

Clustering of Conditional Mutual Information and Quantum Markov Structure at Arbitrary Temperatures

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Recent investigations have unveiled exotic quantum phases that elude characterization by simple bipartite correlation functions. In these phases, long-range entanglement arising from tripartite correlations plays a central role. Consequently, the study of multipartite correlations has become a focal point in modern physics. Here, conditional mutual information (CMI) is one of the most well-established information-theoretic measures, adept at encapsulating the essence of various exotic phases, including topologically ordered ones. Within the realm of quantum many-body physics, it has been a long-sought goal to establish a quantum analog to the Hammersley-Clifford theorem that bridges the two concepts of the Gibbs state and the Markov network. This theorem posits that the correlation length of CMI remains short-range across all thermal equilibrium quantum phases. In this work, we demonstrate that CMI exhibits exponential decay with respect to distance, with its correlation length increasing polynomially with respect to the inverse temperature. While this clustering theorem has previously been believed to hold for high temperatures devoid of thermal phase transitions, it has remained elusive at low temperatures, where genuine long-range entanglement is corroborated to exist by the quantum topological order. Our findings unveil that, even at low temperatures, a broad class of tripartite entanglement cannot manifest in the long-range regime. To achieve the proof, we establish a comprehensive formalism for analyzing the locality of effective Hamiltonians on subsystems, commonly known as the “entanglement Hamiltonian” or “Hamiltonian of mean force.” As one outcome of our analyses, we enhance the prior clustering theorem concerning bipartite entanglement. In essence, we investigate genuine bipartite entanglement that extends beyond the limitations of the positive-partial-transpose class.

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I. INTRODUCTION

In quantum many-body physics, a fundamental challenge is uncovering structures that apply universally, regardless of specific system details. One of the simplest ways to characterize this challenge is by examining the correlation function between two observables, which reveals a distinct short-range behavior in noncritical phases both at finite [1–5] and zero temperatures [6–9]. Recent advances in quantum information science have introduced various methods for comprehending the complexity of quantum phases of matter from an information-theoretic perspective [10–13]. Among these, one particularly

renowned and elegant concept is the area law of quantum entanglement [14]. While the area law has been a conjecture in high dimensions at absolute zero temperature [15–17], it has been rigorously confirmed to hold at nonzero temperatures [18–21]. The area law has had a profound impact on various areas of research. It has dramatically influenced numerical techniques employing the tensor network formalism [22,23]. Additionally, it has paved the way for developing efficiency-guaranteed algorithms that compute physical observables [24,25]. In more recent developments, the study of bipartite quantum entanglement between distant subsystems has emerged as a promising avenue for universally revealing short-range characteristics, even at the thermal critical point [26]. So far, quantum long-range entanglement is believed to manifest primarily in the form of tripartite (or higher-order multipartite) correlations at nonzero temperatures.

Over the past two decades, our understanding of quantum phases has evolved, revealing that they cannot be fully

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characterized solely through bipartite correlation measures. A prominent example of this complexity is found in topologically ordered phases, which exhibit genuinely multipartite correlations [27,28]. Efforts in the field have been directed towards devising comprehensive information-theoretic measures to capture multipartite correlations in quantum many-body systems. Thus far, one of the most established measures for quantifying tripartite correlations is conditional mutual information (CMI) [29–31]. The CMI has found versatile applications, including the definition of topological entanglement entropy [32–34]. Recent studies have uncovered additional uses for the CMI, such as characterizing information scrambling [35–38], identifying measurement-induced quantum phase transitions [39–41], and studying entanglement in conformal field theory [42–45], among others. In different contexts, researchers have extensively investigated the operational significance of the CMI. Remarkably, they have rigorously clarified the relationship between the CMI and the error of the recovery map [46–51].

Here, our fundamental question is whether or not we can establish a universal theorem on the tripartite correlation based on the CMI (Fig. 1). To provide context for this query, we draw upon the well-known Hammersley-Clifford theorem [52] within classical many-body systems, which establishes an equivalence between classical Gibbs states and the concept of a Markov network (or Markov random field). This network’s characteristics are a finite correlation length of the CMI. Then, one might envision a similar relationship in quantum systems in the analogy of classical Gibbs states. Although this equivalence breaks down in a strict sense, conjectures have emerged suggesting that a modified form of the theorem could hold approximately, with CMI decaying rapidly as the distance [see Eq. (7)]. To date, mathematical proofs have been restricted in two scenarios: (i) commuting Hamiltonians [53–55] and (ii) 1D Gibbs states at arbitrary temperatures [56]. Beyond one dimension, the problem is still open even at high temperatures, where the absolute convergence of the cluster expansion breaks down [57,58]. When temperatures are low, the structure of entanglement becomes exceptionally complex, exhibiting long-range entanglement even at $\mathcal{O}(1)$ temperatures [59–62]. Up to this point, no theoretical efforts have overcome this challenge, and the clustering of the CMI remains an inaccessible problem.

The resolution of this conjecture holds substantial significance in various ways. In the realm of quantum many-body physics, it enables us to impose stricter constraints on the nature of long-range entanglement that persists at nonzero temperatures. Additionally, it assures us that the recovery map for the quantum Gibbs state can be constructed using local quantum channels, even when dealing with low temperatures. From a practical standpoint, the Markov property is one of the foundational assumptions when dealing with unknown probability distributions in

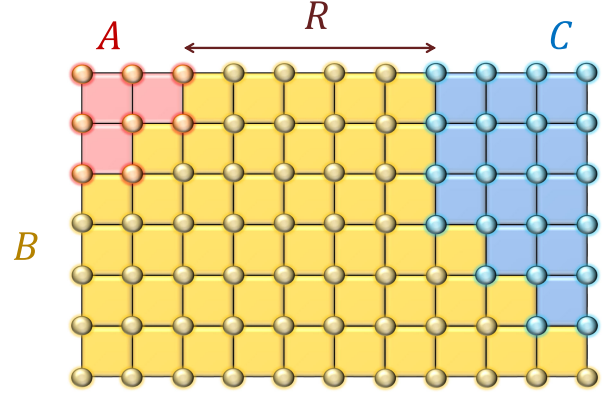


FIG. 1. Illustration of the main problem. We consider a finite-dimensional lattice system, depicted in 2D. The system is partitioned into three subsystems: A, B, and C. We examine the CMI between A and C, conditioned on B. When the system Hamiltonian is classical or commuting, the Hammersley-Clifford theorem indicates that subsystems A and C are conditionally independent. This independence implies that the CMI is zero when the distance between regions A and C exceeds the interaction length. The quantum analog suggests a rapid decay of the CMI with respect to the distance, as conjectured in Eq. (7). A positive resolution of this conjecture would imply that generic quantum Gibbs states form approximate Markov networks, which, in turn, implies the existence of a local recovery map capable of reconstructing the global Gibbs state from the reduced density matrices [47]. Our main result offers a partial solution to this conjecture, encapsulated in the primary statements of Eqs. (8) and (9).

classical theory [63,64]. In the quantum world, the concept of the Markov property plays a fundamental role in various quantum technologies. Some notable applications include quantum Hamiltonian learning [65,66], efficiency-guaranteed quantum Gibbs sampling [67,68], quantum marginal problems [69,70], and more. The establishment of the quantum version of the Hammersley-Clifford theorem has long been an ambitious goal in quantum information science.

In the present paper, we report an unconditional proof of the conjecture regarding the decay of CMI at arbitrary temperatures and in arbitrary finite dimensions. At this stage, it is important to note that our result does not provide a complete resolution of the quantum version of the Hammersley-Clifford theorem, which will be remarked after the main theorem. Nevertheless, it furnishes strong evidence that the Markov property generally holds, even at extremely low temperatures.

II. RESULTS

A. Setup and notations

We consider a quantum system on a D -dimensional lattice, where Λ represents the set of all sites. For any subset $L \subset \Lambda$, let $|L|$ denote the number of sites in L . For two disjoint subsets L and L' , we define the distance $d_{L,L'}$ as the

length of the shortest path connecting L and L' . We also denote the Hilbert space dimension of L by $\mathcal{D}_L$.

