Spectral Small-Incremental Entangling: Breaking Quasipolynomial Complexity Barriers in Long-Range Interacting Systems

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(Received 23 October 2025; revised 23 June 2026; accepted 13 August 2026; published 9 November 2026)

How the detailed entanglement structure emerges from quantum dynamics remains a fundamental challenge motivated by recent advances in quantum simulators and information processing. As a central milestone, the small-incremental-entangling (SIE) theorem bounds the entanglement-entropy growth rate but does not control the entanglement spectrum itself. In this work, we define the spectral-entangling strength, which quantifies how strongly an operator can reshape the distribution of Schmidt coefficients across a bipartition. We then prove a spectral-SIE theorem: for Rényi index $\alpha \geq 1/2$, the growth rate of Rényi-entanglement entropies admits a universal bound. Remarkably, our bound at $\alpha = 1/2$ is both qualitatively and quantitatively optimal; below this threshold ($\alpha < 1/2$), no universal speed limit on entanglement growth can exist. This result yields a sharp $1/s^2$ threshold in the tail of the ordered squared Schmidt coefficients, or equivalently in the entanglement-spectrum weights (with s the Schmidt index). It enables rigorous truncation-based-error control and establishes a quantitative link between entanglement-spectrum structure and computational complexity. As a practical highlight, for one-dimensional power-law interactions $1/r^{\eta}$ with $\eta > 2$, this implies matrix-product-state approximations with bond dimension polynomial in (n/ϵ) for ground states, real-time evolved states, and Gibbs states, thereby closing the quasipolynomial gap, where n is the system size, and ϵ denotes the target approximation error in the relevant state or operator norm. By controlling Rényi entanglement, we further obtain a rigorous *a priori* truncation-error bound for the time-dependent density-matrix renormalization-group scheme analyzed here. Overall, we extend the SIE paradigm from bounding entanglement entropies to constraining the entanglement spectrum itself.

DOI: 10.1103/8fcr-5tf4

Subject Areas: Condensed Matter Physics, Quantum Physics, Quantum Information

I. INTRODUCTION

Understanding how complex structures emerge in the dynamics of quantum systems over time is a fundamental problem in quantum many-body physics. With the rapid progress of quantum simulation platforms and quantum information processing, unveiling such dynamical complexity has become increasingly relevant both theoretically and experimentally. This problem is deeply connected to several essential topics, including the foundation of thermalization in quantum many-body systems [1,2], the characterization of quantum phases of matter [3–5], the efficiency guarantee of quantum many-body algorithms [6–9], and the theoretical framework of entanglement itself, as well as the role of entanglement as a fundamental resource in quantum information science [10–12]. Among the known universal principles governing quantum dynamics, the Lieb-Robinson bound stands out as a cornerstone result [13–15]. It provides an effective light-cone structure in nonrelativistic quantum systems, setting a fundamental limit on how quickly information and entanglement can propagate through local interactions. However, the Lieb-Robinson bound inherently relies on geometric locality, and its applicability becomes significantly limited when

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dealing with systems with long-range interactions or those defined on infinite-dimensional graphs [16–20]. In such cases, the standard notions of locality and causal structure no longer suffice to capture the nontrivial entanglement dynamics.

Another well-known information-theoretic constraint that does not rely on geometric locality is the small-incremental-entangling (SIE) theorem [21]. This theorem considers a bipartition of a quantum system into subsystems A and B, and asserts that the rate at which entanglement entropy is generated between them can be upper bounded solely in terms of the interaction strength and the Hilbert-space dimensions of the directly coupled subsystems $A_0 \subset A$ and $B_0 \subset B$. In systems with spatial locality, the SIE bound can be derived from the Lieb-Robinson bound [14]. However, its general validity across arbitrary quantum systems was first conjectured by Kitaev and Bravyi, and later fully proven in 2013 [22–25]. Because the rate of entanglement growth depends only on the interaction across the boundary between A and B, the SIE theorem is often referred to as the dynamical area law. This term emphasizes the fact that entanglement generation in time is governed by the surface—rather than the volume—of the interacting regions. The SIE theorem has served as a key methodological tool in a wide range of fundamental problems, such as supporting the area-law conjecture for noncritical ground states in higher-dimensional systems [22,26], analyzing complexity growth in quantum circuits [27,28], and characterizing measurement-induced phase transitions [29]. These applications demonstrate the power and universality of the SIE framework. Nevertheless, as we will discuss in the next section, the entanglement-generation rate alone is often insufficient to fully capture the complexity of quantum many-body dynamics.

Despite the theoretical strength of existing approaches such as the Lieb-Robinson bound and the SIE theorem, most of these frameworks focus primarily on the amount of information—for example, how much entanglement is generated—rather than accessing the structure of that information. The Lieb-Robinson bound provides no insight into the internal structure within the light cone, and the SIE theorem says nothing about the detailed features of the entanglement being generated. This shift from quantifying the amount to characterizing the structure represents a major open challenge in modern quantum many-body physics. In particular, it is known that entanglement entropy alone is insufficient to meaningfully describe the quantitative complexity of quantum systems [30,31]. Therefore, if we wish to utilize the SIE theorem as a foundational tool for understanding quantum many-body complexity, and to apply it to more intricate scenarios such as quantum algorithms or nonequilibrium phenomena, it becomes essential to introduce new formalisms that can capture the structure of entanglement itself.

In this work, we move beyond these limitations by establishing a spectral formulation of entanglement growth. Our framework not only provides rigorous dynamical constraints on the entanglement spectrum itself, but also achieves both qualitative and quantitative optimality: at the critical threshold, we establish a universal upper bound corresponding to a $1/s^2$ decay of squared Schmidt coefficients, which is sharp and cannot be improved. This perspective sharpens the dynamical area law into a more fine-grained structural principle and extends its reach to broader consequences, including generalized area laws of gapped ground states, efficient matrix-product-state approximations for long-range systems, and rigorous *a priori* performance guarantees for numerical algorithms such as tDMRG (time-dependent density-matrix renormalization group). Conceptually, the ability to address the entanglement spectrum directly opens the door to a more refined understanding of quantum complexity, enabling applications that go well beyond entropy-based diagnostics.

II. OVERVIEW OF THE MAIN RESULTS

Figure 1 summarizes the main results and their relationships.

A. Spectral-entangling strength

We consider a general bipartite quantum system composed of subsystems A and B, where the Hamiltonian takes the form $H = H_A + H_B + V_{AB}$. The time evolution generated by e^{-iHt} induces nontrivial changes in the entanglement spectrum, namely, the structure of the Schmidt coefficients, and our goal is to understand how this spectrum evolves. To address this, we introduce a quantity called the spectral-entangling (SE) strength defined for a general operator Φ_{AB}. This quantity plays a central role in the present work, as it quantifies the entangling power of an operator.

SE strength (Def. 1, informal): For any product state $|\phi\rangle$, let us denote the Schmidt coefficients of $\Phi_{AB}|\phi\rangle$ by $\{\lambda_s(\phi)\}_s$. Then, the operator Φ_{AB} has the SE strength $\bar{\mathcal{J}}(\Phi_{AB})$ which is defined as supremum over product states: $\sup_\phi \sum_s \lambda_s(\phi)$.

Intuitively, this quantity measures the maximal ability of Φ_{AB} to generate spectral weight across the bipartition. A simple characterization can be obtained when Φ_{AB} admits a decomposition $\Phi_{AB} = \sum_j J_j \Phi_{A,j} \otimes \Phi_{B,j}$ with $\|\Phi_{A,j}\| = \|\Phi_{B,j}\| = 1$, in which case one finds $\bar{\mathcal{J}}(\Phi_{AB}) \leq \sum_j |J_j|$.

B. Spectral SIE

This framework allows us to investigate how the entanglement spectrum $\{|\lambda_s(t)|^2\}$ of an initially product state $|\phi\rangle$ evolves under time evolution into $|\phi_t\rangle$. For a normalized product initial state, the squared Schmidt coefficients obey

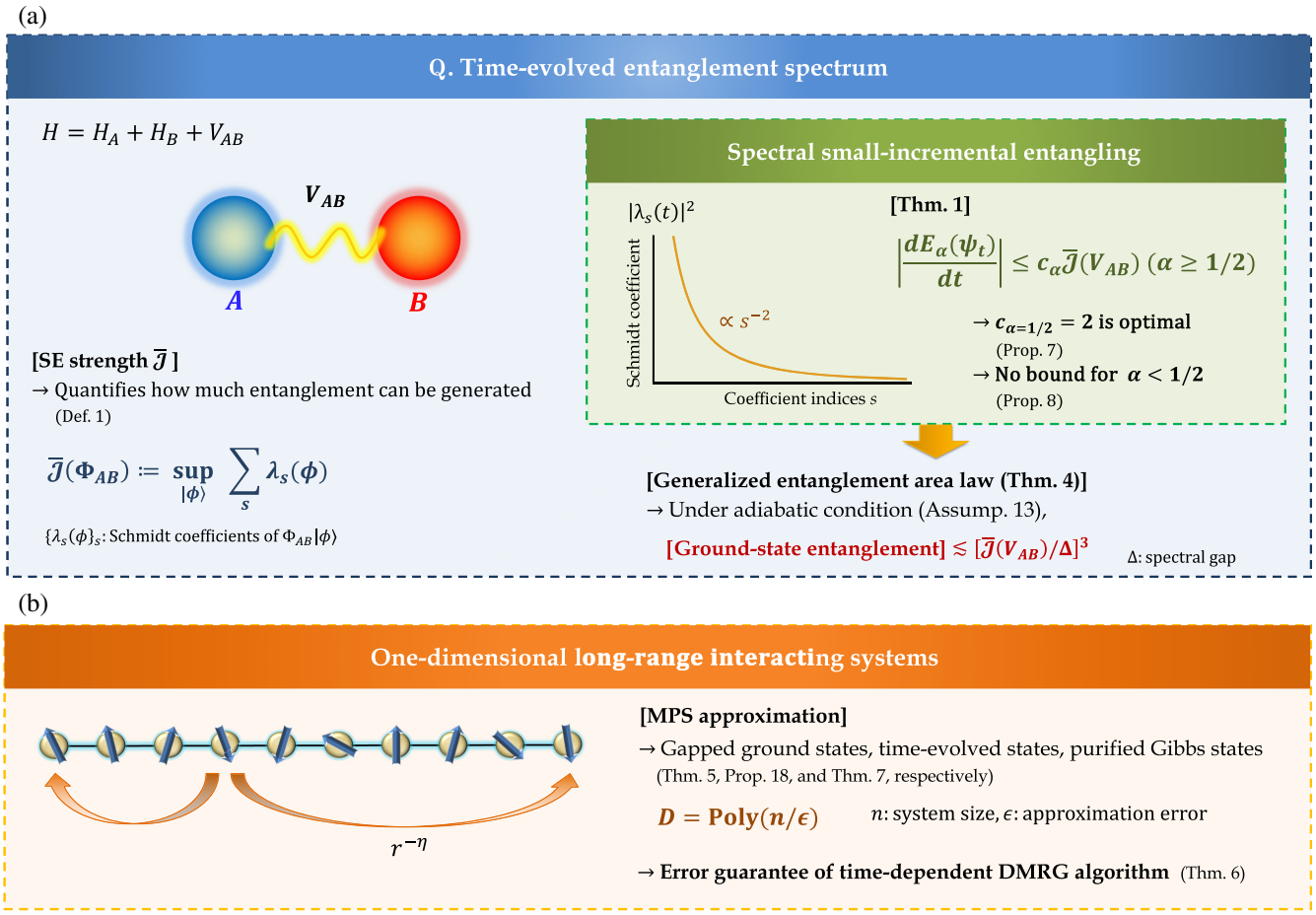


FIG. 1. Overview of the main results. The main conceptual message is that controlling the Rényi-1/2 entanglement controls the Schmidt tail directly, which in turn provides quantitative truncation-error bounds for tensor-network approximations. (a) SIE: introducing the spectral-entangling strength $\bar{\mathcal{J}}$, we establish that Rényi-entanglement entropies satisfy $|dE_\alpha/dt| \le c_\alpha \bar{\mathcal{J}}(V_{AB})$ for $\alpha \ge 1/2$, with the constant $c_{1/2} = 2$ being sharp, while for $\alpha < 1/2$ the growth rate can diverge. For product initial states, the $\alpha = 1/2$ bound gives an s^{-2} upper bound on the squared Schmidt coefficients; the curve illustrates its shape. The cubic area-law bound uses the uniform path bound $\tilde{g}$ of Assumption 13, with fixed S_0 and $\tilde{c}_0$. (b) Applications to 1D long-range interacting systems: when interactions decay faster than r^{-2}, ground states, real-time dynamics, and Gibbs states admit polynomial-bond-dimension matrix-product-state (MPS) and matrix-product-operator (MPO) approximations. The bounds hold at fixed local parameters, with a size-independent positive ground-state gap, fixed evolution time and product initial states, or fixed inverse temperature, respectively. In addition, our results yield rigorous error guarantees for tDMRG simulations.

$$|\lambda_s(t)|^2 \le e^{2\bar{\mathcal{J}}(V_{AB})|t|}/s^2.$$

In general, this uniform upper bound cannot be further improved without imposing additional geometric assumptions such as finite-dimensionality or short-range interactions [8,9,32]. The result can be derived from an upper bound on the growth rate of the α-Rényi entanglement E_α, as defined precisely in Eq. (3). In order to emphasize the spectral nature of the bound, we refer to this extension as the spectral-SIE theorem.

Theorem 1 (informal): For $\alpha \ge 1/2$, the generation rate of the α-Rényi entanglement is bounded from above by $c_\alpha \bar{\mathcal{J}}(V_{AB})$, where c_α is an $O(1)$ constant satisfying $c_{1/2} = 2$, $c_1 = 4/e$, and $c_\infty = 2$.

The conceptual difference from the standard Bravyi-Kitaev SIE framework lies in the choice of the control parameter. In the conventional setting, one considers a decomposition $A = A_1A_2$, $B = B_2B_1$, where the interaction is supported only on the directly coupled subsystems A_2B_2. For the von Neumann entropy, the SIE bound controls the entanglement rate by a quantity of the form $\|V_{A_2B_2}\| \log \mathcal{D}_{A_2|B_2}$, where $\mathcal{D}_{A_2|B_2} := \min(\mathcal{D}_{A_2}, \mathcal{D}_{B_2})$. This dimension-based form is special to the entropy case $\alpha = 1$; for Rényi entropies with $\alpha > 1$, known extensions involve stronger dimension dependences, such as powers of $\mathcal{D}_{A_2|B_2}$ [33,34]. In contrast, the spectral-SIE theorem replaces this subsystem-dimension-based control by the SE strength $\bar{\mathcal{J}}(V_{AB})$, which measures how much Schmidt spectral weight the boundary interaction can actually generate

across the bipartition. Thus, the spectral formulation controls the redistribution of Schmidt weights directly, rather than controlling only the von Neumann entropy and then extracting a Schmidt-tail estimate from it. The role of the SE strength in the optimality of the spectral-SIE bounds is further discussed in Sec. V B.

An important consequence is that the case $\alpha = 1/2$ gives direct control of the Schmidt tail. Indeed, $E_{1/2}(\psi) = 2\log\sum_s \lambda_s$, so bounding $E_{1/2}$ bounds the ℓ_1 size of the Schmidt spectrum and hence yields quantitative Schmidt-rank truncation estimates [30,31]. This control of the Schmidt tail provides a direct link between entanglement-spectrum dynamics and the computational complexity of tensor-network approximations. The optimality of the constant $c_{\alpha=1/2}$ and of the threshold $\alpha = 1/2$ is established in Propositions 7 and 8, respectively.

Finally, we clarify how the threshold $\alpha = 1/2$ can be interpreted in the presence of geometric locality. The simple mechanism behind the instantaneous threshold is already visible in the two-qubit example of Lemma 21 in Sec. IX. Consider an infinitesimal time evolution from a state in which a Schmidt coefficient is initially zero. A generic entangling generator creates such a new coefficient linearly in time; denoting it by $\lambda_{\mathrm{new}}(dt)$, one has $\lambda_{\mathrm{new}}(dt) = \mathcal{O}(dt)$. Its contribution to the Rényi-α functional is therefore of order $\lambda_{\mathrm{new}}(dt)^{2\alpha} = O(dt^{2\alpha})$, and the corresponding instantaneous rate scales as $dt^{2\alpha-1}$. This rate is finite only for $\alpha \geq 1/2$. Hence, the threshold $\alpha = 1/2$ is not a proof artifact but reflects an intrinsic singularity of the instantaneous Rényi-entanglement rate under the birth of infinitesimal Schmidt weights.

This instantaneous obstruction cannot be removed merely by imposing geometric locality: Lemma 21 gives a strictly local example showing that the threshold value $\alpha = 1/2$ cannot be improved. On the other hand, if one considers the time-averaged entanglement rate over a finite time interval, the situation changes [8,9,32,34,35]. Under geometric locality, the short-time singularity can be compensated by local approximation arguments, and a meaningful averaged entanglement-rate bound can be obtained even for $\alpha < 1/2$ (Lemma 22 in Sec. IX). By contrast, without geometric locality, even such an averaged bound fails in general: Proposition 8 shows that the averaged entanglement growth can remain unbounded for $\alpha < 1/2$. Thus, the instantaneous divergence has a simple local origin, whereas its persistence at the level of finite-time averages is a genuinely nontrivial feature of nonlocal dynamics.

C. Low-Schmidt-rank approximation of general operators

As discussed above, states evolved from product initial states admit efficient low-rank approximations. A natural question is whether a similar low-rank approximation exists at the level of the time-evolution operator itself. In conclusion, approximating an operator with respect to the operator norm is significantly more challenging. To address this, we demonstrate that there exists a complexity separation between Schmidt-rank approximability of time-evolved quantum states and that of the corresponding time-evolution unitary.

Proposition 9 (informal): There exist Hamiltonians H_{AB} with $\bar{\mathcal{J}}(V_{AB}) = 1$ for which the unitary $e^{-iH_{AB}t}$ does not admit any low-rank approximation in a certain (but constant) time regime.

At fixed time, $\bar{\mathcal{J}}(V_{AB}) = 1$ ensures low-Schmidt-rank approximability for states evolved from product initial states or from states whose initial Rényi-1/2 entanglement is bounded independently of dimension, whereas the corresponding time-evolution operator need not admit such an approximation. This separation is derived from classical lower bounds on Kolmogorov width [36–38].

Although a general relationship between entanglement generation and operator-level low-rank approximability remains elusive, one possible direction is proposed in Conjecture 2, which is formulated using a generalized version of the SE strength (see Definition 2). Furthermore, we provide sufficient conditions for the existence of operator low-rank approximations, which are formalized in Theorem 3. These results highlight the subtle but fundamental difference between approximability at the state level and at the operator level, and suggest that the spectral-entangling framework may offer a path toward resolving this gap.

D. Generalized entanglement area law

The entanglement area law refers to the property that, for a ground state of a finite-dimensional lattice system partitioned into two regions A and B, the entanglement entropy scales with the size of the boundary between the two regions [11,39]. In particular, the area-law conjecture argues that the area law always holds for noncritical ground states with a spectral gap, and it remains one of the central open problems in quantum many-body physics [40–49]. The notion of an area law can be extended to more general bipartite interaction settings [50]. Specifically, when a system is divided into A and B, the Schmidt rank of the interaction operator between A and B can be interpreted as an effective boundary size, thereby allowing a generalized formulation of the area law. For example, in a four-partite setup (A_1, A_0, B_0, B_1) where the interaction exists only between A_0 and B_0, the relevant boundary size is determined by the dimensions of A_0 and B_0. Typical examples include generalized area laws for local Hamiltonians defined on graphs, which have been widely discussed in the literature. According to the conventional SIE theorem [22,23], the generalized area law can be shown to hold for entanglement dynamics even in such general settings. It is also widely known that generalized area laws for mutual information hold for quantum Gibbs states at finite

temperature [51,52]. However, whether the same statement applies to gapped ground states has long been debated, and this question was resolved in the negative by Aharonov *et al.* in 2014 [50]. At present, aside from special classes [53], it remains a fundamental open problem to determine the precise conditions under which a generalized area law holds.

In this work, we establish that the generalized area law holds under the assumption that an adiabatic path exists [54]. Given a parameter-dependent Hamiltonian $H(s)$, we define an adiabatic path as a continuous trajectory in parameter space that connects a reference Hamiltonian with a known area-law-satisfying ground state to a target Hamiltonian, while maintaining a finite spectral gap throughout. This condition is standard in proofs of the area law for spatially local systems [22,26,55], where it is combined with the Lieb-Robinson bound, and where the SIE theorem provides a key estimate on entanglement growth along the adiabatic evolution. In more general settings, such as systems defined on infinite-dimensional graphs or general two-body systems, the Lieb-Robinson bound is unavailable, and consequently, the adiabatic continuation operator lacks a simple geometric structure. In these regimes, the conventional SIE theorem ceases to be effective. To overcome this difficulty, we combine the adiabatic theorem [56] with two key tools: the approximate-ground-state-projection formalism [41] and the spectral-SIE framework developed in this work. This combination enables us to prove a generalized area law without imposing any assumption of locality.

Our main result can be summarized as follows.

Theorem 4 (informal): If the target ground state is connected to a trivial ground state that satisfies the generalized area law via an adiabatic path (Assumption 13), then the entanglement entropy between A and B is $\mathcal{O}(1+(\tilde{g}/\Delta)^3)$, where $\tilde{g}$ is the uniform path bound and Δ the spectral gap, with fixed S_0 and path-regularity parameter $\tilde{c}_0$.

Furthermore, analogous statements can be made regarding low-Schmidt-rank approximations of the ground state for the bipartition. To the best of our knowledge, this constitutes the first proof of a generalized area law for ground states in the most general settings without assuming geometric locality.

We stress that, except in one-dimensional or geometrically local settings where the effective boundary quantity reduces to the usual boundary area, this result should not be interpreted as a conventional geometric area law. Rather, it is a generalized area-law-type bound in which the role of the boundary size is played by an effective boundary measure controlled by the SE strength of V_{AB}.

E. Polynomial complexity of the long-range interacting systems

Up to this point, our results have addressed the most general settings, including physical systems without any underlying geometric structure. Nevertheless, the framework we have developed also applies in scenarios where geometric structure is present. Among such cases, a particularly important and nontrivial application concerns the computational complexity of simulating quantum systems with long-range interactions. These systems are typically characterized by interactions that decay with spatial distance r as a power law $r^{-\eta}$ with $\eta > 0$, yet a comprehensive understanding of their entanglement properties and simulation complexity remains elusive.

One of the central open questions in this domain is whether the simulation complexity of general long-range interacting systems can be improved from quasipolynomial scaling $e^{\mathrm{polylog}(n/\epsilon)}$ [45,57,58] to polynomial scaling $\mathrm{poly}(n/\epsilon) = (n/\epsilon)^{\mathcal{O}(1)}$, where n denotes the system size. Achieving such an improvement would suggest that long-range interacting systems and short-range systems belong to the same computational complexity class, a possibility that is both highly counterintuitive and profoundly nontrivial.

The difficulty of this problem can be illustrated by considering the approximation of operators such as e^{-iHt} or $e^{-\beta H}$ by polynomial functions of the Hamiltonian H. When H^m is represented as a matrix product operator (MPO), the bond dimension generally grows with m as $n^{\mathcal{O}(m)}$ for a long-range interacting Hamiltonian [57]. If m depends even weakly on the system size—for instance, if $m \propto \log\log(n)$—the required bond dimension quickly exceeds polynomial bounds. This stands in sharp contrast to the case of short-range Hamiltonians, for which an MPO representation with constant bond dimension $\mathcal{O}(1)$ suffices. Consequently, physical quantities that cannot be captured by low-degree polynomial approximations in H inevitably lead to quasipolynomial simulation complexity in the presence of long-range interactions.

Building on our framework, we demonstrate that one-dimensional long-range interacting systems can, in fact, be represented as MPOs with polynomial bond dimension. The essential observation is that the SE strength admits an $\mathcal{O}(1)$ upper bound in such systems provided the interaction decays faster than r^{-2} (Lemma 17):

Lemma 17 (informal): For the 1D long-range Hamiltonians considered here with $\eta > 2$, the interaction across every single cut $A = \{1,\ldots,s\}$, $B = \{s+1,\ldots,n\}$ has SE strength at most $\tilde{J} = \mathcal{O}(1)$, for fixed interaction parameters, locality k, and on-site dimension.

Notably, $\eta > 2$ includes widely used experimental settings such as dipolar ($\eta = 3$) and van der Waals ($\eta = 6$) interactions, e.g., in Rydberg-atom and polar-molecule platforms. Related dynamical area-law results for long-range interacting systems were obtained by Gong *et al.* using the conventional SIE theorem [59]. Their result controls the von Neumann entanglement entropy and shows that the entanglement growth rate is bounded by a boundary-area contribution for power-law interactions

$1/r^{\eta}$ when $\eta > D_{\mathrm{lat}} + 1$, where D_{lat} denotes the spatial dimension. In particular, for one-dimensional systems, this condition reduces to $\eta > 2$, matching the decay threshold appearing in our one-dimensional applications.

The $\mathcal{O}(1)$ bound on the boundary SE strength has two complementary implications. First, it gives a physical constraint on entanglement-spectrum generation: for $\eta > 2$, the total spectral-entangling strength of the interaction crossing any one-dimensional cut remains $\mathcal{O}(1)$, so Schmidt weight cannot be redistributed across the cut faster than a constant rate. Second, it clarifies why earlier polynomial-approximation approaches lead to quasipolynomial bounds. In those approaches, functions of the Hamiltonian, such as e^{-iHt} or $e^{-\beta H}$, are approximated by polynomials in H, and the complexity is then estimated by counting the MPO bond dimension generated by powers H^m. For long-range Hamiltonians, this direct MPO-rank-counting estimate can grow as $n^{\mathcal{O}(m)}$, even though the actual Schmidt weight generated in the state across a cut may remain much smaller. The present spectral approach avoids this operator-level rank-counting route and instead controls the accumulated weighted generation of Schmidt coefficients directly (see also Sec. VIII B).

In this work, we apply this viewpoint to three fundamental classes of quantum states: ground states, time-evolved states, and thermal equilibrium states. For these states, only quasi-polynomial simulation complexity had previously been established [45,57,58]. Our results thus improve the known bounds by showing that, under the above decay condition, these physically relevant states can be captured within polynomial complexity, thereby closing the gap between long-range and short-range interacting systems in one dimension.

1. Ground states

We begin by considering matrix-product-state (MPS) approximations of gapped ground states in one-dimensional long-range interacting systems. For such ground states, the entanglement area law has already been established [45,60], and it is known that the required bond dimension scales as $e^{\log^{5/2}(n)}$, thereby imposing a quasipolynomial overhead. This limitation arises because most existing approaches rely on constructing an approximate ground-state projector (AGSP) using the Chebyshev polynomials of the Hamiltonian [41]. As discussed earlier, polynomial approximations of long-range Hamiltonians typically induce quasipolynomial complexity, rendering high-precision approximations prohibitively costly. In this work, we overcome this bottleneck by employing time-evolution operators together with the spectral-SIE framework. In particular, we estimate the SE strength of a Gaussian-filter AGSP and demonstrate that such filters yield asymptotically better approximations than conventional polynomial-based AGSPs. As a result, we prove that the ground state can be approximated by an MPS with polynomial bond dimension.

Theorem 5 (informal): For the 1D long-range Hamiltonians considered here, a unique ground state with spectral gap $\Delta > 0$ admits an MPS approximation with vector-norm error ϵ and bond dimension $D = \lceil (128 n D_0/\epsilon^2)^{1+2\tilde{J}/\Delta} \rceil$, with D_0 and $\tilde{J}$ defined below in Theorem 5 and Eq. (288). This bond dimension is polynomial in n/ϵ for a size-independent positive lower bound on Δ and fixed local dimension, locality k, and interaction parameters.

For this uniformly gapped class with a unique ground state, polynomially specified local terms, and an inverse-polynomial energy promise gap, the ground-energy decision problem is in NP: polynomial-size MPS witnesses allow polynomial-time verification of the energy.

2. Quantum dynamics

In a similar manner, the spectral-SIE formalism enables polynomially efficient approximation of time-evolved states by MPSs.

Proposition 18 (informal): For any product state $|\phi\rangle$, the time-evolved state can be approximated by an MPS up to an error ϵ, where the bond dimension D scales as $e^{\mathcal{O}(\tilde{J}t)}(n/\epsilon^2)$.

While Proposition 18 establishes efficient approximability of time-evolved states, the time complexity of simulating quantum dynamics on a classical computer remains considerably more challenging. Previous approaches have estimated this complexity by explicitly constructing MPO representations of the time-evolution operator e^{-iHt} [8,9,32,57]. At present, the best-known methods achieve only quasipolynomial simulation cost [57] for long-range interacting systems.

In our approach, instead of considering the explicit MPO construction of the time-evolution operator itself, we certify the accuracy of time-evolution simulation using the tDMRG algorithm [61,62]. This strategy overcomes the expensive task of explicitly constructing the full MPO representation. A key difficulty, previously highlighted by Osborne [32], is that naive error analysis of tDMRG suffers from exponential error amplification at each time step, undermining reliability (Sec. VIII E 2). In particular, while truncation-based diagnostics (e.g., the discarded weight) provide *a posteriori* accuracy checks during the simulation [61], a general *a priori* upper bound on the total error derived solely from the dynamics of a given local Hamiltonian has not been established for standard tDMRG.

In this work, we address this problem by monitoring the $(1/2)$-Rényi-entanglement proxy of the approximated state at each step, thereby guaranteeing controlled error propagation throughout the simulation:

Theorem 6 (informal): For a normalized product initial state, the Euler-and-truncation scheme with bond dimension D has total error at most ϵ for D of order $e^{\mathcal{O}(\tilde{J}t)} n^5/\epsilon^4$,

choosing the number of steps appropriately. Each Euler update is followed by left-to-right exact Schmidt truncations of the current state, without intermediate normalization.

We believe that the same analytical method may serve as a general paradigm for certifying the accuracy of other DMRG-based algorithms in the study of complex quantum systems.

3. Quantum Gibbs states

A natural question is whether similar techniques apply to imaginary-time evolution, which is fundamental for studying thermal equilibrium states. This setting, however, presents intrinsic challenges. The key difficulty is that imaginary-time evolution does not preserve the norm of the quantum state. Because imaginary-time evolution is nonunitary, even a product-form propagator (e.g., $e^{-\tau H_A} \otimes \hat{1}_B$) can, after normalization, significantly modify the entanglement spectrum across $A|B$. This illustrates in a striking way why complex-time evolution is qualitatively more difficult than real-time evolution [8,63]. Consequently, a direct application of the spectral-SIE framework allows efficient MPO (or purified MPS) approximations only in high-temperature regimes, where the thermal state remains close to the identity and entanglement is naturally limited.

To overcome this limitation, instead of relying on spectral SIE, we employ the operator low-rank approximation result established in Theorem 3 and adapt it to quantum Gibbs states. In this approach, unlike in the case of real-time evolution, only the existence of an efficient MPS approximation can be guaranteed.

Theorem 7 (informal): A purified one-dimensional Gibbs state can be approximated to error ϵ by an MPS whose bond dimension scales as $(n/\epsilon)^{\mathcal{O}(1+\beta)}$.

This result qualitatively improves upon the previously known quasipolynomial complexity bounds for quantum Gibbs states [57,58]. By contrast, designing an explicit and efficient algorithm to generate such an MPO remains open.

4. Remark on high-dimensional cases

The spectral-SIE theorem is geometry-free and therefore applies to arbitrary bipartite systems, including those arising from genuinely higher-dimensional lattices. The polynomial tensor-network consequences presented in this manuscript should be viewed as a concrete 1D instantiation: for 1D long-range interactions with sufficiently fast decay, the boundary SE strength across any cut can be bounded by $\mathcal{O}(1)$, enabling efficient and certifiable MPS or MPO truncation.

In higher dimensions, the bottleneck is not the spectral-SIE bound itself, but rather the current lack of comparably general and rigorous tensor-network certification and provably efficient contraction or optimization frameworks (e.g., analogs of the 1D theory for projected entangled-pair states [PEPS] or projected entangled-pair operators [PEPO]). Nevertheless, whenever the SE strength of the boundary interaction can be bounded for a chosen bipartition, our theorem yields dimension-independent constraints on the spectral tail. Developing corresponding higher-dimensional approximation and certification tools remains an important open direction.

III. SETUP

A. General framework

Table I summarizes the notation used throughout this paper.

We consider a general many-body qudit system, with the total set of qudits denoted by Λ. Each qudit is assumed to have a finite Hilbert-space dimension. We bipartition the system into two subsystems A and B such that $\Lambda = A \sqcup B$. For notational simplicity, $A \sqcup B$ is frequently abbreviated as AB, particularly when it appears in a subscript. Additionally, we often use A_0, A_1 and B_0, B_1 to denote decompositions of A and B into two subsets, such as $A = A_0 \sqcup A_1$ and $B = B_0 \sqcup B_1$. For any subset $X \subseteq \Lambda$, we denote the Hilbert-space dimension by $\mathcal{D}_X$. A product state $|\psi_A\rangle \otimes |\psi_B\rangle$ is often abbreviated as $|\psi_A, \psi_B\rangle$.

