

Generalized Statistics on Lattices

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The statistics of particles and extended excitations, such as loops and membranes, are fundamental to modern condensed matter physics, high-energy physics, and quantum information science, yet a comprehensive lattice-level framework for computing them remains elusive. In this work, we develop a universal microscopic method to determine the generalized statistics of Abelian excitations on lattices of arbitrary dimension and demonstrate it by deriving the statistics of particles, loops, and membranes in up to three spatial dimensions. Our approach constructs a sequence of local unitary operators whose many-body Berry phase encodes the desired statistical invariant. The required sequence is generated automatically from the Smith normal form of locality constraints and therefore needs no extra physical input. We prove that the resulting invariants are quantized, provide an algorithm that computes them efficiently, and show how they unify familiar braiding and fusion data of particles while also uncovering new self- and mutual statistics of loop and membrane excitations. We further demonstrate that each statistical invariant corresponds to an 't Hooft anomaly of a generalized symmetry; we show that a nontrivial invariant both (i) obstructs gauging that symmetry and (ii) forbids any short-range-entangled (symmetry-preserving) ground state. This establishes a precise connection between microscopic lattice anomalies and many-body dynamics, providing a generalization of the Lieb-Schultz-Mattis theorem that constrains a wide class of quantum lattice systems.

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I. INTRODUCTION

Statistics of excitations are fundamental to quantum physics and underlie many phenomena in condensed matter and high-energy physics. Nontrivial statistics forbid their condensation. For instance, bosons can condense and lead to superfluidity or superconductivity [1–6], while fermions cannot condense in isolation [7–10]. In two spatial dimensions, the particle statistics broadens to anyons, enabling fractional quantum Hall physics [11–14] and fault-tolerant quantum computation [15–21]. Braided fusion categories provide a complete mathematical description of particle statistics in two spatial dimensions [22–26].

Extended excitations, such as loops and membranes naturally arising in three and higher dimensions, are still under active development. Their statistics govern confinement in gauge theories, constrain possible phase transitions, and control the logical operators of higher-dimensional quantum codes [27–39]. Yet, unlike the particle case, a general and practical framework for defining and computing these statistics on a lattice has remained elusive. Recent studies show that loop statistics in $(3+1)\text{D}$ $\mathbb{Z}_2$ gauge theories with fermionic particles, appearing in certain superconducting phases, realize discrete gravitational anomalies [40–46]. This anomaly is believed to prevent any lattice realization of the gauge theory in three spatial dimensions. Remarkably, the fermionic loop statistics also give rise to a nontrivial quantum cellular automaton, a locality-preserving unitary that plays a central role in the classification of unitary operators on lattice quantum systems [47–49].

The statistics of topological excitations also provide an 't Hooft anomaly of the corresponding symmetry, which constrains the dynamics of the quantum systems and distinguishes different phases of matter. For instance, the

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1-form symmetry in $(2+1)\text{D}$ is generated by Abelian anyons [50,51], and the statistics determines whether the 1-form symmetry can be gauged. In lattice models with a tensor-product Hilbert space, higher-form symmetries are typically emergent. The associated symmetry generators become genuinely topological only once the Gauss law is energetically enforced at low energies. At the same time, families of symmetry operators supported on closed submanifolds can often be identified as exact microscopic symmetries commuting with the full lattice Hamiltonian. Any quantum system with an anomalous higher-form symmetry cannot be a trivially gapped phase, as demonstrated in various lattice models [52–57]. The 't Hooft anomaly matching condition also allows us to distinguish different phases of matter by the statistics of topological excitations, as used extensively in Refs. [58–66].

In this work, we develop a general framework on lattices to compute all possible statistics for excitations of arbitrary dimensions, including particles and extended excitations such as loop and membrane excitations. Statistics are extracted from multistep operations on lattices that transform excitations back to themselves, such as moving particle excitations using string operators, and the statistics arise as the Berry phase of the operation. At each step, microscopic phase ambiguities may arise, modifying the state locally; the operations are designed so that these ambiguities cancel, yielding well-defined, universal statistics. We refer to such constructions as statistical processes. Examples of statistical processes have been discussed in Refs. [67,68] for particle excitations and in Ref. [45] for loop excitations, but these studies do not treat the processes systematically or extend them to excitations in arbitrary dimensions.

We develop a systematic method based on the Smith normal form to construct processes (sequences of unitary operators) with clear physical interpretations. Using this method, we unify and improve known statistical processes for both particle and loop excitations and also uncover new self- and mutual statistics of membrane excitations. The generalized statistics in this framework encompass not only conventional braiding processes but also statistics arising from distinct fusion pathways, such as the fusion of particles in one dimension, loops in two dimensions, and membranes in three dimensions.

Moreover, the statistics of excitations often imply a nontrivial low-energy spectrum, as particles with nontrivial statistics cannot condense. We find that these microscopic definitions of statistics are directly related to 't Hooft anomalies in lattice models, and we demonstrate that these statistics prevent the realization of a short-range-entangled state. This connection provides insight into the dynamical consequences of 't Hooft anomalies in microscopic lattice models. We focus on invertible excitations, whose fusion follows group multiplication

rules. The generalization to noninvertible excitations will be explored in future work.

A. Summary of results

This work introduces a unified framework for defining the statistics of p -dimensional invertible excitations generated by unitary operators supported on $(p+1)$ -dimensional submanifolds, in lattices of arbitrary spatial dimension d . Invertibility guarantees that the corresponding unitary circuit has finite depth [69]; for example, Abelian anyons arise at the end points of finite-depth string operators. Causality imposes locality constraints on these operators: For example, the commutator of two unitaries supported on regions A and B is localized within a neighborhood of their intersection $A \cap B$. These locality constraints determine the possible structure of generalized statistics on lattices.

1. Review of detecting particle statistics on lattices

To illustrate our approach, we begin with the well-known case of particle statistics. From quantum mechanics, we know that the statistics of particles is captured by the phase acquired when two identical particles are exchanged. The challenge is to define such an exchange precisely on a lattice without being obscured by microscopic details. For concreteness, here we focus on particles obeying $\mathbb{Z}_2$ fusion, where two identical particles can fuse into the trivial (vacuum) particle.

Consider two identical particles located at sites 1 and 2 on the lattice, as shown in Fig. 1(b). We perform the following steps:

- (1) Move the particle from site 1 to site 0.
- (2) Move the particle from site 2 to site 1.
- (3) Move the particle from site 0 to site 2.

At the end of this sequence, the particles at sites 1 and 2 have been exchanged. If U_{ij} denotes the string operator that moves a particle from site i to site j , this process is represented by

$$U_{02}U_{21}U_{10}. \quad (1)$$

While this operator exchanges the two particles, it can also introduce unwanted phases into the quantum state, unlike in a classical exchange process. For example, moving the particle from site 1 to site 0 may produce an additional phase in the state, potentially altering the outcome of the entire process. Therefore, we seek a process that captures the particle statistics while remaining insensitive to such phase ambiguities, which generically arise in lattice systems.

We label each site in the particle-number basis, with 0 denoting no particle and 1 denoting a particle; a total configuration state $|a\rangle$ on a finite lattice is thus specified by a 0-chain, a set of $\mathbb{Z}_2$ values assigned to the vertices that indicate particle occupancy at each site. Let $U(s)$ be the

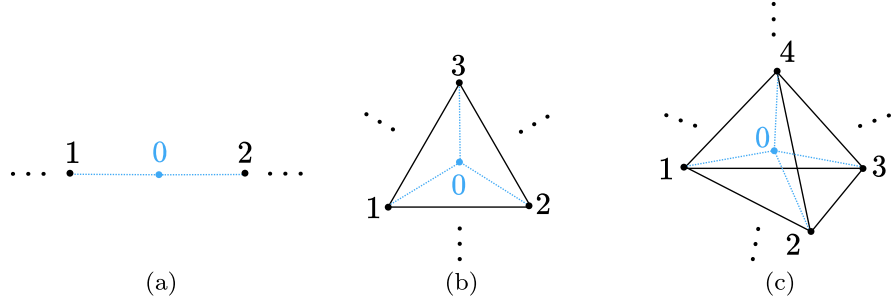


FIG. 1. In each dimension, we consider different simplicial complexes. (a) In $(1+1)\text{D}$, we use a segment subdivided by a vertex, in (b) $(2+1)\text{D}$, a triangle with a central vertex, and in (c) $(3+1)\text{D}$, a tetrahedron subdivided by a central vertex. These complexes are embedded into a larger spatial manifold (omitted by $\dots$), with edge lengths chosen to be much larger than the system's correlation length. For computational convenience, we often compactify the manifold, wrapping the segment into a circle S^1 , the triangle into a two-sphere S^2 , and the tetrahedron into a three-sphere S^3 . We assume that for any simplex Δ , there exists a finite-depth unitary $U(g)_\Delta$ that creates invertible g -excitations on its boundary $\partial\Delta$. For example, $U(g)_{01}$ is a string operator that produces a g particle at vertex 1 and a g^{-1} particle at vertex 0. Similarly, on the 2-simplex Δ_{012} , the membrane operator $U(h)_{012}$ generates an h -loop excitation along the boundary edges of Δ_{012} . Here, elements g and h belong to the fusion groups of particles and loops, respectively.

unitary string operator that moves a particle along a string s , which can also be interpreted as creating two particles at the end points ∂s . Explicit examples of $|a\rangle$ and $U(s)$ are shown in Fig. 2. In general,

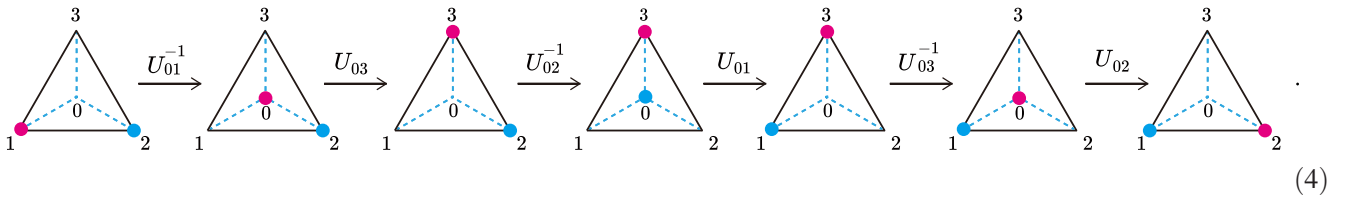
$$U(s)|a\rangle \propto |a + \partial s\rangle, \quad (2)$$

where $|a\rangle$ is the initial state, and $|a + \partial s\rangle$ is the state with two additional particles at ∂s . The phase difference between $U(s)|a\rangle$ and $|a + \partial s\rangle$ is gauge dependent: It depends on the chosen overall phases of $|a\rangle$ and $|a + \partial s\rangle$, as well as on the specific convention used to define $U(s)$. We can construct

an exchange process that is immune to such phase ambiguities. Consider the lattice in Fig. 1(b), where the two particles at sites 1 and 2 are exchanged via the operators [67]

$$U_{02}U_{03}^{-1}U_{01}U_{02}^{-1}U_{03}U_{01}^{-1}. \quad (3)$$

When acting on the initial state $|\triangle\rangle$, where particles occupy vertices 1 and 2, this T -junction process can be visualized as follows:



In contrast to Eq. (1), each string operator here appears together with its inverse, ensuring that the microscopic phase contributions cancel exactly. This construction thus yields the particle statistics without contamination from local, lattice-dependent phase factors.

In this work, we generalize this construction to detect the statistics of extended excitations. We formulate lattice processes that map between excitation configurations in a nontrivial manner, analogous to particle exchange, while ensuring that microscopic phase ambiguities cancel throughout the process. Additional examples of such statistical processes involving particles, loops, and membranes in various spatial dimensions are presented in Sec. II.

2. Universal microscopic description for statistics

We now present the general setting for our construction. This work develops a universal framework for defining generalized statistics on lattices across different dimensions. We consider systems with invertible excitations, whose loci can be moved by finite-depth quantum circuits. An excitation may correspond to a violation of certain Hamiltonian terms, a defect that locally modifies the Hamiltonian, or, more generally, a symmetry defect. Hence, this framework applies to both gapped and gapless systems. An invertible excitation possesses an inverse, forming a fusion group G . For simplicity, we first summarize the results for Abelian groups G , with the understanding that the framework extends to non-Abelian fusion

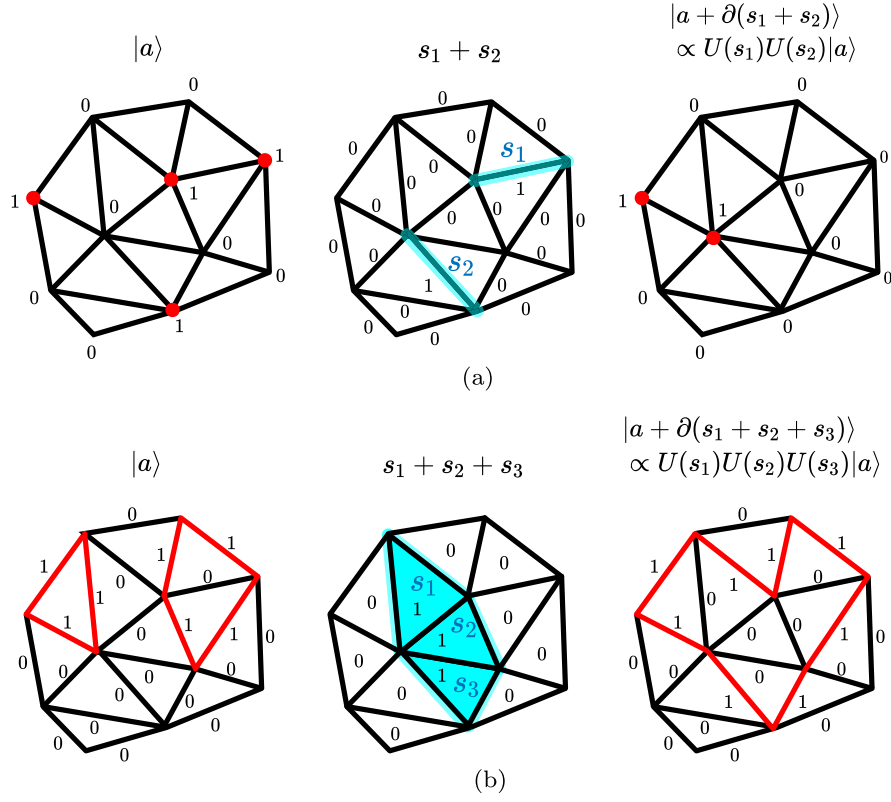


FIG. 2. (a) Particle excitations: for simplicity, we consider particles with the fusion group $G = \mathbb{Z}_2$. The configuration state $|a\rangle$ is labeled by a $\mathbb{Z}_2$ field at each vertex, indicating the presence or absence of particle excitations. Edges s_1 and s_2 have coefficients $1 \in \mathbb{Z}_2$. The unitary operator $\prod_i U(s_i)$ is the string operator supported on these edges, creating particles at the end points. (b) Loop excitations: for loop excitations with the fusion group $G = \mathbb{Z}_2$, the configuration state $|a\rangle$ is labeled by a $\mathbb{Z}_2$ field on each edge, with edges labeled by $1 \in \mathbb{Z}_2$ forming closed loops. Faces s_1, s_2, s_3 have coefficients $1 \in \mathbb{Z}_2$. The unitary operator $\prod_i U(s_i)$ is the membrane operator supported on these faces, creating loop excitations along the boundaries. Applying $\prod_i U(s_i)$ to $|a\rangle$ in both cases yields a configuration state proportional to $|a + \partial(\sum_i s_i)\rangle$. These examples can be generalized to any Abelian group G , with additional considerations for the branching structure.

groups as well. These excitations may also be spatially extended, spanning p dimensions; for example, $p = 0$ corresponds to quasiparticles, $p = 1$ to loop excitations such as magnetic flux loops, and higher p to more extended objects.

Our framework begins by specifying the possible configurations of excitations denoted by $\mathcal{A}$, on a simplicial complex X embedded in space. We consider the Hilbert space basis labeled by excitations $a \in \mathcal{A}$, where each p -dimensional simplex is associated with an element of the group G . Each configuration is represented by the configuration state $|a\rangle$. This state is not uniquely defined, as it can be redefined through the action of unitary operators supported at the locations of the excitations. To address this ambiguity, we fix a particular choice of states $|a\rangle$ and construct invariants that remain independent of these definitions.

Excitations are generated by local unitary operators $U(s)$ supported on a $(p+1)$ -dimensional simplex, where s represents the $(p+1)$ -simplex with a coefficient in G . We assume these excitations are deconfined, meaning that

the operator $U(s)$ creates excitations localized only around its boundary ∂s [70]. When a sum of s , such as $s_1 + \dots + s_n$, is closed (having no boundary), the product operator $U(s_n) \dots U(s_1)$ preserves excitations and generates a generalized symmetry of the system. Since the excitations are assumed to be invertible, this generalized symmetry forms a group. In general, the symmetries can form a higher group [71,72], and they are symmetries supported on subsystems of the lattice [39].

We assume that the unitaries $U(s)$ generating the symmetry can be realized by a finite-depth quantum circuit; for example, they do not involve lattice isometries such as translations or rotations. This is always the case when U generates an internal symmetry, and we will restrict our attention to such cases from now on. This assumption will play an important role in our later formulation of invariants for generalized statistics.

The unitaries move the configuration of the excitations. In general, a unitary operator can be used to connect two states $|a\rangle$ and $|a'\rangle$ through the relation $U(s)|a\rangle \propto |a'\rangle$, where $a + \partial s = a'$, and $+$ represents the fusion of Abelian

excitations. Once the states $|a\rangle$ and the unitary operators $U(s)$ are chosen, the phase factor $\theta(s, a)$ can be specified as

$$U(s)|a\rangle = e^{i\theta(s,a)}|a + \partial s\rangle, \quad (5)$$

with $\theta(s, a)$ depending on both the unitary operator $U(s)$ and the initial state $|a\rangle$. From the relation $U^{-1}U = 1$, it follows that

$$U(s)^{-1}|a\rangle = e^{-i\theta(s,a-\partial s)}|a - \partial s\rangle. \quad (6)$$

These phase factors encode essential information about the system. As an illustration, the overall phase in the T -junction process (3) acting on the initial state $|\triangle\rangle$ is given by

$$e^{i\Theta} = \langle \triangle | U_{02}U_{03}^{-1}U_{01}U_{02}^{-1}U_{03}U_{01}^{-1} | \triangle \rangle, \quad (7)$$

with

$$\begin{aligned} \Theta = & \theta(e_{02}, \triangle) - \theta(e_{03}, \triangle) + \theta(e_{01}, \triangle) \\ & - \theta(e_{02}, \triangle) + \theta(e_{03}, \triangle) - \theta(e_{01}, \triangle), \end{aligned}$$

where the simplices s correspond to edges e_{ij} in Fig. 1(b). A single phase factor $\theta(s, a)$ does not by itself have a direct physical meaning, as it depends on the specific choices of $\{|a\rangle\}$ and $\{U(s)\}$. Changes in these choices can be viewed as “gauge transformations,” under which the “Berry connection” $\theta(s, a)$ varies. However, certain linear combinations of these phases, such as Θ , can be constructed to be independent of the choices of $\{|a\rangle\}$ and $\{U(s)\}$. These gauge-invariant quantities are physically meaningful and will be referred to as generalized statistics. In this sense, generalized statistics encode the intrinsic properties of the excitations, independent of arbitrary conventions.

We then consider a sequence of unitaries starting with the specific configuration of excitations back to itself,

$$\begin{aligned} & \langle a_0 | U(s_{n-1}), \dots, U(s_j), \dots, U(s_0) | a_0 \rangle \\ & = \exp \left(i \sum_{j=0}^{n-1} \theta(s_j, a_j) \right), \end{aligned} \quad (8)$$

with $a_{j+1} := a_j + \partial s_j$. The above sequence can be regarded as the “closed path” in the Hilbert space, along which we measure the Berry phase associated with the family of states. For practical purposes, it is convenient to treat the above phase as the formal sum of the objects $\theta(s, a)$:

$$E = \bigoplus_{s,a} \mathbb{Z} \theta(s, a). \quad (9)$$

The phase defined above may or may not remain invariant under possible deformations of the states $|a\rangle$ and the unitaries $U(s)$, such as the redefinition by phases or local perturbations modifying the unitaries $U \rightarrow UU'$ at each step for unitaries U' close to the identity. The sequence needs to be carefully designed to be invariant under such deformations. We will see that the necessary and sufficient conditions for the invariance can be formulated as the constraints on the $\mathbb{Z}$ coefficients in Eq. (9). We define the subgroup $E_{\text{inv}} \subset E$, which contains the elements that qualify as invariants associated with the excitations.

Some elements of E_{inv} correspond to trivial invariants that reduce to the identity operator. A trivial sequence of unitaries arises from the locality properties of the unitary operators. For instance, if the supports of two operators $U(s_1)$ and $U(s_2)$ do not overlap, i.e., $s_1 \cap s_2 = \emptyset$, their commutator is trivial

$$[U(s_2), U(s_1)] = 1, \quad (10)$$

where $[A, B] := A^{-1}B^{-1}AB$. This property extends to higher commutators involving multiple operators, such as

$$[U(s_n), [\dots, [U(s_2), U(s_1)]]] = 1, \quad (11)$$

if $s_1 \cap s_2 \cap \dots \cap s_n = \emptyset$. This property follows from each unitary being a finite-depth circuit, which ensures that the commutator $[U(s_2), U(s_1)]$ has support only near $s_1 \cap s_2$, since the circuit depth is much smaller than the length scale of the complexes under consideration. These higher commutators acting on any initial configuration state $|a_0\rangle$ form the subgroup $E_{\text{id}} \subset E_{\text{inv}}$. The genuine invariants of the excitations are characterized by the quotient group

$$T := E_{\text{inv}}/E_{\text{id}}, \quad (12)$$

where the trivial phases arising from locality have been factored out. We show in general that, for a finite symmetry group G , the invariants T form a finite Abelian group. This implies that the invariants T must be quantized into the discrete values. This group of invariants can be explicitly computed using a computer, given the possible configurations of the excitations and the unitaries, as well as the group G describing the fusion of excitations.

Remark 1.1. The genuine invariants may appear to depend on the choice of underlying cellulation, and indeed they can vary for small complexes. However, based on our numerical verification, we conjecture that the invariants T become universal once the d -dimensional complex is a cellulation of a d -dimensional manifold that is sufficiently large to embed $\partial\Delta^{d+1}$, the boundary of a $(d+1)$ -simplex, as in the triangulations of spheres illustrated in Fig. 1. In other words, once the underlying complex is refined enough to support the relevant statistical processes (such as particle exchange), any triangulation or cellulation yields

TABLE I. The cohomology of the Eilenberg-MacLane space $B^n G := K(G, n)$ for the finite Abelian group $G = \prod_i \mathbb{Z}_{N_i}$ [73]. The notation $(N_i, N_j, \dots)$ denotes the greatest common divisor among the integers. This cohomology classifies the anomaly as an obstruction to gauging the higher-form G symmetry, which corresponds to SPT phases in one higher dimension. We conjecture that these data precisely match the generalized statistics of particle, loop, and membrane excitations as defined by Eq. (58) in Sec. III A. Specifically, p -dimensional excitations in $(d+1)$ -dimensional spacetime have generalized statistics characterized by $H^{d+2}[B^{d-p}G, U(1)]$. In Sec. IV B, we verify this conjecture for small groups G (specifically, $G = \mathbb{Z}_N$ with $N \leq 12$ or $\mathbb{Z}_N \times \mathbb{Z}_N$ with $N \leq 5$), with explicit unitary operator sequences for the generalized statistics given in Sec. II.

	G -particles with $G = \prod_i \mathbb{Z}_{N_i}$	G -loops with $G = \prod_i \mathbb{Z}_{N_i}$	G -membranes with $G = \prod_i \mathbb{Z}_{N_i}$
$(1+1)\text{D}$	$H^3[BG, U(1)]$ $= \prod_i \mathbb{Z}_{N_i} \prod_{i < j} \mathbb{Z}_{(N_i, N_j)}$ $\prod_{i < j < k} \mathbb{Z}_{(N_i, N_j, N_k)}$		
$(2+1)\text{D}$	$H^4[B^2G, U(1)]$ $= \prod_i \mathbb{Z}_{(N_i, 2) \times N_i} \prod_{i < j} \mathbb{Z}_{(N_i, N_j)}$	$H^4[BG, U(1)]$ $= \prod_{i < j} \mathbb{Z}_{(N_i, N_j)}^2 \prod_{i < j < k} \mathbb{Z}_{(N_i, N_j, N_k)}^2$ $\prod_{i < j < k < l} \mathbb{Z}_{(N_i, N_j, N_k, N_l)}$	
$(3+1)\text{D}$	$H^5[B^3G, U(1)]$ $= \prod_i \mathbb{Z}_{(N_i, 2)}$	$H^5[B^2G, U(1)]$ $= \prod_i \mathbb{Z}_{(N_i, 2)} \prod_{i < j} \mathbb{Z}_{(N_i, N_j)}$	$H^5[BG, U(1)]$ $= \prod_i \mathbb{Z}_{N_i} \prod_{i < j} \mathbb{Z}_{(N_i, N_j)}^2$ $\prod_{i < j < k} \mathbb{Z}_{(N_i, N_j, N_k)}^4$ $\prod_{i < j < k < l} \mathbb{Z}_{(N_i, N_j, N_k, N_l)}^3$ $\prod_{i < j < k < l < m} \mathbb{Z}_{(N_i, N_j, N_k, N_l, N_m)}$

the same genuine invariants. We further conjecture that these universal invariants T coincide with the cohomology of the Eilenberg-MacLane space, as summarized in Table I.

3. Statistics as anomalies and their dynamical consequences

The invariant $\Theta \in T$ provides a microscopic definition of the statistics of excitations. At the same time, Θ can be interpreted as a microscopic definition of an 't Hooft anomaly of the global symmetry, which represents an obstruction to gauging the symmetry. This can be understood as follows.

To gauge a symmetry, one introduces gauge fields and promotes the global symmetry to a local gauge symmetry generated by Gauss law operators G_Δ supported on local regions Δ . The physical Hilbert space is then restricted to states satisfying $G_\Delta = 1$. If this gauge-invariant Hilbert space is empty, the gauging procedure is obstructed; namely, an 't Hooft anomaly is present. In Sec. V, we further show that the generalized statistics can be expressed as a product of Gauss law operators,

$$e^{i\Theta} = \prod G_\Delta. \quad (13)$$

A nontrivial phase Θ obtained in this way signals an obstruction to enforcing the constraint $G_\Delta = 1$ everywhere. In this sense, a nontrivial Θ characterizes an 't Hooft anomaly.