We then define the Hamiltonian as follows:

$$H = \sum_{i,i' \in \Lambda} h_{i,i'} + \sum_{i \in \Lambda} h_i, \quad (1)$$

$$\|h_{i,i'}\| \leq \bar{J}_\ell := \bar{J}_0 e^{-\mu \ell},$$

for $d_{i,i'} = \ell$, where the operator $h_{i,i'}$ represents an interaction term acting on the two sites i and i' . The condition for $\|h_{i,i'}\|$ implies that the interaction is short-range (or exponentially decaying). Here, $\|\cdot\|$ denotes the operator norm, which is equal to the largest singular value of the given operators. We can generalize all the results to generic k -local Hamiltonians, i.e., $H = \sum_{Z \subset \Lambda: |Z| \leq k} h_Z$, where h_Z is an interaction term acting on the subset Z ($|Z| \leq k$). For an arbitrary subset $L \subseteq \Lambda$, we define the subset Hamiltonian as

$$H_L = \sum_{i,i' \in L} h_{i,i'} + \sum_{i \in L} h_i. \quad (2)$$

Throughout this paper, we consider the quantum Gibbs state defined as

$$\rho_\beta := \frac{e^{\beta H}}{Z_\beta}, \quad Z_\beta = 1, \quad (3)$$

where $Z_\beta := \text{tr}(e^{\beta H})$, and we shift the energy origin so that $Z_\beta = 1$. For simplicity, we omit the minus sign, so $e^{-\beta H}$ becomes $e^{\beta H}$ without loss of generality. For a given quantum state ρ , we denote the reduced density matrix on the subregion L by ρ_L :

$$\rho_L = \text{tr}_{L^c}(\rho), \quad L^c := \Lambda \setminus L, \quad (4)$$

where tr_{L^c} means the partial trace for the Hilbert space on the complementary region L^c . In particular, for the Gibbs state, we denote the reduced density matrix by $\rho_{\beta,L}$.

For any three sets A , B , and C within Λ , we start by defining the CMI $\mathcal{I}_\rho(A:C|B)$ between A and C , conditioned on B , for a density matrix ρ :

$$\mathcal{I}_\rho(A:C|B) := S_\rho(AB) + S_\rho(BC) - S_\rho(ABC) - S_\rho(B), \quad (5)$$

where, for $\forall L \subseteq \Lambda$, we define $S_\rho(L)$ as the von Neumann entropy of the reduced density matrix ρ_L . Using the mutual information $\mathcal{I}_\rho(A:B) := S_\rho(A) + S_\rho(B) - S_\rho(AB)$, we can also describe the CMI by

$$\mathcal{I}_\rho(A:C|B) = \mathcal{I}_\rho(A:BC) - \mathcal{I}_\rho(A:B). \quad (6)$$

The quantum Markov conjecture regarding the CMI is now presented as follows: For an arbitrary quantum Gibbs

state ρ_β , the CMI $\mathcal{I}_{\rho_\beta}(A:C|B)$, where $\Lambda = A \sqcup B \sqcup C$, rapidly decays with the distance between A and C :

$$[\text{Conjecture}] \quad \mathcal{I}_{\rho_\beta}(A:C|B) \leq \mathcal{G}_T(R), \quad (7)$$

with $R = d_{A,C}$, where $\mathcal{G}_T(R)$ is a superpolynomially decaying function depending on β , $\{A, B, C\}$, and the system details. It is important to note that the condition $\Lambda = A \sqcup B \sqcup C$ is crucial. If $A \sqcup B \sqcup C \subset \Lambda$, even for commuting Hamiltonians, there exists a counterexample to Eq. (7) at low temperatures [59,60].

In accordance with the subsystem-size dependence (e.g., $|A|$ or $|C|$), the Markov property can be classified into three levels [71,72]:

- (1) Global Markov property: This is the strongest level, implying that the CMI decays even when A and C are macroscopic, i.e., $|A|, |C| = \mathcal{O}(|\Lambda|)$.
- (2) Local Markov property: At this level, a small subsystem size for either A or C is required, i.e., $\min(|A|, |C|) = \mathcal{O}(1)$.
- (3) Pairwise Markov property: The weakest level, where both subsystems A and C must be small, i.e., $|A|, |C| = \mathcal{O}(1)$.

In classical theory, provided the probability distribution is strictly positive, these three concepts are equivalent. However, in quantum scenarios, the conditions for such equivalence remain unclear. To establish the global Markov property in a quantum system, the decay rate $\mathcal{G}_T(R)$ of the CMI should depend on the subsystem sizes $|A|$ and $|C|$ at most polynomially.

B. Exponential clustering of the CMI

The central achievement of this work lies in the unconditional proof of the conjecture (7), which is summarized in the following statement (see Theorems 4 and 5 of Ref. [73]).

Theorem. For an arbitrary quantum Gibbs state, the conditional mutual information is upper bounded as in Eq. (7) with the following expression for $\mathcal{G}_T(R)$:

$$\mathcal{G}_T(R) = \mathcal{D}_{AC} e^{-c_1 R / (\beta^{D+1} \log(R)) + c_2 \log(\beta |A|)}, \quad (8)$$

where $\mathcal{D}_{AC}$ is the Hilbert space dimension on the subsystems $A \sqcup C$. In particular, for one-dimensional systems, the upper bound is improved to

$$\mathcal{G}_T(R) = e^{-c_3 R / \beta + c_4 \beta \log(\beta R)} \quad (D = 1). \quad (9)$$

Here, the parameters c_1 , c_2 , c_3 , and c_4 are $\mathcal{O}(1)$ constants that depend only on the fundamental parameters of the system (see Table I in Ref. [73]). These results ensure the (almost) exponential decay of the conditional mutual information at arbitrary temperatures.

We summarize several key points. For $D = 1$, the result does not depend on the subsystem sizes $|A|$ and $|C|$, and

hence they can be macroscopically large as $|A|, |C| = \mathcal{O}(|\Lambda|)$. We thus conclude that the global Markov property holds in one-dimensional quantum Gibbs states at arbitrary temperatures. This result significantly improves upon the previous one in Ref. [56], which relies on the exponential clustering for bipartite correlations and yields subexponential decay of $\mathcal{G}_I(R) = \exp(-e^{-\Omega(\beta)}\sqrt{R})$. Additionally, for some classes of matrix product density operators (MPDO) with translation invariance, the exponential decay of the CMI has been proven under the assumption of the gap of the transfer matrix [110,111]. Our result leads to the unconditional proof of the CMI decay of MPDOs that have a quasilocal parent Hamiltonian [112].

On the other hand, for $D \geq 2$, we have achieved the exponential decay of the CMI at arbitrary temperatures. However, the current bound is insufficient to prove the global Markov property due to the growth of the coefficient $\mathcal{D}_{AC}$, which increases as $e^{\Omega(|A|+|C|)}$. Essentially, if we compare two large subregions (with sizes on the order of $|\Lambda|$), the influence of $\mathcal{D}_{AC}$ dominates over the decay factor $e^{-\Omega[R/\log(R)]}$. In this sense, the statement in Eq. (8) implies the pairwise Markov property at this stage, and the complete solution of the conjecture in Eq. (7) is still open in high dimensions.

We further mention the temperature dependence of the CMI. The CMI correlation length increases at most polynomially as $\mathcal{O}(\beta)$ and $\mathcal{O}(\beta^{D+1})$ for $D = 1$ and $D \geq 2$, respectively. The different scaling for β between 1D and higher dimensions arises from the use of different analytical techniques for high-dimensional cases (see Sec. V B). Notably, our results imply that the correlation length of the CMI is much smaller than that of the bipartite correlation function, which can be infinitely large at critical points in high dimensions or at least exponentially increases with β as $e^{\mathcal{O}(\beta)}$ in one dimension [1,113].

Finally, we discuss the possibility of a qualitative improvement in the CMI decay rate. Based on numerical tests [114], it appears that the decay rate does not improve beyond exponential decay, even in 1D systems with finite-range interactions. This observation contrasts with the findings in Ref. [115], where the authors reported a superexponential decay of conditional information based on the Belavkin-Staszewski relative entropy [116], an alternative metric for characterizing conditional independence in quantum Gibbs states.

1. Bipartite vs tripartite measures

Our findings demonstrate a fundamental difference between CMI and standard bipartite correlation measures such as mutual information (MI). In high-dimensional quantum systems, it is well known that two-point correlations typically decay as a power law near criticality. Even in one dimension, the correlation length associated with MI

often grows exponentially with inverse temperature, i.e., as $e^{\mathcal{O}(\beta)}$ [1,113].

By contrast, the CMI remains short-ranged even at or near criticality. As we have shown in the main theorem, the CMI decays exponentially with distance at all positive temperatures, and its correlation length grows at most polynomially with β . This locality stems from the structural nature of CMI: It quantifies correlations between A and C that remain after conditioning on an intermediate region B . Because B acts as an information “shield,” it effectively blocks any long-range correlations between A and C unless information about B is available. This type of locality is even more transparent in classical thermal equilibrium states, where it can be rigorously shown that $I(A:C|B) = 0$ whenever A and C are separated beyond the interaction length. In other words, classical CMI exhibits strict locality.

