusive.

Various strategies have been theoretically proposed in a relentless effort to solve this puzzle. Among these, transport probes have emerged as the most intuitive approach. Current proposals in this direction focus on detecting topological signatures that either arise from the driven-induced Floquet edge channels [68–76] or from the bulk response of these out-of-equilibrium systems [21,77–79]. Approaches beyond transport are currently restricted to a very specific class of Floquet systems, where all the bulk states are strongly localized by disorder [80,81]. These phases are predicted to exhibit a quantized bulk orbital magnetization density reflecting the number of anomalous edge channels [82–84].

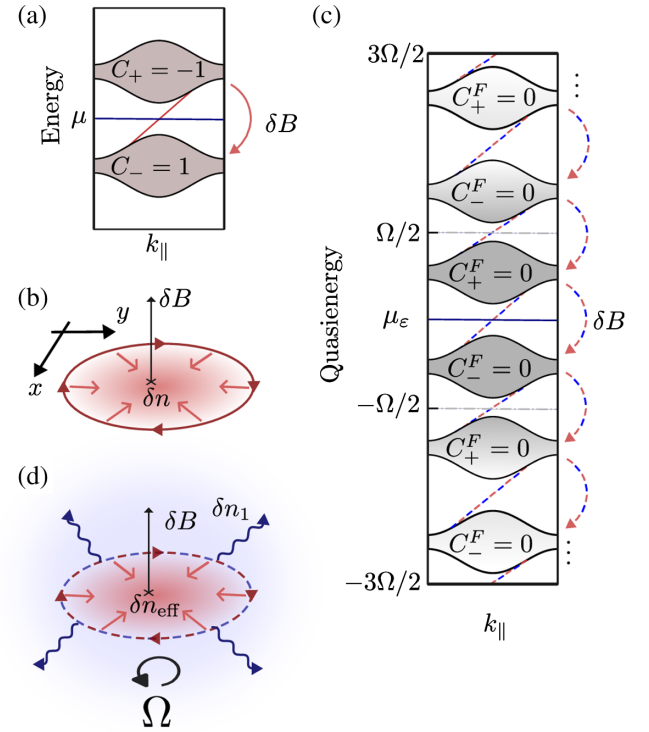


FIG. 1. (a) Sketch of the energy spectrum of a Chern insulator, defined on a semi-infinite ribbon geometry, where $k_{\parallel}$ denotes the quasimomentum along the direction of the ribbon. A chiral edge mode bridges the gap between the two bands, whose Chern numbers are $C_{\pm} = \mp 1$. Perturbing the system with a magnetic field δB generates a spectral flow between the upper and lower bands, physically carried by the chiral channel. (b) Redistribution of the particle density (color gradient) of an open-boundary system, filled up to the chemical potential μ [blue line in (a)], when subjected to the magnetic perturbation. The arrows depict the flow of states from the edge to the bulk, where the density increases according to $\delta n = C \delta B / \Phi_0$. (c) Quasienergy spectrum of an unbounded Floquet Hamiltonian, describing a periodically driven lattice at frequency Ω . Even though all the Chern indices are equal to zero $C_{\pm}^F = 0$, the system is topologically nontrivial, with chiral edge channels bridging all the gaps. The arrows depict the spectral flow between Floquet bands under the δB field insertion. (d) Sketch of the two contributions to the Floquet spectral flow: the normal flow δn_{eff} , dictated by the Chern number of the Floquet bands and describing the flow of states between the edge and the bulk, and the “anomalous” flow δn_1 , encoding the quantized energy flow between the system and the driving field. In the two-band model in (c), the normal flow $\delta n_{\text{eff}} = 0$ because of the trivial Chern numbers.

A. Theoretical approach and main results

This work introduces a physical framework that sheds light on the bulk-boundary correspondence of two-dimensional periodically driven systems. As a key outcome, it connects the abstract winding numbers used to classify Floquet topological phases to physically tractable response functions. Our construction is entirely based on the elementary notion of spectral flow [85], which lies at

the heart of the Středa formula introduced in the context of equilibrium systems [86–89].

The Středa formula is a remarkable thermodynamic relation that links the Hall conductivity of insulating states of matter to the magnetic response of their particle number $N(\mu)$ at a fixed chemical potential μ :

$$\frac{\sigma_H}{\sigma_0} = \Phi_0 \frac{\partial N(\mu)}{\partial \Phi}, \quad (1)$$

where $\sigma_0 = e^2/h$ is the conductance quantum, $\Phi_0 = hc/e$ the normal flux quantum, and $\Phi = BA$ the total flux through a system of area A threaded by a constant magnetic field B . Derived from nonperturbative thermodynamic arguments [87,90], the Středa relation has proven to be an extremely powerful tool to identify quantized bulk responses in a variety of different contexts [41,91–97]. The guiding principle behind its validity is sketched in Figs. 1(a) and 1(b): Piercing a two-dimensional Chern insulator with a perturbing magnetic field induces a chiral spectral flow of states between Bloch bands of different Chern number. In an open-boundary sample, this flow is physically carried by the one-way edge channels bridging the gap between these bands, as shown in the schematic picture in Fig. 1(a). If such a lattice model is filled with particles up to the chemical potential μ , a net flow of states will occur between the edge and the bulk upon activating the magnetic perturbation, leading to a redistribution of the particle density in the sample, as illustrated in Fig. 1(b). In the context of particle physics, this phenomenon is known as *anomaly inflow* or Callan-Harvey mechanism [98–100]. The particle flow into the insulating bulk, as dictated by Eq. (1), is quantized according to the net number of chiral edge modes crossing the chemical potential. The quantized Hall conductivity in Eq. (1) being dictated by the Chern number of the filled Bloch bands [57], one naturally recovers the bulk-boundary correspondence through the Středa relation.

These arguments can be readily extended to two-dimensional Floquet systems, by making use of the aforementioned mapping between periodically driven systems and their corresponding (unbounded) time-independent Hamiltonian [18,19]: the so-called Sambe representation. At this level, the only crucial difference with the spectral flow that occurs at equilibrium in response to a magnetic perturbation is that it now takes place in an *unbounded spectrum*, consisting of an infinite number of Floquet-Bloch bands. As illustrated in Fig. 1(c), the net number of driven-induced chiral edge channels that are located at quasienergy μ_ϵ can be determined by evaluating the number of states that have crossed this quasienergy via a “bucket brigade mechanism.” In this work, we demonstrate that this spectral flow, or Floquet-Středa response, can be formally decomposed as a sum of two contributions: the *normal* and *anomalous* flows. The normal flow is equivalent to the

Středa response exhibited by undriven systems; as previously mentioned, it represents the quantized flow of charge between the edge and the bulk of the system. This flow can be entirely deduced from the Chern numbers of the Floquet-Bloch bands. In the special case represented in Fig. 1(c), all these Chern numbers are zero, such that the normal contribution vanishes. In contrast, the anomalous flow encodes the infinite summation that is required to accurately evaluate the net spectral flow at a given quasienergy. As illustrated in Fig. 1(d), we associate this striking phenomenon, inherent to out-of-equilibrium Floquet systems, to the emergence of a quantized energy flow between the system and the driving field, exclusively induced by the magnetic perturbation. Interestingly, this picture connects to recent developments [101,102], which suggest that anomalous topological phases could find their origin in such energy flows. We also note that quantized energy pumps were recently explored in zero- and one-dimensional Floquet systems [103–107].

The core of our approach relies on the regularization of the Floquet-Středa response defined in Sambe space, which turns out to be a mathematically ill-defined quantity. Notably, the anomalous flow can be identified as an integral version of a nonconvergent Grandi-type series, which we regularize by means of a Cesàro summation method [108,109]. This procedure leads to a series of key results, which we now list and summarize below.

1. Classification of Floquet topological phases from their Floquet-Středa response

We show that the regularized Floquet spectral flow can be entirely obtained from the magnetic response of the Floquet density of states within a given Floquet zone [see Eq. (47)]. This fundamental result not only pinpoints this quantity as a relevant physical observable to extract topological quantized responses, it also highlights that the classification of Floquet topological phases does not require the knowledge of the full time dynamics within a driving cycle, also known as micromotion, as long as the stroboscopic time evolution is known in the presence of a perturbing magnetic field.

2. Bulk-boundary correspondence from the bulk properties of the Floquet-Bloch states

Our theory expresses the topological winding number of noninteracting periodically driven lattice systems in terms of simple band properties, all deriving from the Floquet-Bloch Hamiltonian expressed in the Sambe representation. Indeed, we find a closed analytical and gauge-invariant formula for these topological indices, which is entirely expressed in terms of the quasienergies, Berry curvatures, and intrinsic orbital magnetic moments of the Floquet-Bloch states defined in a given Floquet zone [see Eq. (53)].

3. Winding numbers in terms of the orbital magnetization of Floquet states

We derive an explicit relation between the Floquet-Středa response and the orbital magnetization density associated to the bulk Floquet states [see Eqs. (62) and (63)]. Remarkably, we find that the quantization of the anomalous spectral flow reflects the quantization of the total orbital magnetization density of the Floquet states within a Floquet zone. This key finding extends the result of Ref. [82] beyond Floquet systems with fully localized bulk states, demonstrating its general applicability as long as the Floquet bulk gap at the zone edge remains open. In the case of translationally invariant lattices, we demonstrate that the orbital magnetization density is strictly quantized upon fully occupying the Bloch-Floquet bands, in sharp contradiction with the results reported in Refs. [110,111].

4. Generalized Floquet-Středa formula: Quantized charge and energy pump

The anomalous flow is explicitly identified as the derivative of a first-order winding number $N_1[R]$ with respect to a magnetic field [see Eqs. (72) and (77)]. This mathematical observation allows us to draw a concrete parallelism between the Floquet-Středa response stemming from our approach and the original Středa formula in Eq. (1). The generalized Floquet-Středa formula [see Eq. (79)] contains two physically distinguishable contributions [see Fig. 1(d)]: the normal-flow contribution, which is captured by the Chern numbers of all the Floquet-Bloch bands that are located between the bottom edge of the Floquet-Brillouin zone and the gap at quasienergy μ_c . This term is physically interpreted as the flow of particles (dressed by the driving field) between the edge and the bulk of the system. In the undriven limit, this contribution reduces to Eq. (1). The second contribution corresponds to the anomalous flow $\Phi_0 \partial N_1[R] / \partial \Phi$, which diagnoses the emergence of a quantized energy pump between the system and the driving field, an effect which is inextricably linked to the existence of resonant processes.

5. A Floquet-Středa sum rule

We elucidate a protocol to extract Floquet topological invariants from particle-density-response measurements, in an experimentally realistic setting that involves an engineered heat bath. This proposal is based on a sum-rule scheme [see Eq. (89)], which can be seen as the counterpart of the Floquet sum rule that was previously introduced in the context of edge transport [69,73,74].

6. Real-space markers for Floquet winding numbers

Our formalism naturally leads to the definition of local Floquet winding numbers, which can be used to identify topological phases in different regions of space, e.g., in inhomogeneous or disordered systems. Specifically, this is

achieved by introducing real-space-resolved expressions for the normal and anomalous contributions to the Floquet spectral flow, which are constructed from the magnetic response of the local Floquet density of states [see Eqs. (92) and (93)].

B. Outline

The manuscript is structured as follows. Section II is devoted to a study of the Středa response at thermodynamic equilibrium. Considering systems of noninteracting fermions on a lattice, we derive a general expression that relates the magnetic response of the density of states to bulk properties. An emphasis is set on the quantum geometry of Bloch bands, which is exquisitely probed by this energy-resolved response function. We also obtain insightful expressions for the net spectral flow of states at a given chemical potential, upon threading an external flux through the lattice, and show how this physical quantity connects to well-defined topological invariants. Section III generalizes this approach to the case of periodically driven Floquet systems. We introduce the Sambe formalism in view of mapping these out-of-equilibrium problems to static (time-independent) Hamiltonian systems defined in an infinite-dimensional space. We define the net spectral flow of the corresponding unbounded spectrum, within a given Floquet spectral gap, identify the normal and anomalous flows, and introduce the Cesàro summation method to regularize the mathematically ill-defined anomalous contribution. In Sec. IV, we discuss how our approach relates to other formalisms, previously introduced in the literature. We also provide a physical interpretation of our results and elucidate the explicit connection between the regularized Floquet-Středa response and the orbital magnetization density associated to the Floquet states. Then, in Sec. V, we exhaustively analyze how the proposed formalism applies to a particular model that supports trivial, conventional, and anomalous Floquet topological phases as a function of its parameters. In Sec. VI, we introduce a protocol to measure winding numbers through particle-density measurements, in a setting that involves an engineered heat bath, and show how it applies to a concrete model. In Sec. VII, we provide real-space-resolved markers for Floquet winding numbers, and we validate these definitions by performing numerical simulations for clean inhomogeneous systems (junctions) as well as for disordered samples with open-boundary conditions. Finally, Sec. VIII outlines possible directions for future research. The Appendixes provide technical aspects of our derivations and alternative derivations of some of our results.

II. EQUILIBRIUM STŘEDA RESPONSE

As a warm-up, we first investigate the Středa response of a generic clean static system, consisting of noninteracting fermions on a lattice and described by a Hamiltonian $\hat{H}$.

We note that a local Středa response can be evaluated in the presence of disorder, as we discuss in Sec. VII. The particle density at finite temperature is given by

$$n = \int_{-\infty}^{\infty} d\omega f(\omega) \rho(\omega), \quad (2)$$

where $f(\omega)$ is the Fermi-Dirac distribution function and $\rho(\omega)$ stands for the density of states (DOS) of the system:

$$\rho(\omega) = -\frac{1}{\pi A} \text{Im Tr}[\hat{G}(\omega + i0^+)]. \quad (3)$$

Here, A is the area of the sample, $\hat{G}(\omega)$ is the single-particle Green's function, and $\text{Tr}[\dots]$ traces over spatial and internal degrees of freedom. Hence, evaluating the Středa response in Eq. (1) boils down to determining how $\rho(\omega)$ gets modified upon applying a magnetic perturbation. In the presence of a small magnetic field and considering the real-space representation, the matrix elements of the Hamiltonian $\hat{H}$ are modified as

$$H_{\mathbf{R}_\nu \mathbf{R}'_\nu} = e^{i(2\pi/\Phi_0)\varphi_{\mathbf{R}_\nu \mathbf{R}'_\nu}} H_{\mathbf{R}_\nu \mathbf{R}'_\nu}^0, \quad (4)$$

where

$$\varphi_{\mathbf{R}_\nu \mathbf{R}'_\nu} = \int_{\mathbf{R}_\nu}^{\mathbf{R}'_\nu} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r} \quad (5)$$

are the Peierls phase factors [112,113]. Here, the coordinates $\mathbf{R}_\nu$ denote the position of the site ν within the unit cell at $\mathbf{R}$, $\mathbf{A}(\mathbf{r})$ is the vector potential, and $\hat{H}^0$ is the Hamiltonian in the absence of the perturbing field.

Relevant physical quantities can be identified in the magnetic response of the DOS by imposing periodic boundary conditions in the lattice Hamiltonian. In this case, it is convenient to adopt the formalism used in Refs. [97,114–116] by means of which a gauge-invariant propagator is introduced by factoring out the Peierls phase factors from the matrix elements of the Green's function

$$G_{\mathbf{R}_\nu \mathbf{R}'_\nu}^{(B)}(\omega) = e^{-i(2\pi/\Phi_0)\varphi_{\mathbf{R}_\nu \mathbf{R}'_\nu}} G_{\mathbf{R}_\nu \mathbf{R}'_\nu}(\omega). \quad (6)$$

In the noninteracting limit, the propagator reduces to $\hat{G}(\omega) = (\omega\hat{I} - \hat{H})^{-1}$, and the corresponding $\hat{G}^{(B)}(\omega)$ satisfies the modified Dyson equation of motion

$$\sum_{\mathbf{R}''_\nu} \left(\omega \delta_{\mathbf{R}_\nu \mathbf{R}'_\nu} - H_{\mathbf{R}_\nu \mathbf{R}'_\nu}^0 \right) G_{\mathbf{R}'_\nu \mathbf{R}''_\nu}^{(B)}(\omega) \times e^{i(\pi/\Phi_0)\mathbf{B} \cdot (\mathbf{R}'_\nu - \mathbf{R}_\nu) \times (\mathbf{R}''_\nu - \mathbf{R}'_\nu)} = \delta_{\mathbf{R}_\nu \mathbf{R}''_\nu}, \quad (7)$$

where $\mathbf{B} = \nabla \times \mathbf{A}(\mathbf{r})$ is the external magnetic field. Interestingly, Eq. (7) indicates that $\hat{G}^{(B)}(\omega)$ is manifestly

gauge and translationally invariant. In a translationally invariant lattice, the propagator $\hat{G}^{(B)}(\omega)$ can then be Fourier transformed to quasimomentum space as

$$G_{\mathbf{R}_\nu \mathbf{R}'_\nu}^{(B)}(\omega) = \frac{1}{N_c} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot (\mathbf{R}_\nu - \mathbf{R}'_\nu)} [\hat{G}_{\mathbf{k}}^{(B)}(\omega)]_{\nu\nu'}, \quad (8)$$

with N_c the number of unit cells in the lattice. We note that the DOS in Eq. (3) can be expressed in terms of this gauge-invariant propagator as

$$\begin{aligned} \rho(\omega) &= -\frac{1}{\pi A} \text{Im Tr} [\hat{G}^{(B)}(\omega + i0^+)] \\ &= -\frac{1}{\pi} \int_{\text{BZ}} \frac{d^2 k}{(2\pi)^2} \text{Im tr} [\hat{G}_{\mathbf{k}}^{(B)}(\omega + i0^+)], \end{aligned} \quad (9)$$

where the trace $\text{tr}[\dots]$ is restricted to summing over the internal degrees of freedom. Expanding Eq. (7) up to first order in the magnetic field, we find

$$\hat{G}_{\mathbf{k}}^{(B)}(\omega) = \hat{G}_{\mathbf{k}}(\omega) + \frac{i\pi}{\Phi_0} B^i \epsilon^{ijl} \hat{G}_{\mathbf{k}}(\omega) \frac{\partial \hat{G}_{\mathbf{k}}^{-1}(\omega)}{\partial k_j} \frac{\partial \hat{G}_{\mathbf{k}}(\omega)}{\partial k_l}, \quad (10)$$

where $\hat{G}_{\mathbf{k}}(\omega)$ is the Bloch Green's function in the absence of the applied magnetic field, namely, $\hat{G}_{\mathbf{k}}(\omega) = (\omega\hat{I}_{\mathbf{k}} - \hat{H}_{\mathbf{k}})^{-1}$, where $\hat{H}_{\mathbf{k}}$ is the Bloch Hamiltonian. Here, ϵ^{ijl} is the Levi-Civita tensor, and a summation over indices is implicit.

Considering, without loss of generality, a two-dimensional sample in the x - y plane with the magnetic field applied along the z direction and inserting Eq. (10) into Eq. (9), we find that the magnetic response of the DOS evaluated at zero magnetic field is given by

$$\begin{aligned} \Phi_0 \frac{\partial \rho(\omega)}{\partial B} &= \int_{\text{BZ}} \frac{d^2 k}{2\pi} \sum_{\alpha} \left[\mathcal{F}_{xy}^{\alpha}(\mathbf{k}) \delta(\omega - \varepsilon_{\alpha\mathbf{k}}) \right. \\ &\quad \left. + \frac{\Phi_0}{2\pi} m_z^{\alpha}(\mathbf{k}) \frac{\partial}{\partial \omega} \delta(\omega - \varepsilon_{\alpha\mathbf{k}}) \right], \end{aligned} \quad (11)$$

where we identified the Berry curvature

$$\mathcal{F}_{xy}^{\alpha}(\mathbf{k}) = i(\langle \partial_{k_x} u_{\alpha\mathbf{k}} | \partial_{k_y} u_{\alpha\mathbf{k}} \rangle - \langle \partial_{k_y} u_{\alpha\mathbf{k}} | \partial_{k_x} u_{\alpha\mathbf{k}} \rangle) \quad (12)$$

and the intrinsic orbital magnetic moment

$$m_z^{\alpha}(\mathbf{k}) = \frac{2\pi}{\Phi_0} \text{Im}[\langle \partial_{k_x} u_{\alpha\mathbf{k}} | \hat{H}_{\mathbf{k}} - \varepsilon_{\alpha\mathbf{k}} | \partial_{k_y} u_{\alpha\mathbf{k}} \rangle] \quad (13)$$

of the α th Bloch eigenstate $|u_{\alpha\mathbf{k}}\rangle$ with energy $\varepsilon_{\alpha\mathbf{k}}$. The intrinsic orbital magnetic moment defined in Eq. (13) is semiclassically associated to the self-rotation of a Bloch wave packet in band α around its center of mass [117,118].

From hereon, we refer to Eq. (11) as the *energy-resolved Středa response*. To the best of our knowledge, Eq. (11) has not been reported in the existing literature. We note that it can be alternatively derived from the modified phase-space density introduced in Ref. [118]; see Appendix A.

Interestingly, the energy-resolved Středa response can be used to identify “hot spots” of Berry curvature, as we illustrate below based on a specific example (see Fig. 2). In this sense, the physical quantity in Eq. (11) provides an appealing and elegant way to probe the geometric fine structure of Bloch bands [119].

Throughout this derivation, we have assumed that the lattice Hamiltonian is defined on a torus geometry, such that the total flux threading the system $\Phi = BA$ should be quantized according to the Dirac quantization condition: $\Phi/\Phi_0 \in \mathbb{Z}$. As a corollary, the minimum magnetic-field perturbation that can be applied corresponds to $\Delta B = \Phi_0/A$. In this sense, the magnetic-field derivative in Eq. (11) should be interpreted as such only in the thermodynamic limit (i.e., for a macroscopic system area).

The spectral flow induced by the magnetic perturbation, at fixed chemical potential μ , is given by the integral

$$\mathcal{W}(\mu) = \Phi_0 \int_{-\infty}^{\infty} d\omega f(\omega) \frac{\partial \rho(\omega)}{\partial B} = \Phi_0 \left. \frac{\partial N(\mu)}{\partial \Phi} \right|_{B=0}. \quad (14)$$

The spectral flow, thus, corresponds to the *Středa response*: It quantifies the variation of the total number of particles at fixed chemical potential, $N(\mu)$, upon applying an external magnetic flux. Using Eq. (11), we readily find the expression

$$\mathcal{W}(\mu) = \int_{\text{BZ}} \frac{d^2 k}{2\pi} \sum_{\alpha} \left[\mathcal{F}_{xy}^{\alpha}(\mathbf{k}) f(\varepsilon_{\alpha\mathbf{k}}) + \frac{\Phi_0}{2\pi} m_z^{\alpha}(\mathbf{k}) \frac{\partial f(\varepsilon_{\alpha\mathbf{k}})}{\partial \mu} \right]. \quad (15)$$

The first term in Eq. (15) encodes the topological information of the filled Fermi sea, while the second term, which involves only the states near the Fermi surface, has a nontopological nature and is generically finite for a metal. We also note that Eq. (15) is in agreement with the modern theory of orbital magnetization in crystalline solids [117,118,120–124]. Indeed, it can be alternatively obtained by making use of the thermodynamic Maxwell relation $\partial N(\mu)/\partial \Phi = \partial \mathcal{M}(\mu)/\partial \mu$, where $\mathcal{M}(\mu)$ denotes the orbital magnetization density.

In the limit of zero temperature, and for an insulating state, the topological term is the only surviving contribution in Eq. (15), and we recover the well-known result

$$\begin{aligned} \mathcal{W}(\mu) &= \frac{1}{2\pi} \int_{\text{BZ}} d^2 k \sum_{\alpha} \Theta(\mu - \varepsilon_{\alpha\mathbf{k}}) \mathcal{F}_{xy}^{\alpha}(\mathbf{k}) \\ &= \sum_{\alpha \in \text{occ}} C_{\alpha}, \end{aligned} \quad (16)$$

where C_{α} denotes the Chern number of the α th Bloch band and where the sum runs over all occupied bands. Equation (16) quantifies the quantized Hall conductivity of noninteracting insulating states of matter [57]. Importantly, when considering a system with a boundary, the spectral flow is carried by the chiral modes that propagate around the edge of the system. In this realistic geometry, the quantized response in Eq. (16) reflects the number of edge channels at the chemical potential μ , as dictated by the bulk-edge correspondence.