The presence of 't Hooft anomalies constrains the low-energy spectrum of the theory. In particular, they forbid the

existence of a unique gapped ground state. We find that the invariants of microscopic lattice systems Θ directly lead to such a dynamical consequence: When the invariant Θ with the symmetry defects is nontrivial, the state cannot be a short-range-entangled (SRE) state preserving the symmetry. We show this statement in full generality, assuming the tensor network representation of the state with excitations. This is reminiscent of the Lieb-Schultz-Mattis theorem [74–80], which constrains the low-energy spectrum of the state based on a given action of the internal and crystalline symmetries.

A large class of 't Hooft anomalies can be described through group cohomology by employing the group cohomology symmetry-protected topological (SPT) phase in the bulk via bulk-boundary correspondence [81–83]. The mathematical results presented in Table I show the group cohomology of higher-form symmetry, i.e., the cohomology of the Eilenberg-MacLane space, for finite Abelian groups [73]. Our generalized statistics provides a microscopic perspective of 't Hooft anomalies on the lattice, and in all examples we have evaluated on a computer, its classification matches the results in Table I. Although the computational power limits us from verifying arbitrarily large groups G , the agreement in the small cases suggests the correspondence. This consistency leads us to conjecture that our generalized statistics are classified by the group cohomology of higher groups. In the following section, we explicitly demonstrate examples of generalized statistics for small groups G , illustrating their correspondence to the group cohomology of higher-form symmetries.

The paper is organized as follows: Section II describes a variety of examples of generalized statistics for particles, loops, and membranes in up to three spatial dimensions, including exactly solvable models whose domain-wall excitations demonstrate the nontrivial statistics of each process. Section III defines generalized statistics from axioms for invertible excitations and shows that they take quantized values. Section IV presents an algorithm that computes these statistics by constructing microscopic processes as sequences of lattice operators, suitable for direct computer implementation. Using this framework, Sec. V gives a microscopic definition of 't Hooft anomalies as obstructions to gauging symmetries, and Sec. VI proves that nontrivial statistics forbid short-range-entangled states, revealing their dynamical consequences. We conclude in Sec. VII with possible future directions. Background concepts and extended discussions are provided in the appendixes.

II. EXAMPLES OF GENERALIZED STATISTICS

To illustrate our construction, we compute explicit statistical invariants for particle, loop, and membrane excitations in dimensions $d \leq 3$ using representative Abelian fusion groups. A systematic derivation valid for arbitrary excitations in any dimension will be presented in Sec. IV.

A. Particle excitations

We consider particle excitations in $(1+1)$, $(2+1)$, and $(3+1)$ spacetime dimensions.

1. Particles in $(1+1)\text{D}$

Let the fusion groups of the particles be a finite group G , where each particle is labeled by an element $g \in G$. While G can be non-Abelian in general, we first focus on the Abelian case for demonstration. We define the hopping operator $U(g)_{ij}$ that creates a particle g at vertex j and a particle g^{-1} at vertex i . The statistics of g -particles are defined by

$$Z_3(g) := [U(g)_{02}, U(g)_{01}^{|g|}], \quad (14)$$

where the vertices 0, 1, and 2 are positioned along a segment, as shown in Fig. 1(a), and we define the group commutator by

$$[A, B] := A^{-1}B^{-1}AB. \quad (15)$$

Here, $|g|$ denotes the order of the element $g \in G$. This index corresponds to the degree-3 cohomology class $H^3(BG, U(1))$, as shown in Table I. The statistics $Z_3(g)$ can be interpreted as the partition function of the corresponding cohomology class on a lens space [84,85]:

$$Z_3(g) = \prod_{n=1}^{|g|} F(g, g^n, g), \quad (16)$$

where the F symbol $F(g_1, g_2, g_3)$ is a sequence of unitary operators defined microscopically in Refs. [68,80]. Notably, when particles are treated as symmetry defects, the expression in Eq. (14) coincides with the topological invariants computed from the boundaries of $(2+1)\text{D}$ SPT phases [86], which capture the Else-Nayak index [83]. The nontrivial statistics described by Eq. (14) signal an obstruction to satisfying $U(g)_{ij}^{|g|} = 1$, thereby reflecting an anomalous symmetry action. We also emphasize that the statistics in Eq. (14) apply to non-Abelian groups. For instance, when $G = S_3$ is the symmetric group, which contains the Abelian subgroups $\mathbb{Z}_2$ and $\mathbb{Z}_3$, we can substitute their generators into Eq. (14) to derive the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ invariants, separately. These results are consistent with the cohomology class $H^3(S_3, U(1)) = \mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$.

Now, we illustrate the simplest example of nontrivial fusion statistics. Consider the anomalous $\mathbb{Z}_2$ symmetry on a one-dimensional qubit chain [83,87–89]

$$S := \prod_i X_i \prod_i CZ_{i,i+1}, \quad (17)$$

where X_i and Z_i are Pauli matrices on site i , and $CZ_{i,i+1}$ is the controlled- Z gate between adjacent qubits. This anomalous symmetry action originates at the boundary of the $(2+1)\text{D}$ Levin-Gu $\mathbb{Z}_2$ SPT phase [58]. The symmetry-preserving operators are generated by

$$W_{i+\frac{1}{2}} := Z_i Z_{i+1}, \quad U_{i-\frac{1}{2} \rightarrow i+\frac{1}{2}} := X_i CZ_{i-1,i} CZ_{i,i+1}. \quad (18)$$

We interpret the $W_{i+\frac{1}{2}}$ as the domain-wall excitation on the edge $\langle i, i+1 \rangle$, and $U_{i-\frac{1}{2} \rightarrow i+\frac{1}{2}}$ is the hopping operator of the domain-wall excitation from $i - \frac{1}{2}$ to $i + \frac{1}{2}$. We omit the group label since $\mathbb{Z}_2$ has only one nontrivial element. Applying Eq. (14), the fusion statistic of this domain wall is

$$\begin{aligned} [U_{i+\frac{1}{2} \rightarrow i+\frac{3}{2}}, U_{i-\frac{1}{2} \rightarrow i+\frac{1}{2}}^2] &= [X_{i+1} CZ_{i,i+1} CZ_{i+1,i+2}, Z_{i-1} Z_{i+1}] \\ &= -1. \end{aligned}$$

The value -1 confirms that the $\mathbb{Z}_2$ symmetry in Eq. (17) is anomalous.

2. Particles in $(2+1)\text{D}$ and $(3+1)\text{D}$

Invertible particles in $(2+1)$ - and $(3+1)$ -dimensional spacetime must have an Abelian fusion group G . As reviewed in Sec. IA 1, the statistics of a particle labeled by a group element $g \in G$ can be detected using the T -junction process [45,67,68]

$$e^{i\Theta(g)} := U(g)_{02} U(g)_{03}^{-1} U(g)_{01} U(g)_{02}^{-1} U(g)_{03} U(g)_{01}^{-1}, \quad (19)$$

where the vertices 0, 1, 2, and 3 are shown in Figs. 1(b) and 1(c). When $G = \mathbb{Z}_N$, the possible values of Θ are

$$e^{i\Theta} = \begin{cases} \exp\left(2\pi i \frac{k}{\gcd(2,N) \times N}\right) & \text{in } (2+1)\text{D}, \\ \pm 1 & \text{in } (3+1)\text{D}, \end{cases} \quad (20)$$

where $0 \leq k < \gcd(2, N) \times N$ is an integer. These values are characterized by the degree-4 cohomology class $H^4(B^2G, U(1))$ in $(2+1)\text{D}$ and the degree-5 cohomology class $H^5(B^3G, U(1))$ in $(3+1)\text{D}$ associated with the 1-form and 2-form G symmetry, respectively (see Table I). In $(3+1)\text{D}$ and higher dimensions, the $\pm$ sign

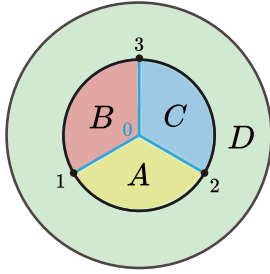
corresponds to boson or fermion statistics, which are related to the second Stiefel-Whitney class w_2 [9,53–55,90–92].

B. Loop excitations

Loop excitations can be interpreted as defect lines or flux loops, and we consider loop excitations in $(2+1)$ and $(3+1)$ spacetime dimensions.

1. Loops in $(2+1)\text{D}$

In this dimension, the “fusion” of loops also defines statistics, analogous to the F symbol for particles. The simplest examples of nontrivial statistics arise when the fusion group is $G = \mathbb{Z}_N \times \mathbb{Z}_N$ (with generators labeled by a and b), and there are two invariants:



$$\begin{aligned} Z_4^I(a, b) &:= (U(a)_{B+C})^{-N} \left(U(a)_{B+C} [U(a)_B, [U(a)_A, U(b)_{A+B+C+D}]] \right)^N, \\ Z_4^{II}(a, b) &:= (U(b)_{B+C})^{-N} \left(U(b)_{B+C} [U(b)_B, [U(b)_A, U(a)_{A+B+C+D}]] \right)^N, \end{aligned} \quad (21)$$

where $U(i)_R$ denotes the membrane operator that creates loop excitations labeled by i on the boundaries of region R (with $i = a, b$ as generators of the fusion group $G = \mathbb{Z}_N \times \mathbb{Z}_N$). The notation $U(i)_{I+J+\dots}$ refers to the membrane operator acting on the union $I \cup J \cup \dots$ defined as the product $U(i)_I U(i)_J \dots$ [93]. The formulas for $N = 2, 3, 4, 5$ are obtained from our algorithm by computer calculation (Sec. IV B), and we expect them to extend to arbitrary N .

Similar to particle fusion, we can show that the nontrivial statistics $Z_4(a)$ and $Z_4(b)$ are obstructions for

$$U(a)_f^N = U(b)_f^N = [U(a)_f, U(b)_f] = 1, \quad \forall f, \quad (22)$$

indicating an anomalous symmetry. We also highlight that the statistics in Eq. (21) correspond to the degree-4 cohomology class $H^4(BG, U(1))$ of the global symmetry G , as shown in Table I.

For a concrete example of nontrivial loop fusion statistics, consider an anomalous $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry on a two-dimensional square lattice. Each vertex v hosts two qubits a, b with Pauli operators X_v^a, Z_v^a and X_v^b, Z_v^b . We label their Z eigenvalues by $a_v, b_v \in \{0, 1\}$ via

$$Z_v^a = (-1)^{a_v}, \quad Z_v^b = (-1)^{b_v}. \quad (23)$$

The anomalous $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is generated by

$$\begin{aligned} S^a &:= \prod_v X_v^a \prod_{f=\square_{1234}} \left(\text{diagram of a square with vertices 1, 2, 3, 4 and loops labeled CCZ} \right) = \prod_v X_v^a \prod_{f=\square_{1234}} (-1)^{b_1 b_2 b_4 + b_1 b_3 b_4}, \\ S^b &:= \prod_v X_v^b, \end{aligned} \quad (24)$$

where CCZ denotes the controlled-controlled-Z gate. This anomalous $(2+1)$ D system can be derived from the boundary of a $(3+1)$ D SPT phase with cocycle [94]

$$\frac{1}{2}A_1 \cup B_1 \cup B_1 \cup B_1 \in H^4(B\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)). \quad (25)$$

In Appendix A 1, we show that, within the symmetric subspace, the domain walls of this anomalous $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry become loop excitations with nontrivial fusion rules.

2. Loops in $(3+1)$ D

In $(3+1)$ D, the loop statistics is determined by the loop-flipping process for $G = \mathbb{Z}_2$. We introduce a novel 24-step process to detect this statistic:

$$\begin{aligned} \mu_{24} := & U_{014}U_{034}U_{023}U_{014}^{-1}U_{024}^{-1}U_{012}U_{023}^{-1}U_{013}^{-1} \\ & \times U_{024}U_{014}U_{013}U_{024}^{-1}U_{034}^{-1}U_{023}U_{013}^{-1}U_{012}^{-1} \\ & \times U_{034}U_{024}U_{012}U_{034}^{-1}U_{014}^{-1}U_{013}U_{012}^{-1}U_{023}^{-1}, \end{aligned} \quad (26)$$

which is shown in Fig. 3 explicitly. Each line in Eq. (26) is obtained from the previous one by applying the substitutions $1 \rightarrow 2$, $2 \rightarrow 3$, and $3 \rightarrow 1$ [96]. The space of single-loop configurations forms an $\mathbb{RP}^2$ structure, as illustrated in Fig. 4. The 24-step process μ_{24} is represented on this $\mathbb{RP}^2$, explicitly manifesting the C_3 rotational symmetry $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$. We demonstrate that the 24-step process yields the same statistics as the 36-step process defined in Ref. [45], while being more efficient. Computational verification confirms that this 24-step sequence is the shortest way to obtain the loop statistics.

We emphasize that if all U_f are Pauli operators, such that $[U_{f_3}, [U_{f_2}, U_{f_1}]] = 1$ for all faces f_1, f_2 , and f_3 , then the above statistics simplify to

$$\mu_{24}^{\text{Pauli}} = [U_{012}, U_{034}]^2 [U_{013}, U_{024}]^2 [U_{014}, U_{023}]^2. \quad (27)$$

The statistic μ_{24} corresponds to a degree-5 cohomology class in $H^5(B^2G, U(1))$ for the 1-form G symmetry and is also related to the third Stiefel-Whitney class w_3 for $G = \mathbb{Z}_2$:

$$\frac{1}{2}w_3 \cup B_2 = \frac{1}{2}B_2 \cup (B_2 \cup_1 B_2) \in H^5(B^2\mathbb{Z}_2, U(1)). \quad (28)$$

To confirm the relation between μ_{24} and w_3 , we insert the unitary operators describing the loop excitation on the $(3+1)$ D boundary of the beyond-cohomology $(4+1)$ D topological quantum field theory with action $S = \frac{1}{2}w_2w_3$ in Ref. [44] into the 24-step procedure and obtain $\mu_{24} = -1$. This demonstrates that the statistic μ_{24} precisely detects the w_3 gravitational anomaly. This process remains valid for the fusion group $G = \mathbb{Z}_N$ for even N , by choosing the operators U_{ijk} that create a loop labeled by $(N/2) \in \mathbb{Z}_N$ representing an element of order two.

C. Membrane excitations

1. Membranes in $(3+1)$ D

Now we consider membrane excitations in $(3+1)$ spacetime dimensions. For $G = \mathbb{Z}_N$, the fusion of membranes gives rise to the $\mathbb{Z}_N$ statistics [97]

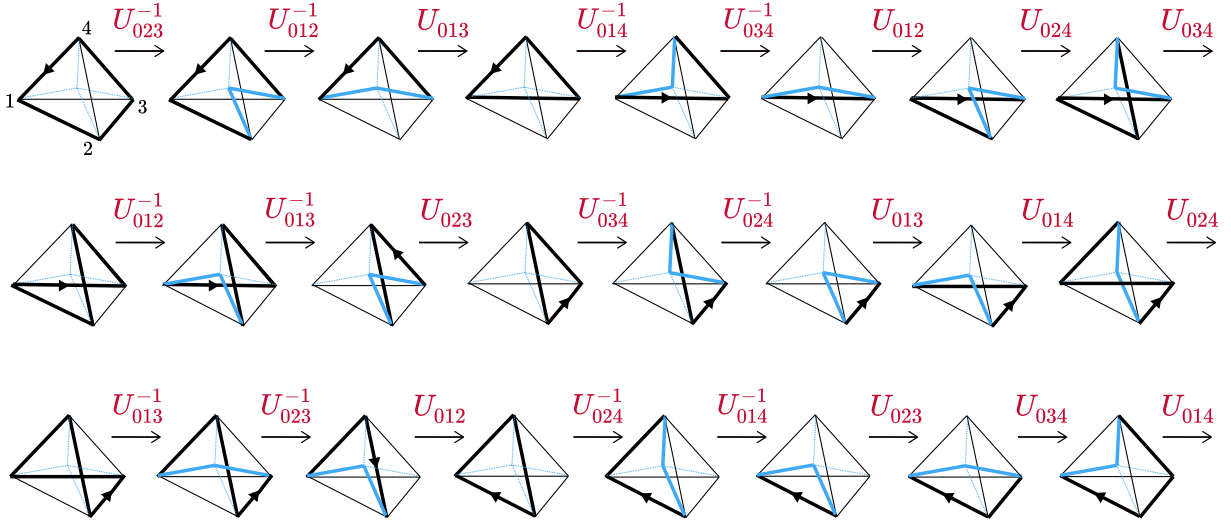


FIG. 3. The 24-step process for detecting the statistics of loops with $G = \mathbb{Z}_2$ fusion in $(3+1)$ D. For $\mathbb{Z}_2$ loops, different orientations correspond to the same configuration state, indicating that the initial and final configurations are reversed and illustrating the loop-flipping process. This unitary sequence yields the same invariant as the 36-step unitary process proposed in Ref. [45]. We prove that this 24-step process is optimal, as no shorter sequence can achieve the same invariant.

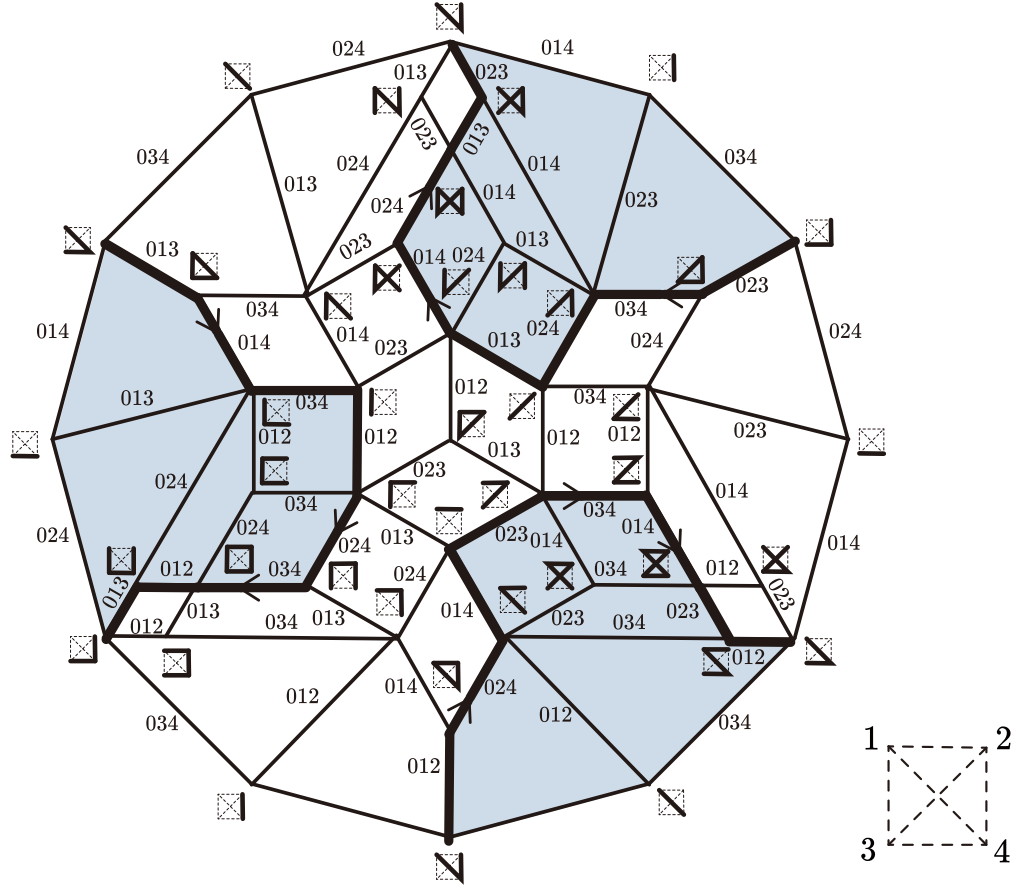
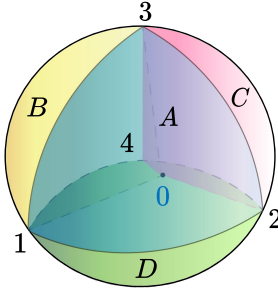


FIG. 4. The space of single-loop configuration states. Each vertex represents a single-loop configuration, where the loop is drawn only on edges between vertices 1, 2, 3, and 4 in Fig. 1(c) since the edges adjacent to vertex 0 can be inferred from these edges. Each edge of this space is labeled by $\langle 0ij \rangle$, and we can apply the operator U_{0ij} or U_{0ij}^{-1} to create or annihilate the loop on edge ij . This configuration space forms an $\mathbb{RP}^2$ structure [45], where antipodal vertices represent the same configuration state. The 24-step process is illustrated by the black directed line exhibiting the C_3 rotational symmetry and corresponding to the nontrivial element of $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$.



$$Z_5(g) := (U(g)_{C+D})^{-N} (U(g)_{C+D} [U(g)_C, [U(g)_B, U(g)_A^N]])^N, \quad (29)$$

where $U(g)_R$ denotes the volume operator on region R that creates a g -membrane excitation on its boundary, and $U(g)_{I+J+\dots}$ denotes the product $U(g)_I U(g)_J \dots$ acting on the union $I \cup J \cup \dots$. The formulas for $N \leq 8$ are obtained explicitly from our algorithm in Sec. IV B, and we expect that the expressions can be extended to arbitrary N . Similar to the $(2+1)$ D fusion in Eq. (21),

the nontrivial statistic $Z_5(g)$ represents the obstruction to $U(g)_t^N = 1$ for all tetrahedra t . The statistic $Z_5(g)$ corresponds to the degree-5 cohomology $H^5(BG, U(1))$ in Table I.

To illustrate a nontrivial $Z_5(g)$, consider a cubic lattice with one qubit at each vertex. The anomalous $\mathbb{Z}_2$ global symmetry is defined by

$$\begin{aligned}
S &:= \prod_v X_v \prod_c \left(\text{Diagram of a cube with vertices 1-8 and colored loops} \right) \\
&= \prod_v X_v \prod_{c=\begin{smallmatrix} 7 & 8 \\ 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{smallmatrix}} (-1)^{a_1 a_2 a_4 a_8 + a_1 a_2 a_6 a_8 + a_1 a_3 a_4 a_8 + a_1 a_3 a_7 a_8 + a_1 a_5 a_6 a_8 + a_1 a_5 a_7 a_8},
\end{aligned} \tag{30}$$

where CCCZ denotes the controlled-controlled-controlled-Z gate. The anomalous $(3+1)$ D theory resides on the boundary of the $(4+1)$ D SPT phase with the cocycle

$$\frac{1}{2} A_1 \cup A_1 \cup A_1 \cup A_1 \cup A_1 \in H^5(B\mathbb{Z}_2, U(1)) \tag{31}$$

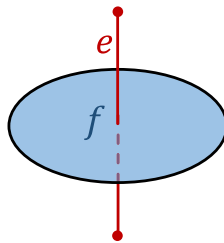
known as the generalized double semion model in Refs. [98,99]. In Appendix A 2, we show that within the symmetric subspace, the domain-wall excitation forms a closed membrane exhibiting nontrivial membrane fusion statistics $Z_5(g)$.

D. Mutual statistics

1. Particle-loop statistics in $(3+1)$ D

In addition to examples in which all excitations share the same dimension, one can examine the mutual statistics between excitations of different dimensionalities. The simplest instance in $(3+1)$ D gauge theories is the braiding of a point charge around a flux loop: As the charge encircles the loop, the wave function acquires a nontrivial phase.

On the lattice, this process is described by the commutator between the string operator that moves the particle and the membrane operator that creates the loop:



$$Z_5^{\text{particle-loop}} := [U_f, U_e], \tag{32}$$

where U_f is the membrane operator acting on a face f that generates the loop excitation, U_e is the string operator acting on an edge e that generates the particle excitation, and the edge e pierces the face f . This defines a $\mathbb{Z}_N$ mutual statistics satisfying $(Z_5^{\text{particle-loop}})^N = 1$. This braiding statistics corresponds to the 5-cocycle

$$\frac{1}{N} A_2 \cup B_3 \in H^5(B^2\mathbb{Z}_N \times B^3\mathbb{Z}_N, U(1)). \tag{33}$$

2. Loop-membrane statistics in $(3+1)$ D

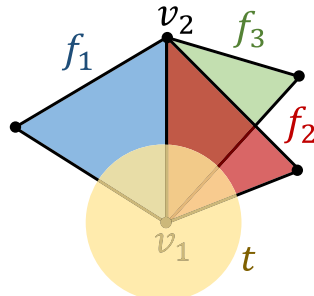
We now present two distinct types of mutual $\mathbb{Z}_2$ statistics between a $\mathbb{Z}_2$ loop and a $\mathbb{Z}_2$ membrane in $(3+1)$ D. These cases are anticipated from the cohomology classification

$$H^5(B\mathbb{Z}_2 \times B^2\mathbb{Z}_2, U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \tag{34}$$

where two $\mathbb{Z}_2$ factors correspond to the self-statistics of particles and loops, respectively, and the remaining two correspond to mutual statistics. A $\mathbb{Z}_2$ loop along the boundary of a face f is generated by the membrane operator U_f , which realizes the 1-form $\mathbb{Z}_2$ symmetry. A $\mathbb{Z}_2$ membrane on the boundary surface of a tetrahedron t is generated by the volume operator U_t , which realizes the 0-form $\mathbb{Z}_2$ symmetry.

(i) $\frac{1}{2} A_1 \cup B_2 \cup B_2$ loop-membrane statistics:

We first consider the following setting with three faces and one tetrahedron:



$$Z_5^{\text{loop-membrane-I}} := [U_t, [U_{f_3}^{-1} U_{f_2} U_{f_1}^{-1} U_{f_3} U_{f_2}^{-1} U_{f_1}]], \tag{35}$$

where f_1 , f_2 , and f_3 are faces adjacent to the edge $\langle v_1 v_2 \rangle$, and t is a tetrahedron (more precisely, a 3-cell) whose bulk contains the vertex v_1 . Equation (35) can be viewed as a T -junction process of loop excitations on faces f_1 , f_2 , and f_3 , with the commutator with U_t isolating the contribution from the surface of tetrahedron t , thereby capturing the statistics of the loop-membrane intersection.