We consider an arbitrary Hamiltonian supported on A and B, which takes the form

$$H = H_A + H_B + V_{AB}, \tag{1}$$

where H_A and H_B act only on A and B, respectively, and V_{AB} represents the interaction term between A and B. At this stage, we do not impose any specific locality constraints on the Hamiltonian. In studying the entanglement generation, we typically assume that H_A and H_B may take arbitrary form (e.g., their norms are unbounded), while only the boundary interaction V_{AB} is restricted (see Definition 1 below).

For a given bipartition $\Lambda = A \sqcup B$ and a quantum state $|\psi\rangle$, we consider its Schmidt decomposition:

$$|\psi\rangle = \sum_{s=1}^{\mathcal{D}_{A|B}} \lambda_s |\psi_{A,s}\rangle \otimes |\psi_{B,s}\rangle. \tag{2}$$

Here, for a bipartition $A|B$, we denote by

$$\mathcal{D}_{A|B} \coloneqq \min(\mathcal{D}_A, \mathcal{D}_B)$$

the maximal possible Schmidt rank across the cut. When ancillas are attached to A and B, the corresponding ancilla-extended dimensions are used.

The Rényi-entanglement entropy of order α is then defined as

$$E_\alpha(\psi) = \frac{1}{1-\alpha} \log\left(\sum_s \lambda_s^{2\alpha}\right) = \frac{1}{1-\alpha} \log[\mathrm{tr}_A(\rho_A^\alpha)], \tag{3}$$

TABLE I. Summary of notation frequently used in the manuscript.

Symbol	Meaning
Λ	Total set of sites or qudits
n	Number of sites in Λ
$\mathcal{D}_X$	Hilbert-space dimension of subsystem X
$A\|B$	Bipartition of the system
$\mathrm{SR}(O)$	Schmidt rank of an operator O for bipartition $A\|B$
$\mathcal{D}_{A\|B}$	Maximal possible Schmidt rank across $A\|B$, $\min(\mathcal{D}_A, \mathcal{D}_B)$
λ_s	Schmidt coefficients in descending order
α	Rényi index
E_α	Rényi-entanglement entropy of order α
$\tilde{\mathcal{J}}(\Phi_{AB})$	Spectral-entangling strength of an operator Φ_{AB}
V_{AB}	Boundary interaction across $A\|B$
Δ	Spectral gap above the ground state
$\|\Omega\rangle$	Ground state of the Hamiltonian
$\|\psi\rangle$, $\|\psi_t\rangle$	Physical input state and its time-evolved state
$\|\phi\rangle$, $\|\phi_D\rangle$	Generic product state and Schmidt-rank-D approximation to a target state
ρ_β	Gibbs state at inverse temperature β
η	Power-law decay exponent, $J(r) \sim r^{-\eta}$
$\tilde{J}$	Uniform upper bound on the boundary SE strength in 1D applications
D	Bond dimension or Schmidt-rank truncation parameter
ϵ	Target approximation error
ε	Auxiliary small parameter often taken to zero or chosen sufficiently small

where $\rho_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|)$ is the reduced density matrix of $|\psi\rangle$ on subsystem A, and tr_B denotes the partial trace over B. Furthermore, for an arbitrary operator O, we define the Schmidt rank $\mathrm{SR}(O)$ as the minimum integer such that

$$O = \sum_{m=1}^{\mathrm{SR}(O)} O_{A,m} \otimes O_{B,m}, \tag{4}$$

where $O_{A,m}$ and $O_{B,m}$ are supported on the subsystems A and B, respectively.

As a measure of operator norm, we frequently employ the Schatten p-norm defined by

$$\|O\|_p \coloneqq [\mathrm{tr}(|O|^p)]^{1/p}, \tag{5}$$

where O is an arbitrary operator and $|O| \coloneqq \sqrt{O^\dagger O}$. In particular, $\|O\|_1$ corresponds to the trace norm, and $\|O\|_\infty$ is the operator norm (i.e., the maximum singular value), which we simply denote by $\|O\|$.

We use $|\psi_t\rangle$ to denote the time-evolved quantum state defined as

$$|\psi_t\rangle \coloneqq e^{-iHt}|\psi\rangle, \tag{6}$$

while, in ground-state statements, $|\Omega\rangle$ denotes the normalized, nondegenerate ground state of H and $\Delta = E_1 - E_0 > 0$, with eigenvalues ordered with multiplicity. Furthermore, we study the quantum Gibbs state at inverse temperature β:

$$\rho_\beta \coloneqq \frac{1}{Z_\beta} e^{-\beta H}, \qquad Z_\beta \coloneqq \mathrm{tr}(e^{-\beta H}). \tag{7}$$

Throughout the paper, tr_X denotes the trace over the Hilbert space associated with a subsystem $X \subseteq \Lambda$. When $AB = \Lambda$, we write tr_{AB} for the full trace over the bipartite system AB; this full trace may also be denoted simply by tr. We shall use the following standard notational convention. Whenever an operator acting nontrivially only on A, such as $\rho_A^{\alpha-1}$, is placed inside tr_{AB}, it is implicitly identified with its trivial extension to AB, namely, $\rho_A^{\alpha-1} \otimes \hat{1}_B$.

B. One-dimensional systems with power-law decaying interactions

In several applications, we consider a one-dimensional spatial geometry. Specifically, we use Λ to denote a chain of n sites, i.e., $\Lambda \coloneqq \{1, 2, \ldots, n\}$. We focus on k-local Hamiltonians of the form

$$H = \sum_{|Z| \le k} h_Z, \qquad \max_{i \in \Lambda} \sum_{Z \ni i} \|h_Z\| \le g, \tag{8}$$

where $Z \subseteq \Lambda$ denotes the support of the interaction term h_Z, and $|Z|$ is its cardinality.

We characterize the decay of interactions using a function $J(r)$ defined by

$$\sum_{Z \supseteq \{i,i'\}} \|h_Z\| \le J(r), \quad \forall\, i, i' \in \Lambda, \tag{9}$$

where $r := |i - i'|$ is the distance between the site i and i'. In contrast, a system is said to have finite-range interactions if there exists a positive integer l_H such that

$$J(r) = 0 \quad \text{for } r > l_H, \tag{10}$$

for some finite integer $l_H > 0$ (see Sec. IX). Conversely, a system exhibits long-range (power-law decaying) interactions if

$$J(r) = J_0 r^{-\eta}, \quad \eta > 2, \tag{11}$$

where the decay exponent η determines the strength of long-range couplings (see Sec. VIII).

For a given subset $A \subseteq \Lambda$, the local Hamiltonian H_A includes all terms fully supported within A:

$$H_A := \sum_{Z \subseteq A} h_Z. \tag{12}$$

The boundary interaction term between A and its complement $B := \Lambda \backslash A$ is then

$$V_{AB} := \sum_{\substack{Z: Z\cap A\neq\emptyset,\\ Z\cap B\neq\emptyset}} h_Z, \tag{13}$$

such that the full Hamiltonian is again written as $H = H_A + H_B + V_{AB}$.

We aim to approximate states such as $|\psi(t)\rangle$, $|\Omega\rangle$, and ρ_β using MPSs or MPOs. For MPSs, we use the form

$$|M_D\rangle = \sum_{s_1,\ldots,s_n} \mathrm{tr}\left(M_1^{[s_1]} M_2^{[s_2]} \ldots M_n^{[s_n]}\right) |s_1, \ldots, s_n\rangle, \tag{14}$$

where D is the bond dimension. We use open boundaries: $M_1^{[\cdot]}$ is a $1 \times D$ row, $M_n^{[\cdot]}$ a $D \times 1$ column, and the other matrices are $D \times D$; the trace acts on their 1×1 product. We note that any local observables for $|M_D\rangle$ can be computed in at most $\mathcal{O}(nD^3)$ computational time.

C. Spectral-entangling strength

A central problem in this work is efficiently characterizing the capacity of an operator to generate entanglement spectra across subsystems. However, within existing methodologies, there has not been an established quantity that efficiently and directly quantifies the ability of a general operator to generate the entanglement spectrum. To address this gap, we introduce a new quantity, which we refer to as the SE strength.

The SE strength introduced here has several desirable features:

(i) Its upper bound can be easily calculated, and in many cases, the quantity itself can be evaluated exactly.

(ii) It provides an optimal upper bound on the efficiency of generating the entanglement spectrum.

In this sense, the SE strength serves as an effective and analytically tractable measure of the spectral-entangling capability of general operators. The formal definition is given below (see also Sec. VI C for an extension of this definition).

Definition 1. (SE strength). Let Φ_{AB} be an arbitrary operator acting across subsystems A and B. For any product state $|\phi\rangle = |\phi_{AA'}\rangle \otimes |\phi_{BB'}\rangle$ with ancillas A' and B', we let the Schmidt decomposition of $\Phi_{AB}|\phi\rangle$ be

$$\Phi_{AB}|\phi\rangle = \sum_s \lambda_s(\phi) |\phi_{AA',s}\rangle \otimes |\phi_{BB',s}\rangle. \tag{15}$$

Then, the SE strength of Φ_{AB} is defined as

$$\bar{\mathcal{J}}(\Phi_{AB}) := \sup_{|\phi\rangle} \sum_s \lambda_s(\phi), \tag{16}$$

where the supremum is taken over all product states $|\phi\rangle = |\phi_{AA'}\rangle \otimes |\phi_{BB'}\rangle$.

Remark. Because of the concave-roof optimization (or supremum over product states), computing $\bar{\mathcal{J}}(\Phi_{AB})$ rigorously is not straightforward. In the subsequent sections, we will consider several cases in Eqs. (81) and (125), where Eqs. (91) and (126) give the analytical solutions, respectively.

On the other hand, one can easily calculate an upper bound for $\bar{\mathcal{J}}(\Phi_{AB})$. For example, if Φ_{AB} admits a decomposition of the form

$$\Phi_{AB} = \sum_j J_j \Phi_{A,j} \otimes \Phi_{B,j} \quad \text{with} \quad \|\Phi_{A,j} \otimes \Phi_{B,j}\| = 1, \tag{17}$$

then the SE strength is trivially bounded by

$$\bar{\mathcal{J}}(\Phi_{AB}) \leq \sum_j |J_j|. \tag{18}$$

We emphasize that the SE strength is a structural property of general quantum operators, not limited to Hamiltonians (see Lemma 4 below). It is well defined for a wide range of operators, including time-evolution unitaries, the AGSP, and general quantum channels. This versatility allows the SE strength to systematically quantify their influence on the entanglement spectrum of quantum states, thereby establishing a unifying framework for entanglement dynamics and structure across diverse physical systems and computational settings.

Finally, we remark on the presence of ancilla systems, which can significantly change the value of the SE strength. For example, the two-qubit swap operator S_{AB} maps any product state $|\phi_A\rangle \otimes |\phi_B\rangle$ to another product state; hence,

$\bar{\mathcal{J}}(S_{AB}) = 1$ if no ancilla is allowed. However, when additional qubits A' and B' are attached, consider the product state

$$|0_A 0_{A'}\rangle \otimes \frac{1}{\sqrt{2}}(|0_B 0_{B'}\rangle + |1_B 1_{B'}\rangle). \tag{19}$$

After applying S_{AB}, the resulting state across the $AA'|BB'$ cut has two equal Schmidt coefficients $1/\sqrt{2}$, implying $\bar{\mathcal{J}}(S_{AB}) \geq \sqrt{2}$. Thus, the existence of ancillas reveals a nontrivial entangling capability that would otherwise remain hidden.

IV. FUNDAMENTAL LEMMAS

In this section, we present a collection of elementary but fundamental lemmas that serve as the technical dictionary for the rest of the paper. The purpose is to explain how control of the ℓ_1 size of the Schmidt spectrum $\sum_s \lambda_s$ can be converted into quantitative Schmidt-tail bounds and truncation-error estimates. While the results themselves are derived via straightforward arguments, they capture the basic mechanism behind our approach: spectral information, rather than entropy alone, controls tensor-network approximability.

First, Lemma 1 and its Corollary 2 provide a basic but powerful statement on the decay of the Schmidt coefficients. These results, when combined with the definition of SE strength, allow us to rigorously discuss the truncation error of the Schmidt rank for an arbitrary operator Φ_{AB} acting on bipartite product states, as detailed in Corollary 3.

Next, Lemma 4 addresses a fundamental property of the SE strength, namely, its subadditivity under certain linear combinations of operators. This property will play a key role in establishing upper bounds for more general operators appearing in our framework.

Finally, Lemma 5 elucidates a general relationship between the decay of the Schmidt coefficients and the Rényi entanglement. This lemma highlights how the entanglement spectrum constrains the scaling of the largest Schmidt coefficients, thereby linking spectral properties to entanglement measures in a quantitative manner.

Together, these lemmas provide the conversion rules used later in the paper: from SE strength to Schmidt-spectrum control, and from Schmidt-spectrum control to truncation-error estimates.

A. Decay of the Schmidt coefficients

We begin by establishing a general upper bound on the sum of overlaps between a nonorthogonal product-state expansion and an orthonormal basis, which underlies our analysis of the entanglement spectrum.

Lemma 1. Let $|\Psi\rangle$ be an arbitrary unnormalized quantum state of the form

$$|\Psi\rangle = \sum_{j=1}^{\infty} g_j |A_j\rangle \otimes |B_j\rangle, \tag{20}$$

where $\{|A_j\rangle\}_j$ and $\{|B_j\rangle\}_j$ are normalized but not necessarily orthogonal states; only normalization is required:

$$\langle A_j|A_j\rangle = \langle B_j|B_j\rangle = 1 \quad \forall\, j. \tag{21}$$

Let $\{|a_s\rangle\}_s$ and $\{|b_s\rangle\}_s$ be arbitrary orthonormal bases on subsystems A and B, respectively. Then the following inequality holds:

$$\sum_s |\langle a_s, b_s|\Psi\rangle| \leq \sum_{j=1}^{\infty} |g_j| =: \mathfrak{g}. \tag{22}$$

1. Proof of Lemma 1

The proof follows immediately from the inequality as follows:

$$\sum_s |\langle a_s, b_s|\phi_A, \phi_B\rangle| \leq 1, \tag{23}$$

which holds for any product state $|\phi_A\rangle \otimes |\phi_B\rangle$. Indeed, using linearity and the above inequality, we obtain

$$\begin{aligned}\sum_s |\langle a_s, b_s|\Psi\rangle| &= \sum_s \left|\sum_j g_j \langle a_s, b_s|A_j, B_j\rangle\right| \\ &\leq \sum_j |g_j| \sum_s |\langle a_s, b_s|A_j, B_j\rangle| \\ &\leq \sum_j |g_j|. \end{aligned} \tag{24}$$

An application of the Cauchy-Schwarz inequality yields inequality (23):

$$\begin{aligned}&\sum_s |\langle a_s|\phi_A\rangle| \times |\langle b_s|\phi_B\rangle| \\ &\leq \left(\sum\nolimits_s |\langle a_s|\phi_A\rangle|^2\right)^{1/2} \left(\sum\nolimits_s |\langle b_s|\phi_B\rangle|^2\right)^{1/2} = 1. \end{aligned} \tag{25}$$

This completes the proof. ∎

The lemma implies the following corollary regarding the Schmidt coefficients:

Corollary 2. Let $|\Psi\rangle$ be as defined in Eq. (20), and let its Schmidt decomposition be given by

$$|\Psi\rangle = \sum_{s=1}^{\mathcal{D}_{A|B}} \lambda_s |\phi_{A,s}\rangle \otimes |\phi_{B,s}\rangle, \tag{26}$$

where the Schmidt coefficients satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. Then we have

$$\sum_{s=1}^{\mathcal{D}_{A|B}} \lambda_s \le \mathfrak{g}, \tag{27}$$

where $\mathfrak{g}$ is as defined in Eq. (22).

Remark. From the ordering $\lambda_1 \ge \cdots \ge \lambda_{\mathcal{D}_{A|B}}$, it follows that for any $s_0 \in \mathbb{N}$,

$$\mathfrak{g} \ge \sum_{s=1}^{s_0} \lambda_s \ge s_0 \lambda_{s_0} \Rightarrow \lambda_{s_0} \le \frac{\mathfrak{g}}{s_0}. \tag{28}$$

Thus, for all $s \ge 1$,

$$\lambda_s \le \frac{\mathfrak{g}}{s}. \tag{29}$$

We now use this to evaluate the approximation error due to Schmidt-rank truncation.

Corollary 3. Let Φ_{AB} be an arbitrary operator acting between subsystems A and B. For any product state $|\phi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$, let $\Phi_{AB}|\phi\rangle$ have Schmidt decomposition

$$\Phi_{AB}|\phi\rangle = \sum_s \lambda_s(\phi)|\phi_{A,s}\rangle \otimes |\phi_{B,s}\rangle, \tag{30}$$

with $\lambda_s(\phi)$ in descending order. Define the truncated state

$$|\phi_D\rangle := \sum_{s=1}^{D} \lambda_s(\phi)|\phi_{A,s}\rangle \otimes |\phi_{B,s}\rangle. \tag{31}$$

Then the approximation error satisfies

$$\|\Phi_{AB}|\phi\rangle - |\phi_D\rangle\| \le \frac{\bar{\mathcal{J}}(\Phi_{AB})}{\sqrt{D}}. \tag{32}$$

Proof of Corollary 3. Using Eq. (29) for each $\lambda_s(\phi)$, and applying the definition of the SE strength, we deduce the inequality

$$\begin{aligned}\|\Phi_{AB}|\phi\rangle - |\phi_D\rangle\|^2 = \sum_{s>D} \lambda_s(\phi)^2 &\le \sum_{s>D} \left(\frac{\bar{\mathcal{J}}(\Phi_{AB})}{s}\right)^2 \\ &\le \bar{\mathcal{J}}(\Phi_{AB})^2 \sum_{s>D} \frac{1}{s^2} \le \frac{\bar{\mathcal{J}}(\Phi_{AB})^2}{D}.\end{aligned} \tag{33}$$

Taking the square root on both sides yields the desired bound. ∎

B. Upper bound on the SE strength for general operators

We consider an arbitrary operator of the form

$$O = \int_{-\infty}^{\infty} f(x)\Phi_{AB}(x)dx, \tag{34}$$

where $\Phi_{AB}(x)$ is an arbitrary operator that depends on $x \in \mathbb{R}$. We then analyze how the entanglement spectrum varies with the choice of the operator O. We prove the following lemma:

Lemma 4. (Subadditivity of the SE strength). For any operator of the form given by Eq. (34), the inequality

$$\bar{\mathcal{J}}(O) \le \int_{-\infty}^{\infty} |f(x)|\bar{\mathcal{J}}[\Phi_{AB}(x)]dx \tag{35}$$

is satisfied.

Proof of Lemma 4. From Definition 1, we define $|\tilde{\phi}\rangle$ such that

$$|\tilde{\phi}\rangle = \arg\sup_{|\phi\rangle} \left(\sum_s \tilde{\lambda}_s(\phi)\right). \tag{36}$$

For the quantum state $|\tilde{\phi}\rangle$, we consider the Schmidt decomposition of

$$O|\tilde{\phi}\rangle = \sum_s \tilde{\lambda}_s(\tilde{\phi})|\tilde{\phi}_{AA',s}\rangle \otimes |\tilde{\phi}_{BB',s}\rangle. \tag{37}$$

Then, our task is to estimate $\sum_s \tilde{\lambda}_s(\tilde{\phi}) = \bar{\mathcal{J}}(O)$.

For an arbitrary $\Phi_{AB}(x)$ and the state $|\tilde{\phi}\rangle$, we define the Schmidt decomposition as follows:

$$\Phi_{AB}(x)|\tilde{\phi}\rangle = \sum_s \tilde{\lambda}_s(\tilde{\phi}, x)|\tilde{\phi}_{AA',s}(x)\rangle \otimes |\tilde{\phi}_{BB',s}(x)\rangle, \tag{38}$$

which also yields

$$O|\tilde{\phi}\rangle = \int_{-\infty}^{\infty} dx f(x) \sum_s \tilde{\lambda}_s(\tilde{\phi}, x)|\tilde{\phi}_{AA',s}(x)\rangle \otimes |\tilde{\phi}_{BB',s}(x)\rangle, \tag{39}$$

where we use Eq. (34). From the Schmidt decomposition (37), we obtain an upper bound on $\bar{\mathcal{J}}(O)$ as

$$\begin{aligned}\bar{\mathcal{J}}(O) &= \sum_{s_0} \langle \tilde{\phi}_{AA',s_0}, \tilde{\phi}_{BB',s_0}|O|\tilde{\phi}\rangle \\ &= \sum_{s_0} \int_{-\infty}^{\infty} dx f(x) \sum_s \tilde{\lambda}_s(\tilde{\phi}, x)\langle \tilde{\phi}_{AA',s_0}, \tilde{\phi}_{BB',s_0}|\tilde{\phi}_{AA',s}(x), \tilde{\phi}_{BB',s}(x)\rangle \\ &\le \int_{-\infty}^{\infty} dx |f(x)| \sum_s \tilde{\lambda}_s(\tilde{\phi}, x) \sum_{s_0} |\langle \tilde{\phi}_{AA',s_0}, \tilde{\phi}_{BB',s_0}|\tilde{\phi}_{AA',s}(x), \tilde{\phi}_{BB',s}(x)\rangle|.\end{aligned} \tag{40}$$

By using inequality (22) in Lemma 1 with $g_1 = 1$ and $g_j = 0$ $(j \geq 2)$, we have

$$\sum_{s_0} |\langle \tilde{\phi}_{AA',s_0}, \tilde{\phi}_{BB',s_0} | \tilde{\phi}_{AA',s}(x), \tilde{\phi}_{BB',s}(x) \rangle| \leq 1 \quad (41)$$

for an arbitrary x and s. Furthermore, applying Eq. (16) to $\bar{\mathcal{J}}[\Phi_{AB}(x)]$ implies

$$\bar{\mathcal{J}}[\Phi_{AB}(x)] \geq \sum_s \tilde{\lambda}_s(\tilde{\phi}, x). \quad (42)$$

By combining the inequalities (41) and (42) with Eq. (40), we arrive at the desired inequality of

$$\begin{aligned} \bar{\mathcal{J}}(O) &\leq \int_{-\infty}^{\infty} dx |f(x)| \sum_s \tilde{\lambda}_s(\tilde{\phi}, x) \\ &\leq \int_{-\infty}^{\infty} dx |f(x)| \bar{\mathcal{J}}[\Phi_{AB}(x)]. \end{aligned} \quad (43)$$

This completes the proof. ∎

C. Schmidt coefficients vs Rényi entanglement

Lemma 5. Let $|\psi\rangle$ be an arbitrary quantum state with the Schmidt decomposition as follows:

$$|\psi\rangle = \sum_{s=1}^{\mathcal{D}_{A|B}} \lambda_s |\psi_{A,s}\rangle \otimes |\psi_{B,s}\rangle, \quad (44)$$

where the descending order $(\lambda_1 \geq \lambda_2 \geq \cdots)$ is assumed. Then, for any natural number s_0 and $\alpha \in (0,1)$, λ_{s_0} satisfies the following inequality:

$$|\lambda_{s_0}| \leq \left(\frac{e^{(1-\alpha)E_\alpha(\psi)}}{s_0} \right)^{1/(2\alpha)}, \quad (45)$$

where $E_\alpha(\psi)$ is the Rényi entanglement (3).

Proof of Lemma 5. From the definition (3), we obtain

$$\sum_{s=1}^{\mathcal{D}_{A|B}} \lambda_s^{2\alpha} = e^{(1-\alpha)E_\alpha(\psi)}. \quad (46)$$

Then, we have

$$\sum_{s=1}^{\mathcal{D}_{A|B}} \lambda_s^{2\alpha} \geq \sum_{s=1}^{s_0} \lambda_s^{2\alpha} \geq s_0 \lambda_{s_0}^{2\alpha}, \quad (47)$$

where we use $\lambda_s \geq \lambda_{s_0}$ for $s \leq s_0$. Therefore, by combining the above two relations, we get

$$s_0 \lambda_{s_0}^{2\alpha} \leq e^{(1-\alpha)E_\alpha(\psi)}, \quad (48)$$

which yields the main inequality (45). This completes the proof. ∎

V. SPECTRAL SIE

The purpose of this section is to prove the spectral-SIE theorem and to clarify its optimality. The key idea is to estimate the time derivative of the Rényi entanglement E_α directly in the instantaneous Schmidt basis. Since the intraregion terms H_A and H_B generate only local rotations on the two sides of the bipartition, the change of the Schmidt spectrum is governed by the boundary interaction V_{AB}. We show that this redistribution of Schmidt weights can be controlled by the SE strength $\bar{\mathcal{J}}(V_{AB})$, which leads to the dimension-independent bound $|dE_\alpha/dt| \leq c_\alpha \bar{\mathcal{J}}(V_{AB})$ for $\alpha \geq 1/2$ (Theorem 1). After proving this bound, we demonstrate that the threshold $\alpha = 1/2$ is optimal and that the constant $c_{\alpha=1/2} = 2$ cannot be improved in general (Propositions 7 and 8). These results provide the main dynamical input for the later tensor-network approximation results, where control of $E_{1/2}$ is converted into Schmidt-tail and truncation-error estimates.

A. Entanglement rate for Rényi entanglement ($\alpha \geq 1/2$)

Theorem 1. For an arbitrary time-evolved quantum state $|\psi_t\rangle = e^{-iHt}|\psi\rangle$ and the α-Rényi entanglement $E_\alpha(\psi_t)$ in Eq. (3), i.e.,

$$E_\alpha(\psi_t) = \frac{1}{1-\alpha} \log[\mathrm{tr}_A(\rho_{t,A}^\alpha)], \qquad \rho_{t,A} = \mathrm{tr}_B(|\psi_t\rangle\langle\psi_t|), \quad (49)$$

we obtain the upper bound on the entanglement rate as

$$\left| \frac{dE_\alpha(\psi_t)}{dt} \right| \leq c_\alpha \bar{\mathcal{J}}(V_{AB}) \quad (50)$$

for $\alpha \geq 1/2$. The bound holds wherever the derivative exists, and almost everywhere. Here, c_α is defined by the following formula, with limiting values at $\alpha = 1/2, 1, \infty$ (see Fig. 2):

$$c_\alpha := \frac{2\alpha}{1-\alpha} [(2\alpha-1)^{(2\alpha-1)/(2-2\alpha)} - (2\alpha-1)^{1/(2-2\alpha)}]. \quad (51)$$

In particular, from $c_{1/2} = 2$, inequality (50) for $\alpha = 1/2$ can be rewritten as

$$\left| \frac{dE_{\alpha=1/2}(\psi_t)}{dt} \right| \leq 2\bar{\mathcal{J}}(V_{AB}). \quad (52)$$

The statement remains valid even when the system is extended by attaching arbitrary ancillas to A and B.

Remark. As a specific example, let us consider the case where V_{AB} is given by

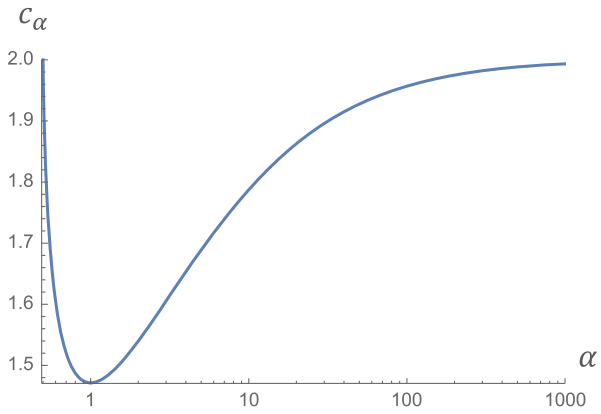


FIG. 2. Plot of c_α with respect to α. We have $c_{\alpha=1/2}=2$, $c_{\alpha=1}=4/e$, and $c_{\alpha=\infty}=2$.

$$V_{AB}=\sum_j J_{j,AB}h_{j,A}\otimes h_{j,B},\qquad \|h_{j,A}\|=\|h_{j,B}\|=1. \tag{53}$$

Then, the upper bound (50) can be simplified as

$$\left|\frac{dE_\alpha(\psi_t)}{dt}\right|\le c_\alpha\sum_j|J_{j,AB}|, \tag{54}$$

where we use inequality (18) to obtain $\bar{\mathcal{J}}(V_{AB})\le\sum_j|J_{j,AB}|$.

B. Optimality of the entanglement generation

Before showing the proof, we emphasize the optimality of the entanglement rate bound for Rényi entanglement. The main points are listed below.

(i) Tightness at $\alpha=1/2$: In the special case $\alpha=1/2$, the coefficient $c_{1/2}=2$ is tight. There exists an explicit quantum dynamics for which the entanglement-generation rate saturates the upper bound over a finite-time interval, demonstrating that the result cannot be improved in general (see Sec. V D).

(ii) Threshold of $\alpha=1/2$: The value $\alpha=1/2$ serves as a sharp threshold for the existence of a meaningful upper bound on the generation rate of Rényi entanglement. For $\alpha<1/2$, no universal bound exists; that is, there are examples where the entanglement generation in an $\mathcal{O}(1)$ time can become arbitrarily large depending on the Hilbert-space dimension. Conversely, for $\alpha\ge1/2$, the bound in Theorem 1 provides a dimension-independent constraint (see Sec. V E).

We also make two remarks on optimality.

(i) For $\alpha>1/2$, the optimality of the entanglement rate bound remains an open problem. In particular, for the case $\alpha=1$ (the von Neumann entropy), the qualitative behavior of the entanglement rate differs from the case $\alpha=1/2$ (see below).

(ii) Even if we consider systems with spatial structure, such as those with short-range interactions, the unbounded entanglement rate below $\alpha=1/2$ can appear instantaneously. However, in such cases, qualitatively different behavior can emerge for the entanglement generation over a finite time. We will discuss this point in detail in Sec. IX.

Regarding the first additional point, the standard SIE theorem [21–23,25] for $\alpha=1$ depends on $\|V_{AB}\|$, while our generalized bound depends on the SE strength $\bar{\mathcal{J}}(V_{AB})$. Here, we can see a qualitative difference between $\bar{\mathcal{J}}(V_{AB})$ and $\|V_{AB}\|$.

As a concrete example, let us consider the case where V_{AB} is supported on A_0B_0 ($A_0\subset A$, $B_0\subset B$) and $\mathcal{D}_{A_0}=\mathcal{D}_{B_0}=M+1$. We then consider an interaction of the form

$$V_{A_0B_0}=\sum_{j=1}^{M}J(|j_{A_0},j_{B_0}\rangle\langle 0_{A_0},0_{B_0}|+\text{H.c.}), \tag{55}$$

where $\{|j_{A_0}\rangle\}_{j=0}^M$ and $\{|j_{B_0}\rangle\}_{j=0}^M$ are orthonormal bases on A_0 and B_0, respectively. For this specific example, we obtain precisely [see also Eq. (91)]

$$\bar{\mathcal{J}}(V_{A_0B_0})=JM,\qquad \|V_{A_0B_0}\|=J\sqrt{M}. \tag{56}$$

From the above estimation, we get the entanglement rate for the $\alpha=1$ case [22]:

$$\begin{aligned}\left|\frac{dE_{\alpha=1}(\psi_t)}{dt}\right|&\le 18\|V_{A_0B_0}\|\log(\mathcal{D}_{A_0})\\&=18J\sqrt{M}\log(M+1),\end{aligned} \tag{57}$$

whereas we can explicitly find a dynamics such that

$$\left|\frac{dE_{\alpha=1/2}(\psi_t)}{dt}\right|\ge 2\bar{\mathcal{J}}(V_{AB})-\varepsilon=2JM-\varepsilon, \tag{58}$$

where ε approaches zero in the limit as the Hilbert-space dimension becomes large (see Proposition 7). Therefore, the entanglement rates for $\alpha=1$ and $\alpha=1/2$ show a qualitative difference in the limit of $M\to\infty$.

It is an open problem to unify the current spectral-SIE and the standard SIE theorems.

C. Proof of Theorem 1

Passing to the interaction picture, we may set $H_A=H_B=0$. First take $\alpha>1/2$, $\alpha\ne1$. For the coefficient matrix M_t, $\mathrm{tr}(\rho_{t,A}^\alpha)=\|M_t\|_{2\alpha}^{2\alpha}$ is differentiable even at zero singular values, and the support-restricted expression below gives its derivative. First, we consider

$$\frac{d}{dt}E_\alpha(\psi_t)=\frac{1}{1-\alpha}\frac{d}{dt}\log\left[\mathrm{tr}_A(\rho_{t,A}^\alpha)\right]$$
$$=\frac{\alpha}{(1-\alpha)\mathrm{tr}_A(\rho_{t,A}^\alpha)}\mathrm{tr}_{AB}\left(\rho_{t,A}^{\alpha-1}[-iH,\rho_t]\right), \quad (59)$$

where $\rho_t=|\psi_t\rangle\langle\psi_t|$.