In this noninteracting framework, the number of particles $N(\mu)$ can be expressed as a first-order winding number of the μ -dependent propagator $\hat{G}(\omega) = [(i\omega + \mu)\hat{I} - \hat{H}]^{-1}$ defined in frequency space [125]

$$\begin{aligned} N(\mu) &= N_1[G] \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega 0^+} \text{Tr} \left[\hat{G}^{-1}(\omega) \frac{\partial \hat{G}(\omega)}{\partial \omega} \right]. \end{aligned} \quad (17)$$

In this formalism, the Středa response in Eq. (14) can be written in the form of a higher-order winding number of the Bloch-Green's function in the absence of magnetic field, namely [89,97],

$$\begin{aligned} \mathcal{W}(\mu) &= N_3[G] \\ &= \frac{\varepsilon^{zjl}}{8\pi^2} \int_{\text{BZ}} d^2 k \int_{-\infty}^{\infty} d\omega e^{i\omega 0^+} \text{tr} \left[\hat{G}_{\mathbf{k}}^{-1}(i\omega) \frac{\partial \hat{G}_{\mathbf{k}}(i\omega)}{\partial \omega} \right. \\ &\quad \times \hat{G}_{\mathbf{k}}^{-1}(i\omega) \frac{\partial \hat{G}_{\mathbf{k}}(i\omega)}{\partial k_j} \hat{G}_{\mathbf{k}}^{-1}(i\omega) \left. \frac{\partial \hat{G}_{\mathbf{k}}(i\omega)}{\partial k_l} \right], \end{aligned} \quad (18)$$

where $\hat{G}_{\mathbf{k}}(i\omega) = [(i\omega + \mu)\hat{I}_{\mathbf{k}} - \hat{H}_{\mathbf{k}}]^{-1}$. This winding number remains quantized as long as this propagator does not have poles at zero frequency, i.e., as long as the Hamiltonian does not have eigenvalues at the chemical potential μ . We remark that Eq. (18) reduces to Eq. (15) upon performing the frequency integration. Altogether, the Středa response in Eq. (14) can be seen as a fundamental relation between a higher-order winding number and the flux derivative of a first-order winding number [89,97]:

$$\mathcal{W}(\mu) = \Phi_0 \left. \frac{\partial N_1[G]}{\partial \Phi} \right|_{B=0} = N_3[G]. \quad (19)$$

Interestingly, similar relations are established below in the context of nonequilibrium Floquet systems; see Sec. IV C.

A. Numerical analysis of the Středa responses

To illustrate the information encoded in both the energy-resolved Středa response and the (integrated) Středa response, we now numerically evaluate the analytical expressions in Eqs. (11) and (15) using the Haldane model [126]: a two-dimensional honeycomb lattice with on-site energies $\pm\Delta$ on each sublattice, first nearest-neighbor hoppings J , and

complex next-nearest-neighbor hoppings $J'e^{\pm i\phi}$, which enable the breaking of time-reversal symmetry. At half filling, the model has a topological phase transition at

$$J' = J_c = \left| \frac{\Delta}{3\sqrt{3} \sin(\phi)} \right|, \quad (20)$$

the critical hopping strength separating a trivial insulator phase ($J' < J_c$), with zero Chern number bands, from a topological Chern insulator phase ($J' > J_c$), with the valence band having $C = \pm 1$ depending on the sign of ϕ . For $J' = 0$ and $\Delta \neq 0$, the system's spectrum has two equal-gap valleys exhibiting a finite Berry curvature, which is the same in magnitude but opposite in sign. As J' is increased from zero, the gap in one of the valleys decreases, while the other one increases monotonically, up to the critical point J_c where the band gap closes and the topological phase transition occurs.

This phenomenology is well captured by the energy-resolved Středa response [Eq. (11)], which is displayed in Fig. 2 as a function of J' , for $\phi = \pi/2$ and $\Delta = 0.8J$. The bifurcation of Berry curvature “hot spots” of opposite sign is clearly visible as the next-nearest-neighbor tunneling is increased. Indeed, when $J' < J_c$, the geometric fine structure of the topologically trivial bands is physically revealed by the flow of states from one valley to the other generated by the external magnetic field. The response is also highly sensitive to Van Hove singularities occurring at higher energies. For $J' > J_c$, the gap reopens but there is a relative change of sign of the low-energy response, a clear signature of the band-inversion phenomenology that occurs after the topological phase transition.

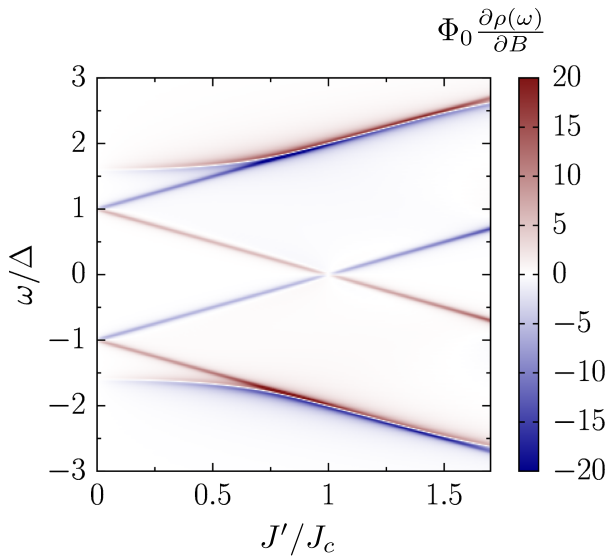


FIG. 2. Energy-resolved Středa response (in units of $1/J$) of the Haldane model with periodic boundary conditions [see Eq. (11)] as a function of the next-nearest-neighbor hopping strength J'/J_c . The parameters are such that $\Delta = 0.8J$ and $\phi = \pi/2$.

In the nontrivial topological regime, the magnetic field generates a net spectral flow from the upper band to the lower band, hence leading to a finite Středa response [Eq. (15)] when setting the chemical potential in the gap; this is shown in Figs. 3(e) and 3(f). In an open-boundary sample, this flow of states is generated via the chiral edge states that bridge the gap [Fig. 3(b)], revealing the deep connection between Středa's formula and the bulk-boundary correspondence.

Although this analysis has been carried out by using the Bloch representation, the Středa response of sufficiently large finite-size systems still preserves the features predicted by Eqs. (11) and (15). In Figs. 3(a) and 3(b), we show the spectrum of a Haldane zigzag ribbon as a function of the quasimomentum in its longitudinal direction. In Fig. 3(a), we have chosen $J' = J_c/2$ such that the system is in the trivial phase, while in Fig. 3(b) the hopping strength is $J' = 1.5J_c$, so that the system is in a nontrivial Chern insulating phase with chiral edge channels bridging the gap.

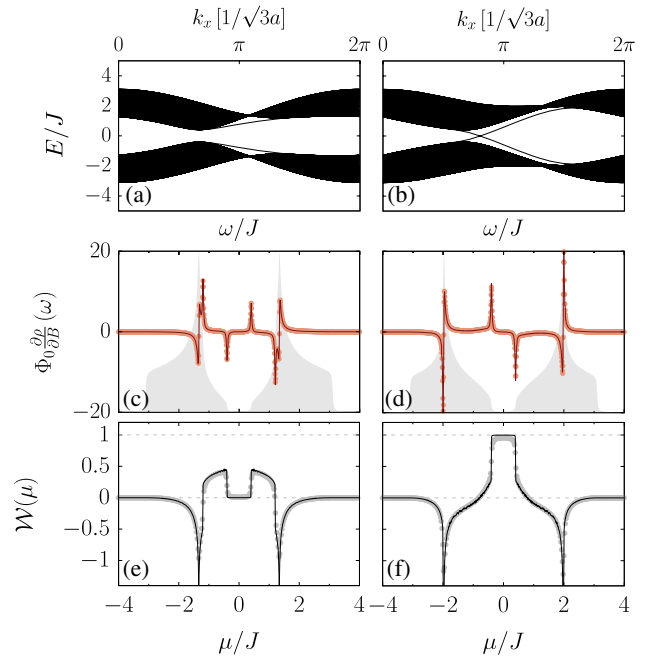


FIG. 3. Energy spectrum of a zigzag ribbon of the Haldane model with $N_y = 150$ cells as a function of quasimomentum k_x in (a) the trivial phase ($J' = J_c/2$) and (b) the topological phase ($J' = 1.5J_c$). Here, $\phi = \pi/2$ and $\Delta = 0.8J$. (c) and (d) show, respectively, the energy-resolved Středa response of the ribbon (filled points) and the same response computed with periodic boundary conditions (solid lines) [see Eq. (11)], in units of $1/J$. To improve the visualization of the curves, we have chosen to broaden the poles of the propagator with an imaginary part of $0^+ = 0.02J$. The filled gray area indicates the DOS at $B = 0$ in arbitrary units. (e) and (f) show, respectively, the Středa response as a function of chemical potential μ for the ribbon (filled points) and for the system with periodic boundary conditions (solid lines). The temperature has been here set to be zero, such that $f(\omega) = \Theta(\mu - \omega)$.

The energy-resolved Středa response of this open-boundary sample is shown for these two configurations with filled points in Figs. 3(c) and 3(d), respectively, presenting an excellent agreement with the analytical response previously shown in Fig. 2 (solid lines). In Figs. 3(e) and 3(f), we show its integrated version, the Středa response $\mathcal{W}(\mu)$, as a function of the chemical potential in the limit of zero temperature. In the gapped regions, this response function approaches the Chern number of the occupied states ($C = 0$ in the trivial phase and $C = 1$ in the topological phase). The small difference between the response of the ribbon and that obtained from Eq. (14) is due to finite-size effects and the broadening of the propagator poles.

The spectral flow under flux insertion, as defined by Eq. (14), presents itself as a fundamental quantity for characterizing the topological properties of two-dimensional Chern insulators in thermodynamic equilibrium. It is the aim of the following section to generalize this powerful notion to out-of-equilibrium systems subjected to a time-periodic drive.

III. FLOQUET-STŘEDA RESPONSE IN SAMBE SPACE

In this section, we generalize the formalism developed in Sec. II with the aim of evaluating the Středa response of a lattice system set out of equilibrium by a periodic drive of frequency Ω . In Sec. III A, we introduce the Sambe formalism that maps driven Floquet problems to static ones in an infinite-dimensional extended (Sambe) space $\mathcal{S}$. In Sec. III B, we formulate the problem of evaluating the magnetic response of the Floquet density of states in Sambe space under flux insertion. Since this spectrum is unbounded from below, the introduction of regularization schemes is required to compute this quantity. A truncation approach akin to the one used in Ref. [23] is discussed in Sec. III C. In Sec. III D, we introduce a regularization procedure based on Cesàro summation methods, which better highlights the low-energy nature of the spectral flow. Finally, in Sec. III E, we particularize the results in Sec. III D for a translationally invariant lattice system.

A. The Sambe space formulation

The Schrödinger equation of Hamiltonians with discrete time-translational symmetry $\hat{H}(t) = \hat{H}(t + T)$, where $T = 2\pi/\Omega$, admits stationary solutions called *Floquet states* [18,19] $|\psi_a(t)\rangle = e^{-i\varepsilon_a t} |u_a(t)\rangle$; we take $\hbar \equiv 1$ from here on. Here, ε_a stands for the *quasienergy*, and $|u_a(t)\rangle$ is a T -periodic vector in Hilbert space, which satisfies

$$\hat{H}^F(t) |u_a(t)\rangle = \varepsilon_a |u_a(t)\rangle, \quad (21)$$

where

$$\hat{H}^F(t) = \hat{H}(t) - i\partial_t \quad (22)$$

is known as the Floquet Hamiltonian or quasienergy operator. The Floquet modes $|u_a(t)\rangle$ can be decomposed into Fourier harmonics as

$$|u_a(t)\rangle = \sum_{n=-\infty}^{\infty} e^{-in\Omega t} |u_a^{(n)}\rangle, \quad (23)$$

where n are integer numbers. They satisfy the time-independent equation

$$\sum_{n=-\infty}^{\infty} (\hat{H}_{m-n} - m\Omega \delta_{mn} \hat{I}) |u_a^{(n)}\rangle = \varepsilon_a |u_a^{(m)}\rangle, \quad \forall m, \quad (24)$$

where $\hat{H}_{m-n} = (1/T) \int_0^T dt e^{i(m-n)\Omega t} \hat{H}(t)$. Note that if the set $\{|u_a^{(n)}\rangle\}$ satisfies Eq. (24) with eigenvalue ε_a , then there is an infinite number of other solutions with eigenvalues $\varepsilon_a + s\Omega$ given by the set $\{|u_{as}^{(n)}\rangle \equiv |u_a^{(n+s)}\rangle\}$, with s being an arbitrary integer number. Since $|\psi_{as}(t)\rangle = |\psi_a(t)\rangle$, all these solutions are physically equivalent, and, hence, it would be sufficient to consider only those that lie inside a given quasienergy window of width Ω , also called a Floquet zone. Nevertheless, for convenience, one can also adopt an extended-zone picture and write all the solutions to Eq. (24) as the ones satisfying

$$\sum_{n=-\infty}^{\infty} \hat{H}_{mn}^F |u_a^{(n+s)}\rangle = \varepsilon_{as} |u_a^{(m+s)}\rangle, \quad \forall m, s, \quad (25)$$

where $\hat{H}_{mn}^F = \hat{H}_{m-n} - m\Omega \delta_{mn} \hat{I}$, $\varepsilon_{as} = \varepsilon_a + s\Omega$, and a labels the solutions within a given Floquet zone. Throughout this work, we choose the convention $\varepsilon_a \in (\varepsilon_\pi - \Omega, \varepsilon_\pi]$, which corresponds to the natural Floquet zone (NFZ) defined in Ref. [59]. A special energy gap can be present at the edge of the NFZ, the so-called zone-edge gap [59]; it is centered around ε_π and becomes infinitely large in the limit $\Omega \rightarrow \infty$. In systems with particle-hole or chiral symmetry, one naturally sets $\varepsilon_\pi = \Omega/2$ [60].

This set of equations [Eq. (25)] can be written in the form of a time-independent Schrödinger equation by introducing the infinite-dimensional Sambe representation, namely,

$$\hat{H}^F |u_{as}\rangle = \varepsilon_{as} |u_{as}\rangle. \quad (26)$$

Here, $|u_{as}\rangle$ are eigenstates of the Floquet Hamiltonian $\hat{H}^F$ in the extended Sambe space $\mathcal{S} = \mathcal{H} \otimes \mathcal{T}$, given by the product of the original Hilbert space $\mathcal{H}$ with the space of square-integrable T -periodic functions [18,19]. Namely,

$$|u_{as}\rangle = (\dots, |u_{as}^{(1)}\rangle, |u_{as}^{(0)}\rangle, |u_{as}^{(-1)}\rangle, \dots)^T, \quad (27)$$

or, equivalently, the n th block element of $|u_{as}\rangle$ is given by $|u_{as}\rangle_n = |u_{as}^{(n)}\rangle = |u_a^{(n+s)}\rangle$. The orthogonality relation in this space is given by

$$\langle\langle u_{as}|u_{bs'}\rangle\rangle = \sum_{n=-\infty}^{\infty} \langle u_{as}^{(n)}|u_{bs'}^{(n)}\rangle = \delta_{ab}\delta_{ss'}. \quad (28)$$

In the following, we use the shorthanded notation $|u_a\rangle \equiv |u_{a0}\rangle$ to denote states in the NFZ.

In the presence of discrete translational symmetry, the Floquet Hamiltonian can be Fourier transformed to quasimomentum space, such that the quantum number $a = (\alpha, \mathbf{k})$, with $\mathbf{k}$ being the quasimomentum and α the Bloch band index. In this case, we have that

$$\hat{H}_k^F |u_{\alpha k}\rangle = \varepsilon_{\alpha k} |u_{\alpha k}\rangle, \quad (29)$$

where $\hat{H}_k^F$ is the corresponding Floquet-Bloch Hamiltonian in $\mathcal{S}$. Here, $\varepsilon_{\alpha k} = \varepsilon_{\alpha k} + s\Omega$ stand for the quasienergies of the Bloch system. In the following, we use the notation $|u_{\alpha k}\rangle \equiv |u_{\alpha 0 k}\rangle$ to denote the α th Floquet-Bloch state in the NFZ, i.e., the $s = 0$ state.

B. Spectral flow of an unbounded operator

The core of this work relies on the evaluation of the magnetic response of the Ω -periodic Floquet density of states, which can be generically expressed in terms of the single-particle Floquet Green's function as

$$\rho^F(\omega) = -\frac{1}{\pi A} \text{Im} \text{Tr}[\hat{G}^F(\omega + i0^+)], \quad (30)$$

where we have introduced the bold trace $\text{Tr}[\cdot]$ to define the trace in $\mathcal{S}$. In the absence of interactions, $\hat{G}^F(\omega) = (\omega \mathbf{I} - \hat{H}^F)^{-1}$, such that Eq. (30) takes the simple form

$$\rho^F(\omega) = \frac{1}{A} \sum_{a,s} \delta(\omega - \varepsilon_{as}), \quad (31)$$

where we remind that the sum over a is restricted such that $\varepsilon_a \in \text{NFZ}$ and the sum over s runs along all $\mathbb{Z}$.

We now introduce the central quantity of the present work: the net spectral flow of the unbounded Floquet spectrum under flux insertion. The *Floquet spectral flow* at quasienergy μ_ε is formally defined as the following integral:

$$\mathcal{W}(\mu_\varepsilon) = \Phi_0 \int_{-\infty}^{\mu_\varepsilon} d\omega \frac{\partial \rho^F(\omega)}{\partial B}. \quad (32)$$

We note that $\mathcal{W}(\mu_\varepsilon) = \mathcal{W}(\mu_\varepsilon + \Omega)$, such that one can restrict μ_ε to the NFZ, without loss of generality. We remark that Eq. (32) is well defined both on a torus and for samples with open-boundary conditions (clean or disordered); see Sec. VII.

For the particular case of a translationally invariant system, the Floquet propagator in Eq. (30) can be expanded

up to first order in the magnetic field by generalizing the formalism used in Sec. II to Sambe space (see Appendix B). As a result, one finds that the magnetic response of the Floquet DOS in Eq. (31), evaluated at zero magnetic field, is given by

$$\Phi_0 \frac{\partial \rho^F(\omega)}{\partial B} = \int_{\text{BZ}} \frac{d^2 k}{2\pi} \sum_{\alpha,s} \left[\mathcal{F}_{xy}^\alpha(\mathbf{k}) \delta(\omega - \varepsilon_{\alpha k}) + \frac{\Phi_0}{2\pi} m_z^\alpha(\mathbf{k}) \frac{\partial}{\partial \omega} \delta(\omega - \varepsilon_{\alpha k}) \right], \quad (33)$$

where we identified the Floquet Berry curvature

$$\mathcal{F}_{xy}^\alpha(\mathbf{k}) = i(\langle\langle \partial_{k_x} u_{\alpha k} | \partial_{k_y} u_{\alpha k} \rangle\rangle - \langle\langle \partial_{k_y} u_{\alpha k} | \partial_{k_x} u_{\alpha k} \rangle\rangle) \quad (34)$$

and the Floquet intrinsic orbital magnetic moment [127]

$$m_z^\alpha(\mathbf{k}) = \frac{2\pi}{\Phi_0} \text{Im}[\langle\langle \partial_{k_x} u_{\alpha k} | \hat{H}_k^F - \varepsilon_{\alpha k} | \partial_{k_y} u_{\alpha k} \rangle\rangle] \quad (35)$$

of the α th Floquet-Bloch eigenstate in the NFZ. Here, we have used that $\mathcal{F}_{xy}^{\alpha s}(\mathbf{k}) = \mathcal{F}_{xy}^\alpha(\mathbf{k})$ and $m_z^{\alpha s}(\mathbf{k}) = m_z^\alpha(\mathbf{k})$. From here on, we hence refer to Eq. (33) as the *energy-resolved Floquet-Středa response*.

In such a clean Bloch lattice setting, Eq. (32) can equally be expressed in terms of the winding number of the unbounded Floquet-Bloch Green's function, namely,

$$\mathcal{W}(\mu_\varepsilon) = N_3[\mathbf{G}^F], \quad (36)$$

with

$$N_3[\mathbf{G}^F] = \frac{e z_{jl}}{8\pi^2} \int_{\text{BZ}} d^2 k \int_{-\infty}^{\infty} d\omega e^{i\omega 0^+} \text{tr} \left[\hat{\mathbf{G}}_k^{F-1}(i\omega) \frac{\partial \hat{\mathbf{G}}_k^F(i\omega)}{\partial \omega} \times \hat{\mathbf{G}}_k^{F-1}(i\omega) \frac{\partial \hat{\mathbf{G}}_k^F(i\omega)}{\partial k_j} \hat{\mathbf{G}}_k^{F-1}(i\omega) \frac{\partial \hat{\mathbf{G}}_k^F(i\omega)}{\partial k_l} \right], \quad (37)$$

where $\text{tr}[\cdot]$ traces over both the Sambe indices and the internal degrees of freedom. In Eq. (37), we have implicitly defined a μ_ε -dependent propagator $\hat{\mathbf{G}}_k^F(i\omega) = [(i\omega + \mu_\varepsilon) \hat{\mathbf{I}}_k - \hat{\mathbf{H}}_k^F]^{-1}$. We note that Eqs. (33) and (37) have exactly the same form as Eqs. (11) and (18), with the sole difference being the use of the $\mathcal{S}$ space.

An important remark is in order: Both Eqs. (32) and (37) are mathematically ill defined, as they are formally expressed as nonconvergent integrals in frequency space. Indeed, the integrand of Eq. (32) is an Ω -periodic function, such that its integral down to $-\infty$ is not well defined. This statement also holds true for the kernel in Eq. (37). This conundrum can be restated as a subtraction-of-infinities problem, since Eq. (32) can be recast in the form

$$\mathcal{W}(\mu_\varepsilon) = A \left(\int_{-\infty}^{\mu_\varepsilon} d\omega \rho^F(\omega, \Phi_0) - \int_{-\infty}^{\mu_\varepsilon} d\omega \rho^F(\omega, 0) \right), \quad (38)$$

where $\rho^F(\omega, \Phi_0)$ stands for the Floquet density of states of a system threaded by one flux quantum and $\rho^F(\omega, 0)$ for the same quantity in the absence of flux. Here, we have considered the minimum difference $\Delta B = \Phi_0/A$ to evaluate the derivative in Eq. (32) as a discrete finite difference in a torus geometry; see Sec. II.

In the following, we present different approaches to regularize these ill-defined quantities.

C. The truncation approach

One possible method to evaluate Eq. (32) would be to truncate the infinite-dimensional Floquet Hamiltonian and Green's function to finite-dimensional operators [23]. These truncated operators contain only a finite number of multiplicities (or Floquet replicas) in Sambe space, such that their spectrum is bounded. In this scenario, the Floquet DOS would no longer be a strictly periodic function.