This statistic is relevant for describing the boundary theory of the $(4+1)$ D beyond-cohomology $\mathbb{Z}_2$ SPT phase with the topological action $S = \frac{1}{2} A_1 \cup w_2 \cup w_2$ [44,100], where w_2 denotes the second Stiefel-Whitney class. Reference [44] shows that this beyond-cohomology SPT phase can be obtained by gauging a higher-form SPT phase with the cocycle containing

$$\frac{1}{2} w_2 \cup A_1 \cup B_2 = \frac{1}{2} A_1 \cup B_2 \cup B_2 \in H^5(B\mathbb{Z}_2 \times B^2\mathbb{Z}_2, U(1)). \quad (36)$$

It has been predicted that the intersection of a membrane (described by the A_1 field) and a loop (described by the B_2 field) has the fermionic particle statistics due to the presence of w_2 . However, an explicit derivation of this phenomenon on a finite lattice has been lacking. Our novel statistics in Eq. (35) resolves this gap by providing a concrete lattice definition. In Appendix A3, we show that the $(3+1)$ D boundary of the $(4+1)$ D beyond-cohomology $\mathbb{Z}_2$ SPT phase exhibits the nontrivial statistics $Z_5^{\text{loop membrane I}} = -1$. More precisely, consider the anomalous 0-form $\mathbb{Z}_2$ symmetry S^a and the 1-form $\mathbb{Z}_2$ symmetry S_v^b , where a qubit a_v is placed on each vertex v and a qubit b_e on each edge e :

$$S^a := \prod_v X_v^a \prod_{\substack{1 \quad 2 \\ 3 \quad 4 \\ 5 \quad 6 \\ 7 \quad 8}} (-1)^{b_{12}(b_{24}+b_{48}+b_{26}+b_{68})+b_{13}(b_{34}+b_{48}+b_{37}+b_{78})+b_{15}(b_{56}+b_{68}+b_{57}+b_{78})},$$

diagrammatically: $\prod_v X_v^a \prod_c \left(\begin{array}{c} \text{CZ} \\ \text{CZ} \\ \text{CZ} \end{array} \right),$

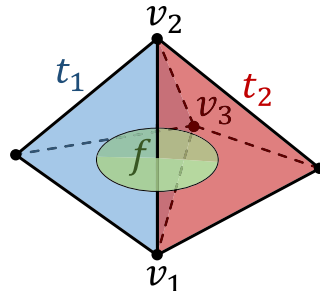
with $\begin{array}{c} e_4 \quad e_5 \\ \text{CZ} \\ e_2 \quad e_3 \\ e_1 \end{array} \quad \begin{array}{c} e_5 \\ \text{CZ} \\ e_3 \\ e_4 \\ e_1 \end{array} \quad \begin{array}{c} e_5 \\ \text{CZ} \\ e_3 \\ e_4 \\ e_1 \end{array} := (-1)^{b_{e_1}(b_{e_2}+b_{e_3}+b_{e_4}+b_{e_5})},$

$$S_v^b := \prod_{e \supset v} X_e^b, \quad \text{diagrammatically: } \begin{array}{c} X_e^b \\ X_e^b \\ X_e^b \\ X_e^b \\ X_e^b \end{array}.$$

We demonstrate that the domain-wall excitations of these symmetries exhibit the nontrivial mutual loop-membrane statistics given in Eq. (35).

(ii) $\frac{1}{2} A_1 \cup A_1 \cup A_1 \cup B_2$ loop-membrane statistics:

We next consider a different structure involving one face and two tetrahedra:

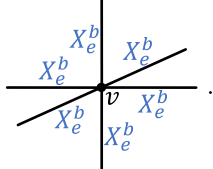


The diagram shows a tetrahedron with vertices v_1 , v_2 , and v_3 . The faces are labeled t_1 (blue), t_2 (red), and f (green). The face f is a central face, and the tetrahedron is composed of two smaller tetrahedra sharing the face f .

$$Z_5^{\text{loop-membrane-II}} := [U_f, [U_{t_2}, U_{t_1}^2]], \quad (38)$$

where t_1 and t_2 are tetrahedra adjacent to the face $\langle v_1 v_2 v_3 \rangle$, and the edge $\langle v_1 v_2 \rangle$ pierces the face f . In Appendix A 4, we show that for the anomalous 0-form $\mathbb{Z}_2$ symmetry S^a and 1-form $\mathbb{Z}_2$ symmetry S_v^b defined below, the corresponding domain-wall excitations exhibit the nontrivial statistics $Z_5^{\text{loop membrane II}} = -1$:

$$S^a := \prod_v X_v^a \prod_{\text{diagram}} (-1)^{a_1 a_2 (b_{24} + b_{48} + b_{26} + b_{68}) + a_1 a_3 (b_{34} + b_{48} + b_{37} + b_{78}) + a_1 a_5 (b_{56} + b_{68} + b_{57} + b_{78})},$$

$$S_v^b := \prod_{e \supset v} X_e^b, \quad \text{diagrammatically: } \text{diagram} \quad (39)$$


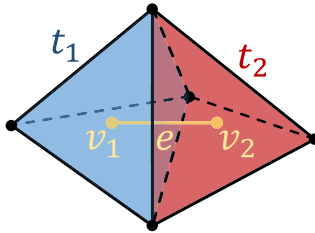
These anomalous symmetries correspond to the SPT phase associated with the degree-5 cohomology class

$$\frac{1}{2} A_1 \cup A_1 \cup A_1 \cup B_2 \in H^5(B\mathbb{Z}_2 \times B^2\mathbb{Z}_2, U(1)). \quad (40)$$

This cocycle provides an intuitive way to interpret Eq. (38). First, the term $A_1 \cup A_1 \cup A_1$ represents the 3-cocycle associated with the fusion statistics in one dimension [Eq. (14)], motivating a structure of the form $[U_{t_2}, U_{t_1}^2]$. Taking the cup product with B_2 corresponds to intersecting with the world sheet of the loop excitation, which motivates the introduction of the additional commutator with U_f to capture this intersection.

3. Particle-membrane statistics in (3+1)D

Finally, we consider the mutual statistics between a $\mathbb{Z}_2$ particle and a $\mathbb{Z}_2$ membrane in three spatial dimensions. Let U_e denote the string operator acting on an edge e that creates particles at its end points, and let U_t denote the volume operator acting on a tetrahedron t that generates a membrane excitation on its boundary surface. The mutual statistics is defined in the following structure:



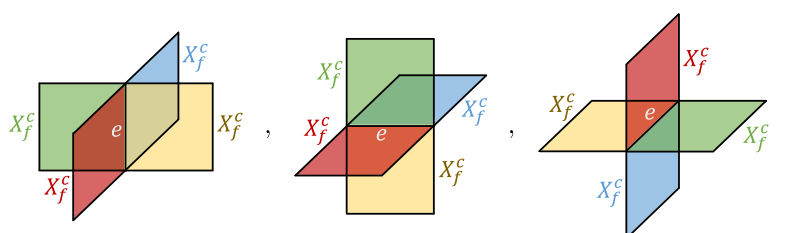
$$Z_5^{\text{particle-membrane}} := [U_e^2, U_{t_1}] [U_{t_2}, [U_{t_1}, U_e]]$$

$$= \left(U_{t_2}^{-1} [U_e, U_{t_1}] U_{t_2} \right) \left(U_e^{-1} [U_e, U_{t_1}] U_e \right), \quad (41)$$

where t_1 and t_2 are tetrahedra adjacent to opposite sides of a common face, v_1 is the vertex contained in t_1 , v_2 is the vertex contained in t_2 , and the edge $e = \langle v_1 v_2 \rangle$ pierces the common face. We present two equivalent expressions for the statistics, both derived from our algorithm; either form can be used for computation, depending on convenience.

Consider the anomalous 0-form $\mathbb{Z}_2$ symmetry S^a and 2-form $\mathbb{Z}_2$ symmetry S_e^c defined below, with a qubit a_v placed on each vertex v and a qubit c_f on each face f :

$$S^a := \prod_v X_v^a \prod_{\text{diagram}} (-1)^{a_1 (c_{1234} + c_{1256} + c_{1357} + c_{2468} + c_{3478} + c_{5678})},$$

$$S_e^c := \prod_{f \supset e} X_f^c, \quad \text{diagrammatically: } \text{diagram} \quad (42)$$


In Appendix A 5, we show that the domain-wall excitations associated with these symmetries exhibit the nontrivial statistics $Z_5^{\text{particle membrane}} = -1$. These symmetries arise from the $(3+1)\text{D}$ boundary of a $(4+1)\text{D}$ SPT phase characterized by the degree-5 cohomology class

$$\frac{1}{2}A_1 \cup A_1 \cup C_3 \in H^5(B\mathbb{Z}_2 \times B^3\mathbb{Z}_2, U(1)), \quad (43)$$

where A_1 and C_3 are the background gauge fields for the 0-form $\mathbb{Z}_2$ and 2-form $\mathbb{Z}_2$ symmetries, respectively.

III. FRAMEWORK FOR GENERALIZED STATISTICS

A. Framework of invariants in generic dimensions

We describe the invariants associated with extended excitations in generic dimensions. The T -junction process, previously discussed for particle excitations, is generalized to extended excitations in any dimension through a sequence of unitaries that create these excitations. Consider a system with a finite, invertible $(d-p-1)$ -form symmetry group G in a d -dimensional space. The group G is generated by unitary operators U supported on $(p+1)$ -dimensional submanifolds. If the support has a boundary, the operator U creates a p -dimensional extended excitation at that boundary. For a 0-form symmetry with $p = d-1$, G can be non-Abelian; in all other cases, G must be Abelian. We first restrict attention to the Abelian fusion group G , and address the non-Abelian case in Sec. III B.

1. Excitations and unitaries on simplicial complex

Following the description of the T -junction process, the construction of the invariant starts by fixing the possible configurations of the excitations in space. We define the excitation model as follows:

Definition III.1. An excitation model defined on a tensor-product Hilbert space $\mathcal{H}$ over a d -dimensional spatial manifold M consists of the following components:

- (1) A finite Abelian group $\mathcal{A}$. Each group element $a \in \mathcal{A}$ corresponds to a configuration of the excitations in the space M . For each configuration $a \in \mathcal{A}$, there exists a state $|a\rangle \in \mathcal{H}$, such that the states that correspond to different configurations are orthogonal [101].
- (2) A finite set $\mathcal{S}$ and a map $\partial: \mathcal{S} \rightarrow \mathcal{A}$, such that $\text{Im } \partial$ generates $\mathcal{A}$. Each element $s \in \mathcal{S}$ represents a unitary operator that creates the excitation ∂s . For each $s \in \mathcal{S}$, there exists a unitary operator $U(s)$ such that $U(s)|a\rangle \propto |a + \partial s\rangle$ for all $a \in \mathcal{A}$. The support of the unitary $U(s)$ is the $(p+1)$ -dimensional locus $\text{supp}(s) \subset M$.

We note that the above definition assumes the invertible symmetry with a grouplike fusion rule, and further that the

fusion group is finite and Abelian. One can extend the above definition to the case of invertible non-Abelian group symmetries, which will be studied in Sec. III B.

Let us take a concrete example of the excitation model based on the T -junction reviewed in Sec. I A 1. In that case, $\{U(s)|s \in \mathcal{S}\}$ is a set of operators that generate all string operators with the fusion group G supported on the fixed edges connecting the vertices 0,1,2,3 of Fig. 1(b). There are six edges $\langle jk \rangle$ with $0 \leq j < k \leq 3$, so $\mathcal{S}$ is a set of generators of G^6 . $\mathcal{A}$ is taken to be all configurations of anyons created by sequences of operators in $\{U(s)\}$ acting on the vacuum. This is isomorphic to $\mathcal{A} = G^3$, which corresponds to the configurations of anyons at four vertices fusing into the vacuum. In the case of the T -junction, $\mathcal{S}$ can be regarded as a set of generators of the group of 1-chains $C_1(X, G)$ in the simplicial complex X shown in Fig. 1(b), ∂ as the boundary map of the simplicial complex, and $\mathcal{A}$ as the boundary group $\mathcal{A} = B_0(X, G)$.

More generally, one can construct an excitation model on any finite simplicial complex X embedded in the space M , which can describe extended excitations in arbitrary dimensions. To illustrate these ideas, consider the excitation model on a simplicial complex. In this case, let $\mathcal{A}$ be the group of p -dimensional simplicial boundaries

$$\mathcal{A} = B_p(X, G). \quad (44)$$

We take $\mathcal{S}$ to be a minimal set of generators of the $(p+1)$ -chains of X with coefficients in G , i.e., generators of $C_{p+1}(X, G)$. Concretely, each element $s \in \mathcal{S}$ is given by $s = g\sigma_{p+1}$, where $g \in G$ is one of the generators of G , and σ_{p+1} is a single $(p+1)$ -simplex of X . Viewing X as a topological space embedded in M , $\text{supp}(s)$ gives the image of the simplex σ_{p+1} in M under the embedding map. The unitary $U(s)$ can then be expressed as $U(s) = U_g(\sigma_{p+1})$, namely, the operator generating the $g \in G$ symmetry at the simplex σ_{p+1} embedded in M . The map $\partial: \mathcal{S} \rightarrow \mathcal{A}$ is the homological boundary map of X , and we have $U(s)|a\rangle \propto |a + \partial s\rangle$ with $a \in \mathcal{A}$, $s \in \mathcal{S}$.

In general, different choices of the simplicial complex X yield different invariants. To extract the complete set of invariants observable in the spatial manifold M , one can choose X as a simplicial decomposition of M . Meanwhile, the generalized statistics of excitations are typically local properties that can be extracted from a simplicial complex X supported on a ball embedded in the whole space M . Such invariants include, for example, the T -junction of anyons, and are insensitive to the global topology of the space.

2. Invariant Berry phases from unitary sequences

Now we are ready to construct the invariants out of the sequence of unitaries $\{U(s)\}$ acting on the excited states $\{|a\rangle\}$. From now, we assume that the excitation model is

constructed on the simplicial complex X embedded in the space, according to Sec. III A 1.

For $a \in \mathcal{A}$, $s \in \mathcal{S}$, the unitary operator $U(s)$ transforms the state $|a\rangle$ into $|a + \partial s\rangle$ up to a phase:

$$U(s)|a\rangle = \exp[i\theta(s, a)]|a + \partial s\rangle. \quad (45)$$

Accordingly, the inverse of the unitary $U(s)$ acts on the states by

$$U(s)^{-1}|a + \partial s\rangle = \exp[-i\theta(s, a)]|a\rangle. \quad (46)$$

The invariant is then expressed as a sequence of unitary operators starting and terminating with the same configuration of excitations

$$\langle a_0 | U(s_{n-1})^\pm, \dots, U(s_j)^\pm, \dots, U(s_0)^\pm | a_0 \rangle, \quad (47)$$

where $\pm$ is the sign which can be chosen for each unitary. This Berry phase will become the sum over the phases $\theta(s, a)$ with $s \in \mathcal{S}$, $a \in \mathcal{A}$.

To qualify the above phase as an invariant, one needs to establish the invariance of the above quantity against possible deformations of the states as well as unitaries. For the formulation of the invariants, it is convenient to express the above Berry phase as the element of the formal sum of the objects $\theta(s, a)$:

$$E = \bigoplus_{s \in \mathcal{S}, a \in \mathcal{A}} \mathbb{Z}\theta(s, a), \quad (48)$$

which we call the expression group associated with the excitation model. Each element $e \in E$ is expressed as $e = \bigoplus_{(s,a)} \epsilon(s, a)\theta(s, a)$ with the integer coefficients $\epsilon(s, a) \in \mathbb{Z}$. The condition for e being the invariant will be compiled into a set of equations that the coefficients $\{\epsilon(s, a) | s \in \mathcal{S}, a \in \mathcal{A}\}$ need to satisfy. The invariants then correspond to a specific subgroup $E_{\text{inv}} \subset E$, which we now characterize.

The first condition for E_{inv} simply requires that the Berry phase corresponds to a sequence of unitaries initiating and terminating with the same state. This is equivalent to requiring the invariance under the redefinition of the states $|a\rangle \rightarrow e^{i\phi(a)}|a\rangle$ for $a \in \mathcal{A}$, which shifts the phases as $\theta(s, a) \rightarrow \theta(s, a) - \phi(a) + \phi(a + \partial s)$. The element $e \in E$ is invariant under such a redefinition of phases if and only if the coefficients $\{\epsilon(s, a)\}$ satisfy

$$\sum_{s \in \mathcal{S}} \epsilon(s, a) - \sum_{s \in \mathcal{S}} \epsilon(s, a - \partial s) = 0, \quad \text{for any } a \in \mathcal{A}. \quad (49)$$

This gives the first necessary condition for $e \in E_{\text{inv}}$.

The second condition for E_{inv} is that the Berry phase is invariant under the redefinition of unitaries $U(s)$ by a phase $U(s) \rightarrow e^{i\phi(s)}U(s)$ for $s \in \mathcal{S}$, shifting the phases as $\theta(s, a) \rightarrow \theta(s, a) + \phi(s)$. The element $e \in E$ is invariant

under such a redefinition if and only if the coefficients $\{\epsilon(s, a)\}$ satisfy

$$\sum_{a \in \mathcal{A}} \epsilon(s, a) = 0, \quad \text{for any } s \in \mathcal{S}. \quad (50)$$

This gives the second necessary condition for $e \in E_{\text{inv}}$.

The rest of the conditions for E_{inv} are that the Berry phase is invariant under the deformations of the unitary $U(s)$ by a local operator near the boundary of the support of $U(s)$. Suppose that we add a local deformation at a point contained in the j -simplex $\sigma_j \in X$ embedded in the space M . This has the effect of locally modifying the symmetry operator $U(s)$ to the other unitary $U'(s)$, when the support $(p+1)$ -simplex $\sigma_{p+1} = \text{supp}(s)$ contains the simplex σ_j . Such an inclusion of a simplex is denoted by $\sigma_j \subset \sigma_{p+1}$, meaning that a set of vertices of σ_j is a subset of that of σ_{p+1} .

The phase $\theta(s, a)$ gets shifted according to the redefinition of the unitary. Since $U'(s)$ differs from $U(s)$ locally, the operator $U'(s)^\dagger U(s)$ is a local operator supported on a local region r within σ_j . The excited states $|a\rangle, |a + \partial s\rangle$ possibly get modified by $|a\rangle' = V|a\rangle, |a + \partial s\rangle' = \tilde{V}|a + \partial s\rangle$ with some local unitary $V, \tilde{V}$ with the same support r as $U'(s)^\dagger U(s)$. $|a\rangle$ is then an eigenstate of the operator $O(r) := V^\dagger U'(s)^\dagger \tilde{V} U(s)$. Let us denote the eigenvalue as the phase $\exp(i\phi(s, a))$,

$$e^{i\phi(s, a)}|a\rangle = O(r)|a\rangle. \quad (51)$$

Because of the locality of the perturbation, $V, \tilde{V}$ depend on the excitation a only through its restriction to the set of p -simplices satisfying $\sigma_j \subset \sigma_p$. Furthermore, the eigenvalue $\exp(i\phi(s, a))$ of the local operator $O(r) = V^\dagger U'(s)^\dagger \tilde{V} U(s)$ also depends on a only through the p -simplices with $\sigma_j \subset \sigma_p$. To see this, we express $\exp(i\phi(s, a))$ as

$$\begin{aligned} e^{i\phi(s, a)} &= \langle a | O(r) | a \rangle \\ &= \langle 0 | U^\dagger(s_a) O(r) U(s_a) | 0 \rangle, \end{aligned} \quad (52)$$

where $s_a \in \mathcal{S}$ satisfies $a = \partial s_a$, and $0 \in \mathcal{A}$ denotes the trivial element of $\mathcal{A}$. Since $U(s_a)$ is a finite-depth circuit, the part of the circuit away from the region r commutes with $O(r)$. This implies that $U^\dagger(s_a) O(r) U(s_a)$ is again a local operator supported on r , and is independent of the configuration of a away from the perturbation at r , which lies within a simplex σ_j .

Now let us define $a|_{\sigma_j}$ as a G -valued p -cochain defined by $a|_{\sigma_j} = a$ at σ_p satisfying $\sigma_j \subset \sigma_p$, otherwise zero. Because of the above argument, the action of $O(r) = V^\dagger U'(s)^\dagger \tilde{V} U(s)$ can be expressed as $V^\dagger U'(s)^\dagger \tilde{V} U(s)|a\rangle = \exp(i\phi(s, a|_{\sigma_j}))|a\rangle$; namely, the

dependence on a is only through the configuration of a near the simplex σ_j where the deformation occurs.

This implies that the action of $U'(s)$ is shifted from $U(s)$ by $\theta'(s, a) = \theta(s, a) + \phi(s, a|_{\sigma_j})$ when $\sigma_j \subset \text{supp}(s)$. Requiring the invariance of $e \in E$ under this shift of θ is equivalent to requiring that the coefficients $\{\epsilon(s, a)\}$ satisfy

$$\sum_{\substack{a \in \mathcal{A}^{(j)} \\ a|_{\sigma_j} = a_*}} \epsilon(s, a) = 0 \quad (53)$$

for any possible choice of $s \in \mathcal{S}, \sigma_j \in X$ satisfying $\sigma_j \subset \text{supp}(s)$ and any $a_*^{(j)}$, with $0 \leq j \leq p$.

One can further see that Eq. (53) labeled by $0 \leq j \leq p$ gives a redundant set of constraints. To see this, pick a j -simplex σ_j and a 0-simplex (vertex) σ_0 of σ_j , i.e., $\sigma_0 \subset \sigma_j$. The chain $a_*^{(0)}$ then satisfies $(a_*^{(0)})|_{\sigma_j} = a_*^{(j)}$. We get

$$\sum_{\substack{a \in \mathcal{A}^{(j)} \\ a|_{\sigma_j} = a_*^{(j)}}} \epsilon(s, a) = \sum_{\substack{a_*^{(0)} \\ (a_*^{(0)})|_{\sigma_j} = a_*^{(j)}}} \left(\sum_{\substack{a \in \mathcal{A}^{(j)} \\ a|_{\sigma_0} = a_*^{(0)}}} \epsilon(s, a) \right), \quad (54)$$

which implies that the equations for $j = 0$ are sufficient to fully characterize the constraints in Eq. (53). Thus, a refined version of the condition equivalent to Eq. (53) can be written as follows:

$$\sum_{\substack{a \in \mathcal{A}^{(0)} \\ a|_{\sigma_0} = a_*^{(0)}}} \epsilon(s, a) = 0 \quad (55)$$

for any possible choice of $s \in \mathcal{S}$, a 0-simplex $\sigma_0 \in X$ satisfying $\sigma_0 \subset \text{supp}(s)$ and any $a_*^{(0)}$.

The above requirements complete the description of E_{inv} ; the group $E_{\text{inv}} \subset E$ is defined as the group of $e \in E$ whose integral coefficients satisfy the conditions (49), (50), (55).

3. Equivalence of Berry phases and genuine invariants

Some elements of E_{inv} correspond to trivial invariants. One class of trivial invariants arises from the commutators of the unitaries. For instance, if two unitaries $U(s_1), U(s_2)$ have no overlap of their support $\text{supp}(s_1) \cap \text{supp}(s_2) = \emptyset$, the commutator (10) $[U(s_2), U(s_1)] = 1$ is trivial. This implies that the element of E_{inv} given by

$$\theta(s_1, a) + \theta(s_2, a + \partial s_1) - \theta(s_1, a + \partial s_2) - \theta(s_2, a) = 0 \pmod{2\pi} \quad (56)$$

gives a trivial invariant. Hence, the set $\{\theta(s, a)\}$ is not entirely independent. More generally, let us take multiple unitaries $U(s_1), \dots, U(s_n)$ without common overlaps $\text{supp}(s_1) \cap \dots \cap \text{supp}(s_n) = \emptyset$. Noting that each unitary $U(s_j)$ is a local finite-depth circuit, the higher commutator

(11), $[U(s_n), [\dots, [U(s_2), U(s_1)]]]$, becomes trivial. Taking the expectation value of these higher commutators with any state $|a\rangle$ gives an element of E_{inv} that is a trivial invariant [102]:

$$\langle a | [U(s_n), [\dots, [U(s_2), U(s_1)]]] | a \rangle = 1, \quad \forall a \in \mathcal{A}, \forall s_1, s_2, \dots, s_n \in \mathcal{S} | s_1 \cap s_2 \cap \dots \cap s_n = \emptyset. \quad (57)$$

This equation follows from the fact that the commutator of two local finite-depth circuits are supported in the neighborhood of their intersection.

We define the subgroup $E_{\text{id}} \subset E_{\text{inv}}$ as the group generated by the locality identities in Eq. (57). It can be verified that E_{id} includes all invariants arising from the higher commutators of operators of the form $V = \prod_{s \in \mathcal{S}} (U(s))^{n_s}$, with $n_s \in \mathbb{Z}$ [103]. To classify the genuine invariants, we introduce the following quotient group:

Definition III.2. (Generalized statistics) The generalized statistics is defined as the quotient of E_{inv} by the subgroup E_{id} ,

$$T := E_{\text{inv}} / E_{\text{id}}, \quad (58)$$

which forms an Abelian group.

We now establish the following theorem about the property of this quotient group T :

Theorem III.3. (Initial-state independence) The generalized statistics is uniquely determined by the sequence of unitaries, independent of the initial state $|a_0\rangle$ it acts upon,

$$\langle a_0 | \prod U(s_j)^{\pm} | a_0 \rangle = \langle a'_0 | \prod U(s_j)^{\pm} | a'_0 \rangle \quad (59)$$

for all $\langle a_0 | \prod U(s_j)^{\pm} | a_0 \rangle \in E_{\text{inv}}$ and $a'_0 \in \mathcal{A}$.

To prove this, we first conjugate the entire sequence of unitaries in Eq. (47) by another unitary operator $U(s')$. Specifically, consider the equality of invariants

$$\langle a_0 | \prod U(s_j)^{\pm} | a_0 \rangle = \langle a_0 + \partial s' | \prod \tilde{U}(s_j)^{\pm} | a_0 + \partial s' \rangle, \quad (60)$$

with $s' \in \mathcal{S}$ and $\tilde{U}(s_j) := U(s')U(s_j)U(s')^\dagger$. One can then write $\tilde{U}(s_j)$ as

$$\tilde{U}(s_j) = U(s_j)O(\partial s_j), \quad (61)$$

Here, an operator $O(\partial s_j) := [U(s_j), U(s')^\dagger]$ is supported at the neighborhood of ∂s_j . This is because for $s' \neq s_j$, the commutator of $U(s'), U(s_j)$ has support at $s_j \cap s' \subset \partial s_j$. For $s' = s_j$, the symmetry operators of the Abelian fusion group is commutative in the bulk, so $O(\partial s_j)$ again has support within ∂s_j . These $O(\partial s_j)$ can be treated as local perturbations acting on the boundary of each unitary $U(s)$. Then, by the definition of E_{inv} , we have

$$\begin{aligned} & \langle a_0 + \partial s' | \prod (U(s_j) O(\partial s_j))^{\pm} | a_0 + \partial s' \rangle \\ &= \langle a_0 + \partial s' | \prod (U(s_j))^{\pm} | a_0 + \partial s' \rangle. \end{aligned} \quad (62)$$

Combining Eq. (60), and noting that $\text{Im } \partial$ generates $\mathcal{A}$, this completes the proof of Theorem III.3. As a result, the ratio

$$\frac{\langle a_0 | \prod U(s_j)^{\pm} | a_0 \rangle}{\langle a'_0 | \prod U(s_j)^{\pm} | a'_0 \rangle} \quad (63)$$

with $\langle a_0 | \prod U(s_j)^{\pm} | a_0 \rangle \in E_{\text{inv}}$ and $a'_0 \in \mathcal{A}$ becomes a trivial phase. In Appendix D, we show that the above ratio (63) becomes an element of E_{id} by explicitly checking that it is given by a product of higher commutators.