It is also important to emphasize that CMI and MI are fundamentally incomparable quantities: There is no general monotonicity relation between them. In some scenarios, the MI between two regions A and C can vanish while the CMI conditioned on B remains nonzero, indicating correlations that only emerge upon conditioning. Conversely, there are cases where the CMI vanishes but the MI remains finite, typically reflecting indirect correlations mediated by B . These contrasting behaviors highlight that CMI captures distinct and often more subtle aspects of correlation structure than MI.

The robustness of short-range behavior in CMI at low temperatures reflects a general physical phenomenon: Certain types of quantum correlations remain fragile even within strongly correlated regimes at low temperatures. This feature highlights the ability of the CMI to detect structural simplicity in quantum correlations: It tends to vanish even in regimes where standard correlation measures remain large. As such, the CMI serves as a sensitive probe for uncovering quasilocal structures or disentangling patterns that remain hidden in bipartite measures such as mutual information. A similar contrast emerges in the context of bipartite entanglement measures such as the entanglement of formation (EOF). As we discuss in the subsequent section, the EOF also tends to decay rapidly even when MI remains large.

III. ENTANGLEMENT CLUSTERING BEYOND THE PPT CLASS

Regarding the applications of the main inequalities (8) and (9), they allow us to address the clustering of genuine bipartite entanglement. Previous work [26] demonstrated that exponential clustering for bipartite entanglement holds at arbitrary temperatures. However, a severe drawback still remains—it could not exclude the possibility of the existence of bound entanglement [117,118], which has been a primary open problem. More precisely, the statement was limited to the positive-partial-transpose (PPT)

relative entanglement [119–122], which has similar properties to entanglement negativity [123].

To capture bound entanglement, we need to analyze a faithful entanglement measure, meaning that the measure should be zero if and only if the target quantum state is separable. We call a bipartite quantum state ρ_{AB} separable if it can be decomposed into a mixed state of product states, i.e.,

$$\rho_{AB} = \sum_s p_s \rho_{A,s} \otimes \rho_{B,s}. \quad (10)$$

Conveniently, the CMI serves as an upper bound for bipartite entanglement in the form of squashed entanglement, which is a faithful measure [124,125]. For an arbitrary quantum state ρ_{AB} composed of systems A and B , squashed entanglement is defined as follows:

$$E_{\text{sq}}(\rho_{AB}) := \inf_E \left\{ \frac{1}{2} \mathcal{I}_{\rho_{ABE}}(A:B|E) \mid \text{tr}_E(\rho_{ABE}) = \rho_{AB} \right\}, \quad (11)$$

where $\inf_E$ is taken over all extensions of ρ_{AB} such that $\text{tr}_E(\rho_{ABE}) = \rho_{AB}$. Assuming that inequality (7) holds, we can derive a clustering theorem for squashed entanglement as follows:

$$E_{\text{sq}}(\rho_{\beta,AB}) \leq \frac{1}{2} \mathcal{I}_{\rho_{\beta}}(A:B|C) \leq \frac{1}{2} \mathcal{G}_{\mathcal{I}}(R), \quad (12)$$

with $R = d_{A,B}$. In the inequality, the ancilla system E is chosen as the residual system to AB , i.e., $E = C = \Lambda \setminus (AB)$. This aspect enables us to prove entanglement clustering for genuine bipartite entanglement, including EOF.

To capture the broadest class of bipartite entanglement measures, we adopt the EOF since it provides an upper bound for other entanglement measures [126], such as the relative entanglement [127], the entanglement cost [128], and the squashed entanglement [29]. The EOF for an arbitrary bipartite quantum state ρ_{AB} is defined as follows:

$$E_F(\rho_{AB}) := \inf_{\{p_s, |\psi_{s,AB}\rangle\}} \sum_s p_s S_{|\psi_{s,AB}\rangle}(A), \quad (13)$$

where $S_{|\psi_{s,AB}\rangle}(A)$ is the von Neumann entropy for the reduced density matrix onto the subsystem A . The convex roof $\inf_{\{p_s, |\psi_{s,AB}\rangle\}}$ is taken over all decompositions $\rho_{AB} = \sum_s p_s |\psi_{s,AB}\rangle \langle \psi_{s,AB}|$, with $p_s > 0$.

In general, we can prove the following statement as a consequence of the main results in Eqs. (8) and (9) (see Corollary 42 and Proposition 43 in Ref. [73]).

Clustering theorem for the EOF. Let A and B be subsystems that are separated by a distance R . Then, for the reduced density matrix $\rho_{\beta,AB}$, the entanglement of formation obeys the following clustering theorem:

$$E_F(\rho_{\beta,AB}) \leq \mathcal{D}_{AB} e^{-c'_1 R / (\beta^{D+1} \log(R)) + c'_2 \log(R)}, \quad (14)$$

which holds in arbitrary dimensions. In particular, for one-dimensional systems, we can improve the upper bound to

$$E_F(\rho_{\beta,AB}) \leq e^{c'_3 \beta \log(\beta) - c'_4 R / \beta^2}. \quad (15)$$

Here, c'_1 , c'_2 , c'_3 , and c'_4 are $\mathcal{O}(1)$ constants.

The general upper bound (14) has the same limitation as Eq. (8) in that it is meaningful only when the subsystem sizes of A and B are much smaller than the length R . The dependence on the Hilbert space dimension $\mathcal{D}_{AB}$ has been removed in the one-dimensional case. We emphasize that the inequality (15) provides a nontrivial upper bound even in the thermodynamic limit (i.e., $|A|, |B| \rightarrow \infty$). This point is a significant advantage compared to the similar result in Ref. [26] (Theorem 12 therein) for the PPT relative entanglement, which becomes meaningless in the thermodynamic limit.

IV. QUASILOCALITY OF THE 1D ENTANGLEMENT HAMILTONIAN

Analyzing the effective Hamiltonian on subsystems poses the most significant challenge in our work, which is also known as the entanglement Hamiltonian [129–135] or Hamiltonian of mean force [136–139]. The latter terminology is more commonly used in the fields of statistical mechanics and nonequilibrium physics. When the subsystem is given by $L \subset \Lambda$, we define the entanglement Hamiltonian by

$$\rho_{\beta,L} = e^{\beta H_L^*}, \quad H_L^* = \frac{1}{\beta} \log(\rho_{\beta,L}). \quad (16)$$

The quasilocality of the entanglement Hamiltonian implies the decay of the CMI [see Eq. (20)], while the converse does not hold, in general [140]. Hence, it is a stronger concept than the conjecture (7). The entanglement Hamiltonian of the quantum Gibbs state has attracted much attention in modern quantum technologies [66,138,141].

Quasilocality of entanglement Hamiltonian. Let $\tilde{H}_{L,R}^*$ be the approximate entanglement Hamiltonian such that the effective interaction terms are localized around the boundary within a distance R (see Fig. 2). Then, if the interaction length is finite, i.e., $\tilde{J}_{\ell} = 0$ for $\ell > \text{const}$ in Eq. (1), the approximation error is given by

$$\|H_L^* - \tilde{H}_{L,R}^*\| \leq e^{\xi_{\beta} - c_5 R / \beta}, \quad (17)$$

where $\xi_{\beta} = e^{\mathcal{O}(\beta)}$ and c_5 is a constant of order $\mathcal{O}(1)$.

From this theorem, we can ensure that the nonlocality of the effective interaction terms is limited to a distance of at most $e^{\mathcal{O}(\beta)}$. In general, it is much larger than the correlation length of $e^{\mathcal{O}(\beta)}$ [113]. One reason is that the result (17)

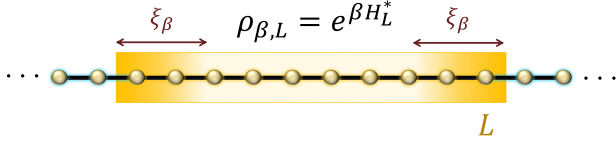


FIG. 2. Illustration of 1D entanglement Hamiltonian. For a contiguous region $L \subset \Lambda$, we define the entanglement Hamiltonian H_L^* by $\beta^{-1} \log(\rho_{\beta,L})$ as in Eq. (16). The effective interaction terms in H_L^* (i.e., $H_L^* - H_L$) are expected to be quasilocal around the boundary of L , with the length scale of the quasilocality given by ξ_β . The quasilocality of the entanglement Hamiltonian imposes a much stronger structural constraint than the CMI decay in Eq. (7). In one-dimensional systems with finite-range interactions, we rigorously prove the error bound in Eq. (17), which yields $\xi_\beta = e^{e^{\mathcal{O}(\beta)}}$.

relies on analyses based on imaginary time evolution, while the 1D CMI decay (9) is based on the quantum belief-propagation (BP) technique [142,143] (see Secs. VB and VE). The assumption of the finite-range interactions also arises from this approach; for exponentially decaying interactions, Ref. [144] shows that the imaginary-time Lieb-Robinson bound intrinsically diverges below a temperature threshold in 1D. We conjecture that the quasilocality of the entanglement Hamiltonian follows a similar inequality to that of the CMI in Eq. (9).