In particular, in a Bloch lattice, Eq. (33) would get modified to

$$\Phi_0 \frac{\partial \tilde{\rho}^F(\omega)}{\partial B} = \int_{\text{BZ}} \frac{d^2 k}{2\pi} \sum_{\alpha} \sum_{s=-S}^S \left[\tilde{\mathcal{F}}_{xy}^{\alpha s}(\mathbf{k}) \delta(\omega - \tilde{\varepsilon}_{\alpha s k}) + \frac{\Phi_0}{2\pi} \tilde{m}_z^{\alpha s}(\mathbf{k}) \frac{\partial}{\partial \omega} \delta(\omega - \tilde{\varepsilon}_{\alpha s k}) \right], \quad (39)$$

where $\tilde{\mathcal{F}}_{xy}^{\alpha s}(\mathbf{k})$ and $\tilde{m}_z^{\alpha s}(\mathbf{k})$ are, respectively, the Berry curvature and the intrinsic orbital magnetic moment of the modified Floquet-Bloch eigenstate $|\tilde{u}_{\alpha s k}\rangle$ of the truncated Hamiltonian, with quasienergy $\tilde{\varepsilon}_{\alpha s k}$. In Eq. (39), the index s runs over only $2S + 1$ terms, and Eq. (39) reduces to Eq. (33) in the limit $S \rightarrow \infty$. Since the components $|\tilde{u}_{\alpha s k}^{(n)}\rangle$ decay exponentially to zero when $|n - s| \gg 1$, the low-energy sector (i.e., the Floquet zones around $s = 0$) of the truncated Hamiltonian provides a reliable approximation for describing the low-energy physics of the exact Floquet Hamiltonian, and, hence, the presence of topological edge modes, as long as the truncation is performed using a large enough S [23]. Specifically, S should be such that $\langle u_{\alpha k}^{(\pm S)} | u_{\alpha k}^{(\pm S)} \rangle \ll 1$. In this case, one can safely state that, when μ_ε lies in a spectral gap, the Floquet spectral flow can be obtained as

$$\begin{aligned} \mathcal{W}(\mu_\varepsilon) &= \Phi_0 \int_{-\infty}^{\mu_\varepsilon} d\omega \frac{\partial \tilde{\rho}^F(\omega)}{\partial B} \\ &= \int_{\text{BZ}} \frac{d^2 k}{2\pi} \sum_{\alpha} \sum_{s=-S}^S \tilde{\mathcal{F}}_{xy}^{\alpha s}(\mathbf{k}) \Theta(\mu_\varepsilon - \tilde{\varepsilon}_{\alpha s k}) \\ &= \sum_{\alpha} \sum_{s=-S}^S \tilde{\mathcal{C}}_{\alpha s}, \end{aligned} \quad (40)$$

where the prime indicates that the summations involve only indices such that $\tilde{\varepsilon}_{\alpha s k} < \mu_\varepsilon$ and where $\tilde{\mathcal{C}}_{\alpha s}$ denotes the Chern number associated with the modified Floquet-Bloch eigenstates $|\tilde{u}_{\alpha s k}\rangle$. While this procedure gives the correct answer for large enough S [23,128], it relies on the computation of the modified Chern numbers $\tilde{\mathcal{C}}_{\alpha s}$ of *all* the quasienergy bands of the truncated Floquet Hamiltonian that are located below μ_ε . Quite generally, in the presence of anomalous Floquet modes, the Chern numbers associated with the lower part of the truncated spectrum (i.e., bands with $s \sim -S$) turn out to be the more relevant ones to correctly quantify $\mathcal{W}(\mu_\varepsilon)$.

In Sec. III D, we develop a different method of regularization that does not rely on the truncation of the Floquet Hamiltonian and that genuinely reflects the low-energy properties of the bulk quasienergy bands of the system. This alternative approach provides substantial physical intuition of the spectral flow of Floquet systems under flux insertion, as we discuss in Sec. IV.

D. Cesàro regularization of the Floquet-Středa response

It is useful to rewrite the Floquet spectral flow, given by Eq. (32), as the sum of two contributions:

$$\mathcal{W}(\mu_\varepsilon) = \mathcal{W}^N(\mu_\varepsilon) + \mathcal{W}^A, \quad (41)$$

where

$$\mathcal{W}^N(\mu_\varepsilon) = \Phi_0 \int_{\varepsilon_\pi - \Omega}^{\mu_\varepsilon} d\omega \frac{\partial \rho^F(\omega)}{\partial B} \quad (42)$$

and

$$\mathcal{W}^A = \Phi_0 \int_{-\infty}^{\varepsilon_\pi - \Omega} d\omega \frac{\partial \rho^F(\omega)}{\partial B}, \quad (43)$$

and we remind that μ_ε is restricted to the interval $(\varepsilon_\pi - \Omega, \varepsilon_\pi]$ (NFZ). Throughout this derivation, we assume that there is a well-defined zone-edge gap located at ε_π [129]. Importantly, the NFZ is *fixed* and remains unchanged as one activates the B field, such that the derivative with respect to the magnetic field commutes with the integrals above. Hereafter, we refer to $\mathcal{W}^N(\mu_\varepsilon)$ as the *normal spectral flow* and $\mathcal{W}^A$ as the *anomalous spectral flow*. To progress, it is important to note that the kernel of Eq. (43) integrates to zero within any Floquet zone:

$$\Phi_0 \int_{\varepsilon_{\pi, s-1}}^{\varepsilon_{\pi, s}} d\omega \frac{\partial \rho^F(\omega)}{\partial B} = 0, \quad \forall s \in \mathbb{Z}, \quad (44)$$

where we defined $\varepsilon_{\pi, s} = \varepsilon_\pi + s\Omega$. This means that the kernel of the integral defining the anomalous spectral flow oscillates with a zero mean down to minus infinity, such that the integral in Eq. (43) is formally not convergent.

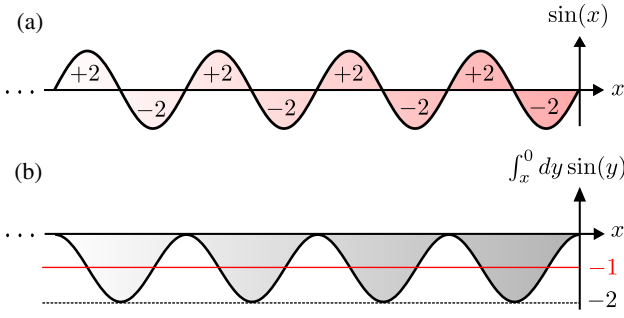


FIG. 4. (a) Illustration of $\sin(x)$, a periodic function with zero mean. The values ± 2 indicate the area below each of the intervals $[(n-1)\pi/2, n\pi/2]$ for $n \in \mathbb{Z}$ and $n \leq 0$. The integral of this function between $-\infty$ and 0 can then be thought of as the discrete Grandi series $2 \sum_{n=1}^{\infty} (-1)^n$. (b) Integral of the sine function in the interval $[x, 0]$ as a function of x . The value -1 , marked with a solid red line, corresponds to the Cesàro regularization of the nonconvergent integral $\int_{-\infty}^0 dy \sin(y)$.

A similar pathological situation occurs for the sine function when considering the integral $\int_{-\infty}^0 dy \sin(y)$. As a matter of fact, this type of integral corresponds to an integral version of a discrete Grandi-type series, such as $\sum_{n=0}^{\infty} (-1)^n$ (see Fig. 4). Mathematically, these nonconvergent series are treated by different summation methods, by means of which it is possible to define and assign values to them [108]. In particular, a series $\sum_{n=1}^{\infty} a_n$ is called “Cesàro summable,” with value $\mathcal{Q}$, if the arithmetic mean $\lim_{n \rightarrow \infty} (1/n) \sum_{k=1}^n s_k = \mathcal{Q}$, with $s_k = a_1 + \dots + a_k$ the k th partial sum. In the case represented in Fig. 4, the partial sums give $(-2, 0, -2, 0, \dots)$ so that $\mathcal{Q} = -1$. More generally, a nonconvergent integral is said to be $(C, 1)$ summable if

$$\int_{-\infty}^0 dy g(y) \stackrel{(C,1)}{=} \lim_{\lambda \rightarrow \infty} \int_{-\lambda}^0 dy g(y) \left(1 + \frac{y}{\lambda}\right) \quad (45)$$

converges. For $g(y) = \sin(y)$, Eq. (45) recovers the $\mathcal{Q}$ value -1 .

In order to properly define the anomalous spectral flow in Eq. (43), we now perform a $(C, 1)$ Cesàro regularization of the integral [108,109], namely,

$$\begin{aligned} \mathcal{W}^A &\stackrel{(C,1)}{=} \lim_{\lambda \rightarrow \infty} \Phi_0 \int_{-\lambda}^{\varepsilon_\pi - \Omega} d\omega \frac{\partial \rho^F(\omega)}{\partial B} \left(1 + \frac{\omega}{\lambda}\right) \\ &= \lim_{S \rightarrow \infty} \Phi_0 \sum_{s=1}^S \int_{\varepsilon_\pi - (s+1)\Omega}^{\varepsilon_\pi - s\Omega} d\omega \frac{\partial \rho^F(\omega)}{\partial B} \left(1 + \frac{\omega}{(S+1)\Omega - \varepsilon_\pi}\right) \\ &= \lim_{S \rightarrow \infty} \Phi_0 \int_{\text{NFZ}} d\omega \frac{\partial \rho^F(\omega)}{\partial B} \omega \frac{S}{(S+1)\Omega - \varepsilon_\pi} \\ &= \Phi_0 \int_{\text{NFZ}} d\omega \frac{\partial \rho^F(\omega)}{\partial B} \frac{\omega}{\Omega}. \end{aligned} \quad (46)$$

Note that we have taken the limit of $\lambda \rightarrow \infty$ by considering that an integer number of Floquet zones are contained within the interval $(-\lambda, \varepsilon_\pi - \Omega]$. We have also used the Ω periodicity of $\rho^F(\omega)$ and Eq. (44). Even though the last integral in Eq. (46) is performed within the NFZ, we remark that one can shift the interval of integration by any arbitrary multiple of the driving frequency and obtain the same result.

Importantly, the Cesàro regularization procedure performed in Eq. (46) leads to an insightful expression for the net Floquet spectral flow in Eq. (41), which highlights two physically distinguishable contributions:

$$\mathcal{W}(\mu_\varepsilon) = \Phi_0 \int_{\varepsilon_\pi - \Omega}^{\mu_\varepsilon} d\omega \frac{\partial \rho^F(\omega)}{\partial B} + \frac{\Phi_0}{\Omega} \int_{\text{NFZ}} d\omega \left(\frac{\partial \rho^F(\omega)}{\partial B} \omega \right). \quad (47)$$

The first term in Eq. (47) represents a flow of *dressed states* in response to the perturbing magnetic field, while the second term, which stems from the anomalous response, characterizes a flow of *energy* within a given Floquet zone. Indeed, the regularized anomalous flow $\mathcal{W}^A$ in Eq. (46) can be equally interpreted as the total orbital magnetization density of the Floquet states within a zone, measured in units of Ω/Φ_0 . This key result, yet simple, has profound physical consequences, which are further studied in Sec. IV.

We also note that the anomalous spectral flow can be expressed as

$$\mathcal{W}^A = -\frac{1}{\Omega} \int_{\text{NFZ}} d\mu_\varepsilon \mathcal{W}^N(\mu_\varepsilon) \equiv -\overline{\langle \mathcal{W}^N \rangle}_{\text{NFZ}}; \quad (48)$$

that is to say, it can be obtained as an average of the normal spectral flow within the natural Floquet zone. We note that Eq. (48) implies a conservation law: $\int_{\text{NFZ}} d\mu_\varepsilon \mathcal{W}(\mu_\varepsilon) = 0$.

In an open-boundary sample with a bulk gap at quasienery μ_ε , the Floquet spectral flow is physically carried by the edge channels that bridge the Floquet spectral gaps (Fig. 1). As a corollary, Eq. (47) fully quantifies the net number of chiral edge modes that are located in the gaps of periodically driven settings. Importantly, this result does not rely on having a translationally invariant system and, hence, remains valid even in the presence of disorder or inhomogeneities, as long as the bulk gaps remain open.

Evaluating the Floquet spectral flow precisely at the zone-edge gap ($\mu_\varepsilon = \varepsilon_\pi$) yields

$$\mathcal{W}(\varepsilon_\pi) = \mathcal{W}^A. \quad (49)$$

Indeed, the normal flow vanishes $\mathcal{W}^N(\varepsilon_\pi) = 0$ due to Eq. (44), and only the anomalous flow contributes to the Floquet-Středa response. This general result indicates that the *anomalous flow* $\mathcal{W}^A$ in Eq. (46) quantifies the number

of *anomalous edge states* that are located at the edge of the Floquet Brillouin zone.

E. Evaluation of the Floquet spectral flow in a translationally invariant system

In a lattice system of Bloch particles, one can use Eq. (33) to explicitly perform the integrals in Eqs. (42) and (46). This yields

$$\mathcal{W}^N(\mu_\varepsilon) = \sum_{\alpha} \int_{\text{BZ}} \frac{d^2k}{2\pi} \left[\mathcal{F}_{xy}^{\alpha}(\mathbf{k}) \Theta(\mu_\varepsilon - \varepsilon_{\alpha\mathbf{k}}) + \frac{\Phi_0}{2\pi} m_z^{\alpha}(\mathbf{k}) \delta(\mu_\varepsilon - \varepsilon_{\alpha\mathbf{k}}) \right] \quad (50)$$

and

$$\mathcal{W}^A \stackrel{(C,1)}{=} \sum_{\alpha} \int_{\text{BZ}} \frac{d^2k}{2\pi\Omega} \left[\mathcal{F}_{xy}^{\alpha}(\mathbf{k}) \varepsilon_{\alpha\mathbf{k}} - \frac{\Phi_0}{2\pi} m_z^{\alpha}(\mathbf{k}) \right], \quad (51)$$

where the Floquet Berry curvatures and intrinsic orbital magnetic moments are defined in Eqs. (34) and (35). We note that the form of Eq. (51) is independent of the choice of the Floquet Brillouin zone, since $\mathcal{F}_{xy}^{\alpha}(\mathbf{k}) = \mathcal{F}_{xy}^{\alpha}(\mathbf{k})$, $m_z^{\alpha}(\mathbf{k}) = m_z^{\alpha}(\mathbf{k})$, and $\sum_{\alpha} \mathcal{F}_{xy}^{\alpha}(\mathbf{k}) = 0$. Even though we have chosen the $(C, 1)$ regularization scheme to obtain this result, different regularizations also lead to the same conclusion, as discussed in Appendix C.

Assuming that μ_ε lies within a gap of the Floquet spectrum, one can express the normal spectral flow $\mathcal{W}^N(\mu_\varepsilon)$ in Eq. (50) as

$$\mathcal{W}^N(\mu_\varepsilon) = \sum_{\alpha} \int_{\text{BZ}} \frac{d^2k}{2\pi} \mathcal{F}_{xy}^{\alpha}(\mathbf{k}) \Theta(\mu_\varepsilon - \varepsilon_{\alpha\mathbf{k}}) = \sum_{\alpha}' C_{\alpha}^F, \quad (52)$$

where we have taken ε_{π} to be located in a well-defined zone-edge gap. In the last equality in Eq. (52), we have introduced the Chern numbers C_{α}^F of the Floquet bands within the NFZ, and we indicated a prime in the summation on the right-hand side to stress that only the Chern numbers of the bands located below μ_ε are considered.

Combining Eqs. (47), (51), and (52) leads us to a general result: The Floquet spectral flow occurring in a spectral gap upon flux insertion is given by

$$\mathcal{W}(\mu_\varepsilon) = \sum_{\alpha}' C_{\alpha}^F + \sum_{\alpha} \int_{\text{BZ}} \frac{d^2k}{2\pi\Omega} \left[\mathcal{F}_{xy}^{\alpha}(\mathbf{k}) \varepsilon_{\alpha\mathbf{k}} - \frac{\Phi_0}{2\pi} m_z^{\alpha}(\mathbf{k}) \right]. \quad (53)$$

Equation (53) is a central result of the present work. It is worthy to emphasize that this formula relies on only the knowledge of low-energy bulk properties of the Floquet Hamiltonian in Sambe space, such as the quasienergies, Berry curvatures, and intrinsic orbital magnetic moments of

Floquet-Bloch bands within a given Floquet zone. With these quantities at hand, it is possible to characterize the anomaly $\mathcal{W}^A$ that must be added to the Chern numbers of the Floquet-Bloch bands in order to correctly quantify the topology of these driven systems.

IV. QUANTIZED ANOMALY AS AN ORBITAL MAGNETIZATION DENSITY AND A GENERALIZED ŠTŘEDA FORMULA FOR FLOQUET SYSTEMS

This section is primarily devoted to the physical interpretation of the formal results obtained in Sec. III D. In Sec. IV A, we first discuss how our classification scheme of two-dimensional Floquet systems relates to the more conventional approach used in the literature and which relies on the properties of the time-evolution operator [9,23]. Then, in Sec. IV B, we present an explicit relation between Eq. (47) and the orbital magnetization density of the Floquet states within a Floquet zone. We also show the correspondence between Eq. (51) and the recently derived expressions for the orbital magnetization density of Floquet-Bloch lattices. In Sec. IV C, we derive a way to reformulate Eq. (47) as a generalized Floquet-Štředa formula, which better highlights the physical interpretation of the normal and anomalous contributions to the Floquet spectral flow. Finally, in Sec. IV D, we illustrate how our theory applies in a simple paradigmatic model of an anomalous Floquet phase.

A. From Sambe to Hilbert space: Stroboscopic dynamics and micromotion

The time-evolution operator of any periodically driven system can always be decomposed as [130–133]

$$\hat{U}(t, t') = \hat{R}(t) e^{-i\hat{H}_{\text{eff}}(t-t')} \hat{R}^{\dagger}(t'), \quad (54)$$

where $\hat{R}(t)$ is a unitary T -periodic operator, also known as *micromotion operator*, and $\hat{H}_{\text{eff}}$ is a time-independent Hamiltonian known as the *effective Hamiltonian*. This Hamiltonian characterizes the stroboscopic time evolution of Floquet-driven systems, since $U(T, 0) = e^{-i\hat{H}_{\text{eff}}T}$ where we choose $t' = 0$ such that $\hat{R}(0) = \hat{I}$; see Refs. [131,132] for more details. Conversely, $\hat{R}(t)$ characterizes the micromotion or short-time behavior within each period. Using Eq. (54) and the equation of motion for the time-evolution operator, it is straightforward to show that

$$\begin{aligned} \hat{H}_{\text{eff}} &= \hat{R}^{\dagger}(t) \hat{H}(t) \hat{R}(t) - i\hat{R}^{\dagger}(t) \partial_t \hat{R}(t) \\ &= \hat{R}^{\dagger}(t) \hat{H}^F(t) \hat{R}(t). \end{aligned} \quad (55)$$

Namely, $\hat{R}(t)$ is the time-dependent unitary transformation that relates the effective Hamiltonian in Eq. (54) to the original (time-dependent) Hamiltonian $\hat{H}(t)$ [130–133].

In Sambe space, this transformation corresponds to the unitary transformation $\hat{R}$ that block diagonalizes $\hat{H}^F$ in Eq. (26) [132]. We note that $\hat{H}_{\text{eff}}$ can always be chosen to have its eigenvalues within the NFZ, which is the convention used throughout this work. Furthermore, it is easy to verify that if $|u_a^{\text{eff}}\rangle$ is an eigenvector of $\hat{H}_{\text{eff}}$ with eigenvalue $\varepsilon_a \in \text{NFZ}$, then $|u_a(t)\rangle = \hat{R}(t)|u_a^{\text{eff}}\rangle$ is an eigenvector of $\hat{H}^F(t)$ with quasienergy ε_a and vice versa.

The normal spectral flow in Eq. (42) can be generically written as a Středa response of $\hat{H}_{\text{eff}}$, namely,

$$\begin{aligned}\mathcal{W}^N(\mu_\varepsilon) &= \Phi_0 \frac{\partial}{\partial \Phi} \int_{\varepsilon_\pi - \Omega}^{\mu_\varepsilon} d\omega \sum_{a,s} \delta(\omega - \varepsilon_{as}) \\ &= \Phi_0 \frac{\partial}{\partial \Phi} \int_{\varepsilon_\pi - \Omega}^{\mu_\varepsilon} d\omega \text{Tr}[\delta(\omega - \hat{H}_{\text{eff}})] \\ &= \Phi_0 \frac{\partial N_{\text{eff}}(\mu_\varepsilon)}{\partial \Phi},\end{aligned}\quad (56)$$

where $N_{\text{eff}}(\mu_\varepsilon)$ simply counts the number of eigenstates that $\hat{H}_{\text{eff}}$ exhibits between $\varepsilon_\pi - \Omega$ and μ_ε . In complete analogy with the equilibrium context, one can reexpress the number of states $N_{\text{eff}}(\mu_\varepsilon)$ as the winding number of an effective propagator in frequency space [see Eq. (17)], namely,

$$\begin{aligned}N_{\text{eff}}(\mu_\varepsilon) &= N_1[G_{\text{eff}}] \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega 0^+} \text{Tr} \left[\hat{G}_{\text{eff}}^{-1}(i\omega) \frac{\partial \hat{G}_{\text{eff}}(i\omega)}{\partial \omega} \right],\end{aligned}\quad (57)$$

where we have introduced the μ_ε -dependent single-particle Green's function $\hat{G}_{\text{eff}}(\omega) = [(i\omega + \mu_\varepsilon)\hat{I} - \hat{H}_{\text{eff}}]^{-1}$. As long as $\hat{G}_{\text{eff}}(i\omega)$ does not have poles at zero frequency (or, equivalently, $\hat{H}_{\text{eff}}$ does not have eigenvalues at μ_ε), Eq. (57) remains quantized.

It is well known that knowledge of the unperturbed stroboscopic time evolution, as dictated by $\hat{H}_{\text{eff}}$, is not enough to characterize anomalous topological Floquet phases—see discussion at the end of this section. Nevertheless, the formalism presented in Sec. III D offers a method for determining the number of anomalous edge channels solely based on the knowledge of the magnetic response of $\hat{H}_{\text{eff}}$. Indeed, by inserting Eq. (31) into Eq. (46), we find that

$$\mathcal{W}^A = \frac{\Phi_0}{\Omega} \frac{\partial}{\partial \Phi} \sum_a \varepsilon_a = \frac{\Phi_0}{\Omega} \frac{\partial}{\partial \Phi} \text{Tr}[\hat{H}_{\text{eff}}], \quad (58)$$

where the sum is restricted to quasienergies $\{\varepsilon_a\}$ in a given Floquet zone. We remark that Eq. (58) indeed signals an anomaly: The expression on the right-hand side should vanish whenever $\hat{H}_{\text{eff}}$ represents a conventional (local) Hamiltonian describing a closed physical system.

A couple of remarks are in order. The equality in Eq. (58) is independent of the boundary conditions or the presence of inhomogeneities in the system, with the only caveat being that, in a torus geometry, the flux derivative must be interpreted as a finite difference due to the quantization of the magnetic monopole charge. In a realistic open-boundary sample, the flux can take continuous values. In this scenario, it is crucial to perform the summation over quasienergies in Eq. (58) *prior* to taking the flux derivative. Indeed, the restricted sum over a given Floquet zone and the magnetic-field derivative are not commuting operations, due to a nontrivial boundary term that appears under their exchange. This term leads to discontinuities or jumps (in multiples of Ω/Φ_0) generated by edge states that enter or leave the Floquet zone. This is essential to maintain the quantization of $\mathcal{W}^A$ in open-boundary samples. An example of this behavior is shown in Fig. 6 and discussed in more detail in Sec. IV D.

We point out that expressions similar to Eq. (58) were found in the context of Anderson localized Floquet phases [82] and in the analysis of the spectral flow of the Floquet Hofstadter butterfly [134]. In this sense, our approach offers a unifying framework, establishing the relation in Eq. (58) for generic two-dimensional Floquet systems. For the particular case of a system of Bloch particles, it is worth recalling that Eq. (51) provides a closed analytical formula for $\mathcal{W}^A$, which is entirely written in terms of bulk properties in the absence of the applied field.

Using Eqs. (56) and (58), we can, therefore, express the total Floquet spectral flow at quasienergy μ_ε as

$$\mathcal{W}(\mu_\varepsilon) = \Phi_0 \frac{\partial N_{\text{eff}}(\mu_\varepsilon)}{\partial \Phi} + \frac{\Phi_0}{\Omega} \frac{\partial}{\partial \Phi} \text{Tr}[\hat{H}_{\text{eff}}]. \quad (59)$$

Equation (59) clearly indicates that it is not necessary to have information on the micromotion operator $\hat{R}(t)$ to classify two-dimensional Floquet topological phases, provided the stroboscopic evolution is known in the presence of a small magnetic field. This is in full agreement with the conclusion of Ref. [134].

