Interacting Topological Insulators with Synthetic Dimensions

Chao-Ming Jian^{1,2} and Cenke Xu³

¹*Station Q, Microsoft Research, Santa Barbara, California 93106-6105, USA*

²*Kavli Institute of Theoretical Physics, Santa Barbara, California 93106, USA*

³*Department of Physics, University of California, Santa Barbara, California 93106, USA*

(Received 25 April 2018; revised manuscript received 19 September 2018; published 20 November 2018)

Recent developments of experimental techniques have given us unprecedented opportunities of studying topological insulators in high dimensions, while some of the dimensions are “synthetic,” in the sense that the effective lattice momenta along these synthetic dimensions are controllable periodic tuning parameters. In this work, we study interaction effects on topological insulators with synthetic dimensions. We show that, although the free fermion band structure of high-dimensional topological insulators can be precisely simulated with the “synthetic techniques,” the generic interactions in these effective synthetic topological insulators are qualitatively different from the local interactions in ordinary condensed-matter systems. And we show that these special but generic interactions have unexpected effects on topological insulators; namely, they change (or reduce) the classification of topological insulators *differently* from the previously extensively studied local interactions.

DOI: [10.1103/PhysRevX.8.041030](https://doi.org/10.1103/PhysRevX.8.041030)

Subject Areas: Condensed Matter Physics,
Strongly Correlated Materials,
Topological Insulators

I. INTRODUCTION

Ever since the proposal of the analogue of quantum Hall effects in four spatial dimensions [1], the topological states of matter in higher dimensions have attracted a great deal of theoretical interest. The classification of free fermion topological insulators (TIs) and superconductors (TSCs) in all dimensions (the so-called “tenfold way”) was a milestone in our understanding of noninteracting fermionic systems [2–4]. Later, great progress in understanding strongly interacting bosonic states of matter was achieved through the classification and description of bosonic symmetry-protected topological states, which can also be generalized to higher dimensions [5–18].

Until recently, the study of higher-dimensional TIs and their bosonic analogues was of pure theoretical interest only. However, the study of higher-dimensional TIs has gained important experimental relevance recently. For example, the four-dimensional quantum Hall insulator (or the four-dimensional Chern insulator) was successfully constructed experimentally [19,20] with two out of its four spatial dimensions “synthetic.” In fact, the synthetic dimensions are experimentally realized as periodic tuning

parameters that can be identified as the corresponding lattice momenta in these dimensions. This experiment [19,20] demonstrates a general experimental strategy for synthetic dimensions, which is to directly realize a Hamiltonian of the form $\hat{H}(\vec{p}) = \sum_{(i,j)} \hat{H}_{i,j}(\vec{p}) c_i^\dagger c_j$, where $\vec{p}$ is a $\delta = (D - d)$ -component tuning parameter and i and j label the sites on the d -dimensional physical (optical) lattice. Once we identify $\vec{p}$ as the lattice momenta in the synthetic dimensions, the entire system can be viewed as a D -dimensional noninteracting fermion system. For example, the experiment in Ref. [20] constructs a system on a two-dimensional square optical superlattice ($d = 2$), which consists of two one-dimensional superlattices along the x and y directions, each formed by superimposing two lattices with different lattice constants. The phase shifts between the two superimposed lattices along the x and y directions become the $\delta = 2$ tunable parameters, and they are identified as the lattice momenta along the extra synthetic dimensions. This system is shown to exhibit the $D = 4$ generalization of the quantum Hall effect. Such a synthetic construction, in principle, can give us an experimental realization of all classes of noninteracting fermion TIs in any dimension. The same perspective, i.e., viewing lattice momenta in certain dimensions as tuning parameters, is used by theorists to make connections between TIs in different dimensions [21] and also to construct topological semimetals [22–28]. Synthetic topological systems have also been experimentally explored by using internal atomic spin states as the synthetic dimension [29,30], which is a different strategy from the synthesis of lattice

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

FIG. 1. (a) The $D = 3$ synthetic TI with $d = 2$, and the lattice momentum p is physically a tuning parameter. We always consider the boundary parallel with the synthetic dimensions. Here, the exposed boundary is in the (x, p) plane with a fixed y coordinate. (b) The vortex line of the complex bosonic order parameter ϕ in Eq. (13), and there is one nonchiral $1d$ Majorana fermion mode localized in the core of the vortex line [Eq. (14)].

momenta. The size of the synthetic dimension in this realization, limited by the number of internal spin states, is not so close to the thermodynamical limit. In contrast, the size of synthetic dimensions realized as synthetic lattice momenta is effectively infinite, since the synthetic momenta are continuously tunable parameters. In the following, we focus on only the synthetic TIs realized by the synthetic-momentum technique.

Interestingly, interactions can drastically change the classification of TIs and TSCs with certain symmetries. It was first discovered in Refs. [31,32] that some TIs nontrivial in the noninteracting limit can be trivialized by interactions, which exemplifies the importance of interaction in altering or, more precisely, “reducing” the classification of TIs. Subsequent studies have shown the same effect of interaction, i.e., the reduction of classification, in TIs and TSCs in all dimensions [14,17,18,33–42]. These conclusions were made under the assumptions of *spatially local* interactions that preserve the crucial symmetries which define the TIs.

In contrast, the generic interaction on a synthetic TI is fundamentally different from that on the ordinary TIs given the synthetic nature of the extra dimensions. The interaction must be local in the d -dimensional real-space lattice and also local in the $\delta = (D - d)$ -dimensional synthetic *momentum* space; i.e., under adiabatic tuning, at each value of the tuning parameter $\vec{p}$, the system has an interaction in the d -dimensional real space labeled by $\vec{p}$: $H_{\text{int}}(\vec{p})$. This specific form of interaction actually allows us to use the powerful “dimensional reduction” procedure [21], which is a method that we exploit in this work.

A generic interaction $H_{\text{int}}(\vec{p})$ in a synthetic TI is no longer completely local in the effective D -dimensional real space: It is local in the physical dimensions but nonlocal in the synthetic spatial dimensions [43]. It appears that a TI with a nonlocal interaction is not even definable, since a nonlocal interaction would easily mix the edge states at two opposite boundaries of the TIs. But in the current situation, the interaction is nonlocal only along the synthetic

directions in the effective D -dimensional space. And as long as we are considering boundaries parallel to the $\delta = (D - d)$ -dimensional synthetic space—namely, the δ -component synthetic momenta are still conserved at the boundary—the two opposite boundaries will not be mixed by the nonlocal interaction [Fig. 1(a)]. This is the most natural choice of boundary in the effective D -dimensional synthetic TI, as we simply need to choose the physical boundary of the system while keeping the tuning parameter $\vec{p}$ periodic. Then one can still study the fate of the edge states at one single boundary, and the effective D -dimensional system can be called a nontrivial TI as long as this boundary remains gapless under the interaction.

(D, δ)	(1,0)	(1,1)	(3,0)	(3,1)	(3,2)
Classification, $U(1) \times Z_2^T$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_4$	$\mathbb{Z}_2$

(D, δ)	(2,0)	(2,1)	(4,0)	(4,1)	(4,2)
Classification, $U(1) \times Z_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_4$	$\mathbb{Z}_2$

Our main results are the following: (i) The classification for *interacting* TIs with symmetry $U(1) \times Z_2^T$ at total odd dimensions $D = 2n + 1$ and synthetic dimensions δ is given by $\mathbb{Z}_{2^{n+2-\delta}}$. (ii) The classification for interacting *nonchiral* TIs with symmetry $U(1) \times Z_2$ at total even dimensions $D = 2n$ and synthetic dimensions δ is given by $\mathbb{Z}_{2^{n+1-\delta}}$. We select these symmetries because TIs defined with these symmetries, as we discuss in the next few sections, are strongly affected by the interaction. In the table above, we list the results for $D = 2, 3, 4$ explicitly, which are discussed in detail in Secs. II–IV. In Secs. V and VI, we discuss the classification of interacting synthetic TIs in general higher dimensions.