One straightforward application of the inequality in Eq. (17) is to establish the logarithmic sample complexity of Hamiltonian learning with respect to the system size $|\Lambda|$. In Hamiltonian learning [65,145,146], given N copies of the same quantum Gibbs state, we reconstruct the Hamiltonian using N measurement data sets. Using the inequality in Eq. (17) along with the method described in Ref. [66]—which is designed for commuting Hamiltonians—we derive the following expression for sample complexity (see Corollary 48 of Ref. [73]):

$$N = e^{\xi_\beta} (1/\epsilon)^{c_6 \beta^2} \log(|\Lambda|), \quad (18)$$

with $c_6 = \mathcal{O}(1)$ to ensure the estimation error of $\epsilon = \max_{Z \subset \Lambda} \|h_Z - \tilde{h}_Z\|$, where $\{\tilde{h}_Z\}_{Z \subset \Lambda}$ are the reconstructed interaction terms. This result is the first to achieve logarithmic sample complexity for 1D Hamiltonian learning at arbitrary temperatures. For the Hamiltonian learning problem, the recent flat polynomial approximation techniques [145,147] might be more effective than using the quasilocality of the effective Hamiltonian. Nevertheless, we emphasize that the quasilocal structure of the entanglement Hamiltonian reveals more fundamental aspects of quantum many-body physics beyond its application to Hamiltonian learning.

The fundamental difficulty in accessing the entanglement Hamiltonian, i.e., the inequality in Eq. (17), stems from the analytical instability of the operator logarithm. In particular, even exponentially small features in the

spectrum of the reduced density matrix (e.g., on the order of $e^{-\Omega(|L|)}$) can contribute significantly to the structure of the entanglement Hamiltonian. To faithfully capture such fine structure, one must approximate the reduced density matrix with extremely high precision, typically up to errors of order $e^{-\Omega(|L|)}$. However, achieving this level of precision generally requires using global information or nonlocal methods, which in turn destroys any quasilocal structure in the approximating Hamiltonian. This difficulty reflects a deeper conceptual gap between the nature of the CMI decay and the quasilocal structure of entanglement Hamiltonians.

To make this more concrete, we show that one can construct a quasilocal Hamiltonian $\tilde{H}_L^*$ supported around the subsystem L such that

$$e^{\beta \tilde{H}_L^*} \approx \rho_{\beta,L}, \quad (19)$$

but this approximation does not imply that $\tilde{H}_L^*$ is close to the true entanglement Hamiltonian H_L^* . While such an approximation is sufficient to establish the exponential decay of the CMI (see Sec. VD), it does not grant us access to the exact structure of H_L^* . This limitation presents an obstacle for applications such as achieving polylogarithmic sample complexity in Hamiltonian learning [66]. Nevertheless, in the one-dimensional case (see Theorem 6 in Ref. [73]), it is possible to partially bridge this gap under more favorable conditions.

V. PROOF TECHNIQUES

A. Overviews

As outlined in Sec. IV, our main objective is to demonstrate the quasilocal nature of the entanglement Hamiltonian as suggested in Eq. (19). Using the notation introduced in Eq. (16), the CMI is represented as

$$\begin{aligned} \mathcal{I}_{\rho_\beta}(A:C|B) &= \text{tr}[\rho_\beta H_{\rho_\beta}(A:C|B)], \\ H_{\rho_\beta}(A:C|B) &= -\beta(H_{AB}^* + H_{BC}^* - H - H_B^*), \end{aligned} \quad (20)$$

where we have set $H_{ABC}^* = H$. Our significant technical contribution lies in developing a systematic methodology to analyze the entanglement Hamiltonian, denoted as H_L^* .

In the following, we choose to work with the effective Hamiltonian on L^c (i.e., $H_{L^c}^*$) since taking the partial trace over L leads to a simpler notation. This notational simplification is made solely for presentation purposes and does not affect the core of the argument.

Our approach is structured into three steps:

- (1) Reduction to connected exponential operators: We first approximate the reduced density matrix ρ_{β,L^c} as an exponential operator:

$$e^{\beta H_{L^c}^*} \approx \mathcal{T} e^{\int_0^\tau V_x dx} e^{\beta H_0} \left(\mathcal{T} e^{\int_0^\tau V_x dx} \right)^\dagger, \quad (21)$$

where V_x ($0 \leq x \leq \tau$) is a quasilocal operator around region L , $\mathcal{T}$ denotes the time-ordering operator, and H_0 is a k -local Hamiltonian that may differ from the original H . The precision of this approximation is dependent upon τ and the quasilocal nature of V_x .

The motivation for this step lies in the perturbative treatment of the reduced density matrix via the formalism of connected exponential operators. Rather than working directly with $\log(\rho_{\beta, L^c})$, we consider an iterative update from

$$e^{\beta H_{L^c, x_0}^*} := \mathcal{T} e^{\int_0^{x_0} V_x dx} e^{\beta H_0} \left(\mathcal{T} e^{\int_0^{x_0} V_x dx} \right)^\dagger$$

to

$$e^{\beta H_{L^c, x_0+dx}^*} := e^{V_{x_0+dx} dx} e^{\beta H_{L^c, x_0}^*} e^{V_{x_0+dx}^\dagger dx},$$

in a perturbative manner. This representation allows us to handle the operator logarithm indirectly while keeping the evolution quasilocal at each infinitesimal step.

- (2) Logarithm of exponential operators: We next consider the logarithm of the approximated exponential operators as in Eq. (21):

$$\hat{H}_\tau = \log \left[\mathcal{T} e^{\int_0^\tau V_x dx} e^{\beta H_0} \left(\mathcal{T} e^{\int_0^\tau V_x dx} \right)^\dagger \right]. \quad (22)$$

Our analysis needs to establish that the effective interaction terms $\hat{H}_\tau - H_0$ are localized around the region L . Additionally, we need to demonstrate that these effective interactions are predominantly determined by the interaction terms within the neighboring region of L .

- (3) Linking approximations to CMI: To derive the CMI decay, we connect the approximation in Eq. (21) with the quasilocality of $\hat{H}_\tau$. The challenge lies in that a close approximation between two operators cannot translate to the closeness of their logarithms. The qualitatively optimal bound between $H_{L^c}^*$ and $\hat{H}_\tau$ is expressed as

$$\|H_{L^c}^* - \hat{H}_\tau\| \leq \frac{c\delta_\tau}{\lambda_{\min}(\rho_{\beta, L^c})}, \quad (23)$$

with c a constant of $\mathcal{O}(1)$, where δ_τ represents the approximation error for Eq. (21), and $\lambda_{\min}(\rho_{\beta, L^c})$ is the minimum eigenvalue of ρ_{β, L^c} . In general, $\lambda_{\min}(\rho_{\beta, L^c}) = e^{-\mathcal{O}(|L^c|)}$; hence, we need to make the error δ_τ exponentially small with the system size, i.e., $\delta_\tau = e^{-\Omega(|\Lambda|)}$. Achieving an exponentially small error δ_τ spoils the quasilocality of the effective interactions. Hence, we approach the estimation of

the CMI without directly analyzing the true entanglement Hamiltonian.

In the following, we demonstrate how these steps are taken. To explain the essence of the analyses, we omit most of the detailed calculations. Further details and specific calculations are reserved for the Supplemental Material (see Secs. III–VI of Ref. [73]).

B. First step

In the initial phase of our methodology, we adapt two distinct formalisms based on the dimensional characteristics of the system:

- (i) The BP formalism for one-dimensional systems, as detailed in Sec. III B of Ref. [73].
- (ii) The partial-trace-projection (PTP) formalism for higher-dimensional systems, described in Sec. III C of Ref. [73].