At this stage, it is important to relate our results to the theoretical framework that initially guided the classification of Floquet topological systems [15,23,60], in the absence of magnetic perturbations. In that context, the bulk-boundary correspondence was established by introducing a spatiotemporal winding number entirely built from the micromotion operator $\hat{R}(t)$, which we denote as $N_3[R]$. First, we note that the anomalous flow identified in Eq. (51) implicitly contains all the information about the micromotion, since it is written in terms of quantities that are defined in Sambe space. Importantly, as shown in detail in Appendix D, one can rigorously prove that the anomalous flow in Eq. (51) exactly corresponds to this spatiotemporal winding number:

$$\mathcal{W}^A = \frac{e^2 j l}{8\pi^2} \int_0^T dt \int_{\text{BZ}} d^2 k \text{tr} \left[\hat{R}_k^\dagger(t) \frac{\partial \hat{R}_k(t)}{\partial t} \right. \\ \left. \times \hat{R}_k^\dagger(t) \frac{\partial \hat{R}_k(t)}{\partial k_l} \hat{R}_k^\dagger(t) \frac{\partial \hat{R}_k(t)}{\partial k_j} \right] \equiv N_3[R]. \quad (60)$$

Here, we have introduced $\hat{R}_k(t)$, the Fourier transform of $\hat{R}(t)$ to quasimomentum space in the absence of the perturbing magnetic field. Equation (60) can be equally expressed as the winding number of the two-point micromotion operator $\hat{P}_k(t, t') = \hat{R}_k(t) \hat{R}_k^\dagger(t')$ [13]; see Appendix D. This result is in perfect agreement with Refs. [15, 23, 60], where Eq. (60) was obtained entirely based on mathematical considerations.

In Appendix E, we also show that the Chern numbers associated to the Bloch bands of $\hat{H}_{\text{eff}}$ are the same as the Chern numbers C_α^F , which characterize the normal flow within a spectral gap [see Eq. (52)].

B. Floquet orbital magnetization density

In analogy with equilibrium systems, the orbital magnetization density associated to Floquet states with quasienergies $\epsilon_a \in (\epsilon_\pi - \Omega, \mu_\epsilon]$ can be defined as

$$\mathcal{M}^F(\mu_\epsilon) = - \int_{\text{NFZ}} d\omega \frac{\partial \rho^F(\omega)}{\partial B} (\omega - \mu_\epsilon) \Theta(\mu_\epsilon - \omega). \quad (61)$$

It is straightforward to verify that the normal and anomalous spectral flows, introduced in Eqs. (42) and (46), can then be expressed in terms of $\mathcal{M}^F(\mu_\epsilon)$ as

$$\mathcal{W}^N(\mu_\epsilon) = \Phi_0 \frac{\partial \mathcal{M}^F(\mu_\epsilon)}{\partial \mu_\epsilon} \quad (62)$$

and

$$\mathcal{W}^A = - \frac{\Phi_0}{\Omega} \int_{\text{NFZ}} d\mu_\epsilon \frac{\partial \mathcal{M}^F(\mu_\epsilon)}{\partial \mu_\epsilon} = - \frac{\Phi_0}{\Omega} \mathcal{M}^F(\epsilon_\pi), \quad (63)$$

where we used the fact that $\mathcal{M}^F(\epsilon_\pi - \Omega) = 0$, with our current conventions. Altogether, the total Floquet spectral flow can be rewritten in terms of the Floquet magnetization density as

$$\mathcal{W}(\mu_\epsilon) = \Phi_0 \frac{\partial \mathcal{M}^F(\mu_\epsilon)}{\partial \mu_\epsilon} - \frac{\Phi_0}{\Omega} \mathcal{M}^F(\epsilon_\pi). \quad (64)$$

Equation (63) stands as a remarkable relation, which generically states that the quantized anomalous spectral flow $\mathcal{W}^A$, intrinsic to Floquet systems, reflects the quantization of the total orbital magnetization density of the Floquet-Sambe states within a Floquet zone, in units of $\Omega/\Phi_0 = e/cT$. Through Eq. (58), we can also interpret $\mathcal{M}^F(\epsilon_\pi)$ in Eq. (63) as being the orbital magnetization density of the *fully occupied* spectrum of $\hat{H}_{\text{eff}}$. In Sec. IV C, we derive a series of relations which better highlight the

physical interpretation of $\mathcal{W}^A$ as a measure of the exchange of quantized orbital angular momentum per cycle between the system of dressed particles, described by $\hat{H}_{\text{eff}}$, and the driving field [72, 128]. Such an effect can take place only in the resonant regime (i.e., when Ω is of the order of the energy scales of the time-averaged Hamiltonian [9, 23, 128, 135]), since virtual processes cannot transfer angular momentum. This is consistent with the bulk-boundary correspondence principle: Edge modes at $\mu_\epsilon = \epsilon_\pi$ can emerge only through a gap closing at the boundary of the Floquet zones.

The quantization of orbital magnetization density in Floquet systems has previously been associated with anomalous Floquet Anderson insulator (AFAI) phases, as demonstrated in Ref. [82]. These phases occur in strongly disordered systems, where topological chiral edge modes coexist with a bulk of fully localized Floquet states [80]. In this sense, Eq. (63) extends the result of Ref. [82] beyond Floquet systems with fully localized bulk states, demonstrating its general applicability provided there is a well-defined bulk gap at the Floquet zone edge. This statement is further supported by our real-space study of inhomogeneous systems in Sec. VII.

Interestingly, a general expression for the orbital magnetization density of Floquet systems with lattice translational symmetry, valid for any arbitrary stationary occupation of the Floquet-Bloch bands, was only recently derived in Refs. [110, 127]. In order to connect these new developments to our spectral flow analysis, which entirely relies on properties of the Floquet states, we consider the stationary occupation of the αk th Floquet mode to be given by a zero-temperature Fermi-Dirac distribution function with chemical potential $\mu = \mu_\epsilon$, namely, $\Theta(\mu_\epsilon - \epsilon_{\alpha k})$. In the original Hilbert space $\mathcal{H}$, the Floquet-Bloch orbital magnetization density can then be written as [110]

$$\mathcal{M}^F(\mu_\epsilon) = \mathcal{M}_1^F(\mu_\epsilon) + \mathcal{M}_2^F(\mu_\epsilon), \quad (65)$$

where

$$\mathcal{M}_1^F(\mu_\epsilon) = \frac{1}{\Phi_0} \sum_\alpha \int_{\text{BZ}} \frac{d^2 k}{2\pi} \Theta(\mu_\epsilon - \epsilon_{\alpha k}) \\ \times \frac{1}{T} \int_0^T dt \text{Im} [\langle \partial_{k_x} u_{\alpha k}(t) | \hat{H}_k^F(t) - \epsilon_{\alpha k} | \partial_{k_y} u_{\alpha k}(t) \rangle] \quad (66)$$

and

$$\mathcal{M}_2^F(\mu_\epsilon) = \frac{1}{\Phi_0} \sum_\alpha \int_{\text{BZ}} \frac{d^2 k}{\pi} \Theta(\mu_\epsilon - \epsilon_{\alpha k}) \\ \times \frac{1}{T} \int_0^T dt \text{Im} [\langle \partial_{k_x} u_{\alpha k}(t) | \epsilon_{\alpha k} - \mu_\epsilon | \partial_{k_y} u_{\alpha k}(t) \rangle], \quad (67)$$

where $|u_{ak}(t)\rangle$ are the eigenmodes of the quasienergy operator with eigenenergy $\varepsilon_{ak} \in \text{NFZ}$ and $\hat{H}_k^F(t) = \hat{H}_k(t) - i\partial_t$. In Eq. (65), we introduced the standard decomposition of the orbital magnetization of Bloch systems; see Refs. [118,121,123,124]. The $\mathcal{M}_1^F(\mu_\varepsilon)$ term is solely determined by the intrinsic orbital magnetic moment of the Floquet-Bloch bands, while the $\mathcal{M}_2^F(\mu_\varepsilon)$ contribution can be interpreted as the one stemming from the Berry curvature correction to the phase-space Floquet density of states [118,127]. We remind that μ_ε plays the role of the chemical potential μ in the present context.

The two contributions given by Eqs. (66) and (67) can be alternatively written in terms of the Berry curvatures, quasienergies, and intrinsic orbital magnetic moments of the Floquet-Bloch bands in Sambe space as

$$\mathcal{M}_1^F(\mu_\varepsilon) = \sum_\alpha \int_{\text{BZ}} \frac{d^2k}{(2\pi)^2} \Theta(\mu_\varepsilon - \varepsilon_{ak}) m_z^\alpha(\mathbf{k}) \quad (68)$$

and

$$\mathcal{M}_2^F(\mu_\varepsilon) = -\sum_\alpha \int_{\text{BZ}} \frac{d^2k}{2\pi\Phi_0} \Theta(\mu_\varepsilon - \varepsilon_{ak}) (\varepsilon_{ak} - \mu_\varepsilon) \mathcal{F}_{xy}^\alpha(\mathbf{k}). \quad (69)$$

We note that the $\mathbf{k}$ -space Floquet orbital magnetization density computed on a torus geometry would physically represent the spatially averaged orbital magnetization density of an open-boundary system, which generically has nontrivial contributions stemming both from the bulk and from the edge of the sample [120]. Using Eqs. (68) and (69), it is easy to verify that the relations determined in Eqs. (62) and (63) are in full agreement with the results found in Sec. III E. In particular, the net number of chiral anomalous edge channels in a clean lattice, as given by Eq. (51), is in one-to-one correspondence with the quantization of the total Floquet-Bloch orbital magnetization density, namely,

$$\begin{aligned} \mathcal{W}^A &= \sum_\alpha \int_{\text{BZ}} \frac{d^2k}{2\pi\Omega} \left[\mathcal{F}_{xy}^\alpha(\mathbf{k}) \varepsilon_{ak} - \frac{\Phi_0}{2\pi} m_z^\alpha(\mathbf{k}) \right] \\ &= -\frac{\Phi_0}{\Omega} \left[\mathcal{M}_1^F(\varepsilon_\pi) + \mathcal{M}_2^F(\varepsilon_\pi) \right]. \end{aligned} \quad (70)$$

We note that this fundamental relation stands in sharp contradiction with the findings of Refs. [110,111], which report nonuniversal or trivial magnetic responses in the anomalous regime [136].

C. Generalized Floquet-Středa formula: From first-order to higher-order winding numbers

We now demonstrate that the anomalous flow in Eq. (46) can be expressed as the flux derivative of a first-order

winding number built from the micromotion operator. This allows one to rewrite Eq. (47) in the form of a generalized Floquet-Středa response, which has the same mathematical form as Eq. (19) and which reduces to that expression in the undriven limit.

Taking the trace of Eq. (55) and time averaging the result over one driving cycle, one finds that

$$\text{Tr}[\hat{H}_{\text{eff}}] = \frac{1}{T} \int_0^T dt \text{Tr}[\hat{H}(t)] + N_1[R]\Omega, \quad (71)$$

where we have identified $N_1[R]$ as the first-order winding number of the micromotion operator $\hat{R}(t)$ over one period of the driving cycle:

$$N_1[R] = -\frac{i}{2\pi} \int_0^T dt \text{Tr}[\hat{R}^\dagger(t) \partial_t \hat{R}(t)], \quad (72)$$

which guarantees that $N_1[R] \in \mathbb{Z}$. The first term on the right-hand side in Eq. (71) is nothing but the total mean energy of the driven system, which can be written as

$$\begin{aligned} \frac{1}{T} \int_0^T dt \text{Tr}[\hat{H}(t)] &= \frac{1}{T} \int_0^T dt \text{Tr}[\hat{H}^F(t) + i\partial_t] \\ &= \sum_a \left(\varepsilon_a + \Omega \sum_n n \langle u_a^{(n)} | u_a^{(n)} \rangle \right), \end{aligned} \quad (73)$$

where we remind that $\hat{H}^F(t)$ is the Floquet Hamiltonian in $\mathcal{H}$ space, defined in Eq. (22), and where the trace was taken in the eigenbasis of $\hat{H}^F(t)$ [see Eq. (21)]. Comparing Eq. (73) with Eq. (71), one obtains the insightful identification

$$N_1[R] = -\sum_a \sum_{n=-\infty}^{\infty} n \langle u_a^{(n)} | u_a^{(n)} \rangle. \quad (74)$$

Moreover, using the eigenvectors $|u_a^{\text{eff}}\rangle$ of $\hat{H}_{\text{eff}}$ to take the trace in Eq. (72) and their relation with the eigenvectors of $\hat{H}^F(t)$, $|u_a(t)\rangle = \hat{R}(t)|u_a^{\text{eff}}\rangle$, one arrives at the alternative rewriting of this winding number as

$$\begin{aligned} N_1[R] &= -\frac{i}{2\pi} \sum_a \int_0^T dt \langle u_a(t) | \partial_t u_a(t) \rangle \\ &= -\frac{1}{2\pi} \sum_a \gamma_{AA}^a, \end{aligned} \quad (75)$$

where γ_{AA}^a denotes the Aharonov-Anandan phase of the time-periodic Floquet modes in the NFZ [137], also known as nonadiabatic Berry phase [138,139]. Interestingly, Ref. [101] identified γ_{AA}^a as the average “position” of the a th Floquet state in the frequency domain and has, therefore, interpreted this phase as the polarization of these states in frequency space. Quite notably, our identification

of Eqs. (74) and (75) with the first-order winding number of $\hat{R}(t)$ [see Eq. (72)] imposes that the sum of Aharonov-Anandan phases over a Floquet Brillouin zone should be quantized in integer numbers. These results are in complete analogy with the modern theory of electric polarization $\mathbf{P}$ [123,140–143]. Here, $N_1[R]$ takes the role of $\mathbf{P}$ and the time domain that of quasimomentum space [144]. Similarly to $\mathbf{P}$, the value of $N_1[R]$ depends on the choice of the Floquet zone (see Appendix F). In particular, our choice of the NFZ [59] implies that $N_1[R] = 0$ for $\Phi = 0$.

The total magnetic response of the effective Hamiltonian can be easily related to the winding number $N_1[R]$ by taking the derivative of Eq. (71) with respect to flux. Importantly, the energy spectral flow of the time-averaged Hamiltonian is zero:

$$\frac{\Phi_0}{T} \frac{\partial}{\partial \Phi} \int_0^T dt \text{Tr}[\hat{H}(t)] = \Phi_0 \frac{\partial}{\partial \Phi} \text{Tr}[\hat{H}_0] = 0. \quad (76)$$

Indeed, the dc component of the time-dependent Hamiltonian $\hat{H}_0$ has essentially the same form as the undriven Hamiltonian, possibly with renormalized parameters. As such, its associated angular momentum operator $\propto \partial \hat{H}_0 / \partial \Phi$ is indeed traceless. Therefore, using Eq. (58), we conclude that

$$\mathcal{W}^A = \frac{\Phi_0}{\Omega} \frac{\partial}{\partial \Phi} \text{Tr}[\hat{H}_{\text{eff}}] = \Phi_0 \frac{\partial N_1[R]}{\partial \Phi}; \quad (77)$$

that is to say, the anomalous spectral flow $\mathcal{W}^A$ is given by the derivative with respect to the magnetic flux of the winding number $N_1[R]$. We stress that, even though the value of $N_1[R]$ is defined modulo an integer number (see Appendix F), its variation with magnetic flux is gauge independent and, hence, physically meaningful.

As discussed in Sec. II, the flux derivative of first-order winding numbers (formally understood as a finite difference) is known to generate higher-order topological invariants [89,97]. Indeed, Eq. (77) demonstrates that in a Bloch lattice

$$\Phi_0 \frac{\partial N_1[R]}{\partial \Phi} = N_3[R], \quad (78)$$

where $N_3[R]$ was introduced in Eq. (60). This observation, which naturally derives from our Středa approach, provides yet another route to derive the winding numbers of Refs. [15,23,60].

Finally, we can use Eqs. (56), (57), and (77) to express the total Floquet spectral flow as

$$\begin{aligned} \mathcal{W}(\mu_\epsilon) &= \Phi_0 \left(\frac{\partial N_{\text{eff}}(\mu_\epsilon)}{\partial \Phi} + \frac{\partial N_1[R]}{\partial \Phi} \right) \\ &= \Phi_0 \left(\frac{\partial N_1[G_{\text{eff}}]}{\partial \Phi} + \frac{\partial N_1[R]}{\partial \Phi} \right), \end{aligned} \quad (79)$$

which has a particularly appealing form as it is reminiscent of the equilibrium Středa formula in Eq. (19). As a matter of fact, the number of particles $N(\mu)$, or, equivalently, the winding number $N_1[G]$, is replaced here by the sum of two contributions, $N_1[G_{\text{eff}}] + N_1[R]$, arising from the normal and anomalous flow terms, respectively. We refer to Eq. (79) as the *generalized Floquet-Středa formula for periodically driven systems*, which trivially reduces to Eq. (19) in the absence of the driving field.

The first term in Eq. (79) is analogous to the spectral flow exhibited by equilibrium systems, as it can be entirely obtained from the Chern numbers of the Floquet-Bloch bands; see Eq. (52). In a finite-size system with boundaries, we interpret this term as the one stemming from the flow of *dressed states*, between the edge and the bulk of the sample. In contrast, the second term in Eq. (79) is specific to periodically driven settings and represents a quantized flow of *energy* between the system and the driving field; see Eq. (77). The quantization of the orbital magnetization density found in Eq. (63) is nothing but a reflection of the quantization of this magnetic-field-induced energy flow; see also Eq. (47).

In this framework, the driving field is treated classically, such that it can absorb or emit energy in a continuous fashion. Nonetheless, our results suggest that the magnetic-field-induced energy exchange between the system and the driving field, averaged over one period, is quantized. In this sense, this effect can be physically understood as a quantized energy pump.

Inserting the relation between the anomalous flow and the Floquet orbital magnetization density [Eq. (63)] into Eq. (77) and using Eq. (78), one readily finds that

$$\frac{\Phi_0}{\Omega} \mathcal{M}^F(\epsilon_\pi) + \hbar N_3[R] = 0, \quad (80)$$

where we momentarily restored $\hbar$. Equation (80) is suggestive of a conservation law, reflecting the exchange of quantized angular momentum between the driving field and the system.

Equation (80) further suggests that the anomaly of these exotic Floquet phases could be canceled if one properly incorporates the dynamics of the driving field into the global description of the driven system. This can be thought of as the generalization of the Callan-Harvey mechanism [98–100] for periodically driven Floquet systems. This interpretation is consistent with the results found in Refs. [72,128], where the number of anomalous edge channels was found to be in one-to-one correspondence with the total number of photons involved in the resonant processes that are responsible for the opening of the Floquet gaps. A recent study, which fully incorporates the quantum mechanical description of the driving field, further supports this interpretation [145].

D. A simple illustrative example

To grasp the physical content behind the results presented in this section, let us consider the Rudner-Lindner-Berg-Levin (RLBL) tight-binding model, which was introduced in Ref. [23] as a paradigmatic model for the anomalous Floquet phase. The system consists of a bipartite square lattice with nearest-neighbor hoppings, which are turned on and off in time in a cyclic manner (see Fig. 5). For an appropriate selection of parameters, the bulk dynamics *during* one cycle is such that a particle initially located at a given site will return with probability one to the initial site after performing a closed counterclockwise loop throughout a lattice plaquette. Under these circumstances, the stroboscopic evolution operator reduces to the identity and the corresponding effective Hamiltonian $\hat{H}_{\text{eff}} = 0$. Clearly, this effective Hamiltonian is topologically trivial and cannot distinguish the nontrivial dynamics described above from one where the hopping terms are always off and particles do not move at all. Nevertheless, it is well known [23] that this driving protocol generates chiral states, which are localized at the edges of open-boundary samples. This suggests that the knowledge of the micromotion (i.e., the time evolution during a period) is required to properly characterize the topology of such a phase. However, this situation drastically changes when one applies a small perpendicular magnetic field to the lattice. In this scenario, a negatively charged particle that makes a counterclockwise closed loop acquires an Aharonov-Bohm phase equal to $-2\pi\Phi_p/\Phi_0$, where $\Phi_p = \Phi/2N_c \ll \Phi_0$ is the magnetic flux per plaquette and N_c denotes the number of unit cells. In a torus geometry (i.e., without edges), $\hat{H}_{\text{eff}} = (\Omega/2N_c)(\Phi/\Phi_0)\hat{I} \neq 0$, such that $\text{Tr}[\hat{H}_{\text{eff}}] = \Omega\Phi/\Phi_0$. Interestingly enough, this $\hat{H}_{\text{eff}}$ with a

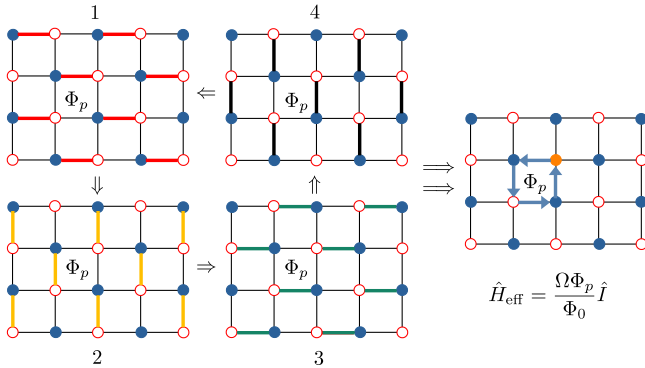


FIG. 5. RLBL model of an anomalous Floquet insulator phase [23]. Nearest-neighbor hoppings are turned on in a sequential manner (1 – 2 – 3 – 4) as indicated by the thick colored lines. The parameters are such that a particle has probability one to jump between two connected sites on each step of the cycle. After a period, a particle initially on a given site (orange dot) makes a counterclockwise loop around a lattice plaquette. If a magnetic flux Φ_p per plaquette is present, such a particle acquires a $-2\pi\Phi_p/\Phi_0$ phase after one period.

nonzero trace cannot be obtained from a high-frequency expansion to any finite order. Finally, from Eqs. (58) and (56), one obtains $\mathcal{W}^A = 1$ and $\mathcal{W}^N(\mu_e) = 0 \forall \mu_e \neq 0$, respectively, such that $\mathcal{W}(\mu_e) = 1$ within all spectral gaps, in agreement with Ref. [23].

In this specific example, all bulk states are localized and the quasienergies are independent of $\mathbf{k}$; in particular, they are equal to zero for $\Phi = 0$. Therefore, the quantization of $\mathcal{W}^A = 1$ implies, by virtue of Eq. (51), or, equivalently, Eq. (63), that the total orbital magnetization density is also quantized (in units of Ω/Φ_0):

$$\mathcal{M}^F(\varepsilon_\pi) = \mathcal{M}_1^F(\varepsilon_\pi) = \sum_{\alpha} \int_{\text{BZ}} \frac{d^2k}{(2\pi)^2} m_z^{\alpha}(\mathbf{k}) = -\frac{\Omega}{\Phi_0}, \quad (81)$$

since, in this fine-tuned case, the term $\mathcal{M}_2^F(\varepsilon_\pi) = 0$. This is consistent with the result found in Refs. [82] for the AFAI phase, where strong disorder was introduced to localize all bulk Floquet states.

At this point, it is important to emphasize that the quantization derived in Eq. (63) does not require localized bulk states, which generically quenches $\mathcal{M}_2^F(\varepsilon_\pi)$. In Sec. V, we present a deeper analysis of the behavior of the two contributions, $\mathcal{M}_1^F(\varepsilon_\pi)$ and $\mathcal{M}_2^F(\varepsilon_\pi)$, in a more general situation (see Fig. 10). In Sec. VII, we additionally study the anomaly $\mathcal{W}^A$ in settings without translational invariance and in the presence of disorder.