II. SYNTHETIC TI WITH $D = 2$

We start with the example of a $D = 2$ synthetic TI, with $d = \delta = 1$; namely, one of the two dimensions is synthetic. The synthetic noninteracting Chern insulator with $D = 2$ and $\delta = 1$ has been studied previously [45]. One particular type of two-dimensional TIs that we know will be strongly influenced by interaction and reduce its classification is the nonchiral TI with $U(1) \times Z_2$ symmetry [39,40], where physically the Z_2 is usually the reflection symmetry about the z axis, which becomes an on-site symmetry in the two-dimensional plane. The minimal version of this TI is basically two copies of Chern insulators with opposite Chern numbers (which makes the entire system “nonchiral”) and also opposite eigenvalues (± 1) under the Z_2 symmetry operation. Its $1d$ edge state has the Hamiltonian

$$H = \int dx v (\psi_1^\dagger i \partial_x \psi_1 - \psi_2^\dagger i \partial_x \psi_2). \quad (1)$$

The charge conservation $U(1)$ symmetry and the Z_2 symmetry act on the boundary fermions, respectively, as

$$\begin{aligned} U(1): (\psi_1, \psi_2) &\rightarrow e^{i\theta}(\psi_1, \psi_2), \\ Z_2: (\psi_1, \psi_2) &\rightarrow (\psi_1, -\psi_2). \end{aligned} \quad (2)$$

One can see that, for arbitrary copies of the edge states, any fermion bilinear mass term at the edge mixes left- and right-moving fermions and, hence, break the Z_2 symmetry, while any fermion-pairing mass gap breaks the $U(1)$ symmetry. Hence, without an interaction, the classification of this TI is $\mathbb{Z}$. It was shown that under a local interaction the classification of this $U(1) \times Z_2$ TI is reduced to $\mathbb{Z}_4$ [39,40]. This conclusion has two related implications.

- (i) Four copies of the one-dimensional edge states can be gapped out by local interactions without breaking either $U(1)$ or Z_2 symmetry, which is directly shown in Ref. [39].
- (ii) For four copies of the system, the TI-to-trivial phase transition in the two-dimensional bulk can be avoided under an interaction; namely, the noninteracting TI phase and the trivial phase can be adiabatically connected with an interaction while keeping the bulk gap open. This adiabatic connection can be understood using the Chalker-Coddington network picture of the bulk topological transition [46,47]: The transition between two phases with the same symmetry and no bulk ground-state degeneracy can be interpreted as the percolation of their interface (domain wall). If the interface is fully gapped, then these two phases can be adiabatically connected without closing the bulk gap.

Let us now investigate what if the dimension along the boundary is synthetic. In this case, the boundary Hamiltonian including the interaction is

$$H = \sum_p H_p, \quad H_p = v p (\psi_{1,p}^\dagger \psi_{1,p} - \psi_{2,p}^\dagger \psi_{2,p}) + H_{\text{int}}(p). \quad (3)$$

Because p is the synthetic momentum, it is actually a tuning parameter, and $H_{\text{int}}(p)$ is local in the synthetic momentum space; i.e., for each value of p , there is an interaction labeled by p . Then, for each p , solving H_p is equivalent to solving a zero-dimensional problem with finite-dimensional Hilbert space. The original gapless point $p = 0$ becomes a level crossing of two states: One state has $N_{1,p} = \psi_{1,p}^\dagger \psi_{1,p} = 1$, $N_{2,p} = 0$, and the other state has $N_{1,p} = 0$, $N_{2,p} = 1$. If a level crossing of the edge states between the two limits $p \rightarrow \pm\infty$ is inevitable even under an interaction, then the synthetic TI has effective gapless edge states, and it remains topologically nontrivial; if a level crossing can be avoided under an interaction, then the synthetic TI is trivialized by the interaction. Strictly speaking, the boundary Hamiltonian

Eq. (3) without $H_{\text{int}}(p)$ is only an effective description up to an momentum cutoff Λ , which is determined by the bulk band gap. For Eq. (3) to be valid in the presence of $H_{\text{int}}(p)$, we naturally require $H_{\text{int}}(p)$ to have an energy scale smaller than $v\Lambda$. As we take the cutoff to infinity, i.e., $\Lambda \rightarrow +\infty$, the same requirement becomes that the term $H_{\text{int}}(p)$, which can, in principle, depend on p , should be smaller than $v|p|$ when $p \rightarrow \pm\infty$.

If $H_{\text{int}}(p)$ preserves the $U(1) \times Z_2$ symmetry, then for a single copy of Eq. (3), there is no way to avoid the level crossing, because $H_{\text{int}}(p)$ can always be recombined into a function of $N_{1,p}$ and $N_{2,p}$, and changing the filling of $N_{1,p}$ and $N_{2,p}$ will lead to a level crossing. Also, because the ground states in the two opposite limits with $p \rightarrow \pm\infty$ carry different Z_2 charges, a level crossing, in principle, cannot be avoided.

More generally, let us consider another tuning parameter m : When $m = 1$, the system is a nontrivial $D = 2$ synthetic TI with $U(1) \times Z_2$, and when $m = -1$, the system is a trivial $D = 2$ synthetic insulator. With a fixed m , the nontrivial $2D$ synthetic TI is equivalent to a series of $1d$ systems parameterized by p_y , which have a quantized Z_2 charge pump from one side of the $1d$ system to the other when p_y is tuned adiabatically from 0 to 2π ; and a trivial $2D$ synthetic insulator corresponds to a series of $1d$ systems without such a Z_2 charge pump while tuning p_y . A Z_2 charge pump can happen only when there is a level crossing at the $0d$ boundary of the $1d$ systems while tuning p_y . And when we further tune m from $m = 1$ to $m = -1$, this level crossing can vanish only through a gap closing at a certain parameter (m^*, p_y^*) in the one-dimensional bulk, when the $0d$ level crossings at the two opposite boundaries annihilate each other through the gapless modes in the $1d$ bulk.

In the language of the $2D$ synthetic space, this gap closing at (m^*, p_y^*) corresponds to an unavoidable phase transition between the nontrivial $2D$ synthetic TI and a trivial $2D$ synthetic insulator, and this phase transition must still exist under a synthetic interaction, because in our case even with an interaction the entire Hamiltonian is parameterized by m and p_y , and the boundary level crossing can still vanish only through a bulk gap closing.

But for two copies of Eq. (3), the story is different. First of all, the ground states in the two opposite limits with $p \rightarrow \pm\infty$ now carry the same Z_2 charge, and then, in principle, a level crossing can be avoided under an interaction. To demonstrate this effect in a specific interacting Hamiltonian, let us perform a basis transformation of the second copy of the system so that the edge state Hamiltonian reads

$$H_p = v p (\psi_{1,p}^\dagger \tau^z \psi_{1,p} - \psi_{2,p}^\dagger \tau^z \psi_{2,p}) + H_{\text{int}}(p). \quad (4)$$

Our goal is to show, that for a certain choice of $H_{\text{int}}(p)$, the ground state in the limit $p > 0$ and $p < 0$ of Eq. (4) can be

FIG. 2. The exact many-body spectrum (including the ground state and the first excited state) of the Hamiltonian Eq. (4) with the interaction chosen to be Eq. (5). The ground state is non-degenerate and is separated from the excited states by a finite energy gap for all values of p .

connected adiabatically without closing the ground-state gap. The following choice of $H_{\text{int}}(p)$ suffices:

$$H_{\text{int}}(p) = J \vec{S}_{p,+} \cdot \vec{S}_{p,-}, \quad \vec{S}_{p,\pm} = \psi_{p,\pm}^\dagger \vec{\sigma} \psi_{p,\pm}, \quad (5)$$

where $\vec{\sigma}$ are three Pauli matrices that act on the fermion index (1,2) and $\psi_{p,\pm}$ are fermion modes with eigenvalues ± 1 of τ^z . This interaction preserves the entire $U(1) \times Z_2$ symmetry, and a direct computation of the spectrum of Eq. (4) plotted in Fig. 2 confirms that the interaction H_{int} can indeed ensure a finite gap for all values of p for the Hamiltonian Eq. (4).