Precisely speaking, since $\rho_{t,A}$ may have zero eigenvalues, we interpret $\rho_{t,A}^{\alpha-1}$ as an operator acting on the support of $\rho_{t,A}$, i.e., the subspace spanned by eigenvectors of $\rho_{t,A}$ with strictly positive eigenvalues. Let Π_A denote the orthogonal projector onto $\mathrm{supp}(\rho_{t,A})$. Throughout, we implicitly work with the restricted operator on $\mathrm{supp}(\rho_{t,A})$, i.e.,

$$\Pi_A\rho_{t,A}^{\alpha-1}\Pi_A, \quad (60)$$

so that contributions from eigenstates with zero eigenvalues are excluded and $\rho_{t,A}^{\alpha-1}$ is well defined.

Then, we obtain

$$\left|\frac{d}{dt}E_\alpha(\psi_t)\right|\le\frac{\alpha}{|1-\alpha|\mathrm{tr}_A(\rho_{t,A}^\alpha)}\left|\mathrm{tr}_{AB}(\rho_{t,A}^{\alpha-1}[H,\rho_t])\right|. \quad (61)$$

In the following, for an arbitrary quantum state $\rho=|\psi_1\rangle\langle\psi_1|$, we generally consider the upper bound of $|\mathrm{tr}_{AB}(\rho_A^{\alpha-1}[H,\rho])|$, where $\rho_A=\mathrm{tr}_B(\rho)$.

For this purpose, we use the Schmidt decomposition of

$$|\psi_1\rangle=\sum_{s=1}^{\mathcal{D}_{A|B}}\lambda_s|\psi_{s,A}\rangle\otimes|\psi_{s,B}\rangle,\qquad \rho_A=\sum_s\lambda_s^2|\psi_{s,A}\rangle\langle\psi_{s,A}|. \quad (62)$$

From the expression, we obtain

$$\begin{aligned}&\mathrm{tr}_{AB}(\rho_A^{\alpha-1}[H,\rho])\\&=\sum_{s,s''}\lambda_s^{2\alpha-2}\langle\psi_{s,A},\psi_{s'',B}|[V_{AB},\rho]|\psi_{s,A},\psi_{s'',B}\rangle\\&=\sum_{s,s''}\lambda_s^{2\alpha-2}(\langle\psi_{s,A},\psi_{s'',B}|V_{AB}|\psi_1\rangle\langle\psi_1|\psi_{s,A},\psi_{s'',B}\rangle-\mathrm{c.c.})\\&=\sum_{s,s'}\lambda_s^{2\alpha-2}(\langle\psi_{s,A},\psi_{s,B}|V_{AB}\lambda_s\lambda_{s'}|\psi_{s',A},\psi_{s',B}\rangle-\mathrm{c.c.})\\&=\sum_{s,s'}\lambda_s^{2\alpha-1}\lambda_{s'}(\langle\psi_{s,A},\psi_{s,B}|V_{AB}|\psi_{s',A},\psi_{s',B}\rangle-\mathrm{c.c.})\\&=\sum_{s,s'}(\lambda_s^{2\alpha-1}\lambda_{s'}-\lambda_{s'}^{2\alpha-1}\lambda_s)\langle\psi_{s,A},\psi_{s,B}|V_{AB}|\psi_{s',A},\psi_{s',B}\rangle,\end{aligned} \quad (63)$$

where we use Eq. (62) in the step from the second to the third line.

Next, we consider the relations

$$|\lambda_s^{2\alpha-1}\lambda_{s'}-\lambda_{s'}^{2\alpha-1}\lambda_s|=\lambda_s^{2\alpha}|x-x^{2\alpha-1}|\le\tilde{c}_\alpha\lambda_s^{2\alpha} \quad (64)$$

for $\lambda_s>\lambda_{s'}$ (or $s<s'$), where $x=\lambda_{s'}/\lambda_s\in[0,1]$ and the constant $\tilde{c}_\alpha$ is defined by

$$\tilde{c}_\alpha:=\left|(2\alpha-1)^{1/(2-2\alpha)}-(2\alpha-1)^{(2\alpha-1)/(2-2\alpha)}\right|, \quad (65)$$

where we set $\tilde{c}_{1/2}=1$. The constant $\tilde{c}_\alpha$ is obtained by maximizing the elementary function $f_\alpha(x):=|x-x^{2\alpha-1}|$ over $x\in[0,1]$. For $\alpha>1/2$, $\alpha\neq1$, this maximum is attained at $x=(2\alpha-1)^{1/(2-2\alpha)}$, which gives Eq. (65). For $\alpha=1/2$, we use the limiting value $\tilde{c}_{1/2}=1$.

Introducing $\zeta_{s,s'}$ as

$$\zeta_{s,s'}:=|\langle\psi_{s,A},\psi_{s,B}|V_{AB}|\psi_{s',A},\psi_{s',B}\rangle|, \quad (66)$$

the term on the extreme left-hand side of Eq. (63) admits an upper bound as follows:

$$|\mathrm{tr}_{AB}(\rho_A^{\alpha-1}[V_{AB},\rho])|\le\tilde{c}_\alpha\left(\sum_{s<s'}\lambda_s^{2\alpha}\zeta_{s,s'}+\sum_{s>s'}\lambda_{s'}^{2\alpha}\zeta_{s,s'}\right)$$
$$=2\tilde{c}_\alpha\sum_{s<s'}\lambda_s^{2\alpha}\zeta_{s,s'}. \quad (67)$$

From Definition 1 for the SE strength, $V_{AB}|\psi_{s,A},\psi_{s,B}\rangle$ can be rewritten as

$$V_{AB}|\psi_{s,A},\psi_{s,B}\rangle=\sum_{s'}\lambda_{s'}^{(s)}|\tilde{\psi}_{s',A},\tilde{\psi}_{s',B}\rangle,\quad \sum_{s'}\lambda_{s'}^{(s)}\le\bar{\mathcal{J}}(V_{AB}). \quad (68)$$

Therefore, applying inequality (22) in Lemma 1, we obtain

$$\sum_{s':\,s'>s}\zeta_{s,s'}\le\sum_{s'}|\langle\psi_{s',A},\psi_{s',B}|V_{AB}|\psi_{s,A},\psi_{s,B}\rangle|\le\bar{\mathcal{J}}(V_{AB}) \quad (69)$$

for any s. The above upper bound reduces inequality (67) to

$$|\mathrm{tr}_{AB}(\rho_A^{\alpha-1}[V_{AB},\rho])|\le2\tilde{c}_\alpha\bar{\mathcal{J}}(V_{AB})\sum_s\lambda_s^{2\alpha}$$
$$=2\tilde{c}_\alpha\bar{\mathcal{J}}(V_{AB})\mathrm{tr}_A(\rho_A^\alpha). \quad (70)$$

By applying the above inequality to Eq. (61), we obtain

$$\left|\frac{d}{dt}E_\alpha(\psi_t)\right|\le\frac{2\tilde{c}_\alpha\alpha}{|1-\alpha|}\bar{\mathcal{J}}(V_{AB})=c_\alpha\bar{\mathcal{J}}(V_{AB}). \quad (71)$$

Integrating and using finite-dimensional continuity as $\alpha\downarrow1/2$, $\alpha\to1$, or $\alpha\to\infty$ gives, for $s\le t$,

$$|E_\alpha(\psi_t)-E_\alpha(\psi_s)|\le c_\alpha\int_s^t\bar{\mathcal{J}}(V_{AB}(u))\,du,$$

with $c_{1/2}=2$, $c_1=4/e$, and $c_\infty=2$. These bounds imply the end-point derivative estimates almost everywhere. This completes the proof. ∎

From Theorem 1, we can deduce the following corollary, which will be useful in subsequent discussions:

Corollary 6. For a Hamiltonian H of the form given in Eq. (1), the SE strength $\bar{\mathcal{J}}(e^{-iHt})$ is upper-bounded by

$$\bar{\mathcal{J}}(e^{-iHt}) \le e^{\bar{\mathcal{J}}(V_{AB})|t|}. \tag{72}$$

We note that the upper bound is generalized to arbitrary time-dependent Hamiltonians.

Proof of Corollary 6. It suffices to take $t \ge 0$; replacing H by $-H$ covers negative times. Let us consider an arbitrary product state $|\phi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$. Then, the $(1/2)$-Rényi entanglement of the time-evolved state $|\phi_t\rangle = e^{-iHt}|\phi\rangle$ satisfies the upper bound

$$E_{1/2}(\phi_t) \le \int_0^t \left|\frac{d}{dt_1} E_{1/2}(\phi_{t_1})\right| dt_1 \le 2\bar{\mathcal{J}}(V_{AB})t, \tag{73}$$

where we use $c_{1/2}=2$, i.e., $|dE_{1/2}(\phi_t)/dt| \le 2\bar{\mathcal{J}}(V_{AB})$. We then define the Schmidt decomposition of $|\phi_t\rangle$ as follows:

$$|\phi_t\rangle = \sum_s \lambda_s(t) |\phi^{(t)}_{A,s}\rangle \otimes |\phi^{(t)}_{B,s}\rangle, \tag{74}$$

from which we obtain $E_{1/2}(\phi_t)$ as

$$E_{1/2}(\phi_t) = 2\log\left(\sum_s \lambda_s(t)\right). \tag{75}$$

Therefore, we obtain

$$\sum_s \lambda_s(t) = e^{E_{1/2}(\phi_t)/2} \le e^{\bar{\mathcal{J}}(V_{AB})t}. \tag{76}$$

Since this holds for any product initial state $|\phi\rangle$, the main inequality (72) follows from the definition (16), i.e., $\bar{\mathcal{J}}(\Phi_{AB}) := \sup_\phi \sum_s \lambda_s(\phi)$. This completes the proof. ∎

D. Tightness at $\alpha = 1/2$

We show here that the coefficient 2 in Theorem 1 at $\alpha = 1/2$ is tight. We prove the following proposition:

Proposition 7. Let A and B each consist of $n+1$ qudits of dimension $M+1$. Then, in the limit of $n \to \infty$, there exists quantum dynamics such that the average entanglement rate saturates the spectral-SIE bound (52) over a finite-time interval:

$$\left|\frac{E_{1/2}(\psi_t) - E_{1/2}(\psi_0)}{t}\right| \ge 2\bar{\mathcal{J}}(V_{A_0B_0}) - \frac{2[\bar{\mathcal{J}}(V_{A_0B_0})]^2 t}{n} \tag{77}$$

for

$$0 < t \le \frac{n\sqrt{M}}{2\bar{\mathcal{J}}(V_{A_0B_0})}. \tag{78}$$

In inequality (77), the first term dominates when $n \gg \bar{\mathcal{J}}(V_{A_0B_0})t$, or, equivalently, $t \ll n/\bar{\mathcal{J}}(V_{A_0B_0})$.

1. Proof of Proposition 7

From the given assumptions, the subsystems A and B are composed of $n+1$ qudits with a dimension $M+1$, respectively:

$$A = A_0 \sqcup A_1 \sqcup \cdots \sqcup A_n, \qquad B = B_0 \sqcup B_1 \sqcup \cdots \sqcup B_n \tag{79}$$

with

$$\begin{aligned} \mathcal{D}_{A_0} &= \mathcal{D}_{A_1} = \cdots = \mathcal{D}_{A_n} = M+1, \\ \mathcal{D}_{B_0} &= \mathcal{D}_{B_1} = \cdots = \mathcal{D}_{B_n} = M+1. \end{aligned} \tag{80}$$

Here, take $J > 0$ and consider the interaction

$$V_{A_0B_0} = \sum_{j=1}^{M} J(|j_{A_0}, j_{B_0}\rangle\langle 0_{A_0}, 0_{B_0}| + \text{H.c.}). \tag{81}$$

Since this operator is diagonalizable, we obtain

$$\begin{aligned} e^{-iV_{A_0B_0}t}|0_{A_0}, 0_{B_0}\rangle &= \cos(\sqrt{M}Jt)|0_{A_0}, 0_{B_0}\rangle \\ &\quad - i\sum_{j=1}^{M} \frac{\sin(\sqrt{M}Jt)}{\sqrt{M}} |j_{A_0}, j_{B_0}\rangle \end{aligned} \tag{82}$$

for any t. Also, we consider the Hamiltonian $H_{A,s}$ and $H_{B,s}$ that realizes a swap operation between A_0 (resp., B_0) and A_s (resp., B_s). Because the norm of $H_{A,s}$ and $H_{B,s}$ can be arbitrarily large, we can let

$$\begin{aligned} e^{-iH_{A,s}\varepsilon} &= \sum_{j,j'=0}^{M} |j'_{A_0}, j_{A_s}\rangle\langle j_{A_0}, j'_{A_s}|, \\ e^{-iH_{B,s}\varepsilon} &= \sum_{j,j'=0}^{M} |j'_{B_0}, j_{B_s}\rangle\langle j_{B_0}, j'_{B_s}| \end{aligned} \tag{83}$$

for an infinitesimally small ε. We recall that there are no constraints on the form of H_A and H_B in the Hamiltonian (1).

Then, we construct the quantum dynamics using the time-dependent Hamiltonian as follows:

$$U_{0\to t'} := \prod_{s=1}^{n} e^{-i(H_{A,s}+H_{B,s})\varepsilon} e^{-iV_{A_0B_0}t/n}, \tag{84}$$

where $t' = t + n\varepsilon$, which can be made arbitrarily close to t by taking the limit $\varepsilon \to +0$. We choose the initial state $|\psi_0\rangle$ as

$$|\psi_0\rangle = \bigotimes_{s=0}^{n} |0_{A_s}, 0_{B_s}\rangle. \tag{85}$$

By applying the unitary time evolution $U_{0\to t'}$, we have

$$\begin{aligned} |\psi_t\rangle &= U_{0\to t'}|\psi_0\rangle = |0_{A_0}, 0_{B_0}\rangle \otimes \prod_{s=1}^{n} |\psi_{A_s,B_s}(t/n)\rangle, \\ |\psi_{A_s,B_s}(x)\rangle &= \cos\left(\sqrt{M}Jx\right)|0_{A_s}, 0_{B_s}\rangle \\ &\quad - i\sum_{j=1}^{M} \frac{\sin(\sqrt{M}Jx)}{\sqrt{M}} |j_{A_s}, j_{B_s}\rangle, \end{aligned} \tag{86}$$

where we use Eqs. (82) and (83). Each of the quantum states $\{|\psi_{A_s,B_s}(t/n)\rangle\}_{s=1}^{n}$ is given in the form of the Schmidt decomposition.

Then, in the limit of $\varepsilon \to +0$, the (1/2)-Rényi-entanglement entropy for $|\psi_t\rangle$ is given by

$$\begin{aligned} E_{1/2}(\psi_t) &= \sum_{s=1}^{n} E_{1/2}(\psi_{A_s,B_s}(t/n)) \\ &= 2n\log\left(\cos(\sqrt{M}Jt/n) + M\frac{|\sin(\sqrt{M}Jt/n)|}{\sqrt{M}}\right) \\ &= 2n\log(\cos(z) + \sqrt{M}\sin(z)), \end{aligned} \tag{87}$$

where we set $z = \sqrt{M}Jt/n$ which satisfies $0 \le z \le 1/2$ from the condition (78). Note that the conditions $\cos(z), \sin(z) \ge 0$ for $0 \le z \le 1/2$ were used in the above equation. By differentiating with respect to z, we obtain

$$\frac{d}{dz}\log(\cos(z) + \sqrt{M}\sin(z)) = \frac{\sqrt{M}\cos(z) - \sin(z)}{\cos(z) + \sqrt{M}\sin(z)} \ge 0, \tag{88}$$

where we use $M \ge 1$ and $z \le 1/2$. Hence, by using

$$\begin{aligned} &\frac{\sqrt{M}\cos(z) - \sin(z)}{\cos(z) + \sqrt{M}\sin(z)} \ge \sqrt{M} - (M+1)z \\ &\qquad \text{for } 0 \le z \le 1/2, \quad M \ge 1, \end{aligned} \tag{89}$$

we obtain

$$\begin{aligned} E_{1/2}(\psi_t) &= 2n\log\left(\cos(z) + \sqrt{M}\sin(z)\right) \\ &\ge 2n\left(\sqrt{M}z - \frac{M+1}{2}z^2\right) \\ &\ge 2MJt - \frac{2M^2J^2t^2}{n}. \end{aligned} \tag{90}$$

Finally, we prove the equation

$$\bar{\mathcal{J}}(V_{A_0B_0}) = MJ, \tag{91}$$

which reduces inequality (90) to the desired form (77).

We recall that the SE strength is defined by taking the supremum over arbitrary product states after attaching arbitrary ancillas A' and B'. Hence, let

$$|\phi\rangle = |\phi_{A_0A'}\rangle \otimes |\phi_{B_0B'}\rangle \tag{92}$$

be an arbitrary product state across the bipartition $A_0A'|B_0B'$. Expanding the two factors in the fixed bases used in $V_{A_0B_0}$, we write

$$\begin{aligned} |\phi_{A_0A'}\rangle &= \sum_{j=0}^{M} a_j |j_{A_0}\rangle \otimes |\xi_{j,A'}\rangle, \\ |\phi_{B_0B'}\rangle &= \sum_{j=0}^{M} b_j |j_{B_0}\rangle \otimes |\zeta_{j,B'}\rangle, \end{aligned} \tag{93}$$

where $\sum_{j=0}^{M} |a_j|^2 = \sum_{j=0}^{M} |b_j|^2 = 1$, and the conditional states $|\xi_{j,A'}\rangle$ and $|\zeta_{j,B'}\rangle$ are normalized whenever the corresponding coefficients are nonzero. Here, no orthogonality of the conditional ancilla states is assumed.

Applying $V_{A_0B_0}$ of Eq. (81), we obtain

$$\begin{aligned} V_{A_0B_0}|\phi\rangle &= J\sum_{j=1}^{M} a_j b_j (|0_{A_0}\rangle \otimes |\xi_{j,A'}\rangle) \otimes (|0_{B_0}\rangle \otimes |\zeta_{j,B'}\rangle) \\ &\quad + J\sum_{j=1}^{M} a_0 b_0 (|j_{A_0}\rangle \otimes |\xi_{0,A'}\rangle) \otimes (|j_{B_0}\rangle \otimes |\zeta_{0,B'}\rangle). \end{aligned} \tag{94}$$

This is a product-state expansion across $A_0A'|B_0B'$, not necessarily a Schmidt decomposition. Therefore, by Lemma 1, the sum of the Schmidt coefficients of $V_{A_0B_0}|\phi\rangle$ is upper-bounded by the sum of the absolute values of the coefficients in the above expansion:

$$\sum_s \lambda_s \le J\sum_{j=1}^{M} |a_j b_j| + J\sum_{j=1}^{M} |a_0 b_0|. \tag{95}$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\sum_s \lambda_s \le J\left(\sum_{j=1}^{M}|a_j|^2+\sum_{j=1}^{M}|a_0|^2\right)^{1/2}\times\left(\sum_{j=1}^{M}|b_j|^2+\sum_{j=1}^{M}|b_0|^2\right)^{1/2} = J[1+(M-1)|a_0|^2]^{1/2}[1+(M-1)|b_0|^2]^{1/2} \le MJ. \tag{96}$$

Since this holds for arbitrary ancillas A', B' and arbitrary product states $|\phi\rangle = |\phi_{A_0A'}\rangle \otimes |\phi_{B_0B'}\rangle$, we have

$$\bar{\mathcal{J}}(V_{A_0B_0}) \le MJ. \tag{97}$$

Conversely, the upper bound is attained by choosing

$$|\phi_{A_0A'}\rangle = |0_{A_0}\rangle \otimes |\xi_{A'}\rangle, \qquad |\phi_{B_0B'}\rangle = |0_{B_0}\rangle \otimes |\zeta_{B'}\rangle, \tag{98}$$

with arbitrary normalized ancilla states $|\xi_{A'}\rangle$ and $|\zeta_{B'}\rangle$. Then,

$$V_{A_0B_0}|\phi_A,\phi_B\rangle = J\sum_{j=1}^{M}(|j_{A_0}\rangle \otimes |\xi_{A'}\rangle) \otimes (|j_{B_0}\rangle \otimes |\zeta_{B'}\rangle). \tag{99}$$

The states $\{|j_{A_0}\rangle \otimes |\xi_{A'}\rangle\}_{j=1}^{M}$ and $\{|j_{B_0}\rangle \otimes |\zeta_{B'}\rangle\}_{j=1}^{M}$ are orthonormal sets. Hence, the above expression is already a Schmidt decomposition with M Schmidt coefficients all equal to J. Therefore,

$$\bar{\mathcal{J}}(V_{A_0B_0}) \ge MJ. \tag{100}$$

Combining the two bounds gives Eq. (91). This reduces inequality (90) to the main lower bound (77). Moreover, the condition $z = \sqrt{M}Jt/n \le 1/2$ becomes

$$t \le \frac{n\sqrt{M}}{2\bar{\mathcal{J}}(V_{A_0B_0})}. \tag{101}$$

E. Optimality of the threshold: Unbounded entanglement rate for $\alpha < 1/2$

We next show that the threshold $\alpha = 1/2$ cannot, in principle, be removed. As shown below, we explicitly demonstrate that the average entanglement rate is unbounded for $\alpha < 1/2$ for a finite-time interval.

Proposition 8. For $0 < \alpha < 1/2$, let A and B each consist of two qudits $\{A_0, A_1\}$ and $\{B_0, B_1\}$, respectively, where $\mathcal{D}_{A_0} = \mathcal{D}_{B_0} = 3$ and $\mathcal{D}_{A_1} = \mathcal{D}_{B_1} = \mathcal{D}_0 + 1$. Then, there exists a quantum dynamics such that

$$|E_\alpha(\psi_t) - E_\alpha(\psi_{t=0})| \ge \frac{1-2\alpha}{1-\alpha}\log(\mathcal{D}_0) + \frac{1}{1-\alpha}\log\left[\alpha\left(\frac{\bar{\mathcal{J}}(V_{A_0B_0})t}{2}\right)^{2\alpha}\right] \tag{102}$$

under an appropriate initial state $|\psi_{t=0}\rangle$ [see Eq. (112) below], where we assume

$$0 < t \le \frac{1}{\bar{\mathcal{J}}(V_{A_0B_0})}. \tag{103}$$

Remark. By setting $\bar{\mathcal{J}}(V_{A_0B_0}) = 1$ and $t = 1$, inequality (102) yields

$$\frac{|E_\alpha(\psi_t) - E_\alpha(\psi_{t=0})|}{t} \ge \frac{1-2\alpha}{1-\alpha}\log(\mathcal{D}_0) + \frac{1}{1-\alpha}\log(\alpha 2^{-2\alpha}). \tag{104}$$

From the above bound, as long as $\alpha < 1/2$, the average entanglement rate can be made arbitrarily large regardless of $\bar{\mathcal{J}}(V_{A_0B_0}) = 1$. Thus, information about the boundary interaction alone does not suffice to upper-bound the Rényi entanglement for $\alpha < 1/2$. This point is compared to the geometrically local interacting systems, which will be discussed in Sec. IX.

1. Proof of Proposition 8

For $J > 0$, define the interaction between A_0 and B_0 by

$$V_{A_0B_0} := J(|1_{A_0}, 1_{B_0}\rangle\langle 0_{A_0}, 0_{B_0}| + \text{H.c.}). \tag{105}$$

In this setup, we prove that $\bar{\mathcal{J}}(V_{A_0B_0})$ is equal to J:

$$\bar{\mathcal{J}}(V_{A_0B_0}) = J, \tag{106}$$

where the proof proceeds in a manner analogous to that of Eq. (91). From the condition (103), we get

$$Jt \le 1. \tag{107}$$

We introduce $\mathcal{D}_0$ sets of Hamiltonians $\{H_{s,A} + H_{s,B}\}_{s=1}^{\mathcal{D}_0}$, where $H_{s,A}$ and $H_{s,B}$ are defined as

$$H_{s,A} = -J_A(|s_{A_1}, 2_{A_0}\rangle\langle 0_{A_1}, 1_{A_0}| + \text{H.c.}),$$
$$H_{s,B} = -J_B(|s_{B_1}, 2_{B_0}\rangle\langle 0_{B_1}, 1_{B_0}| + \text{H.c.}). \tag{108}$$

Here, J_A and J_B can be chosen arbitrarily large, and hence we set them such that

$$e^{-iH_{s,A}\varepsilon}|0_{A_1}, 1_{A_0}\rangle = i|s_{A_1}, 2_{A_0}\rangle,$$
$$e^{-iH_{s,B}\varepsilon}|0_{B_1}, 1_{B_0}\rangle = i|s_{B_1}, 2_{B_0}\rangle, \tag{109}$$

where we choose $\varepsilon J_A = \varepsilon J_B = \pi/2$. Also, from Eq. (105), we get

$$e^{-iV_{A_0B_0}\Delta t}|0_{A_0}, 0_{B_0}\rangle = \cos(J\Delta t)|0_{A_0}, 0_{B_0}\rangle - i\sin(J\Delta t)|1_{A_0}, 1_{B_0}\rangle. \tag{110}$$

We subdivide the time interval into $\mathcal{D}_0$ segments as $t/\mathcal{D}_0 = \Delta t$ and construct the unitary time evolution by

$$U_{0\to t'} := \prod_{s=1}^{\mathcal{D}_0} e^{-i(H_{s,A}+H_{s,B})\varepsilon} e^{-iV_{A_0B_0}\Delta t}, \tag{111}$$

where $t' = t + \mathcal{D}_0\varepsilon$. By construction, t' can be made arbitrarily close to t. We begin with the quantum state of

$$|\psi_0\rangle = |0_{A_1}, 0_{A_0}, 0_{B_0}, 0_{B_1}\rangle. \tag{112}$$

As the first process, we have

$$\begin{aligned}|\psi_1\rangle &:= e^{-i(H_{1,A}+H_{1,B})\varepsilon} e^{-iV_{A_0B_0}\Delta t}|\psi_0\rangle \\ &= e^{-i(H_{1,A}+H_{1,B})\varepsilon}(\cos(J\Delta t)|0_{A_1}, 0_{A_0}, 0_{B_0}, 0_{B_1}\rangle \\ &\quad - i\sin(J\Delta t)|0_{A_1}, 1_{A_0}, 1_{B_0}, 0_{B_1}\rangle) \\ &= \cos(J\Delta t)|\psi_0\rangle + i\sin(J\Delta t)|1_{A_1}, 2_{A_0}, 2_{B_0}, 1_{B_1}\rangle.\end{aligned} \tag{113}$$

The subsequent unitary dynamics $e^{-i(H_{s,A}+H_{s,B})\varepsilon} e^{-iV_{A_0B_0}\Delta t}$ ($s \geq 2$) does not change $|1_{A_1}, 2_{A_0}, 2_{B_0}, 1_{B_1}\rangle$, i.e.,

$$\begin{aligned}&e^{-i(H_{s,A}+H_{s,B})\varepsilon} e^{-iV_{A_0B_0}\Delta t}|1_{A_1}, 2_{A_0}, 2_{B_0}, 1_{B_1}\rangle \\ &= |1_{A_1}, 2_{A_0}, 2_{B_0}, 1_{B_1}\rangle\end{aligned} \tag{114}$$

for $s \geq 2$. In the next step, we have

$$\begin{aligned}&e^{-i(H_{2,A}+H_{2,B})\varepsilon} e^{-iV_{A_0B_0}\Delta t}|\psi_1\rangle \\ &= \cos(J\Delta t) e^{-i(H_{2,A}+H_{2,B})\varepsilon} e^{-iV_{A_0B_0}\Delta t}|\psi_0\rangle \\ &\quad + i\sin(J\Delta t)|1_{A_1}, 2_{A_0}, 2_{B_0}, 1_{B_1}\rangle \\ &= \cos^2(J\Delta t)|\psi_0\rangle + i\cos(J\Delta t)\sin(J\Delta t)|2_{A_1}, 2_{A_0}, 2_{B_0}, 2_{B_1}\rangle \\ &\quad + i\sin(J\Delta t)|1_{A_1}, 2_{A_0}, 2_{B_0}, 1_{B_1}\rangle.\end{aligned} \tag{115}$$

Repeating this type of process iteratively in an analogous manner, we obtain

$$U_{0\to t'}|\psi_0\rangle = \cos^{\mathcal{D}_0}(J\Delta t)|0_{A_1}, 0_{A_0}, 0_{B_0}, 0_{B_1}\rangle + i\sum_{s=1}^{\mathcal{D}_0}\cos^{s-1}(J\Delta t)\sin(J\Delta t)|s_{A_1}, 2_{A_0}, 2_{B_0}, s_{B_1}\rangle. \tag{116}$$

The above expression gives the Schmidt decomposition of $U_{0\to t'}|\psi_0\rangle$, and hence the Rényi entanglement is given by

$$E_\alpha(\psi_t) = \frac{1}{1-\alpha}\log\left(\cos^{2\mathcal{D}_0\alpha}(J\Delta t) + \sum_{s=1}^{\mathcal{D}_0}\cos^{2(s-1)\alpha}(J\Delta t)\sin^{2\alpha}(J\Delta t)\right), \tag{117}$$

where we take the limit of $\varepsilon \to +0$ in Eq. (116).

Equation (107) imposes the constraint $J\Delta t \leq Jt \leq 1$. Because of $\cos(x) \geq 1 - x/2$ and $\sin(x) \geq x/2$ for $0 \leq x \leq 1$, the following relation holds:

$$\begin{aligned}E_\alpha(\psi_t) &\geq \frac{1}{1-\alpha}\log\left(\sum_{s=1}^{\mathcal{D}_0+1}(1 - J\Delta t/2)^{2(s-1)\alpha}(J\Delta t/2)^{2\alpha}\right) \\ &\geq \frac{1}{1-\alpha}\log\left((J\Delta t/2)^{2\alpha}\frac{1-(1-J\Delta t/2)^{2\mathcal{D}_0\alpha}}{J\Delta t/2}\right).\end{aligned} \tag{118}$$

Here, we note that the condition (107) yields

$$\frac{J\Delta t}{2} = \frac{Jt}{2\mathcal{D}_0} \leq \frac{1}{2\mathcal{D}_0} \leq \frac{1}{2\mathcal{D}_0\alpha} \quad \text{for } 0 < \alpha < 1. \tag{119}$$

Replacing $J\Delta t/2$ with x and treating x as a variable on the rhs of Eq. (118), we are led to consider the function of

$$x^{2\alpha}\frac{1-(1-x)^{2\mathcal{D}_0\alpha}}{x} \quad \text{for } 0 \leq x \leq \frac{1}{2\mathcal{D}_0\alpha}. \tag{120}$$

Since $(1-x)^{2\mathcal{D}_0\alpha} \leq 1 - \mathcal{D}_0\alpha x$, we obtain

$$x^{2\alpha}\frac{1-(1-x)^{2\mathcal{D}_0\alpha}}{x} \geq \mathcal{D}_0\alpha x^{2\alpha}. \tag{121}$$

Therefore, we reduce the lower bound (118) to the main inequality:

$$\begin{aligned}E_\alpha(\psi_t) &\geq \frac{1}{1-\alpha}\log\left[\mathcal{D}_0\alpha\left(\frac{Jt}{2\mathcal{D}_0}\right)^{2\alpha}\right] \\ &= \frac{1-2\alpha}{1-\alpha}\log(\mathcal{D}_0) + \frac{1}{1-\alpha}\log\left[\alpha\left(\frac{\bar{\mathcal{J}}(V_{A_0B_0})t}{2}\right)^{2\alpha}\right],\end{aligned} \tag{122}$$

where we use Eq. (106) in the last equation. Note that $E_\alpha(\psi_{t=0}) = 0$ from the definition (112). This completes the proof. ∎

VI. OPERATOR APPROXIMATION VS ENTANGLEMENT GENERATION

Here, we consider the following question: "Does a small spectral-entangling capability imply an efficient low-Schmidt-rank approximation of the corresponding operator?" In general, the answer to this question is no.

Consequently, we obtain a complexity separation—measured in terms of the Schmidt rank required for a given approximation error—between approximating the time-evolution unitary and approximating time-evolved states.

This distinction is physically important because entanglement generation and operator approximability are not equivalent notions. For quantum states, control of the Schmidt spectrum directly leads to efficient low-Schmidt-rank truncation. By contrast, a small spectral-entangling capability of an operator constrains the entanglement that it can generate in states, but does not necessarily constrain the Schmidt-rank complexity of the operator itself. Hence, efficient approximation at the state level does not automatically lift to an efficient MPO or low-Schmidt-rank approximation of the full time-evolution operator.