In the fine-tuned RLBL model, considering a torus geometry and using Eq. (77), one finds that $N_1[R] = \Phi/\Phi_0 \in \mathbb{Z}$, the quantized magnetic monopole charge. Interestingly, the definition of the winding number in Eq. (72) does not rely on the use of periodic boundary conditions, such that its behavior can also be studied in an inhomogeneous system. In Fig. 6(a), we plot $N_1[R]$ as a function of the flux (now a continuous variable) in an open-boundary sample of the RLBL model. The values of $N_1[R]$ are obtained by using Eq. (77), with the effective Hamiltonian being numerically extracted from the stroboscopic time-evolution operator. The parameters correspond to the driving protocol illustrated in Fig. 5, such that all bulk states are localized while chiral edge modes propagate along the system boundaries. The winding number $N_1[R]$ has a clear staircase behavior, with its coarse-grained slope being directly given by $N_3[R] = \mathcal{W}^A = +1$, as predicted by Eqs. (77) and (78). The fine-tuned parameters of the model allow for an unambiguous separation of the contributions to $N_1[R]$ stemming from the bulk and from the edge of the sample, which we clearly discriminate in Fig. 6(a). For this range of magnetic flux, the bulk state contribution increases monotonically in a smooth fashion, while the edge contribution is a discontinuous sawtooth function with period Φ_0 . When both contributions are smooth, their slope is the same in magnitude but opposite in sign, reflecting the fact that the edge modes have the opposite direction of circulation with respect to the bulk states. This difference

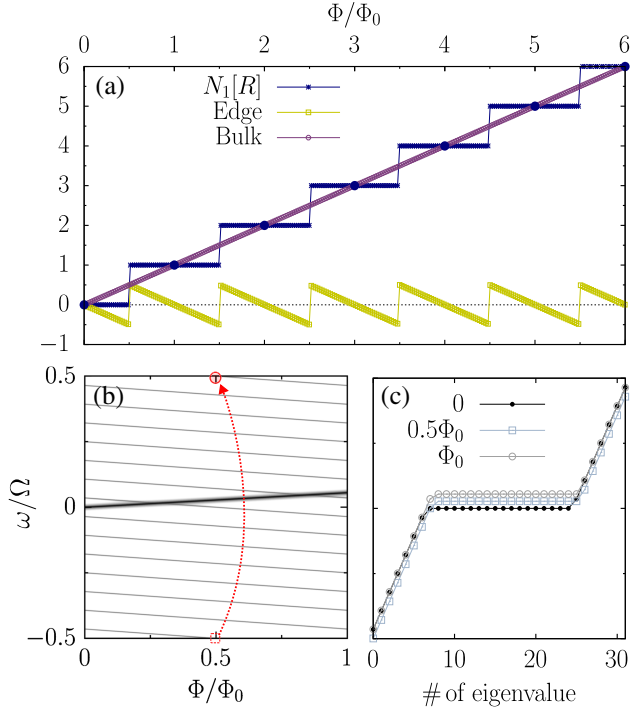


FIG. 6. (a) Winding number $N_1[R]$ for the RLBL model as a function of the total flux threading an open-boundary sample (20×10 unit cells) in units of Φ_0 . The parameters are the fine-tuned ones illustrated in Fig. 5. The coarse-grained slope of this steplike function is given by $N_3[R] = \mathcal{W}^A = +1$. We plot separately the contributions to $N_1[R]$ coming from the states localized around the edge of the system and that of the bulk states. (b) Flux dependence of the quasienergies on the NFZ for a smaller sample (4×4 unit cells). At $\Phi/\Phi_0 = 0.5$, an edge mode “leaves” the NFZ from below and reenters from above, as indicated by the arrow. (c) Quasienergy spectrum of the 4×4 sample for three different values of flux $\Phi/\Phi_0 = 0, 0.5$, and 1.

in the spectral flow of the two contributions is more clearly seen in Fig. 6(b), where we plot the quasienergies of a smaller sample in the NFZ as a function of Φ/Φ_0 . We remind that, within our analysis, the NFZ is fixed to be the one at zero field ($\Phi = 0$). Here, the thick line corresponds to the bulk states, while the rest of the lines correspond to the edge modes. When Φ/Φ_0 reaches half an odd-integer value, the quasienergy corresponding to one of the edge modes leaves the NFZ boundary from below and “reappears” from above, leading to a jump of +1 in $N_1[R]$. This mechanism leads to a corresponding jump of Ω in $\text{Tr}(\hat{H}_{\text{eff}})$ [see Eq. (77)], reflecting the quantized energy pump discussed in Sec. IV C. In Fig. 6(c), we plot the quasienergy spectrum for three different values of $\Phi/\Phi_0 = 0, 0.5, 1$. The degenerate quasienergies near the center of the NFZ correspond to bulk states. It is worth noticing the following: (i) The total number of states inside the NFZ is independent of Φ/Φ_0 as a consequence of Eq. (44); (ii) the number of bulk states is also constant, since the Chern number is zero and so there is no normal flow; (iii) the quasienergies of the

edge modes for $\Phi/\Phi_0 = 1$ have the same value as for $\Phi = 0$, while the quasienergies of the bulk states are shifted up.

This simple example illustrates several of our key results: the possibility of extracting the winding number from the stroboscopic time evolution of a lattice system; the correspondence between the quantization of the total orbital magnetization density of the Floquet bands on a torus geometry and the number of anomalous edge channels; and the magnetic-field-induced energy-pump mechanism leading to discontinuities in the behavior of $N_1[R]$, which are crucial to maintain the quantization of its coarse-grained slope as a function of flux in an open-boundary sample.

V. KITAGAWA MODEL AS A CASE OF STUDY

In order to illustrate how the general formulas derived in Secs. III B, III D, III E, and IV B apply in a concrete setting, we consider a Kitagawa-type model [9] as a case of study [Fig. 7(a)]. The Hamiltonian is described within a tight-binding description of a honeycomb lattice:

$$\hat{H}(t) = \sum_{R \in \mathcal{A}} \sum_{\nu=1}^3 \left(J_{\nu}(t) \hat{c}_R^{\dagger} \hat{c}_{R+\delta_{\nu}} + \text{H.c.} \right) + \Delta \left(\sum_{R \in \mathcal{A}} \hat{c}_R^{\dagger} \hat{c}_R - \sum_{R \in \mathcal{B}} \hat{c}_R^{\dagger} \hat{c}_R \right), \quad (82)$$

where $\delta_1 = (0, a_0)$, $\delta_2 = (-\sqrt{3}a_0/2, -a_0/2)$, and $\delta_3 = (\sqrt{3}a_0/2, -a_0/2)$. The distance between neighboring sites is denoted as a_0 . The second line in Eq. (82) corresponds to an inversion-symmetry-breaking term, an on-site energy

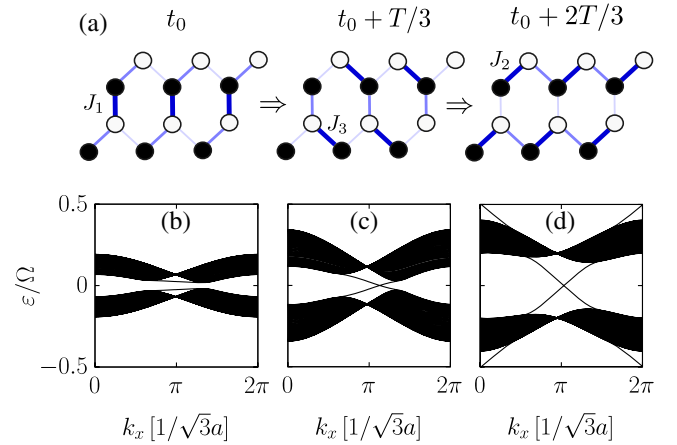


FIG. 7. (a) Driving protocol described by Eq. (83). The width and color of the lines connecting each lattice site represent the strength of the hopping amplitudes for different times $t_0, t_0 + T/3$, and $t_0 + 2T/3$. In the lower panels, we show the quasienergy spectrum of a zigzag ribbon as a function of the quasimomentum along the translationally invariant direction. The parameters are such that $\Omega/J = 20$, $\Delta/J = 0.5$, and (b) $F = 1.0$, (c) $F = 2.0$, and (d) $F = 2.75$.

mass term $+\Delta$ in sublattice $\mathcal{A}$ and $-\Delta$ in sublattice $\mathcal{B}$. The hopping elements are modulated in a continuous fashion [39] as

$$J_\nu(t) = J e^{F \cos(\Omega t + \theta_\nu)}, \quad (83)$$

where F is a dimensionless parameter and $\theta_\nu = -(4\pi/3\sqrt{3}a_0)\delta_\nu \cdot \hat{\mathbf{x}}$. Different variants of this model have been experimentally realized in both photonics [51,52] and ultracold atom systems [39,42].

In Fig. 7, we plot the dispersion relation of the Floquet quasienergies of this model in a ribbon geometry with zigzag terminations. Here, we fix the driving frequency to $\Omega/J = 20$, $\Delta/J = 0.5$, and modify the dimensionless parameter F . For $F = 1.0$ [Fig. 7(b)], the system is a trivial Floquet insulator, with both bands having a Chern number $C_\pm^F = 0$. When $F \approx 1.56$, the gap at $\varepsilon = 0$ closes and the system undergoes a transition to a conventional Floquet topological phase, with bands having $C_\mp^F = \pm 1$ and chiral edge channels bridging the gap [as seen in Fig. 7(c)]. Indeed, the phase transition taking place between Figs. 7(b) and 7(c) emulates the one of the Haldane model presented in Sec. II. As F is further increased, the zone-edge gap at $\varepsilon_\pi = \Omega/2$ vanishes and the transition to the anomalous Floquet phase occurs. As seen in Fig. 7(d), chiral edge channels emerge at both spectral gaps, while the bulk bands have trivial Chern numbers.

In Fig. 8, we plot the energy-resolved Floquet-Středa response of this model [Eq. (33)] as a function of the energy and the parameter F , for $\Omega/J = 20$. Since this quantity is described by an Ω -periodic function, one can safely restrict its analysis to the NFZ ($s = 0$). Throughout the work, we numerically keep a maximum number of Floquet replicas such that $|s| < 5$, which ensures numerical convergence of the results in the NFZ within the range of parameters used. The inset in Fig. 8 shows how the zero-field DOS evolves as a function of F , which essentially controls the bandwidth of the effective model. The physics taking place for $F \lesssim 2.5$ emulates the one discussed in Fig. 2 in Sec. II. Indeed, one can clearly resolve the Berry curvature hot spots coming from the Dirac valleys and the band inversion taking place around $F \simeq 1.56$. In region I, we have a trivial insulator phase, with no spectral flow between the upper and lower bands, while in region II the spectral flow is the one of a topological Chern insulator. When $F \simeq 2.5$, the Floquet quasienergy bands touch the edge of the Floquet Brillouin zone, and the system undergoes a transition to an anomalous Floquet phase III, with chiral edge channels bridging all the gaps in an open-boundary sample [see Fig. 7(d)].

In Fig. 9, we show the energy-resolved Floquet-Středa response for three representative values of F that set the system deep within (I) the trivial insulator phase, (II) the Floquet Chern insulator phase, and (III) the anomalous phase. These are vertical cuts in Fig. 8. We also present its corresponding integrated version, i.e., the normal spectral

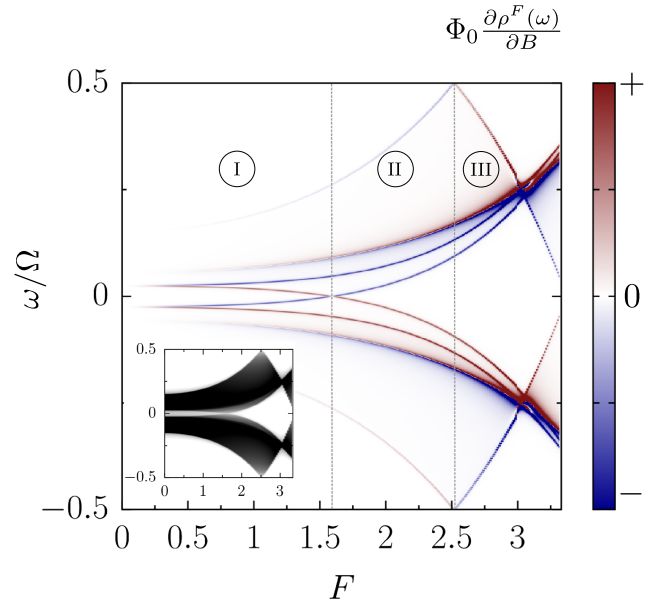


FIG. 8. Energy-resolved Floquet-Středa response [see Eq. (33)] of the Kitagawa model given by Eq. (82) as a function of the dimensionless parameter F . The parameters are such that $\Delta/J = 0.5$ and $\Omega/J = 20$. The color scale is in arbitrary units, with the symbols $\pm$ indicating the sign of the response. In the inset, we show the evolution of the DOS of this model (shaded area) at zero field. The dashed lines separate the different phases of the model: (I) trivial insulator phase, (II) Floquet Chern insulator phase, and (III) anomalous Floquet phase.

flow $\mathcal{W}^N(\mu_\varepsilon)$ in Eq. (50), and the Floquet orbital magnetization density $\mathcal{M}^F(\mu_\varepsilon)$ in units of Ω/Φ_0 [Eq. (65)], for each of these values of F . In each of these plots, the solid red line indicates the average of $\mathcal{W}^N(\varepsilon)$ within the NFZ, $\overline{\langle \mathcal{W}^N \rangle}_{\text{NFZ}} = -\mathcal{W}^A = \Phi_0 \mathcal{M}^F(\Omega/2)/\Omega$; see Eqs. (48) and (63). When μ_ε lies in gapped regions, the normal spectral flow takes quantized values, given by the Chern numbers of the Floquet bands in the interval $(-\Omega/2, \mu_\varepsilon]$; see Eq. (52). These Chern numbers are identically equal to zero in both the trivial and the anomalous phases, as seen from Figs. 9(I) and 9(III). In the Floquet Chern insulator phase, the lower and upper bands have Chern numbers ± 1 , as deduced from Fig. 9(II). The most striking effect is reflected in the finite mean value $\overline{\langle \mathcal{W}^N \rangle}_{\text{NFZ}} = -1$ of the normal spectral flow in Fig. 9(III), highlighted with the red arrow. This signature readily indicates the existence of a finite anomaly, as dictated by Eq. (48), and reflects the existence of a nontrivial topologically quantized Floquet spectral flow, even when the Chern numbers of the Floquet bands are identically equal to zero. The nontrivial quantization of the Floquet orbital magnetization density in units of Ω/Φ_0 is made evident in Fig. 9(III). Indeed, the dashed curve indicates that this quantity takes the half-quantized value of $\mathcal{M}^F(0) = -\Omega/2\Phi_0$ at the center of the Floquet zone and the integer-quantized value $\mathcal{M}^F(\Omega/2) = -\Omega/\Phi_0$ at the zone-edge gap. The origin of the half quantization

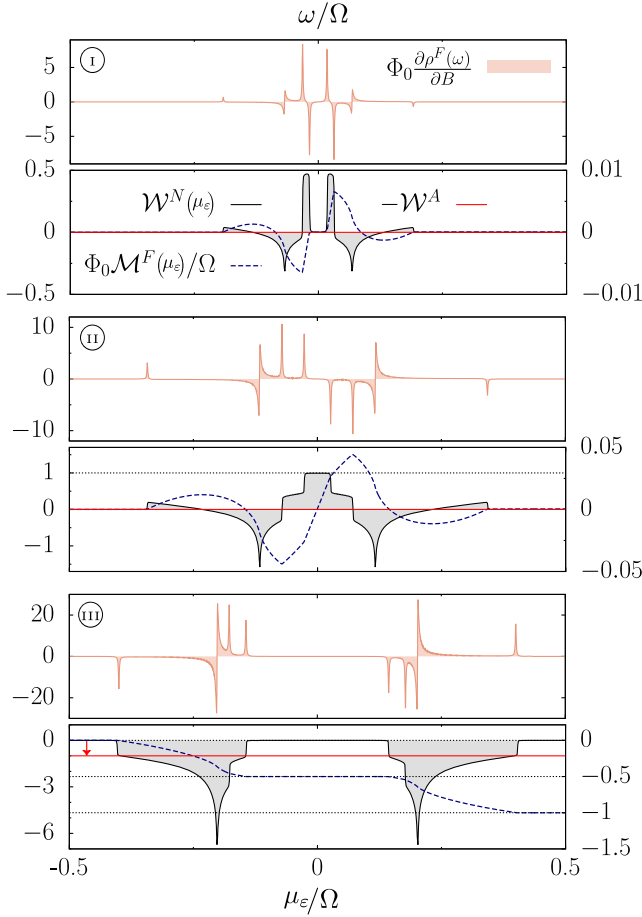


FIG. 9. Energy-resolved Floquet-Středa response [Eq. (33)] (in units of $1/J$) and the corresponding normal spectral flow $\mathcal{W}^N(\mu_\varepsilon)$ (left y axis), given by Eq. (50). We also plot the Floquet orbital magnetization density $\mathcal{M}^F(\mu_\varepsilon)$ (right y axis) with a dashed blue line in units of Ω/Φ_0 [see Eq. (65)]. We have chosen three representative parameters: (I) $F = 1.0$, trivial insulator phase, (II) $F = 2.0$, Floquet Chern insulator phase, and (III) $F = 2.75$, anomalous Floquet phase. The rest of the parameters are the same as in Fig. 8. The red solid line indicates the average of the normal spectral flow within the NFZ, $\overline{\mathcal{W}^N}_{\text{NFZ}} = -\mathcal{W}^A = \Phi_0 \mathcal{M}^F(\Omega/2)/\Omega$ [see Eqs. (48) and (63)]. The red arrow in (III) highlights the anomaly.

occurring at the Floquet-zone center can be traced to the particle-hole symmetry of the problem.

The existence of a finite anomalous flow is further highlighted in Fig. 10. In Fig. 10(a), we plot the Chern numbers of the upper and lower Floquet quasienergy bands ($C_\pm^F$, respectively) within the NFZ as a function of the parameter F . As already anticipated, the Chern numbers are finite only within region II. In region III, the Chern numbers are zero, but the system is still in a topologically nontrivial phase. This is clearly captured by the spectral flow calculated from Eq. (53): In Figs. 10(b) and 10(c), we plot $\mathcal{W}(\mu_\varepsilon)$ for $\mu_\varepsilon = \Omega/2$ and $\mu_\varepsilon = 0$, respectively, as a function of F . Sharp transitions from zero to finite values

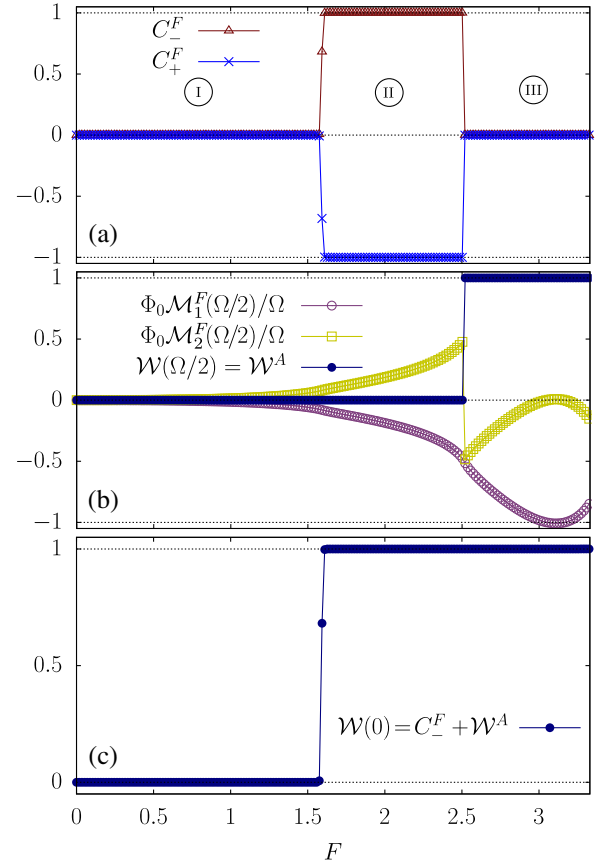


FIG. 10. (a) Chern numbers of the upper and lower quasienergy bands ($C_\pm^F$, respectively) of the Kitagawa model as a function of F . (b),(c) Spectral flow $\mathcal{W}(\mu_\varepsilon)$ as obtained from Eq. (53) for $\mu_\varepsilon = \Omega/2$ and $\mu_\varepsilon = 0$, respectively, as a function of F . In (b), we also plot the two contributions to the anomalous winding number $\mathcal{W}^A$, namely, $\Phi_0 \mathcal{M}_1^F(\Omega/2)/\Omega$ and $\Phi_0 \mathcal{M}_2^F(\Omega/2)/\Omega$ [see Eqs. (68) and (69)]. In all panels, $\Omega/J = 20$ and $\Delta/J = 0.5$.

occur only when the corresponding gap closes and are consistent with the emergence of boundary edge modes in finite-size samples (see Fig. 7).

In Fig. 10(b), we specifically plot the two contributions to the anomalous winding number $\mathcal{W}^A$, namely, $\Phi_0 \mathcal{M}_1^F(\Omega/2)/\Omega$ and $\Phi_0 \mathcal{M}_2^F(\Omega/2)/\Omega$; see Eqs. (68) and (69). Within phases I and II, these two terms are the same in magnitude but have opposite sign, exactly cancelling each other in such a way that $\mathcal{W}^A$ remains zero. Once the zone-edge gap vanishes and the parameter F is within region III, these two contributions do not cancel anymore and exactly sum to an integer number (in this case, -1), which correctly quantifies the quantized orbital magnetization density $\mathcal{M}^F(\Omega/2)$ in units of Ω/Φ_0 [see Eq. (63)]. Within the region where $\mathcal{M}_2^F(\Omega/2)$ is vanishingly small or zero, the Floquet bands become extremely narrow (see the inset in Fig. 8), hence reaching the limit of the fine-tuned RLBL model discussed in Eq. (81) in Sec. IV D. We remark that, when moving away from this finely tuned limit—i.e., for

general parameter values—the bulk Floquet states are extended, while the total orbital magnetization density continues to be quantized.

VI. SUM-RULE PROCEDURE: WINDING NUMBERS FROM THE TIME-AVERAGED FLOQUET DOS

In this section, we present an indirect way to access the quantized spectral flow $\mathcal{W}(\mu_\epsilon)$, which is particularly appealing from the point of view of possible experimental explorations. In particular, this protocol which is based on an engineered-bath scenario, provides a concrete scheme to measure the Floquet-Středa response within a genuine many-body context.

The starting point is the alternative rewriting of the Floquet density of states defined in Eqs. (31) and (30) as the infinite sum

$$\rho^F(\omega) = \sum_{m=-\infty}^{\infty} \rho_0^F(\omega + m\Omega), \quad (84)$$

where $m \in \mathbb{Z}$ and where we introduced

$$\rho_0^F(\omega) = \frac{1}{A} \sum_{a,s} \langle u_a^{(s)} | u_a^{(s)} \rangle \delta(\omega - \epsilon_{as}). \quad (85)$$

Equation (85) defines the well-known *time-averaged* Floquet density of states [16,21,72,146,147], which projects the DOS of the extended Floquet Hamiltonian into its static component (the zeroth harmonic). We remark that $\rho_0^F(\omega)$ in Eq. (85) is localized in frequency space due to the localization of $|u_a^{(s)}\rangle$ in the index s . The Floquet spectral flow at quasienergy μ_ϵ [see Eq. (32)] can, hence, be obtained as

$$\begin{aligned} \mathcal{W}(\mu_\epsilon) &= \Phi_0 \int_{-\infty}^{\mu_\epsilon} d\omega \sum_{m=-\infty}^{\infty} \frac{\partial \rho_0^F(\omega + m\Omega)}{\partial B} \\ &\stackrel{(C,1)}{=} \Phi_0 \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\mu_\epsilon + m\Omega} d\omega \frac{\partial \rho_0^F(\omega)}{\partial B}. \end{aligned} \quad (86)$$

We note that, in going from the first line to the second line in Eq. (86), we have switched the unbounded integral and the summation over the infinite $m \in \mathbb{Z}$ via a Cesàro regularization procedure; see Appendix G. Importantly, even though the first line in Eq. (86) is an ill-defined (nonconvergent) integral, the infinite sum in the second equality is completely convergent, in the sense that only a few values of m around $m = 0$ are needed to obtain $\mathcal{W}(\mu_\epsilon)$ with μ_ϵ in the NFZ. Indeed, the terms with $m \ll -1$ are clearly small, due to the localization of $\rho_0^F(\omega)$ at low frequencies. On the other hand, using Eq. (44), one can straightforwardly show that

$$\int_{-\infty}^{\infty} d\omega \frac{\partial \rho_0^F(\omega)}{\partial B} = 0, \quad (87)$$

which means that the terms with $m \gg 1$ in Eq. (86) will also be vanishingly small.