4. Quantization of generalized statistics

In this section, we show that the above group T with any finite simplicial complex X and finite Abelian group G always becomes a finite Abelian group. In particular, this implies that the invariants $e^{i\theta}$ in T must take the quantized values.

Since the number of generators of T is finite upper bounded by $|\mathcal{S}| \times |\mathcal{A}|$, it suffices to show that T is a torsion, i.e., a direct sum of finite Abelian groups and does not contain a free part. Let us take an invariant $[e] \in E_{\text{inv}}/E_{\text{id}}$. Then take a representative $e \in E_{\text{inv}}$ expressed as $e = \sum_{(s,a)} \epsilon(s,a) \theta(s,a)$. Since Eq. (63) is included in E_{id} , the equivalence class of e is left invariant under the global shift of $a \rightarrow a + a_0$ with $a_0 \in \mathcal{A}$ in phases $\theta(s,a)$, which leads to the expression $e' = \sum_{(s,a)} \epsilon(s,a) \theta(s, a + a_0)$ with $[e] = [e']$. We then get

$$\begin{aligned} |\mathcal{A}|[e] &= \sum_{a_0 \in \mathcal{A}} \sum_{(s,a)} \epsilon(s,a) \theta(s, a + a_0) \\ &= \sum_{a_0 \in \mathcal{A}} \sum_{(s,a)} \epsilon(s, a - a_0) \theta(s, a) \\ &= \sum_{(s,a)} \left(\sum_{a_0 \in \mathcal{A}} \epsilon(s, a_0) \right) \theta(s, a) = 0 \pmod{2\pi}, \end{aligned} \quad (64)$$

where we used Eq. (50) in the last equation. Since $\mathcal{A} = B_p(X, G)$ is finite, this implies that any element of T has finite order which divides $|\mathcal{A}|$. This shows that T is a finite Abelian group.

Later in Secs. V and VI, the invariants in T will be identified as the microscopic definition of the 't Hooft anomalies of the global symmetry G . The above observation implies that the 't Hooft anomalies in the microscopic lattice systems generally become a torsion when G is finite. This is consistent with a generic conjecture that the 't Hooft anomalies realized in symmetry-preserving gapped theories must be a torsion [104]. We comment on the applicability

of generalized statistics in generic gapless systems in Sec. VII.

B. Generalized statistics for non-Abelian 0-form symmetries

1. Generalized statistics for non-Abelian fusion groups

It is straightforward to extend the above formalism to the case of non-Abelian fusion group G , which can happen for the 0-form symmetry. Let us describe the excitation model for the non-Abelian G on a simplicial complex X embedded in a d -dimensional space. $\mathcal{S}$ is again a set of a pair $s = (g_j, \sigma_d)$, where $\{g_j\}$ is a minimal set of generators of G , and σ_d is a d -simplex of X . Each element $s \in \mathcal{S}$ corresponds to the unitary $U(s) = U_{g_j}(\sigma_d)$. Then, $\mathcal{A}$ is identified as the set of the states $\{|a\rangle\}$ obtained by the sequence of unitaries on a fixed G symmetric state $|0\rangle$. The unitary acts on the state by $U(s)|a\rangle = e^{i\theta(s,a)}|a \times \partial s\rangle$ with the group fusion $\times$, i.e., fusing the excitations at the boundary of a d -simplex.

The expression groups $E, E_{\text{inv}}, E_{\text{id}}$ can be readily generalized to the non-Abelian group G .

There are slight modifications to the characteristic equations for $E_{\text{inv}}, E_{\text{id}}$ arising from the non-Abelian nature of fusion groups. The only modification to E_{inv} is that one of the conditions for E_{inv} (49) is expressed according to the fusion of non-Abelian groups as

$$\sum_{s \in \mathcal{S}} \epsilon(s, a) - \sum_{s \in \mathcal{S}} \epsilon(s, a \times (\partial s)^{-1}) = 0, \quad \text{for any } a \in \mathcal{A}. \quad (65)$$

For the definition of E_{id} , the non-Abelian fusion group has the effect of modifying Eq. (59). Instead of Eq. (59), for non-Abelian G we have the equation

$$\langle a_0 | \prod U(s_j)^{\pm} | a_0 \rangle = \langle a_0 \times \partial s' | \prod \rho_{s'}[U(s_j)]^{\pm} | a_0 \times \partial s' \rangle \quad (66)$$

for all $\langle a_0 | \prod U(s_j)^{\pm} | a_0 \rangle \in E_{\text{inv}}$ and any choice of $s' = (g', \sigma'_d)$. Here, we write the conjugation action of $U(s')$ on $U(s)$ with $s = (g, \sigma_d)$ by

$$\rho_{s'}[U(s)] = \begin{cases} U_{g'g^{-1}}(\sigma_d), & \sigma_d = \sigma'_d, \\ U_g(\sigma_d), & \sigma_d \neq \sigma'_d. \end{cases} \quad (67)$$

That is, $U(s')$ acts on $U(s)$ by an automorphism if these operators have the same support. Equation (66) can be derived by a similar discussion to the proof of Eq. (59).

We define the group $E_{\text{id}} \subset E_{\text{inv}}$ as the group generated by the elements in the form of the ratio

$$\frac{\langle a_0 | \prod U(s_j)^{\pm} | a_0 \rangle}{\langle a_0 \times \partial s' | \prod \rho_{s'}[U(s_j)]^{\pm} | a_0 \times \partial s' \rangle} \quad (68)$$

with $\langle a_0 | \prod U(s_j)^\pm | a_0 \rangle \in E_{\text{inv}}$, $s' \in \mathcal{S}$, and the elements given by the higher commutators

$$\langle a | [U(s_n), [\dots, [U(s_2), U(s_1)]]] | a \rangle = 1, \\ \forall a \in \mathcal{A}, \forall s_1, s_2, \dots, s_n \in \mathcal{S} | s_1 \cap s_2 \cap \dots \cap s_n = \emptyset. \quad (69)$$

It is expected that the ratio (68) is again given by a product of higher commutators, and the above definition gives a redundant set of generators. Verifying this expectation is left for future studies.

2. Quantization of invariants for non-Abelian fusion groups

One can also show the quantization of invariants for non-Abelian fusion groups. For simplicity, let us take X to be a triangulation of a sphere S^d .

Consider an element $[e] \in T = E_{\text{inv}}/E_{\text{id}}$. Since Eq. (68) is an element of E_{id} , conjugation by a unitary $U(s')$ leaves the equivalence class $[e]$ invariant. Hence, conjugation by any unitary of the form $\prod_{\sigma' \in X} U_{g'(\sigma')}(\sigma')$ leaves $[e]$ invariant, where $g'(\sigma')$ is any group element $g'(\sigma') \in G$ chosen for each d -simplex σ' . Each $U_{g'(\sigma')}(\sigma')$ is a product of operators $U(s)$ with $s \in \mathcal{S}$ supported on the simplex σ' . Now let us sum over all possible conjugation actions of $[e]$ by $\prod_{\sigma' \in X} U_{g'(\sigma')}(\sigma')$. This corresponds to summing over all choices of group elements $g'(\sigma')$ for each d -simplex, giving $|G|^{N_d}$ possibilities, where N_d is the number of d -simplices in X .

After this summation, we obtain an expression of $|G|^{N_d}[e]$, and we will show that it vanishes. To see this, pick a single d -simplex $\sigma_d \in X$, and focus on the coefficient of the phase $\theta(s, a)$ with s supported on σ_d . Then, in the sum over conjugation actions, first sum over the $N_d - 1$ d -simplices with $\sigma' \neq \sigma_d$,

$$|G|^{N_d-1}[e] \\ = \sum_{\substack{\{g'(\sigma')\} \\ \sigma' \neq \sigma_d}} \sum_{a \in \mathcal{A}} \epsilon(s, a) \theta\left(s, a \times \prod_{\sigma' \neq \sigma_d} g'(\sigma'), \sigma'\right) + \dots,$$

where $\dots$ denotes the other phases θ . Note that while the excitation a is shifted by the conjugation action, s remains invariant under conjugation since the action avoids the simplex σ_d . Here, since X is a triangulation of a sphere, $\text{Im}(\partial)$ with the domain $\sigma' \neq \sigma_d$ generates the entire group $\mathcal{A}$. Therefore, one can rewrite

$$|G|^{N_d-1}[e] = \sum_{a' \in \mathcal{A}} \sum_{a \in \mathcal{A}} \epsilon(s, a) \theta(s, a \times a') + \dots \\ = \sum_{a' \in \mathcal{A}} \sum_{a \in \mathcal{A}} \epsilon(s, a \times a'^{-1}) \theta(s, a) + \dots \\ = \sum_{a \in \mathcal{A}} \left(\sum_{a' \in \mathcal{A}} \epsilon(s, a') \right) \theta(s, a) + \dots \\ = 0 + \dots, \quad (70)$$

implying that the contribution from each $s \in \mathcal{S}$ vanishes when summing over the $N_d - 1$ simplices avoiding s . This derivation shows that $|G|^{N_d}[e]$ becomes zero. Therefore, we conclude that the generalized statistics T forms a finite Abelian group whose order divides $|G|^{N_d}$. The invariants are thus quantized into discrete values.

IV. COMPUTATION OF GENERALIZED STATISTICS

In the previous section, we provided the conditions [Eqs. (49), (50), and (55)] under which a process remains invariant under any local deformation. Solving for $\epsilon(s, a)$ that satisfies these equations is generally challenging. For instance, Ref. [45] uses computational methods to solve similar equations for loop excitations in $(3+1)\text{D}$, with additional constraints (specifically, restricting to processes that flip a loop). However, extending this approach to other types of excitations is not straightforward. Therefore, we adopt an alternative approach. Instead of directly solving the equations, we investigate the “trivial solutions” and use them to construct new solutions for genuine generalized statistics.

Specifically, for each locality identity in Eq. (57) within E_{id} , a linear combination of $\theta(s, a)$ must be zero modulo 2π . The coefficients $\epsilon(s, a)$ satisfy the conditions given in Eqs. (49), (50), and (55) since the locality identities always hold, regardless of how $U(s)$ or $|a\rangle$ is deformed. Since the commutators of unitaries for nonoverlapping simplexes always yield a phase factor of $+1$, the solution $\epsilon(s, a)$ derived from the locality identity is considered a trivial solution.

Next, note that these equations are linear in $\epsilon(s, a)$. Specifically, if $\{\epsilon(s, a)\}$ is a solution, then $\{\epsilon'(s, a) := \alpha \epsilon(s, a)\}$ is also a solution, provided that all $\alpha \epsilon(s, a)$ remain integers. If we find a combination of locality identities such that all coefficients $\{\epsilon(s, a)\}$ are multiples of an integer k , we can obtain a potentially nontrivial solution by dividing these coefficients by k . If this new solution cannot be expressed as a linear combination of identities in E_{id} , we categorize it as a genuine invariant. Moreover, this solution imposes a constraint on the generalized statistics, which can only take the form $\exp[(2\pi i j)/k]$, where $j \in 0, 1, 2, \dots, k-1$.

A. Deriving generalized statistics via locality identities

In this section, we explicitly derive several statistical processes from locality identities. These derivations can be regarded as mathematical proofs carried out by hand (without computer assistance), confirming the correctness of the results presented in the previous sections. Specifically, the $(2+1)\text{D}$ $\mathbb{Z}_2$ particle-exchange statistics is derived in Sec. IV A 1, and the $(3+1)\text{D}$ $\mathbb{Z}_2$ loop-flipping statistics is derived in Sec. IV A 2. In addition, the $(1+1)\text{D}$ $\mathbb{Z}_2$ particle-fusion statistics and the $(3+1)\text{D}$

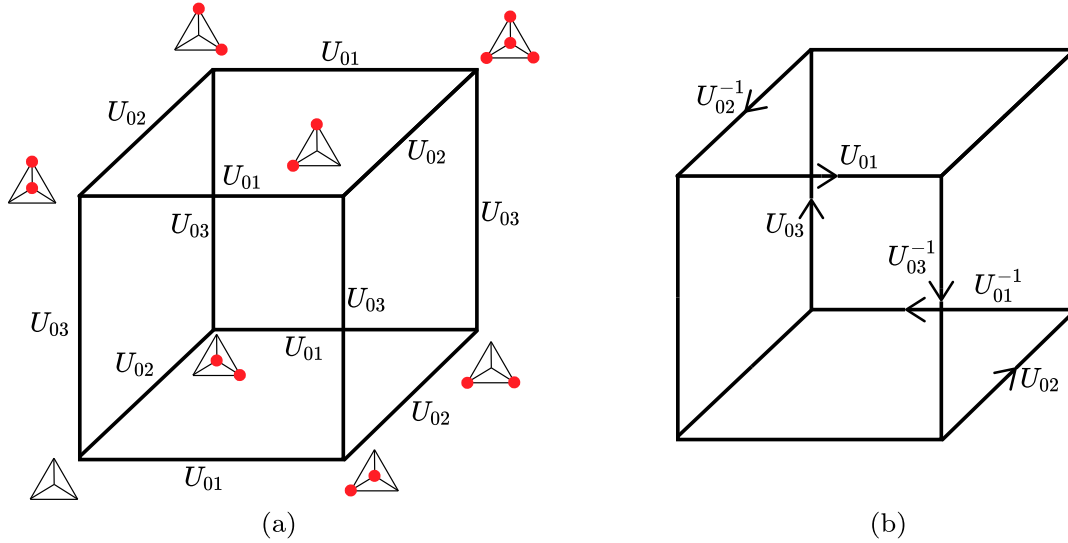


FIG. 5. (a) Configuration space for $\mathbb{Z}_2$ particles on a triangle with a central vertex. We analyze states within the same superselection sector as the vacuum. These states are interconnected through string operators U_{0i} where $i = 1, 2, 3$ delineating the 1-skeleton of a cube. It is important to note that transitions between states can be achieved by applying either U_{0i} or U_{0i}^{-1} , and generally, the operator U_{0i}^2 does not necessarily equal $+1$. Connections between states via U_{ij} are not explicitly shown in this diagram. (b) Visualization of the T -junction process in the configuration space. A specific initial state is selected (the outcome is independent of this choice), demonstrating how the T -junction process swaps the positions two particles.

loop-membrane mutual statistics are derived in Appendixes C3 and C4, respectively. The algorithmic approach for systematically deriving these processes will be presented in the following section.

1. $\mathbb{Z}_2$ particles in $(2+1)\text{D}$

As an example, we begin by considering the T -junction process in $(2+1)\text{D}$ with the fusion group $G = \mathbb{Z}_2$, demonstrating that its anyon statistics must take the form $\exp(i\Theta) = \pm 1, \pm i$. The configuration states are shown in Fig. 5(a). We start by presenting particular locality identities for demonstration:

$$\langle \triangleleft \mid [[U_{02}, U_{03}], U_{12}] \mid \triangleleft \rangle = 1. \quad (71)$$

In terms of $\theta(s, a)$, the above identity can be expanded as

$$\begin{aligned} & \theta(U_{03}, \triangleleft) + \theta(U_{02}, \triangleleft) + \theta(U_{03}^{-1}, \triangleleft) \\ & + \theta(U_{02}^{-1}, \triangleleft) + \theta(U_{02}, \triangleleft) + \theta(U_{03}, \triangleleft) \\ & + \theta(U_{02}^{-1}, \triangleleft) + \theta(U_{03}^{-1}, \triangleleft) = 0 \pmod{2\pi}, \end{aligned} \quad (72)$$

where we have adjusted the notation from $\theta(s, a)$ to $\theta(U(s), a)$. This change allows us to extend the definition of θ for $U(s)^{-1}$ as

$$\theta(U(s)^{-1}, a) := -\theta(U(s), a - \partial s), \quad (73)$$

which follows from Eqs. (45) and (46). More generally, θ can be defined for any sequence of $U(s)$ using the following property:

$$\theta(VU(s)^\pm, a) := \theta(V, a \pm \partial s) + \theta(U(s)^\pm, a) \quad (74)$$

for all unitary operators V . Using Eq. (73), the locality identity (72) can be rewritten as

$$\begin{aligned} & \theta(U_{03}, \triangleleft) + \theta(U_{02}, \triangleleft) - \theta(U_{03}, \triangleleft) \\ & - \theta(U_{02}, \triangleleft) + \theta(U_{02}, \triangleleft) + \theta(U_{03}, \triangleleft) \\ & - \theta(U_{02}, \triangleleft) - \theta(U_{03}, \triangleleft) = 0 \pmod{2\pi}, \end{aligned} \quad (75)$$

where we express θ solely in terms of U , without involving U^{-1} . Besides this locality identity, we can write down the following eight identities:

- (1) $\langle \triangleleft \mid [[U_{01}, U_{02}], U_{13}] \mid \triangleleft \rangle = 1.$
- (2) $\langle \triangleleft \mid [[U_{03}, U_{01}], U_{23}] \mid \triangleleft \rangle = 1.$
- (3) $\langle \triangleleft \mid [[U_{02}^{-1}, U_{03}^{-1}], U_{12}] \mid \triangleleft \rangle = 1.$
- (4) $\langle \triangleleft \mid [[U_{01}^{-1}, U_{02}^{-1}], U_{13}] \mid \triangleleft \rangle = 1.$
- (5) $\langle \triangleleft \mid [[U_{03}^{-1}, U_{01}^{-1}], U_{23}] \mid \triangleleft \rangle = 1.$
- (6) $\langle \triangleleft \mid \left([[U_{02}, U_{03}], U_{23}] \right)^2 \mid \triangleleft \rangle = 1.$
- (7) $\langle \triangleleft \mid \left([[U_{01}, U_{02}], U_{12}] \right)^2 \mid \triangleleft \rangle = 1.$
- (8) $\langle \triangleleft \mid \left([[U_{03}, U_{01}], U_{13}] \right)^2 \mid \triangleleft \rangle = 1.$

In Appendix C 1, we explicitly demonstrate that summing over the nine identities above yields the resulting equation:

$$\begin{aligned}
 & 4 \left(\theta(U_{01}^{-1}, \text{triangle with red dot}) + \theta(U_{03}, \text{triangle with red dot}) + \theta(U_{02}^{-1}, \text{triangle with red dot}) \right. \\
 & \quad \left. + \theta(U_{01}, \text{triangle with red dot}) + \theta(U_{03}^{-1}, \text{triangle with red dot}) + \theta(U_{02}, \text{triangle with red dot}) \right) \\
 & = 0 \pmod{2\pi},
 \end{aligned} \tag{76}$$

where the six θ terms can be combined into a total phase of a sequence of unitaries

$$4\theta(U_{02}U_{03}^{-1}U_{01}U_{02}^{-1}U_{03}U_{01}^{-1}, \text{triangle with red dot}) = 0 \pmod{2\pi}. \tag{77}$$

This sequence of unitaries precisely corresponds to the T -junction process as defined in Eq. (19) and is illustrated within the configuration state space in Fig. 5(b). Note that the coefficient 4 in Eq. (77) is crucial, as it constrains the phases to be

$$\theta(U_{02}U_{03}^{-1}U_{01}U_{02}^{-1}U_{03}U_{01}^{-1}, \text{triangle with red dot}) = 0, \pi, \pm \frac{\pi}{2} \pmod{2\pi}, \tag{78}$$

indicating that the anyon with $G = \mathbb{Z}_2$ fusion can be a boson, a fermion, or an (anti)semion. The coefficient in Eq. (77) depends on both the fusion group G and the spatial dimensions.

This computation can be generalized for $G = \mathbb{Z}_N$ straightforwardly. Specifically, in $(2+1)\text{D}$, the coefficient is $2N$ for even N and N for odd N , while in $(3+1)\text{D}$, the coefficient is 2 for even N and 1 for odd N , as discussed previously in Eq. (20). This is consistent with the topological spin of $\mathbb{Z}_N$ anyons (related to the topological twist by $e^{2\pi i h}$) given by $h = [p/(2N)] \bmod 1$ for integer $p = 0, 1, \dots, 2N-1$ for even N , and $p = 0, 2, 4, \dots, 2N-2$ for odd N in a bosonic theory, i.e., no transparent local fermion [50].

In the following sections, we present an algorithm that systematically determines these coefficients using the Smith normal form. In particular, we have reproduced the above quantization of topological spin for $G = \mathbb{Z}_N$ with $N \leq 10$ using our personal computers, while higher N demands more computational resources.

2. $\mathbb{Z}_2$ loops in $(3+1)\text{D}$

In addition to the well-known T -junction process for detecting the spins of particles, we now proceed to another more intricate example: the loop-flipping process for $\mathbb{Z}_2$ loops in $(3+1)\text{D}$. This process was first proposed in

Ref. [45], where computers found a sequence of 36 unitaries to ensure invariance under any local perturbation. In Appendix C 2, we utilize our new method, which involves summing over locality identities, to derive a sequence of 24 unitaries μ_{24} that satisfy the following equation analogous to Eq. (77):

$$2\theta \left(\mu_{24}, \text{diamond with vertices 1, 2, 3, 4 and center 0} \right) = 0 \pmod{2\pi}, \tag{79}$$

with μ_{24} defined in Eq. (26) and illustrated in Figs. 3 and 4. Therefore, the total phase of this sequence must be 0 or $\pi \bmod 2\pi$, implying that a loop with $G = \mathbb{Z}_2$ fusion can be either bosonic or fermionic [44,45]. We further demonstrate that the total phase generated by the 36-unitary sequence from Ref. [45] matches the phase obtained from our μ_{24} sequence, up to some locality identities. In other words, both sequences yield the same loop statistics, but our process is more efficient, requiring fewer unitaries. We have verified that the 24-step sequence is optimal, as no shorter sequence exists.

B. Computational algorithm using the Smith normal form

In the previous section, we derived the T -junction process in $(2+1)\text{D}$ and introduced the novel 24-step loop-flipping processes in $(3+1)\text{D}$. A natural question arises how to systematically determine specific linear combinations of the local identities such that the overall coefficients $\{\epsilon(s, a)\}$ in the expression $e = \sum_{s,a} \epsilon(s, a) \theta(s, a)$ result in a greatest common divisor (gcd) greater than 1.

For smaller fusion groups, such as $G = \mathbb{Z}_2$, it is often feasible to rely on human intuition to manually determine the appropriate local identities to be summed in order to cancel certain $\theta(s, a)$ terms, ultimately ensuring that the remaining coefficients are all 2 or 4. We demonstrated this process in detail in Appendix C, where we effectively exploited the structure of $\mathbb{Z}_2$ to construct explicit summations that achieve the desired gcd properties.

For larger groups or more complex settings, it becomes challenging to manually determine the appropriate summation of local identities. Meanwhile, it turns out that finding such a summation can be efficiently handled by computational algorithms. The algorithm systematically determines the appropriate summation of locality identities by computing the Smith normal form of the corresponding matrix representation. By expressing the locality identities in matrix form, we can use the Smith normal form to identify combinations of rows that yield nontrivial gcd values for the coefficients $\epsilon(s, a)$. This method provides a systematic approach to finding the appropriate summations

for larger fusion groups beyond $\mathbb{Z}_2$, and allows us to optimize the result by finding the shortest unitary sequence.

The algorithm proceeds as follows: We first construct the matrix M , whose rows represent the locality identities, with entries corresponding to the coefficients of the $\theta(s, a)$ terms in each identity. We then compute the Smith normal form (SNF) of M , which allows us to determine the invariant factors and, subsequently, the linear combinations of the original identities that yield the desired gcd properties for the coefficients $\epsilon(s, a)$.

To illustrate this process, let us consider a toy example with three θ terms, θ_1 , θ_2 , and θ_3 , and three locality identities:

$$\begin{aligned}\theta_1 + 2\theta_2 + 3\theta_3 &= 0 \pmod{2\pi}, \\ 4\theta_1 + 5\theta_2 + 6\theta_3 &= 0 \pmod{2\pi}, \\ 7\theta_1 + 8\theta_2 + 9\theta_3 &= 0 \pmod{2\pi}.\end{aligned}\quad (80)$$

These three locality identities are not independent, as the sum of the first and third identities equals twice the second identity. However, for the sake of demonstration, we proceed without recognizing this redundancy, as it can be challenging to identify such relationships in more extensive systems with thousands of identities in practice.

These locality identities can be represented as the phase relation matrix:

$$M = \begin{array}{c|ccc} & \theta_1 & \theta_2 & \theta_3 \\ \hline & 1 & 2 & 3 \\ & 4 & 5 & 6 \\ & 7 & 8 & 9 \end{array}, \quad (81)$$

where each row represents a locality identity. Row operations involving adding an integer multiple of one row to another do not change the row vector space spanned by these locality identities; they merely redefine the basis. However, column operations require more attention, as they necessitate the redefinition of phase labels for each column. Specifically, adding column k to column j requires a redefinition of phases as $\theta'_j = \theta_j$ and $\theta'_k = (-\theta_j + \theta_k)$:

$$M_{i,j}\theta_j + M_{i,k}\theta_k = (M_{i,j} + M_{i,k})\theta'_j + M_{i,k}\theta'_k \quad \forall i. \quad (82)$$

Next, we calculate the SNF of M , obtaining integer matrices L , R , and A such that

$$M = LAR, \quad (83)$$

where A is a diagonal integer matrix, and L and R are unimodular matrices (i.e., with determinant ± 1). For the matrix M given in Eq. (81), we decompose it as

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 7 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}. \quad (84)$$

We claim that this system exhibits $\mathbb{Z}_3$ generalized statistics corresponding to the second diagonal entry in the matrix A . This corresponds to the process $\theta_2 + 2\theta_3$, which is represented by the second row of the matrix R .

To justify the physical interpretation of this computation, let us proceed step by step. We begin with the matrix M in Eq. (81) and perform row operations over integers to obtain its Hermite normal form (HNF). We start by eliminating the entries below the first pivot, which is 1 in the first row and first column. First, we subtract $4 \times \text{row 1}$ from row 2 and subtract $7 \times \text{row 1}$ from row 3. The matrix then becomes

$$M' = \begin{array}{c|ccc} & \theta_1 & \theta_2 & \theta_3 \\ \hline & 1 & 2 & 3 \\ & 0 & -3 & -6 \\ & 0 & -6 & -12 \end{array}. \quad (85)$$

Next, we eliminate the entries below the second pivot. We focus on the second column, where the pivot is -3 . We add $2 \times \text{row 2}$ to row 3 to eliminate the entry below the pivot. The matrix M' becomes

$$M'' = \begin{array}{c|ccc} & \theta_1 & \theta_2 & \theta_3 \\ \hline & 1 & 2 & 3 \\ & 0 & -3 & -6 \\ & 0 & 0 & 0 \end{array}. \quad (86)$$

To make the second pivot positive, we multiply row 2 by (-1) :

$$M_{\text{SNF}} = \begin{array}{c|ccc} & \theta_1 & \theta_2 & \theta_3 \\ \hline & 1 & 2 & 3 \\ & 0 & 3 & 6 \\ & 0 & 0 & 0 \end{array}, \quad (87)$$

which is the final Hermite normal form.