We emphasize that the BP formalism cannot be directly applied to high-dimensional systems in our setting. Although the quantum belief propagation itself is defined for arbitrary dimensions, in higher dimensions the error grows exponentially with the size of the boundary between regions (see the discussion below), which prevents us from using this approach beyond 1D.

We begin with the belief-propagation formalism to derive the exponential form shown in Eq. (21), which is used to approximate the reduced density matrix. We briefly review the quantum belief propagation [142,143]. In essence, it provides a mechanism to connect two exponential operators, $e^{\beta H_0}$ and $e^{\beta(H_0+V)}$, for any operators H_0 and V , as follows:

$$e^{\beta(H_0+V)} = \Phi_V e^{\beta H_0} \Phi_V^\dagger. \quad (24)$$

The operator Φ_V , known as the quantum-belief-propagation operator, is defined by

$$\Phi_V = \mathcal{T} e^{\int_0^1 \phi_\tau d\tau}, \quad \phi_\tau = \int_{-\infty}^{\infty} f_\beta(t) V(H_\tau, t) dt, \quad (25)$$

$$f_\beta(t) := \frac{2}{\beta\pi} \log \left(\frac{e^{\pi|t|/\beta} + 1}{e^{\pi|t|/\beta} - 1} \right),$$

where $V(H_\tau, t) = e^{iH_\tau t} V e^{-iH_\tau t}$ and $H_\tau = H_0 + \tau V$. The filter function $f_\beta(t)$ exhibits a decay of $e^{-\mathcal{O}(|t|/\beta)}$, emphasizing the dominance of the time integral within $\mathcal{O}(\beta)$. Specifically, when H_0 represents a short-range interacting Hamiltonian, the Lieb-Robinson bound [148,149] can be applied to the time evolution $V(H_\tau, t)$. By selecting V as a local operator within region $L \subset \Lambda$, we can accurately approximate $V(H_\tau, t)$ onto the neighboring region around L .

In what follows, we partition the one-dimensional system into three parts: $\Lambda = L \cup X \cup Y$, where L is the target region and X is a subsystem of length ℓ that connects to L (see Fig. 3). We then decompose the Hamiltonian as

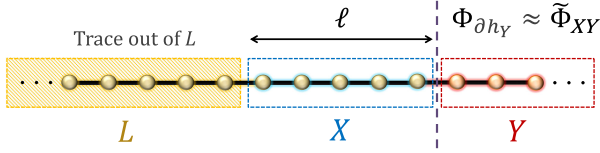


FIG. 3. Belief-propagation formalism for the entanglement Hamiltonian. By tracing out the subsystem L , we aim to derive the connected exponential form given in Eq. (21) to approximate the reduced density matrix ρ_{L^c} . For this purpose, we apply the quantum-belief-propagation method to segregate the boundary interaction term ∂h_Y from the quantum Gibbs state, which is represented as $e^{\beta H} = e^{\beta(H_{LX} + H_Y + \partial h_Y)}$. In this setup, the quantum-belief-propagation operator $\Phi_{\partial h_Y}$ serves as a bridge linking $e^{\beta(H_{LX} + H_Y)}$ to $e^{\beta H}$. By approximating this operator within the region XY with $\tilde{\Phi}_{XY}$, we can manage the partial trace $\text{tr}_L(\dots)$ independently from the quantum-belief-propagation operation. This approximation leads to the desired form of Eq. (28).

$H = H_{LX} + H_Y + \partial h_Y$, with ∂h_Y representing the boundary interaction between the subsystems LX and Y . Employing the quantum belief propagation described in Eq. (24) with $H_0 = H_{LX} + H_Y$ and $V = \partial h_Y$, we derive

$$e^{\beta H} = \Phi_{\partial h_Y} e^{\beta H_{LX}} \otimes e^{\beta H_Y} \Phi_{\partial h_Y}^\dagger. \quad (26)$$

Applying the Lieb-Robinson bound to the time evolution described in Eq. (25), we obtain the following approximation (see Lemmas 9 and 10 of Ref. [73]):

$$\|\Phi_{\partial h_Y}^{-1} \tilde{\Phi}_{XY} - 1\| \leq e^{c\beta\|\partial h_Y\| - c'\ell/\beta}, \quad (27)$$

$$\tilde{\Phi}_{XY} = \mathcal{T} e^{\int_0^1 \tilde{\phi}_{XY,\tau} d\tau},$$

where c and c' are constants of $\mathcal{O}(1)$ and $\tilde{\phi}_{XY,\tau}$ is supported on XY . By applying the approximation $\Phi_{\partial h_Y} \approx \tilde{\Phi}_{XY}$ to Eq. (26), we approximate the reduced density matrix ρ_{β,L^c} as

$$\begin{aligned} \rho_{\beta,L^c} &\approx \tilde{\Phi}_{XY} \text{tr}_L(e^{\beta H_{LX}}) \otimes e^{\beta H_Y} \tilde{\Phi}_{XY}^\dagger \\ &= \mathcal{T} e^{\int_0^1 \tilde{\phi}_{XY,\tau} d\tau} e^{\beta(\tilde{H}_X^* + H_Y)} \left(\mathcal{T} e^{\int_0^1 \tilde{\phi}_{XY,\tau} d\tau} \right)^\dagger, \end{aligned} \quad (28)$$

which is the desired form of Eq. (21), where $\tilde{H}_X^*$ is defined such that $\text{tr}_L(e^{\beta H_{LX}}) = e^{\beta \tilde{H}_X^*}$. The approximation error is dependent on the length ℓ of X and aligns with the error bound in Eq. (27).

In the application of the BP formalism to derive the effective Hamiltonian, it is necessary to select the length ℓ to be larger than $\beta^2 \|\partial h_Y\|$ to ensure a small approximation error from the bound in Eq. (27). In higher dimensions, the norm $\|\partial h_Y\|$ scales with the surface size of X , which is $\mathcal{O}(\ell^{D-1})$. However, for $D \geq 2$ and $\beta \gg 1$, it becomes

challenging to satisfy the condition $\ell \gtrsim \beta^2 \mathcal{O}(\ell^{D-1})$. To address this limitation, we develop an alternative approach. We introduce an ancillary system L_a , which is a copy of the subsystem L , and express the partial trace operation as follows:

$$\begin{aligned} \text{tr}_L(\rho_\beta) &= \mathcal{D}_L \langle \mathcal{P}_L | \rho_\beta \otimes \hat{1}_{L_a} | \mathcal{P}_L \rangle, \\ | \mathcal{P}_L \rangle &:= \frac{1}{\sqrt{\mathcal{D}_L}} \sum_{j=1}^{\mathcal{D}_L} | j_L \rangle \otimes | j_{L_a} \rangle, \end{aligned} \quad (29)$$

where $| \mathcal{P}_L \rangle$ represents a maximally entangled state between the subsystems L and L_a . We define the PTP operator $\mathcal{P}_L$ as the projection onto the state $| \mathcal{P}_L \rangle$, which results in

$$\rho_{\beta,L^c} \otimes \mathcal{P}_L = \mathcal{D}_L \mathcal{P}_L \rho_\beta \otimes \hat{1}_{L_a} \mathcal{P}_L, \quad (30)$$

where the operator $\mathcal{P}_L$ is supported on the doubled Hilbert space on $L \cup L_a$.

To approximate Eq. (30) in the exponential form as Eq. (21), we define the approximate PTP operator $\mathcal{P}_{L,\tau}$ as

$$\mathcal{P}_{L,\tau} := e^{-\tau \mathcal{Q}_L}, \quad \mathcal{Q}_L := 1 - \mathcal{P}_L. \quad (31)$$

Using the approximate PTP operator, we obtain the desired exponential form of the reduced density matrix:

$$\rho_{\beta,L^c} \otimes \mathcal{P}_L \approx \mathcal{D}_L e^{-\tau \mathcal{Q}_L} e^{\beta H} e^{-\tau \mathcal{Q}_L}, \quad (32)$$

where the approximation error is proved to be $\mathcal{D}_L e^{-\mathcal{O}(\tau)}$. Because of the coefficient $\mathcal{D}_L$, which scales as $e^{\Omega(|L|)}$, it is required to make $\tau = \Omega(|L|)$. This requirement results in a problematic dependence on the Hilbert space dimension for the CMI decay as detailed in Eq. (8).

In summary, step 1 allows us to approximate the reduced density matrix in a tractable form, namely, the connected exponential form given in Eq. (22). The next challenge is to analyze the operator logarithm of this form in order to establish that the effective Hamiltonian of the approximate reduced state has a quasilocal structure.