The purpose of this section is therefore twofold. First, we show that bounded SE strength, while sufficient to control state-level entanglement generation, does not automatically imply efficient operator-level approximation in the operator norm (Proposition 9). This disproves a naive equivalence between spectral-entanglement generation and low-Schmidt-rank approximability of the full unitary. Second, motivated by this separation, we introduce the α-SE strength $\bar{\mathcal{J}}_\alpha$ (Definition 2) and formulate Conjecture 2 as a possible sufficient condition for efficient operator approximation. Subsequent results on Conjecture 2 are summarized in the Note added. Here, Theorem 3 uses an explicit structural assumption on the interaction decomposition (Assumption 10). This theorem provides a rigorous positive result in the direction suggested by the conjecture and will later be used to obtain the Gibbs-state approximation bound in Theorem 7.

A. Lower bound on the operator low-rank approximation

Proposition 9. Let us consider a class of quantum dynamics $e^{-iH_{AB}t}$ such that

$$\bar{\mathcal{J}}(V_{AB}) = 1. \tag{123}$$

For each fixed D, choosing the Hilbert-space dimension sufficiently large gives dynamics $e^{-iH_{AB}t}$ such that, for $t \geq 0$,

$$\inf_{U_{AB,D}} \|e^{-iH_{AB}t} - U_{AB,D}\| \geq 1 + \frac{5t}{4} - e^t, \tag{124}$$

where $\inf_{U_{AB,D}}$ is taken over all the operators such that the Schmidt rank is smaller than or equal to D.

Remark. In this setup, for any initial state with dimension-independent Rényi-1/2 entanglement, the corresponding time-evolved state can be well approximated by a finite Schmidt rank D, whereas the full unitary time evolution admits no such approximation.

The lower bound is intended only to be nontrivial on a constant time interval (e.g., for t such that $1 + 5t/4 - e^t > 0$); this suffices to establish the separation between state-level approximability and operator-level approximability.

Here, D is fixed independently of dimension. For the example (125), write $\tilde{V} = \tilde{V}_{A_0B_0}$ and note that $\tilde{V}^2 = \tilde{V}$, so $e^{-it\tilde{V}} = 1 + (e^{-it} - 1)\tilde{V}$. For $2 \leq D \leq N$, Eqs. (142) and (143) give a diagonal W_{D-1} with $\mathrm{SR}(W_{D-1}) \leq D - 1$ and $\|\tilde{V} - W_{D-1}\| \leq 2\sqrt{\log N/(D-1)}$. Thus, $U_{AB,D} = 1 + (e^{-it} - 1)W_{D-1}$ has operator Schmidt rank at most D and error at most $2|e^{-it} - 1|\sqrt{\log N/(D-1)}$. At fixed time, this error bound is $O(\sqrt{\log N/D})$, tending to zero when $D/\log N \to \infty$; the identity accounts for the additional rank one. Low-rank approximations of the core quantum Fourier transform, with bit reversal removed, have also been studied [64,65]. An interesting open question is whether dynamics with SE strength bounded independently of dimension can require an operator Schmidt rank growing as a positive power of $\mathcal{D}_{AB}$ for approximation at a fixed nonzero time and fixed operator-norm accuracy.

B. Proof of Proposition 9

Consider a four-qudit system comprising subsystems A_0, A_1, B_0, and B_1, where $\mathcal{D}_{A_0} = \mathcal{D}_{B_0} = N$ with N chosen sufficiently large. We then consider the Hamiltonian of the form

$$H_{AB} = \tilde{V}_{A_0B_0} = \sum_{s=1}^{N} (|s\rangle\langle s|)_{A_0} \otimes (|s\rangle\langle s|)_{B_0}, \tag{125}$$

where we choose the boundary interaction V_{AB} as the above $\tilde{V}_{A_0B_0}$. By the same calculation as in Eq. (97) used to prove Eq. (91), we obtain

$$\bar{\mathcal{J}}(\tilde{V}_{A_0B_0}) = 1. \tag{126}$$

We then consider a time evolution such that

$$e^{-iH_{AB}t} = 1 - i\tilde{V}_{A_0B_0}t + \sum_{m=2}^{\infty} \frac{(-it)^m}{m!} \tilde{V}^m_{A_0B_0}, \tag{127}$$

which satisfies

$$\|e^{-iH_{AB}t} - (1 - i\tilde{V}_{A_0B_0}t)\| \leq \sum_{m=2}^{\infty} \frac{t^m}{m!} \|\tilde{V}_{A_0B_0}\|^m \leq e^t - 1 - t, \tag{128}$$

where we use $\|\tilde{V}_{A_0B_0}\| = 1$.

We then define $U_{AB,D}$ as the optimal approximation of $e^{-iH_{AB}t}$. Using this, we find the following relations:

$$\begin{aligned}\|e^{-iH_{AB}t} - U_{AB,D}\| &\geq \|1 - i\tilde{V}_{A_0B_0}t - U_{AB,D}\| \\ &\quad - \|e^{-iH_{AB}t} - (1 - i\tilde{V}_{A_0B_0}t)\| \\ &\geq t \inf_{W_{2D}} \|\tilde{V}_{A_0B_0} - W_{2D}\| - (e^t - 1 - t).\end{aligned} \tag{129}$$

Note that we used

$$\begin{aligned}&\|1 - i\tilde{V}_{A_0B_0}t - U_{AB,D}\| \\ &= \|1 - i\tilde{V}_{A_0B_0}t - [\mathrm{Re}(U_{AB,D}) + i\,\mathrm{Im}(U_{AB,D})]\| \\ &\geq t\left\|\tilde{V}_{A_0B_0} + \frac{1}{t}\mathrm{Im}(U_{AB,D})\right\| \\ &\geq t \inf_{W_{2D}} \|\tilde{V}_{A_0B_0} - W_{2D}\|.\end{aligned} \tag{130}$$

Here the infimum is over Hermitian W_{2D} with $\mathrm{SR}(W_{2D}) \leq 2D$. For $t > 0$, $-\mathrm{Im}(U_{AB,D})/t$ belongs to this class, and taking the Hermitian imaginary part is contractive in operator norm; the case $t = 0$ is immediate. We use $\mathrm{Re}(U) = (U + U^\dagger)/2$ and $\mathrm{Im}(U) = (U - U^\dagger)/(2i)$.

The remaining task is to estimate a lower bound for the low-rank approximation of $\tilde{V}_{A_0B_0}$. In detail, we calculate

$$\delta_{2D} \coloneqq \inf_{W_{2D}} \|\tilde{V}_{A_0B_0} - W_{2D}\|. \tag{131}$$

We define $\tilde{W}_{2D}$ as

$$\tilde{W}_{2D} \coloneqq \arg\inf_{W_{2D}} \|\tilde{V}_{A_0B_0} - W_{2D}\|, \tag{132}$$

and prove that the form of $\tilde{W}_{2D}$ should be given by

$$\sum_{s,} \sum_{s'=1}^{N} W_{s,s'}(|s\rangle\langle s|)_{A_0} \otimes (|s'\rangle\langle s'|)_{B_0}, \tag{133}$$

where $W_{s,s'} \in \mathbb{R}$ for all s, s'.

For the admissible operators,

$$W_{2D} = W_{2D}^\dagger = \mathrm{Re}(W_{2D}), \qquad \mathrm{Im}(W_{2D}) = 0. \tag{134}$$

Hence,

$$\|\tilde{V}_{A_0B_0} - W_{2D}\| = \sup_{\|\psi\|=1} |\langle\psi|(\tilde{V}_{A_0B_0} - W_{2D})|\psi\rangle|. \tag{135}$$

The normalized partial traces and local dephasing below preserve Hermiticity, do not increase operator Schmidt rank, and do not increase the error.

Next, we consider the component acting nontrivially on the subset A_1B_1 and prove that it cannot contribute to the error reduction. Let W_{2D} be Hermitian. We decompose the operator W_{2D} into

$$W_{2D} = W_{2D,A_0B_0} + \delta W_{2D,AB}, \qquad \mathrm{tr}_{A_1B_1}(\delta W_{2D,AB}) = 0. \tag{136}$$

Note that W_{2D,A_0B_0} abbreviates $W_{2D,A_0B_0} \otimes \hat{\mathrm{I}}_{A_1B_1}$. We choose a quantum state ρ defined as

$$\rho = |\psi_{A_0B_0}\rangle\langle\psi_{A_0B_0}| \otimes \frac{\hat{\mathrm{I}}_{A_1B_1}}{\mathcal{D}_{A_1B_1}}, \tag{137}$$

where the state $|\psi_{A_0B_0}\rangle$ satisfies

$$\|\tilde{V}_{A_0B_0} - W_{2D,A_0B_0}\| = |\langle\psi_{A_0B_0}|\tilde{V}_{A_0B_0} - W_{2D,A_0B_0}|\psi_{A_0B_0}\rangle|. \tag{138}$$

For any $\delta W_{2D,AB}$, the following inequality holds:

$$\begin{aligned}\|\tilde{V}_{A_0B_0} - W_{2D,A_0B_0}\| &= |\mathrm{tr}[\rho(\tilde{V}_{A_0B_0} - W_{2D,A_0B_0} - \delta W_{2D,AB})]| \\ &\leq \|\tilde{V}_{A_0B_0} - W_{2D,A_0B_0} - \delta W_{2D,AB}\|.\end{aligned} \tag{139}$$

This shows that $\|\tilde{V}_{A_0B_0} - W_{2D,A_0B_0} - \delta W_{2D,AB}\|$ is still larger than or equal to $\|\tilde{V}_{A_0B_0} - W_{2D,A_0B_0}\|$. We thus conclude that the existence of $\delta W_{2D,AB}$ cannot reduce the error of $\|\tilde{V}_{A_0B_0} - W_{2D,A_0B_0}\|$ for an arbitrary choice of W_{2D,A_0B_0}.

Finally, we consider the off-diagonal part of the operator on the subset A_0B_0. We also decompose W_{2D} into

$$W_{2D} = W_{\mathrm{diag}} + W_{\mathrm{off}}, \tag{140}$$

where W_{diag} is given in the form of Eq. (133). For any choice of W_{diag}, we can also prove that the existence of W_{off} does not reduce the error $\|\tilde{V}_{A_0B_0} - W_{\mathrm{diag}}\|$. Suppose that the error $\|\tilde{V}_{A_0B_0} - W_{\mathrm{diag}}\|$ attains its maximum for the quantum state $|s_0, s_0'\rangle$, which is an eigenstate of $\tilde{V}_{A_0B_0} - W_{\mathrm{diag}}$. In this situation, we have $\langle s_0, s_0'|W_{\mathrm{off}}|s_0, s_0'\rangle = 0$; hence, the error $\|\tilde{V}_{A_0B_0} - W_{\mathrm{diag}} - W_{\mathrm{off}}\|$ is nondecreasing when W_{off} is present. In conclusion, we need to consider the class of operators (133) for the solution of $\inf_{W_{2D}}\|\tilde{V}_{A_0B_0} - W_{2D}\|$.

Second, we consider the approximation error δ_{2D} in Eq. (131). For the purpose, using the fact that $\tilde{W}_{2D}$ is given in the form of Eq. (133), we consider

$$\begin{aligned}&\left\|\tilde{V}_{A_0B_0} - \sum_{s,s'=1}^{N} W_{s,s'}(|s\rangle\langle s|)_{A_0} \otimes (|s'\rangle\langle s'|)_{B_0}\right\| \\ &= \max_{1\leq s,s'\leq N} |\delta_{s,s'} - W_{s,s'}|,\end{aligned} \tag{141}$$

where the rank of the matrix $\{W_{s,s'}\}_{s,s'}$ is less than or equal to $2D$. Hence, the problem is equivalent to estimating

$$\inf_{\mathrm{rank}(W)\le 2D}\max_{1\le s,s'\le N}|\delta_{s,s'}-W_{s,s'}|. \tag{142}$$

The above quantity is known to be equal to the Kolmogorov width of the octahedron [36–38,66] denoted by $d_{2D}(B_1^N,\ell_\infty^N)$ (see also Sec. VI B 1 for a detailed review). For $1\le D<N$, from Ref. [36] and Ref. [38] Theorem 10.10 therein, using Gelfand-Kolmogorov width duality, the Kolmogorov width obeys the inequality

$$\begin{aligned}&\frac{1}{4}\min\left[\frac{2}{1+4\ln(9)}\frac{\log(eN/D)}{D},1\right]\\&\le d_D(B_1^N,\ell_\infty^N)\le 2\left(\frac{\log(N)}{D}\right)^{1/2}.\end{aligned} \tag{143}$$

For $D\ge 1$, the rank-one matrix with all entries $1/2$ gives $d_D(B_1^N,\ell_\infty^N)\le 1/2$.

Choose N sufficiently large that the minimum in Eq. (143), with D replaced by $2D$, equals 1. Then,

$$\inf_{\mathrm{rank}(W)\le 2D}\max_{1\le s,s'\le N}|\delta_{s,s'}-W_{s,s'}|\ge\frac{1}{4}, \tag{144}$$

which also provides $\inf_{W_{2D}}\|\tilde{V}_{A_0B_0}-W_{2D}\|\ge 1/4$. By combining this lower bound with inequality (129), we prove the main inequality (124). This completes the proof. ∎

1. A short review of the Kolmogorov width

We now consider $d_D(B_1^N,\ell_\infty^N)$ instead of $d_{2D}(B_1^N,\ell_\infty^N)$ for simplicity of notation. We introduce the unit ball B_1^N as follows:

$$B_1^N=\{\vec{x}\in\mathbb{R}^N:\ \|\vec{x}\|_{\ell_1}\le 1\}, \tag{145}$$

where $\|\vec{x}\|_{\ell_m}$ $(m\in\mathbb{N})$ denotes the ℓ_m norm, i.e., $\|\vec{x}\|_{\ell_m}=(\sum_j|x_j|^m)^{1/m}$. We also denote by ℓ_∞^N the space $\mathbb{R}^N$ equipped with the ℓ_∞ norm. We define the Kolmogorov width as

$$d_D(B_1^N,\ell_\infty^N):=\inf_{\substack{X_D\subset\ell_\infty^N\\ \dim(X_D)\le D}}\sup_{\vec{x}\in B_1^N}\inf_{\vec{y}\in X_D}\|\vec{x}-\vec{y}\|_{\ell_\infty}. \tag{146}$$

To connect $d_D(B_1^N,\ell_\infty^N)$ to Eq. (142), we first denote an arbitrary vector $\vec{y}$ in the space X_D by

$$\begin{aligned}\vec{y}&=\sum_{j=1}^{D}\begin{pmatrix}w_1^{(j)}\\w_2^{(j)}\\\vdots\\w_N^{(j)}\end{pmatrix}\begin{pmatrix}w_1^{(j)}&w_2^{(j)}&\cdots&w_N^{(j)}\end{pmatrix}\begin{pmatrix}x_1'\\x_2'\\\vdots\\x_N'\end{pmatrix}\\&=\sum_{j=1}^{D}\vec{w}^{(j)}\vec{w}^{(j)\mathrm{T}}\vec{x}'=W\vec{x}',\end{aligned} \tag{147}$$

with $x'\in\ell_\infty^N$, where each of the vectors $\vec{w}^{(j)}=\{w_m^{(j)}\}_{1\le m\le N}$ forms an orthonormal basis. The matrix W can be viewed as the projection onto the subspace spanned by the D vectors $\{\vec{w}^{(j)}\}_{j=1}^D$.

Then, we immediately obtain the upper bound for the Kolmogorov width as follows:

$$\begin{aligned}\inf_{\substack{X_D\subset\ell_\infty^N\\ \dim(X_D)\le D}}\sup_{\vec{x}\in B_1^N}\inf_{\vec{y}\in X_D}\|\vec{x}-\vec{y}\|_{\ell_\infty}&\le\inf_{\mathrm{rank}(W)=D}\sup_{\vec{x}\in B_1^N}\|\vec{x}-W\vec{x}\|_{\ell_\infty}\\&=\inf_{\mathrm{rank}(W)=D}\|1-W\|_{\max}\\&=\inf_{\mathrm{rank}(W)=D}\max_{1\le s,s'\le N}|\delta_{s,s'}-w_{s,s'}|,\end{aligned} \tag{148}$$

where we use $\sup_{\vec{x}\in B_1^N}\|M\vec{x}\|_{\ell_\infty}=\|M\|_{\max}:=\max_{1\le s,s'\le N}\times|M_{s,s'}|$ for an arbitrary matrix $M=\{M_{s,s'}\}$.

Next, we prove that the lhs of inequality (148) also provides the lower bound for the Kolmogorov width. We consider the choices of $\vec{x}$ as

$$\vec{x}=\vec{I}_s:=\{\overbrace{0,0,\ldots,0}^{s-1},1,\overbrace{0,\ldots,0}^{N-s}\} \tag{149}$$

for $s=1,2,\ldots,N$. For a given space X_D, we define

$$\arg\inf_{\vec{y}\in X_D}\|\vec{I}_s-\vec{y}\|_{\ell_\infty}=\vec{y}_s. \tag{150}$$

Then, by defining the rank D matrix $W(X_D)$ as

$$W(X_D)=\begin{pmatrix}\vec{y}_1&\vec{y}_2&\cdots&\vec{y}_N\end{pmatrix}, \tag{151}$$

we derive

$$\begin{aligned}&\max_{1\le s,s'\le N}|\delta_{s,s'}-[W(X_D)]_{s,s'}|\le\max_s\left\|\vec{I}_s-\vec{y}_s\right\|_{\ell_\infty}\\&=\max_s\inf_{\vec{y}\in X_D}\left\|\vec{I}_s-\vec{y}\right\|_{\ell_\infty}\\&\le\sup_{\vec{x}\in B_1^N}\inf_{\vec{y}\in X_D}\|\vec{x}-\vec{y}\|_{\ell_\infty}.\end{aligned} \tag{152}$$

This also implies

$$\begin{aligned}&\inf_{\substack{X_D\subset\ell_\infty^N\\ \dim(X_D)\le D}}\sup_{\vec{x}\in B_1^N}\inf_{\vec{y}\in X_D}\|\vec{x}-\vec{y}\|_{\ell_\infty}\\ &\ge \inf_{\substack{X_D\subset\ell_\infty^N\\ \dim(X_D)\le D}}\max_{1\le s,s'\le N}|\delta_{s,s'}-[W(X_D)]_{s,s'}|\\ &\ge \inf_{\mathrm{rank}(W)=D}\max_{1\le s,s'\le N}|\delta_{s,s'}-W_{s,s'}|. \end{aligned}\tag{153}$$

By combining the inequalities (148) and (153), we obtain the desired identity as

$$d_D(B_1^N,\ell_\infty^N)=\inf_{\mathrm{rank}(W)\le D}\max_{1\le s,s'\le N}|\delta_{s,s'}-W_{s,s'}|. \tag{154}$$

C. Conjecture on the operator low-Schmidt-rank approximation

As shown in Proposition 9, the condition that V_{AB} (as well as $e^{-iH_{AB}t}$) has small SE strength [i.e., $\bar{\mathcal{J}}(V_{AB})=1$] does not by itself guarantee that the time-evolution operator can be efficiently approximated by a low-Schmidt-rank operator. It therefore remains an important open problem to determine under what additional conditions such efficient operator approximations become possible. To address this issue, we introduce an extension of the SE strength, namely, the α-SE strength $\bar{\mathcal{J}}_\alpha$ (Definition 2), and conjecture that having small α-SE strength provides a sufficient condition for efficient low-rank approximation of the unitary dynamics $e^{-iH_{AB}t}$ (Conjecture 2 below).

We first generalize the SE strength in Definition 1.

Definition 2. (α-SE strength). Let Φ_{AB} be an arbitrary operator acting across subsystems A and B. For any product state $|\phi\rangle=|\phi_{AA'}\rangle\otimes|\phi_{BB'}\rangle$ with the ancillas A' and B', we denote the Schmidt decomposition of $\Phi_{AB}|\phi\rangle$ by

$$\Phi_{AB}|\phi\rangle=\sum_s\lambda_s(\phi)|\phi_{AA',s}\rangle\otimes|\phi_{BB',s}\rangle. \tag{155}$$

Then, the α-SE strength of Φ_{AB} is defined as

$$\bar{\mathcal{J}}_\alpha(\Phi_{AB}):=\sup_{|\phi\rangle}\left(\sum_s|\lambda_s(\phi)|^{2\alpha}\right)^{1/(2\alpha)}, \tag{156}$$

where the supremum is taken over all product states $|\phi\rangle=|\phi_{AA'}\rangle\otimes|\phi_{BB'}\rangle$.

Using this definition, we propose the following conjecture:

Conjecture 2. For any quantum operator such that

$$\bar{\mathcal{J}}_\alpha(\Phi_{AB})=1,\qquad \alpha<1/2, \tag{157}$$

there exists an efficient approximation of Φ_{AB} by an operator $\Phi_{AB,D}$ with $\mathrm{SR}(\Phi_{AB,D})=D$:

$$\inf_{\Phi_{AB,D}}\|\Phi_{AB}-\Phi_{AB,D}\|\le g_\alpha(D), \tag{158}$$

where the function $g_\alpha(D)$ decays as a power law in D, with the decay rate depending on α.

Remark. A frequently asked question is why the standard operator Schmidt decomposition cannot be directly applied to address Conjecture 2. Recall that the operator Schmidt decomposition expands a given operator O as

$$O=\sum_s\mu_s O_{A,s}\otimes O_{B,s}, \tag{159}$$

where $\{O_{A,s}\}$ and $\{O_{B,s}\}$ form operator bases. It is well known that such a decomposition provides an optimal low-rank approximation with respect to the Frobenius norm:

$$\|O-O_D\|_F=\sqrt{\mathrm{tr}[(O-O_D)^\dagger(O-O_D)]}, \tag{160}$$

where O_D is a low-rank approximation of the operator O.

For unitary time-evolution operators, the operator norm controls the worst-case error over input states. Operator Schmidt truncation is optimal in the Frobenius norm but need not be optimal in the operator norm. For the unnormalized norm in the preceding equation, $\|O-O_D\|\le\|O-O_D\|_F$; hence, an absolute Frobenius-error bound also bounds the operator-norm error. In contrast, a normalized error $\|O-O_D\|_F/\sqrt{\mathcal{D}_{AB}}\le\epsilon$ gives only $\|O-O_D\|\le\sqrt{\mathcal{D}_{AB}}\epsilon$. A relative Frobenius-error bound likewise need not give dimension-independent worst-case accuracy.

The quantum Fourier transform illustrates a separate distinction: its maximal operator entanglement [67–69] depends on the input or output ordering, whereas the small-entanglement core studied in Ref. [64] is obtained by removing bit reversal. These versions of the quantum Fourier transform are different operators across a fixed cut, so their comparison is not an example of a separation between the two norm guarantees.

Under the condition of $\bar{\mathcal{J}}_\alpha(V_{AB})=1$, we can exclude the interaction in Eq. (125), i.e., $\tilde{V}_{A_0B_0}=\sum_{s=1}^N(|s\rangle\langle s|)_{A_0}\otimes(|s\rangle\langle s|)_{B_0}$. A simple computation yields

$$\bar{\mathcal{J}}_\alpha(\tilde{V}_{A_0B_0})\ge N^{1/(2\alpha)-1}, \tag{161}$$

which increases with the local Hilbert-space dimension N.

Finally, in the absence of ancillas, the conjecture is trivially false. In this case, the swap operator S_{AB} maps between A and B serves as a counterexample. It creates no entanglement but cannot be approximated by a finite Schmidt rank, as can be shown using similar arguments to Eq. (144).

D. Theorem on low-rank approximation of operators

Theorem 3 provides a constructive approximation under explicit structural assumptions, complementary to the information-theoretic condition of Conjecture 2.

Subsequent results on that conjecture are summarized in the Note added.

1. Stronger assumption

Instead of the small α-SE strength, we consider the following property:

Assumption 10. Given the decomposition of V_{AB} into operator bases $\{V_j\}_j$, each having small Schmidt rank:

$$V_{AB} = \sum_{j=1}^{\infty} V_j, \qquad \mathrm{SR}(V_j) \le D_0 \quad \forall\, j, \tag{162}$$

where D_0 is a constant of order $\mathcal{O}(1)$ and $\tilde{g}$ below is a chosen upper bound on $\sum_j \|V_j\|$. Then, there exist constants C_0 and κ such that the following inequality holds:

$$\sum_{j\ge D+1} \|V_j\| \le C_0 \tilde{g}(D+1)^{-\kappa}, \qquad (C_0 \ge 1), \qquad \sum_{j=1}^{\infty} \|V_j\| \le \tilde{g}. \tag{163}$$

Remark. The inequality (163) also implies a low-rank approximation of V_{AB}:

$$\|V_{AB} - V_{AB}^{(D)}\| \le C_0 \tilde{g}(D+1)^{-\kappa}, \qquad V_{AB}^{(D)} = \sum_{j=1}^{D} V_j. \tag{164}$$

We also emphasize that under Assumption 10 V_{AB} has a constant α-SE strength with $\alpha < 1/2$ as shown in the following lemma:

Lemma 11. Under assumption 10 with $D_0 = 1$, the interaction V_{AB} satisfies

$$\bar{\mathcal{J}}_\alpha(V_{AB}) \le \frac{2^{1/(2\alpha)-1} C_0 \tilde{g}}{[1 - 2^{1-2\alpha(1+\kappa)}]^{1/(2\alpha)}} \quad \text{for } \frac{1}{2(1+\kappa)} < \alpha \le \frac{1}{2}. \tag{165}$$

Remark. From the lemma, we obtain the converse of Conjecture 2; namely, a low-rank approximation of the operator (Assumption 10) implies a small α-SE strength for $\alpha < 1/2$. For general D_0, one should multiply the right-hand side of Eq. (165) by the factor $D_0^{1/(2\alpha)}$.

Proof of Lemma 11. For an arbitrary product state $|\phi_0\rangle$, $V_{AB}|\phi_0\rangle$ can be expressed as

$$V_{AB}|\phi_0\rangle = \sum_j V_j|\phi_0\rangle = \sum_j g_j |\phi_{A,j}\rangle \otimes |\phi_{B,j}\rangle, \tag{166}$$

where $0 \le g_j \le \|V_j\|$ and $\{|\phi_{A,j}\rangle, |\phi_{B,j}\rangle\}_j$ are normalized states on A and B, respectively, that are not necessarily orthogonal to each other. Next, we consider the Schmidt decomposition of $V_{AB}|\phi_0\rangle$ as

$$V_{AB}|\phi_0\rangle = \sum_s \lambda_s |\phi_{A,s}\rangle \otimes |\phi_{B,s}\rangle. \tag{167}$$

Now, observe that the following inequality holds:

$$\sum_s \lambda_s^{2\alpha} \le \sum_j g_j^{2\alpha} \quad \text{for } 0 < \alpha \le \frac{1}{2}. \tag{168}$$

The above inequality follows directly from the matrix inequality as [70–72]

$$\left\| \sum_j M_j \right\|_p^p \le \sum_j \|M_j\|_p^p \quad \text{for } 0 < p \le 1, \tag{169}$$

where $\|\cdots\|_p$ denotes the Schatten p-quasi norm $\|M\|_p := [\mathrm{tr}(|M|^p)]^{1/p}$ $(p \le 1)$.

We set $N_m = 2^m$ and rewrite $\sum_j g_j^{2\alpha}$ as double sums, which leads to the following inequalities:

$$\begin{aligned} \sum_j g_j^{2\alpha} &= \sum_{m=1}^{\infty} \sum_{j \in [N_{m-1}, N_m)} g_j^{2\alpha} \\ &\le \sum_{m=1}^{\infty} \left(\sum_{j \in [N_{m-1}, N_m)} 1 \right)^{1-2\alpha} \left(\sum_{j \in [N_{m-1}, N_m)} g_j \right)^{2\alpha} \\ &\le \sum_{m=1}^{\infty} N_m^{(1-2\alpha)} \cdot [C_0 \tilde{g} (N_{m-1})^{-\kappa}]^{2\alpha}, \end{aligned} \tag{170}$$

where we apply the Hölder inequality in the first inequality and use the upper bound (163) in the last inequality. Finally, we calculate

$$\begin{aligned} \sum_{m=1}^{\infty} N_m^{(1-2\alpha)} (N_{m-1})^{-2\alpha\kappa} &= 2^{2\alpha\kappa} \sum_{m=1}^{\infty} 2^{m(1-2\alpha) - 2\alpha\kappa m} \\ &= \frac{2^{1-2\alpha}}{1 - 2^{1-2\alpha(1+\kappa)}}. \end{aligned} \tag{171}$$

By applying the inequalities (170) and (171) to Eq. (168), we obtain

$$\sum_s \lambda_s^{2\alpha} \le \frac{2^{1-2\alpha} (C_0 \tilde{g})^{2\alpha}}{1 - 2^{1-2\alpha(1+\kappa)}}, \tag{172}$$

which reduces to the main inequality (165) from Definition 2. This completes the proof. ∎

2. Result on the low-rank approximation

We show the following theorem on the operator approximation:

Theorem 3. Let us introduce a parameter $\mathcal{Q}$ that satisfies the following inequality:

$$\frac{1}{s!}\|\mathrm{ad}^s_{H_0}(V_j)\| \le \mathcal{Q}^s\|V_j\| \quad \forall\, s\in\mathbb{N}_0, \quad \forall\, j, \tag{173}$$

where $\{V_j\}_j$ are operator bases that constitute the boundary interaction terms as in Eq. (162). Then, under Assumption 10, for t, $\beta>0$ and $0<\epsilon_0\le 1$, for both the unitary time evolution e^{-iHt} and the imaginary-time evolution $e^{\beta H}$, we can construct approximate operators $\tilde{U}_t$ and $\tilde{\rho}_\beta$ such that

$$\|e^{-iHt}-\tilde{U}_t\|\le\epsilon_0, \qquad \|e^{\beta H}-\tilde{\rho}_\beta\|_p \le \epsilon_0\|e^{\beta H}\|_p, \tag{174}$$

and

$$\begin{aligned}\mathrm{SR}(\tilde{U}_t) &\le \left(\frac{8\lceil t\mathcal{Q}_0\rceil}{\epsilon_0}\right)^{[6+4/\kappa+\log_2(D_0)]\lceil t\mathcal{Q}_0\rceil},\\ \mathrm{SR}(\tilde{\rho}_\beta) &\le \left(\frac{48\lceil \beta\mathcal{Q}_0\rceil}{\epsilon_0}\right)^{2[6+4/\kappa+\log_2(D_0)]\lceil \beta\mathcal{Q}_0\rceil},\end{aligned} \tag{175}$$

respectively, where $\mathcal{Q}_0$ is defined as

$$\mathcal{Q}_0 := \left[\min\left(\frac{1}{4\mathcal{Q}},\frac{1}{4eC_0\tilde{g}}\right)\right]^{-1}. \tag{176}$$

At $t=0$ the identity suffices; negative times follow by replacing H by $-H$. Here, $p\ge 1$ and $\|\cdots\|_p$ is the Schatten p norm, i.e., $\|O\|_p=[\mathrm{tr}(|O|^p)]^{1/p}$.

Remark. The theorem shows that the required Schmidt rank grows only polynomially with respect to the inverse error $1/\epsilon_0$, with an exponent proportional to $t\mathcal{Q}_0$. Importantly, the result guarantees efficient operator approximation provided that the generator V_{AB} itself admits a suitable low-rank decomposition. Although Assumption 10 may appear rather strong at first sight, we will show in Lemma 20 that it is in fact satisfied by a broad class of physically relevant systems, including long-range or power-law decaying interactions. This observation serves as a key ingredient for the proof of Theorem 7, where we establish polynomially efficient approximations of long-range quantum Gibbs states.

E. Proof of Theorem 3

We first define the following merging operator between the subsets A and B:

$$\begin{aligned}\Psi &:= e^{-zH_0}e^{z(H_0+V_{AB})} = \mathcal{T}\left(e^{z\int_0^1 V_{AB}(x)dx}\right)\\ &= \sum_{s=0}^{\infty} z^s \int_0^1 dx_1 \int_0^{x_1} dx_2\cdots\\ &\quad\times\int_0^{x_{s-1}} dx_s V_{AB}(x_1)V_{AB}(x_2)\cdots V_{AB}(x_s),\end{aligned} \tag{177}$$

where $V_{AB}(x)=e^{-xzH_0}V_{AB}e^{xzH_0}$ and $z\in\mathbb{C}$. Using it, we have

$$e^{z(H_0+V_{AB})}=e^{zH_0}\Psi. \tag{178}$$

We notice that e^{zH_0} is a product operator between A and B. First, we prove that for small $|z|$ the operator Ψ can be approximated by another operator $\tilde{\Psi}_\epsilon$ with a small Schmidt rank.