Subsequently, we perform column operations, using the 1 in the first row to cancel out the entries 2 and 3 in the first row. Similarly, we use the 3 in the second row to cancel out the 6. According to the rule in Eq. (82), we reduce M to its Smith normal form

$$M = \begin{array}{c|ccc} & \theta'_1 & \theta'_2 & \theta'_3 \\ \hline & 1 & 0 & 0 \\ & 0 & 3 & 0 \\ & 0 & 0 & 0 \end{array}, \quad (88)$$

with redefined phase labels as

$$\begin{aligned}\theta'_1 &= \theta_1 + 2\theta_2 + 3\theta_3, \\ \theta'_2 &= \theta_2 + 2\theta_3, \\ \theta'_3 &= \theta_3.\end{aligned}\quad (89)$$

The physical interpretation of Eq. (88) is that all locality identities can be reduced to three equations:

$$1\theta'_1 = 0, \quad 3\theta'_2 = 0, \quad 0\theta'_3 = 0. \quad (90)$$

This means that θ'_1 is fixed, θ'_3 can take any value, and θ'_2 must be $(2\pi j)/3$ for $j = 0, 1, 2$. In other words, this toy system exhibits $\mathbb{Z}_3$ statistics corresponding to the unitary sequence labeled by $\theta'_2 = \theta_2 + 2\theta_3$.

Inspired by this example, we can derive the following theorem:

Theorem IV.1. (Generalized statistics) Let M be the phase relation matrix for a given simplicial complex and a fusion group, and consider its Smith decomposition $M = LAR$. If any diagonal entry a_{ii} in A satisfies $a_{ii} \neq 0, 1$, then it gives rise to generalized statistics of type $\mathbb{Z}_{a_{ii}}$, with the i th row of the matrix R specifying the corresponding unitary sequence. Namely, the generalized statistics T forms an Abelian group characterized by

$$T = \bigoplus_{a_{ii} \neq 0, 1} \mathbb{Z}_{a_{ii}}. \quad (91)$$

Remark IV.1. The sum of θ terms in each row of R must originate from a sequence of unitary operators. This is because, in the original phase relation matrix M , each row corresponds to a (higher) commutator, which forms closed loops connecting the configuration states. Therefore, although each row of R may be divided by the greatest common divisor of the coefficients, it still represents a superposition of multiple closed loops. By the Eulerian path theorem, we can identify a sequence that generates these loops (allowing for retracing edges to cancel their contributions).

Remark IV.2. In the Smith decomposition (83), the matrix A is uniquely determined, while the matrices L and R are not. However, any valid choice of R will yield the same generalized statistics, with the total phases differing only by linear combinations of locality identities. Typically, we choose R such that its entries contain as many zeros as possible, as this simplifies the corresponding unitary sequence.

The phase relation matrix M in Eq. (81) serves as a toy example; in practice, the computation can be much more involved. For example, when determining the SNF, each diagonal element must divide the next, requiring careful manipulation. Obtaining either the HNF or the SNF involves repeated use of the Euclidean algorithm to compute the greatest common divisors between matrix

entries in each row or column. For general dense matrices, this procedure can be both time and memory intensive for most computer implementations of SNF algorithms. However, in our setting the matrices are sparse and most entries are ± 1 , which allows programs such as SHEAFHOM [105] to run significantly faster. In practice, the computation time is mainly determined by the number of columns of M :

$$\dim E = |\text{number of simplexes}| \times |\text{generators of } G| \times |A|.$$

On a standard personal computer, it is feasible to compute the HNF or SNF for $\dim E \sim 10^4$ within a reasonable time.

For example, in the $(2+1)$ D T -junction process with fusion group $G = \mathbb{Z}_N$ on the simplicial complex shown in Fig. 1(b), there are six edges, G has a single generator, and there are N^3 configuration states. Thus, the number of $\theta(s, a)$ terms is $6N^3$. In this situation, we are able to compute the cases with $N \leq 12$.

The loop fusion statistics with fusion group $G = \mathbb{Z}_N \times \mathbb{Z}_N$ is also computed on the simplicial complex shown in Fig. 1(b). In this case, there are four faces, G has two generators, and the number of configuration states is N^6 . Consequently, the number of $\theta(s, a)$ terms is $8N^6$. We can carry out the computation for cases with $N \leq 5$.

Consider another example involving loop statistics with $G = \mathbb{Z}_N$ on the $(3+1)$ D simplicial complex shown in Fig. 1(c). Here, there are ten faces, G has a single generator, and there are N^6 configurations. The total number of $\theta(s, a)$ terms is $10N^6$, which limits our computations to $N \leq 4$.

Lastly, for membrane statistics with $G = \mathbb{Z}_N$ in the $(3+1)$ D simplicial complex shown in Fig. 1(c), there are five tetrahedra (including the outer one), G has a single generator, and there are N^4 configurations. Therefore, the total number of $\theta(s, a)$ terms is $5N^4$, allowing us to compute the cases with $N \leq 8$.

C. Generalized statistics as anomalies: Computational approach

Using the SNF algorithm presented above, we obtain the classification of the generalized statistics in various spatial dimensions, as summarized in Sec. II.

Below, we comment on the implications of several generalized statistics for the algebraic structure of the symmetry operators. We take the simplicial complex X for the excitation model to be the minimal triangulation of S^d embedded in d -dimensional space [106].

- (i) In $(1+1)$ D with finite 0-form symmetry G , we obtain the generalized statistics T generated by the unitary sequence

$$Z_3(g) := [U(g)_{02}, U(g)_{01}^{|g|}], \quad (92)$$

with $g \in G$. This form of the invariant immediately implies that the nontrivial invariant $Z_3(g) = e^{i\theta}$

becomes an obstruction to having

$$U(g)_{01}^{|g|} = 1, \quad (93)$$

which means that the symmetry operator on an interval must violate the original group fusion rule. Note that an on-site symmetry satisfies $U(g)_{01}^{|g|} = 1$, since on-site operators supported on an interval again follow the G fusion rule. This immediately leads to the physical consequence that the operator $U(g)$ with the nontrivial invariant Θ cannot be realized by an on-site symmetry on the lattice. Physically, the invariant indicates that the fusion of the operator $U(g)_{01}^{|g|}$ creates an electric charge of the G symmetry localized at the end of the interval, which is a nontrivial point operator at the boundary.

- (ii) In $(2+1)\text{D}$ with 0-form symmetry $G = \mathbb{Z}_N \times \mathbb{Z}_N$, the invariant is classified by $T = \mathbb{Z}_N \times \mathbb{Z}_N$. The classification is generated by a pair of invariants $Z_4(a), Z_4(b)$ shown in Eq. (21), with a, b the generators of $\mathbb{Z}_N \times \mathbb{Z}_N$. Although not manifest from their form, it turns out that either of the invariants $Z_4(a), Z_4(b)$ becomes an obstruction to the following $\mathbb{Z}_N \times \mathbb{Z}_N$ fusion algebra of symmetry operators:

$$U(a)_{ijk}^N = U(b)_{ijk}^N = [U(a)_{ijk}, U(b)_{ijk}] = 1. \quad (94)$$

This obstruction can be explicitly demonstrated by the SNF algorithm. Specifically, we add the equations of the phases $\theta(s, a)$ originating from the three equations in Eq. (94) to the rows of the phase relation matrix M , and denote the new matrix by M' . Running the SNF algorithm on M' classifies the invariants under these additional algebraic constraints. This invariant turns out to be trivial, which means that the original invariants $Z_4(a), Z_4(b)$ must be trivial under the $\mathbb{Z}_N \times \mathbb{Z}_N$ fusion rule of $U(a), U(b)$. This shows that $Z_4(a), Z_4(b)$ obstruct the fusion rule (94). By a discussion analogous to the $(1+1)\text{D}$ case, this trivialization implies that either of the invariants $Z_4(a), Z_4(b)$ becomes an obstruction to realizing the $\mathbb{Z}_N \times \mathbb{Z}_N$ symmetry by on-site operators.

In the above examples, the generalized statistics define obstructions to a certain group theoretical identity of operators $U(s)$ under its fusion. It turns out that this is a symptom of the 't Hooft anomaly of the global symmetry G defined as obstructions to gauging the symmetry. Roughly speaking, failure of the group identities such as Eq. (94) is directly interpreted as the failure of the Gauss law constraints in the gauge theory, where each unitary $U(s)$ is identified as a product of Gauss law operators. This leads to the absence of a gauge-invariant Hilbert space after an

attempt to promote the global symmetry to the gauge symmetry. Such a perspective of the generalized statistics as the anomalies will be discussed in detail in the following sections.

In (bosonic) quantum field theory, the 't Hooft anomalies of a finite $(d-p-1)$ -form symmetry G in d spatial dimensions are classified by the group cohomology $H^{d+2}(B^{d-p}G, U(1))$, where $B^{d-p}G$ represents the Eilenberg-MacLane space of the group G . In all the examples we evaluated using the SNF algorithm, the generalized statistics T match the group cohomology. This consistency leads us to the following conjecture:

Conjecture IV.2. Consider p -dimensional excitations with a $(d-p-1)$ -form symmetry and fusion group G in d spatial dimensions. Let X be a simplicial complex that triangulates the d -dimensional sphere S^d . Then, the generalized statistics T of p -dimensional excitations on X are classified by the cohomology of the Eilenberg-MacLane space [107]:

$$T = H^{d+2}[B^{d-p}G, U(1)]. \quad (95)$$

We remark that, in principle, any finite simplicial complex X on S^d could be chosen to compute the generalized statistics. We numerically observed that different triangulations of S^d yield the same result. This leads us to the above conjecture that the generalized statistics do not depend on the choice of the triangulation of S^d . Hence, we have decided to use minimal triangulation for convenience.

D. Stability of generalized statistics

Beyond the evidence of Conjecture IV.2 provided above by computational methods, we present additional insights to support our conjecture. Assuming the independence of T under the choice of triangulations, one can see that both T and the group cohomology in Eq. (95) stabilize in the sense that for fixed p , both become independent of d for $d \geq d_{\text{crit}} := 2p + 3$.

Let us first see how the generalized statistics T stabilizes. When we define the generalized statistics of p -dimensional excitations, the excitations and unitaries are supported within the finite $(p+1)$ -skeleton X_{p+1} that collects $0, \dots, (p+1)$ -simplexes of X . According to dimension theory in point-set topology [109], every finite $(p+1)$ -dimensional simplicial complex, with Lebesgue covering dimension of $p+1$, can be embedded in S^{2p+3} . Therefore, any complex X_{p+1} can be embedded in a triangulation of S^{2p+3} . This implies that any invariants of p -dimensional excitations can be realized by a unitary sequence in a suitable triangulation of S^{2p+3} . Assuming the independence on the triangulation of S^d , one can conclude that the generalized statistics T stabilizes for $d \geq d_{\text{crit}} = 2p + 3$.

Let us explicitly check this stability with several examples of small p . For particle statistics ($p = 0$), all

one-dimensional simplicial complexes can be embedded in S^d for $d \geq 3$. In other words, in spatial dimensions $d \geq 3$, particle statistics stabilizes and is characterized by a single- $\mathbb{Z}_2$ invariant distinguishing bosons from fermions. This result aligns with the fact in Lorentz invariant theories, the particles with fractional statistics can exist only in $(1+1)D$ or $(2+1)D$, while the statistics is restricted to bosons or fermions in higher dimensions. Similarly, for loop excitations ($p = 1$), the critical dimension is $d_{\text{crit}} = 2p + 3 = 5$. In spatial dimensions $d \geq 5$, loop statistics stabilizes and is characterized by a single- $\mathbb{Z}_2$ invariant that distinguishes between bosonic and fermionic loops. An analogous result holds for membrane excitations ($p = 2$), where the generalized statistics stabilize at $d_{\text{crit}} = 7$.

Meanwhile, the (co)homology of the Eilenberg-MacLane space also stabilizes.

Theorem IV.3. (13.2.2 of Ref. [110]) For any Abelian group A , there exists a chain complex $Q_*(A)$ whose homology is isomorphic to the stable homology of the Eilenberg-MacLane space:

$$H_n[Q_*(A)] \cong H_{n+k}[K(A, k)], \quad k \geq n + 1.$$

Using the universal coefficient theorem and the long exact sequence associated with $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow 0$, we obtain the stabilization of the cohomology of the Eilenberg-MacLane space with coefficients in $\mathbb{R}/\mathbb{Z} = U(1)$:

$$H^{n+k}(B^k G, U(1)) = H^{2n+1}(B^{n+1} G, U(1)) \quad (96)$$

for all $k \geq n + 1$. By choosing $n = p + 2$ and $k = d - p$, we find that $H^{d+2}(B^{d-p} G, U(1))$ stabilizes with respect to d for $k \geq n + 1$, or equivalently,

$$d \geq d_{\text{crit}} = 2p + 3, \quad (97)$$

which matches precisely with the argument from the embedding theorem. Therefore, this consistency for the pattern of stability supports Conjecture IV.2.

The group cohomology that appears in Conjecture IV.2 classifies the topological responses in $(d + 2)$ dimensions that describe the anomaly inflow. We close this section with comments on the stable generalized statistics and the corresponding topological response:

- (i) For particle statistics ($p = 0$) with the fusion group $G = \mathbb{Z}_N$, the stable generalized statistics for $d \geq 3$ is classified by $\mathbb{Z}_{\text{gcd}(2, N)}$ —boson or fermion for even N . The statistics of the emergent particles are interpreted as a framing anomaly of the topological line operator, which is in turn understood as the 't Hooft anomaly of the $(d - 1)$ -form $\mathbb{Z}_2$ symmetry generated by this line operator. This 't Hooft anomaly is characterized by the response

$$\pi \int w_2 \cup B_d = \pi \int Sq^2(B_d), \quad (98)$$

where B_d is the d -form $\mathbb{Z}_N$ background gauge field with even N , and w_2 is the second Stiefel-Whitney class.

- (ii) For loop statistics ($p = 1$) with the fusion group $G = \mathbb{Z}_N$, the stable generalized statistics for $d \geq 5$ is classified by $\mathbb{Z}_{\text{gcd}(2, N)}$ —bosonic or fermionic loops for even N . The statistics is again interpreted as a framing anomaly of the topological surface operator, which is the 't Hooft anomaly of the $(d - 2)$ -form $\mathbb{Z}_2$ symmetry generated by this surface operator. The 't Hooft anomaly is characterized by the response

$$\begin{aligned} \pi \int w_3 \cup B_{d-1} &= \pi \int w_2 (dB_{d-1}/2) \\ &= \pi \int Sq^2(dB_{d-1}/2), \end{aligned} \quad (99)$$

where B_{d-1} is the $(d - 1)$ -form $\mathbb{Z}_N$ background gauge field with even N , and w_3 is the third Stiefel-Whitney class. The first equality follows from $w_3 = Sq^1 w_2$ on orientable manifolds, and $Sq^1 B_{d-1} = dB_{d-1}/2$ where the right-hand side uses a lift of $\mathbb{Z}_2$ cocycle B_{d-1} to $\mathbb{Z}_4$ cochain.

- (iii) For membrane statistics ($p = 2$) with the fusion group $G = \mathbb{Z}_N$, numerical computations suggest that the stable generalized statistics for $d \geq 7$ is classified by $\mathbb{Z}_{\text{gcd}(2, N)} \times \mathbb{Z}_{\text{gcd}(3, N)}$. We expect that the $\mathbb{Z}_2$ statistics with even N is again associated with the mixed gravitational 't Hooft anomaly involving the Stiefel-Whitney class,

$$\pi \int (w_4 + w_2^2) \cup B_{d-2} = \pi \int Sq^4(B_{d-2}). \quad (100)$$

Meanwhile, for the $\mathbb{Z}_3$ statistics with N multiple of 3, we expect that the anomaly is associated with the Pontryagin class p_1 ,

$$\frac{2\pi}{3} \int p_1 \cup B_{d-2}. \quad (101)$$

This implies that the membrane excitation is chiral; it has chiral central charge $c_- = -8$. See also Ref. [111] for a recent discussion on this response action. This $\mathbb{Z}_{\text{gcd}(3, N)}$ invariant already appears in $d = 4$, with the simplest example given in Ref. [112]:

$$\frac{2\pi}{3} \int p_1 \cup B_2 = \frac{2\pi}{3} \int B_2 \cup B_2 \cup B_2. \quad (102)$$

It would be interesting to verify the above expectations and find the corresponding generalized statistics on lattice models.

V. STATISTICS AS OBSTRUCTION TO GAUGING

Let us make a direct connection between the generalized statistics and the 't Hooft anomaly of the lattice models. Suppose that a lattice model in d spatial dimensions has a finite $(d - p - 1)$ -form symmetry G . We argue that the invariant Θ gives an obstruction to gauging the global symmetry. This implies that Θ generally gives a microscopic definition of the 't Hooft anomaly.

For simplicity, let us study the system on a d -dimensional hypercubic lattice. We assume that the generators of the symmetry $g \in G$ are expressed as a product of local unitaries:

$$U_g(\Sigma_{p+1}) = \prod_{\Delta_{p+1} \in \Sigma_{p+1}} U_{\Delta_{p+1},g}, \quad (103)$$

where $U_{\Delta_{p+1},g}$ is a unitary supported on a $(p + 1)$ -cube Δ_{p+1} . We do not require the operators $\{U_g(\Delta_{p+1})\}$ to commute with each other; the above product is understood as the action of a finite-depth circuit of local unitaries $\{U_g(\Delta_{p+1})\}$.

Let us briefly recall the procedure of gauging G on the lattice. The first step is to enlarge the Hilbert space by adding G gauge fields on the p -cubes. The next step is to impose the Gauss law constraints on the Hilbert space,

$$G_{\Delta_{p+1},g} = 1, \quad (104)$$

with

$$G_{\Delta_{p+1},g} = U_{\Delta_{p+1},g} \prod_{\Delta' \in \partial \Delta_{p+1}} (A_{g,\Delta'})^\pm, \quad (105)$$

where $A_{g,\Delta'}$ is the operator generating the $g \in G$ gauge transformation on the gauge field located on the p -cube Δ' , and the sign $\pm$ is determined by the outgoing or ingoing orientation of the hypercubes Δ' . One can then express the symmetry operator as a product of Gauss law operators,

$$\tilde{U}_g(\Sigma_{p+1}) = \prod_{\Delta_{p+1} \in \Sigma_{p+1}} G_{\Delta_{p+1},g}, \quad (106)$$

which turns on the $g \in G$ gauge field at the boundary of Σ_{p+1} . This operator $\tilde{U}_g(\Sigma_{p+1})$ is regarded as the symmetry operator $U_g(\Sigma_{p+1})$ coupled to the G gauge fields.

One can then construct an excitation model (see Sec. III A 1) from the Gauss law operators. Pick a simplicial complex X embedded in space. Consider a set of operators $\tilde{U}_{g_j}(\sigma_{p+1})$ associated with a fixed set of generators $\{g_j\}$ of G , where σ_{p+1} denotes a $(p + 1)$ -simplex of X . Each unitary is labeled by a pair $s = (\sigma_{p+1}, g_j)$, and we define the set $\mathcal{S} = \{s\}$ consisting of all possible choices of g_j and $\sigma_{p+1} \subset X$. Next, consider the set of states $\mathcal{A} = \{|a\rangle\}$ representing G gauge field configurations a , which are

generated by sequences of unitaries $\tilde{U}(s)$ acting on the fixed G -symmetric state with vanishing gauge field $|0\rangle$.

The pair $(\mathcal{S}, \mathcal{A})$ then forms the excitation model. We define an invariant as the sequence of unitaries

$$e^{i\Theta} = \langle 0 | \prod \tilde{U}(s)^\pm | 0 \rangle = \langle 0 | \prod U(s)^\pm | 0 \rangle, \quad (107)$$

which is an element of E_{inv} . By definition, Θ is the phase obtained from the product of the Gauss law operators $G_{\Delta_{p+1},g}$. Thus, $\Theta \neq 0$ indicates an obstruction to the commuting Gauss law operators $\{G_{\Delta_{p+1},g}\}$. In particular, a given symmetric state $|0\rangle$ cannot be promoted to a gauge-invariant state, since $|0\rangle$ is annihilated when projected onto the Hilbert space satisfying all Gauss law constraints. Therefore, Θ defines an obstruction to gauging the G symmetry of the given G -symmetric state $|0\rangle$.

In our formalism, the invariant Θ is defined by evaluating the sequence of unitaries on any initial state. Consequently,

$$e^{i\Theta} = \prod U(s)^\pm \quad (108)$$

holds as an operator identity, and the invariant characterizes the obstruction to gauging the symmetry in the entire Hilbert space.

VI. STATISTICS AS OBSTRUCTION TO SHORT-RANGE ENTANGLEMENT: DYNAMICAL CONSEQUENCE OF ANOMALIES

Here, let us show that the generalized statistics in $T = E_{\text{inv}}/E_{\text{id}}$ define the obstructions to the short-range entanglement. That is, if the bosonic state $|\Psi\rangle$ preserves the global symmetry with the nontrivial invariant $\Theta \neq 0$ in T , then the state cannot be short-range entangled, $|\Psi\rangle \neq V(|0\rangle^n)$ for any choice of a finite-depth circuit V and a product state $|0\rangle^n$. Given that the invariant Θ can be regarded as a microscopic definition of the 't Hooft anomaly, the obstruction to the SRE state is regarded as a dynamical consequence of the 't Hooft anomaly.

We generally show this statement by checking that SRE states must carry trivial invariants. Suppose that a state $|\Psi\rangle = V(|0\rangle^n)$ and the symmetry operators U have an invariant Θ . We can redefine the symmetry operators by $U' := V^{-1}UV$ and the input state $|\Psi'\rangle = |0\rangle^n$ without changing the invariant. Hence, we take the input state as a product state $|\Psi\rangle = |0\rangle^n$ without loss of generality. Below, we simply write U' as U .

The first step of the proof is to notice that the state with excitations $|a\rangle = U(s)|\Psi\rangle$ with $a = \partial s$ becomes a product state away from the location of the excitations.

We first show this statement when $U(s)$ generates the k -form symmetry with $k \geq 1$; i.e., the excitation has the dimension smaller than $d - 1$. In that case, let us separate the system into $A \cup A^c$, where the subsystem A is the locus

of the excitations a , and A^c is its complement. Let us consider the projector Π_{A^c} onto the product state $|0\rangle^n$ within the subsystem A^c . One can write Π_{A^c} as the product of local projectors within A^c , $\Pi_{A^c} = \prod_{L_j \subset A^c} \Pi_{L_j}$, where the set of local regions $\{L_j\}$ satisfies $\bigcup_j L_j = A^c$. Since the operator $U(s)$ can be topologically deformed on $|\Psi\rangle$ as long as $a = \partial s$ is fixed, one can set s to avoid each region L_j by the deformation. This implies that $U(s)$ and Π_{L_j} are commutative within the state $|\Psi\rangle$, so $\Pi_{L_j} U(s) |\Psi\rangle = U(s) \Pi_{L_j} |\Psi\rangle = U(s) |\Psi\rangle$. This implies that $\Pi_{A^c} U(s) |\Psi\rangle = U(s) |\Psi\rangle$, so the state $U(s) |\Psi\rangle$ is expressed in the form of

$$U(s) |\Psi\rangle = |a\rangle_A \otimes |0\rangle_{A^c}, \quad (109)$$

where $|0\rangle_{A^c}$ is the product state at A^c , and $|a\rangle_A$ is the state whose Hilbert space is localized along the locations of excitations A .

Then, let us show Eq. (109) when $U(s)$ is the 0-form symmetry. In that case, take a symmetry operator $U(\bar{s})$ with $-a = \partial \bar{s}$ so that the product $U(\bar{s})U(s)$ becomes a closed symmetry operator that preserves the state $|\Psi\rangle$, $U(\bar{s})U(s) |\Psi\rangle \propto |\Psi\rangle$. This operator is chosen so that $\text{supp}(s)$, $\text{supp}(\bar{s})$ do not have an overlap on their bulk. We again separate the system into $A \cup A^c$, where A has a support near the boundary of $\text{supp}(s)$. A^c is naturally separated into the two subsystems by $A^c = A_{\text{in}}^c \cup A_{\text{out}}^c$, where A_{in}^c is supported inside $\text{supp}(s)$, while A_{out}^c is on the outside. Let us again write the projector onto the product state as $\Pi_{A^c} = \Pi_{A_{\text{in}}^c} \Pi_{A_{\text{out}}^c}$, and each projector $\Pi_{A_{\text{in}}^c}$, $\Pi_{A_{\text{out}}^c}$ is given by the product of projectors at local regions $\{L_j^{\text{in}}\}$, $\{L_j^{\text{out}}\}$. The projectors $\Pi_{L_j^{\text{out}}}$ obviously commute with $U(s)$ since their support does not overlap. For $\Pi_{L_j^{\text{in}}}$, we have

$$\begin{aligned} U(\bar{s})U(s)\Pi_{L_j^{\text{in}}} |\Psi\rangle &= \Pi_{L_j^{\text{in}}} U(\bar{s})U(s) |\Psi\rangle \\ &= U(\bar{s})\Pi_{L_j^{\text{in}}} U(s) |\Psi\rangle, \end{aligned} \quad (110)$$

so $\Pi_{L_j^{\text{in}}}$ commutes with $U(s)$ on $|\Psi\rangle$, $U(s)\Pi_{L_j^{\text{in}}} |\Psi\rangle = \Pi_{L_j^{\text{in}}} U(s) |\Psi\rangle$. This implies that $\Pi_{A^c} U(s) |\Psi\rangle = U(s) |\Psi\rangle$, so we again obtain Eq. (109) for 0-form symmetry.

We remark that the discussion here assumes a tensor-product Hilbert space. There also exist fermionic SPT phases with physical fermions, which require imposing constraints of the form $\delta E = w_2$ to specify a “spin structure” E on the underlying manifold. Such phases necessarily involve Hilbert spaces that are not tensor-product spaces and therefore lie outside the scope of the present discussion.