Equation (86) can be rewritten in a more appealing fashion, by introducing the time-averaged particle-density

$$\bar{n}|_{\mu_\epsilon} = \int_{-\infty}^{\mu_\epsilon} d\omega \rho_0^F(\omega). \quad (88)$$

In this way, the Floquet spectral flow at quasienergy μ_ϵ , as obtained from Eq. (86), can be written as the following summation:

$$\mathcal{W}(\mu_\epsilon) = \Phi_0 \sum_{m=-\infty}^{\infty} \left. \frac{\partial \bar{n}}{\partial B} \right|_{\mu_\epsilon + m\Omega}, \quad (89)$$

from here on referred to as the *Floquet-Středa sum-rule procedure*.

Using the formalism developed in Appendix B, it is possible to obtain the magnetic response of the time-averaged Floquet density of states on a translationally invariant system as

$$\begin{aligned} \Phi_0 \frac{\partial \rho_0^F(\omega)}{\partial B} &= \text{Im} \int_{\text{BZ}} \frac{d^2 k}{(2\pi)^2} \epsilon^{zjl} \mathbf{tr} \left[i \hat{\mathbf{P}}_{0k} \mathbf{G}_k^F(\omega + i0^+) \frac{\partial \hat{\mathbf{H}}_k^F}{\partial k_j} \right. \\ &\quad \left. \times \mathbf{G}_k^F(\omega + i0^+) \frac{\partial \hat{\mathbf{H}}_k^F}{\partial k_l} \mathbf{G}_k^F(\omega + i0^+) \right], \end{aligned} \quad (90)$$

where $[\hat{\mathbf{P}}_{0k}]_{nm} = \delta_{n0} \delta_{m0} \hat{I}_k$, with $\hat{I}_k$ the identity in $\mathbf{k}$ space. Equation (90) is referred to as the *energy-resolved Floquet-Středa response of the time-averaged DOS*.

By means of Eqs. (88) and (90), we can numerically evaluate the different contributions appearing in Eq. (89) entirely from the knowledge of the bulk Floquet-Bloch Green's functions $\hat{\mathbf{G}}_k^F$ and Hamiltonian $\hat{\mathbf{H}}_k^F$.

The restoration of quantized response functions via Floquet sum-rule schemes was originally discussed in Refs. [69,73,74]. These theoretical works focused on the study of edge transport coefficients in open-boundary samples subjected to a time-periodic drive and showed how quantized conductivities could be recovered by performing a series of measurements where the chemical potential of the leads is shifted in integer multiples of the driving frequency. As opposed to these transport settings, the Floquet-Středa sum-rule procedure introduced in Eq. (89) is entirely based on the measurement of a static response function that generically has nontrivial contributions from the edge and from the bulk of an open-boundary sample.

As shown in Appendix H, the simplest model of a fermionic heat bath, described by featureless Büttiker-type reservoirs [148] connected to each site of the driven lattice,

leads precisely to the steady-state particle density given by Eq. (88) when taking the limit of weak hybridization and zero temperature. In this case scenario, the value μ_e in Eq. (88) is physically determined by the chemical potential μ of the bath, such that the sum rule in Eq. (89) could be explicitly tested by changing this chemical potential in multiples of Ω . Interestingly, this Büttiker bath does not lead to a thermal occupation of the Floquet bands in the NFZ [see Eq. (H7) in Appendix H]. Notwithstanding, this highly nonthermal occupation is precisely the one that is needed to obtain quantization through the Floquet-Středa sum-rule procedure. In this sense, this approach relaxes the typical requirement of preparing an insulatorlike steady state, which fully occupies the Floquet-Bloch bands. Engineering a bath as the one described in Appendix H could be possible with current experimental techniques in cold atom platforms; see Refs. [149,150].

To further illustrate how the Floquet-Středa sum-rule procedure applies in practice, we now perform an explicit evaluation of these expressions for the Kitagawa model studied in Sec. V. In Fig. 11(a), we present the energy-resolved Floquet-Středa response of the time-averaged DOS $\Phi_0[\partial\rho_0^F(\omega)/\partial B]$ [see Eq. (90)], as a function of the parameter F in the Kitagawa model. The color scale is in arbitrary units, with $\pm$ indicating the sign of the response. (b) Time-averaged Floquet DOS $\rho_0^F(\omega)$ as a function of frequency and the parameter F for zero magnetic field (note the logarithmic color scale). The parameters are the same as in Fig. 8. We can appreciate how both $\rho_0^F(\omega)$ and its magnetic field derivative are localized at low energies, with most of their weight in the NFZ for small values of F . As F increases, the power spectrum of the

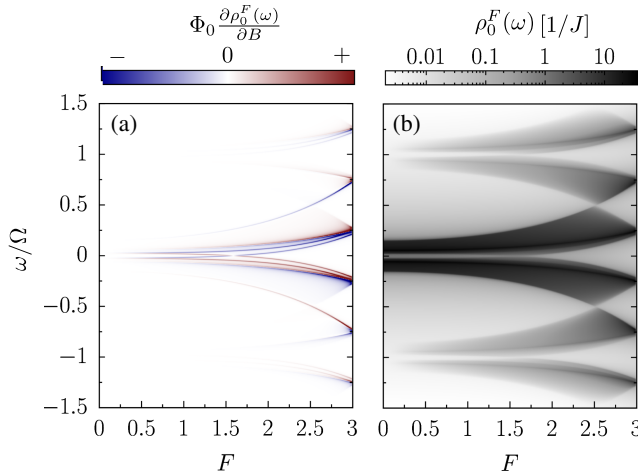


FIG. 11. (a) Energy-resolved Floquet-Středa response of the time-averaged DOS, $\Phi_0[\partial\rho_0^F(\omega)/\partial B]$ [see Eq. (90)], as a function of the parameter F in the Kitagawa model. The color scale is in arbitrary units, with $\pm$ indicating the sign of the response. (b) Time-averaged Floquet DOS $\rho_0^F(\omega)$ as a function of frequency and the parameter F for zero magnetic field. Note the logarithmic color scale.

time-periodic Floquet states is spread over more harmonics, leading to a corresponding spreading of the DOS and its magnetic response over the rest of the Floquet Brillouin zones.

The Floquet spectral flow $\mathcal{W}(\mu_e)$ evaluated at the different spectral gaps of the NFZ, as obtained from the sum rule in Eq. (89), is depicted in Fig. 12. Here, we highlight the evolution of the $m = 0, \pm 1, \pm 2$ contributions to this response function as a function of F . In Fig. 12(a), we fix $\mu_e = 0$, while in Fig. 12(b), we set $\mu_e = \Omega/2$. For this range of parameters, the response is remarkably well converged and quantized by only summing these few values of m . The quantized values obtained for the different phases are in nice agreement with the ones in Fig. 10. We note that the symmetry between the $\pm m$ contributions in Fig. 12(a) is a consequence of particle-hole symmetry of the model at zero field. This also explains the symmetry between the m and $-m-1$ contributions in Fig. 12(b). The increase of the contributions with higher values of m when increasing the parameter F in Fig. 12 is consistent with the redistribution of spectral weight appreciated in Fig. 11.

These results demonstrate that the knowledge of the magnetic response of the time-averaged Floquet DOS is enough to fully characterize the topological properties of Floquet driven systems. This is another central result of the present work, as it provides an interesting route for experimentally accessing Floquet topological invariants in driven-dissipative systems with engineered heat baths. In the next section, we further develop this idea by defining a real-space version of the Floquet spectral flow.

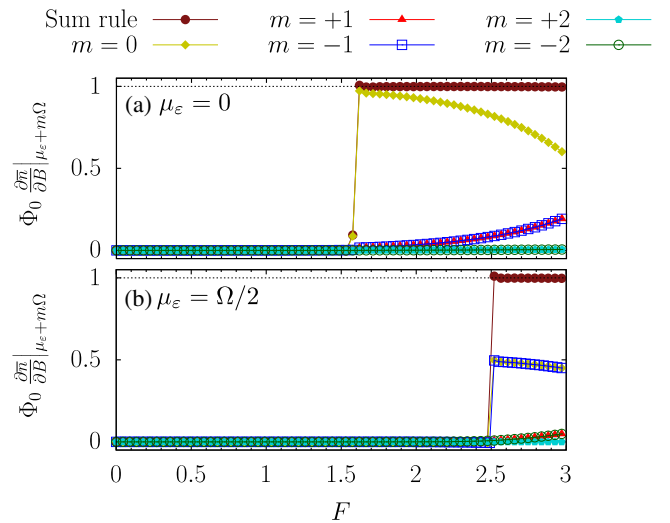


FIG. 12. Floquet spectral flow obtained from the sum-rule procedure in Eq. (89). We present the contributions $m = 0, \pm 1, \pm 2$ as a function of F for (a) $\mu_e = 0$ and (b) $\mu_e = \Omega/2$.

VII. REAL-SPACE APPROACH TO FLOQUET TOPOLOGICAL INVARIANTS

In this section, we address the problem of promoting the bulk winding numbers that characterize the topological structure of two-dimensional Floquet driven systems to real-space-resolved markers. While local topological markers have been extensively studied both in and out of equilibrium [151–156], their extension and generalization to Floquet topological phases still remains elusive. Indeed, the local Chern number identified in Ref. [152] has been proven to be useful only to characterize driven Floquet phases when Ω is larger than the bandwidth of the undriven model and when the initially occupied states evolve under a unitary time evolution that adiabatically populates the Floquet bands [154].

We remark that, until this point, the Floquet spectral flow as obtained from Eq. (53), and its two contributions, $\mathcal{W}^N(\mu_\varepsilon)$ and $\mathcal{W}^A$, as defined from Eqs. (50) and (51), have been explicitly computed only in lattices with discrete translational symmetry. Nevertheless, the notion of spectral flow can be straightforwardly applied in a local fashion by defining the real-space-resolved version of these quantities as

$$\mathcal{W}(\mu_\varepsilon, \mathbf{R}) = \mathcal{W}^N(\mu_\varepsilon, \mathbf{R}) + \mathcal{W}^A(\mathbf{R}), \quad (91)$$

where

$$\mathcal{W}^N(\mu_\varepsilon, \mathbf{R}) = \Phi_0 \int_{\varepsilon_\pi - \Omega}^{\mu_\varepsilon} d\omega \frac{\partial \rho^F(\omega, \mathbf{R})}{\partial B} \quad (92)$$

and

$$\mathcal{W}^A(\mathbf{R}) = \Phi_0 \int_{\text{NFZ}} d\omega \frac{\partial \rho^F(\omega, \mathbf{R})}{\partial B} \frac{\omega}{\Omega}. \quad (93)$$

Here, we have defined the real-space versions of the normal and anomalous spectral flow by means of the local Floquet density of states at the unit cell $\mathbf{R}$, given by

$$\rho^F(\omega, \mathbf{R}) = -\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \sum_{\nu} \text{Im}[\langle \mathbf{R}_\nu | \hat{\mathbf{G}}_{nn}^F(\omega + i0^+) | \mathbf{R}_\nu \rangle], \quad (94)$$

where the block elements of the Floquet Green's function are given in Eq. (B3) in Appendix B. Interestingly, these markers do not deal with the problem of the occupation of the Floquet bands [154] but rely only on its spectral properties. From here on, we refer to $\mathcal{W}(\mu_\varepsilon, \mathbf{R})$ as the local Floquet winding number.

Deep within the gapped bulk of an homogeneous sample, Eqs. (91)–(93) should coincide with the bulk invariants that we previously identified for lattice systems in a torus geometry. This is explicitly shown in Fig. 13, where we compute these quantities along the width of an

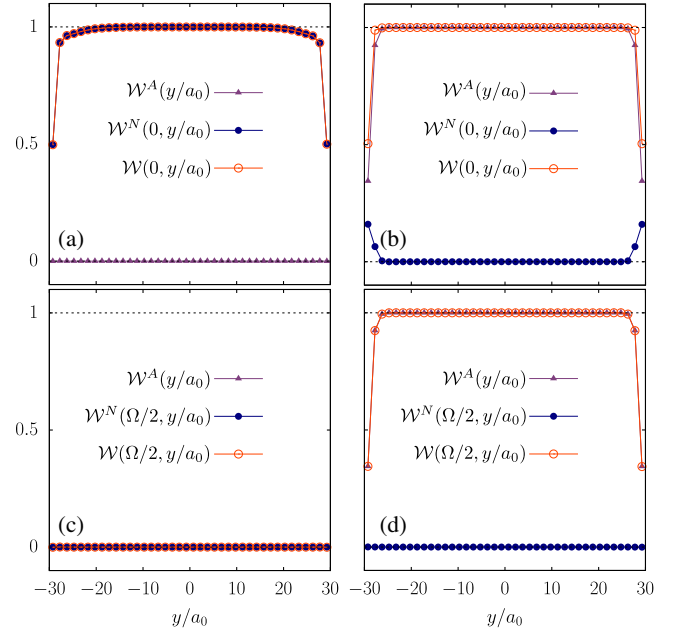


FIG. 13. Local Floquet winding number $\mathcal{W}(\mu_\varepsilon, \mathbf{R})$ in an infinite zigzag ribbon of the Kitagawa model [Eq. (82)] as a function of the y coordinate along its width ($N_y = 40$ unit cells). We also show its two contributions $\mathcal{W}^N(\mu_\varepsilon, \mathbf{R})$ and $\mathcal{W}^A(\mathbf{R})$. (a) $\mu_\varepsilon = 0$ and $F = 2.0$, (b) $\mu_\varepsilon = 0$ and $F = 2.75$, (c) $\mu_\varepsilon = \Omega/2$ and $F = 2.0$, and (d) $\mu_\varepsilon = \Omega/2$ and $F = 2.75$. In all the panels, $\Omega/J = 20$ and $\Delta/J = 0.5$.

infinite ribbon of the Kitagawa model studied in Sec. V. In Figs. 13(a) and 13(b), we focus on the gap at the center of the NFZ ($\mu_\varepsilon = 0$) and evaluate the local winding numbers for $F = 2.0$ (Floquet Chern insulator phase II) and $F = 2.75$ (anomalous Floquet phase III), respectively. The plots nicely confirm that the local winding number approaches the value 1 in the bulk of the ribbon—an indication of the existence of nontrivial topological edge modes for both set of parameters (see Fig. 7)—while it deviates from the quantized value near the boundaries of the sample. In Fig. 13(a), all the contribution comes from $\mathcal{W}^N(0, \mathbf{R})$, while in Fig. 13(b) the main contribution comes from the local anomaly $\mathcal{W}^A(\mathbf{R})$, in agreement with the results in Fig. 10. We note that the local anomaly in Fig. 13(a) is strictly zero across the entire ribbon, in agreement with the fact that the anomalous spectral flow should strictly vanish for Ω larger than the bandwidth of the model (as is the case here). In Figs. 13(c) and 13(d), we focus on the evaluation of the local winding number at the zone-edge gap ($\mu_\varepsilon = \varepsilon_\pi = \Omega/2$) and again consider the parameters $F = 2.0$ and $F = 2.75$, respectively. In Fig. 13(c), we obtain a trivial local response, indicating the absence of edge states at $\Omega/2$ for these parameters. In contrast, in Fig. 13(d), we recover the quantized value of 1 deep in the bulk. Note that, in these last two panels, the spectral flow given by $\mathcal{W}^N(\Omega/2, \mathbf{R})$ is strictly zero.

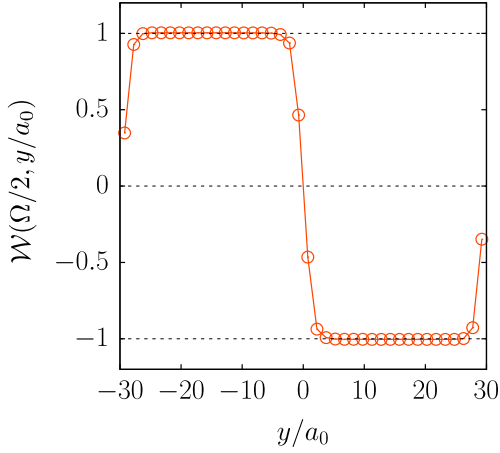


FIG. 14. Local Floquet winding number $\mathcal{W}(\mu_\epsilon, \mathbf{R})$ at $\mu_\epsilon = \Omega/2$ in an infinite zigzag ribbon of the Kitagawa model [Eq. (82)] as a function of the y coordinate along its width. The ribbon is divided into two infinite halves. While both are in the anomalous regime ($F = 2.75$) and with parameters $\Delta/J = 0.5$ and $\Omega/J = 20$, the upper half ($y > 0$) has hopping phases with a relative change of sign with respect to the lower half ($y < 0$), i.e., $\theta_\nu(y < 0) = -\theta_\nu(y > 0)$, hence changing the chirality of the anomalous edge state.

These local quantities can be used as well in the presence of disorder or to identify regions of different topological order in heterojunctions. This latter situation is shown in Fig. 14, where we plot the local Floquet winding number $\mathcal{W}(\mu_\epsilon, \mathbf{R})$ at $\mu_\epsilon = \Omega/2$ in a Kitagawa ribbon that is divided at $y = 0$ into two halves. While both halves are in the anomalous regime ($F = 2.75$) and with parameters $\Delta/J = 0.5$ and $\Omega/J = 20$, the upper half ($y > 0$) has modulated nearest-neighbor hoppings [Eq. (82)] with phases that have a relative change of sign with respect to the lower half ($y < 0$), i.e., $\theta_\nu(y < 0) = -\theta_\nu(y > 0) \forall \nu$, hence changing the chirality of the anomalous edge states. This result nicely confirms that the local winding number in Eq. (91) is able to distinguish phases with different topological order in different bulklike regions. We have verified (not shown) that our approach correctly describes systems with arbitrary large winding numbers, as, for example, graphene in the presence of circularly polarized light with a frequency much smaller than the bandwidth [72,128].

In Fig. 15, we study how disorder affects the local Floquet winding number in the anomalous regime ($F = 2.75$). To that end, we add to the Hamiltonian in Eq. (82) a random time-independent potential of the form $\hat{V}_{\text{disorder}} = \sum_{\mathbf{R} \in \{\mathcal{A}, \mathcal{B}\}} w_{\mathbf{R}} \hat{c}_{\mathbf{R}}^\dagger \hat{c}_{\mathbf{R}}$, with the on-site energies $w_{\mathbf{R}}$ randomly drawn from a uniform distribution in the interval $[-W, W]$. The spatial dependence of the local anomalous winding number $\mathcal{W}^A(\mathbf{R})$ within a finite flake for $W = 0.22\Omega$ and a given disorder realization is shown in Fig. 15(a). It clearly appears that, within the bulk, this quantity fluctuates around a macroscopic value of approximately 1, the quantized value expected for the clean

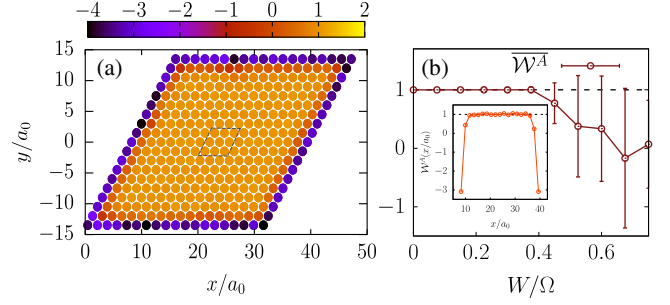


FIG. 15. (a) Local anomalous winding number $\mathcal{W}^A(\mathbf{R})$ in a finite flake (19×19 unit cells) of the Kitagawa model for $F = 2.75$, $\Omega/J = 20$, and a given realization of random diagonal disorder of strength $W = 0.22\Omega$. (b) Average of the local anomalous winding number $\bar{\mathcal{W}}^A$ over the bulk region delimited with dashed gray lines in (a) as a function of disorder strength W . The error bars represent the standard deviation of the local marker within the bulk region. Each point has been obtained by averaging over 100 disorder realizations. The inset shows a horizontal cut ($y = 0$) of the local Floquet marker for a disorder strength of $W = 0.22\Omega$ with the same disorder realization shown in (a).

system ($W = 0$). In Fig. 15(b), we present the averaged local marker within a bulk region as a function of disorder strength W , showing how it remains remarkably well quantized until a critical disorder strength of $W_c \sim 0.4\Omega$. We have verified that, at this critical value, the bulk spectral gap of the flake closes at the Floquet zone edge.

These results confirm that the total bulk orbital magnetization density of Floquet states within a zone is quantized for both clean and disordered systems, provided the dynamical bulk gap at the Floquet zone edge remains open.

VIII. DISCUSSION AND PERSPECTIVES

This work introduced a general framework to classify topological Floquet systems based on the notion of spectral flow inherent to the equilibrium Středa formula, hence offering a unified and coherent approach to the bulk-boundary correspondence of periodically driven systems.

Our theory provides a physical route to derive Floquet winding numbers, elucidating how these topological invariants can be entirely built from simple bulk properties of the Floquet states defined in Sambe space. Importantly, this theory identifies a fundamental mechanism at the origin of anomalous edge states: the emergence of a magnetic-field-induced energy pump, between the system and the driving field, which is inextricably linked to the presence of resonant processes. Furthermore, our approach reveals that the classification of topological Floquet systems directly derives from the magnetic response of their density of states, such that all Floquet winding numbers can be fully extracted from the stroboscopic time evolution of driven systems.

Interestingly, periodically driven systems belonging to certain symmetry classes [60] can exhibit anomalous edge

modes with rich structures, which are not accounted for by the $N_3[R]$ invariant in Eq. (60). This is the case of the driven systems analyzed in Refs. [157–159], where particle-hole symmetry gives rise to helical edge modes that are reminiscent of valley Hall systems, i.e., robust edge modes with opposite chirality that emerge at opposite valleys (gap closing points) in the Brillouin zone. As pointed out in Ref. [158], the bulk-boundary correspondence of these symmetry-protected Floquet settings is correctly captured by a set of topological invariants that properly count the number of counterpropagating edge modes in each gap. In this context, a quantized Středa response could be generated by a perturbation that generates a helical spectral flow of edge states, i.e., a momentum-dependent pseudo-magnetic field that acts on opposite valleys with an opposite sign. In static valley-Hall systems, Ref. [94] demonstrated that such a helical Středa-type response can be generated by strain. Extending this approach to Floquet systems with anomalous helical edge states and, more generally, elucidating generalized Středa responses for all topological symmetry classes [60] constitutes an exciting program.

Another promising direction concerns the generalization of our work to correlated systems. Indeed, the classification of interacting Floquet topological systems and the stabilization of these dynamical phases is a subject of current research interest [81,83,160–171]. Our approach could be employed to characterize many-body Floquet systems in nonequilibrium steady states stabilized by coupling to thermal reservoirs [172,173]. In such settings, a positive-definite Floquet density of states can be defined [174], allowing for the extraction of the magnetic response from the single-particle correlators in the Floquet representation [175]. This could offer a new perspective on the physical interpretation of the topological indices that are relevant to classify correlated Floquet phases. Such a generalization of our approach would also be relevant for the experimental characterization of quantum-engineered Floquet settings, such as drive-induced fractional Chern insulators and chiral spin liquids [7,8,176–180].