Proposition 12. Let $0<\epsilon\le 1$ and choose z such that

$$|z|\le \min\left(\frac{1}{4\mathcal{Q}},\frac{1}{4eC_0\tilde{g}}\right) =: \mathcal{Q}_0^{-1}. \tag{179}$$

Then, there exists an operator $\tilde{\Psi}_\epsilon$ satisfying

$$\|\Psi-\tilde{\Psi}_\epsilon\|\le\epsilon, \tag{180}$$

and

$$\mathrm{SR}(\tilde{\Psi}_\epsilon)\le(4/\epsilon)^{6+4/\kappa+\log_2(D_0)}. \tag{181}$$

We defer the proof of the proposition to Sec. VI E 1 below. Using this proposition, we now give a proof of the main statement, as follows.

We first consider the real-time evolution $z=-it$ $(t>0)$ and decompose the total time t into mt_0 with t_0 satisfying the condition (179). We generally obtain

$$\begin{aligned}&e^{-imt_0(H_0+V_{AB})}-\left(e^{-it_0H_0}\tilde{\Psi}_\epsilon\right)^m\\ &=\sum_{j=0}^{m-1} e^{-i(m-j-1)t_0(H_0+V_{AB})}\left(e^{-it_0(H_0+V_{AB})}-e^{-it_0H_0}\tilde{\Psi}_\epsilon\right)\\ &\quad\times\left(e^{-it_0H_0}\tilde{\Psi}_\epsilon\right)^j,\end{aligned} \tag{182}$$

which gives

$$\begin{aligned}&\left\|e^{-imt_0(H_0+V_{AB})}-\left(e^{-it_0H_0}\tilde{\Psi}_\epsilon\right)^m\right\|\\ &\le\sum_{j=0}^{m-1}\left\|e^{-it_0(H_0+V_{AB})}-e^{-it_0H_0}\tilde{\Psi}_\epsilon\right\|\cdot\left\|e^{-it_0H_0}\tilde{\Psi}_\epsilon\right\|^j.\end{aligned} \tag{183}$$

Using Proposition 12, we obtain

$$\left\|e^{-it_0(H_0+V_{AB})}-e^{-it_0H_0}\tilde{\Psi}_\epsilon\right\|\le\|\Psi-\tilde{\Psi}_\epsilon\|\le\epsilon, \tag{184}$$

where we use $\|e^{-it_0H_0}\| = 1$ and $e^{-it_0(H_0+V_{AB})} = e^{-it_0H_0}\Psi$. This also yields

$$\|e^{-it_0H_0}\tilde{\Psi}_\epsilon\| = \left\| e^{-it_0(H_0+V_{AB})} - e^{-it_0(H_0+V_{AB})} + e^{-it_0H_0}\tilde{\Psi}_\epsilon \right\| \le 1+\epsilon. \tag{185}$$

By applying the inequalities (184) and (185) to Eq. (183), we get

$$\left\| e^{-imt_0(H_0+V_{AB})} - (e^{-it_0H_0}\tilde{\Psi}_\epsilon)^m \right\| \le \sum_{j=0}^{m-1} \epsilon(1+\epsilon)^j \le me^{m\epsilon}\epsilon. \tag{186}$$

We now choose

$$t_0 = \frac{t}{\lceil t\mathcal{Q}_0\rceil}(\le \mathcal{Q}_0^{-1}), \qquad m = \frac{t}{t_0} = \lceil t\mathcal{Q}_0\rceil,$$
$$\epsilon = \frac{\epsilon_0}{2m}, \qquad \tilde{U}_t := (e^{-it_0H_0}\tilde{\Psi}_\epsilon)^m. \tag{187}$$

We then obtain $\|e^{-iHt} - \tilde{U}_t\| \le \epsilon_0/2e^{\epsilon_0/2} \le \epsilon_0$ from $\epsilon_0 \le 1$ and

$$\mathrm{SR}(\tilde{U}_t) \le [\mathrm{SR}(\tilde{\Psi}_\epsilon)]^m \le \left(\frac{8\lceil t\mathcal{Q}_0\rceil}{\epsilon_0}\right)^{[6+4/\kappa+\log_2(D_0)]\lceil t\mathcal{Q}_0\rceil}. \tag{188}$$

This establishes the first main inequality in Eq. (175).

Next, we consider the imaginary-time evolution $z = \beta$ with the decomposition $z = 2m\beta_0$, where β_0 satisfies the condition (179). Using $\tilde{\Psi}_\epsilon$, we examine $e^{\beta_0H_0}\tilde{\Psi}_\epsilon$, which approximates $e^{\beta_0(H_0+V_{AB})}$ by

$$\left\| e^{\beta_0(H_0+V_{AB})} - e^{\beta_0H_0}\tilde{\Psi}_\epsilon \right\|_p = \|e^{\beta_0H_0}(\Psi - \tilde{\Psi}_\epsilon)\|_p \le \|e^{\beta_0H_0}\|_p \|\Psi - \tilde{\Psi}_\epsilon\|_\infty \le \|e^{\beta_0H_0}\|_p \epsilon, \tag{189}$$

where we use the Hölder inequality to obtain $\|O_1O_2\|_p \le \|O_1\|_p\|O_2\|_\infty$. Furthermore, from Ref. [73] Lemma 7 therein, we have

$$\|e^{O_1} - e^{O_1-O_2}\|_p \le e^{\|O_2\|}\|O_2\|\|e^{O_1}\|_p, \tag{190}$$

which gives

$$\left\| e^{\beta_0(H_0+V_{AB})} - e^{\beta_0H_0} \right\|_p \le \left\| e^{\beta_0(H_0+V_{AB})} \right\|_p e^{\|\beta_0V_{AB}\|} \left\| \beta_0V_{AB} \right\| \le \left\| e^{\beta_0(H_0+V_{AB})} \right\|_p e^{\beta_0\tilde{g}}\beta_0\tilde{g}. \tag{191}$$

From $\|e^{\beta_0(H_0+V_{AB})} - e^{\beta_0H_0}\|_p \ge \|e^{\beta_0H_0}\|_p - \|e^{\beta_0(H_0+V_{AB})}\|_p$, we obtain

$$\|e^{\beta_0H_0}\|_p \le \left\| e^{\beta_0(H_0+V_{AB})} \right\|_p (1 + e^{\beta_0\tilde{g}}\beta_0\tilde{g}) \le e^{2\beta_0\tilde{g}} \left\| e^{\beta_0(H_0+V_{AB})} \right\|_p. \tag{192}$$

In total, we derive the following error bound:

$$\|e^{\beta_0(H_0+V_{AB})} - e^{\beta_0H_0}\tilde{\Psi}_\epsilon\|_p \le e^{2\beta_0\tilde{g}}\epsilon\|e^{\beta_0(H_0+V_{AB})}\|_p. \tag{193}$$

Set $T = e^{\beta_0H}$, $A = e^{\beta_0H_0}\tilde{\Psi}_\epsilon$, and $\delta_\epsilon = e^{2\beta_0\tilde{g}}\epsilon$. Applying the preceding estimate with $q = 2mp$, Hölder's inequality and telescoping give

$$\left\| (AA^\dagger)^m - T^{2m} \right\|_p \le [(1+\delta_\epsilon)^{2m} - 1]\left\| T^{2m} \right\|_p \le 2m\delta_\epsilon e^{2m\delta_\epsilon}\left\| T^{2m} \right\|_p.$$

In particular,

$$\left\| e^{\beta(H_0+V_{AB})} - (e^{\beta_0H_0}\tilde{\Psi}_\epsilon\tilde{\Psi}_\epsilon^\dagger e^{\beta_0H_0})^m \right\|_p \le 3\delta_\epsilon me^{3\delta_\epsilon m}\left\| e^{\beta(H_0+V_{AB})} \right\|_p, \tag{194}$$

where we use $e^{2m\beta_0(H_0+V_{AB})} = e^{\beta(H_0+V_{AB})}$ and set $\delta_\epsilon = e^{2\beta_0\tilde{g}}\epsilon$. We then choose

$$\beta_0 = \frac{\beta}{2\lceil\beta\mathcal{Q}_0\rceil}(\le \mathcal{Q}_0^{-1}), \qquad 2m = \frac{\beta}{\beta_0} = 2\lceil\beta\mathcal{Q}_0\rceil,$$
$$\epsilon = \frac{e^{-2\beta_0\tilde{g}}\epsilon_0}{6m}, \qquad \tilde{\rho}_\beta := (e^{\beta_0H_0}\tilde{\Psi}_\epsilon\tilde{\Psi}_\epsilon^\dagger e^{\beta_0H_0})^m. \tag{195}$$

Here, the condition $\beta_0 \le \mathcal{Q}_0^{-1} \le 1/(4eC_0\tilde{g})$ implies $e^{2\beta_0\tilde{g}} \le e^{1/(2eC_0)} \le 2$ because of $C_0 \ge 1$. The above choices give $\|e^{\beta H} - \tilde{\rho}_\beta\|_p \le \epsilon_0\|e^{\beta H}\|_p$ for any $p \ge 1$, and the Schmidt rank is given by

$$\mathrm{SR}(\tilde{\rho}_\beta) \le [\mathrm{SR}(\tilde{\Psi}_\epsilon)]^{2m} \le \left(\frac{48\lceil\beta\mathcal{Q}_0\rceil}{\epsilon_0}\right)^{2[6+4/\kappa+\log_2(D_0)]\lceil\beta\mathcal{Q}_0\rceil}. \tag{196}$$

We thus prove the second main inequality in Eq. (175).

This completes the proof of Theorem 3. ∎

1. Proof of Proposition 12

We introduce the decomposition of V_{AB} as follows:

$$V_{AB} = \sum_{m=0}^{\infty} V_{r_m}, \tag{197}$$

with V_{r_m} defined by

$$V_{r_m} \coloneqq \sum_{j\in[r_m,r_{m+1})} V_j, \qquad r_m = 4^{m/\kappa}. \tag{198}$$

Here, we introduce the notation $\overline{V_{r_m}}$ as

$$\overline{V_{r_m}} = \sum_{j\in[r_m,r_{m+1})} \|V_j\|, \tag{199}$$

and from the assumption (163), we obtain

$$\|V_{r_m}\| \le \overline{V_{r_m}} \le \sum_{j\ge 4^{m/\kappa}}^{\infty} \|V_j\| \le C_0\tilde{g}4^{-m}. \tag{200}$$

We also define $V_{r_m}(x,q)$ as follows:

$$V_{r_m}(x,q) = \frac{(-xz)^q}{q!}\mathrm{ad}^q_{H_0}(V_{r_m}), \tag{201}$$

which gives

$$V_{r_m}(x) = \sum_{q=0}^{\infty}\frac{(-xz)^q}{q!}\mathrm{ad}^q_{H_0}(V_{r_m}) = \sum_{q=0}^{\infty} V_{r_m}(x,q). \tag{202}$$

Using the parameter $\mathcal{Q}$ in Eq. (173), we obtain

$$\begin{aligned}\|V_{r_m}(x,q)\| &\le \frac{|z|^q}{q!}\sum_{j\in[r_m,r_{m+1})}\|\mathrm{ad}^q_{H_0}(V_j)\| \le (|z|\mathcal{Q})^q \\ &\times \sum_{j\in[r_m,r_{m+1})}\|V_j\| \le C_0\tilde{g}(|z|\mathcal{Q})^q 4^{-m},\end{aligned} \tag{203}$$

where we use $x \le 1$.

By using the above notations, we first rewrite Ψ in the following form:

$$\begin{aligned}\Psi \coloneqq \sum_{s=0}^{\infty} z^s \sum_{m_1,m_2,\ldots,m_s=0}^{\infty}\sum_{q_1,q_2,\ldots,q_s=0}^{\infty}\int_0^1 dx_1\int_0^{x_1} dx_2\cdots \\ \times\int_0^{x_{s-1}} dx_s V_{r_{m_1}}(x_1,q_1)V_{r_{m_2}}(x_2,q_2)\cdots V_{r_{m_s}}(x_s,q_s).\end{aligned} \tag{204}$$

Then, we approximate Ψ by Ψ_{Q,M,s_0}:

$$\begin{aligned}\Psi_{Q,M,s_0} \coloneqq \sum_{s=0}^{s_0} z^s \sum_{m_1+m_2+\cdots+m_s\le M}\sum_{q_1+q_2+\cdots+q_s\le Q}\int_0^1 dx_1\cdots \\ \times\int_0^{x_{s-1}} dx_s V_{r_{m_1}}(x_1,q_1)\cdots V_{r_{m_s}}(x_s,q_s).\end{aligned} \tag{205}$$

By using the inequality as

$$\begin{aligned}&\left\|\int_0^1 dx_1\cdots\int_0^{x_{s-1}} dx_s V_{r_{m_1}}(x_1,q_1)\cdots V_{r_{m_s}}(x_s,q_s)\right\| \\ &\le \frac{1}{s!}\prod_{j=1}^{s} C_0\tilde{g}(|z|\mathcal{Q})^{q_j}4^{-m_j},\end{aligned} \tag{206}$$

we obtain

$$\begin{aligned}\|\Psi-\Psi_{Q,M,s_0}\| \le \left(\sum_{s>s_0}\sum_{m_1,m_2,\ldots,m_s=0}^{\infty}\sum_{q_1,q_2,\ldots,q_s=0}^{\infty} + \sum_{s=0}^{\infty}\sum_{m_1+m_2+\cdots+m_s>M}\sum_{q_1,q_2,\ldots,q_s=0}^{\infty} + \sum_{s=0}^{\infty}\sum_{m_1,m_2,\ldots,m_s=0}^{\infty}\sum_{q_1+q_2+\cdots+q_s>Q}\right) \\ \times\frac{|z|^s}{s!}\prod_{j=1}^{s} C_0\tilde{g}(|z|\mathcal{Q})^{q_j}4^{-m_j}.\end{aligned} \tag{207}$$

For the first summation in Eq. (207), we can derive

$$\begin{aligned}\sum_{m_1,m_2,\ldots,m_s=0}^{\infty}\sum_{q_1,q_2,\ldots,q_s=0}^{\infty}\prod_{j=1}^{s} C_0\tilde{g}(|z|\mathcal{Q})^{q_j}4^{-m_j} &= (C_0\tilde{g})^s\left(\sum_{m=0}^{\infty}4^{-m}\right)^s\left(\sum_{q=0}^{\infty}(|z|\mathcal{Q})^q\right)^s \\ &\le \left(\frac{16C_0\tilde{g}}{9}\right)^s \le (2C_0\tilde{g})^s,\end{aligned} \tag{208}$$

where we use the condition $|z| \le 1/(4\mathcal{Q})$. Hence, we derive

$$\sum_{s>s_0}\sum_{m_1,m_2,\ldots,m_s=0}^{\infty}\sum_{q_1,q_2,\ldots,q_s=0}^{\infty}\frac{|z|^s}{s!}\prod_{j=1}^{s}C_0\tilde{g}(|z|\mathcal{Q})^{q_j}4^{-m_j}$$
$$\leq\sum_{s>s_0}\frac{|z|^s}{s!}(2C_0\tilde{g})^s\leq\left(\frac{2eC_0\tilde{g}|z|}{s_0+1}\right)^{s_0+1}e^{2C_0\tilde{g}|z|}\leq 2^{-s_0-1}e^{1/(2e)}, \tag{209}$$

where we use $\sum_{s>s_0}x^s/s!\leq e^x x^{s_0+1}/(s_0+1)!$ in the second inequality and use $|z|\leq 1/(4eC_0\tilde{g})$.

For the second summation in Eq. (207), we obtain

$$\sum_{m_1+m_2+\cdots+m_s>M}\sum_{q_1,q_2,\ldots,q_s=0}^{\infty}\prod_{j=1}^{s}C_0\tilde{g}(|z|\mathcal{Q})^{q_j}4^{-m_j}\leq\left(\frac{4C_0\tilde{g}}{3}\right)^s\sum_{\bar{m}=M+1}^{\infty}\sum_{m_1+m_2+\cdots+m_s=\bar{m}}4^{-\bar{m}}$$
$$=\left(\frac{4C_0\tilde{g}}{3}\right)^s\sum_{\bar{m}=M+1}^{\infty}\binom{s+\bar{m}-1}{\bar{m}}4^{-\bar{m}}$$
$$\leq\left(\frac{8C_0\tilde{g}}{3}\right)^s\sum_{\bar{m}=M+1}^{\infty}2^{-\bar{m}-1}\leq 2^{-M-1}(3C_0\tilde{g})^s, \tag{210}$$

where the number of combinations such that $m_1+m_2+\cdots+m_s=\bar{m}$ is given by $(\!(\begin{smallmatrix}s\\ \bar{m}\end{smallmatrix})\!)=\binom{s+\bar{m}-1}{\bar{m}}$. We thus derive

$$\sum_{s=0}^{\infty}\sum_{m_1+m_2+\cdots+m_s>M}\sum_{q_1,q_2,\ldots,q_s=0}^{\infty}\frac{|z|^s}{s!}\prod_{j=1}^{s}C_0\tilde{g}(|z|\mathcal{Q})^{q_j}4^{-m_j}\leq 2^{-M-1}e^{3C_0\tilde{g}|z|}\leq 2^{-M-1}e^{3/(4e)}, \tag{211}$$

where we use $|z|\leq 1/(4eC_0\tilde{g})$.

Finally, for the third summation in Eq. (207), we deduce

$$\sum_{m_1,m_2,\ldots,m_s=0}^{\infty}\sum_{q_1+q_2+\cdots+q_s>Q}\prod_{j=1}^{s}C_0\tilde{g}(|z|\mathcal{Q})^{q_j}4^{-m_j}\leq\left(\frac{4C_0\tilde{g}}{3}\right)^s\sum_{\bar{q}=Q+1}^{\infty}\sum_{q_1+q_2+\cdots+q_s=\bar{q}}4^{-\bar{q}}$$
$$\leq 2^{-Q-1}(3C_0\tilde{g})^s, \tag{212}$$

where we use the same calculations as in Eq. (210). This yields

$$\sum_{s=0}^{\infty}\frac{|z|^s}{s!}\sum_{m_1,m_2,\ldots,m_s=0}^{\infty}\sum_{q_1+q_2+\cdots+q_s>Q}\prod_{j=1}^{s}C_0\tilde{g}(|z|\mathcal{Q})^{q_j}4^{-m_j}\leq 2^{-Q-1}e^{3/(4e)}. \tag{213}$$

In total, we obtain the approximation bound of

$$\|\Psi-\Psi_{Q,M,s_0}\|\leq 2^{-s_0-1}e^{1/(2e)}+(2^{-M-1}+2^{-Q-1})e^{3/(4e)}. \tag{214}$$

We next estimate the Schmidt rank of Ψ_{Q,M,s_0}, which is upper-bounded by

$$\mathrm{SR}(\Psi_{Q,M,s_0})\leq\sum_{s=0}^{s_0}\sum_{m_1+m_2+\cdots+m_s\leq M}\sum_{q_1+q_2+\cdots+q_s\leq Q}\prod_{j=1}^{s}\mathrm{SR}\left[\mathrm{ad}_{H_0}^{q_j}(V_{r_{m_j}})\right]. \tag{215}$$

By combining the condition (162) and Eq. (198), we obtain

$$\mathrm{SR}(V_{r_m})\leq r_{m+1}D_0=4^{(m+1)/\kappa}D_0. \tag{216}$$

By applying the above inequality to

$$\mathrm{ad}_{H_0}^q(V_{r_m}) = (\mathrm{ad}_{H_A} + \mathrm{ad}_{H_B})^q(V_{r_m}) = \sum_{j=0}^{q} \binom{q}{j} \mathrm{ad}_{H_A}^j \otimes \mathrm{ad}_{H_B}^{q-j}(V_{r_m}), \tag{217}$$

we have

$$\mathrm{SR}\left[\mathrm{ad}_{H_0}^{q_j}(V_{r_{m_j}})\right] \le (q_j+1)4^{(m_j+1)/\kappa} D_0. \tag{218}$$

Therefore, we obtain

$$\prod_{j=1}^{s} \mathrm{SR}\left[\mathrm{ad}_{H_0}^{q_j}(V_{r_{m_j}})\right] \le 4^{(M+s)/\kappa}\left(\frac{Q+s}{s} D_0\right)^s, \tag{219}$$

where we use the inequality of arithmetic and geometric means, with the $s=0$ factor interpreted as the empty product 1.

By applying inequality (219) to Eq. (215), we obtain the upper bound on the Schmidt rank as follows:

$$\begin{aligned}\mathrm{SR}(\Psi_{Q,M,s_0}) &\le \sum_{s=0}^{s_0} \binom{M+s}{s}\binom{Q+s}{s} 4^{(M+s)/\kappa}\left(\frac{Q+s}{s} D_0\right)^s \\ &\le 2^{M+s_0}\cdot 2^{Q+s_0} \times 4^{(M+s_0)/\kappa} D_0^{s_0} e^Q \\ &\le 2^{(1+2/\kappa)M+3Q+[2+2/\kappa+\log_2(D_0)]s_0},\end{aligned} \tag{220}$$

where we use

$$\begin{aligned}\sum_{m_1+m_2+\cdots+m_s\le M} 1 &= \sum_{\tilde{m}=0}^{M} \sum_{m_1+m_2+\cdots+m_s=\tilde{m}} 1 \\ &= \sum_{\tilde{m}=0}^{M} \binom{s+\tilde{m}-1}{\tilde{m}} = \binom{s+M}{s}.\end{aligned} \tag{221}$$

Note that the last equation follows from the hockey-stick identity.

By choosing

$$s_0 = M = Q = \lceil \log_2(2/\epsilon)\rceil \le \log_2(4/\epsilon), \tag{222}$$

we reduce the error bound (214) to

$$\begin{aligned}\|\Psi - \Psi_{Q,M,s_0}\| &\le 2^{-s_0-1} e^{1/(2e)} + (2^{-M-1}+2^{-Q-1}) e^{3/(4e)} \\ &\le 2^{-s_0+1} \le \epsilon.\end{aligned} \tag{223}$$

Moreover, we have

$$\mathrm{SR}(\Psi_{Q,M,s_0}) \le 2^{[6+4/\kappa+\log_2(D_0)]s_0} \le (4/\epsilon)^{6+4/\kappa+\log_2(D_0)}. \tag{224}$$

By choosing Ψ_{Q,M,s_0} as $\tilde{\Psi}_\epsilon$, we prove the main statement. This completes the proof. ∎

VII. GENERALIZED ENTANGLEMENT AREA LAW

In this section, we explain under what additional assumptions a generalized area-law-type statement can be recovered for gapped ground states. We emphasize from the outset that, in the absence of a geometric lattice structure, this should not be interpreted as a conventional geometric area law. Rather, the role of the boundary size is played by an effective boundary measure controlled by the SE strength of the boundary interaction. Since the generalized area law is known to fail for arbitrary gapped ground states, the main question is not whether such a bound holds in full generality, but which additional structure prevents the formation of anomalously large entanglement. Here we impose an adiabatic-path assumption, Assumption 13, for a parametrized Hamiltonian $H^{(\nu)}$ with $0 \le \nu \le 1$: the target Hamiltonian $H^{(1)}$ is connected to a reference Hamiltonian $H^{(0)}$ while keeping the spectral gap open and the boundary SE strength uniformly bounded along the path.

The proof follows a three-stage approximation strategy. First, using an AGSP for the initial Hamiltonian $H^{(0)}$, we convert the assumed $\alpha=\infty$ Rényi-entanglement condition on the reference ground state $|\Omega^{(0)}\rangle$ into a low-Schmidt-rank approximation of $|\Omega^{(0)}\rangle$; this step is formulated in Proposition 15. Second, we evolve this low-rank approximation along the adiabatic path, which produces a state having a constant overlap with the target ground state $|\Omega^{(1)}\rangle$. Third, we apply an AGSP for the target Hamiltonian $H^{(1)}$ to this constant-overlap state in order to amplify the accuracy and obtain a high-precision low-Schmidt-rank approximation of $|\Omega^{(1)}\rangle$. The entanglement generated during the adiabatic evolution and the AGSP filtering is controlled by the spectral-SIE framework through the SE strength of the corresponding boundary interactions. This yields the low-Schmidt-rank approximation in Theorem 4, from which the generalized area-law-type entropy bound follows.

A. Setup and main result

We first recall the conventional generalized area-law setting and then state the boundary-adiabatic assumption under which the bound can be recovered (Fig. 3). In a conventional notation, we consider the Hamiltonian such that

$$H = H_{A_1A_0} + V_{A_0B_0} + H_{B_0B_1}, \qquad \|V_{A_0B_0}\| = 1, \tag{225}$$

where the subsets A and B are given by $A = A_1 \sqcup A_0$ and $B = B_0 \sqcup B_1$, respectively. Assuming a constant spectral gap $\Delta = \mathrm{const}$, we argue that the following upper bound for the entanglement entropy holds:

$$E_{\alpha=1}(\Omega) \le C \log[\min(\mathcal{D}_{A_0}, \mathcal{D}_{B_0})], \tag{226}$$

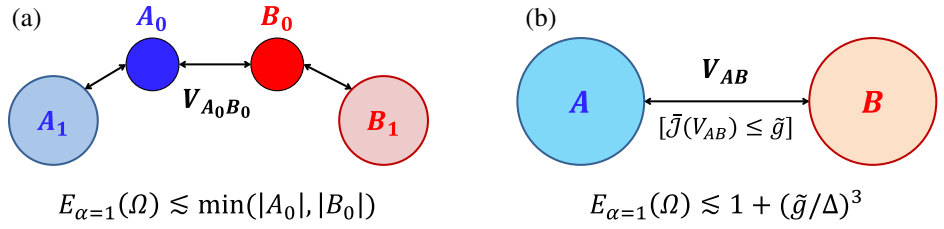


FIG. 3. Schematic picture of the generalized area law. (a) The generalized area law states that the entanglement entropy is bounded in terms of the dimension of the interacting region. However, this bound is known to be violated for general ground states. (b) By assuming an adiabatic path for the boundary interaction, we establish the generalized area law under a condition formulated with the SE strength $\tilde{g}$. Roughly speaking, $\tilde{g}$ corresponds to the effective size of the boundary. For example, in systems defined on graphs, it becomes more evident that $\tilde{g}$ is proportional to the boundary size. If A_0 and B_0 consist of m qubits and the interaction $V_{A_0B_0}$ is described by $\mathcal{O}(m)$ two-qubit interactions, then one obtains $\tilde{g} \propto m = \log(\mathcal{D}_{A_0})$ from inequality (18).

where the gap Δ is defined as the energy difference between the ground energy E_0 and the first excited energy E_1, i.e., $\Delta \coloneqq E_1 - E_0$. Generally, it has been known that this generalized area law does not hold [50].

Here, we impose the following additional assumptions, which are typically considered for the area-law proof under the adiabatic path [22,26,54,55,59]. We restrict ourselves to the following boundary-adiabatic path:

Assumption 13. (boundary-adiabatic path). We consider a family of Hamiltonians parametrized by ν through the boundary interaction $V_{AB}^{(\nu)}$:

$$H^{(\nu)} = H_A + V_{AB}^{(\nu)} + H_B, \qquad \bar{\mathcal{J}}\left[V_{AB}^{(\nu)}\right] \leq \tilde{g}, \tag{227}$$

where we assume that the ground state $|\Omega^{(\nu)}\rangle$ of $H^{(\nu)}$ is nondegenerate for all $\nu \in [0,1]$. We say that there exists a boundary-adiabatic path from $\nu = 0$ to $\nu = 1$ if

$$\Delta^{(\nu)} \geq \Delta, \tag{228}$$

where $\Delta^{(\nu)}$ denotes the spectral gap of $H^{(\nu)}$.

Finally, we introduce a parameter $\tilde{c}_0$ such that

$$\max\left(\left\|\frac{dV_{AB}^{(\nu)}}{d\nu}\right\|, \left\|\frac{d^2V_{AB}^{(\nu)}}{d\nu^2}\right\|\right) \leq \tilde{c}_0\tilde{g}. \tag{229}$$

Remark. In contrast to the conventional notation, the interaction $V_{AB}^{(\nu)}$ is allowed to act more generally across the boundary between A and B, rather than being restricted to A_0 and B_0. The restriction on the interaction region imposed by A_0 and B_0 is effectively absorbed into the condition on the SE strength, i.e., $\bar{\mathcal{J}}[V_{AB}^{(\nu)}] \leq \tilde{g}$.

Additionally, in a more general setting, one may consider an arbitrary parametrization of the Hamiltonian, for example, $H_A^{(\nu)} + V_{AB}^{(\nu)} + H_B^{(\nu)}$. In such a general framework, there may exist adiabatic paths that connect the initial ground state to an anomalous, highly entangled ground state, as demonstrated in Ref. [50]. Although we have not yet identified an explicit counterexample, we conjecture that the standard adiabatic-path condition by itself is insufficient to guarantee the generalized area law.

Under the above notations, we prove the following theorem:

Theorem 4. Let us denote the initial-state entanglement by the ∞-Rényi entanglement:

$$E_\infty(\Omega^{(0)}) = S_0. \tag{230}$$

Under Assumption 13, there exists a state $|\psi_D\rangle$ with Schmidt rank D such that

$$\||\psi_D\rangle - |\Omega^{(1)}\rangle\| \leq C_{\tilde{g},\Delta,S_0} D^{-\kappa_\Delta}, \tag{231}$$

where we define

$$\kappa_\Delta \coloneqq \frac{\Delta}{2\Delta + 4\tilde{g}}, \tag{232}$$

$$C_{\tilde{g},\Delta,S_0} \coloneqq \exp\left[\frac{S_0+4}{4\kappa_\Delta} + \log(12) + \frac{4\tilde{c}_0\tilde{g}^3(3\Delta/\tilde{g} + 7\tilde{c}_0)}{\Delta^3}\right]. \tag{233}$$

Moreover, the entanglement entropy $E_{\alpha=1}(\Omega^{(1)})$ for the ground state $|\Omega^{(1)}\rangle$ is upper-bounded by

$$E_{\alpha=1}(\Omega^{(1)}) \leq c_{\kappa_\Delta,1} \log(C_{\tilde{g},\Delta,S_0}) + c_{\kappa_\Delta,2}, \tag{234}$$

where

$$c_{\kappa_\Delta,1} \coloneqq \frac{2 - 2^{-2\kappa_\Delta}}{\kappa_\Delta(1 - 2^{-2\kappa_\Delta})}, \qquad c_{\kappa_\Delta,2} \coloneqq \frac{(6 + 2\kappa_\Delta)\log(2)}{(1 - 2^{-2\kappa_\Delta})^2}. \tag{235}$$

Remark. Since $E_\infty(\Omega^{(0)}) \leq E_\alpha(\Omega^{(0)})$, the initial condition involving the ∞-Rényi entanglement is the least restrictive, enabling the broadest applicability.

In our setup, the boundary size is roughly estimated as $\tilde{g}/\Delta$ since $\tilde{g}$ characterizes the dynamical entanglement rate

from the spectral SIE. Indeed, in a graph system with finite-range interactions, $\tilde{g}$ is indeed proportional to the boundary size between A and B. For fixed S_0 and $\tilde{c}_0$, the sharper estimate at the end of the proof yields

$$E_{\alpha=1}(\Omega^{(1)}) = \mathcal{O}(1 + (\tilde{g}/\Delta)^3). \tag{236}$$

It is an important open question whether the upper bound (234) can be improved to a bound of the form $E_{\alpha=1}(\Omega^{(1)}) = \mathcal{O}(1 + \tilde{g}/\Delta)$.

B. AGSP

In the proof of Theorem 4, we combine the quantum adiabatic theorem [56] with the so-called approximate-ground-state projection approach [40,41,44,45].

We consider Schmidt-rank truncation for nondegenerate gapped ground states. To this end, we introduce an AGSP operator K defined by the conditions

$$\|(K-1)|\Omega\rangle\| \approx 0, \qquad \|K|\Omega_\perp\rangle\| \approx 0 \tag{237}$$

for any state $|\Omega_\perp\rangle$ orthogonal to the ground state $|\Omega\rangle$. Thus, K serves as an approximation to the ground-state projector, while allowing us to impose constraints on the Schmidt-rank structure.

In contrast to previous studies, the present use of the AGSP has the following distinctive feature:

(1) In the standard AGSP framework, the primary focus is to bound the Schmidt rank of the AGSP operator.
(2) In our setting, however, the focus shifts to the entanglement generation induced by the AGSP operator. This distinction originates from the fact that there is a fundamental difference between (i) low-rank approximation of the operator and (ii) the entanglement generated by the action of the operator (see Proposition 9).

Proposition 14. For a Hamiltonian $H = H_A + H_B + V_{AB}$ with a unique ground state $|\Omega\rangle$ and gap $\Delta > 0$, there exists an AGSP operator K_β with the following properties:

$$\|(K_\beta - 1)|\Omega\rangle\| \le e^{-\beta\Delta^2}, \qquad \|K_\beta|\Omega_\perp\rangle\| \le 2e^{-\beta\Delta^2}, \tag{238}$$

and

$$\bar{\mathcal{J}}(K_\beta) \le e^{2\beta\Delta\bar{\mathcal{J}}(V_{AB})}, \tag{239}$$

where $\beta > 0$ is arbitrary and $|\Omega_\perp\rangle$ is any normalized state orthogonal to $|\Omega\rangle$.