A. Review: Anyons imply long-range entanglement

Let us illustrate the obstruction to SRE states when the excitations are particles with fusion rule $G = \mathbb{Z}_2$. This

follows from the results reviewed in Refs. [113,114]; see also Ref. [115] for recent developments. Because of the structure of excited states in Eq. (109), each state $|jk\rangle$ with a pair of particle excitations at vertices j and k has the following form [116]:

$$|jk\rangle = |j\rangle \otimes |k\rangle \otimes |0\rangle_{\overline{j,k}}, \quad (111)$$

where $|j\rangle, |k\rangle$ are local states around the excitations, and $|0\rangle_{\overline{j,k}}$ is the product state on the complement. This immediately implies that when j, k, l, m are distinct positions in the space, we have $\langle jl|U_{kj}|kl\rangle = \langle jm|U_{kl}|km\rangle$. Hence,

$$\theta(U_{kj}, kl) = \theta(U_{kl}, km) \quad (112)$$

for distinct j, k, l, m . This immediately shows that the following invariant of the particles must be trivial:

$$\begin{aligned} \Theta &= \theta(U_{02}U_{03}^{-1}U_{01}U_{02}^{-1}U_{03}U_{01}^{-1}, 12) \\ &= -\theta(U_{01}, 02) + \theta(U_{03}, 02) - \theta(U_{02}, 03) \\ &\quad + \theta(U_{01}, 03) - \theta(U_{03}, 01) + \theta(U_{02}, 01) = 0. \end{aligned} \quad (113)$$

Therefore, the SRE state cannot support Abelian anyons of nontrivial self-statistics.

B. Example: Fermionic loops imply long-range entanglement

Next, let us study the cases where the excitations are loops. In a 3D space, we demonstrate that SRE states cannot support the fermionic loops with the nontrivial invariant

$$\begin{aligned} e^{i\Theta} &= U_{014}U_{034}U_{023}U_{014}^{-1}U_{024}^{-1}U_{012}U_{023}^{-1}U_{013}^{-1} \\ &\quad \times U_{024}U_{014}U_{013}U_{024}^{-1}U_{034}^{-1}U_{023}U_{013}^{-1}U_{012}^{-1} \\ &\quad \times U_{034}U_{024}U_{012}U_{034}^{-1}U_{014}^{-1}U_{013}U_{012}^{-1}U_{023}^{-1}, \end{aligned} \quad (114)$$

where U_{jkl} is the generator of the $\mathbb{Z}_2$ 1-form symmetry supported on the a triangle with vertices j, k, l .

1. Matrix product state representation of excitations in SRE states

For SRE states, one can assume that the excited state $|a\rangle$ has an expression $|a\rangle = |a\rangle_A \otimes |0\rangle_{A^c}$ with the 1D state $|a\rangle_A$ localized at the position of excitations. One can generally express the 1D state $|a\rangle_A$ using the matrix product state (MPS) representation.

Let us consider an MPS $|a\rangle_A$ on the 1D subsystem A , with the bipartition into $A = A_1 \sqcup A_2$. Accordingly, the excitation a allows a decomposition $a = a_1 + a_2$. The state then has the expression

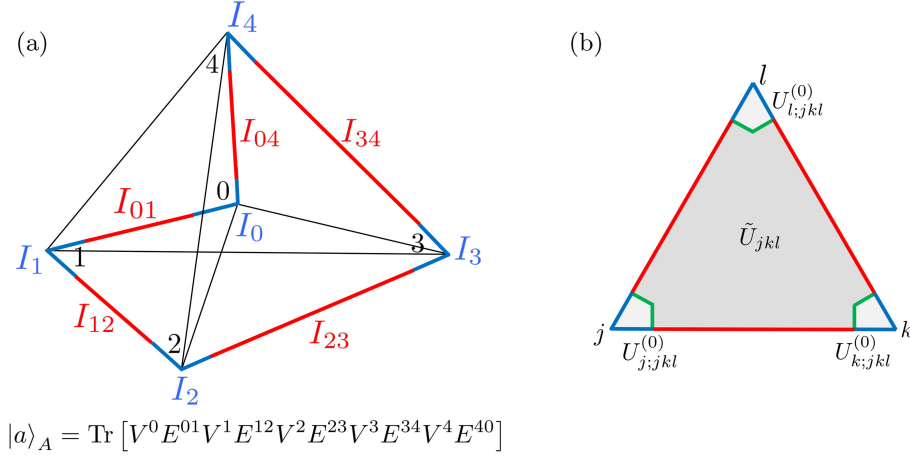


FIG. 6. (a) The excited state $|a\rangle_A$ is expressed as the patchwork of the MPS at the intervals I_j, I_{jk}, I_j supports an MPS $|V\rangle^j$, and I_{jk} supports an MPS $|E\rangle^{jk}$. (b) The support of each unitary on the triangle jkl . $U_{j,jkl}^{(0)}$ acts near the vertex j whose boundary contains I_j .

$$|a\rangle_A = \sum_{\mu,\nu} |\psi\rangle_{\mu\nu}^{A_1} |\psi\rangle_{\nu\mu}^{A_2}, \quad (115)$$

where μ, ν are bond indices. Let us consider a symmetry operator $U(s)$ with $\partial s = -a_2 + a'_2$. That is, the unitary acts on the 2D subsystem S that contains A_2 on its boundary, and transforms $|a\rangle$ into a state $|a'\rangle$ with excitations at $A' = A_1 \sqcup A'_2$. The unitary acts on the state $|a\rangle$ by

$$U(s)|a\rangle = \sum_{\mu,\nu} |\psi\rangle_{\mu\nu}^{A_1} \otimes U(s)[|\psi\rangle_{\nu\mu}^{A_2} |0\rangle^{S \cap A_2^c}] \otimes |0\rangle^{S^c}. \quad (116)$$

Given that $U(s)|a\rangle$ again allows an expression $U(s)|a\rangle = |a'\rangle_{A'} \otimes |0\rangle_{A'^c}$, the state $U(s)[|\psi\rangle_{\nu\mu}^{A_2} |0\rangle^{S \cap A_2^c}]$ has an expression

$$U(s)[|\psi\rangle_{\nu\mu}^{A_2} |0\rangle^{S \cap A_2^c}] = |\psi\rangle_{\nu\mu}^{A'_2} |0\rangle^{S \cap A_2'^c}. \quad (117)$$

That is, the operator $U(s)$ transforms the MPS $|\psi\rangle_{\nu\mu}^{A_2}$ into the new MPS $|\psi\rangle_{\nu\mu}^{A'_2}$ at the different location.

This motivates us to express the generic $|a\rangle_A$ in terms of the “patchwork” of MPS; see Fig. 6(a). Each state $|a\rangle_A$ is expressed in terms of the following MPS of two types:

- (i) $|V\rangle^j$ is an MPS on a small interval $I_j \subset A$ that contains the vertex j .
- (ii) $|E\rangle^{jk}$ is an MPS on an interval $I_{jk} \subset A$ contained in the edge $e = \langle jk \rangle$. I_{jk} is adjacent to I_j, I_k .

If the excitation a has a support at the collection of edges $\langle jk \rangle, \langle kl \rangle, \dots$ that forms a closed loop, we have $A = I_j \sqcup I_{jk} \sqcup I_k \sqcup I_{kl} \sqcup I_l \dots$, and the state $|a\rangle_A$ has an MPS representation as

$$|a\rangle_A = \text{Tr}[V^j E^{jk} V^k E^{kl} V^l \dots]. \quad (118)$$

The MPS $\{V^j, E^{jk}\}$ associated with the vertices and edges gives the canonical choice of the states $\{|a\rangle\}$.

2. Decomposition of phases

According to the expression of states $|a\rangle_A$ in terms of the patch of the MPS, it is convenient to separate the symmetry generator into the operators supported near the corner vertices j, k, l , and the rest,

$$U_{jkl} = U_{j,jkl}^{(0)} U_{k,jkl}^{(0)} U_{l,jkl}^{(0)} \times \tilde{U}_{jkl}, \quad (119)$$

where $U_{j,jkl}^{(0)}$ is a small defect operator acting on the region I_j together with the subsystem of the surface jkl near the vertex. $\tilde{U}_{jkl}$ acts on the intervals I_{jk}, I_{kl}, I_{jl} , and the bulk of the surface jkl except near the vertices; see Fig. 6(b).

Consequently, the phase $U_{jkl}|a\rangle = \theta(U_{jkl}, a)|a'\rangle$ admits an expression [117]

$$\begin{aligned} \theta(U_{jkl}, a) &= \theta(U_{j,jkl}^{(0)}, a) + \theta(U_{k,jkl}^{(0)}, a) \\ &\quad + \theta(U_{l,jkl}^{(0)}, a) + \theta(\tilde{U}_{jkl}, a). \end{aligned} \quad (120)$$

Given that each state is given in the form of the patch of the MPS, the phases θ achieve the following properties:

- (i) The dependence of $\theta(\tilde{U}_{jkl}, a)$ on a is through a restricted to edges jk, kl, jl . In other words, $\theta(\tilde{U}_{jkl}, a) = \theta(\tilde{U}_{jkl}, a')$ as long as $a = a'$ on edges jk, kl, jl . This is because this phase depends only on the MPS patch E^{jk}, E^{kl}, E^{jl} [and the small MPS patch for the intermediate state $\tilde{U}_{jkl}|a\rangle$ connecting between I_{jk}, I_{kl}, I_{jl} , which is expressed as the green lines in Fig. 6(b)]. Note that this property does not hold for the original phase $\theta(U_{jkl}, a)$ with the whole operator U_{jkl} , since this would depend on other edges that end at vertices j, k , or l through the MPS patch V^j, V^k, V^l .
- (ii) The dependence of $\theta(U_{j,jkl}^{(0)}, a)$ on a is through a restricted to edges ending at the vertex j . In other

words, $\theta(U_{j:kl}^{(0)}, a) = \theta(U_{j:kl}^{(0)}, a')$ as long as $a = a'$ on edges ending at j . This is because this phase depends only on the MPS patch V^j (and the choice of the surface $ijkl$ where we act U_{jkl} , together with the small MPS patch for the intermediate state).

This allows us to write the above phases as $\theta(U, \{e\})$ where $\{e\}$ is the set of edges with excitations on which θ depends. We can then express the invariant as

$$\begin{aligned}
\Theta = & -\theta(U_{0:023}^{(0)}, 03, 02) - \theta(U_{3:023}^{(0)}, 03, 34) - \theta(U_{2:023}^{(0)}, 02, 12) - \theta(\tilde{U}_{023}, 03, 02) \\
& - \theta(U_{0:012}^{(0)}, 01, 03) - \theta(U_{1:012}^{(0)}, 01, 14) - \theta(U_{2:012}^{(0)}) - \theta(\tilde{U}_{012}, 01) \\
& + \theta(U_{0:013}^{(0)}, 03, 01) + \theta(U_{3:013}^{(0)}, 03, 34) + \theta(U_{1:013}^{(0)}, 01, 14) \\
& + \theta(\tilde{U}_{013}, 03, 01) - \theta(U_{0:014}^{(0)}, 01, 04) \\
& - \theta(U_{1:014}^{(0)}, 01, 13) - \theta(U_{4:014}^{(0)}, 04, 34) - \theta(\tilde{U}_{014}, 01, 04) \\
& - \theta(U_{0:034}^{(0)}, 01, 03) - \theta(U_{3:034}^{(0)}, 03, 13) - \theta(U_{4:034}^{(0)}) - \theta(\tilde{U}_{034}, 03) \\
& + \theta(U_{0:012}^{(0)}, 01, 03) + \theta(U_{1:012}^{(0)}, 01, 13) + \theta(U_{2:012}^{(0)}) + \theta(\tilde{U}_{012}, 01) \\
& + \theta(U_{0:024}^{(0)}, 02, 03) + \theta(U_{2:024}^{(0)}, 02, 12) + \theta(U_{4:024}^{(0)}) + \theta(\tilde{U}_{024}, 02) \\
& + \theta(U_{0:034}^{(0)}, 03, 04) + \theta(U_{3:034}^{(0)}, 03, 13) + \theta(U_{4:034}^{(0)}, 04, 24) + \theta(\tilde{U}_{034}, 03, 04) \\
& - \theta(U_{0:012}^{(0)}, 02, 01) - \theta(U_{2:012}^{(0)}, 02, 24) - \theta(U_{1:012}^{(0)}, 01, 31) - \theta(\tilde{U}_{012}, 02, 01) \\
& - \theta(U_{0:013}^{(0)}, 03, 02) - \theta(U_{3:013}^{(0)}, 03, 34) - \theta(U_{1:013}^{(0)}) - \theta(\tilde{U}_{013}, 03) \\
& + \theta(U_{0:023}^{(0)}, 02, 03) + \theta(U_{2:023}^{(0)}, 02, 24) + \theta(U_{3:023}^{(0)}, 03, 34) + \theta(\tilde{U}_{023}, 02, 03) \\
& - \theta(U_{0:034}^{(0)}, 03, 04) - \theta(U_{3:034}^{(0)}, 03, 32) - \theta(U_{4:034}^{(0)}, 04, 24) - \theta(\tilde{U}_{034}, 03, 04) \\
& - \theta(U_{0:024}^{(0)}, 03, 02) - \theta(U_{2:024}^{(0)}, 02, 32) - \theta(U_{4:024}^{(0)}) - \theta(\tilde{U}_{024}, 02) \\
& + \theta(U_{0:013}^{(0)}, 03, 02) + \theta(U_{3:013}^{(0)}, 03, 32) + \theta(U_{1:013}^{(0)}) + \theta(\tilde{U}_{013}, 03) \\
& + \theta(U_{0:014}^{(0)}, 01, 02) + \theta(U_{1:014}^{(0)}, 01, 31) + \theta(U_{4:014}^{(0)}) + \theta(\tilde{U}_{014}, 01) \\
& + \theta(U_{0:024}^{(0)}, 02, 04) + \theta(U_{2:024}^{(0)}, 02, 32) + \theta(U_{4:024}^{(0)}, 04, 14) + \theta(\tilde{U}_{024}, 02, 04) \\
& - \theta(U_{0:013}^{(0)}, 01, 03) - \theta(U_{1:013}^{(0)}, 01, 14) - \theta(U_{3:013}^{(0)}, 03, 23) - \theta(\tilde{U}_{013}, 01, 03) \\
& - \theta(U_{0:023}^{(0)}, 02, 01) - \theta(U_{2:023}^{(0)}, 02, 24) - \theta(U_{3:023}^{(0)}) - \theta(\tilde{U}_{023}, 02) \\
& + \theta(U_{0:012}^{(0)}, 01, 02) + \theta(U_{1:012}^{(0)}, 01, 14) + \theta(U_{2:012}^{(0)}, 02, 24) + \theta(\tilde{U}_{012}, 01, 02) \\
& - \theta(U_{0:024}^{(0)}, 02, 04) - \theta(U_{2:024}^{(0)}, 02, 21) - \theta(U_{4:024}^{(0)}, 04, 14) - \theta(\tilde{U}_{024}, 02, 04) \\
& - \theta(U_{0:014}^{(0)}, 02, 01) - \theta(U_{1:014}^{(0)}, 01, 21) - \theta(U_{4:014}^{(0)}) - \theta(\tilde{U}_{014}, 01) \\
& + \theta(U_{0:023}^{(0)}, 02, 01) + \theta(U_{2:023}^{(0)}, 02, 21) + \theta(U_{3:023}^{(0)}) + \theta(\tilde{U}_{023}, 02) \\
& + \theta(U_{0:034}^{(0)}, 03, 01) + \theta(U_{3:034}^{(0)}, 03, 23) + \theta(U_{4:034}^{(0)}) + \theta(\tilde{U}_{034}, 03) \\
& + \theta(U_{0:014}^{(0)}, 01, 04) + \theta(U_{1:014}^{(0)}, 01, 21) + \theta(U_{4:014}^{(0)}, 04, 34) + \theta(\tilde{U}_{014}, 01, 04). \tag{121}
\end{aligned}$$

One can explicitly check that the above phases cancel out. Therefore, we get $\Theta = 0$, and the SRE state cannot support fermionic loops.

C. Nontrivial statistics imply long-range entanglement

One can extend the above discussion to the generic statistics of p -dimensional excitations embedded in d spatial dimensions. We assume that the excitation model is defined on a certain simplicial complex X embedded in space. For SRE states, the excited state can be represented by a state $|a\rangle_A$ localized at the position A of the p -dimensional excitations.

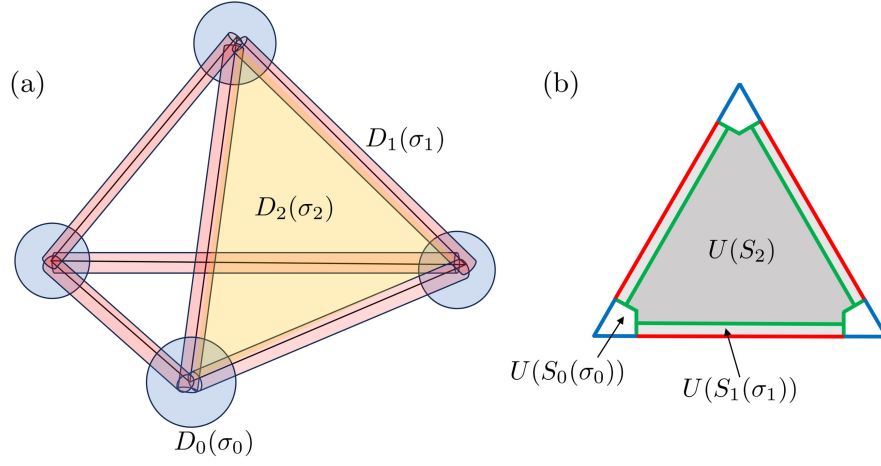


FIG. 7. (a) The disk locus of each simplex is denoted by $D_j(\sigma_j)$. Each TNS patch for the state $|a\rangle_A$ corresponds to extracting the TNS state inside each locus of σ_j with the locus of its boundary simplexes excluded. (b) The decomposition of the unitary operator in the case of $p = 1$.

1. Patchwork of tensor network states at the excitations

As a generalization of the MPS patchwork state for loop excitations, we assume that $|a\rangle_A$ admits an expression in terms of a tensor network state (TNS) at A . Note that A is a locus of a simplicial complex, so it can include, for example, a junction of hypersurfaces and is generally not a manifold.

This assumption leads to the canonical choice of the state $|a\rangle_A$ using a patch of TNS, where each patch is defined in the vicinity of a simplex. To define the support of each TNS patch, we introduce a locus for each simplex σ_j of X in space. Let $D_j(\sigma_j)$ denote the locus of the j -simplex σ_j with radius r_j ; this is given by $\sigma_j \times D^{d-j}$ in d -dimensional space, where D^{d-j} is a $(d-j)$ -ball of radius r_j , and its center point $\sigma_j \times \{0\}$ corresponds to σ_j ; see Fig. 7(a). We also define the collection of these regions as $D_j = \bigcup_{\sigma_j \in \mathcal{T}} D_j(\sigma_j)$ for the j -simplices $\{\sigma_j\}$. For convenience, we set the radii r_j to satisfy $r_{j-1} > \alpha r_j$ with a sufficiently large constant $\alpha \gg 1$. In addition, we require that the radii $\{r_j\}$ are large enough so that $A \subset \bigcup_{0 \leq j \leq p} D_j$.

We then define a TNS patch $T_j^A(\sigma_j, a)$ with $0 \leq j \leq p$ supported on $A \cap D_j(\sigma_j) \cap D_{j-1}^c \cap \dots \cap D_0^c$. This is the tensor network state supported on a single j -simplex σ_j , excluding the vicinity of its boundaries. This TNS has bond indices on the boundary that are contracted with other TNS patches at adjacent simplices. We represent the state $|a\rangle_A$ as

$$|a\rangle_A = \prod_{0 \leq j \leq p} \prod_{\sigma_j \in X} T_j^A(\sigma_j, a), \quad (122)$$

where the product denotes the contraction of bond indices. The conditions $r_{j-1} > \alpha r_j$ ensure that the supports of each pair of TNS patches T_j^A do not overlap.

2. Decomposition of phases

Each $(p+1)$ -simplex $\sigma_{p+1} \in X$ can support a unitary $U(S(\sigma_{p+1}))$ that generates the global symmetry G . Here $S(\sigma_{p+1})$ denotes a neighborhood of the simplex σ_{p+1} that supports the unitary. We require $S(\sigma_{p+1}) \subset D_{p+1}(\sigma_{p+1}) \cup (\bigcup_{0 \leq j \leq p} D_j)$.

Following the argument in Sec. VI B 2, we decompose the unitary $U(S(\sigma_{p+1}))$ into a sequence of smaller unitaries compatible with the TNS patch structure. That is, we decompose the support of the unitary as $S(\sigma_{p+1}) = S_{p+1} \sqcup S_p \sqcup S_{p-1} \sqcup \dots \sqcup S_0$, with

$$\begin{cases} S_{p+1} = S \cap D_{p+1}(\sigma_{p+1}) \cap D_p^c \cap \dots \cap D_0^c, \\ S_j = S \cap D_j \cap D_{j-1}^c \cap \dots \cap D_0^c \end{cases} \quad \text{for } 0 \leq j \leq p.$$

Note that S_j with $0 \leq j \leq p$ consists of disconnected components near each j -simplex of σ_{p+1} as

$$S_j = \bigsqcup_{\sigma_j \subset \sigma_{p+1}} S_j(\sigma_j). \quad (123)$$

We then rewrite the unitary $U(S(\sigma_{p+1}))$ as the sequence

$$U[S(\sigma_{p+1})] = \left(\prod_{\sigma_0 \subset \sigma_{p+1}} U[S_0(\sigma_0)] \right) \times \dots \times \left(\prod_{\sigma_p \subset \sigma_{p+1}} U[S_p(\sigma_p)] \right) \times U(S_{p+1}); \quad (124)$$

see Fig. 7(b). Each unitary $U(S_j(\sigma_j))$ acts on a single TNS patch on the j -simplex σ_j with $0 \leq j \leq p$. Accordingly, the phase $U(S(\sigma_{p+1}))|a\rangle = \theta\{U(S(\sigma_{p+1})), a\}|a'\rangle$ can be expressed as

$$\begin{aligned} \theta(U(S(\sigma_{p+1})), a) = & \left(\sum_{\sigma_0 \subset \sigma_{p+1}} \theta(U(S_0(\sigma_0)), a) \right) + \dots \\ & + \left(\sum_{\sigma_p \subset \sigma_{p+1}} \theta(U(S_p(\sigma_p)), a) \right) \\ & + \theta(U(S_{p+1})). \end{aligned} \quad (125)$$

Given that each small unitary $U(S_j(\sigma_j))$ acts within a single TNS patch, the phases θ satisfy the following properties:

- (i) The dependence of $\theta(U(S_j(\sigma_j)), a)$ (with $0 \leq j \leq p$) on a arises only through a restricted to the set of p -simplices $\{\sigma_p\}$ satisfying $\sigma_j \subset \sigma_p$. This is because the action of the unitary $U(S_j(\sigma_j))$ on A is confined to the TNS patch $T_j^A(\sigma_j, a)$. $\theta(U(S_j(\sigma_j)), a)$ also depends on the choice of unitary $U(\sigma_{p+1})$, namely, the choice of σ_{p+1} together with the group element $g \in G$ that corresponds to U .
- (ii) $\theta(U(S_{p+1}))$ does not depend on a but only on the choice of $U(S_{p+1})$. Note that this creates the intermediate tensor network state on the boundary sphere of S_{p+1} inside the $(p+1)$ -simplex.

3. Trivial invariants in SRE states

Suppose that the sequence of unitaries $\langle a | \prod U(S) | a \rangle$ evaluated on a given excited state becomes the sum of phases θ which gives the element of E_{inv} ,

$$\Theta = \sum_{U[S(\sigma_{p+1})], a} \epsilon(U(S(\sigma_{p+1})), a) \times \theta(U(S(\sigma_{p+1})), a), \quad (126)$$

with $\epsilon((U(S(\sigma_{p+1})), a)) \in \mathbb{Z}$. Recall that this phase $\Theta \in E_{\text{inv}}$ satisfies Eqs. (50) and (53) given by

- (i) the coefficients ϵ that satisfy

$$\sum_a \epsilon(U(S(\sigma_{p+1})), a) = 0 \quad (127)$$

for any choice of $U(S(\sigma_{p+1}))$, and

- (ii) the coefficients ϵ that satisfy

$$\sum_{a|_{\sigma_j=a_*}} \epsilon(U(S(\sigma_{p+1})), a) = 0 \quad (128)$$

with any choice of $U(S(\sigma_{p+1}))$ and a cochain a_* .

With this in mind, one can show that such a Θ satisfying the invariance condition must have $\Theta = 0$ in the SRE state. Let us decompose the phases using Eq. (125) valid for the TNS patchwork state. In the sum of Θ , let us extract the part involving a fixed $U(S(\sigma_{p+1}))$, and then further pick the part of θ involving $S_j(\sigma_j)$ contained in $S(\sigma_{p+1})$ with $0 \leq j \leq p$. The extracted sum has the form of

$$\begin{aligned} & \sum_a \epsilon(U(S(\sigma_{p+1})), a) \times \theta(U(S_j(\sigma_j)), a) \\ & = \sum_{a_*} \sum_{a|_{\sigma_j=a_*}} \epsilon(U(S(\sigma_{p+1})), a) \times \theta(U(S_j(\sigma_j)), a). \end{aligned} \quad (129)$$

Here, recall that the phase $\theta(U(S_j(\sigma_j)), a)$ depends on a only through the reduced one $a|_{\sigma_j}$ which is fixed in the second sum. We then have

$$\begin{aligned} & \sum_a \epsilon(U(S(\sigma_{p+1})), a) \times \theta(U(S_j(\sigma_j)), a) \\ & = \sum_{a_*} \left(\sum_{a|_{\sigma_j=a_*}} \epsilon(U(S(\sigma_{p+1})), a) \right) \theta(U(S_j(\sigma_j)), a) = 0. \end{aligned} \quad (130)$$

So, the extracted sum with each choice of $U(S(\sigma_{p+1}))$ and $S_j(\sigma_j)$ in Θ vanishes due to Eq. (128).

Finally, one can also verify that the phases involving $\theta(U(S_{p+1}))$ also vanish. The part of Θ that contains $\theta(U(S_{p+1}))$ is given by

$$\begin{aligned} & \sum_a \epsilon(U(S(\sigma_{p+1})), a) \times \theta(U(S_{p+1})) \\ & = \left(\sum_a \epsilon(U(S(\sigma_{p+1})), a) \right) \theta(U(S_{p+1})) = 0, \end{aligned} \quad (131)$$

where we used Eq. (127). Therefore, we get $\Theta = 0$ in the SRE state.