This analysis implies that the two-dimensional nonchiral TI with $U(1) \times Z_2$ symmetry has its classification reduced from $\mathbb{Z}$ to $\mathbb{Z}_2$ under an interaction if one of the spatial dimensions is synthetic, which is different from the $\mathbb{Z}$ to $\mathbb{Z}_4$ reduction as discussed previously.

III. SYNTHETIC TI WITH $D=3$

Now let us consider the case with $D=3$ and $\delta=1$; namely, we are considering an effective $D=3$ -dimensional TI with one synthetic dimension. One type of three-dimensional TI whose classification is changed by an interaction is the TI defined by symmetry $U(1) \times Z_2^T$. One tight-binding model of this TI is $\hat{H} = \sum_{\vec{k}} c_{\vec{k}}^\dagger H(\vec{k}) c_{\vec{k}}$, and $H(\vec{k})$ is

$$H(\vec{k}) = -t \left[\sum_{i=1}^3 \Gamma^i \sin k_i - \Gamma^4 \left(h - \sum_{i=1}^3 \cos k_i \right) \right]. \quad (6)$$

Here, we use the following convention of the Gamma matrices: $\Gamma^1 = \sigma^{30}$, $\Gamma^2 = \sigma^{10}$, $\Gamma^3 = \sigma^{22}$, $\Gamma^4 = \sigma^{21}$, $\Gamma^5 = \sigma^{23}$, and $\sigma^{ab} = \sigma^a \otimes \sigma^b$, with $\sigma^0 = \mathbf{1}_{2 \times 2}$. The anti-unitary time-reversal Z_2^T symmetry acts on the fermion

operators as $\mathcal{T}: c_k \rightarrow i\Gamma^5 c_k^\dagger$, combined with a complex conjugation. This model is essentially two copies of the topological superconductor $^3\text{He-B}$ phase. Note that whether $\mathcal{T}^2$ is $+1$ or -1 no longer matters in this case, as the sign of $\mathcal{T}^2$ can be changed by the $U(1)$ rotation. In the literature, this $U(1)$ symmetry is often referred to as the spin $U(1)$ symmetry, since it commutes with the time reversal (e.g., see Ref. [42]). Notice that this choice of time reversal corresponds to the AIII class of the TI in the “tenfold way” language, and Z_2^T symmetry takes c_k to $c_k^\dagger$ up to a unitary transformation, where the “label” k is unchanged, which is compatible with realizing synthetic momenta as tuning parameters of the system: The time reversal does not connect systems at different tuning parameters. The requirement that the synthetic system is $U(1) \times Z_2^T$ symmetric is then to require the same symmetry to hold for any fixed value of the synthetic momenta (or tuning parameters).

At the (for example) XY boundary, the system has a $2d$ gapless Dirac fermion with Hamiltonian

$$H = \int d^2x v \psi^\dagger (-i\sigma^3 \partial_x - i\sigma^1 \partial_y) \psi. \quad (7)$$

The time-reversal symmetry acts on the boundary two-component Dirac fermion as

$$\mathcal{T}: \psi \rightarrow i\sigma^y \psi^\dagger. \quad (8)$$

It is well known that, without any interaction, the classification of this TI is $\mathbb{Z}$ [2–4], while under ordinary local interactions, the classification of this TI is reduced to $\mathbb{Z}_8$ [37,48]; i.e., eight copies of this TI are rendered trivial under a local interaction, or, equivalently, the edge state can be gapped by a local interaction without developing a nonzero expectation value of any fermion bilinear operator, which is forbidden by the $U(1) \times Z_2^T$.

Now let us investigate what happens if the y direction is a synthetic dimension and consider multiple copies of the boundary Hamiltonian. Then the boundary Hamiltonian at each synthetic momentum p is

$$H_p = \int dx \sum_{a=1}^N v \psi_{a,p}^\dagger (-i\sigma^3 \partial_x + \sigma^1 p) \psi_{a,p} + H_{\text{int}}(p), \quad (9)$$

where $H_{\text{int}}(p)$ is a $U(1) \times Z_2^T$ symmetry-allowed flavor-mixing interaction term that is parameterized by p . Notice that the time-reversal symmetry does not mix fermion operators labeled by different momenta; i.e., time reversal does not mix systems with different parameters p . The entire Hamiltonian Eq. (9) can be viewed as N copies of a $1d$ interacting topological insulator with the same $U(1) \times Z_2^T$ symmetry (which again has $\mathbb{Z}$ classification in the noninteracting limit), tuned close to its transition to the trivial insulator, which corresponds to N copies of $1d$

Dirac fermions with Dirac mass p . The synthetic momentum p is precisely the Dirac mass that tunes the system across the topological-trivial transition. Let us emphasize again that this interpretation of the synthetic momentum does not apply to the ordinary interacting TI if all dimensions are physical dimensions, because different physical momenta are mixed by the ordinary local interactions.

Now the problem readily reduces to the interaction effects on the $1d$ TI with $U(1) \times Z_2^T$ symmetry, which has been studied and well understood. It is shown that the $1d$ TI with $U(1) \times Z_2^T$ symmetry, though it has $\mathbb{Z}$ classification without an interaction, is reduced to a $\mathbb{Z}_4$ classification under a local interaction [41,42]. This $\mathbb{Z}$ to $\mathbb{Z}_4$ reduction under an interaction is also consistent with the classification of bosonic symmetry-protected topological phases: Two copies of the TIs under an interaction can be adiabatically connected to the Haldane phase by gapping out the single-particle excitations [38], and it is well known that two copies of coupled Haldane phases become a trivial phase [5,6].

The observation above implies that, when $N = 4$ in Eq. (9), the phase transition between the two limits $p > 0$ and $p < 0$ can be avoided by turning on the interaction; i.e., the two regions in the phase diagrams can be adiabatically connected under an interaction without closing the gap, and the original critical point $p = 0$ is rendered gapped and nondegenerate by the interaction. This observation leads to the conclusion that the classification of the $D = 3$ TI with $U(1) \times Z_2^T$ symmetry is reduced to $\mathbb{Z}_4$ instead of $\mathbb{Z}_8$ if one of the three spatial dimensions is synthetic.