Remark. For any product state $|\phi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$, we apply the upper bound (239) to inequality (32) in Corollary 3. Then, the quantum state $K_\beta|\phi\rangle$ is well approximated by $|\phi_D\rangle$ with Schmidt rank D as follows:

$$\|K_\beta|\phi\rangle - |\phi_D\rangle\| \le \frac{\bar{\mathcal{J}}(K_\beta)}{\sqrt{D}} \le \frac{e^{2\beta\Delta\bar{\mathcal{J}}(V_{AB})}}{\sqrt{D}}. \tag{240}$$

Here, the AGSP K_β itself may not be approximated by an operator with Schmidt rank D (see also Sec. VI). However, in applying the AGSP formalism to the ground state, the Schmidt-rank approximation with respect to $K_\beta|\phi\rangle$ is more crucial. We use this proposition to achieve the ground-state approximation in Theorem 4, as well as in Theorem 5 below, concerning the polynomial complexity of the ground state in 1D long-range interacting systems.

1. Proof of Proposition 14

Let E_0 be the ground energy of the fixed Hamiltonian H being filtered. The scalar phase from this centering changes neither the norm nor the SE strength and does not redefine the adiabatic path. As a candidate for the AGSP operator K_β, we employ the following one:

$$K_\beta = \frac{1}{\sqrt{4\pi\beta}} \int_{-t_c}^{t_c} e^{-t^2/(4\beta)} e^{-i(H-E_0\hat{1})t} dt. \tag{241}$$

First, we demonstrate that the above operator K_β indeed satisfies the properties in Eq. (238).

To this end, we first obtain the following bounds on the norm difference between K_β and $e^{-\beta(H-E_0\hat{1})^2}$:

$$\begin{aligned}
&\left\| K_\beta - e^{-\beta(H-E_0\hat{1})^2} \right\| \\
&\le \frac{1}{\sqrt{\pi\beta}} \int_{t_c}^{\infty} e^{-t^2/(4\beta)} dt \\
&\le \frac{1}{\sqrt{\pi\beta}} e^{-t_c^2/(4\beta)} \int_{t_c}^{\infty} e^{-(t^2-t_c^2)/(4\beta)} dt \\
&\le \frac{1}{\sqrt{\pi\beta}} e^{-t_c^2/(4\beta)} \int_0^{\infty} e^{-(\tilde{t}^2+2\tilde{t}t_c)/(4\beta)} d\tilde{t} \le e^{-t_c^2/(4\beta)}.
\end{aligned} \tag{242}$$

This upper bound immediately implies

$$\|(K_\beta - 1)|\Omega\rangle\| \le e^{-t_c^2/(4\beta)}, \tag{243}$$

where we use $e^{-\beta(H-E_0\hat{1})^2}|\Omega\rangle = |\Omega\rangle$. For an arbitrary quantum state $|\Omega_\perp\rangle$, we also obtain

$$\begin{aligned}
\|K_\beta|\Omega_\perp\rangle\| &\le \left\| e^{-\beta(H-E_0\hat{1})^2}|\Omega_\perp\rangle \right\| + \left\| K_\beta - e^{-\beta(H-E_0\hat{1})^2} \right\| \\
&\le e^{-\beta\Delta^2} + e^{-t_c^2/(4\beta)}.
\end{aligned} \tag{244}$$

By setting

$$t_c = 2\beta\Delta, \tag{245}$$

we obtain the first and second inequalities in Eq. (238).

Next, we estimate the SE strength $\bar{\mathcal{J}}(K_\beta)$ based on Definition 1 to prove inequality (239). We use Lemma 4 and obtain the inequality

$$\bar{\mathcal{J}}(K_\beta) \le \frac{1}{\sqrt{4\pi\beta}} \int_{-t_c}^{t_c} e^{-t^2/(4\beta)} \bar{\mathcal{J}}(e^{-i(H-E_0\hat{1})t}) dt. \tag{246}$$

From Corollary 6, we get $\bar{\mathcal{J}}(e^{-i(H-E_0\hat{1})t}) \le e^{|t|\bar{\mathcal{J}}(V_{AB})}$, and hence the desired inequality yields as follows:

$$\begin{aligned}\bar{\mathcal{J}}(K_\beta) &\le \frac{1}{\sqrt{4\pi\beta}} \int_{-t_c}^{t_c} e^{-t^2/(4\beta)+|t|\bar{\mathcal{J}}(V_{AB})} dt \\ &\le \frac{e^{t_c\bar{\mathcal{J}}(V_{AB})}}{\sqrt{4\pi\beta}} \int_{-t_c}^{t_c} e^{-t^2/(4\beta)} dt \le e^{2\beta\Delta\bar{\mathcal{J}}(V_{AB})},\end{aligned} \tag{247}$$

where we use Eq. (245) in the last inequality. This completes the proof. ∎

C. Proof of Theorem 4

We begin by considering the Schmidt-rank truncation problem for the initial ground state $|\Omega^{(\nu)}\rangle$ with $\nu = 0$. In general, any restrictions to Rényi entanglement with $\alpha \ge 1$ cannot ensure the efficiency guarantee of the truncation [30,31]. However, by utilizing the fact that $|\Omega^{(0)}\rangle$ is the gapped ground state, we can prove the following proposition, which plays a key role in the proof:

Proposition 15. For the reference Hamiltonian $H^{(0)}$ in Eq. (227), we assume the gap condition (228) and the restriction (230) to the ∞-Rényi entanglement. Then, there exists a quantum state $|\psi_D\rangle$ with Schmidt rank D such that

$$\| |\Omega^{(0)}\rangle - |\psi_D\rangle \| \le 4e^{S_0/2} D^{-\kappa_\Delta}, \tag{248}$$

with

$$\kappa_\Delta := \frac{\Delta}{2\Delta + 4\tilde{g}}. \tag{249}$$

Recall that $\tilde{g}$ is the upper bound for the SE strength for all $V_{AB}^{(\nu)}$ $(0 \le \nu \le 1)$ as in Eq. (227).

Proof of Proposition 15. We first denote the Schmidt decomposition of $|\Omega^{(0)}\rangle$ by

$$|\Omega^{(0)}\rangle = \sum_s \lambda_s |\phi_{A,s}\rangle \otimes |\phi_{B,s}\rangle. \tag{250}$$

Then, we have the following general equation:

$$\lambda_1 = e^{-S_0/2}, \tag{251}$$

where we use $-\log(\lambda_1^2) = E_\infty(\Omega^{(0)}) = S_0$. We cannot get further information on λ_s with $s \ge 2$ only from the information on the ∞-Rényi entanglement.

To circumvent the difficulty stemming from this limitation, we recover the decay rate of $\{\lambda_s\}_s$ from the condition (251) using the AGSP formalism. To proceed along this line, we decompose $|\phi_{A,1}, \phi_{B,1}\rangle$ in Eq. (250) into

$$|\phi_{A,1}, \phi_{B,1}\rangle =: |\phi_1\rangle = \lambda_1 |\Omega^{(0)}\rangle + \sqrt{1-\lambda_1^2}\, |\Omega_\perp^{(0)}\rangle, \tag{252}$$

where $\langle\Omega^{(0)}|\Omega_\perp^{(0)}\rangle = 0$. To recover $|\Omega^{(0)}\rangle$ from $|\phi_{A,1}, \phi_{B,1}\rangle$, we utilize the AGSP K_β constructed in Proposition 14, which yields

$$\begin{aligned}\left\| \lambda_1^{-1} K_\beta |\phi_1\rangle - |\Omega^{(0)}\rangle \right\| &\le \left\| (K_\beta - 1)|\Omega^{(0)}\rangle \right\| \\ &\quad + \frac{\sqrt{1-\lambda_1^2}}{\lambda_1} \left\| K_\beta |\Omega_\perp^{(0)}\rangle \right\| \\ &\le \frac{3}{\lambda_1} e^{-\beta\Delta^2}.\end{aligned} \tag{253}$$

On the other hand, from inequality (240), there exists an (unnormalized) quantum state $|\phi_D\rangle$ satisfying

$$\| K_\beta |\phi_1\rangle - |\phi_D\rangle \| \le \frac{e^{2\beta\Delta\tilde{g}}}{D^{1/2}}, \tag{254}$$

where we use the parameter $\tilde{g}$, which upper-bounds the SE strength as in Eq. (227). Therefore, by letting $|\psi_D\rangle := \lambda_1^{-1}|\phi_D\rangle$ and combining the inequalities (253) and (254), we obtain

$$\left\| |\Omega^{(0)}\rangle - |\psi_D\rangle \right\| \le \frac{3}{\lambda_1} e^{-\beta\Delta^2} + \frac{e^{2\beta\Delta\tilde{g}}}{\lambda_1 D^{1/2}}. \tag{255}$$

By choosing β such that $e^{-\beta\Delta^2} = e^{2\beta\Delta\tilde{g}} D^{-1/2}$, or equivalently,

$$\beta = \frac{1}{2\Delta(\Delta + 2\tilde{g})} \log(D), \tag{256}$$

we have the main inequality as

$$\left\| |\Omega^{(0)}\rangle - |\psi_D\rangle \right\| \le \frac{4}{\lambda_1} D^{-\Delta/(2\Delta+4\tilde{g})} = 4e^{S_0/2} D^{-\kappa_\Delta}, \tag{257}$$

where we use Eq. (251) and the notation κ_Δ introduced in Eq. (249). This completes the proof. ∎

To analyze the target ground state $|\Omega^{(\nu)}\rangle$ with $\nu = 1$, we next utilize the adiabatic time evolution using the parametrized Hamiltonian $H^{(\nu)}$ $(0 \le \nu \le 1)$ as follows:

$$U_{0\to 1/\varepsilon} := \mathcal{T} e^{-i\int_0^{1/\varepsilon} H^{(ex)} dx}, \tag{258}$$

where ε is sufficiently small as will be chosen in Eq. (269). We then approximate the target ground state $|\Omega^{(1)}\rangle$ by $U_{0\to 1/\varepsilon}|\Omega^{(0)}\rangle$, which we denote by $|\psi_\varepsilon\rangle$:

$$|\psi_\varepsilon\rangle \coloneqq U_{0\to 1/\varepsilon}|\Omega^{(0)}\rangle. \tag{259}$$

Choose the phase of $|\Omega^{(1)}\rangle$ to minimize the distance to $|\psi_\varepsilon\rangle$. The projector estimate of Ref. [56] Theorem 3, including the second-derivative integral, bounds this vector distance with an additional factor $\sqrt{2}$:

$$\left\||\psi_\varepsilon\rangle - |\Omega^{(1)}\rangle\right\| \le \frac{\sqrt{2}\tilde{c}_0\tilde{g}\varepsilon}{\Delta^2}\left(3+\frac{7\tilde{c}_0\tilde{g}}{\Delta}\right). \tag{260}$$

Let us denote the Schmidt decomposition of $|\Omega^{(0)}\rangle$ by

$$|\Omega^{(0)}\rangle = \sum_s \lambda_s |\phi_{A,s}\rangle \otimes |\phi_{B,s}\rangle. \tag{261}$$

We then define $|\psi_{\ell,t}\rangle$ as follows:

$$|\psi_{\ell,t}\rangle \coloneqq a_\ell^{-1} U_{0\to t}\sum_{s=1}^{\ell}\lambda_s|\phi_{A,s}\rangle\otimes|\phi_{B,s}\rangle, \qquad a_\ell^2 = \sum_{s\le\ell}\lambda_s^2, \tag{262}$$

where the truncation number ℓ will be appropriately chosen afterward, and a_ℓ is a normalization factor. We obtain the overlap between $|\psi_\varepsilon\rangle$ and $|\psi_{\ell,1/\varepsilon}\rangle$ by

$$\begin{aligned}\langle\psi_{\ell,1/\varepsilon}|\psi_\varepsilon\rangle &= \left(a_\ell^{-1}\sum_{s=1}^{\ell}\lambda_s\langle\phi_{A,s},\phi_{B,s}|\right)|\Omega^{(0)}\rangle \\ &= \left(\sum\nolimits_{s\le\ell}\lambda_s^2\right)^{1/2} = a_\ell,\end{aligned} \tag{263}$$

where we use the definition (259) for $|\psi_\varepsilon\rangle$.

We next estimate the lower bound of a_ℓ. Applying the Eckart-Young theorem [74] and employing the quantum state $|\psi_D\rangle$ described in Proposition 15, we obtain

$$\sum_{s>\ell}\lambda_s^2 \le \left\||\Omega^{(0)}\rangle - |\psi_{D=\ell}\rangle\right\|^2 \le 16e^{S_0}\ell^{-2\kappa_\Delta}, \tag{264}$$

where we use the error bound (248) with $D=\ell$. Hence, we obtain the lower bound of a_ℓ^2 in the form of

$$a_\ell^2 = 1-\sum_{s>\ell}\lambda_s^2 \ge 1-16e^{S_0}\ell^{-2\kappa_\Delta}. \tag{265}$$

Therefore, to achieve $a_\ell^2 \ge 1/2$, we need to choose ℓ such that

$$\ell = \left\lceil (32e^{S_0})^{1/(2\kappa_\Delta)}\right\rceil \le e^{(S_0+4)/(2\kappa_\Delta)}. \tag{266}$$

We further estimate the overlap between $|\psi_{\ell,1/\varepsilon}\rangle$ and the target ground state $|\Omega^{(1)}\rangle$. We use the decomposition of

$$\langle\psi_{\ell,1/\varepsilon}|\psi_\varepsilon\rangle = \langle\psi_{\ell,1/\varepsilon}|(|\psi_\varepsilon\rangle - |\Omega^{(1)}\rangle) + \langle\psi_{\ell,1/\varepsilon}|\Omega^{(1)}\rangle. \tag{267}$$

By combining the error bound (260) and Eq. (263) with $a_\ell^2 \ge 1/2$, we derive the inequality

$$\left|\langle\psi_{\ell,1/\varepsilon}|\Omega^{(1)}\rangle\right| \ge \frac{1}{\sqrt{2}} - \frac{\sqrt{2}\tilde{c}_0\tilde{g}\varepsilon}{\Delta^2}\left(3+\frac{7\tilde{c}_0\tilde{g}}{\Delta}\right). \tag{268}$$

We thus choose the parameter ε as

$$\varepsilon = \frac{\Delta^3}{4\tilde{c}_0\tilde{g}(3\Delta+7\tilde{c}_0\tilde{g})}, \tag{269}$$

which gives

$$\left|\langle\psi_{\ell,1/\varepsilon}|\Omega^{(1)}\rangle\right| \ge \frac{1}{2\sqrt{2}} \ge \frac{1}{3}. \tag{270}$$

We then construct an approximation of the target ground state $|\Omega^{(1)}\rangle$ based on the state $|\psi_{\ell,1/\varepsilon}\rangle$. After an irrelevant phase adjustment of $|\psi_{\ell,1/\varepsilon}\rangle$, we decompose

$$|\psi_{\ell,1/\varepsilon}\rangle = c_1|\Omega^{(1)}\rangle + \sqrt{1-c_1^2}|\Omega_\perp^{(1)}\rangle, \qquad c_1 \ge \frac{1}{3}, \tag{271}$$

where $\langle\Omega^{(1)}|\Omega_\perp^{(1)}\rangle = 0$. To recover $|\Omega^{(1)}\rangle$ from $|\psi_{\ell,t}\rangle$, we utilize the same inequality as Eq. (253) that is based on the AGSP K_β in Proposition 14:

$$\left\|c_1^{-1}K_\beta|\psi_{\ell,1/\varepsilon}\rangle - |\Omega^{(1)}\rangle\right\| \le \frac{3}{c_1}e^{-\beta\Delta^2} \le 9e^{-\beta\Delta^2}. \tag{272}$$

We consider the approximation of $c_1^{-1}K_\beta|\psi_{\ell,1/\varepsilon}\rangle$ using a quantum state with a small Schmidt rank. We denote the Schmidt decomposition of the quantum states $|\psi_{\ell,1/\varepsilon}\rangle$ and $c_1^{-1}K_\beta|\psi_{\ell,1/\varepsilon}\rangle$ by

$$\begin{aligned}|\psi_{\ell,1/\varepsilon}\rangle &= \sum_{s=1}^{\infty}\tilde{\lambda}_s|\tilde{\phi}_{A,s}\rangle\otimes|\tilde{\phi}_{B,s}\rangle, \\ c_1^{-1}K_\beta|\psi_{\ell,1/\varepsilon}\rangle &= \sum_{s=1}^{\infty}\tilde{\lambda}_{\beta,s}|\tilde{\phi}_{A,\beta,s}\rangle\otimes|\tilde{\phi}_{B,\beta,s}\rangle,\end{aligned} \tag{273}$$

respectively. The Rényi entanglement $E_{\alpha=1/2}(\psi_{\ell,1/\varepsilon})$ is upper-bounded by the spectral-SIE bound (52) as follows:

$$E_{\alpha=1/2}(\psi_{\ell,1/\varepsilon}) \le E_{\alpha=1/2}(\psi_{\ell,t=0}) + \int_0^{1/\varepsilon} \frac{d}{dt} E_{\alpha=1/2}(\psi_{\ell,t}) dt \le \log(\ell) + \frac{2\tilde{g}}{\varepsilon}, \tag{274}$$

where $|\psi_{\ell,t=0}\rangle$ has the Schmidt rank ℓ [see Eq. (262)] and hence, $E_{\alpha=1/2}(\psi_{\ell,t=0}) \le \log(\ell)$ as a trivial bound. Hence, we obtain

$$\begin{aligned}\sum_{s=1}^{\infty} \tilde{\lambda}_{\beta,s} &\le \sum_{s=1}^{\infty} \tilde{\lambda}_s \bar{\mathcal{J}}(K_\beta/c_1) \\ &= \bar{\mathcal{J}}(K_\beta/c_1) e^{E_{\alpha=1/2}(\psi_{\ell,1/\varepsilon})/2} \\ &\le 3\ell^{1/2} e^{2\beta\Delta\tilde{g}+\tilde{g}/\varepsilon},\end{aligned} \tag{275}$$

where we use inequality (168) and the upper bound (239) for the SE strength of K_β in Proposition 14. We can thus truncate the Schmidt rank of $c_1^{-1}K_\beta|\psi_{\ell,t}\rangle$ up to D and construct a quantum state $|\psi_{\beta,D}\rangle$, which satisfies

$$\| c_1^{-1} K_\beta |\psi_{\ell,1/\varepsilon}\rangle - |\psi_{\beta,D}\rangle \| \le \frac{3\ell^{1/2} e^{(2\beta\Delta+1/\varepsilon)\tilde{g}}}{D^{1/2}}, \tag{276}$$

where we use an inequality similar to Eq. (33).

By applying the above inequality to Eq. (272), we obtain

$$\left\| |\psi_{\beta,D}\rangle - |\Omega^{(1)}\rangle \right\| \le 9e^{-\beta\Delta^2} + \frac{3\ell^{1/2} e^{(2\beta\Delta+1/\varepsilon)\tilde{g}}}{D^{1/2}}. \tag{277}$$

By choosing the parameter β such that $e^{-\beta\Delta^2} = e^{2\beta\Delta\tilde{g}} D^{-1/2}$, or equivalently,

$$\beta = \frac{1}{2\Delta(\Delta + 2\tilde{g})} \log(D), \tag{278}$$

we have

$$\begin{aligned}\left\| |\psi_{\beta,D}\rangle - |\Omega^{(1)}\rangle \right\| &\le (9 + 3\ell^{1/2} e^{\tilde{g}/\varepsilon}) D^{-\Delta/(2\Delta+4\tilde{g})} \\ &\le 12\ell^{1/2} e^{\tilde{g}/\varepsilon} D^{-\kappa_\Delta},\end{aligned} \tag{279}$$

where we use the definition of κ_Δ in Eq. (232).

Finally, we upper-bound the coefficient $12\ell^{1/2}e^{\tilde{g}/\varepsilon}$ in Eq. (279). As in Eqs. (266) and (269), ℓ and ε satisfy

$$\ell \le e^{(S_0+4)/(2\kappa_\Delta)}, \qquad \varepsilon = \frac{\Delta^3}{4\tilde{c}_0\tilde{g}(3\Delta + 7\tilde{c}_0\tilde{g})}, \tag{280}$$

and hence,

$$\begin{aligned}\log(12\ell^{1/2} e^{\tilde{g}/\varepsilon}) &\le \frac{S_0+4}{4\kappa_\Delta} + \log(12) \\ &\quad + \frac{4\tilde{c}_0\tilde{g}^3(3\Delta/\tilde{g} + 7\tilde{c}_0)}{\Delta^3} = \log(C_{\tilde{g},\Delta,S_0}).\end{aligned} \tag{281}$$

This reduces inequality (279) to the first main inequality (231) by setting $|\psi_D\rangle$ to be $|\psi_{\beta,D}\rangle$.

To derive the second main inequality (234), we prove the following lemma:

Lemma 16. Let us consider a quantum state $|\psi\rangle$ that is approximated by another unnormalized quantum state $|\psi_D\rangle$ with Schmidt rank D as follows:

$$\| |\psi\rangle - |\psi_D\rangle \| \le \frac{C}{D^\kappa} \quad \forall\, D \in \mathbb{N}, \tag{282}$$

where $C \ge 1$ and $\kappa > 0$ are constants. Then, the entanglement entropy is upper-bounded by

$$E_{\alpha=1}(\psi) \le c_{\kappa,1} \log(C) + c_{\kappa,2}, \tag{283}$$

where the constants $c_{\kappa,1}$ and $c_{\kappa,2}$ are defined by

$$c_{\kappa,1} := \frac{2 - 2^{-2\kappa}}{\kappa(1 - 2^{-2\kappa})}, \qquad c_{\kappa,2} := \frac{(6 + 2\kappa)\log(2)}{(1 - 2^{-2\kappa})^2}. \tag{284}$$

By applying this lemma with the first main inequality (231), i.e., $\kappa \to \kappa_\Delta$ and $C \to C_{\tilde{g},\Delta,S_0}$, we immediately prove the second main inequality (234). This completes the proof of Theorem 4. ∎

Proof of Lemma 16. We consider a set of $\{D_p\}_{p=0}^{\infty}$ such that $D_p = 2^{p+p_0}$ and define δ_p as

$$\delta_p := \frac{C}{D_p^\kappa} = C 2^{-(p+p_0)\kappa}, \tag{285}$$

where the value of p_0 will be chosen appropriately to satisfy a condition required for the proof. Temporarily, we leave p_0 free and proceed. Then, by using Ref. [45] Supplemental Proposition 3, we obtain

$$\begin{aligned}E_{\alpha=1}(\psi) &\le \log(D_0) + \sum_{p=0}^{\infty} \delta_p^2 \log\left(\frac{3D_{p+1}}{\delta_p^2}\right) = p_0\log(2) + C^2 \sum_{p=0}^{\infty} 2^{-2(p+p_0)\kappa} \log\left(\frac{6}{C^2} 2^{(p+p_0)(2\kappa+1)}\right) \\ &\le p_0\log(2) + \frac{2^{-2p_0\kappa}C^2}{1 - 2^{-2\kappa}} \log\left(\frac{6 \cdot 2^{p_0(2\kappa+1)}}{C^2}\right) + \frac{C^2 2^{-2\kappa(p_0+1)}(2\kappa+1)\log(2)}{(1 - 2^{-2\kappa})^2},\end{aligned} \tag{286}$$

where we use $\sum_{p=0}^{\infty} px^p = x(d/dx)\sum_{p=0}^{\infty} x^p = x(d/dx)1/(1-x) = x/(1-x)^2$. Finally, by choosing $p_0 = \lceil (2\kappa)^{-1}\log_2(C^2)\rceil \le (2\kappa)^{-1}\log_2(C^2)+1$, we obtain $2^{-2p_0\kappa} \le C^{-2}$, and hence, the main inequality is proven as follows:

$$\begin{aligned} E_{\alpha=1}(\psi) &\le p_0\log(2) + \frac{1}{1-2^{-2\kappa}}\log(6\times 2^{p_0+2\kappa}) \\ &\quad + \frac{2^{-2\kappa}(2\kappa+1)\log(2)}{(1-2^{-2\kappa})^2} \\ &\le \frac{2-2^{-2\kappa}}{\kappa(1-2^{-2\kappa})}\log(C) + \log(2)\frac{6+2\kappa}{(1-2^{-2\kappa})^2}. \end{aligned} \tag{287}$$

This completes the proof. ∎

To obtain the cubic scaling, put $\kappa = \kappa_\Delta$ and $L = \sqrt{\ell}e^{\tilde{g}/\epsilon}$. For $D > L^2$, choose $\beta = (\log D - 2\log L)/[2\Delta(\Delta+2\tilde{g})]$ in Eq. (277). The error is at most $C_* D^{-\kappa}$ with $C_* = 12L^{2\kappa}$; for $D \le L^2$, the zero-vector approximation suffices. A Schmidt tail $\sum_{j>D} p_j \le C^2 D^{-2\kappa}$, $C \ge 1$ implies

$$S(p) \le (1+\kappa)[\log 2 + \kappa^{-1}\log C + (2\kappa)^{-1}] + \log(1+\kappa^{-1}).$$

Indeed, integrating $\Pr(j > e^x) \le \min\{1, C^2\lfloor e^x\rfloor^{-2\kappa}\}$ bounds $\sum_j p_j \log j$ by the bracketed quantity; comparison with $j^{-1-\kappa}/\zeta(1+\kappa)$ using relative entropy gives the displayed bound. With $(C = C_*)$, this entropy bound reads $S \le 2(1+\kappa)\log L + O(\kappa^{-1})$. Equations (266) and (269) give $\log L = O(1+R^3)$ and $\kappa^{-1} = 2+4R$ for $R = \tilde{g}/\Delta$, proving $S = O(1+R^3)$ at fixed $S_0, \tilde{c}_0$.

VIII. SIMULATING 1D LONG-RANGE INTERACTIONS WITH POLYNOMIAL COMPLEXITY

In this section, we investigate the simulation of one-dimensional quantum systems with long-range interactions. Our focus is on the computational complexity of simulating ground states, quantum dynamics, and quantum thermal states on a classical computer. The best-known results to date achieve only quasipolynomial complexity requiring computational cost of the form $e^{\mathrm{poly}\log(n/\epsilon)}$ in terms of the system size n and error tolerance ϵ. The goal of this work is to improve this to polynomial complexity, namely, $\mathrm{poly}(n/\epsilon)$ or $(n/\epsilon)^{O(1)}$.

More concretely, we establish the following results:

(i) Ground states and thermal states (Theorems 5 and 7): We prove the existence of efficient descriptions in terms of MPSs and MPOs with polynomial bond dimension.

(ii) Quantum dynamics: We prove polynomial-bond-dimension approximability of real-time evolved states (Proposition 18) and provide a rigorous *a priori* truncation-error guarantee for tDMRG-type simulations based on MPS representations (Theorem 6).

We also comment on the relation to numerical tensor-network simulations. Numerical DMRG and time-evolution studies have long suggested that many physically relevant one-dimensional long-range systems, including power-law interacting systems such as dipolar and van der Waals systems, can often be efficiently represented by MPS in practice [75,76]. In parallel, several MPO- and MPS-based approaches have been developed to represent long-range Hamiltonians and simulate their real-time dynamics [77–82]. Our result is complementary to these numerical observations. Rather than proposing a new heuristic algorithm, we provide an *a priori* rigorous explanation for why polynomial bond dimension suffices when $\eta > 2$, and we give explicit Schmidt-tail bounds that certify the truncation error.

A. SE strength for *k*-local Hamiltonians

We first show a simple lemma on the SE strength of the 1D long-range interacting Hamiltonians [see Eq. (11)]. This lemma shows that the SE strength is given by an $\mathcal{O}(1)$ constant as long as the power-law decay is faster than r^{-2}.

Lemma 17. Let H be an arbitrary 1D Hamiltonian defined by Eq. (8) with Eqs. (9) and (11). Then, for an arbitrary decomposition $\Lambda = A \sqcup B$ with $A = \{1,2,\ldots,i\}$ and $B = \{i+1, i+2, \ldots, n\}$, the SE strength $\bar{\mathcal{J}}(V_{AB})$ is upper-bounded by

$$\bar{\mathcal{J}}(V_{AB}) \le \tilde{J} := d^{\lfloor k/2\rfloor}\frac{\eta J_0}{\eta-2}, \tag{288}$$

where $d = \max_i \mathcal{D}_{\{i\}}$, and J_0 and η characterize the interaction decay as in Eq. (11).

Remark. When the interaction decay is slower than r^{-2}, the entanglement generation is known to be highly enhanced even in the context of the standard SIE theorem [29]. Most of the locality properties break down for $\eta < 2$ [20,45,57,73,83] with a few exceptions [52,84]

Proof of Lemma 17. The action of h_Z on a normalized product state, including local ancillas, has norm at most $\|h_Z\|$ and Schmidt rank at most $d^{2\ \min(|Z\cap A|,|Z\cap B|)}$. Cauchy-Schwarz gives $\bar{\mathcal{J}}(h_Z) \le d^{\lfloor k/2\rfloor}\|h_Z\|$. Subadditivity then gives

$$\bar{\mathcal{J}}(V_{AB}) \le d^{\lfloor k/2\rfloor}\sum_{\substack{Z: Z\cap A\neq\emptyset,\\ Z\cap B\neq\emptyset}} \|h_Z\|. \tag{289}$$

The above quantity was simply treated using inequality (11) in Ref. [45] Supplementary Lemma 1, which reduces the above inequality to the desired one (288). This completes the proof. ∎

B. Bottleneck in the previous studies and our ideas

The origin of the improvement can be understood from the way low-rank approximations are controlled. In many previous approaches, one approximates functions of the long-range Hamiltonian, such as time-evolution or filtering operators, by polynomials in H. If the polynomial degree is m, then the Schmidt rank of the resulting operator is typically governed by the Schmidt-rank growth of powers of the Hamiltonian. For one-dimensional long-range interactions, this growth can scale as $n^{O(m)}$. Therefore, unless the required degree m is independent of the system size n and the target precision, the resulting bond dimension is not polynomial in general; for example, even a mildly growing degree such as $m = \mathcal{O}[\log(n)]$ already leads to quasipolynomial complexity [45,57,58]. This rank growth is the main bottleneck of polynomial-approximation-based methods.

Our approach avoids this bottleneck by not treating all interaction terms as contributing equally to the Schmidt rank. Instead, the key point is to keep track of the weights of the interaction terms that actually contribute to entanglement generation and to low-rank approximation. The spectral-entangling strength provides precisely such a weighted measure: it quantifies how much each boundary interaction can contribute to the generation of the Schmidt spectrum. Thus, rather than estimating the Schmidt rank only by counting how many interaction terms appear in powers of H, the spectral-SIE framework controls the accumulated entangling weight of those terms.

This weighted viewpoint is the essential mechanism behind the improvement from quasipolynomial to polynomial complexity. In the low-rank approximation result used below, strong interaction components are retained explicitly, while the accumulated contribution of sufficiently weak components is controlled as an approximation error. Consequently, the approximation cost is governed by the weighted tail of the interaction decomposition rather than by the formal Schmidt rank generated by polynomial powers of the Hamiltonian.

C. MPS description of the long-range ground state

In this subsection, we discuss the MPS representation of gapped ground states in one-dimensional long-range interacting systems. The best-known result to date shows that approximating the ground state within error ϵ requires an MPS with bond dimension $e^{\log^{5/2}(n/\epsilon)}$, where n denotes the system size.

Here, we improve upon this bound by extending the result of Ref. [45]. In particular, building on Proposition 14, which establishes an AGSP construction tailored to long-range systems, we show that the ground state admits an efficient polynomially bounded MPS approximation:

Theorem 5. Let $|\Omega\rangle$ be the normalized, unique ground state, with gap $\Delta > 0$, of a long-range interacting Hamiltonian H as in Eq. (8) with Eqs. (9) and (11). For an arbitrary cut of the total system as $\Lambda = A_s \sqcup B_s$ (i.e., $A_s = \{1, 2, \ldots, s\}$), we consider the Schmidt decomposition of $|\Omega\rangle$ as follows:

$$|\Omega\rangle = \sum_{j=1} \lambda_j^{(s)} |\Omega_{A_s,j}\rangle \otimes |\Omega_{B_s,j}\rangle. \tag{290}$$

Then, the truncation error of the Schmidt rank is bounded above by

$$\sum_{j>D} \left(\lambda_j^{(s)}\right)^2 \leq 32 D_0 D^{-\Delta/(2\tilde{J}+\Delta)}, \tag{291}$$

where $D_0 \geq 1$ is a seed constant, uniform in the cut, ensuring Eq. (296). It can be chosen with

$$\log D_0 \leq c^* \log^2(d) \left(\frac{\log d}{\delta}\right)^{1+2/\bar{\eta}} \log^{3+3/\bar{\eta}}\left(2 + \frac{\log d}{\delta}\right), \tag{292}$$

where $\bar{\eta} = \eta - 2$, $d := \max_{i\in\Lambda} \mathcal{D}_{\{i\}}$, and $\delta := \min\{\Delta/g, 1/2\}$ is a dimensionless gap lower bound for H/g. The constant c^* depends only on $k, \eta, J_0/g$. The area law of Ref. [45] allows this choice with $E_1(\Omega) \leq \log(2D_0)$ across every cut. Thus $D_0 = O(1)$ for fixed local parameters and a size-independent positive gap. Note that the constant $\tilde{J}$ has been defined in Eq. (288) as an upper bound for the SE strength of V_{AB}.