C. Second step

The logarithm of connected exponential operators, such as $\log(e^{H_0} e^V)$, is commonly analyzed using the Baker-Campbell-Hausdorff formula or the Magnus expansion [150]. If a small-order truncation of the Magnus expansion is employed, it can ensure the quasilocality of the effective Hamiltonian $\hat{H}_\tau$ in Eq. (22). However, a significant challenge arises due to the fact that the Magnus expansion is not absolutely convergent [151–153].

To establish the quasilocality of the effective Hamiltonian, we utilize the following theoretical framework outlined in Lemma 18 and Corollary 19 in Ref. [73].

Effective Hamiltonian Theory. Let H_0 and V_{τ_1} ($0 \leq \tau_1 \leq \tau$) be arbitrary operators (possibly interaction operators). By defining the effective operators $\hat{H}_\tau$ as

$$\hat{H}_\tau := \log \left[\mathcal{T} e^{\int_0^\tau V_{\tau_1} d\tau_1} e^{\beta H_0} \left(\mathcal{T} e^{\int_0^\tau V_{\tau_1} d\tau_1} \right)^\dagger \right], \quad (33)$$

we can simplify it to the form

$$\hat{H}_\tau = U_\tau (\beta H_0 + \hat{V}_\tau) U_\tau^\dagger, \quad (34)$$

with

$$\begin{aligned} U_\tau &:= \mathcal{T} e^{-i \int_0^\tau \mathcal{C}_{\tau_1} d\tau_1}, \\ \hat{V}_\tau &:= 2 \int_0^\tau U_{\tau_1}^\dagger V_{\tau_1} U_{\tau_1} d\tau_1, \\ \mathcal{C}_{\tau_1} &:= \frac{2}{\beta} \int_{-\infty}^\infty g_\beta(t) e^{iH_{\tau_1}t} V_{\tau_1} e^{-iH_{\tau_1}t} dt, \end{aligned} \quad (35)$$

where $g_\beta(t) := -\text{sign}(t)(e^{2\pi|t|/\beta} - 1)^{-1}$.

In the Supplemental Material [73], we use the symbols $\mathcal{A}$ and $\mathcal{B}$ for the corresponding operators H_0 and V .

This expression significantly simplifies the form of the effective Hamiltonian. For simplicity, we assume $V_\tau = V$ without τ dependence in the following analysis. We then point out that from the definition (33), the operator $\hat{H}_\tau$ originates from the differential equation

$$\frac{d}{d\tau} \hat{H}_\tau = 2V - i[\mathcal{C}_\tau, \hat{H}_\tau]. \quad (36)$$

Assuming the effective operator H_τ satisfies quasilocality, the Lieb-Robinson bound can be applied to ensure the quasilocality of $\mathcal{C}_\tau$ in Eq. (35), which further implies the quasilocality of $dH_\tau/d\tau$. Note that $g_\beta(t)$ approximately decays as $e^{-\mathcal{O}(|t|/\beta)}$, making the time integral in the range of $|t| = \mathcal{O}(\beta)$ dominant in the definition of $\mathcal{C}_\tau$.

The challenge emerges from the fact that Eq. (36) includes double-bracket flows, i.e., terms like $[[V, \hat{H}_\tau], \hat{H}_\tau]$. According to Ref. [154], the Lieb-Robinson bound for double-bracket flow does not generally preserve

quasilocality, unlike the standard Heisenberg equation. To illustrate this, consider

$$-i[\mathcal{C}_\tau, \hat{H}_\tau] = \frac{-2i}{\beta} \int_{-\infty}^\infty g_\beta(t) \left[e^{iH_\tau t} V e^{-iH_\tau t}, \hat{H}_\tau \right] dt, \quad (37)$$

applying the definition of $\mathcal{C}_\tau$ from Eq. (35). For small $\delta t \ll 1$, we approximate $g_\beta(\delta t) \approx -\beta/(2\pi\delta t)$, resulting in the double-bracket flow appearing as

$$\frac{-ig_\beta(\delta t)}{\beta} [e^{iH_\tau \delta t} V e^{-iH_\tau \delta t}, \hat{H}_\tau] \approx \frac{[[V, \hat{H}_\tau], \hat{H}_\tau]}{2\pi}, \quad (38)$$

up to an $\mathcal{O}(\delta t)$ error. This analysis does not necessarily imply a breakdown in the quasilocality of the effective Hamiltonian. First, for sufficiently small δt , although double-bracket terms appear in the integral of $\mathcal{C}_\tau$ in Eq. (35), their contribution (i.e., the integral for $|t| \leq \delta t$) is proportional to δt . Moreover, it is observed that the integrand in Eq. (38) deviates from exact double-bracket flows for large t . By analyzing this deviation more carefully, we can still ensure the quasilocality of the effective interaction terms.

In the present analyses, we take $V = V_{i_0}$, for example, where the Hamiltonian H_0 is a short-range interacting Hamiltonian as specified in Eq. (1), and V_{i_0} is a local operator located at site $i_0 \in \Lambda$. We focus on estimating the effective Hamiltonian $\hat{H}_\tau = \log(e^{\tau V_{i_0}} e^{\beta H_0} e^{\tau V_{i_0}})$. Notably, V_{i_0} can also be generalized to a quasilocal operator around any subsystem, as discussed in Sec. V in Ref. [73]. A crucial aspect we examine is the quasilocality of the unitary operator U_τ , particularly through the commutator norm $\|[U_\tau, u_i]\|$, where u_i is any local unitary operator on a site $i \in \Lambda$. The dynamic property of this commutator is given by

$$\frac{d}{d\tau} \|[U_\tau, u_i]\| \leq \|[\mathcal{C}_\tau, u_i]\|. \quad (39)$$

To estimate $\|[\mathcal{C}_\tau, u_i]\|$, we upper bound the commutator norm between the time-evolved V_{i_0} and u_i (see Lemma 22 in Ref. [73]):

$$\left\| \left[e^{i\hat{H}_\tau t} V_{i_0} e^{-i\hat{H}_\tau t}, u_i \right] \right\| \leq \| [V_{i_0}, u_i(-t)] \| + 2 \| V_{i_0} \| \left(\| [U_\tau, u_i] \| + \| [U_\tau, u_i(-t)] \| + \int_0^t \| [\hat{V}_\tau, u_i(-t_1)] \| dt_1 \right), \quad (40)$$

where we denote $u_i(-t) = e^{-iH_0 t} u_i e^{iH_0 t}$. In analyzing the growth of the commutator norm, we employ the upper bound $\|[U_\tau, u_i]\| \leq Q(\tau, r)$, where $r = d_{i_0, i}$, and $Q(\tau, r)$ is defined as

$$Q(\tau, r) = e^{\kappa_0 \tau + \kappa_1 \tau \log(r + \tau + e) - \kappa_\beta r}, \quad (41)$$

with κ_0, κ_1 , and κ_β being free parameters chosen afterward.

To ensure consistency between the upper bounds in Eq. (39) and $\|[U_\tau, u_i]\| \leq Q(\tau, r)$, we can prove that it is sufficient to satisfy the condition

$$\|[\mathcal{C}_\tau, u_i]\| \leq [\kappa_0 + \kappa_1 \log(r + \tau + e)] Q(\tau, r). \quad (42)$$

Applying the Lieb-Robinson bound to $u_i(-t)$ and utilizing the upper bound in Eq. (41) for $\|[U_\tau, u_i]\|$, we can upper

bound the right-hand side of the inequality in Eq. (40). After integrating Eq. (40) with the filter function $g_\beta(t)$, we further upper bound $\|[\mathcal{C}_\tau, u_i]\|$ using the parameters κ_0 , κ_1 , and κ_β . Under appropriate choices of parameters such as

$$\begin{aligned}\kappa_0 &\propto \beta^D \|V_{i_0}\| \log(\beta \|V_{i_0}\|), \\ \kappa_1 &\propto \beta^D \|V_{i_0}\|, \\ \kappa_\beta &\propto 1/\beta,\end{aligned}\quad (43)$$

we can satisfy the condition in Eq. (42). This estimation shows that the unitary operator U_τ maintains quasilocality around the site i_0 with exponentially decaying tails, expressed as $e^{\mathcal{O}(\beta^D \|V_{i_0}\| - r/\beta)}$. The detailed calculations are complex, and we defer all specifics to Subtheorem 1 in Ref. [73]. In the one-dimensional case, the dependence of κ_0 and κ_1 on β disappears entirely [i.e., they are $\mathcal{O}(1)$ with respect to β], as shown in Theorem 3 of Ref. [73]. We conjecture that a similar improvement to a β^{D-1} dependence may also be possible in higher dimensions.