If we choose $D = 3$, $\delta = 2$ —namely, two out of the three directions are synthetic dimensions—and we consider a two-dimensional boundary whose both directions are synthetic (let us label them as the x and y directions), then at each p_x and p_y the edge state is a two-component complex fermion $\psi_{\vec{p}}$. Again, the problem at the boundary reduces to solving a zero-dimensional system with two tuning parameters p_x and p_y . For $N = 2$ copies of the edge states, at every synthetic momentum $\vec{p}$, the free fermion part of the boundary Hamiltonian reads

$$H_{\vec{p}} = v\psi_{\vec{p}}^\dagger(\sigma^3 p_x + \sigma^1 p_y) \otimes \tau^z \psi_{\vec{p}}, \quad (10)$$

where again we perform a basis transformation for the $\tau^z = -1$ component of the edge state. Then the same interaction as Eq. (5), which is also $U(1) \times Z_2^T$ invariant, $H_{\text{int}}(p) = J\vec{S}_{p,+} \cdot \vec{S}_{p,-}$ can gap out the entire boundary without leading to any ground-state degeneracy (see Fig. 3). Hence, when there are two synthetic dimensions, the $D = 3$ TI with $U(1) \times Z_2^T$ is reduced to a $\mathbb{Z}_2$ classification.

To summarize this section, for a $D = 3$ interacting synthetic TI with $U(1) \times Z_2^T$ symmetry and different choices of $\delta = D - d$, its classification is

$$\mathbb{Z}_8(\delta = 0); \quad \mathbb{Z}_4(\delta = 1); \quad \mathbb{Z}_2(\delta = 2). \quad (11)$$

FIG. 3. The exact many-body spectrum (including the ground state and the first excited state) of the Hamiltonian $\psi_{\vec{p}}^\dagger(\sigma^3 p_x + \sigma^1 p_y) \otimes \tau^z \psi_{\vec{p}} + \vec{S}_{p,+} \cdot \vec{S}_{p,-}$. The ground state is non-degenerate and is separated from the excited states by a finite energy gap for all values of p_x and p_y .

One can also discuss the simplest example of the $D = 1$ TI with $U(1) \times Z_2^T$ symmetry. If the only dimension is synthetic—namely, the lattice momentum along this dimension is actually a tuning parameter—although it is unnatural to discuss the edge state of the synthetic dimension, the TI can still be defined as whether there must be a gap-closing transition between the TI and the trivial insulator or not. In the noninteracting limit, near the critical point between the TI and trivial insulator, the bulk Hamiltonian takes exactly the same form as Eq. (3), and the time-reversal symmetry acts as $\mathcal{T}:\psi_p \rightarrow i\sigma^y \psi_p^\dagger$. The mass term that tunes the topological-to-trivial transition is $m\psi_p^\dagger \sigma^x \psi_p$. Then, one can show that, with two copies of the system, an interaction $H_{\text{int}}(p)$ similar to Eq. (5) adiabatically connects the original topologically nontrivial TI to the trivial phase. Thus, the interaction reduces the classification of the synthetic $D = 1$ TI with $U(1) \times Z_2^T$ symmetry to $\mathbb{Z}_2$.

IV. SYNTHETIC TI WITH $D = 4$

Now, we discuss the higher-dimensional (total dimension $D = 4$) TI that cannot be realized in the lab without using the synthetic techniques. First of all, the classification of the $D = 4$ quantum Hall insulator itself is not reduced by an interaction, due to the chiral anomaly at its three-dimensional boundary and the anomaly matching condition [49,50]. Thus, we consider the $D = 4$ nonchiral TI with a $U(1) \times Z_2$ symmetry. Like its $D = 2$ analogue, this TI is simply two copies of the original $D = 4$ quantum Hall state with opposite Chern numbers, and they have eigenvalues $+1$ and -1 , respectively, under the Z_2 symmetry operation.

Since this $D = 4$ TI was not discussed much in the past, we first consider the case where all the dimensions are physical, i.e., $D = d = 4$. The $3d$ boundary of this TI is two copies of Weyl fermions with opposite chiralities and also opposite eigenvalues under the Z_2 symmetry:

$$H = \int d^3x v\psi_L^\dagger(i\vec{\sigma} \cdot \vec{\partial})\psi_L - v\psi_R^\dagger(i\vec{\sigma} \cdot \vec{\partial})\psi_R. \quad (12)$$

The charge conservation $U(1)$ symmetry and the Z_2 symmetry act on the boundary fermions as $U(1):\psi_L, \psi_R \rightarrow e^{i\theta}\psi_L, e^{i\theta}\psi_R$, $Z_2:\psi_L \rightarrow \psi_L, \psi_R \rightarrow -\psi_R$. With this $U(1) \times Z_2$ symmetry action, this TI has a $\mathbb{Z}$ classification without an interaction, because any fermion bilinear mass term should either break the Z_2 symmetry (which corresponds to the ordinary Dirac mass term that mixes the left- and right-handed Weyl fermion) or break the $U(1)$ symmetry, which is a fermion-pairing term (the so-called Majorana mass term).

To study the interaction effects on the TI, we follow the similar procedure of Ref. [37]. We first couple the left- and right-handed Weyl fermions both to a complex bosonic order parameter ϕ :

$$\phi(\psi_L^\dagger \sigma^y \psi_L + \psi_R^\dagger \sigma^y \psi_R) + \text{H.c.} \quad (13)$$

When ϕ condenses, i.e., $\langle \phi \rangle \neq 0$, the charge $U(1)$ symmetry is spontaneously broken and the fermions are all gapped. Then we try to restore the $U(1)$ symmetry by proliferating the vortex loops of the $U(1)$ order parameter ϕ . Along the vortex loop, by directly solving the Dirac equation with a background vortex configuration of ϕ , we find there are nonchiral 1d Majorana fermion described by the effective Hamiltonian [Fig. 1(b)]

$$H_{1d} = \int dx i v (\chi_L \partial_x \chi_L - \chi_R \partial_x \chi_R). \quad (14)$$

Again, the left- and right-moving Majorana fermions carry eigenvalue ± 1 under the Z_2 symmetry, respectively.

The 1d Hamiltonian Eq. (14) and its Z_2 symmetry are exactly the same as the 1d edge states of a 2d $p_x \pm i p_y$ TSC, where the $p_x + i p_y$ and $p_x - i p_y$ superconductors carry eigenvalues ± 1 , respectively, under the Z_2 symmetry. It is known that, for eight copies of Eq. (14), a local interaction that preserves the Z_2 symmetry can gap out Eq. (14) without spontaneously breaking the Z_2 symmetry [33–35, 51], and, hence, eight copies of the 1d system Eq. (14) along the vortex line can be rendered gapped and nondegenerate under an interaction. Thus, the $U(1) \times Z_2$ symmetry can be restored for eight copies of the 3d edge states Eq. (12) by proliferating the fully gapped vortex loops. Hence, the classification of the $D = 4$ TI described above is reduced from $\mathbb{Z}$ to $\mathbb{Z}_8$ under local interactions.

Now, let us consider the case with $D = 4, d = 3$; namely, three out of the four dimensions are physical, while one dimension is synthetic. We still look at the most natural boundary of the synthetic TI which is parallel to the synthetic dimensions. Then, for N copies of the system, at each fixed synthetic momentum p , the Hamiltonian for the boundary states reads

$$H_p = \int d^2x \sum_{a=1}^N v \psi_{a,L}^\dagger (i\sigma^x \partial_x + i\sigma^y \partial_y - p\sigma^z) \psi_{a,L} - v \psi_{a,R}^\dagger (i\sigma^x \partial_x + i\sigma^y \partial_y - p\sigma^z) \psi_{a,R} + H_{\text{int}}(p). \quad (15)$$

The Hamiltonian Eq. (15) can be viewed as the previously discussed two-dimensional nonchiral TI with a $U(1) \times Z_2$ symmetry tuned close to the topological-trivial transition point $p = 0$, and p serves as the tuning parameter for this transition. As we already mentioned before, the classification of this two-dimensional nonchiral TI is reduced from $\mathbb{Z}$ to $\mathbb{Z}_4$ under an interaction [39, 40]. Hence, for $N = 4$, the critical point $p = 0$ can be avoided under an interaction without breaking the Z_2 symmetry, and the two regions with $p < 0$ and $p > 0$ can be smoothly connected without closing the gap. This result implies that the classification of the $D = 4$ and $d = 3$ synthetic TI is reduced to $\mathbb{Z}_4$ rather than $\mathbb{Z}_8$ under the generic interaction of the synthetic TI.