VII. DISCUSSION

In this paper, we established a universal microscopic description for the statistics of excitations in generic spacetime dimensions. The invariants are expressed as the Berry phase associated with families of excited states transformed by sequences of unitaries that generate finite global symmetries and move the excitations. These invariants are generally quantized into discrete values characterizing the generalized statistics of excitations in microscopic lattice models. For instance, this framework leads to the quantization of spins for Abelian anyons, which are microscopically defined via T -junctions. These invariants can be computed using algorithms with inputs such as the symmetry group and the configurations of excitations. This approach allows us to identify new invariants in microscopic lattice models, as well as a simplified expression for the invariants of fermionic loops in $(3+1)\text{D}$. These invariants can naturally be interpreted as obstructions to gauging the finite global symmetry G in microscopic lattice models, providing a microscopic definition of 't Hooft anomalies. We show that these anomalies have dynamical consequences, as nontrivial invariants forbid the existence of short-range-entangled states.

We close this paper with a number of future directions. One immediate direction for future work is to extend our framework of generalized statistics to include mixtures of $(d - p - 1)$ -form symmetries with distinct degrees p , which generally form higher-group symmetries. Even the simplest gapped phases, such as the $\mathbb{Z}_2$ toric code, exhibit a rich structure of higher-group symmetries with 't Hooft anomalies [38,39,118,119]. For example, in $(3 + 1)\text{D}$, string operators that create particle excitations may coexist with membrane operators that create loop excitations, and these excitations can exhibit nontrivial interplay; for instance, the intersection of loops may generate an additional particle. Determining the invariants associated with such higher-group structures in topologically ordered phases is an interesting direction for future work.

While we mainly studied the finite invertible symmetries in this paper, there are other important classes of symmetries which require further study, such as the continuous symmetries. For instance, the $(2 + 1)\text{D}$ gapped phases with continuous symmetries can exhibit electric Hall conductance, which is not associated with the 't Hooft anomaly of global symmetry in the usual sense. It would be interesting to see if the Hall conductance admits a similar microscopic characterization to our current paper, e.g., through the spin of the vortex of $U(1)$ symmetry. This would help clarify if the nontrivial Hall conductance gives obstructions to gauging continuous symmetries.

The other important class of symmetries not studied in this paper is the noninvertible symmetries, which includes the non-Abelian anyons. The microscopic definitions of the anyon data of the non-Abelian topological order in $(2 + 1)\text{D}$ has been discussed in Ref. [68]. Given that the spins of Abelian anyons are shown to be quantized using the locality identities, it would be interesting to see if the anyon data of non-Abelian anyon systems are also quantized due to the locality of topological operators.

The “Borromean ring” in $(2 + 1)\text{D}$ topological order is intrinsic to the non-Abelian anyons including the D_8 gauge theory, which is beyond the scope of our setup. Also, the three-loop braiding of loop excitations in $(3 + 1)\text{D}$ is expected to require either noninvertible loop excitations or a nontrivial higher-group structure involving Abelian loop and particle excitations. Both scenarios lie beyond the scope of the present work, but it would be interesting to characterize the corresponding invariant for three-loop braiding.

Recent developments on noninvertible symmetries have revealed a variety of new invariants associated with their underlying algebraic structures. For example, defects of noninvertible symmetries can carry invariants such as the Frobenius-Schur indicator and its generalizations [120–122]. It would be intriguing to investigate how such invariants can be realized through sequences of symmetry operators. Noninvertible symmetries also give rise to other invariants, including three-loop braiding of

vortices in $(3 + 1)\text{D}$ topological order [27], and generalizations of Hall conductance associated with continuous noninvertible symmetries [123]. Exploring the characterization of these invariants remains an important direction for future work.

While the symmetries considered in this work are assumed to be implemented by finite-depth circuits, many lattice symmetries cannot be expressed in this form. Examples include crystalline symmetries such as lattice translations, and more generally, symmetries generated by quantum cellular automata [47,49,124–127]. It would be interesting to investigate how the corresponding invariants can be characterized for such symmetries.

Several open issues have arisen from our work, which deserve further study. The genuine Berry phase invariants T are conjectured to correspond to the cohomology of the Eilenberg-MacLane space $H^{d+2}(B^{d-p}G, U(1))$ classifying the anomalies of higher-form symmetries. It would be desirable to prove this correspondence using the explicit model of the Eilenberg-MacLane space. A related formal mathematical treatment appears in Ref. [108].

Also, while the invariants in this paper were defined with an explicit reference to a specific state in the Hilbert space, but it is expected that the anomalies can be characterized solely by the algebra of symmetry operators without explicit reference to the states [83]. It would be interesting to see if our invariants are promoted to operator equations independent of the choice of any states in the Hilbert space.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: ANOMALOUS SYMMETRIES AND GENERALIZED STATISTICS

In this appendix, we present several concrete examples exhibiting nontrivial generalized statistics. For each (anomalous) global symmetry, we construct the operators that generate codimension-1 domain-wall excitations. The nontrivial fusion statistics of these domain walls serve as indicators of the underlying symmetry anomaly. To our knowledge, the constructions of membrane operators for the global $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry in two dimensions and of volume operators for the global $\mathbb{Z}_2$ symmetry in three dimensions are novel. Consequently, these generalized statistics provide an effective diagnostic for distinguishing anomalous symmetries.

1. Anomalous 0-form $\mathbb{Z}_2 \times \mathbb{Z}_2$ in two spatial dimensions

As an explicit illustration of the loop fusion statistics in Eq. (21), we consider a two-dimensional square lattice with an anomalous global $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. At each vertex v , there are two qubits with Pauli operators X_v^a, Z_v^a and X_v^b, Z_v^b . For convenience, we label the eigenvalues of Z_v^a and Z_v^b by elements of $\mathbb{Z}_2 = \{0, 1\}$ via

$$Z_v^a := (-1)^{a_v}, \quad Z_v^b := (-1)^{b_v}. \quad (\text{A1})$$

The anomalous $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is generated by

$$S^a := \prod_v X_v^a \prod_{f=\square_{1234}} \left(\text{CCZ}_{\text{triangle}} \right) = \prod_v X_v^a \prod_{f=\square_{1234}} (-1)^{b_1 b_2 b_4 + b_1 b_3 b_4}, \quad (\text{A2})$$

$$S^b := \prod_v X_v^b,$$

where the CCZ term represents applying a controlled-controlled-Z gate on the vertices of each triangle of a chosen triangulation of the square lattice. Focusing on the symmetric operators, we first define the domain-wall operators as

$$W(a)_{\langle ij \rangle} := Z_i^a Z_j^a, \quad W(b)_{\langle ij \rangle} := Z_i^b Z_j^b. \quad (\text{A3})$$

Consider the square patch

$$, \quad (\text{A4})$$

where the membrane operator that creates a closed-loop domain wall around vertex 5 is

$$U(a)_5 = X_5^a, \quad U(b)_5 = X_5^b (-1)^{\gamma_5(a,b)}, \quad (\text{A5})$$

with

$$\gamma_5(a,b) := a_5(b_1 b_2 + b_2 b_6 + b_6 b_9 + b_1 b_4 + b_4 b_8 + b_8 b_9). \quad (\text{A6})$$

Here, each product $a_5 b_i b_j$ represents a CCZ gate acting on the three qubits at vertices $\{5, i, j\}$ of each triangle adjacent to vertex 5. Membrane operators at other vertices are defined by translation; they create loop excitations around each vertex and commute with the global symmetry in Eq. (A2).

We compute their statistics $Z_4^H(a,b)$ in Eq. (21). Choosing region $A = \{5\}$, $B = \{2\}$, $C = \{1, 4\}$, and D as the set of all other vertices (including those outside the displayed patch), we start with

$$U(a)_{A+B+C+D} = \prod_v X_v^a, \quad U(b)_A = X_5^b (-1)^{\gamma_5(a,b)}, \quad (\text{A7})$$

and compute their commutator

$$[U(b)_A, U(a)_{A+B+C+D}] = (-1)^{b_1 b_2 + b_2 b_6 + b_6 b_9 + b_1 b_4 + b_4 b_8 + b_8 b_9}. \quad (\text{A8})$$

Next, with $U(b)_B = X_2^b (-1)^{\gamma_2(a,b)}$, we find

$$[U(b)_B, [U(b)_A, U(a)_{A+B+C+D}]] = (-1)^{b_1 + b_6} = Z_1^b Z_6^b. \quad (\text{A9})$$

Finally, since $U(b)_{B+C}$ contains $X_1^b X_2^b X_4^b$ up to a phase that commutes with $Z_1^b Z_6^b$, only the anticommutation between X_1^b and Z_1^b contributes to the final statistics:

$$Z_4^I(a, b) = (U(b)_{B+C})^{-2} (U(b)_{B+C} [U(b)_B, [U(b)_A, U(a)_{A+B+C+D}]]^2 = -1. \quad (\text{A10})$$

This nontrivial loop fusion statistic confirms that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is anomalous.

2. Anomalous 0-form $\mathbb{Z}_2$ in three spatial dimensions

In this section, we give an example exhibiting nontrivial membrane fusion statistics $Z_5(g)$ defined in Eq. (29). Consider a cubic lattice with one qubit at each vertex. As before, we define

$$Z_v := (-1)^{a_v}. \quad (\text{A11})$$

The anomalous $\mathbb{Z}_2$ global symmetry on this lattice is generated by

$$S := \prod_v X_v \prod_{c=\square_{12345678}} \left(\text{CCCZ} \right) \quad (\text{A12})$$

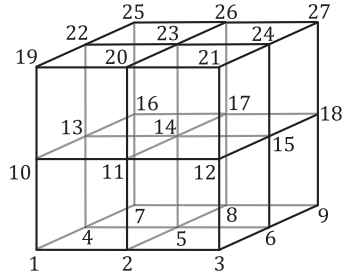
$$= \prod_v X_v \prod_{c=\square_{12345678}} (-1)^{a_1 a_2 a_4 a_8 + a_1 a_2 a_6 a_8 + a_1 a_3 a_4 a_8 + a_1 a_3 a_7 a_8 + a_1 a_5 a_6 a_8 + a_1 a_5 a_7 a_8},$$

where CCCZ denotes the controlled-controlled-controlled-Z gate. It is more transparent to visualize this symmetry on a triangulated three-dimensional manifold, where the CCCZ simply acts on the four vertices of each tetrahedron. To adapt this construction to the cubic lattice, we have chosen a subdivision of the cube into tetrahedra.

Similarly, we introduce the symmetric domain-wall operators on each edge $\langle ij \rangle$,

$$W_{\langle ij \rangle} = Z_i Z_j. \quad (\text{A13})$$

On the cube patch



$$, \quad (\text{A14})$$

the volume operator that creates a closed-membrane domain wall can be written as

$$U_{14} := X_{14} (-1)^{\gamma_{14}}, \quad (\text{A15})$$

where γ_{14} collects the phases from all CCCZ gates acting on tetrahedra containing vertex 14:

$$\begin{aligned} \gamma_{14} = & (a_1 a_2 a_5 a_{14} + a_1 a_2 a_{11} a_{14} + a_1 a_4 a_5 a_{14} + a_1 a_4 a_{13} a_{14} + a_1 a_{10} a_{11} a_{14} + a_1 a_{10} a_{13} a_{14}) \\ & + (a_{14} a_{15} a_{18} a_{27} + a_{14} a_{15} a_{24} a_{27} + a_{14} a_{17} a_{18} a_{27} + a_{14} a_{17} a_{26} a_{27} + a_{14} a_{23} a_{24} a_{27} + a_{14} a_{23} a_{26} a_{27}) \\ & + (a_2 a_5 a_{14} a_{15} + a_2 a_{11} a_{14} a_{15}) + (a_4 a_5 a_{14} a_{17} + a_4 a_{13} a_{14} a_{17}) + (a_5 a_{14} a_{15} a_{18} + a_5 a_{14} a_{17} a_{18}) \\ & + (a_{10} a_{11} a_{14} a_{23} + a_{10} a_{13} a_{14} a_{23}) + (a_{11} a_{14} a_{15} a_{24} + a_{11} a_{14} a_{23} a_{24}) + (a_{13} a_{14} a_{17} a_{26} + a_{13} a_{14} a_{23} a_{26}), \end{aligned} \quad (\text{A16})$$

which acts as CCCZ gates on the vertices of tetrahedra adjacent to vertex 14. It is straightforward to verify that U_{14} commutes with the global symmetry S in Eq. (A12) [128].

To extract the membrane fusion statistics, we select regions

$$A = \{14\}, \quad B = \{11\}, \quad C = \{10, 13\}, \quad D = \{1, 2, 4, 5\}. \quad (\text{A17})$$

First, observe

$$U_A^2 = U_{14}^2 = (-1)^{\gamma_{14}/a_{14}}. \quad (\text{A18})$$

Next, the commutator gives

$$\begin{aligned} [U_B, U_A^2] &= [X_{11}, (-1)^{a_1 a_2 a_{11} + a_1 a_{10} a_{11} + a_2 a_{11} a_{15} + a_{10} a_{11} a_{23} + a_{11} a_{15} a_{24} + a_{11} a_{23} a_{24}}] \\ &= (-1)^{a_1 a_2 + a_1 a_{10} + a_2 a_{15} + a_{10} a_{23} + a_{15} a_{24} + a_{23} a_{24}}, \end{aligned} \quad (\text{A19})$$

and hence,

$$[U_C, [U_B, U_A^2]] = [X_{10} X_{13}, (-1)^{a_1 a_2 + a_1 a_{10} + a_2 a_{15} + a_{10} a_{23} + a_{15} a_{24} + a_{23} a_{24}}] = (-1)^{a_1 + a_{23}} = Z_1 Z_{23}. \quad (\text{A20})$$

Since Z_1 anticommutes with the X_1 factor in U_{C+D} , the resulting fusion statistic is

$$Z_5 = (U_{C+D})^{-2} (U_{C+D} [U_C, [U_B, U_A^2]])^2 = -1. \quad (\text{A21})$$

3. Boundary anomaly of the beyond-cohomology (4+1)D $\mathbb{Z}_2$ SPT phase

According to Ref. [44], the boundary theory of the beyond-cohomology (4+1)D $\mathbb{Z}_2$ SPT phase has the anomalous symmetry described in Eq. (37) (the boundary also hosts additional particle excitations, which we omit in the following discussion). In the cup product notation [48], the symmetry operators can be expressed as

$$S^a = \left(\prod_v X_v^a \right) (-1)^{\int \mathbf{b} \cup \delta \mathbf{b}}, \quad S_v^b = \prod_{e|\delta \mathbf{v}(e)=1} X_e^b. \quad (\text{A22})$$

The corresponding domain-wall operators are defined in the usual way:

$$W_{e_{ij}}^a := Z_i^a Z_j^a, \quad W_f^b = \prod_{e \in \partial f} Z_e^b. \quad (\text{A23})$$

On the patch cube (A14), the associated volume operator U_v^a and membrane operator U_e^b are given by

$$\begin{aligned} U_5^a &= X_5^a, \\ U_{\langle 14, 15 \rangle}^b &= X_{\langle 14, 15 \rangle}^b (-1)^{a_{14}(b_{\langle 15, 18 \rangle} + b_{\langle 18, 27 \rangle} + b_{\langle 15, 24 \rangle} + b_{\langle 24, 27 \rangle}) + a_2(b_{\langle 2, 5 \rangle} + b_{\langle 5, 14 \rangle} + b_{\langle 2, 11 \rangle} + b_{\langle 11, 14 \rangle})}, \\ U_{\langle 14, 17 \rangle}^b &= X_{\langle 14, 17 \rangle}^b (-1)^{a_{14}(b_{\langle 17, 18 \rangle} + b_{\langle 18, 27 \rangle} + b_{\langle 17, 26 \rangle} + b_{\langle 26, 27 \rangle}) + a_4(b_{\langle 4, 5 \rangle} + b_{\langle 5, 14 \rangle} + b_{\langle 4, 13 \rangle} + b_{\langle 13, 14 \rangle})}, \\ U_{\langle 14, 23 \rangle}^b &= X_{\langle 14, 23 \rangle}^b (-1)^{a_{14}(b_{\langle 23, 24 \rangle} + b_{\langle 24, 27 \rangle} + b_{\langle 23, 26 \rangle} + b_{\langle 26, 27 \rangle}) + a_{10}(b_{\langle 10, 11 \rangle} + b_{\langle 11, 14 \rangle} + b_{\langle 10, 13 \rangle} + b_{\langle 13, 14 \rangle})}, \end{aligned} \quad (\text{A24})$$

and U_v^a and U_e^b on other vertices and edges are obtained by translation. Both U_v^a and U_e^b commute with the symmetries. The operator U_v^a excites the domain walls W_e^a on all edges e adjacent to v , producing a membrane excitation in the dual lattice, while U_e^b excites the domain walls W_f^b on all faces f adjacent to e , generating a loop excitation in the dual lattice. These operators can be derived from the cup product formalism as

$$U_v^a = X_v^a, \quad U_e^b = X_e^b (-1)^{\int a \cup (e \cup \delta \mathbf{b} + \delta \mathbf{b} \cup e)}. \quad (\text{A25})$$

To verify the nontrivial statistics, we choose the faces f_1, f_2 , and f_3 to be the Poincaré duals of the edges $\langle 14, 15 \rangle$, $\langle 14, 17 \rangle$, and $\langle 17, 18 \rangle$, respectively,

$$U_{f_3}^{-1} U_{f_2} U_{f_1}^{-1} U_{f_3} U_{f_2}^{-1} U_{f_1} = Z_5^a Z_{14}^a. \quad (\text{A26})$$

We take U_t to be the operator U_5^a , which can be viewed as the volume operator on the cube formed by the vertices 5, 6, 8, 9, 14, 15, 17, 18, as determined by the framing implicitly chosen when defining the symmetry. This U_t includes one of the end points of the common edge shared by f_1 , f_2 , and f_3 . With this choice, we find

$$Z_5^{\text{loop membrane I}} = [U_t, [U_{f_3}^{-1} U_{f_2} U_{f_1}^{-1} U_{f_3} U_{f_2}^{-1} U_{f_1}]] = -1. \quad (\text{A27})$$

Therefore, the boundary theory of the beyond-cohomology $(4+1)\text{D } \mathbb{Z}_2$ SPT phase exhibits a nontrivial loop-membrane mutual statistics.

4. Anomalous 0-form $\mathbb{Z}_2$ and 1-form $\mathbb{Z}_2$ symmetries in three spatial dimensions

For the anomalous symmetries in Eq. (39), the domain-wall operators take the form

$$W_{e_{ij}}^a := Z_i^a Z_j^a, \quad W_f^b = \prod_{e \in \partial f} Z_e^b. \quad (\text{A28})$$

On the patch cube (A14), the corresponding volume operator U_v^a and membrane operator U_e^b are

$$\begin{aligned} U_{14}^a &= X_{14}^a (-1)^{\zeta_{14}}, \\ U_{\langle 14,15 \rangle}^b &= X_{\langle 14,15 \rangle}^b (-1)^{a_2(a_5+a_{11})(a_{14}+1)}, \\ U_{\langle 14,17 \rangle}^b &= X_{\langle 14,17 \rangle}^b (-1)^{a_4(a_5+a_{13})(a_{14}+1)}, \\ U_{\langle 14,23 \rangle}^b &= X_{\langle 14,23 \rangle}^b (-1)^{a_{10}(a_{11}+a_{13})(a_{14}+1)}, \end{aligned} \quad (\text{A29})$$

with

$$\begin{aligned} \zeta_{14} &:= a_{14}a_{15}(b_{\langle 15,18 \rangle} + b_{\langle 18,27 \rangle} + b_{\langle 15,24 \rangle} + b_{\langle 24,27 \rangle}) \\ &+ a_{13}a_{14}(b_{\langle 14,17 \rangle} + b_{\langle 17,26 \rangle} + b_{\langle 14,23 \rangle} + b_{\langle 23,26 \rangle}) \\ &+ a_{14}a_{17}(b_{\langle 17,18 \rangle} + b_{\langle 18,27 \rangle} + b_{\langle 17,26 \rangle} + b_{\langle 26,27 \rangle}) \\ &+ a_{11}a_{14}(b_{\langle 14,15 \rangle} + b_{\langle 15,24 \rangle} + b_{\langle 14,23 \rangle} + b_{\langle 23,24 \rangle}) \\ &+ a_{14}a_{23}(b_{\langle 23,24 \rangle} + b_{\langle 24,27 \rangle} + b_{\langle 23,26 \rangle} + b_{\langle 26,27 \rangle}) \\ &+ a_5a_{14}(b_{\langle 14,15 \rangle} + b_{\langle 15,18 \rangle} + b_{\langle 14,17 \rangle} + b_{\langle 17,18 \rangle}). \end{aligned} \quad (\text{A30})$$

The operators U_v^a and U_e^b at other vertices and edges are obtained by translation and commute with the anomalous symmetries (39).

To verify the statistics in Eq. (38), we choose the cells t_1 , t_2 , and f in the patch cube (A14) as

$$t_1 = \{14\}, \quad t_2 = \{13\}, \quad f = \{\langle 14,23 \rangle\}. \quad (\text{A31})$$

More specifically, U_{14}^a is the volume operator on the cube with vertices 14, 15, 17, 18, 23, 24, 26, 27, and

U_{13}^a is the volume operator on the cube with vertices 13, 14, 16, 17, 22, 23, 25, 26. The operator $U_{\langle 14,23 \rangle}^b$ is the membrane operator acting on the face Poincaré dual to the edge $\langle 14,23 \rangle$. With this choice, we find

$$(U_{t_1})^2 = (U_{14}^a)^2 = (-1)^{\zeta_{14}/a_{14}}. \quad (\text{A32})$$

Then,

$$\begin{aligned} [U_{t_2}, U_{t_1}^2] &= [U_{13}^a, (U_{14}^a)^2] \\ &= [X_{13}^a, (-1)^{a_{13}(b_{\langle 14,17 \rangle} + b_{\langle 17,26 \rangle} + b_{\langle 14,23 \rangle} + b_{\langle 23,26 \rangle})}] \\ &= Z_{\langle 14,17 \rangle}^b Z_{\langle 17,26 \rangle}^b Z_{\langle 14,23 \rangle}^b Z_{\langle 23,26 \rangle}^b. \end{aligned} \quad (\text{A33})$$

Finally,

$$\begin{aligned} [U_f, [U_{t_2}, U_{t_1}^2]] &= [X_{\langle 14,23 \rangle}^b, Z_{\langle 14,17 \rangle}^b Z_{\langle 17,26 \rangle}^b Z_{\langle 14,23 \rangle}^b Z_{\langle 23,26 \rangle}^b] \\ &= -1. \end{aligned} \quad (\text{A34})$$

Thus, the domain-wall excitations realize the nontrivial loop-membrane statistics described in Eq. (38).

5. Anomalous 0-form $\mathbb{Z}_2$ and 2-form $\mathbb{Z}_2$ symmetries in three spatial dimensions

In the cup product notation [44,48], the anomalous symmetries defined in Eq. (42) can be written as

$$S^a := \left(\prod_v X_v^a \right) (-1)^{\int a \cup \delta c}, \quad S_e^c := \prod_{f | \delta e(f)=1} X_f^c. \quad (\text{A35})$$

The associated domain-wall operators are

$$W_{e_{ij}}^a := Z_i^a Z_j^a, \quad W_t^c = \prod_{f \in \partial t} Z_f^c, \quad (\text{A36})$$

where $W_{e_{ij}}^a$ acts on each edge e and W_t^c acts on each tetrahedron (3-cell) t . The corresponding volume and string operators, which commute with the symmetry (A35), are

$$U_v^a = X_v^a (-1)^{\int v \cup a \cup \delta c}, \quad U_f^c = X_f^c (-1)^{\int a \cup \delta a \cup f}. \quad (\text{A37})$$

The volume operator U_v^a excites the domain walls W_e^a on all edges e adjacent to v , producing a membrane excitation in the dual lattice, while the string operator U_f^c excites the domain walls W_t^c on two tetrahedra (3-cells) t adjacent to f , generating two particle excitations in the dual lattice.

On the patch cube (A14), the corresponding volume operator U_v^a and string operator U_f^b are

$$\begin{aligned}
U_{14}^a &= X_{14}^a (-1)^{a_{14}(c_{(14,15,17,18)} + c_{(14,15,23,24)} + c_{(14,17,23,26)} + c_{(15,18,24,27)} + c_{(17,18,26,27)} + c_{(23,24,26,27)})}, \\
U_{\langle 14,15,17,18 \rangle}^c &= X_{\langle 14,15,17,18 \rangle}^c (-1)^{a_5(a_{14}+1)}, \\
U_{\langle 14,15,23,24 \rangle}^c &= X_{\langle 14,15,23,24 \rangle}^c (-1)^{a_{11}(a_{14}+1)}, \\
U_{\langle 14,17,23,26 \rangle}^c &= X_{\langle 14,17,23,26 \rangle}^c (-1)^{a_{13}(a_{14}+1)},
\end{aligned} \tag{A38}$$

where $\langle i, j, k, l \rangle$ denotes the face (square) formed by the four vertices i, j, k , and l . The operators U_v^a and U_f^c at other vertices and faces are obtained by translation.

To verify the statistics in Eq. (41), we choose the cells t_1 , t_2 , and e in the patch cube (A14) as

$$t_1 = \{14\}, \quad t_2 = \{13\}, \quad e = \{\langle 14, 17, 23, 26 \rangle\}. \tag{A39}$$

U_{14}^a is the volume operator on the cube with vertices 14, 15, 17, 18, 23, 24, 26, 27, and U_{13}^a is the volume operator on the cube with vertices 13, 14, 16, 17, 22, 23, 25, 26. The operator $U_{\langle 14,17,23,26 \rangle}^c$ is the string operator acting on the edge Poincaré dual to the face $\langle 14, 17, 23, 26 \rangle$.

First, note that $U_e^2 = 1$ in our model, implying $[U_e^2, U_{t_1}] = 1$. The nontrivial contribution therefore comes from the second term in Eq. (41). Evaluating the first commutator, we find

$$[U_{t_1}, U_e] = (-1)^{a_{13}+a_{14}} = Z_{13}^a Z_{14}^a. \tag{A40}$$

Then, the double commutator is

$$[U_{t_2}, [U_{t_1}, U_e]] = [X_{13}^a, Z_{13}^a Z_{14}^a] = -1. \tag{A41}$$

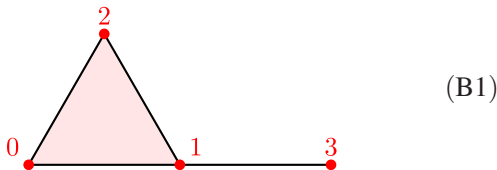
Combining these results, the statistics is

$$[U_e^2, U_{t_1}][U_{t_2}, [U_{t_1}, U_e]] = -1, \tag{A42}$$

showing that the domain walls carry a nontrivial $\mathbb{Z}_2$ mutual statistics.