Moreover, for every contiguous interval X, there exists a normalized MPS $|M(D, X)\rangle$ of bond dimension at most D satisfying

$$\begin{aligned} &\| \mathrm{tr}_{X^c}(|\Omega\rangle\langle\Omega|) - \mathrm{tr}_{X^c}[|M(D,X)\rangle\langle M(D,X)|] \|^2 \\ &\leq 64(|X|+1) D_0 D^{-\Delta/(2\tilde{J}+\Delta)}. \end{aligned} \tag{293}$$

The approximating MPS may depend on X.

Remark. The global truncation estimate [85] Lemma 1 gives a normalized, phase-aligned MPS with $\| |\Omega\rangle - |M(D)\rangle \|^2 \leq 2\sum_s \delta_s^2(D) \leq 64 n D_0 D^{-\Delta/(2\tilde{J}+\Delta)}$. Thus, vector-norm accuracy ϵ is ensured by

$$D = \left\lceil \left(\frac{128 n D_0}{\epsilon^2}\right)^{1+2\tilde{J}/\Delta} \right\rceil. \tag{294}$$

1. Proof of Theorem 5

For the proof, we utilize the Eckart-Young theorem [74]:

$$\sum_{j>D} (\lambda_j^{(s)})^2 \leq \| |\Omega\rangle - |\tilde{\phi}_D\rangle \|^2, \tag{295}$$

where $|\tilde{\phi}_D\rangle$ is an arbitrary (unnormalized) quantum state with Schmidt rank D between A_s and B_s. Apply the area law in Ref. [45] Supplementary Theorem 1 to H/g with gap

lower bound δ. Its inter-region constant can be taken as $g_0 = \max\{1, \eta J_0/[g(\eta-2)]\}$ by Eq. (9). Enlarging c^* in Eq. (292) if necessary, choose D_0 with $E_1(\Omega) \le \log(2D_0)$. Since $-\log[(\lambda_1^{(s)})^2] \le E_1(\Omega)$, the leading Schmidt product state $|\phi\rangle$ satisfies

$$|\langle\phi|\Omega\rangle| \ge \frac{1}{\sqrt{2D_0}}. \tag{296}$$

We then apply the AGSP K_β to $|\phi\rangle$ and estimate the error of

$$\|\langle\Omega|\phi\rangle|\Omega\rangle - K_\beta|\phi\rangle\|. \tag{297}$$

By expanding $|\phi\rangle$ as

$$|\phi\rangle = \langle\Omega|\phi\rangle|\Omega\rangle + \sqrt{1-|\langle\Omega|\phi\rangle|^2}|\Omega_\perp\rangle, \qquad \langle\Omega|\Omega_\perp\rangle = 0, \tag{298}$$

we obtain

$$\begin{aligned}
&\|\langle\Omega|\phi\rangle|\Omega\rangle - K_\beta|\phi\rangle\| \\
&\le \left\|\langle\Omega|\phi\rangle(1-K_\beta)|\Omega\rangle + \sqrt{1-|\langle\Omega|\phi\rangle|^2}K_\beta|\Omega_\perp\rangle\right\| \\
&\le |\langle\Omega|\phi\rangle| \times \|(1-K_\beta)|\Omega\rangle\| \\
&\quad + \sqrt{1-|\langle\Omega|\phi\rangle|^2}\|K_\beta|\Omega_\perp\rangle\| \le 3e^{-\beta\Delta^2},
\end{aligned} \tag{299}$$

where we use inequality (238) in Proposition 14.

Moreover, using inequality (240) in Proposition 14, we can also prove the following approximation:

$$\|K_\beta|\phi\rangle - |\phi_D\rangle\| \le \frac{e^{2\beta\Delta\tilde{J}}}{\sqrt{D}}, \tag{300}$$

where $|\phi_D\rangle$ has the Schmidt rank of D for the bipartition A_s and B_s. Combining inequality (299) with this approximation gives

$$\|\langle\Omega|\phi\rangle|\Omega\rangle - |\phi_D\rangle\| \le 3e^{-\beta\Delta^2} + \frac{e^{2\beta\Delta\tilde{J}}}{\sqrt{D}}. \tag{301}$$

Finally, by applying the lower bound (296) to the above inequality, we arrive at the error bound of

$$\left\||\Omega\rangle - \frac{1}{\langle\Omega|\phi\rangle}|\phi_D\rangle\right\| \le 3\sqrt{2D_0}e^{-\beta\Delta^2} + \frac{\sqrt{2D_0}e^{2\beta\Delta\tilde{J}}}{\sqrt{D}}. \tag{302}$$

The two terms on the rhs of inequality (302) have opposite dependence on β. The first term $e^{-\beta\Delta^2}$ is the AGSP filtering error and decreases with β, whereas the second term $e^{2\beta\Delta\tilde{J}}/\sqrt{D}$ is the Schmidt-truncation error of the filtered state and increases with β. We therefore choose β by balancing these two contributions:

$$e^{-\beta\Delta^2} = \frac{e^{2\beta\Delta\tilde{J}}}{\sqrt{D}},$$

which gives

$$\beta = \frac{1}{2\Delta(2\tilde{J}+\Delta)}\log(D). \tag{303}$$

This choice further reduces inequality (302) to

$$\left\||\Omega\rangle - \frac{1}{\langle\Omega|\phi\rangle}|\phi_D\rangle\right\| \le 4\sqrt{2D_0}e^{-\frac{\Delta}{2(2\tilde{J}+\Delta)}\log(D)}. \tag{304}$$

Therefore, by letting $|\tilde{\phi}_D\rangle = |\phi_D\rangle/\langle\Omega|\phi\rangle$ in inequality (295), we prove the first main inequality (291).

Define the Schmidt-tail errors of $|\Omega\rangle$ by

$$\delta_s(D) = \left\|\sum_{j>D}\lambda_j^{(s)}|\Omega_{A_s,j}\rangle \otimes |\Omega_{B_s,j}\rangle\right\| = \left[\sum_{j>D}(\lambda_j^{(s)})^2\right]^{1/2}. \tag{305}$$

For $X = \{i_0, \ldots, i_1\}$, block the two exterior regions and sequentially Schmidt-truncate across cuts $s = i_0 - 1, \ldots, i_1$. The nested retained subspaces give $\langle\Omega|\phi\rangle = \|\phi\|^2$ and $1 - \|\phi\|^2 \le \sum_{s=i_0-1}^{i_1}\delta_s^2(D)$. For $\chi = \phi/\|\phi\|$, contraction of trace distance yields $\|\rho_X - \chi_X\|^2 \le 1 - \|\phi\|^2$ in operator norm, where ρ_X and χ_X are the reduced states of Ω and χ. Each exterior block has support dimension at most D; a unitary within it maps a support basis to at most D computational-basis product vectors, giving bond dimension at most D everywhere without changing χ_X. Thus, there is an MPS $|M(D,X)\rangle$ with

$$\begin{aligned}
&\|\mathrm{tr}_{X^c}(|\Omega\rangle\langle\Omega|) - \mathrm{tr}_{X^c}[|M(D,X)\rangle\langle M(D,X)|]\|^2 \\
&\le 2\sum_{s=i_0-1}^{i_1}\delta_s^2(D).
\end{aligned} \tag{306}$$

Here, $i_1 - i_0 = |X| - 1$ and absent boundary tails are zero. If the tail sum is at least 1, any normalized product state suffices. By combining the inequalities (295) and (304), we obtain

$$\delta_s^2(D) \le 32D_0D^{-\Delta/(2\tilde{J}+\Delta)}, \tag{307}$$

which reduces the upper bound (306) to the second main inequality (293). This completes the proof. ∎

D. Quantum dynamics

We next consider the quantum dynamics. We begin with an initial product state $|\phi\rangle$ and consider the efficient computation of the time-evolved state $|\phi_t\rangle = e^{-iHt}|\phi\rangle$.

We first prove the existence of a good MPS approximation for the time-evolved state as a simple application of Theorem 1.

Proposition 18. We consider the 1D long-range Hamiltonian in Eq. (8) with Eqs. (9) and (11). For $t \geq 0$ and any given product state $|\phi\rangle$, there exists an MPS $|M_t(D)\rangle$ with the bond dimension D that approximates $e^{-iHt}|\phi\rangle$, satisfying the following error bound:

$$\||e^{-iHt}|\phi\rangle - |M_t(D)\rangle\|^2 \leq \frac{2e^{2\tilde{J}t}}{D}n, \tag{308}$$

where $\tilde{J}$ was defined in Eq. (288).

1. Proof of Proposition 18

Let us denote the Schmidt decomposition of $|\phi_t\rangle$ by

$$|\phi_t\rangle = \sum_j \lambda_j^{(s)} |\phi_{A_s,j}\rangle \otimes |\phi_{B_s,j}\rangle, \tag{309}$$

where the index s characterizes the decomposition of the total system ($\Lambda = A_s \sqcup B_s$). The global truncation estimate [85] Lemma 1 gives a normalized, phase-aligned MPS. For this purpose, we estimate the parameter $\delta_s(D)$ for the Schmidt decomposition:

$$\delta_s(D) = \left[\sum\nolimits_{j>D} (\lambda_j^{(s)})^2\right]^{1/2}, \tag{310}$$

which gives

$$\||\phi_t\rangle - |M_t(D)\rangle\|^2 \leq 2\sum_{s=1}^{n-1} \delta_s^2(D). \tag{311}$$

Using Theorem 1 with the parameter $\tilde{J}$, we can upper-bound the (1/2)-Rényi-entanglement entropy between A_s and B_s as

$$E_{1/2}(\phi_t) \leq 2\tilde{J}t, \tag{312}$$

and hence, we can derive from inequality (45) in Lemma 5

$$\lambda_j^{(s)} \leq \frac{e^{E_{1/2}(\phi_t)/2}}{j} \leq \frac{e^{\tilde{J}t}}{j}, \tag{313}$$

where we choose $\alpha = 1/2$. Therefore, we obtain

$$\delta_s^2(D) = \sum_{j>D} (\lambda_j^{(s)})^2 \leq e^{2\tilde{J}t} \sum_{j>D} j^{-2} \leq \frac{e^{2\tilde{J}t}}{D}, \tag{314}$$

which reduces the upper bound (311) to the main inequality (308). This completes the proof. ∎

E. Precision guarantee for the tDMRG algorithm

In this section, we prove that the MPS in Proposition 18 can be efficiently computed with a precision guarantee by using the tDMRG algorithm. As has been pointed out in Ref. [32] (see also inequality (341) below), a rigorous precision guarantee for the tDMRG algorithm has been an open problem.

1. Review of tDMRG

We review the tDMRG algorithm [61,62,82]. The purpose of the tDMRG algorithm is to construct an MPS that approximates the time evolution

$$e^{-iHt}|\psi_0\rangle.$$

We start from a normalized product state $|M_0(D)\rangle = |\psi_0\rangle$, divide $t \geq 0$ into $\mathcal{N}$ steps, and set $\Delta t = t/\mathcal{N}$. Each Euler update $1 - iH\Delta t$ is followed by left-to-right exact Schmidt truncations of the current state to bond dimension D, without intermediate normalization:

(1) First, we apply the first-order expansion

$$e^{-iH\Delta t}|\psi\rangle \approx (1 - iH\Delta t)|\psi\rangle. \tag{315}$$

(2) The resulting state is then approximated by an MPS truncated up to bond dimension D:

$$(1 - iH\Delta t)|\psi\rangle \approx |M_1(D)\rangle, \tag{316}$$

where $|M_1(D)\rangle$ is generally unnormalized; its norm is bounded by the estimate following the definition of ϵ_m below. Repeating the same procedure, we approximate

$$e^{-iH\Delta t}|M_m(D)\rangle \approx |M_{m+1}(D)\rangle.$$

(3) After $\mathcal{N}$ steps, we obtain the MPS $|M_{\mathcal{N}}(D)\rangle$ as an approximation to $e^{-iHt}|\psi_0\rangle$.

For nonzero $|M_{\mathcal{N}}(D)\rangle$, normalization gives the following error bound:

$$\begin{aligned}
&\left\| \frac{|M_{\mathcal{N}}(D)\rangle}{\||M_{\mathcal{N}}(D)\rangle\|} - e^{-iHt}|\psi_0\rangle \right\| \\
&\leq \||M_{\mathcal{N}}(D)\rangle - e^{-iHt}|\psi_0\rangle\| + \left\| \frac{|M_{\mathcal{N}}(D)\rangle}{\||M_{\mathcal{N}}(D)\rangle\|} - |M_{\mathcal{N}}(D)\rangle \right\| \\
&\leq \||M_{\mathcal{N}}(D)\rangle - e^{-iHt}|\psi_0\rangle\| + |1 - \||M_{\mathcal{N}}(D)\rangle\||.
\end{aligned} \tag{317}$$

Using the reverse triangle inequality

$$|1 - \||M_{\mathcal{N}}(D)\rangle\|| \leq \|e^{-iHt}|\psi_0\rangle - |M_{\mathcal{N}}(D)\rangle\|,$$

we obtain the bound

$$\left\| \frac{|M_{\mathcal{N}}(D)\rangle}{\||M_{\mathcal{N}}(D)\rangle\|} - e^{-iHt}|\psi_0\rangle \right\| \le 2\epsilon,$$
$$\epsilon := \||M_{\mathcal{N}}(D)\rangle - e^{-iHt}|\psi_0\rangle\|. \tag{318}$$

2. Error guarantee and the challenging point

Here, we consider the error accumulation in each of the processes. We prove the following lemma:

Lemma 19. The number of steps $\mathcal{N}$ is chosen to satisfy

$$\frac{(gnt)^2}{\mathcal{N}} \le 1, \qquad \Delta t = t/\mathcal{N}. \tag{319}$$

Then, the total error in the tDMRG algorithm is bounded by

$$\|e^{-iHt}|\phi\rangle - |M_{\mathcal{N}}(D)\rangle\| \le \frac{(gnt)^2}{\mathcal{N}} + \sqrt{2n}\sum_{m=0}^{\mathcal{N}-1}\bar{\delta}_m(D), \tag{320}$$

where $\bar{\delta}_m(D)$ is defined by Eqs. (330) and (331) below.

Proof of Lemma 19. We first note that the condition (319) implies

$$\Delta t\|H\| \le gn\Delta t \le 1 \tag{321}$$

because of

$$\|H\| \le \sum_{i\in\Lambda}\sum_{Z:Z\ni i}\|h_Z\| \le \sum_{i\in\Lambda} g = gn, \tag{322}$$

where we use the parameter g in Eq. (8).

For the proof, we define the error ϵ_m as

$$\epsilon_m := \|e^{-iHm\Delta t}|\phi\rangle - |M_m(D)\rangle\|. \tag{323}$$

Schmidt truncation does not increase the norm. Hence, for $0 \le m \le \mathcal{N}$,

$$\||M_m(D)\rangle\| \le [1+(gn\Delta t)^2]^{m/2} \le e^{(gnt)^2/(2\mathcal{N})} \le \sqrt{e}.$$

We aim to derive an upper bound for $\epsilon_{\mathcal{N}}$. Then, we analyze how the error increases with the approximation step. Given the error ϵ_m and the MPS $|M_m(D)\rangle$, we consider ϵ_{m+1} and $|M_{m+1}(D)\rangle$. We start from

$$\left\| e^{-iH\Delta t}|M_m(D)\rangle - e^{-iH(m+1)\Delta t}|\phi\rangle \right\|$$
$$\le \||M_m(D)\rangle - e^{-iHm\Delta t}|\phi\rangle\| \le \epsilon_m. \tag{324}$$

Hence, we get

$$\epsilon_{m+1} = \left\| e^{-iH(m+1)\Delta t}|\phi\rangle - |M_{m+1}(D)\rangle \right\| \le \epsilon_m + \|e^{-iH\Delta t}|M_m(D)\rangle - |M_{m+1}(D)\rangle\|. \tag{325}$$

In the following, we estimate the error $\|e^{-iH\Delta t}\times |M_m(D)\rangle - |M_{m+1}(D)\rangle\|$. For Hermitian H, the integral Taylor remainder gives

$$\|e^{-iH\Delta t} - (1-iH\Delta t)\| \le \frac{1}{2}\Delta t^2\|H\|^2 \le \frac{1}{2}(gn\Delta t)^2, \tag{326}$$

which, together with $\||M_m(D)\rangle\| \le \sqrt{e}$ and $\sqrt{e}/2 < 1$, yields

$$\|e^{-iH\Delta t}|M_m(D)\rangle - (1-iH\Delta t)|M_m(D)\rangle\| \le (gn\Delta t)^2 = \frac{(gnt)^2}{\mathcal{N}^2}. \tag{327}$$

To bound the second term

$$\|(1-iH\Delta t)|M_m(D)\rangle - |M_{m+1}(D)\rangle\|, \tag{328}$$

we define the Schmidt decomposition of $(1-iH\Delta t)|M_m(D)\rangle$ between A_s and B_s as follows:

$$(1-iH\Delta t)|M_m(D)\rangle = \sum_j \lambda_{m,j}^{(s)}|\phi_{A_s,j}^{(m)}\rangle \otimes |\phi_{B_s,j}^{(m)}\rangle. \tag{329}$$

Sequential singular-value truncation, without normalization, gives the following bound in terms of the original cut tails:

$$\|(1-iH\Delta t)|M_m(D)\rangle - |M_{m+1}(D)\rangle\|^2 \le 2\sum_{s=1}^{n-1}\delta_{m,s}^2(D) \le 2n\bar{\delta}_m^2(D) \tag{330}$$

with

$$\bar{\delta}_m(D) := \max_{s\in[n-1]}[\delta_{m,s}(D)],$$
$$\delta_{m,s}(D) = \left[\sum\nolimits_{j>D}(\lambda_{m,j}^{(s)})^2\right]^{1/2}. \tag{331}$$

Note that the explicit form of $|M_{m+1}(D)\rangle$ is systematically constructed by using the canonical form of the matrix-product state [85,86].

By combining the inequalities (327) and (330), we derive

$$\|e^{-iH\Delta t}|M_m(D)\rangle - |M_{m+1}(D)\rangle\| \le \frac{(gnt)^2}{\mathcal{N}^2} + \sqrt{2n}\bar{\delta}_m(D). \tag{332}$$

Therefore, from the upper bound (325), we obtain

$$\epsilon_{m+1} \le \epsilon_m + \frac{(gnt)^2}{\mathcal{N}^2} + \sqrt{2n}\bar{\delta}_m(D). \tag{333}$$

By solving the above recurrence inequality and considering $\epsilon_{\mathcal{N}}$, we arrive at the desired inequality (320). This completes the proof. ∎

The remaining problem is to estimate the upper bound of $\bar{\delta}_m(D)$. The most frequently used way is to utilize the Eckart-Young theorem as follows. Let $|\phi_{s,D}\rangle$ be an arbitrary quantum state with Schmidt rank D between A_s and B_s. Then, we upper-bound the approximation error $\delta_{m,s}(D)$ in Eq. (331) by

$$\delta_{m,s}(D) \le \|(1 - iH\Delta t)|M_m(D)\rangle - |\phi_{s,D}\rangle\|. \tag{334}$$

This reduces the problem to finding a good state $|\phi_{s,D}\rangle$ to approximate $(1 - iH\Delta t)|M_m(D)\rangle$.

For this purpose, we consider $e^{-iH(m+1)\Delta t}|\phi\rangle$ and consider the Schmidt decomposition between A_s and B_s as follows:

$$e^{-iH(m+1)\Delta t}|\phi\rangle = \sum_j \tilde{\lambda}^{(s)}_{j,m+1}|\tilde{\phi}^{(m)}_{A_s,j}\rangle \otimes |\tilde{\phi}^{(m)}_{B_s,j}\rangle, \tag{335}$$

which is approximated by $|\psi_{s,D}\rangle$ as

$$|\psi_{s,D}\rangle := \sum_{j=1}^{D} \tilde{\lambda}^{(s)}_{j,m+1}|\tilde{\phi}^{(m)}_{A_s,j}\rangle \otimes |\tilde{\phi}^{(m)}_{B_s,j}\rangle. \tag{336}$$

Then, by following the proof of Proposition 18, we derive

$$\left\| e^{-iH(m+1)\Delta t}|\phi\rangle - |\psi_{s,D}\rangle \right\|^2 \le \frac{e^{2\tilde{J}t}}{D}. \tag{337}$$

Therefore, we also obtain

$$\begin{aligned}
&\|(1 - iH\Delta t)|M_m(D)\rangle - |\psi_{s,D}\rangle\| \\
&\le \|e^{-iH\Delta t}|M_m(D)\rangle - |\psi_{s,D}\rangle\| \\
&\quad + \|(1 - iH\Delta t - e^{-iH\Delta t})|M_m(D)\rangle\| \\
&\le \|e^{-iH\Delta t}(|M_m(D)\rangle - e^{-iHm\Delta t}|\phi\rangle)\| \\
&\quad + \left\| e^{-iH(m+1)\Delta t}|\phi\rangle - |\psi_{s,D}\rangle \right\| + \frac{(gnt)^2}{\mathcal{N}^2} \\
&\le \epsilon_m + \frac{e^{\tilde{J}t}}{D^{1/2}} + \frac{(gnt)^2}{\mathcal{N}^2},
\end{aligned} \tag{338}$$

which, when applied to Eq. (334), implies

$$\delta_{m,s}(D) \le \epsilon_m + \frac{e^{\tilde{J}t}}{D^{1/2}} + \frac{(gnt)^2}{\mathcal{N}^2}. \tag{339}$$

Combining Eq. (339) with Eq. (333) yields

$$\begin{aligned}
\epsilon_{m+1} &\le \epsilon_m + \frac{(gnt)^2}{\mathcal{N}^2} + \sqrt{2n}\bar{\delta}_m(D) \\
&\le (1 + \sqrt{2n})\epsilon_m + \frac{\sqrt{2n}e^{\tilde{J}t}}{\sqrt{D}} + (1 + \sqrt{2n})\frac{(gnt)^2}{\mathcal{N}^2}.
\end{aligned} \tag{340}$$

Solving the linear recurrence, we obtain for all $m \ge 0$,

$$\begin{aligned}
\epsilon_m \le (1 + \sqrt{2n})^m \epsilon_0 + \frac{(1 + \sqrt{2n})^m - 1}{\sqrt{2n}} \\
\times \left(\frac{\sqrt{2n}e^{\tilde{J}t}}{\sqrt{D}} + (1 + \sqrt{2n})\frac{(gnt)^2}{\mathcal{N}^2} \right).
\end{aligned} \tag{341}$$

In particular, this error bound grows exponentially in the number of steps m [with rate $\log(1 + \sqrt{2n})$], highlighting the difficulty of establishing an efficiency guarantee for tDMRG [32].

3. Main theorem on the precision guarantee of tDMRG

Here, we resolve the problem of obtaining a meaningful precision guarantee for the tDMRG method: indeed, we prove the following theorem:

Theorem 6. Let H be the 1D long-range Hamiltonian defined by Eq. (8) with Eqs. (9) and (11). Start from a normalized product state $|\psi_0\rangle = |M_0(D)\rangle$, impose Eq. (319), and use the Euler updates followed by left-to-right exact Schmidt truncations of the current state to bond dimension D, without intermediate normalization. Let $\bar{\delta}_m(D)$ be defined by Eq. (331). Then, for any step $m \in \{0, \ldots, \mathcal{N} - 1\}$, we obtain

$$\bar{\delta}_m(D) \le \frac{e^{\tilde{J}t + (gnt)^2/\mathcal{N}}}{\sqrt{D}}. \tag{342}$$

Combining Lemma 19, the following inequality holds

$$\|e^{-iHt}|\psi_0\rangle - |M_{\mathcal{N}}(D)\rangle\| \le \frac{(gnt)^2}{\mathcal{N}} + \frac{\mathcal{N}}{\sqrt{D/(2n)}} e^{\tilde{J}t + (gnt)^2/\mathcal{N}}. \tag{343}$$

Remark. According to the theorem, to achieve accuracy ϵ, one may set $\mathcal{N} \propto t^2n^2/\epsilon$ and $D \propto e^{\mathcal{O}(t)}n^5/\epsilon^4$. This choice leads to a time complexity that is merely polynomial in n/ϵ.

4. Key idea: Rényi-entanglement monitoring

The monitoring procedure is illustrated in Fig. 4.

Our goal is to obtain a rigorous and practically meaningful precision guarantee for tDMRG. The core strategy is

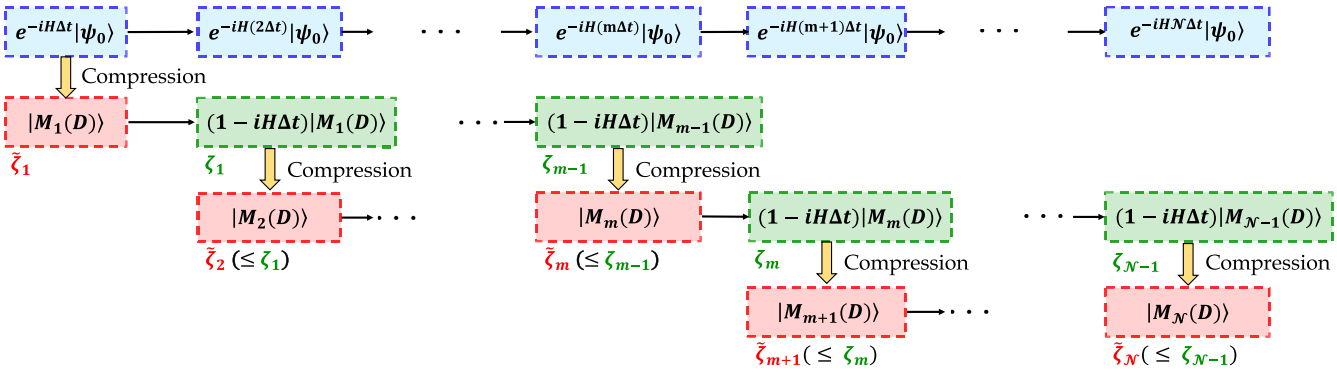


FIG. 4. Monitoring the Euler-and-truncation scheme. The exact evolution is shown in blue. Starting from $|M_0(D)\rangle = |\psi_0\rangle$, each green Euler-update state $(1 - iH\Delta t)|M_m(D)\rangle$ is compressed to the red MPS $|M_{m+1}(D)\rangle$. Compression cannot increase the sum of unnormalized Schmidt coefficients across any cut. Tracking this sum controls the accumulated truncation error.

to monitor the sum of the unnormalized Schmidt coefficients, which controls the truncation error. The argument proceeds in three steps.

(1) Step 1: Monitor the unnormalized Schmidt sum. Let the Schmidt decompositions of the (generally unnormalized) MPS $|M_m(D)\rangle$ at step m across a fixed cut (A_s, B_s) be

$$|M_m(D)\rangle = \sum_{j=1}^{D} \tilde{\lambda}_{m,j}^{(s)} |\psi_{A_s,j}^{(m)}\rangle \otimes |\psi_{B_s,j}^{(m)}\rangle. \tag{344}$$

In the same way, we denote the Schmidt decomposition of $(1 - iH\Delta t)|M_m(D)\rangle$ by

$$(1 - iH\Delta t)|M_m(D)\rangle = \sum_{j=1}^{\infty} \lambda_{m,j}^{(s)} |\phi_{A_s,j}^{(m)}\rangle \otimes |\phi_{B_s,j}^{(m)}\rangle. \tag{345}$$

We track

$$\zeta_m := \sum_{j\geq 1} \lambda_{m,j}^{(s)}, \qquad \tilde{\zeta}_m := \sum_{j=1}^{D} \tilde{\lambda}_{m,j}^{(s)}. \tag{346}$$

For any nonzero vector $|\xi\rangle$, $\sum_j \lambda_j(\xi) = \||\xi\rangle\| \exp[E_{1/2}(\xi/\||\xi\rangle\|)/2]$. We track the unnormalized sums themselves to control the truncation error.

(2) Step 2: Show how tDMRG controls $\tilde{\zeta}_m$ step by step. We split one time step into a local part and an across-the-cut part [see Eq. (352) below] and derive a precise upper bound on the increase of the Schmidt-coefficient sum after applying $(1 - iH\Delta t)$ and truncation:

$$\zeta_m \leq [1 + \tilde{J}\Delta t + (gn\Delta t)^2]\tilde{\zeta}_m. \tag{347}$$

Schmidt truncation does not increase the sum of unnormalized Schmidt coefficients across any cut. Each truncation is an orthogonal projection representable on either side of its cut; for any other cut, one representation acts on one side only, so the coefficient-matrix nuclear norm cannot increase. No intermediate renormalization is performed. This proves the following inequality; sequential singular-value truncation also gives Eq. (330):

$$\tilde{\zeta}_m \leq \zeta_{m-1}. \tag{348}$$

(3) Step 3: Convert entanglement control into truncation-error control. Given ζ_m for the pretruncation state $(1 - iH\Delta t)|M_m(D)\rangle$ at step m, the tail satisfies

$$\delta_{m,s}(D) := \left(\sum\nolimits_{j>D} |\lambda_{m,j}^{(s)}|^2\right)^{1/2} \leq \frac{\zeta_m}{\sqrt{D}}, \tag{349}$$

where we use the same analysis as in Eq. (314). Thus, an upper bound on ζ_m (hence, on $\tilde{\zeta}_m$) yields a quantitative bound on the truncation error at that step. Summing over steps and cuts gives the final precision guarantee for tDMRG.

Controlling the unnormalized Schmidt sums along the algorithmic trajectory gives explicit trade-offs among Δt, $\mathcal{N}$, and D, and hence a precision guarantee.

5. Proof of Theorem 6

We use the notation introduced in the previous section. Our goal is to derive an upper bound for $\delta_{m,s}(D)$ in Eq. (331) that holds for any s:

$$\delta_{m,s}^2(D) = \sum_{j>D} \left(\lambda_{m,j}^{(s)}\right)^2. \tag{350}$$

Note that $\bar{\delta}_m(D)$ is given by $\bar{\delta}_m(D) := \max_{s\in[n-1]}[\delta_{m,s}(D)]$. For this purpose, we estimate the sum of the unnormalized Schmidt coefficients in Eq. (346).

In the following, we upper-bound ζ_m based on ζ_{m-1} (or $\tilde{\zeta}_m$). For this purpose, we utilize

$$\zeta_m \le \bar{\mathcal{J}}(1 - iH\Delta t)\tilde{\zeta}_m. \tag{351}$$

Note that ζ_m is connected to the (1/2)-Rényi entanglement of $(1 - iH\Delta t)|M_m(D)\rangle$. A simple estimate gives $\bar{\mathcal{J}}(1 - iH\Delta t) \propto 1 + \mathcal{O}(\|H\|\Delta t)$, which is too loose and leads to an exponential increase in $\tilde{\zeta}_m$ with m. To refine this, we use the following expansion:

$$\begin{aligned} 1 - iH\Delta t &= e^{-i(H_{A_s}+H_{B_s})\Delta t} - iV_{A_sB_s}\Delta t \\ &\quad + [1 - i(H_{A_s} + H_{B_s})\Delta t - e^{-i(H_{A_s}+H_{B_s})\Delta t}] \\ &= e^{-i(H_{A_s}+H_{B_s})\Delta t} - iV_{A_sB_s}\Delta t \\ &\quad - \sum_{m=2}^{\infty} \frac{(-i\Delta t)^m}{m!}(H_{A_s} + H_{B_s})^m, \end{aligned} \tag{352}$$

where we decompose the total Hamiltonian into $H = H_{A_s} + H_{B_s} + V_{A_sB_s}$. Using Lemma 4, we obtain

$$\begin{aligned} \bar{\mathcal{J}}(1 - iH\Delta t) &\le 1 + \tilde{J}\Delta t + \sum_{m=2} \frac{(\Delta t)^m}{m!} \\ &\quad \times \sum_{m_1=0}^{m} \binom{m}{m_1} \bar{\mathcal{J}}(H_{A_s}^{m_1} \otimes H_{B_s}^{m-m_1}), \end{aligned} \tag{353}$$

where we use $\bar{\mathcal{J}}(e^{-i(H_{A_s}+H_{B_s})\Delta t}) = 1$ and $\bar{\mathcal{J}}(V_{A_sB_s}) \le \tilde{J}$.