If we instead choose $D = 4, d = 2$ in the first place—namely, two out of the four dimensions are synthetic dimensions—then, at every synthetic momentum $\vec{p} = (p_x, p_y)$, the Hamiltonian reads

$$H_p = \int dx \psi_L^\dagger (i\sigma^x \partial_x - p_y \sigma^y - p_z \sigma^z) \psi_L - \psi_R^\dagger (i\sigma^x \partial_x - p_y \sigma^y - p_z \sigma^z) \psi_R. \quad (16)$$

Now, at each p_z , the edge state can be viewed as the $D = 2, d = 1$ synthetic TI with $U(1) \times Z_2$ symmetry tuned close to the topological-trivial transition, while one of the dimensions is a synthetic dimension p_y . p_z is precisely the tuning parameter that tunes the two-dimensional synthetic system across the transition. As we have argued before, the classification of this $D = 2$ synthetic TI is reduced to $\mathbb{Z}_2$ under a generic synthetic interaction. This reduction implies that, for two copies of Eq. (16), a properly designed $H_{\text{int}}(\vec{p})$ gaps out the edge system Eq. (16) without ground-state degeneracy. Then the classification reduces from $\mathbb{Z}$ to $\mathbb{Z}_2$ when $\delta = d = 2$.

To summarize this section, for a $D = 4$ nonchiral interacting synthetic TI with $U(1) \times Z_2$ symmetry and different choices of $\delta = D - d$, its classification is

$$\mathbb{Z}_8(\delta = 0); \quad \mathbb{Z}_4(\delta = 1); \quad \mathbb{Z}_2(\delta = 2). \quad (17)$$

V. SYNTHETIC TI IN GENERAL EVEN DIMENSIONS $D = 2n$

In general even spatial dimensions $D = 2n$, we can consider the generalized version of the nonchiral TI with $U(1) \times Z_2$ symmetry discussed in Secs. II and IV. Such TIs in the noninteracting limit can always be constructed by putting the fermions with Z_2 eigenvalue $+1$ into a $2n$ -dimensional Chern insulator with a nontrivial Chern-Simons response while simultaneously putting fermions with Z_2 eigenvalue -1 also into a Chern insulator but with an opposite Chern-Simons response. In the noninteracting limit, the classification of this type of TI is given by $\mathbb{Z}$.

Let us first consider the case with all of the dimensions physical, namely, $D = d = 2n$ and $\delta = 0$. The boundary state of this nonchiral TI can be described by the Hamiltonian

$$H = \int d^{2n-1}x \sum_{k=1}^{2n-1} v [\psi_L^\dagger (i\gamma^k \partial_k) \psi_L - \psi_R^\dagger (i\gamma^k \partial_k) \psi_R], \quad (18)$$

where $\{\gamma^k\}$ with $k = 1, 2, \dots, 2n-1$ are 2^{n-1} -dimensional matrices that generate the complex Clifford algebra. The product of all γ matrices $\prod_{k=1}^{2n-1} \gamma^k$ is a constant. The symmetries act on the boundary fermions as $U(1): \psi_L, \psi_R \rightarrow e^{i\theta} \psi_L, e^{i\theta} \psi_R$ and $Z_2: \psi_L \rightarrow \psi_L, \psi_R \rightarrow -\psi_R$. The $U(1) \times Z_2$ symmetry forbids any fermion bilinear mass terms, because all ordinary mass terms break the Z_2 symmetry due to the mixing of ψ_L and ψ_R fermions and all possible Majorana mass terms break the $U(1)$ symmetry. Therefore, this TI has a $\mathbb{Z}$ classification in the noninteracting limit.

In the following, we show that the classification of $U(1) \times Z_2$ nonchiral TIs in $D = 2n$ spatial dimensions with $\delta = 0$ reduces from $\mathbb{Z}$ to $\mathbb{Z}_{2^{n+1}}$ when we consider interactions. To show this reduction, we apply the method of the fermion sigma model [14]. We first consider $\nu = 2^n$ copies of the boundary states described in Eq. (18), rewrite them using Majorana fermions χ , and couple the Majorana fermions to a $(2n+2)$ -component dynamical vector field n_a ($a = 1, \dots, 2n+2$) via the mass terms:

$$H_{\nu} = \int d^{2n-1}x \chi^\dagger \left(\sum_{k=1}^{2n-1} i v \alpha^k \partial_k + \sum_{a=1}^{2n+2} n_a \beta^a \right) \chi. \quad (19)$$

Here, the set of matrices $\{\alpha^1, \dots, \alpha^{2n-1}; i\beta^1, \dots, i\beta^{2n+2}\}$ generates the real Clifford algebra $\mathcal{C}\ell_{2n-1, 2n+2}$ in its 2^{2n+1} -dimensional representation. The matrices α^i are real symmetric matrices, while the matrices β^a are imaginary antisymmetric ones. All the α^i and β^a matrices square to the identity matrix. In this representation, the $U(1)$ and Z_2 symmetry actions on the Majorana fermions χ are generated by $i\beta^1 \beta^2$ and $\prod_{i=1}^{2n-1} \alpha^i$, respectively. The symmetry transformation of the vector field n_a is given by

$$\begin{aligned} U(1): (n_1 + i n_2) &\rightarrow e^{i\theta} (n_1 + i n_2); \\ n_a &\rightarrow n_a \quad \text{for } a = 3, \dots, 2n+2; \\ Z_2: n_a &\rightarrow -n_a \quad \text{for all } a. \end{aligned} \quad (20)$$

Assuming that the vector field n_a is a slowly varying field with unit modulus, we can integrate out the Majorana fermions χ and obtain an effective action of n_a which can be identified as a $O(2n+2)$ nonlinear sigma model (NL σ M) in $2n-1$ spatial dimensions with a level-1 (or level-1) Wess-Zumino-Witten (WZW) term [14,52]. Such an effective action can be identified as that of the

boundary state of a bosonic symmetry-protected topological (SPT) state with $U(1) \times Z_2$ symmetry. The bulk of such a bosonic SPT should be described by an $O(2n+2)$ NL σ M in $2n$ spatial dimensions with a $2\pi\Theta$ term. Now, we can conclude that ν copies of the TI whose boundary state is described in Eq. (18) can be adiabatically connected to this bosonic SPT state. This bosonic SPT has a $\mathbb{Z}_2$ classification, because we can consider two copies of the $O(2n+2)$ NL σ M s, each with a $2\pi\Theta$ term, and couple them such that their n_a vector fields align in the first $2n+1$ components and antialign in the $2n+2$ th component. The coupling effectively merges the two copies into a single $O(2n+2)$ NL σ M which is free of a net Θ term and, hence, is topologically trivial. Therefore, we can conclude that 2ν copies of the boundary state described in Eq. (18) together are topologically equivalent to the boundary state of a trivial state. The classification of the $U(1) \times Z_2$ nonchiral TI is hence reduced from $\mathbb{Z}$ in the noninteracting limit to $\mathbb{Z}_{2\nu} = \mathbb{Z}_{2^{n+1}}$ in $D = 2n$ spatial dimensions (with $\delta = 0$) when we take interactions into account.