At this stage, we have established only the quasilocality of the unitary operator U_τ . However, this does not necessarily imply that U_τ is determined solely by the neighboring region around the site i_0 (Fig. 4). To address this issue mathematically, we consider the subset Hamiltonian $H_{0,i_0[r]}$ on the ball region centered at the site i_0 with radius r and define

$$\begin{aligned}\log\left(e^{\tau V_{i_0}} e^{\beta H_{0,i_0[r]}} e^{\tau V_{i_0}}\right) \\ = \beta U_{\tau,i_0[r]} \left(H_{0,i_0[r]} + \hat{V}_{\tau,i_0[r]}\right) U_{\tau,i_0[r]}^\dagger,\end{aligned}\quad (44)$$

where the unitary $U_{\tau,i_0[r]}$ is determined only by the subset Hamiltonian $H_{0,i_0[r]}$. If $U_\tau \approx U_{\tau,i_0[r]}$, we can infer that the unitary U_τ is approximately constructed from $H_{0,i_0[r]}$. Utilizing the quasilocality from Eq. (41), we can also derive the approximation error between U_τ and $U_{\tau,i_0[r]}$, which qualitatively follows the same upper bound as in Eq. (41) (see Theorem 2 in Ref. [73]).

So far, we have shown that when the connected exponential form approximates the reduced density matrix, its operator logarithm can be controlled, and it exhibits a quasilocal structure. In step 3, we connect this quasilocality to the decay of the CMI, thereby deriving our main results on CMI decay, Eqs. (8) and (9).

D. Third step

Here, we introduce $\tilde{\rho}_{\beta,L^c}$, which denotes an approximate reduced density matrix for ρ_{β,L^c} . Concretely, $\tilde{\rho}_{\beta,L^c}$ is constructed via the connected exponential: in one dimension, using the BP formalism in Eq. (27) and, in higher dimensions, using the PTP formalism in Eq. (32). Throughout this section, the tilde notation consistently

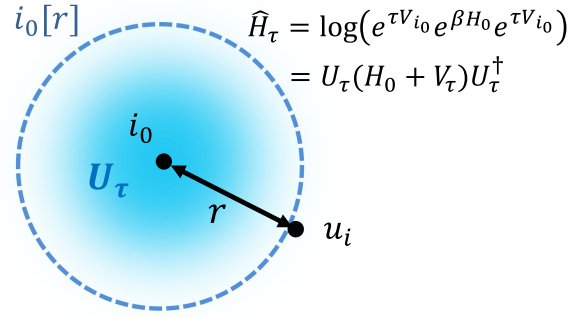


FIG. 4. Quasilocality of the effective interaction terms. We examine the operator logarithm given by $\log(e^{\tau V_{i_0}} e^{\beta H_0} e^{\tau V_{i_0}})$ and begin by establishing the quasilocality of the unitary operator U_τ in Eq. (35), which is instrumental in defining the effective Hamiltonian presented in Eq. (33). The quasilocality of U_τ is evaluated based on the norm $\|U_\tau, u_i\|$, where u_i represents any arbitrary unitary operator located on a site i ($d_{i,i_0} = r$). However, confirming the quasilocality of U_τ alone does not fulfill all our requirements. We further stipulate that U_τ should primarily be influenced by the neighboring region around the site i_0 . This requirement leads us to approximate U_τ by $U_{\tau,i_0[r]}$, as expressed in Eq. (44). Here, $U_{\tau,i_0[r]}$ is specifically determined by the ball region centered at the site i_0 with radius r (inside the dashed circle).

indicates a quasilocal approximation, in contrast to the exact reduced states used earlier.

Leveraging the quasilocality of the effective Hamiltonian described in step 2, we can estimate the operator norm of

$$\begin{aligned}\tilde{H}(A:C|B) &:= -\beta[\log(\tilde{\rho}_{\beta,AB}) + \log(\tilde{\rho}_{\beta,BC}) \\ &\quad - \log(\tilde{\rho}_{\beta,ABC}) - \log(\tilde{\rho}_{\beta,B})].\end{aligned}\quad (45)$$

Employing the quasilocality function from Eq. (41), we are able to approximate $\|\tilde{H}(A:C|B)\| \lesssim Q(\tau, R)$, where $R = d_{A,C}$ and τ follows from the approximation in Eq. (21). From the definition of CMI in Eq. (20), we derive

$$\begin{aligned}\mathcal{I}_{\rho_\beta}(A:C|B) - \text{tr}(\rho_\beta \tilde{H}(A:C|B)) \\ = -S(\rho_{\beta,AB} \| \tilde{\rho}_{\beta,AB}) - S(\rho_{\beta,BC} \| \tilde{\rho}_{\beta,BC}) \\ + S(\rho_{\beta,ABC} \| \tilde{\rho}_{\beta,ABC}) + S(\rho_{\beta,B} \| \tilde{\rho}_{\beta,B}),\end{aligned}\quad (46)$$

where $S(\rho \| \tilde{\rho})$ is the relative entropy, defined as $S(\rho \| \tilde{\rho}) := \text{tr}[\rho \log(\rho) - \rho \log(\tilde{\rho})]$.

In order to complete the argument, it is necessary to upper bound the right-hand side of Eq. (46). At this stage, however, the techniques required in the BP formalism and in the PTP formalism differ from each other. Since the notation becomes rather cumbersome in this part of the proof, we refer interested readers to the detailed statements in Lemma 14 and Corollary 17 of Ref. [73]. The approaches allow us to demonstrate the decay of CMI without directly accessing the true entanglement

Hamiltonian H_L^* , addressing the more complex challenge described in Eq. (23).

This result leads to the proof of our main results on CMI decays in Eqs. (8) and (9) (see Theorems 4 and 5 in Ref. [73]).

As a side note, the following simple upper bound cannot be used:

$$\mathcal{I}_{\rho_\beta}(A:C|B) \leq \|\tilde{H}(A:C|B)\| + 4 \max_{L \subseteq \Lambda} [S(\rho_{\beta,L^c} \|\tilde{\rho}_{\beta,L^c})]. \quad (47)$$

Even for $\rho_{\beta,L^c} \approx \tilde{\rho}_{\beta,L^c}$, the continuity inequality does not suffice for estimating $S(\rho_{\beta,L^c} \|\tilde{\rho}_{\beta,L^c})$. Utilizing the Alicki-Fannes inequality [155], we find the following inverse Pinsker's inequality:

$$S(\rho_{\beta,L^c} \|\tilde{\rho}_{\beta,L^c}) \lesssim |L^c| \cdot \|\rho_{\beta,L^c} - \tilde{\rho}_{\beta,L^c}\|_1. \quad (48)$$

However, the dependency on the size of $|L^c|$ suggests that the simple inequality (47) is not applicable to the thermodynamic limit as $|\Lambda| \rightarrow \infty$. Notably, our main results, encapsulated in Eqs. (8) and (9), exclude the $|\Lambda|$ dependency. Since the continuity inequality is already optimal, we must explore alternative methods to establish the independence of the relative entropy from $|\Lambda|$. Thus, we consider a refined approach in Lemma 14 and Corollary 17 of Ref. [73].

As an outlook, it remains an interesting question to identify cases where the inverse Pinsker's inequality holds without dependence on the logarithm of the Hilbert space dimension, possibly by extending the current analytical techniques.

E. Analyses of the 1D entanglement Hamiltonian

Finally, we provide a technical overview of accessing the true entanglement Hamiltonian in a one-dimensional case. The central approach relies on the continuity of the operator logarithm concerning the relative error, as stated in Theorem 7 in Ref. [73].

Continuity of the operator logarithm in terms of the relative error. Consider two density operators ρ and σ . We define the relative error $\delta_R(\rho, \sigma)$ as

$$\delta_R(\rho, \sigma) := \sup_{|\psi\rangle} \frac{|\langle\psi|(\rho - \sigma)|\psi\rangle|}{\langle\psi|\rho|\psi\rangle}, \quad (49)$$

where $\sup_{|\psi\rangle}$ is taken over all quantum states. The continuity inequality for $\log(\rho) - \log(\sigma)$ is then given by

$$\|\log(\sigma) - \log(\rho)\| \lesssim \delta_R(\rho, \sigma) \log[\lambda_{\min}^{-1}(\rho)], \quad (50)$$

where $\lambda_{\min}(\rho)$ is the minimum eigenvalue of ρ .