A protocol to dynamically detect the local Floquet density of states and to measure the quasienergy spectrum was recently introduced in Ref. [181]. Having access to the magnetic response of these quantities represents a promising route to measure the energy-resolved Floquet-Středa response and the local Floquet winding numbers introduced in this paper. Recent advances in the engineering of heat baths in quantum simulator settings could also open the possibility of extracting the winding numbers from density measurements by means of the sum-rule scheme proposed in our work.

From a more fundamental point of view, it would be interesting to further explore the quantized energy-pump mechanism encoded in the anomalous flow, by theoretically considering the full quantum-mechanical description

of the driving field [145,182–192]. Within our semiclassical Floquet picture, the first-order winding number of the micromotion operator $N_1[R]$ is defined modulo an arbitrary integer number, but its absolute value could be meaningful in a fully quantized theory, where it should represent the average number of quanta of the driving field and, hence, be bounded from below. Establishing a rigorous description of this topological index and its magnetic field derivative in the context of quantum Floquet-engineered systems could provide new insight on the physical origin of anomalous edge states in periodically driven settings. In particular, the interpretation of $\hbar N_3[R]$ as being the angular momentum exchanged between the system and the driving field [Eq. (80)] should naturally emerge from a description in terms of driving-field quanta.

Let us conclude with a more formal remark, regarding the use of Cesàro summation in this physical context. The identification of a Grandi-type series at the very core of Floquet topological physics but also the fact that its formal evaluation gives rise to a physical result are, at the very least, intriguing. In particular, the fact that the infinite summation of apparent zeros (i.e., the trivial Chern numbers in Fig. 1) leads to a *finite* (and quantized) result highlights the exquisite geometric fine structure that is hidden within each Floquet-Bloch band. We are unaware of other applications of the Cesàro summation scheme as a method to regularize winding numbers of operators with an unbounded spectrum, as was the case for the Floquet Green's function index $N_3[G^F]$ defined in Eq. (37). It would be interesting to explore whether this mathematical method could be further employed to identify new classes of topological indices and phenomena in physical settings.

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L. P. G. conceived the project and developed the theoretical formalism with inputs from G. U. and N. G. All authors contributed to the analysis and interpretation of the results and to the writing of the manuscript.

DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: DERIVATION OF EQ. (11) FROM THE MODIFIED DENSITY OF STATES IN PHASE SPACE

In Ref. [118], the authors showed, using semiclassical arguments, that the phase-space density of states of Bloch electrons is modified in the presence of both a finite Berry curvature and a weak external magnetic field. In a two-dimensional crystal, up to first order in the field, the phase-space DOS can be expressed as

$$g_\alpha(\mathbf{k}) = \frac{1}{(2\pi)^2} \left(1 + \frac{2\pi}{\Phi_0} B \mathcal{F}_{xy}^\alpha(\mathbf{k}) \right), \quad (\text{A1})$$

with the index α indicating the Bloch band under consideration. In this appendix, we show that the energy-resolved density of states response we report in Eq. (11) is in full agreement with Eq. (A1). By definition,

$$\rho(\omega) = \int_{\text{BZ}} d^2k \sum_\alpha g_\alpha(\mathbf{k}) \delta(\omega - \varepsilon_{\alpha\mathbf{k}}), \quad (\text{A2})$$

such that

$$\begin{aligned} \Phi_0 \frac{\partial \rho(\omega)}{\partial B} \Big|_{B=0} &= \int_{\text{BZ}} \frac{d^2k}{2\pi} \sum_\alpha \mathcal{F}_{xy}^\alpha \delta(\omega - \varepsilon_{\alpha\mathbf{k}}) \\ &- \Phi_0 \int_{\text{BZ}} \frac{d^2k}{(2\pi)^2} \sum_\alpha \frac{\partial \delta(\omega - \varepsilon_{\alpha\mathbf{k}})}{\partial \omega} \frac{\partial \varepsilon_{\alpha\mathbf{k}}}{\partial B}. \end{aligned} \quad (\text{A3})$$

Using the semiclassical equation for the intrinsic orbital magnetic moment of a wave packet centered at $\mathbf{k}$ in the α th Bloch band

$$m_z^\alpha(\mathbf{k}) = -\frac{\partial \varepsilon_{\alpha\mathbf{k}}}{\partial B} = \frac{2\pi}{\Phi_0} \text{Im}[\langle \partial_{k_x} u_{\alpha\mathbf{k}} | \hat{H}_{\mathbf{k}} - \varepsilon_{\alpha\mathbf{k}} | \partial_{k_y} u_{\alpha\mathbf{k}} \rangle], \quad (\text{A4})$$

and replacing this expression in Eq. (A3), we recover Eq. (11) in the main text.

APPENDIX B: FLOQUET GREEN'S FUNCTIONS IN THE PRESENCE OF A PERTURBING MAGNETIC FIELD

In this appendix, we generalize the formalism used in Sec. II to obtain the first-order correction of the Floquet-Green's function in the presence of a perturbing magnetic field. The two-time propagators of systems subjected to a time-periodic drive are expressed as [173]

$$\hat{G}(t, t') = \sum_{m,n} \int_{\text{NFZ}} \frac{d\omega}{2\pi} e^{-i(\omega+m\Omega)t} e^{i(\omega+n\Omega)t'} \hat{\mathbf{G}}_{mn}^F(\omega), \quad (\text{B1})$$

where $\hat{\mathbf{G}}^F(\omega)$ stands for the time-independent propagator written in Sambe representation. In the absence of interactions,

$$\hat{\mathbf{G}}^F(\omega) = (\omega \hat{\mathbf{I}} - \hat{\mathbf{H}}^F)^{-1}, \quad (\text{B2})$$

with $\hat{\mathbf{H}}^F$ the infinite-dimensional Floquet Hamiltonian. We note that

$$\hat{\mathbf{G}}_{mn}^F(\omega) = \sum_{a,s} \frac{|u_a^{(m+s)}\rangle \langle u_a^{(n+s)}|}{\omega - \varepsilon_{as}}, \quad (\text{B3})$$

where we remind that the index a is restricted such that the quasienergies ε_a lie in the NFZ and the sum over s runs over all $\mathbb{Z}$. The Dyson equations of motion coupling these blocks are given by

$$\sum_{m'} [\omega \delta_{mm'} \hat{\mathbf{I}} - \hat{\mathbf{H}}_{mm'}^F] \hat{\mathbf{G}}_{m'n}^F = \delta_{mn} \hat{\mathbf{I}}. \quad (\text{B4})$$

In the presence of a small magnetic field, the matrix elements of the Floquet Hamiltonian in real space are modified as

$$[\hat{\mathbf{H}}_{mn}^F]_{\mathbf{R}_i \mathbf{R}'_j} = e^{i(2\pi/\Phi_0)\varphi_{\mathbf{R}_i \mathbf{R}'_j}} [\hat{\mathbf{H}}_{mn}^{F0}]_{\mathbf{R}_i \mathbf{R}'_j}, \quad (\text{B5})$$

with the Peierls phase factors defined as in Eq. (5) and with $\hat{\mathbf{H}}^{F0}$ being the Floquet Hamiltonian in the absence of the field. In the same spirit of the derivation in Sec. II, we can introduce a gauge-invariant Floquet propagator with matrix elements defined as

$$[\hat{\mathbf{G}}_{mn}^{F(B)}(\omega)]_{\mathbf{R}_i \mathbf{R}'_j} = e^{-i(2\pi/\Phi_0)\varphi_{\mathbf{R}_i \mathbf{R}'_j}} [\hat{\mathbf{G}}_{mn}^F(\omega)]_{\mathbf{R}_i \mathbf{R}'_j}. \quad (\text{B6})$$

Introducing Eq. (B6) into Eq. (B4), we find that the gauge-invariant Floquet propagator satisfies the following modified Dyson's equation of motion:

$$\begin{aligned} \sum_{\mathbf{R}'_j} \sum_{m'} \left(\omega \delta_{\mathbf{R}_i \mathbf{R}'_j} \delta_{mm'} - [\hat{\mathbf{H}}_{mm'}^{F0}]_{\mathbf{R}_i \mathbf{R}'_j} \right) [\hat{\mathbf{G}}_{m'n}^{F(B)}(\omega)]_{\mathbf{R}'_j \mathbf{R}''_j} \\ \times e^{i(\pi/\Phi_0) \mathbf{B} \cdot (\mathbf{R}'_j - \mathbf{R}_i) \times (\mathbf{R}''_j - \mathbf{R}'_j)} = \delta_{mn} \delta_{\mathbf{R}_i \mathbf{R}''_j}. \end{aligned} \quad (\text{B7})$$

Fourier transforming Eq. (B7) to quasimomentum space and expanding up to first order in the magnetic perturbation, we find that

$$\hat{\mathbf{G}}_k^{F(B)}(\omega) = \hat{\mathbf{G}}_k^F(\omega) + \frac{i\pi}{\Phi_0} B^i \epsilon^{ijl} \hat{\mathbf{G}}_k^F(\omega) \frac{\partial \hat{\mathbf{G}}_k^{F-1}(\omega)}{\partial k_j} \frac{\partial \hat{\mathbf{G}}_k^F(\omega)}{\partial k_l}, \quad (\text{B8})$$

where

$$\hat{\mathbf{G}}_k^F(\omega) = (\omega \hat{\mathbf{I}}_k - \hat{\mathbf{H}}_k^F)^{-1} \quad (\text{B9})$$

is the Floquet-Bloch Green's function in the absence of the field. We have here introduced the Floquet-Bloch Hamiltonian $\hat{\mathbf{H}}_k^F$ in $\mathbf{k}$ space, related to $\hat{\mathbf{H}}^{F0}$ in Eq. (B5) via

$$[\hat{\mathbf{H}}_{mn}^{F0}]_{\mathbf{R},\mathbf{R}'} = \frac{1}{N_c} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot (\mathbf{R}_\nu - \mathbf{R}'_{\nu'})} [\hat{\mathbf{H}}_{k,mn}^F]_{\nu\nu'}. \quad (\text{B10})$$

Considering that the external field $\mathbf{B}$ is applied along the z direction, the magnetic response of the Floquet DOS is, hence, given by

$$\begin{aligned} \Phi_0 \frac{\partial \rho^F(\omega)}{\partial B} &= -\frac{\Phi_0}{\pi} \frac{\partial}{\partial B} \text{Im Tr}[\hat{\mathbf{G}}^F(\omega + i0^+)] \\ &= -\frac{\Phi_0}{\pi} \frac{\partial}{\partial B} \text{Im Tr}[\hat{\mathbf{G}}^{F(B)}(\omega + i0^+)] \\ &= -\frac{\Phi_0}{\pi} \int_{\text{BZ}} \frac{d^2 k}{(2\pi)^2} \text{Im tr} \left[\frac{\partial \hat{\mathbf{G}}_k^{F(B)}(\omega + i0^+)}{\partial B} \right], \end{aligned} \quad (\text{B11})$$

where we have used that $\text{Tr}[\hat{\mathbf{G}}^F] = \text{Tr}[\hat{\mathbf{G}}^{F(B)}]$, on account of the cancellation of the Peierls phase factors when performing the trace operation. Inserting Eq. (B8) into Eq. (B11) and using the spectral decomposition of the Floquet Hamiltonian and Green's function, we finally arrive at Eq. (33) in the main text.

APPENDIX C: ALTERNATIVE REGULARIZATION SCHEMES

In Eq. (46) in the main text, we have chosen to present a regularization of the divergent integral defining $\mathcal{W}^A$ [Eq. (43)] via a Cesàro $(C, 1)$ method. Notably, once an integral is $(C, 1)$ summable, it is also (C, ζ) summable with ζ an integer larger than 1. Even more, the result of this higher-order regularization procedure is exactly the same as the one obtained through the $(C, 1)$ regularization [109], such that

$$\begin{aligned} \mathcal{W}^A &\stackrel{(C,\zeta)}{=} \lim_{\lambda \rightarrow \infty} \Phi_0 \int_{-\infty}^{\varepsilon_\pi - \Omega} d\omega \frac{\partial \rho^F(\omega)}{\partial B} \left(1 + \frac{\omega}{\lambda}\right)^\zeta \\ &= \sum_{\alpha} \int_{\text{BZ}} \frac{d^2 k}{2\pi\Omega} \left[\mathcal{F}_{xy}^{\alpha}(\mathbf{k}) \varepsilon_{\alpha\mathbf{k}} - \frac{\Phi_0}{2\pi} m_z^{\alpha}(\mathbf{k}) \right]. \end{aligned} \quad (\text{C1})$$

We could have also chosen to regularize Eq. (43) by considering Abel's summation prescription, which consists on taking the following limit:

$$\begin{aligned} \mathcal{W}^A &= \lim_{\kappa \rightarrow 0} \Phi_0 \int_{-\infty}^{\varepsilon_\pi - \Omega} d\omega \frac{\partial \rho^F(\omega)}{\partial B} e^{\kappa\omega} \\ &= \lim_{\kappa \rightarrow 0} \sum_{\alpha} \sum_{s=-\infty}^{s=-1} \int_{\text{BZ}} \frac{d^2 k}{2\pi} \left[\mathcal{F}_{xy}^{\alpha}(\mathbf{k}) - \kappa \frac{\Phi_0}{2\pi} m_z^{\alpha}(\mathbf{k}) \right] e^{\kappa \varepsilon_{\alpha\mathbf{k}}}. \end{aligned} \quad (\text{C2})$$

Explicitly performing the infinite summation over Floquet Brillouin zones,

$$\sum_{s=-\infty}^{s=-1} e^{\kappa s \Omega} = \frac{e^{-\kappa \Omega}}{e^{-\kappa \Omega} - 1}, \quad (\text{C3})$$

replacing Eq. (C3) in Eq. (C2), and then taking the limit $\kappa \rightarrow 0$, we also reobtain the result in Eq. (C1). In this sense, the result obtained in Eq. (46) in the main text, expressed in terms of physical observables such as the Berry curvatures, quasienergies, and intrinsic orbital magnetic moments of Floquet-Bloch bands, remains remarkably independent of the regularization scheme introduced to perform the calculation.

APPENDIX D: EQUIVALENCE OF $\mathcal{W}^A$ AND THE WINDING NUMBER OF THE MICROMOTION OPERATOR

The anomalous winding number $\mathcal{W}^A$ in Eq. (51) can be expressed in the extended Sambe space as

$$\begin{aligned} \mathcal{W}^A &= \frac{ie^{zj}}{2\pi\Omega} \sum_{\alpha,\beta,s} \left\{ \int_{\text{BZ}} d^2 k \varepsilon_{\alpha\mathbf{k}} \frac{\langle\langle u_{\alpha\mathbf{k}} | \frac{\partial \hat{\mathbf{H}}_k^F}{\partial k_j} | u_{\beta s \mathbf{k}} \rangle\rangle \langle\langle u_{\beta s \mathbf{k}} | \frac{\partial \hat{\mathbf{H}}_k^F}{\partial k_l} | u_{\alpha\mathbf{k}} \rangle\rangle}{(\varepsilon_{\alpha\mathbf{k}} - \varepsilon_{\beta s \mathbf{k}})^2} \right. \\ &\quad \left. - \frac{1}{2} \int_{\text{BZ}} d^2 k \frac{\langle\langle u_{\alpha\mathbf{k}} | \frac{\partial \hat{\mathbf{H}}_k^F}{\partial k_j} | u_{\beta s \mathbf{k}} \rangle\rangle \langle\langle u_{\beta s \mathbf{k}} | \frac{\partial \hat{\mathbf{H}}_k^F}{\partial k_l} | u_{\alpha\mathbf{k}} \rangle\rangle}{\varepsilon_{\alpha\mathbf{k}} - \varepsilon_{\beta s \mathbf{k}}} \right\}, \end{aligned} \quad (\text{D1})$$

where we have explicitly used the form of the Berry curvature and orbital magnetization given by Eqs. (34) and (35). We recall that the notation $|\cdots\rangle\rangle$ indicates a vector in the extended Sambe space, $\varepsilon_{\beta s \mathbf{k}} = \varepsilon_{\beta \mathbf{k}} + s\Omega$, that $|u_{\alpha\mathbf{k}}\rangle\rangle \equiv |u_{\alpha 0 \mathbf{k}}\rangle\rangle$ and that the sum over $s \in \mathbb{Z}$ represents the infinite number of Floquet Brillouin zones. It is straightforward to verify that the two main terms in Eq. (D1) cancel out for $s = 0$ (NFZ)—this is easily seen by symmetrizing the first term with respect to the indexes α and β . Therefore, the sum over s can be restricted to $s \neq 0$. In such a case, one can safely write

$$\langle\langle u_{\alpha\mathbf{k}} | \frac{\partial \hat{\mathbf{H}}_k^F}{\partial k_j} | u_{\beta s \mathbf{k}} \rangle\rangle = (\varepsilon_{\alpha\mathbf{k}} - \varepsilon_{\beta s \mathbf{k}}) \langle\langle \frac{\partial u_{\alpha\mathbf{k}}}{\partial k_j} | u_{\beta s \mathbf{k}} \rangle\rangle. \quad (\text{D2})$$

Let us introduce a new basis set $\{|v_{\xi sk}\rangle\}$ that block diagonalizes $\hat{H}_k^F$. This new basis can be related to a $\mathbf{k}$ -independent basis $\{|pm\rangle\}$ through a unitary transformation defined by the operator $\hat{R}_k$, that is, $\{|pm\rangle\} \xrightarrow{\hat{R}_k} \{|v_{\xi sk}\rangle\}$, so that

$$\begin{aligned} \langle\langle qm|\hat{R}_k^\dagger \hat{H}_k^F \hat{R}_k|pm'\rangle\rangle \\ = \delta_{mm'}(\langle q|\hat{H}_{\text{eff}}(\mathbf{k})|p\rangle - \delta_{qp}m\Omega). \end{aligned} \quad (\text{D3})$$

This defines the effective Hamiltonian $\hat{H}_{\text{eff}}(\mathbf{k})$ in the Hilbert space whose matrix elements in the $\{|p\rangle\}$ basis are $\langle q|\hat{H}_{\text{eff}}(\mathbf{k})|p\rangle$. The matrix elements in Eq. (D2) can be rewritten in terms of $\hat{R}_k$. Indeed, we note that

$$\langle\langle \frac{\partial u_{ak}}{\partial k_j}|u_{\beta sk}\rangle\rangle = \sum_{\xi, s'} \langle\langle \frac{\partial u_{ak}}{\partial k_j}|v_{\xi s'k}\rangle\rangle \langle\langle v_{\xi s'k}|u_{\beta sk}\rangle\rangle \quad (\text{D4})$$

$$= -\sum_{\xi, s'} \langle\langle u_{ak}|\frac{\partial v_{\xi s'k}}{\partial k_j}\rangle\rangle \langle\langle v_{\xi s'k}|u_{\beta sk}\rangle\rangle \quad (\text{D5})$$

$$= -\langle\langle u_{ak}|\frac{\partial \hat{R}_k}{\partial k_j} \hat{R}_k^\dagger|u_{\beta sk}\rangle\rangle. \quad (\text{D6})$$

In Eq. (D4), we have simply introduced an identity operator, and in Eq. (D5) we used that $s \neq 0$ so that $\langle\langle u_{ak}|v_{\xi sk}\rangle\rangle = 0$. Finally, in Eq. (D6), we have used that, because $\{|pm\rangle\}$ is independent of $\mathbf{k}$,

$$\frac{\partial \hat{R}_k}{\partial k_j} \hat{R}_k^\dagger = \sum_{\xi, s} |\frac{\partial v_{\xi sk}}{\partial k_j}\rangle \langle\langle v_{\xi sk}|. \quad (\text{D7})$$

Replacing Eq. (D6) in Eq. (D2), we then find that

$$\langle\langle u_{ak}|\frac{\partial \hat{H}_k^F}{\partial k_j}|u_{\beta sk}\rangle\rangle = (\varepsilon_{\beta sk} - \varepsilon_{ak}) \langle\langle u_{ak}|\frac{\partial \hat{R}_k}{\partial k_j} \hat{R}_k^\dagger|u_{\beta sk}\rangle\rangle. \quad (\text{D8})$$

Replacing Eq. (D8) in Eq. (D1) and using that $\langle\langle u_{ak}|(\partial \hat{R}_k/\partial k_j) \hat{R}_k^\dagger|u_{\beta sk}\rangle\rangle$ is invariant under the translation of the Floquet BZ index $\langle\langle u_{ak}|(\partial \hat{R}_k/\partial k_j) \hat{R}_k^\dagger|u_{\beta sk}\rangle\rangle = \langle\langle u_{\alpha(-s)k}|(\partial \hat{R}_k/\partial k_j) \hat{R}_k^\dagger|u_{\beta k}\rangle\rangle$, one gets, after some simple but tedious algebra, that

$$\begin{aligned} \mathcal{W}^A = & -\frac{i\epsilon^{zjl}}{4\pi\Omega} \int_{\text{BZ}} d^2k \sum_{\alpha, \beta, s} s\Omega \\ & \times \langle\langle u_{ak}|\frac{\partial \hat{R}_k}{\partial k_j} \hat{R}_k^\dagger|u_{\beta sk}\rangle\rangle \langle\langle u_{\beta sk}|\frac{\partial \hat{R}_k}{\partial k_l} \hat{R}_k^\dagger|u_{ak}\rangle\rangle. \end{aligned} \quad (\text{D9})$$

We now define a new operator $\tilde{R}_k$ such that $\tilde{R}_k^\dagger|u_{\beta sk}\rangle = -is\Omega \hat{R}_k^\dagger|u_{\beta sk}\rangle$. Then,

$$\mathcal{W}^A = \frac{\epsilon^{zjl}}{4\pi\Omega} \int_{\text{BZ}} d^2k \sum_{\alpha} \langle\langle u_{ak}|\frac{\partial \hat{R}_k}{\partial k_j} \tilde{R}_k^\dagger \frac{\partial \hat{R}_k}{\partial k_l} \hat{R}_k^\dagger|u_{ak}\rangle\rangle. \quad (\text{D10})$$

In order to proceed, we now use the fact that any operator $\hat{O}_k$ in $\mathcal{S}$ that is translationally invariant with respect to the Floquet index m can be related to a periodic operator $\hat{O}_k(t)$ acting on $\mathcal{H}$ by the relation [132]

$$\hat{O}_k(t) = \sum_{p, q} \sum_m |qk\rangle \langle\langle qkm|\hat{O}_k|pk0\rangle\rangle \langle pk| e^{-im\Omega t}. \quad (\text{D11})$$

Hence, we can make the following identification: $\hat{R}_k \rightarrow \hat{R}_k(t)$ and $\tilde{R}_k \rightarrow -\partial \hat{R}_k(t)/\partial t$, where $\hat{R}_k(t)$ is nothing but a unitary operator that relates the time-dependent Hamiltonian $\hat{H}_k(t)$ with the time-independent effective Hamiltonian [132]

$$\hat{H}_{\text{eff}}(\mathbf{k}) = \hat{R}_k^\dagger(t) \hat{H}_k(t) \hat{R}_k(t) - i \hat{R}_k^\dagger(t) \frac{d\hat{R}_k(t)}{dt}. \quad (\text{D12})$$

In addition, from Eq. (D11), we can write

$$\frac{1}{T} \int_0^T dt \text{tr}(\hat{O}_k(t)) = \sum_{\alpha} \langle\langle u_{ak}|\hat{O}_k|u_{ak}\rangle\rangle, \quad (\text{D13})$$

while it can be easily shown that a product of operators in $\mathcal{S}$ translates on the same product of the corresponding operators in $\mathcal{H}$, $\hat{O}_1 \hat{O}_2 \rightarrow \hat{O}_1(t) \hat{O}_2(t)$. Hence, after some rearranging, the final expression for $\mathcal{W}^A$ in terms of $\hat{R}_k(t)$ reads

$$\begin{aligned} \mathcal{W}^A = & \frac{\epsilon^{zjl}}{8\pi^2} \int_0^T dt \int_{\text{BZ}} d^2k \text{tr} \left[\hat{R}_k^\dagger(t) \frac{\partial \hat{R}_k(t)}{\partial t} \right. \\ & \times \left. \hat{R}_k^\dagger(t) \frac{\partial \hat{R}_k(t)}{\partial k_l} \hat{R}_k^\dagger(t) \frac{\partial \hat{R}_k(t)}{\partial k_j} \right] \\ = & N_3[R], \end{aligned} \quad (\text{D14})$$

which is the higher-order winding number $N_3[R]$ associated to the micromotion operator defined by $\hat{R}_k(t)$ in time-quasimomenta space. This invariant can also be equally written in terms of the two-point micromotion operator $\hat{P}_k(t, t') = \hat{R}_k(t) \hat{R}_k^\dagger(t')$ [131,132]:

$$\begin{aligned} \mathcal{W}^A = & \frac{\epsilon^{zjl}}{8\pi^2} \int_0^T dt \int_{\text{BZ}} d^2k \text{tr} \left[\hat{P}_k^\dagger(t, t') \frac{\partial \hat{P}_k(t, t')}{\partial t} \right. \\ & \times \left. \hat{P}_k^\dagger(t, t') \frac{\partial \hat{P}_k(t, t')}{\partial k_l} \hat{P}_k^\dagger(t, t') \frac{\partial \hat{P}_k(t, t')}{\partial k_j} \right] \\ = & N_3[P], \end{aligned} \quad (\text{D15})$$

because the winding number of the product of two operators is given by the sum of the winding numbers of each operator [193], $N_3[O_1 O_2] = N_3[O_1] + N_3[O_2]$ and $\partial \hat{R}_k^\dagger(t')/\partial t = 0$. Equation (D15) is the topological invariant

introduced in Refs. [15,23,60] and, hence, proves the equivalence between both approaches.