Now, we consider the case with one of the spatial dimension synthetic, i.e., $D = 2n$, $d = 2n-1$, and $\delta = 1$. We can start with the boundary state Hamiltonian

$$\begin{aligned} H_p = \int d^{2n-2}x &\left\{ \psi_L^\dagger \left(\sum_{k=1}^{2n-2} i v \gamma^k \partial_k - p \gamma^{2n-1} \right) \psi_L \right. \\ &\left. - \psi_R^\dagger \left(\sum_{k=1}^{2n-2} i v \gamma^k \partial_k - p \gamma^{2n-1} \right) \psi_R \right\}, \end{aligned} \quad (21)$$

where p represents the synthetic momentum along the synthetic dimension. We notice that this boundary Hamiltonian Eq. (21) can be identified as the bulk Hamiltonian of the $U(1) \times Z_2$ symmetric nonchiral TI in $D' = d' = 2n-2$ spatial dimensions tuned to the vicinity of a topological-trivial transition point at $p = 0$. As we just discussed, the classification of the $D' = d' = 2n-2$ nonchiral TI is reduced from $\mathbb{Z}$ to $\mathbb{Z}_{2^n}$ under an interaction. That means, when we consider 2^n copies of the model Eq. (21), the critical point at $p = 0$ can be avoided under interactions without breaking the $U(1) \times Z_2$ symmetry. In other words, the classification of the $U(1) \times Z_2$ symmetric nonchiral TI in $D = 2n$ dimensions with $\delta = 1$ is reduced to $\mathbb{Z}_{2^n}$.

For $\delta \geq 1$, we can always perform a similar analysis by studying the boundary state Hamiltonians that are counterparts of Eqs. (18) and (21). Interestingly, such boundary state Hamiltonians can always be identified as the bulk Hamiltonians of the $U(1) \times Z_2$ symmetric nonchiral TI in $D' = 2n-2$ dimensions with $\delta' = \delta - 1$ near the topological-trivial critical point. This identification is done by choosing one of the synthetic momenta as the tuning parameter for the topological-trivial transition while leaving the other $\delta' = \delta - 1$ synthetic momenta still representing the synthetic dimensions among the total $D' = 2n-2$

dimensions. With this identification, we can directly conclude that, for general δ , the classification of the $U(1) \times Z_2$ synthetic nonchiral TI becomes $\mathbb{Z}_{2n+1-\delta}$.

To summarize this section, for a $2n$ -dimensional nonchiral interacting synthetic TI with $U(1) \times Z_2$ symmetry, the classification, in the presence of δ synthetic dimensions, is given by $\mathbb{Z}_{2n+1-\delta}$. This result is consistent with the analysis in Secs. II and IV.

VI. SYNTHETIC TI IN GENERAL ODD DIMENSIONS $D = 2n + 1$

In general odd spatial dimensions $D = 2n + 1$, we can consider the generalized version of the TI with $U(1) \times Z_2^T$ symmetry discussed in Sec. III. In the noninteracting limit, such systems belong to class AIII in the tenfold way classification [2,3] and have a classification of $\mathbb{Z}$.

Let us first consider the case with all of the dimensions physical, namely, $D = d = 2n + 1$ and $\delta = 0$. The boundary state of the $U(1) \times Z_2^T$ symmetric TI that can generate the whole $\mathbb{Z}$ class can be described by the Hamiltonian

$$H = \int d^2x \sum_{k=1}^{2n} v\psi^\dagger(i\gamma^k \partial_k)\psi, \quad (22)$$

where $\{\gamma^k\}_{k=1}^{2n}$ are a set of 2^n -dimensional complex matrices that forms the complex Clifford algebra. The symmetries act on the boundary fermions as $U(1): \psi \rightarrow e^{i\theta}\psi$ and $Z_2^T: \psi \rightarrow (\prod_{i=1}^{2n} \gamma^i)\psi^\dagger$. One can check that the $U(1) \times Z_2^T$ symmetry forbids any fermion bilinear mass terms even when there are multiple copies of such boundary states. Therefore, this TI has a $\mathbb{Z}$ classification in the noninteracting limit.

In the following, we show that the classification of the $U(1) \times Z_2^T$ TI in $D = 2n + 1$ spatial dimensions with $\delta = 0$ reduces from $\mathbb{Z}$ to $\mathbb{Z}_{2n+2}$ when we take interactions into account. To show this reduction, we again use the method of the fermion sigma model. The following discussion is largely parallel to Sec. V. We first consider $\nu' = 2^{n+1}$ copies of the boundary states described in Eq. (22), rewrite them using Majorana fermions χ , and couple the Majorana fermions to a $(2n + 3)$ -component dynamical vector field n_a via the mass terms

$$H_{\nu'} = \int d^2x \chi^\dagger \left(\sum_{k=1}^{2n} i v \alpha^k \partial_k + \sum_{a=1}^{2n+3} n_a \beta^a \right) \chi. \quad (23)$$

Here, the set of matrices $\{\alpha^1, \dots, \alpha^{2n}, i\beta^1, \dots, i\beta^{2n+3}\}$ generates the real Clifford algebra $\mathcal{Cl}_{2n,2n+3}$ in its 2^{2n+2} -dimensional representation. In this representation, the $U(1)$ symmetry action on the Majorana fermions is generated by $i\beta^1\beta^2$. The antiunitary time-reversal symmetry acts as $Z_2^T: \chi \rightarrow (\prod_{i=1}^{2n} \alpha^i)\chi$, combined with a complex conjugation. The symmetry transformation of the vector field n_a is given by

$$\begin{aligned} U(1): (n_1 + in_2) &\rightarrow e^{i\theta}(n_1 + in_2); \\ n_a &\rightarrow n_a \quad \text{for } a = 3, \dots, 2n + 3; \\ Z_2^T: n_a &\rightarrow -n_a \quad \text{for all } a. \end{aligned} \quad (24)$$

As we discussed in Sec. V, when the vector field n_a is slowly varying and has a unit modulus, we can integrate out the Majorana fermion χ and obtain a $O(2n + 3)$ NLM with a level-1 (or level-1) WZW term. Here, the action of $n_a \rightarrow -n_a$ in $O(2n + 3)$ should be identified as the time-reversal symmetry Z_2^T . Such an effective action in fact describes the boundary state of a $U(1) \times Z_2^T$ symmetric bosonic SPT which itself has a $\mathbb{Z}_2$ classification. Therefore, by the same argument given in Sec. V, the classification of the $U(1) \times Z_2^T$ symmetric TI is reduced from $\mathbb{Z}$ in the noninteracting limit to $\mathbb{Z}_{2\nu'} = \mathbb{Z}_{2n+2}$ in $D = 2n + 1$ spatial dimensions (with $\delta = 0$) when we take interactions into account.

When synthetic dimensions are present—i.e., $\delta \geq 1$, we can identify one of the synthetic momenta as the tuning parameter of a topological-trivial transition—the boundary state Hamiltonian can then be identified as the bulk Hamiltonian of a $U(1) \times Z_2^T$ TI in the vicinity of the topological-trivial critical point in $D' = 2n - 1$ spatial dimensions and with $\delta' = \delta - 1$ synthetic dimensions. With this identification, we can conclude that the classification of the $U(1) \times Z_2^T$ symmetric interacting TI is given $\mathbb{Z}_{2n+2-\delta}$.

To summarize this section, for a $2n + 1$ -dimensional interacting synthetic TI with $U(1) \times Z_2^T$ symmetry, the classification, in the presence of δ synthetic dimensions, is given by $\mathbb{Z}_{2n+2-\delta}$. This result is consistent with the analysis in Sec. III.