With this framework, by analyzing the relative error between $\rho_{\beta,L}$ and its approximation $\tilde{\rho}_{\beta,L}$, we establish the quasilocalty of the true entanglement Hamiltonian

$\log(\rho_{\beta,L})$ compared to $\log(\tilde{\rho}_{\beta,L})$. Nevertheless, two primary challenges arise: (i) estimating the minimum eigenvalue of the reduced density matrix and (ii) estimating the relative error $\delta_R(\rho_{\beta,L}, \tilde{\rho}_{\beta,L})$.

While naively straightforward, the estimation of $\lambda_{\min}^{-1}(\rho_{\beta,L})$, expected to be proportional to $\beta|L|$, presents difficulties. This proportionality is established in commuting cases, as shown in Ref. [66]. Extending this to noncommuting cases requires strict k locality of the Hamiltonian, utilizing technical methods from Refs. [65,156]. Generally, it can be shown (see Proposition 12 in Ref. [73]) that

$$\log[\lambda_{\min}^{-1}(\rho_{\beta,L})] \leq \beta \bar{J}_0 |L| + \log(16 \bar{J}_0 |L| \mathcal{D}_L) \quad (51)$$

for two-body interacting Hamiltonians, where we can choose $\bar{J}_0 = 3J_0 + 4J_0|L^c|^{-1} \log(8\mathcal{D}_{L^c}) + 1$ using the Hamiltonian parameter J_0 in Eq. (1). This approach can be generalized to any k -local Hamiltonians.

To address the second challenge, i.e., the estimation of $\delta_R(\rho_{\beta,L}, \tilde{\rho}_{\beta,L})$, we first note that the relative error for the reduced density matrices is always less than that for the global ones, i.e., $\delta_R(\rho_{\beta,L}, \tilde{\rho}_{\beta,L}) \leq \delta_R(\rho_\beta, \tilde{\rho}_\beta)$. In one-dimensional cases, this comparison involves two density matrices derived using the quantum belief propagation, as specified in Eqs. (26) and (28):

$$\begin{aligned} \rho_\beta &= \Phi_{\partial h_Y} e^{\beta H_{LX}} \otimes e^{\beta H_Y} \Phi_{\partial h_Y}^\dagger, \\ \tilde{\rho}_\beta &= \Phi_{XY} e^{\beta H_{LX}} \otimes e^{\beta H_Y} \Phi_{XY}^\dagger. \end{aligned} \quad (52)$$

The upper bound for the relative error can then be derived as

$$\delta_R(\rho_\beta, \tilde{\rho}_\beta) \leq \|1 - \mathcal{W}\mathcal{W}^\dagger\|, \quad (53)$$

where $H_0 = H_{LX} + H_Y$ and

$$\mathcal{W} := e^{-\beta H_0/2} \Phi_{\partial h_Y}^{-1} \Phi_{XY} e^{\beta H_0/2}. \quad (54)$$

From the inequality (27), we have $\Phi_{\partial h_Y}^{-1} \Phi_{XY} \approx 1$. Hence, the primary issue is the amplification by the imaginary time evolution of $e^{-\beta H_0/2}$.

The analysis of this imaginary time evolution reveals that the difficulty stems from the potential exponential amplification of norms, as expressed by

$$\|e^{-\beta H_0/2} O_X e^{\beta H_0/2}\| \leq e^{c_7 \beta |X| + e^{c_8 \beta}} \|O_X\|,$$

with $c_7, c_8 = \mathcal{O}(1)$ independent of $|\Lambda|$, where O_X is an operator supported on $X \subset \Lambda$. On the other hand, utilizing the Lieb-Robinson bound, we can decompose

$$\Phi_{\partial h_Y}^{-1} \Phi_{XY} = 1 + \sum_{\ell=1}^{\infty} \Phi_\ell, \quad (55)$$

where each Φ_ℓ is an ℓ -local operator with a norm that decays exponentially with ℓ/β . However, the amplification by the imaginary time evolution for each decomposed term Φ_ℓ can be significant, leading to $\|e^{-\beta H_0/2}\Phi_\ell e^{\beta H_0/2}\| = e^{\Omega(\beta\ell) + e^{\Omega(\beta)}}\|\Phi_\ell\|$. Such amplification substantially breaks down the approximation $\mathcal{W} \approx 1$ for large β . To derive a better bound, we need to fully utilize the structure of the Hamiltonian in more elaborate ways (see Subtheorem 2 in Ref. [73]).

VI. OUTLOOK

In this work, we have addressed the resolution of the conjecture stated in Eq. (7), proposing that every quantum Gibbs state approximates a Markov network at any temperature. Our principal findings are encapsulated in the inequalities presented in Eqs. (8) and (9). Because of the dependency on the Hilbert space dimension $\mathcal{D}_{AC}$, our analysis only concludes that quantum Gibbs states constitute approximately pairwise Markov networks, rather than forming a global Markov network. This study significantly advances the understanding of the entanglement Hamiltonian by systematically approximating its properties. As applications of our results, we have identified a clustering property of genuine bipartite entanglement beyond the PPT class and established the logarithmic sample complexity required for learning 1D Hamiltonians.

Despite these advancements, various open questions remain. One of the most pressing challenges is improving the subsystem-size dependence (i.e., $|A|, |C|$) of the CMI decay to a polynomial form $\text{poly}(|A|, |C|)$, which would enable its application to macroscopically large subsystems. The current analyses are highly intricate, whereas point-by-point calculations have been optimized. Hence, we believe that further improvement is likely to require a theoretical leap, which could involve leveraging the quasilocalty of the Hamiltonians more effectively. One potential approach is to impose additional constraints on the quantum Gibbs states, such as clustering of bipartite correlations, as discussed in Ref. [56]. Even under these constraints, the improvement is still beneficial for applications like quantum Gibbs sampling, as detailed in Ref. [68].

The second crucial problem we address is identifying the types of long-range entanglement that robustly persist at finite temperatures. Despite extensive studies, computing entanglement in quantum Gibbs states remains a significant challenge, even at the numerical level [59,157–166]. Our approaches are twofold. First, we aim to refine the subsystem-size dependence as described in Eq. (14) for bipartite-entanglement decay in high dimensions. From the relation (12), this case is closely linked to the first open problem regarding the CMI decay. The second approach considers the presence of multipartite entanglement.

So far, understanding multipartite entanglement is challenging due to the need for established methods for its characterization [167–172]. We still anticipate that certain types of tripartite entanglement may be absent in general many-body systems, as the CMI effectively describes tripartite correlations.

The third open problem focuses on improving the quasilocalty analyses of the entanglement Hamiltonian. In our studies of the CMI decay, we have only needed to address the approximate entanglement Hamiltonian in terms of Eq. (19). However, more than this approximation is required in order to access the true entanglement Hamiltonian. In the 1D case, this issue is partially resolved, as demonstrated in Eq. (17), although the temperature dependence remains doubly exponential. By improving the temperature dependence of the polynomial forms, we can access the 1D entanglement Hamiltonian at the ground state, where $\beta = \log(n)$ is taken for approximating the Gibbs state to the ground state. This method may further lead to the solution of the Li-Haldane-Poilblanc conjecture [129,173,174] in a generic setup beyond the CFT class [135,175]. Our present approach relies on a new continuity inequality for the operator logarithm, indicated by Eq. (50), suggesting that a small relative error between two operators implies their logarithms are similarly close. Future improvements could involve developing a better bound by introducing an alternative measure of operator distance, distinct from the relative error. The essential obstacle to accessing the entanglement Hamiltonian stems from the difficulty of connecting the closeness of density matrices and the closeness of the logarithms of those matrices. This issue has been a central challenge in the Hamiltonian learning problem [65,145,176]. We expect that recent technical advances in that area will contribute significantly to resolving the quasilocalty of the entanglement Hamiltonian in more general settings.

Finally, the primary method used to analyze the behavior of exponential operators extends beyond studying the quantum Gibbs states. This technique is particularly valuable for examining the effective Hamiltonians in systems undergoing open quantum dynamics, which involve interactions with the environment, as also reviewed in Ref. [136]. A thorough analysis of quantum effects arising from environmental interactions is essential for a better understanding of quantum thermalization processes as well as for characterizing mixed quantum phases [177–179]. Our current work will pave the way for new analytical approaches to tackle these complex issues.

Note added. Recently, the CMI decay was revisited using a completely different approach, constructing the recovery map [180,181] with the combination of Fawzi-Renner inequality [47]. In particular, the result in Ref. [181] partially overcomes one of the critical open problems regarding subsystem-size (i.e., $|A|, |C|$) dependence at

any temperature. Still, the method presented here has a strong advantage in that it provides a route to access the structure of the entanglement Hamiltonian.

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