Using the parameter g in Eq. (8), we have

$$\begin{aligned} \bar{\mathcal{J}}(H_{A_s}^{m_1} \otimes H_{B_s}^{m-m_1}) &\le \|H_{A_s}\|^{m_1} \times \|H_{B_s}\|^{m-m_1} \\ &\le g^m |A_s|^{m_1} |B_s|^{m-m_1}, \end{aligned} \tag{354}$$

which reduces the upper bound (353) to

$$\begin{aligned} \bar{\mathcal{J}}(1 - iH\Delta t) &\le 1 + \tilde{J}\Delta t + \sum_{m=2} \frac{(gn\Delta t)^m}{m!} \\ &= 1 + \tilde{J}\Delta t + (e^{gn\Delta t} - gn\Delta t - 1) \\ &\le 1 + \tilde{J}\Delta t + (gn\Delta t)^2, \end{aligned} \tag{355}$$

where we use $|A_s| + |B_s| = |\Lambda| = n$ and $e^x - 1 - x \le x^2$ for $0 \le x \le 1$, using Eq. (319).

By combining the inequalities (351) and (355), we obtain

$$\zeta_m \le [1 + \tilde{J}\Delta t + (gn\Delta t)^2]\tilde{\zeta}_m \le [1 + \tilde{J}\Delta t + (gn\Delta t)^2]\zeta_{m-1}, \tag{356}$$

where we use Eq. (348) in the last inequality. Starting from $\tilde{\zeta}_0 = 1$ and using $\tilde{\zeta}_{m+1} \le \zeta_m$, we obtain, for $0 \le m \le \mathcal{N} - 1$,

$$\zeta_m \le [1 + \tilde{J}\Delta t + (gn\Delta t)^2]^{m+1} \le e^{\tilde{J}t + (gnt)^2/\mathcal{N}}. \tag{357}$$

Therefore, from the upper bound (357) and inequality (349), we arrive at the Schmidt-rank truncation error for $(1 - iH\Delta t)|M_m(D)\rangle$ as follows:

$$\delta_{m,s}(D) \le \frac{e^{\tilde{J}t + (gnt)^2/\mathcal{N}}}{\sqrt{D}}. \tag{358}$$

We thus prove the main inequality (342) from $\bar{\delta}_m(D) := \max_{s \in [n-1]}[\delta_{m,s}(D)]$. This completes the proof. ∎

F. Quantum Gibbs states

In considering the quantum Gibbs states, we utilize the following purification:

$$|\rho_\beta\rangle = \frac{1}{\sqrt{Z_\beta}}(e^{-\beta H/2} \otimes \hat{1}_{\Lambda'}) \sum_{j=1}^{\mathcal{D}_\Lambda} |j_\Lambda\rangle \otimes |j_{\Lambda'}\rangle, \tag{359}$$

with Λ' a copied system and Z_β the partition function. Use corresponding sitewise product bases, so that $|I\rangle = \otimes_{i \in \Lambda} \sum_{a=1}^{d_i} |a_i\rangle|a_{i'}\rangle$. Here, $d_i = \mathcal{D}_{\{i\}}$. Each MPS site is the pair (i, i'), and spatial cuts are $A_sA_s'|B_sB_s'$. Note that $\mathrm{tr}_{\Lambda'}(|\rho_\beta\rangle\langle\rho_\beta|) = \rho_\beta$.

Our task is to implement the imaginary-time evolution to $\sum_{j=1}^{\mathcal{D}_\Lambda} |j_\Lambda\rangle \otimes |j_{\Lambda'}\rangle$ [8]. However, we cannot directly apply the analyses of the real-time evolution. The primary challenge occurs in the analyses of Eq. (352), which characterized the SE strength of $\bar{\mathcal{J}}(1 - iH\Delta t)$. In extending it to the imaginary-time evolution, we need to consider $\bar{\mathcal{J}}(1 - H\Delta\tau)$ with $\Delta\tau$ the decomposed unit of the total imaginary time β. We then replace Eq. (352) with

$$\begin{aligned} 1 - H\Delta\tau &= e^{-(H_{A_s}+H_{B_s})\Delta\tau} - V_{A_sB_s}\Delta\tau \\ &\quad - \sum_{m=2}^{\infty} \frac{(-\Delta\tau)^m}{m!}(H_{A_s} + H_{B_s})^m. \end{aligned} \tag{360}$$

Then, the problem is that the SE strength of $\bar{\mathcal{J}}(e^{-(H_{A_s}+H_{B_s})\Delta\tau})$ is not equal to 1 unlike the real-time case; instead, we have $\bar{\mathcal{J}}(e^{-(H_{A_s}+H_{B_s})\Delta\tau}) \propto 1 + n\Delta\tau$. This also modifies inequality (357) by

$$\zeta_m \le e^{\tilde{J}\beta + (gn\beta)^2/\mathcal{N} + \mathcal{O}(n\beta)}, \tag{361}$$

which is meaningless for a large system size n.

Currently, we can prove only the existence of an efficient MPS approximation of $|\rho_\beta\rangle$ as follows:

Theorem 7. We consider the 1D long-range Hamiltonian defined in the form of Eq. (8) with Eqs. (9) and (11). For $\beta > 0$, there exists an efficient MPS approximation $|M_\beta(D)\rangle$ for $|\rho_\beta\rangle$ in Eq. (359) such that

$$\||\rho_\beta\rangle - |M_\beta(D)\rangle\| \le 960n\lceil\beta\mathcal{Q}_0/4\rceil D^{-1/\kappa_\beta}, \tag{362}$$

where we defined κ_β and $\mathcal{Q}_0$ as

$$\kappa_\beta := 4\left[6+\frac{4(k+1)}{\eta-2}+2k\log_2(d)\right]\lceil\beta\mathcal{Q}_0/4\rceil, \tag{363}$$

and

$$\mathcal{Q}_0 := \left[\min\left(\frac{1}{8gk}, \frac{\eta-2}{16eJ_0(\eta-1)^2 2^{\eta-2}}\right)\right]^{-1}. \tag{364}$$

Remark. From the definition, we have $\kappa_\beta = O(1+\beta\mathcal{Q}_0)$, and hence to achieve an error ϵ, we need to set the bond dimension D to be as large as $(n/\epsilon)^{\mathcal{O}(1+\beta)}$. This is qualitatively better than the state-of-the-art result of $e^{\mathcal{O}(\beta)\log^3(n/\epsilon)}$ in Ref. [57]. On the other hand, it is still an open question whether one can also prove the polynomial-time complexity to find such an MPS approximation. Even under the high-temperature condition (i.e., $\beta \ll 1$), the current best time complexity is given by $e^{\log^2(n/\epsilon)}$ [58].

1. Proof of Theorem 7

We utilize the global estimate (311) as in the proof of Proposition 18. Let us denote the Schmidt decomposition of $|\rho_\beta\rangle$ across $A_sA_s'|B_sB_s'$ by

$$|\rho_\beta\rangle = \sum_j \lambda_j^{(s)} |\phi_{A_sA_s',j}\rangle \otimes |\phi_{B_sB_s',j}\rangle, \tag{365}$$

where the index s characterizes the decomposition of the total system, i.e., $\Lambda = A_s \sqcup B_s$ with $A_s = \{1,2,\ldots,s\}$ and $B_s = \Lambda\backslash A_s$. Then, we obtain the same inequality as Eq. (311):

$$\||\rho_\beta\rangle - |M_\beta(D)\rangle\|^2 \le 2\sum_{s=1}^{n-1}\delta_s^2(D) \tag{366}$$

and

$$\delta_s^2(D) = \sum_{j>D}\left(\lambda_j^{(s)}\right)^2. \tag{367}$$

To estimate Eq. (367), we utilize the Eckart-Young theorem as follows:

$$\sum_{j>D}(\lambda_j^{(s)})^2 \le \||\rho_\beta\rangle - |\psi_D\rangle\|^2, \tag{368}$$

where $|\psi_D\rangle$ is an arbitrary quantum state with Schmidt rank D for the cut $A_sA_s'|B_sB_s'$. To construct $|\psi_D\rangle$, we apply Theorem 3 to $-H$ and approximate $e^{-\beta H/4}$ by $\tilde{\rho}_{\beta/4}$ such that

$$\|e^{-\beta H/4} - \tilde{\rho}_{\beta/4}\|_p \le \bar{\epsilon}\|e^{-\beta H/4}\|_p. \tag{369}$$

Set $A = e^{-\beta H/4}$ and $B = \tilde{\rho}_{\beta/4}$. The normalized vectorization is

$$|\tilde{\rho}_\beta\rangle = \frac{B^\dagger B \otimes \hat{1}_{\Lambda'}}{\|B^\dagger B\|_2}|I\rangle. \tag{370}$$

For $0 < \bar{\epsilon} \le 1/2$, Eq. (369) with $p=4$ gives $\|B^\dagger B - A^2\|_2 \le \bar{\epsilon}(2+\bar{\epsilon})\|A^2\|_2$. Normalizing their vectorizations therefore yields

$$\||\rho_\beta\rangle - |\tilde{\rho}_\beta\rangle\|^2 \le [2\bar{\epsilon}(2+\bar{\epsilon})]^2 \le 36\bar{\epsilon}^2. \tag{371}$$

By applying $|\tilde{\rho}_\beta\rangle$ to inequality (368), we need to set $D = [\mathrm{SR}(\tilde{\rho}_{\beta/4})]^2$, which yields

$$\sum_{j>[\mathrm{SR}(\tilde{\rho}_{\beta/4})]^2}\left(\lambda_j^{(s)}\right)^2 \le 36\bar{\epsilon}^2. \tag{372}$$

The remaining problem is to get the Schmidt rank $\mathrm{SR}(\tilde{\rho}_{\beta/4})$ for the approximation (369). For this purpose, we use Theorem 3. Here, the boundary interaction $V_{A_sB_s}$ is decomposed as

$$V_{A_sB_s} = \sum_{Z:Z\cap A_s\neq\emptyset, Z\cap B_s\neq\emptyset} h_Z = \sum_{j=1}^{\infty} V_j. \tag{373}$$

Here we enumerate the terms h_Z by nondecreasing interaction diameter and associate them with V_j, padding by zero terms if necessary. To apply the theorem, we first prove that the boundary interaction $V_{A_sB_s}$ satisfies Assumption 10 in the following sense:

Lemma 20. Under the decomposition of Eq. (373), the operator $V_{A_sB_s}$ ($\forall\, s\in[1,n-1]$) satisfies the conditions (162) and (163) by choosing κ, D_0, $\tilde{g}$, and C_0 as follows:

$$\kappa = \frac{\eta-2}{k+1}, \qquad D_0 = d^{2k},$$
$$\tilde{g} = 4J_0\left(1+\frac{1}{\eta-2}\right), \qquad C_0 = (\eta-1)2^{\eta-2}. \tag{374}$$

Note that η was set to be larger than 2 as in Eq. (11).

Moreover, by using Ref. [87] Lemma 3 therein, we have

$$\|\mathrm{ad}_{H_0}^s(h_Z)\| \le (2gk)^s s!\|h_Z\|, \tag{375}$$

and hence, we also obtain the parameter $\mathcal{Q}$ in Eq. (173) as

$$\mathcal{Q} = 2gk. \tag{376}$$

We recall that the parameter g is defined as one-site energy scale as in Eq. (8). Equations (374) and (376) make

$$\mathcal{Q}_0 := \left[\min\left(\frac{1}{4\mathcal{Q}}, \frac{1}{4eC_0\tilde{g}}\right)\right]^{-1} = \left[\min\left(\frac{1}{8gk}, \frac{\eta-2}{16eJ_0(\eta-1)^2 2^{\eta-2}}\right)\right]^{-1}. \quad (377)$$

Therefore, from inequality (175), we obtain

$$\mathrm{SR}(\tilde{\rho}_{\beta/4}) \le \left(\frac{48\lceil\beta\mathcal{Q}_0/4\rceil}{\bar{\epsilon}}\right)^{2[6+4(k+1)/(\eta-2)+2k\log_2(d)]\lceil\beta\mathcal{Q}_0/4\rceil} = \left(\frac{48\lceil\beta\mathcal{Q}_0/4\rceil}{\bar{\epsilon}}\right)^{\kappa_\beta/2}, \quad (378)$$

where we use the notation κ_β in Eq. (363). Write $m = \lceil\beta\mathcal{Q}_0/4\rceil$ and choose

$$\bar{\epsilon} = 48mD^{-1/\kappa_\beta}. \quad (379)$$

If $\bar{\epsilon} \le 1/2$, the rank estimate gives $[\mathrm{SR}(B)]^2 \le D$ and

$$\delta_s^2(D) \le 36(48m)^2 D^{-2/\kappa_\beta}. \quad (380)$$

Equation (366) then gives $\||\rho_\beta\rangle - |M_\beta(D)\rangle\| \le 288\sqrt{2n}mD^{-1/\kappa_\beta} \le 960nmD^{-1/\kappa_\beta}$. If $\bar{\epsilon} > 1/2$, the distance to any normalized product state is at most 2 and satisfies Eq. (362). At $\beta = 0$, the paired-site purification is a bond-one MPS. This completes the proof. ∎

2. Proof of Lemma 20

For simplicity, we denote A_s and B_s by A and B, respectively. We aim to derive the inequalities of

$$V_{AB} = \sum_{j=1}^{\infty} V_j, \qquad \mathrm{SR}(V_j) \le D_0 \quad \forall j, \quad (381)$$

and

$$\sum_{j\ge D+1} \|V_j\| \le C_0\tilde{g}(D+1)^{-\kappa} \quad (C_0 \ge 1), \qquad \sum_{j=1}^{\infty}\|V_j\| \le \tilde{g}, \quad (382)$$

under the choice of Eq. (374).

We first upper-bound the Schmidt rank $\mathrm{SR}(h_Z)$ for an arbitrary interaction term h_Z with $|Z| \le k$. For an arbitrary local operator defined on a subset X, the total number of operator bases is upper-bounded by $d^{2|X|}$, and hence,

$$\mathrm{SR}(h_Z) \le d^{2k}. \quad (383)$$

Therefore, we can choose D_0 as

$$D_0 = d^{2k}. \quad (384)$$

Second, we introduce the following decomposition of V_{AB}:

$$V_{AB} = \sum_{r=1}^{\infty} V_r, \qquad V_r := \sum_{\substack{Z: Z\cap A\neq\emptyset, Z\cap B\neq\emptyset \\ \mathrm{diam}(Z)=r}} h_Z. \quad (385)$$

We also denote the set $\mathcal{S}_r$ such that

$$\mathcal{S}_r := \{Z \subset \Lambda : |Z| \le k, \mathrm{diam}(Z) = r, Z\cap A \neq \emptyset, Z\cap B \neq \emptyset\}. \quad (386)$$

Using it, we obtain

$$V_r := \sum_{Z\in\mathcal{S}_r} h_Z. \quad (387)$$

We define the subset X_r ($r \in \mathbb{N}$) to span the region within the distance r from the boundary between A and B. Then, any subset $Z \in \mathcal{S}_r$ is supported on X_r, and hence we get

$$|\mathcal{S}_r| \le |X_r|^k \le (2r)^k. \quad (388)$$

For the counting bound, encode each nonempty subset of size at most k by its increasing list, padded to length k by repeating its last element. We next estimate the norm sum

$$\bar{V}_r := \sum_{Z\in\mathcal{S}_r}\|h_Z\|. \quad (389)$$

By using inequality (9) with Eq. (11), we obtain

$$\sum_{\substack{Z: Z\cap A\neq\emptyset, Z\cap B\neq\emptyset \\ \mathrm{diam}(Z)=r}} \|h_Z\| \le \sum_{i\in X_r}\sum_{j: d_{i,j}=r}\sum_{Z: Z\ni\{i,j\}}\|h_Z\| \le 2J_0\sum_{i\in X_r} r^{-\eta} \le 4J_0 r^{-\eta+1}, \quad (390)$$

where we use $|X_r| \le 2r$ in the last inequality.

After straightforward calculations, we obtain

$$\sum_{r=1}^{\infty}\sum_{Z\in\mathcal{S}_r}\|h_Z\| \le \sum_{r=1}^{\infty} 4J_0 r^{-\eta+1} \le 4J_0\left(1+\frac{1}{\eta-2}\right) =: \tilde{g}_0 = \tilde{g} \quad (391)$$

and

$$\sum_{r\ge R}^{\infty}\sum_{Z\in\mathcal{S}_r}\|h_Z\| \le \sum_{r\ge R} 4J_0 r^{-\eta+1} \le \frac{4J_0}{\eta-2}(\eta-1)R^{-(\eta-2)}, \quad (392)$$

where the tail is bounded by its first term plus an integral, for $R \geq 1$. The total number of interaction operators $\{h_Z\}_{Z\in\mathcal{S}_r, r<R}$ is smaller than

$$\sum_{r<R}|\mathcal{S}_r| \leq \sum_{r<R}(2r)^k \leq 2^k(R-1)^{k+1}, \tag{393}$$

and hence, we formally obtain

$$\begin{aligned}\sum_{j\geq D+1}\|V_j\| &\leq \sum_{r\geq R}^{\infty}\sum_{Z\in\mathcal{S}_r}\|h_Z\| \leq \frac{4J_0}{\eta-2}(\eta-1)R^{-(\eta-2)} \\ &\leq \tilde{g}_0(\eta-1)R^{-(\eta-2)} \quad \text{for } 2^k(R-1)^{k+1} \\ &\leq D < 2^kR^{k+1}.\end{aligned} \tag{394}$$

Since $D < 2^kR^{k+1}$ and these quantities are integers, $D+1 \leq 2^kR^{k+1}$. Thus, $R \geq 2^{-k/(k+1)}(D+1)^{1/(k+1)} \geq \frac{1}{2}(D+1)^{1/(k+1)}$, giving

$$\sum_{j\geq D+1}\|V_j\| \leq (\eta-1)2^{\eta-2}\times\tilde{g}_0\times(D+1)^{-(\eta-2)/(k+1)}. \tag{395}$$

Therefore, we can derive inequality (382) under the choice of Eq. (374). This completes the proof. ∎

IX. SYSTEMS WITH SHORT-RANGE INTERACTIONS

In this section, we clarify how the spectral-SIE picture changes when genuine geometric locality is imposed. We focus on geometrically local systems governed by short-range interactions corresponding to the Hamiltonian defined in Eq. (10). The first purpose is to show that, for the instantaneous Rényi-entanglement rate, the threshold in Theorem 1 cannot be improved even in this local setting. The reason is that the singular behavior responsible for the threshold $\alpha = 1/2$ already appears in a strictly local two-site entangling process. This is made explicit in Lemma 21, which saturates Eq. (52) as $t\downarrow 0$ and has a divergent rate for $0 < \alpha < 1/2$.

The second purpose is to contrast this instantaneous obstruction with finite-time entanglement generation. Over a finite-time interval, geometric locality can be used to obtain stronger bounds than those available in the general nonlocal setting [8,9,32,34,35]. We summarize this improvement in Lemma 22, which provides a representative finite-time entanglement-generation bound and illustrates how locality can improve the behavior of time-averaged entanglement growth.

We begin with the following statement:

Lemma 21. Let us consider the following simple case in a two-qubit system:

$$|\psi_0\rangle = |0,0\rangle, \qquad H = |0,0\rangle\langle 1,1| + \text{H.c.}. \tag{396}$$

Note that $\bar{\mathcal{J}}(H) = 1$. Then, the Rényi-entanglement rate satisfies

$$\lim_{t\to+0}\left|\frac{dE_\alpha(t)}{dt}\right| = \begin{cases}\infty & \text{for } 0<\alpha<1/2,\\ 2 & \text{for } \alpha = 1/2,\\ 0 & \text{for } \alpha > 1/2.\end{cases} \tag{397}$$

Proof of Lemma 21. By solving the eigenproblem of H, we obtain

$$|\psi_t\rangle = e^{-iHt}|0,0\rangle = \cos(t)|0,0\rangle - i\sin(t)|1,1\rangle, \tag{398}$$

which, for $0 < t < \pi/2$ and $\alpha \neq 1$, gives the derivative (the case $\alpha = 1$ follows by continuity)

$$\begin{aligned}\frac{dE_\alpha(t)}{dt} &= \frac{2\alpha}{1-\alpha}\frac{\cos(t)\sin^{2\alpha-1}(t) - \sin(t)\cos^{2\alpha-1}(t)}{\cos^{2\alpha}(t)+\sin^{2\alpha}(t)} \\ &\approx \frac{2\alpha}{1-\alpha}(t^{2\alpha-1} - t).\end{aligned} \tag{399}$$

The above equation immediately leads to the main statement (397). This completes the proof. ∎

The following estimate bounds finite-time Rényi-entanglement growth for every fixed positive index:

Lemma 22. (A modified version of Corollary 8 in Ref. [8]). Let H_{AB} be a time-independent, bounded-strength, finite-range 1D Hamiltonian with fixed local dimension and range, and take a single spatial cut $A|B$. Set $T = \max\{1, |t|\}$ and $0 < \epsilon \leq 1/2$. For arbitrary time-evolution operator $e^{-iH_{AB}t}$, there exists an operator $U_{AB,D}$ such that $\|U_{AB,D} - e^{-iH_{AB}t}\| \leq \epsilon$, where $U_{AB,D}$ has the Schmidt rank D with

$$D = e^{\tilde{O}\left[\sqrt{T\log(1/\epsilon)+T^2}\right]}. \tag{400}$$

Here, $\tilde{O}(x) = O[x\log(2+x)]$, with constants depending only on the fixed local parameters.

Remark. For a normalized product initial state across $A|B$, inversion of Eq. (400) gives, for $s \geq 2$,

$$\lambda_s(t) \leq \exp\left[C_1T - \frac{C_2(\log s)^2}{T[\log(2+\log s)]^2}\right]. \tag{401}$$

For any pure initial state and $0 < \alpha \leq 1/2$, the argument below gives

$$|E_\alpha(t) - E_\alpha(0)| \leq \frac{CT}{\alpha}\left[\log\frac{2+T}{\alpha}\right]^2. \tag{402}$$

For $\alpha > 1/2$, Theorem 1 applies. This is not a uniform linear-in-$|t|$ bound near $t = 0$. Higher-dimensional extensions require corresponding rank estimates.

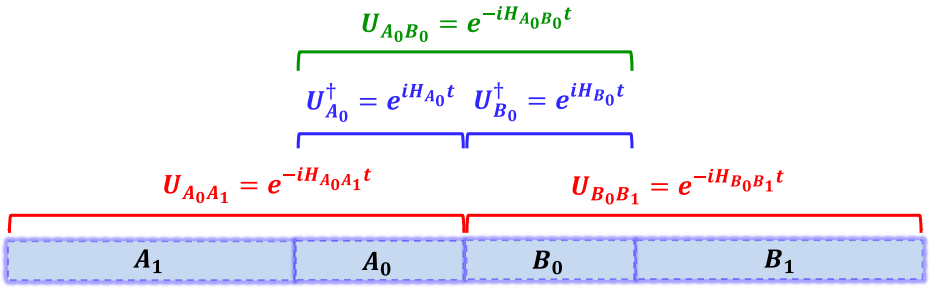


FIG. 5. Schematic picture of the Haah-Hastings-Kothari-Low approximation along the cut $A_0A_1 \sqcup B_0B_1$.

Proof sketch. For the reader's convenience, we sketch the proof in the spirit of Ref. [8]. The key ideas are (i) decomposing the time evolution using the Haah-Hastings-Kothari-Low (HHKL) approximation [7], and (ii) applying the Schmidt-rank bounds of Refs. [40,41] to the Taylor expansion of the time-evolution operator restricted to the central block.

We first decompose the total system into $\Lambda = A_1 \sqcup A_0 \sqcup B_0 \sqcup B_1$, where $|A_0| = |B_0| = \ell$. Then, one can obtain the Haah-Hastings-Kothari-Low approximation of the time evolution $e^{-iH_{AB}t}$ as follows [7] (Fig. 5):

$$\left\| e^{-iH_{AB}t} - e^{-iH_{A_0B_0}t} e^{i(H_{A_0}+H_{B_0})t} e^{-i(H_A+H_B)t} \right\| \le Ce^{-\mu\ell+v|t|}, \tag{403}$$

where $C \ge 1$, v, and μ are determined by the Lieb-Robinson bound. We recall that $H_{A_0B_0}$, H_{A_0}, and H_{B_0} are the subset Hamiltonians on $A_0 \sqcup B_0$, A_0, and B_0, respectively. Note that if the higher dimension is considered, the above upper bound depends on the surface area of A.

Choose $\ell = \lceil [\log(2C/\epsilon) + v|t|]/\mu \rceil$, so the HHKL error is at most $\epsilon/2$. The two rightmost factors act separately on A and B and do not increase operator Schmidt rank. It remains to approximate $e^{-iH_{A_0B_0}t}$ to error $\epsilon/2$. By considering the Taylor expansion as

$$e^{-iH_{A_0B_0}t} = \sum_{m=0}^{\infty} \frac{(-it)^m}{m!} H_{A_0B_0}^m, \tag{404}$$

the Taylor polynomial $P_M = \sum_{m=0}^{M} (-it)^m H_{A_0B_0}^m/m!$ has error at most $\epsilon/2$ for a sufficiently large choice $M = O[|t|\|H_{A_0B_0}\| + \log(1/\epsilon)] = O[T\log(1/\epsilon) + T^2]$, using $\|H_{A_0B_0}\| = O(\ell)$ and $\ell = O[\log(1/\epsilon) + T]$. The rank bounds of Refs. [40,41] give $\mathrm{SR}(P_M) \le \exp[O\{\sqrt{M}\log(2+M)\}] \le \exp[\tilde{O}\{\sqrt{T\log(1/\epsilon) + T^2}\}]$. Combining the two error contributions proves Eq. (400).

For Eq. (402), choose approximants U_m to $U = e^{-iH_{AB}t}$ with errors e^{-m} and ranks D_m from Eq. (400), and write $U = U_1 + \sum_{m\ge1}(U_{m+1} - U_m)$. For $q = 2\alpha \le 1$, Schatten q-power subadditivity bounds the output coefficient sum for every normalized product input by

$$G_\alpha = 2^{2\alpha} D_1^{1-\alpha} + \sum_{m\ge1} (D_m + D_{m+1})^{1-\alpha} (2e^{-m})^{2\alpha}.$$

The rank estimate gives $\log G_\alpha \le (CT/\alpha)[\log\{(2+T)/\alpha\}]^2$. Applying the same subadditivity to the initial Schmidt expansion bounds the increase of E_α by $(1-\alpha)^{-1}\log G_\alpha$. Applying the argument to $U^\dagger$ gives the absolute increment bound (402), after adjusting C. This completes the proof. ∎

X. CONCLUSION AND PERSPECTIVE

In this work, we aimed to establish universal principles governing the entanglement spectrum. The key notions introduced and developed are the SE strength (Definition 1) and the spectral SIE (Theorem 1). They provide a refined and quantitatively optimal framework for dynamical entanglement structure, thereby extending the celebrated SIE theorem to address richer many-body phenomena. As an application, we have analyzed long-range interacting systems and have demonstrated that, contrary to previous expectations, the quasipolynomial complexity of their simulation can in fact be improved to polynomial complexity. These main achievements are summarized in Fig. 1.

At the same time, our analysis naturally brings to light a variety of fundamental open problems. These problems indicate both the limitations of the present results and promising avenues for further exploration. In the following, we list and discuss several of these open directions.

(1) Unification of spectral and standard SIEs: The spectral SIE developed in this work has the distinct advantage that it provides bounds on the generation rate of Rényi entanglement for $\alpha < 1$, a feature not available in the standard SIE. This refinement allows for a more precise control of computational complexity. However, an important limitation arises: the spectral SIE, when specialized to $\alpha = 1$, does not straightforwardly recover the standard SIE. As emphasized in Ref. [34], the case $\alpha = 1$ exhibits special properties that set it apart from the regime $\alpha < 1$. Consequently, a major open challenge is to develop a unified theorem that fully integrates the spectral- and standard SIE frameworks into a comprehensive theory of dynamical entanglement

growth. At the present stage of our understanding, our results show that the bound is sharp only for $\alpha = 1/2$, whereas the optimal bound for $\alpha > 1/2$ remains unknown.

(2) Limits of operator approximability: Proposition 9 shows that small SE strength does not guarantee an efficient approximation of the time-evolution operator, thereby revealing a complexity separation between the full unitary operator and time-evolved states. However, once we allow the Schmidt rank to depend on the Hilbert-space dimension $\mathcal{D}_{AB}$, it remains unresolved whether the required scaling is polynomial in $\mathcal{D}_{AB}$ or only logarithmic in $\mathcal{D}_{AB}$. Clarifying this point is crucial for achieving a more quantitative understanding of complexity separation in many-body quantum dynamics.

(3) Sufficient conditions for operator-level low-rank approximation (Conjecture 2): Subsequent work described in the Note added establishes the Hermitian case of Conjecture 2 for $\alpha < 1/2$ and gives non-Hermitian counterexamples for $\alpha \geq 1/3$. Further questions concern analogous approximation guarantees for general non-Hermitian operators.

(4) Precise conditions for generalized area laws: The generalized area law currently relies on the assumption of boundary-adiabatic paths (Assumption 13). It remains unknown whether this assumption is indeed indispensable or can be relaxed. Determining the minimal assumptions under which area laws hold is essential for both condensed matter physics and quantum information theory.

(5) Tightening the area-law upper bound: We obtained $E_{\alpha=1}(\Omega) = O[1 + (\tilde{g}/\Delta)^3]$ (Theorem 4), with $\tilde{g}$ the uniform path bound in Assumption 13 and fixed $S_0, \tilde{c}_0$. It is natural to seek $E_{\alpha=1}(\Omega) \lesssim 1 + \tilde{g}/\Delta$. Improving the exponent in the upper bound is therefore an important quantitative challenge, with implications for the efficiency of tensor-network representations of gapped ground states.

(6) Time-efficient algorithms for ground states and Gibbs states: Current results provide existence proofs of MPS (or purified MPS) approximations to gapped ground states and Gibbs states in 1D long-range interacting systems (Theorems 5 and 7). However, explicit and efficient algorithms to actually construct such approximations are lacking. Developing such algorithms would make rigorous complexity results directly applicable to numerical simulation. More importantly, it would represent a genuine complexity-theoretic breakthrough, implying that the computational complexity of classical simulation of 1D long-range interacting systems lies in the class P.

(7) Generalization of rigorous error certification for DMRG-type algorithms: The certification scheme introduced for tDMRG (Theorem 6) may serve as a paradigm for broader classes of variational methods, including higher-dimensional tensor networks. Extending the framework, improving its dependence on system size, time, and error tolerance, and making it practical for real simulations are natural directions for future work. In particular, achieving rigorous error certification for imaginary-time evolution remains one of the most important open problems.

(8) Dynamical entanglement structure in short-range interacting systems: For one-dimensional finite-range systems, Lemma 22 gives Eq. (402), finite for every fixed positive Rényi index. Conformal-field-theory analyses motivate a $1/\alpha$ dependence [88–90] (see also Ref. [91]). Removing the additional logarithmic factors and determining the optimal joint time and index dependence remain open questions.

(9) High-dimensional entanglement structures: The spectral-SIE theorem is intrinsically bipartite and does not rely on one-dimensional geometry. Thus, it provides a dimension-independent constraint on the Schmidt spectrum across any cut for which the boundary SE strength can be controlled. In higher dimensions, however, converting such bipartite spectral information into efficient tensor-network descriptions is substantially more subtle. Results such as those of Ge and Eisert indicate that bipartite entanglement constraints alone are not sufficient to characterize the complexity of many-body states [92]. One should therefore expect that higher-dimensional applications require entanglement constraints that are adapted to the spatial geometry, rather than bounds for isolated bipartitions only. Determining the appropriate geometry-respecting spectral condition remains an important open problem.

Altogether, the resolution of these open problems would mark significant progress toward a unified and quantitative theory of entanglement dynamics, with far-reaching implications for the simulation of quantum many-body systems and for the broader interface between physics and computer science.

Note added. Recently, Conjecture 2 was resolved in our subsequent work [93]. As originally formulated for arbitrary bipartite operators, the conjecture does not hold over its full proposed parameter range: a non-Hermitian rank-one family provides a counterexample for $\alpha \geq 1/3$. In contrast, for Hermitian operators and $\alpha < 1/2$, the conjectured operator-norm low-rank approximation bound holds with the sharp constant.

ACKNOWLEDGMENTS

We are grateful to Daniel Toniolo for useful comments on the manuscript and for pointing out a minor issue in the

proof of Theorem 1 concerning the zero eigenvalues of the density matrix. All the authors acknowledge the Hakubi projects of RIKEN. T. K. is supported by JST PRESTO (Grant No. JPMJPR2116), Exploratory Research for Advanced Technology (Grant No. JPMJER2302), and JSPS Grants-in-Aid for Scientific Research (Grants No. JP23H01099 and No. JP24H00071), Japan. Y. K. was supported by the JSPS Grant-in-Aid for Scientific Research (Grant No. JP24K06909), Japan. The research of C. R. is supported by the DFG through Grant No. TRR 352—Project-ID 470903074. This research was conducted during the internship of H. M. and C. R. at RIKEN, under the supervision of T. K.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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