VII. SUMMARY

In this work, we have analyzed the interaction effects on the effective TIs simulated with the newly developed synthetic techniques. We demonstrate that, unlike ordinary interacting TIs, the interaction causes a different classification reduction of the simulated TI, due to the generic while special form of the interaction in systems with synthetic dimensions. We need to point out that the analysis used the fact that the system at every synthetic momentum $\vec{p}$ is a lower-dimensional system with the same symmetry as the desired effective D -dimensional TI. This analysis no longer naturally applies (although not impossible) if the system involves time-reversal symmetry that does not commute with the charge $U(1)$ symmetry, because in this case time-reversal symmetry would bring c_k to c_{-k} , and, hence, it mixes systems labeled by different parameters $\vec{p}$.

Topological semimetals have also attracted enormous research interest and efforts in the past few years [22–28,53]. One can also construct semimetals in the lab using the same synthetic techniques, and these semimetals could be vulnerable to interactions; i.e., the bulk of the system

can be driven into an insulator with a nondegenerate ground state due to an interaction. For example, the long-wavelength effective Hamiltonians Eqs. (7) and (15) can also be viewed as the bulk Hamiltonian expanded near the gapless momenta of synthetic semimetals with $D = 2$ and $D = 3$; again, p is the tuning parameter which is viewed as an extra synthetic momentum. And our previous analysis indicates that, for $N = 4$, both cases can be gapped out by interaction $H_{\text{int}}(p)$ without leading to ground-state degeneracy, and the Dirac semimetal becomes an insulator with a nondegenerate ground state.

It has also been shown that under a strong interaction the boundary of a nontrivial 3d TI can be a 2d topological order with anomalous quantum number fractionalizations [36,54–61]. As we explained before, at each parameter p , the edge Hamiltonian Eq. (15) describes an interacting 2d system at the boundary of a $D = 4$ system. Thus, under a strong interaction, it is possible that at each parameter (effective momentum) p the edge is driven into topological order. This situation could correspond to a very exotic “topological order” at the boundary of the $D = 4$ TI. We leave this topic for future studies.

ACKNOWLEDGMENTS

C.-M. J. is partly supported by the Gordon and Betty Moore Foundations EPIQS Initiative through Grant No. GBMF4304. C. X. is supported by the David and Lucile Packard Foundation and National Science Foundation Grant No. DMR-1151208.

-
- [1] S.-C. Zhang and J. Hu, *A Four-Dimensional Generalization of the Quantum Hall Effect*, *Science* **294**, 823 (2001).
 - [2] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, *Classification of Topological Insulators and Superconductors*, *AIP Conf. Proc.* **1134**, 10 (2009).
 - [3] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, *Topological Insulators and Superconductors: Tenfold Way and Dimensional Hierarchy*, *New J. Phys.* **12**, 065010 (2010).
 - [4] A. Kitaev, *Periodic Table for Topological Insulators and Superconductors*, *AIP Conf. Proc.* **1134**, 22 (2009).
 - [5] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, *Symmetry Protected Topological Orders and the Group Cohomology of their Symmetry Group*, *Phys. Rev. B* **87**, 155114 (2013).
 - [6] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, *Symmetry-Protected Topological Orders in Interacting Bosonic Systems*, *Science* **338**, 1604 (2012).
 - [7] A. Kapustin, *Bosonic Topological Insulators and Paramagnets: A View from Cobordisms*, *arXiv:1404.6659*.
 - [8] A. Kapustin and R. Thorngren, *Anomalies of Discrete Symmetries in Various Dimensions and Group Cohomology*, *arXiv:1404.3230*.
 - [9] D. S. Freed, *Anomalies and Invertible Field Theories*, *arXiv:1404.7224*.
 - [10] D. S. Freed, *Short-Range Entanglement and Invertible Field Theories*, *arXiv:1406.7278*.
 - [11] X.-G. Wen, *Construction of Bosonic Symmetry-Protected-Trivial States and Their Topological Invariants via $G \times SO(\infty)$ Nonlinear σ Models*, *Phys. Rev. B* **91**, 205101 (2015).
 - [12] Z. Bi and C. Xu, *Construction and Field Theory of Bosonic-Symmetry-Protected Topological States beyond Group Cohomology*, *Phys. Rev. B* **91**, 184404 (2015).
 - [13] C. Xu and Y.-Z. You, *Bosonic Short-Range Entangled States beyond Group Cohomology Classification*, *Phys. Rev. B* **91**, 054406 (2015).
 - [14] Y.-Z. You and C. Xu, *Symmetry-Protected Topological States of Interacting Fermions and Bosons*, *Phys. Rev. B* **90**, 245120 (2014).
 - [15] M. F. Lapa, C.-M. Jian, P. Ye, and T. L. Hughes, *Topological Electromagnetic Responses of Bosonic Quantum Hall, Topological Insulator, and Chiral Semimetal Phases in All Dimensions*, *Phys. Rev. B* **95**, 035149 (2017).
 - [16] X.-G. Wen, *A Lattice Non-perturbative Definition of an $SO(10)$ Chiral Gauge Theory and Its Induced Standard Model*, *Chin. Phys. Lett.* **30**, 111101 (2013).
 - [17] Y.-Z. You and C. Xu, *Interacting Topological Insulator and Emergent Grand Unified Theory*, *Phys. Rev. B* **91**, 125147 (2015).
 - [18] Y.-Z. You, Y. BenTov, and C. Xu, *Interacting Topological Superconductors and Possible Origin of 16n Chiral Fermions in the Standard Model*, *arXiv:1402.4151*.
 - [19] Y. E. Kraus, Z. Ringel, and O. Zilberberg, *Four-Dimensional Quantum Hall Effect in a two-Dimensional Quasicrystal*, *Phys. Rev. Lett.* **111**, 226401 (2013).
 - [20] M. Lohse, C. Schweizer, H. M. Price, O. Zilberberg, and I. Bloch, *Exploring 4D Quantum Hall Physics with a 2D Topological Charge Pump*, *Nature (London)* **553**, 55 (2018).
 - [21] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, *Topological Field Theory of Time-Reversal Invariant Insulators*, *Phys. Rev. B* **78**, 195424 (2008).
 - [22] X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, *Topological Semimetal and Fermi-Arc Surface States in the Electronic Structure of Pyrochlore Iridates*, *Phys. Rev. B* **83**, 205101 (2011).
 - [23] A. A. Burkov and L. Balents, *Weyl Semimetal in a Topological Insulator Multilayer*, *Phys. Rev. Lett.* **107**, 127205 (2011).
 - [24] H. Weng, C. Fang, Z. Fang, B. A. Bernevig, and X. Dai, *Weyl Semimetal Phase in Noncentrosymmetric Transition-Metal Monophosphides*, *Phys. Rev. X* **5**, 011029 (2015).
 - [25] N. P. Armitage, E. J. Mele, and A. Vishwanath, *Weyl and Dirac Semimetals in Three-Dimensional Solids*, *Rev. Mod. Phys.* **90**, 015001 (2018).
 - [26] S.-Y. Xu, I. Belopolski, N. Alidoust, M. Neupane, G. Bian, C. Zhang, R. Sankar, G. Chang, Z. Yuan, C.-C. Lee, S.-M. Huang, H. Zheng, J. Ma, D. S. Sanchez, B. Wang, A. Bansil, F. Chou, P. P. Shibayev, H. Lin, S. Jia *et al.*, *Discovery of a Weyl Fermion Semimetal and Topological Fermi Arcs*, *Science* **349**, 613 (2015).
 - [27] S.-M. Huang, S.-Ya. Xu, I. Belopolski, C.-C. Lee, G. Chang, B. Wang, N. Alidoust, G. Bian, M. Neupane, C. Zhang, S. Jia, A. Bansil, H. Lin, and M. Z. Hasan, *A Weyl Fermion Semimetal with Surface Fermi Arcs in the Transition Metal Monopnictide TaAs Class*, *Nat. Commun.* **6**, 7373 (2015).