APPENDIX E: RELATION BETWEEN THE BERRY CURVATURE DEFINED IN SAMBE SPACE AND THAT ASSOCIATED WITH $\hat{H}_{\text{eff}}$

Here, we provide an explicit formula relating the Berry curvature calculated in $\mathcal{S}$ space, given by Eq. (34) in the main text, and the corresponding expression obtained from the eigenvectors of the effective Hamiltonian $\hat{H}_{\text{eff}}$. Any matrix element of an operator $\hat{O}$ in $\mathcal{S}$ can be written as

$$\langle\langle u_a | \hat{O} | u_b \rangle\rangle = \frac{1}{T} \int_0^T dt \langle u_a(t) | \hat{O}(t) | u_b(t) \rangle, \quad (\text{E1})$$

where $\hat{O}(t)$ is the associated time-dependent operator in the Hilbert space and $|u_a(t)\rangle$ are the Floquet modes; see Eq. (21). In the particular case of the Floquet Berry curvature in Eq. (34), we have

$$\begin{aligned} \mathcal{F}_{xy}^\alpha(\mathbf{k}) &= i\epsilon^{zjl} \langle\langle \partial_{k_j} u_{\alpha k} | \partial_{k_l} u_{\alpha k} \rangle\rangle \\ &= i\epsilon^{zjl} \frac{1}{T} \int_0^T dt \langle \partial_{k_j} u_{\alpha k}(t) | \partial_{k_l} u_{\alpha k}(t) \rangle. \end{aligned} \quad (\text{E2})$$

Using Eq. (55) to relate the eigenvectors of the Floquet Hamiltonian in $\mathbf{k}$ space, $\hat{H}_k^F(t)$, with those of $\hat{H}_{\text{eff}}(\mathbf{k})$ via the relation $|u_{\alpha k}(t)\rangle = \hat{R}_k(t) |u_{\alpha k}^{\text{eff}}\rangle$, with $\hat{R}_k(t)$ satisfying $\hat{H}_{\text{eff}}(\mathbf{k}) = \hat{R}_k^\dagger(t) \hat{H}_k^F(t) \hat{R}_k(t)$, we readily find that

$$\mathcal{F}_{xy}^\alpha(\mathbf{k}) = \mathcal{F}_{xy}^{\alpha,\text{eff}}(\mathbf{k}) + \mathcal{D}_{xy}^{\alpha,\text{eff}}(\mathbf{k}). \quad (\text{E3})$$

In Eq. (E3), we have defined the Berry curvature of the α th band of the effective Hamiltonian:

$$\mathcal{F}_{xy}^{\alpha,\text{eff}}(\mathbf{k}) = i\epsilon^{zjl} \langle u_{\alpha k}^{\text{eff}} | \partial_{k_j} u_{\alpha k}^{\text{eff}} | \partial_{k_l} u_{\alpha k}^{\text{eff}} \rangle \quad (\text{E4})$$

and

$$\mathcal{D}_{xy}^{\alpha,\text{eff}}(\mathbf{k}) = i\epsilon^{zjl} \partial_{k_j} \left(\langle u_{\alpha k}^{\text{eff}} | \frac{1}{T} \int_0^T dt \hat{R}_k^\dagger(t) \partial_{k_l} \hat{R}_k(t) | u_{\alpha k}^{\text{eff}} \rangle \right). \quad (\text{E5})$$

Notice that the difference between $\mathcal{F}_{xy}^\alpha(\mathbf{k})$ and $\mathcal{F}_{xy}^{\alpha,\text{eff}}(\mathbf{k})$, given by Eq. (E5), is written in terms of total derivatives with respect to different quasimomenta of a $\mathbf{k}$ -periodic function and, hence, vanishes when integrated over the Bloch Brillouin zone. Namely,

$$\begin{aligned} C_\alpha^F &= \frac{1}{2\pi} \int_{\text{BZ}} d^2k \mathcal{F}_{xy}^\alpha(\mathbf{k}) \\ &= \frac{1}{2\pi} \int_{\text{BZ}} d^2k \left(\mathcal{F}_{xy}^{\alpha,\text{eff}}(\mathbf{k}) + \mathcal{D}_{xy}^{\alpha,\text{eff}}(\mathbf{k}) \right) \\ &= \frac{1}{2\pi} \int_{\text{BZ}} d^2k \mathcal{F}_{xy}^{\alpha,\text{eff}}(\mathbf{k}) = C_{\alpha,\text{eff}}^F. \end{aligned} \quad (\text{E6})$$

This guarantees that the Chern number of a given Floquet band, C_α^F , can be equally calculated with $\mathcal{F}_{xy}^\alpha(\mathbf{k})$ or $\mathcal{F}_{xy}^{\alpha,\text{eff}}(\mathbf{k})$. Importantly, this is not the case for the calculation of the anomalous spectral flow $\mathcal{W}^A$, as it involves terms as the product $\epsilon_{ak} \mathcal{F}_{xy}^\alpha(\mathbf{k})$. A similar analysis can be done for the case of the intrinsic orbital magnetic moment [see Eq. (35)], where differences between $m_z^\alpha(\mathbf{k})$ and $m_z^{\alpha,\text{eff}}(\mathbf{k})$ are also found.

Finally, we call attention to the fact that

$$\mathcal{W}_{\text{eff}}^A = \sum_\alpha \int_{\text{BZ}} \frac{d^2k}{2\pi\Omega} \left[\mathcal{F}_{xy}^{\alpha,\text{eff}} \epsilon_{ak} - \frac{\Phi_0}{2\pi} m_z^{\alpha,\text{eff}}(\mathbf{k}) \right] = 0. \quad (\text{E7})$$

That means that, in general, $\mathcal{W}_{\text{eff}}^A \neq \mathcal{W}^A$.

APPENDIX F: GAUGE DEPENDENCE OF $N_1[\mathbf{R}]$

As shown in Eq. (72), the first-order winding number of the micromotion operator $\hat{R}(t)$ over one period of the driving cycle is defined as

$$N_1[\mathbf{R}] = -\frac{i}{2\pi} \int_0^T dt \text{Tr}[\hat{R}^\dagger(t) \partial_t \hat{R}(t)], \quad (\text{F1})$$

which imposes that $N_1[\mathbf{R}] \in \mathbb{Z}$. This quantity is related to the effective Hamiltonian by the equation

$$\text{Tr}[\hat{H}_{\text{eff}}] = \frac{1}{T} \int_0^T dt \text{Tr}[\hat{H}(t)] + N_1[\mathbf{R}]\Omega. \quad (\text{F2})$$

It is important to notice that the value of $N_1[\mathbf{R}]$ has no physical meaning by itself, as it includes trivial contributions that make it gauge dependent. To show this, let us first denote the total time-averaged energy as $\bar{E} = (1/T) \int_0^T dt \text{Tr}[\hat{H}(t)]$. Then, given $\hat{R}(t)$, we can introduce a new, equally valid, time-periodic operator

$$\hat{R}'(t) = e^{-i[\bar{E}/\Omega]\Omega t} \hat{R}(t), \quad (\text{F3})$$

where $[x]$ denotes the closest integer number to x , $\bar{E} = \bar{E}/d_{\mathcal{H}}$, and $d_{\mathcal{H}} = \text{Tr}(\hat{I})$ is the dimension of the Hilbert space. The corresponding effective Hamiltonian is then given by

$$\begin{aligned} \hat{H}'_{\text{eff}} &= \hat{R}'^\dagger(t) \hat{H}(t) \hat{R}'(t) - i\hat{R}'^\dagger(t) \partial_t \hat{R}'(t) \\ &= \hat{H}_{\text{eff}} - [\bar{E}/\Omega]\Omega \hat{I}. \end{aligned} \quad (\text{F4})$$

It is then clear that

$$N_1[R'] = N_1[R] - [\bar{\varepsilon}/\Omega]d_{\mathcal{H}}. \quad (\text{F5})$$

Equation (F5) highlights the fact that $N_1[R']$ can present discrete jumps when a global energy shift is introduced in the problem (which has no physical relevance). The change of $N_1[R']$ is a consequence of the intrinsic indeterminate nature of $\hat{H}'_{\text{eff}}$ up to integer multiples of Ω . Note, however, that $\partial_{\Phi}N_1[R'] = \partial_{\Phi}N_1[R]$, since $\partial_{\Phi}\bar{\varepsilon} = 0$ as discussed in the main text.

This gauge dependence can also be traced back to the properties of the Floquet states. In fact, Eq. (74) shows that the value of $N_1[R]$ depends on the Floquet zone of reference. If we had chosen the effective Hamiltonian as the one having eigenvalues in the s th Floquet zone, namely, $\varepsilon_{as} \in (-\Omega/2 + s\Omega, \Omega/2 + s\Omega]$, then its corresponding winding would be given by

$$\begin{aligned} N_1^s[R] &= -\sum_a \sum_{n=-\infty}^{\infty} n \langle u_{as}^{(n)} | u_{as}^{(n)} \rangle \\ &= -\sum_a \sum_{n=-\infty}^{\infty} n \langle u_a^{(n+s)} | u_a^{(n+s)} \rangle \\ &= N_1^0[R] + sd_{\mathcal{H}} = N_1[R] + sd_{\mathcal{H}}, \end{aligned} \quad (\text{F6})$$

for all values of $s \in \mathbb{Z}$.

The unitary transformation [Eq. (55)], from where Eq. (F2) is derived, does not necessarily implies that all the eigenvalues of $\hat{H}'_{\text{eff}}$ belong to the same Floquet zone for an arbitrary $\hat{R}(t)$. In order to guarantee that, an additional “folding” procedure might be required. As shown below, this folding can always be done using a T -periodic unitary operator, then assuring that there is a $\hat{R}_{\text{fold}}(t)$ that transforms $H(t)$ into $\hat{H}'_{\text{eff}}$ with its eigenvalues inside a single Floquet zone. To prove that, let us denote $\{\varepsilon_a\}$ the set of eigenvalues of $\hat{H}_{\text{eff}}$, obtained after a rotation with a given $\hat{R}(t)$. We can then define the unitary operator $\hat{R}'(t)$, which is diagonal in the eigenbasis of $\hat{H}_{\text{eff}}$ and has the following diagonal matrix elements:

$$[\hat{R}'(t)]_{aa} = \exp\left(-i \left[\frac{\varepsilon_a - \varepsilon}{\Omega} \right] \Omega t\right), \quad (\text{F7})$$

where ε denotes the center of the chosen Floquet zone. Then, $\hat{R}_{\text{fold}}(t) = \hat{R}'(t)\hat{R}(t)$ is the desired unitary transformation that takes $H(t)$ into $\hat{H}'_{\text{eff}}$ with eigenvalues $\varepsilon'_a = \varepsilon_a - [(\varepsilon_a - \varepsilon)/\Omega]\Omega$ such that $|\varepsilon'_a - \varepsilon| \leq \Omega/2 \forall a$. Correspondingly,

$$N_1[R_{\text{fold}}] = N_1[R] - \sum_a \left[\frac{\varepsilon_a - \varepsilon}{\Omega} \right]. \quad (\text{F8})$$

The freedom to chose the Floquet zone around any particular quasienergy can be exploited to set $N_1[R]$ equal

to zero for a fixed value of the system’s parameters. This is done by adopting the “natural Floquet zone” (NFZ) as defined in Ref. [59], namely, a zone such that

$$\text{Tr}[\hat{H}_{\text{eff}}] = \frac{1}{T} \int_0^T dt \text{Tr}[\hat{H}(t)], \quad (\text{F9})$$

with $\hat{H}_{\text{eff}}$ having all its eigenvalues inside such a zone. Making such a choice for $\Phi = 0$ and using Eq. (78), we have that

$$N_1[R, \Phi/\Phi_0] = N_3[R] \frac{\Phi}{\Phi_0}. \quad (\text{F10})$$

As a final remark, we notice that the expression for the time-averaged energy [Eq. (73)] can also be written as [194,195]

$$\frac{1}{T} \int_0^T dt \text{Tr}[\hat{H}(t)] = \sum_a \left(\varepsilon_a - \Omega \frac{\partial \varepsilon_a}{\partial \Omega} \right), \quad (\text{F11})$$

where we assumed that $H(t)$ depends on time though the combination Ωt . Then, using Eq. (71), we have yet another expression for $N_1[R]$, namely,

$$N_1[R] = \sum_a \frac{\partial \varepsilon_a}{\partial \Omega}. \quad (\text{F12})$$

APPENDIX G: DEMONSTRATION OF EQ. (86)

The spectral flow $\mathcal{W}(\mu_{\varepsilon})$ is defined as

$$\begin{aligned} \mathcal{W}(\mu_{\varepsilon}) &= \Phi_0 \int_{-\infty}^{\mu_{\varepsilon}} d\omega \frac{\partial \rho^F(\omega)}{\partial B} \\ &= \Phi_0 \int_{-\infty}^{\mu_{\varepsilon}} d\omega \sum_{m=-\infty}^{\infty} \frac{\partial \rho_0^F(\omega + m\Omega)}{\partial B}, \end{aligned} \quad (\text{G1})$$

where Eq. (84) was used in the second equality. Introducing the Cesàro renormalization, we get

$$\mathcal{W}(\mu_{\varepsilon}) \stackrel{(C,1)}{=} \Phi_0 \lim_{\lambda \rightarrow \infty} \int_{-\lambda}^{\mu_{\varepsilon}} d\omega \sum_{m=-\infty}^{\infty} \frac{\partial \rho_0^F(\omega + m\Omega)}{\partial B} \left(1 + \frac{\omega}{\lambda} \right). \quad (\text{G2})$$

For a given (finite) value of λ , this expression has the form $\int_a^b d\omega \sum_m f_m(\omega)$, which can be rewritten as $\sum_m \int_a^b d\omega f_m(\omega)$ provided $\sum_m \int_a^b d\omega |f_m(\omega)| < \infty$ (Fubini-Tonelli’s theorem [196]). In our case, we have

$$f_m(\omega) = \Phi_0 \frac{\partial \rho_0^F(\omega + m\Omega)}{\partial B} \left(1 + \frac{\omega}{\lambda} \right). \quad (\text{G3})$$

Because of the exponential decay of $\rho_0(\omega)$ away from the NFZ, the integral of $|f_m(\omega)|$ in the interval $[-\lambda, \mu_{\varepsilon}]$

is finite $\forall m$. Furthermore, for the same reason, such an integral goes to zero exponentially fast for large m . Hence, the condition of the Fubini-Tonelli theorem is fulfilled so that

$$\mathcal{W}(\mu_\varepsilon) \stackrel{(C.1)}{=} \Phi_0 \lim_{\lambda \rightarrow \infty} \sum_{m=-\infty}^{\infty} \int_{-\lambda}^{\mu_\varepsilon} d\omega \frac{\partial \rho_0^F(\omega + m\Omega)}{\partial B} \left(1 + \frac{\omega}{\lambda}\right). \quad (\text{G4})$$

We now define

$$A_m(\lambda) = \Phi_0 \int_{-\lambda}^{\mu_\varepsilon} d\omega \frac{\partial \rho_0^F(\omega + m\Omega)}{\partial B} \left(1 + \frac{\omega}{\lambda}\right) \quad (\text{G5})$$

and

$$A_m^\infty = \Phi_0 \int_{-\infty}^{\mu_\varepsilon} d\omega \frac{\partial \rho_0^F(\omega + m\Omega)}{\partial B}. \quad (\text{G6})$$

Since the integral in Eq. (G6) is well defined and finite in the standard sense, the Cesàro renormalization must agree with it so that $\lim_{\lambda \rightarrow \infty} A_m(\lambda) = A_m^\infty$. It is straightforward to verify that, due to Eq. (87) and the fast (exponential) decay of $\rho_0(\omega)$, the series $\sum_m |A_m^\infty| < \infty$. Hence, the sum and the limit in Eq. (G4) can be exchanged [196]:

$$\begin{aligned} \mathcal{W}(\mu_\varepsilon) &\stackrel{(C.1)}{=} \sum_{m=-\infty}^{\infty} \lim_{\lambda \rightarrow \infty} A_m(\lambda) \\ &\stackrel{(C.1)}{=} \sum_{m=-\infty}^{\infty} A_m^\infty \\ &\stackrel{(C.1)}{=} \Phi_0 \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\mu_\varepsilon} d\omega \frac{\partial \rho_0^F(\omega + m\Omega)}{\partial B}, \end{aligned} \quad (\text{G7})$$

which reduces to Eq. (86) after a change of variable.

APPENDIX H: TIME-AVERAGED FLOQUET DOS WITH BÜTTIKER-TYPE RESERVOIRS

In this appendix, we introduce the simplest model of a heat bath that leads to a steady-state occupation which is determined by the time-averaged Floquet density of states introduced in Eq. (85). In particular, we consider a Büttiker-type bath [148]: featureless fermionic reservoirs which are attached to every site of the driven lattice but uncoupled from each other (see Fig. 16).

The steady-state time-dependent particle density of such a driven dissipative system can be obtained by making use of standard Floquet-Keldysh Green's function techniques [172,173]. The particle density can be expressed in terms of its Fourier harmonics as

$$n(t) = \sum_p e^{-ip\Omega t} n_p, \quad (\text{H1})$$

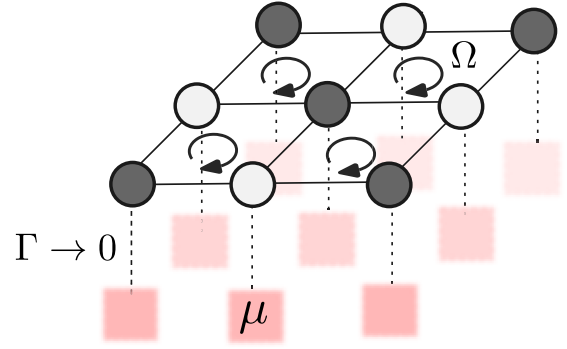


FIG. 16. Scheme of the lattice system, driven with frequency Ω , coupled to a Büttiker-type fermionic heat bath at chemical potential μ . Each lattice site is connected to its own reservoir with a vanishing strength $\Gamma \rightarrow 0$.

where

$$n_p = -\frac{i}{A} \int_{\text{NFZ}} \frac{d\omega}{2\pi} \sum_n \text{Tr} [\hat{G}_{p+n,n}^{F<}(\omega)]. \quad (\text{H2})$$

We have here introduced the lesser Green's function in Floquet representation, given by [173]

$$\hat{G}^{F<}(\omega) = \hat{G}^F(\omega + i\Gamma) \hat{\Sigma}^{F<}(\omega) \hat{G}^F(\omega - i\Gamma). \quad (\text{H3})$$

The parameter Γ characterizes the hybridization with the reservoirs. For simplicity, we have omitted the frequency dependence of the latter, equivalent to considering a bath with a constant density of states in the wideband limit. The lesser self-energy due to these featureless reservoirs is diagonal in Sambe space, namely,

$$\hat{\Sigma}_{mn}^{F,<}(\omega) = \delta_{mn} \hat{I} f(\omega + n\Omega) 2i\Gamma. \quad (\text{H4})$$

Replacing Eqs. (H3) and (H4) in Eq. (H2), making use of Eq. (B3) and the orthogonality relation in Eq. (28), we find that

$$\begin{aligned} n_p &= \frac{\Gamma}{\pi A} \int_{\text{NFZ}} d\omega \sum_a \sum_{l,s} f(\omega + l\Omega) \\ &\times \frac{\langle u_a^{(l+s)} | u_a^{(l+p+s)} \rangle}{[\omega - (\varepsilon_a + s\Omega) + i\Gamma][\omega - (\varepsilon_a + (s+p)\Omega) - i\Gamma]}. \end{aligned} \quad (\text{H5})$$

When considering the weak hybridization limit ($\Gamma \rightarrow 0$), all the higher-order harmonics ($p \neq 0$) vanish, meaning that in the steady state the system reaches a time-independent particle density, given by

$$\bar{n}|_\mu = \lim_{\Gamma \rightarrow 0} n_0 = \sum_a D(\varepsilon_a), \quad (\text{H6})$$

with

$$D(\varepsilon_a) = \sum_l f(\varepsilon_a + l\Omega) \langle u_a^{(l)} | u_a^{(l)} \rangle. \quad (\text{H7})$$

We recall that in the equations above the sum over a runs over states in the NFZ. In Eq. (H6), we emphasize that $\bar{n}$ is computed at the chemical potential μ , which is fixed by the heat bath. The occupation function $D(\varepsilon)$ is self-consistent, in the sense that it depends itself on the Floquet eigenstates and, thus, changes when modifying the driving parameters. We also note that Eqs. (H6) and (H7) have been reported in Refs. [197–199]. In the undriven case, the Floquet states in the natural Floquet Brillouin zone are such that $|u_a^{(l)}\rangle = \delta_{l0} |u_a^{(0)}\rangle$ which makes $D(\varepsilon_a) = f(\varepsilon_a)$.

Interestingly, Eq. (H6) can be written as an integral over frequencies of the equilibrium Fermi-Dirac distribution function times an effective density of states as

$$\bar{n}|_\mu = \int_{-\infty}^{\infty} d\omega f(\omega) \rho_0^F(\omega), \quad (\text{H8})$$

where

$$\rho_0^F(\omega) = \sum_{a,l} \langle u_a^{(l)} | u_a^{(l)} \rangle \delta[\omega - (\varepsilon_a + l\Omega)]. \quad (\text{H9})$$

This is precisely the time-averaged Floquet density of states introduced in Eq. (85) in the main text. Equation (H8) suggests that we can describe the effect of this featureless bath by considering that the system has a thermal occupation given by the Fermi-Dirac function but with an effective density of states described by $\rho_0^F(\omega)$. In the limit of zero temperature, Eq. (H8) reduces to

$$\bar{n}|_\mu = \int_{-\infty}^{\mu} d\omega \rho_0^F(\omega), \quad (\text{H10})$$

which is the time-averaged particle density introduced in Eq. (88) in the main text for $\mu_\varepsilon = \mu$.

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