- [28] Z. K. Liu, B. Zhou, Y. Zhang, Z. J. Wang, H. M. Weng, D. Prabhakaran, S.-K. Mo, Z. X. Shen, Z. Fang, X. Dai, Z. Hussain, and Y. L. Chen, *Discovery of a Three-Dimensional Topological Dirac Semimetal, Na_3Bi* , *Science* **343**, 864 (2014).
- [29] M. Mancini, G. Pagano, G. Cappellini, L. Livi, M. Rider, J. Catani, C. Sias, P. Zoller, M. Inguscio, M. Dalmonte, and L. Fallani, *Observation of Chiral Edge States with Neutral Fermions in Synthetic Hall Ribbons*, *Science* **349**, 1510 (2015).
- [30] B. K. Stuhl, H.-I. Lu, L. M. Ayccock, D. Genkina, and I. B. Spielman, *Visualizing Edge States with an Atomic Bose Gas in the Quantum Hall Regime*, *Science* **349**, 1514 (2015).
- [31] L. Fidkowski and A. Kitaev, *Effects of Interactions on the Topological Classification of Free Fermion Systems*, *Phys. Rev. B* **81**, 134509 (2010).
- [32] L. Fidkowski and A. Kitaev, *Topological Phases of Fermions in One Dimension*, *Phys. Rev. B* **83**, 075103 (2011).
- [33] X.-L. Qi, *A New Class of $(2+1)$ -Dimensional Topological Superconductors with $\mathbb{Z}_8$ Topological Classification*, *New J. Phys.* **15**, 065002 (2013).
- [34] H. Yao and S. Ryu, *Interaction Effect on Topological Classification of Superconductors in Two Dimensions*, *Phys. Rev. B* **88**, 064507 (2013).
- [35] S. Ryu and S.-C. Zhang, *Interacting Topological Phases and Modular Invariance*, *Phys. Rev. B* **85**, 245132 (2012).
- [36] L. Fidkowski, X. Chen, and A. Vishwanath, *Non-Abelian Topological Order on the Surface of a 3D Topological Superconductor from an Exactly Solved Model*, *Phys. Rev. X* **3**, 041016 (2013).
- [37] C. Wang and T. Senthil, *Interacting Fermionic Topological Insulators/Superconductors in Three Dimensions*, *Phys. Rev. B* **89**, 195124 (2014).
- [38] Y.-Z. You, Z. Bi, A. Rasmussen, M. Cheng, and C. Xu, *Bridging Fermionic and Bosonic Short Range Entangled States*, *New J. Phys.* **17**, 075010 (2015).
- [39] H. Isobe and L. Fu, *Theory of Interacting Topological Crystalline Insulators*, *Phys. Rev. B* **92**, 081304 (2015).
- [40] H. Song, S.-J. Huang, L. Fu, and M. Hermele, *Topological Phases Protected by Point Group Symmetry*, *Phys. Rev. X* **7**, 011020 (2017).
- [41] T. Morimoto, A. Furusaki, and C. Mudry, *Breakdown of the Topological Classification $\mathbb{Z}$ for Gapped Phases of Non-interacting Fermions by Quartic Interactions*, *Phys. Rev. B* **92**, 125104 (2015).
- [42] E. Tang and X.-G. Wen, *Interacting One-Dimensional Fermionic Symmetry-Protected Topological Phases*, *Phys. Rev. Lett.* **109**, 096403 (2012).
- [43] A similar nonlocal interaction along the synthetic dimension is noticed in Ref. [44], where, however, the synthetic dimension has a very different physical origin from our case: There, the lattice sites along the synthetic dimension are the different spin components of the atom.
- [44] T.-S. Zeng, C. Wang, and H. Zhai, *Charge Pumping of Interacting Fermion Atoms in the Synthetic Dimension*, *Phys. Rev. Lett.* **115**, 095302 (2015).
- [45] Y. E. Kraus, Y. Lahini, Z. Ringel, M. Verbin, and O. Zeitlinger, *Topological States and Adiabatic Pumping in Quasicrystals*, *Phys. Rev. Lett.* **109**, 106402 (2012).
- [46] J. T. Chalker and P. D. Coddington, *Percolation, Quantum Tunnelling and the Integer Hall Effect*, *J. Phys. C* **21**, 2665 (1988).
- [47] J. B. Marston and S.-W. Tsai, *Chalker-Coddington Network Model Is Quantum Critical*, *Phys. Rev. Lett.* **82**, 4906 (1999).
- [48] L. Fidkowski, X. Chen, and A. Vishwanath, *Non-Abelian Topological Order on the Surface of a 3D Topological Superconductor from an Exactly Solved Model*, *Phys. Rev. X* **3**, 041016 (2013).
- [49] G. 't Hooft, *Recent Developments in Gauge Theories* (Plenum Press, New York, 1980); reprinted in *Unity of Forces in the Universe*, edited by A. Zee (World Scientific, Singapore, 1982), Vol. II, p. 1004.
- [50] A. Zee, *From Baryons to Quarks through π^0 Decay*, *Phys. Lett.* **95B**, 290 (1980).
- [51] Z.-C. Gu and M. Levin, *Effect of Interactions on Two-Dimensional Fermionic Symmetry-Protected Topological Phases with $\mathbb{Z}_2$ Symmetry*, *Phys. Rev. B* **89**, 201113 (2014).
- [52] A. G. Abanov and P. B. Wiegmann, *Theta-Terms in Non-linear Sigma-Models*, *Nucl. Phys.* **B570**, 685 (2000).
- [53] C. H. Lee, S. Imhof, C. Berger, F. Bayer, J. Brehm, L. W. Molenkamp, T. Kiessling, and R. Thomale, *Topoelectrical Circuits*, *arXiv:1705.01077*.
- [54] P. Bonderson, C. Nayak, and X.-L. Qi, *A Time-Reversal Invariant Topological Phase at the Surface of a 3D Topological Insulator*, *J. Stat. Mech.* (2013) P09016.
- [55] C. Wang, A. C. Potter, and T. Senthil, *Gapped Symmetry Preserving Surface State for the Electron Topological Insulator*, *Phys. Rev. B* **88**, 115137 (2013).
- [56] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher, *Symmetry-Respecting Topologically Ordered Surface Phase of Three-Dimensional Electron Topological Insulators*, *Phys. Rev. B* **92**, 125111 (2015).
- [57] S. Sahoo, Z. Zhang, and J. C. Y. Teo, *Coupled Wire Model of Symmetric Majorana Surfaces of Topological Superconductors*, *Phys. Rev. B* **94**, 165142 (2016).
- [58] P. Ye, M. Cheng, and E. Fradkin, *Fractional s -Duality, Classification of Fractional Topological Insulators, and Surface Topological Order*, *Phys. Rev. B* **96**, 085125 (2017).
- [59] C. Wang and M. Levin, *Anomaly Indicators for Time-Reversal Symmetric Topological Orders*, *Phys. Rev. Lett.* **119**, 136801 (2017).
- [60] M. Cheng, *Microscopic Theory of Surface Topological Order for Topological Crystalline Superconductors*, *Phys. Rev. Lett.* **120**, 036801 (2018).
- [61] M. Cheng, M. Zaletel, M. Barkeshli, A. Vishwanath, and P. Bonderson, *Translational Symmetry and Microscopic Constraints on Symmetry-Enriched Topological Phases: A View from the Surface*, *Phys. Rev. X* **6**, 041068 (2016).