APPENDIX B: SIMPLICIAL COMPLEX

This appendix reviews the notion of simplicial complexes used in this work. A simplicial complex is a piecewise-linear topological space obtained by gluing points, line segments, triangles, tetrahedra, and their higher-dimensional analogs along common faces. For example, the simplicial complex



is obtained by gluing a 1-simplex and a 2-simplex at the face 1 [129]. Including all faces, the simplices in this complex are

- (i) (-1) -dimensional simplex: $\emptyset$ (included by convention)
- (ii) 0-simplices: $\{0\}, \{1\}, \{2\}, \{3\}$
- (iii) 1-simplices: $\{0, 1\}, \{0, 2\}, \{1, 2\}, \{1, 3\}$
- (iv) 2-simplices: $\{0, 1, 2\}$

This geometric description motivates a purely combinatorial definition.

Definition B.1. A simplicial complex X of n vertices is a collection of finite subsets of $\{0, \dots, n-1\}$ called simplices, such that

- (1) $\{i\} \in X$ for every $0 \leq i \leq n-1$
- (2) if $\sigma \in X$ and $\tau \subset \sigma$, then $\tau \in X$

A simplex $\sigma \in X$ with $q+1$ vertices is called a q -simplex. We write X_q for the set of all q -simplices of X .

The vertex set $\{0, 1, \dots, n-1\}$ carries its standard total order, which induces an ordering on each simplex. Every p -simplex can be uniquely written as

$$\sigma = \{a_0, \dots, a_p\}, \quad 0 \leq a_0 < a_1 < \dots < a_p \leq n-1,$$

and we use the shorthand notation $\sigma = [a_0, \dots, a_p]$. In this notation, the simplicial complex from Eq. (B1) can be written as

$$X = \{\emptyset, [0], [1], [2], [3], [0, 1], [0, 2], [1, 2], [1, 3], [0, 1, 2]\}. \tag{B2}$$

Let $\sigma = [a_0, \dots, a_p]$ be a simplex. Any subset of σ is called a face of σ . In particular, the $(p-1)$ -dimensional faces are obtained by deleting a single vertex. For each $0 \leq i \leq p$, removing a_i gives the face

$$\partial_i \sigma = [a_0, \dots, \hat{a}_i, \dots, a_p],$$

where the hat indicates omission.

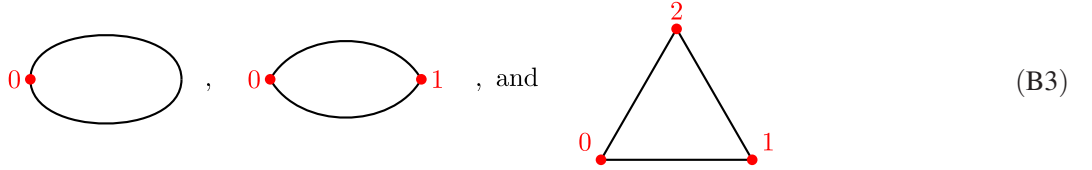
Simplicial complexes are useful tools to study topological spaces. Although a topological space typically contains infinitely many points, its triangulation as a simplicial complex can be described by a finite set of combinatorial data, which greatly simplifies its study. Including all its faces, the $(d+1)$ -simplex Δ^{d+1} is a simplicial complex defined as the set of all subsets of $\{0, \dots, d+1\}$; its boundary $\partial \Delta^{d+1}$ is obtained by deleting the only

$(d+1)$ -simplex. $\partial\Delta^{d+1}$ is the simplest triangulation of the d -sphere S^d , and this triangulation is used in our study of statistics in d -dimensional space.

Homology is naturally defined on simplicial complexes. Let $X(n)$ denote the set of n -simplices of X ; the n th chain group is $C_n(X) = \sum_{\sigma \in X(n)} \mathbb{Z}\sigma$, and the boundary map $\partial: C_n(X) \rightarrow C_{n-1}(X)$ is defined by $\partial\sigma = \sum_{i=0}^n (-1)^i \partial_i \sigma$. For any discrete Abelian group G , we can formally multiply a group element $g \in G$ with a simplex $\sigma \in X(n)$, denoting the result by $g\sigma$. The n th chain group with coefficients in G is then $C_n(X, G) = \bigoplus_{\sigma \in X_n} G\sigma$. An element of $C_n(X, G)$ is written as $\sum_{\sigma \in X_n} g_\sigma \sigma$ with

$g_\sigma \in G$. The homological boundary map $\partial: C_n(X, G) \rightarrow C_{n-1}(X, G)$ is defined by $\partial(g\sigma) = \sum_{i=0}^n (-1)^i g \partial_i \sigma$. The image of ∂ is the $(n-1)$ th boundary chain group denoted by $B_{n-1}(X, G)$; in our excitation models, this group is the configuration group $\mathcal{A}$ for an $(n-1)$ -dimensional excitation model with fusion group G in the simplicial complex X .

We note a related but slightly different concept: the Δ -complex (see Sec. 2 of Ref. [130]). In a simplicial complex, each p -simplex has $(p+1)$ distinct vertices and is uniquely determined by them; in a Δ -complex, vertices of a simplex may coincide, and a simplex may not be uniquely determined by its vertices. For example,



are all Δ -complexes, but only the last is a simplicial complex. The first two do not work well as excitation models for F symbols because they lack locality identities. In general, simplicial complexes have better behavior.

Another subtlety is that manifolds are special topological spaces. In a d -dimensional manifold, every point has a neighborhood homeomorphic to $\mathbb{R}^d$, and this property is reflected in any triangulation. Technically, for any $\sigma \in X$ we define $Lk(\sigma)$, the link of σ , to be the subcomplex of X consisting of all τ such that $\sigma \cap \tau = \emptyset$, and the union of the vertices of σ and τ forms the vertex set of a simplex of X . We say X is a d -dimensional combinatorial manifold if, for any $\sigma \in X(p)$, the link $Lk(\sigma)$ is homeomorphic to S^{d-p-1} .

This property of local topology has important implications for statistics: Trivial local topology implies trivial local statistics, which further implies that the statistics is independent of the choice of operator. This is referred to as the strong operator independence condition in Ref. [108], where the full definition and proof are given.

APPENDIX C: DERIVATIONS FOR GENERALIZED STATISTICS WITH FUSION GROUP $\mathbb{Z}_2$

1. T -junction process in $(2+1)\text{D}$

In this section, we explicitly evaluate the summation of locality identities for the $(2+1)\text{D}$ T -junction with $G = \mathbb{Z}_2$ as outlined in Sec. IV A. This analysis shows that a particular combination of these locality identities results in four iterations of the T -junction process, thereby

verifying the quantization of the T -junction into the fourth root of unity.

For convenience, we represent a configuration of excitations by a vertex of a graph. An edge connecting two vertices represents the action of a unitary operator that changes the configuration from one vertex to the other. The direction of the edge further specifies the initial and final configurations. All possible transformations of the states using the operators U_{0i} can be represented as a cube, as shown in Fig. 5(a).

Since $G = \mathbb{Z}_2$, for a fixed initial configuration of excitations, the actions of the unitaries U and U^{-1} yield the same final state. Each direction of an edge can be associated with two distinct unitaries U or U^{-1} . Thus, each edge corresponds to four possible transformations arising from the choice of edge direction and whether U or U^{-1} is applied. According to

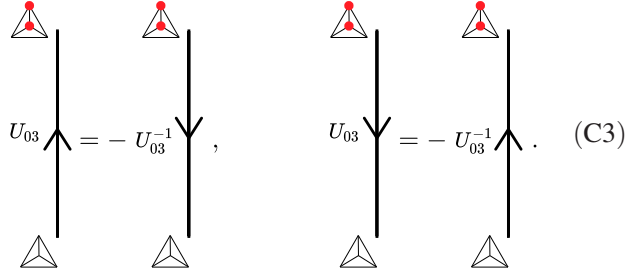
$$\theta(U(s), a) = -\theta(U(s)^{-1}, a + \partial s), \quad (\text{C1})$$

for any $a \in \mathcal{A}$ and $s \in \mathcal{S}$, there are two relations among the four possible phases. Taking the leftmost vertical edge in Fig. 5(a) with U_{03} as an example, we have

$$\begin{aligned} \theta(U_{03}, \triangle) + \theta(U_{03}^{-1}, \triangle) &= 0, \\ \theta(U_{03}, \triangle) + \theta(U_{03}^{-1}, \triangle) &= 0. \end{aligned} \quad (\text{C2})$$

Based on the above discussion, we describe the sum of phases using a cube with operators $\{U_{0j}\}$, where the directed edges indicate the initial and final states.

We can express the phases in Eq. (C2) diagrammatically on the edges of the cube as



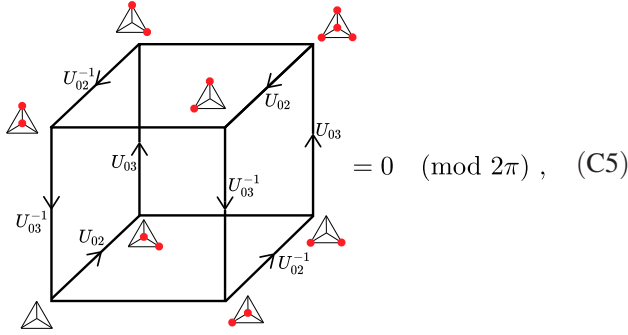
$$U_{03} = -U_{03}^{-1}, \quad U_{03} = -U_{03}^{-1}. \quad (C3)$$

The same relation holds for other edges in different directions.

Next, we consider the following locality identities acting on the vacuum state represented by specific commutators of hopping operators whose supports do not overlap:

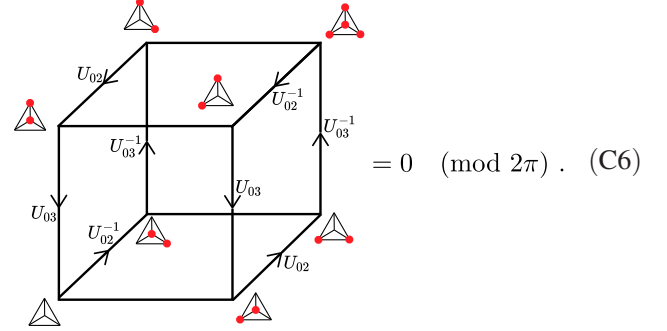
$$\begin{aligned} \text{Type 1: } & \langle \triangle \uparrow | [[U_{02}, U_{03}], U_{12}] | \triangle \uparrow \rangle = 1, \\ \text{Type 2: } & \langle \triangle \uparrow | [[U_{02}^{-1}, U_{03}^{-1}], U_{12}] | \triangle \uparrow \rangle = 1, \\ \text{Type 3: } & \langle \triangle \uparrow | [[U_{03}, U_{02}], U_{23}] | \triangle \uparrow \rangle = 1. \end{aligned} \quad (C4)$$

Each identity involves a total of eight hopping operators within the cube of configuration states, forming two distinct oriented squares. The oriented squares associated with the type-1 locality identity can be represented as follows:



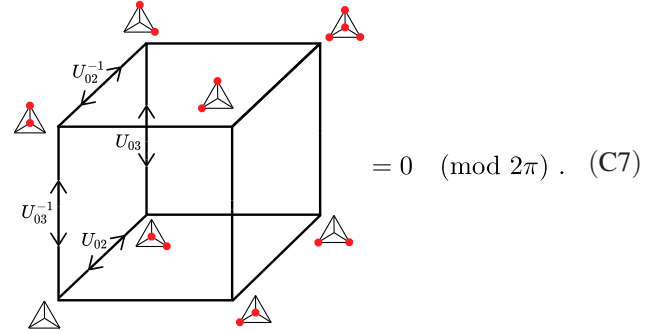
$$= 0 \pmod{2\pi}, \quad (C5)$$

where two squares correspond to the terms $\langle \triangle \uparrow | [U_{02}, U_{03}] | \triangle \uparrow \rangle$ and $-\langle \triangle \uparrow | [U_{02}, U_{03}] | \triangle \uparrow \rangle$, respectively. Similarly, the type-2 locality identity can be visualized on the cube of configuration states as follows:



$$= 0 \pmod{2\pi}. \quad (C6)$$

The type-3 locality identity, on the other hand, is represented by two oriented squares located on the same face of the cube:



$$= 0 \pmod{2\pi}. \quad (C7)$$

Note that each type of identity includes the one depicted above, along with two others that are related by the C_3 rotation along the axis extending from $\triangle \uparrow$ to $\triangle \downarrow$.

Below, we list all identities of the three types, which will be used in the derivation of the T -junction process.

$$\begin{aligned} \text{Type 1: } & \theta(U_{03}, \triangle \uparrow) + \theta(U_{02}, \triangle \downarrow) - \theta(U_{03}, \triangle \downarrow) - \theta(U_{02}, \triangle \uparrow) \\ & - \theta(U_{03}, \triangle \uparrow) - \theta(U_{02}, \triangle \downarrow) + \theta(U_{03}, \triangle \downarrow) + \theta(U_{02}, \triangle \uparrow) = 0 \pmod{2\pi}, \\ & \theta(U_{02}, \triangle \uparrow) + \theta(U_{01}, \triangle \downarrow) - \theta(U_{02}, \triangle \downarrow) - \theta(U_{01}, \triangle \uparrow) \\ & - \theta(U_{02}, \triangle \uparrow) - \theta(U_{01}, \triangle \downarrow) + \theta(U_{02}, \triangle \downarrow) + \theta(U_{01}, \triangle \uparrow) = 0 \pmod{2\pi}, \\ & \theta(U_{01}, \triangle \uparrow) + \theta(U_{03}, \triangle \downarrow) - \theta(U_{01}, \triangle \downarrow) - \theta(U_{03}, \triangle \uparrow) \\ & - \theta(U_{01}, \triangle \uparrow) - \theta(U_{03}, \triangle \downarrow) + \theta(U_{01}, \triangle \downarrow) + \theta(U_{03}, \triangle \uparrow) = 0 \pmod{2\pi}. \end{aligned} \quad (C8)$$

Type 2: $\theta(U_{03}, \text{triangle with red dot at top}) + \theta(U_{02}, \text{triangle with red dot at top}) - \theta(U_{03}, \text{triangle with red dot at bottom}) - \theta(U_{02}, \text{triangle with red dot at bottom})$
 $- \theta(U_{03}, \text{triangle with red dot at left}) - \theta(U_{02}, \text{triangle with red dot at left}) + \theta(U_{03}, \text{triangle with red dot at right}) + \theta(U_{02}, \text{triangle with red dot at right}) = 0 \pmod{2\pi},$
 $\theta(U_{02}, \text{triangle with red dot at top}) + \theta(U_{01}, \text{triangle with red dot at top}) - \theta(U_{02}, \text{triangle with red dot at bottom}) - \theta(U_{01}, \text{triangle with red dot at bottom})$
 $- \theta(U_{02}, \text{triangle with red dot at left}) - \theta(U_{01}, \text{triangle with red dot at left}) + \theta(U_{02}, \text{triangle with red dot at right}) + \theta(U_{01}, \text{triangle with red dot at right}) = 0 \pmod{2\pi},$
 $\theta(U_{01}, \text{triangle with red dot at top}) + \theta(U_{03}, \text{triangle with red dot at top}) - \theta(U_{01}, \text{triangle with red dot at bottom}) - \theta(U_{03}, \text{triangle with red dot at bottom})$
 $- \theta(U_{01}, \text{triangle with red dot at left}) - \theta(U_{03}, \text{triangle with red dot at left}) + \theta(U_{01}, \text{triangle with red dot at right}) + \theta(U_{03}, \text{triangle with red dot at right}) = 0 \pmod{2\pi}.$ (C9)

Type 3: $\theta(U_{03}, \text{triangle with red dot at top}) + \theta(U_{02}, \text{triangle with red dot at top}) - \theta(U_{03}, \text{triangle with red dot at bottom}) - \theta(U_{02}, \text{triangle with red dot at bottom})$
 $- \theta(U_{03}, \text{triangle with red dot at left}) - \theta(U_{02}, \text{triangle with red dot at left}) + \theta(U_{03}, \text{triangle with red dot at right}) + \theta(U_{02}, \text{triangle with red dot at right}) = 0 \pmod{2\pi},$
 $\theta(U_{02}, \text{triangle with red dot at top}) + \theta(U_{01}, \text{triangle with red dot at top}) - \theta(U_{02}, \text{triangle with red dot at bottom}) - \theta(U_{01}, \text{triangle with red dot at bottom})$
 $- \theta(U_{02}, \text{triangle with red dot at left}) - \theta(U_{01}, \text{triangle with red dot at left}) + \theta(U_{02}, \text{triangle with red dot at right}) + \theta(U_{01}, \text{triangle with red dot at right}) = 0 \pmod{2\pi},$
 $\theta(U_{01}, \text{triangle with red dot at top}) + \theta(U_{03}, \text{triangle with red dot at top}) - \theta(U_{01}, \text{triangle with red dot at bottom}) - \theta(U_{03}, \text{triangle with red dot at bottom})$
 $- \theta(U_{01}, \text{triangle with red dot at left}) - \theta(U_{03}, \text{triangle with red dot at left}) + \theta(U_{01}, \text{triangle with red dot at right}) + \theta(U_{03}, \text{triangle with red dot at right}) = 0 \pmod{2\pi}.$ (C10)

By summing over all type-1 identities, type-2 identities, and twice the type-3 identities, we obtain

$$4\theta(U_{03}, \text{triangle with red dot at top}) - 4\theta(U_{02}, \text{triangle with red dot at top}) + 4\theta(U_{01}, \text{triangle with red dot at top})$$

$$- 4\theta(U_{03}, \text{triangle with red dot at bottom}) + 4\theta(U_{02}, \text{triangle with red dot at bottom}) - 4\theta(U_{01}, \text{triangle with red dot at bottom}) = 0 \pmod{2\pi}. \quad (\text{C11})$$

Using Eq. (C1), this expression can be simplified to

$$4\theta(U_{03}, \text{triangle with red dot at top}) + 4\theta(U_{02}^{-1}, \text{triangle with red dot at top}) + 4\theta(U_{01}, \text{triangle with red dot at top})$$

$$+ 4\theta(U_{03}^{-1}, \text{triangle with red dot at bottom}) + 4\theta(U_{02}, \text{triangle with red dot at bottom}) + 4\theta(U_{01}^{-1}, \text{triangle with red dot at bottom})$$

$$= 4\theta(U_{02}U_{03}^{-1}U_{01}U_{02}^{-1}U_{03}U_{01}^{-1}, \text{triangle with red dot at top}) = 0 \pmod{2\pi}. \quad (\text{C12})$$

This implies that four iterations of the T -junction process yield $0 \pmod{2\pi}$. The T -junction process is illustrated in Fig. 5(b).

2. Loop-flipping process in (3+1)D

In this section, we derive the unitary sequence used to detect loop statistics for the fusion group $G = \mathbb{Z}_2$ in (3+1)D. This derivation is inspired by Appendix D of Ref. [45]. Similar to the previous section, we first construct the graph space of configurations. In principle, there are six operators U_{012} , U_{013} , U_{014} , U_{023} , U_{024} , and U_{034} , which create loops along the boundary of these faces, resulting in a total of $2^6 = 64$ configuration states within the same superselection sector as the vacuum state. For simplicity, we restrict our analysis to single-loop configurations, where the loop excitation is connected and does not have self-intersection. There are 37 single-loop configurations represented as vertices in Fig. 8. Interestingly,

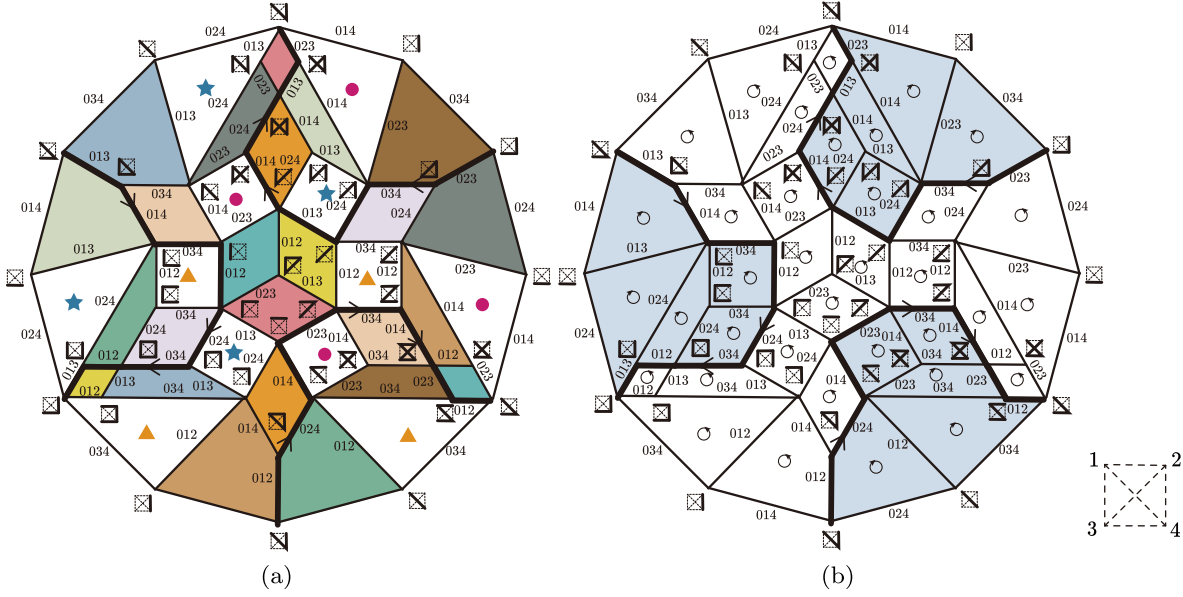


FIG. 8. Each vertex represents a single-loop configuration on the tetrahedron $\langle 1234 \rangle$ with a central vertex 0, as shown in Fig. 1(c). Only the loop configuration along the edges between vertices 1, 2, 3, and 4 is depicted, as the configurations on the remaining edges can be inferred. The operator U_{0ij} creates a loop excitation on edge $\langle ij \rangle$, while U_{0ij}^{-1} annihilates it. This structure forms an $\mathbb{RP}^2$ space, as antipodal vertices represent the same configuration state [45]. (a) Each plaquette is labeled by a color or symbol indicating that summing over plaquettes [according to the orientation in Fig. 8(b)] with the same color or symbol yields a locality identity. (b) The orientation of each plaquette used to express the local identity. Summing over all locality identities results in twice the 24-step process (represented by the black directed line), as the $\theta(U(s), a)$ terms on all other edges are canceled out.

the computation is even more straightforward than in the T -junction case. Since $G = \mathbb{Z}_2$, transitioning from one state to another allows us to apply either U or U^{-1} , as used in the locality identities in Eqs. (C5)–(C7). However, for loop excitations, we further constrain ourselves such that U_{0ij} can create only a loop on edge $\langle ij \rangle$, while U_{0ij}^{-1} can annihilate it only on the same edge.

The locality identities are represented by colors and symbols in Fig. 8(a). For example, the two yellow plaquettes correspond to the following locality identity:

$$\begin{aligned} & \theta(U_{012}^{-1} U_{013}^{-1} U_{012} U_{013}, \partial\langle 023 \rangle) \\ & + \theta(U_{013}^{-1} U_{012}^{-1} U_{013} U_{012}, \partial\langle 023 \rangle + \partial\langle 234 \rangle) \\ & = \theta(U_{012}^{-1} U_{013}^{-1} U_{012} U_{013}, \partial\langle 023 \rangle) \\ & - \theta(U_{012}^{-1} U_{013}^{-1} U_{012} U_{013}, \partial\langle 023 \rangle + \partial\langle 234 \rangle) \\ & = \theta([U_{234}, [U_{012}, U_{013}]], \partial\langle 023 \rangle) = 0 \pmod{2\pi}. \end{aligned} \quad (\text{C13})$$

Each color corresponds to a pair of plaquettes, and summing over two plaquettes with orientations given in Fig. 8(b) results in the locality identity for each color. On the other hand, each symbol (red circles, blue stars, yellow triangles) appears in four plaquettes. For instance, the red circles in Fig. 8(a) are present on four plaquettes spanning $\langle 014 \rangle$ and $\langle 023 \rangle$. The corresponding locality identity across these four plaquettes is more involved:

$$\begin{aligned} & \theta([U_{014}, U_{023}], \partial\langle 012 \rangle) + \theta([U_{014}, U_{023}], \partial\langle 024 \rangle) \\ & + \theta([U_{014}, U_{023}], \partial\langle 034 \rangle) \\ & + \theta([U_{014}, U_{023}], \partial\langle 013 \rangle) = 0 \pmod{2\pi}, \end{aligned} \quad (\text{C14})$$

which results from summing over four plaquettes following the orientations in Fig. 8(b). To show that Eq. (C14) is indeed a locality identity, we expand it as follows:

$$\begin{aligned} & (\theta(U_{023}, \partial\langle 012 \rangle) + \theta(U_{014}, \partial\langle 012 \rangle + \partial\langle 023 \rangle) - \theta(U_{023}, \partial\langle 012 \rangle + \partial\langle 014 \rangle) - \theta(U_{014}, \partial\langle 012 \rangle)) \\ & + (\theta(U_{023}, \partial\langle 024 \rangle) + \theta(U_{014}, \partial\langle 024 \rangle + \partial\langle 023 \rangle) - \theta(U_{023}, \partial\langle 024 \rangle + \partial\langle 014 \rangle) - \theta(U_{014}, \partial\langle 024 \rangle)) \\ & + (\theta(U_{023}, \partial\langle 034 \rangle) + \theta(U_{014}, \partial\langle 034 \rangle + \partial\langle 023 \rangle) - \theta(U_{023}, \partial\langle 034 \rangle + \partial\langle 014 \rangle) - \theta(U_{014}, \partial\langle 034 \rangle)) \\ & + (\theta(U_{023}, \partial\langle 013 \rangle) + \theta(U_{014}, \partial\langle 013 \rangle + \partial\langle 023 \rangle) - \theta(U_{023}, \partial\langle 013 \rangle + \partial\langle 014 \rangle) - \theta(U_{014}, \partial\langle 013 \rangle)). \end{aligned} \quad (\text{C15})$$

Next, the subsequent locality identities are added to the above equation to modify the configuration states a within the $\theta(U(s), a)$ terms appearing in the first three lines:

- (1) $\theta([U_{014}, U_{023}], U_{123}, \partial\langle 012 \rangle) = 0 \pmod{2\pi}$
- (2) $\theta([U_{014}, U_{023}], U_{123}U_{124}, \partial\langle 024 \rangle) = 0 \pmod{2\pi}$
- (3) $\theta([U_{014}, U_{023}], U_{134}, \partial\langle 034 \rangle) = 0 \pmod{2\pi}$

Equation (C15) is transformed to

$$\begin{aligned}
 & \left(\theta(U_{023}, \partial\langle 012 \rangle + \partial\langle 123 \rangle) + \theta(U_{014}, \partial\langle 012 \rangle + \partial\langle 023 \rangle + \partial\langle 123 \rangle) \right. \\
 & \quad \left. - \theta(U_{023}, \partial\langle 012 \rangle + \partial\langle 014 \rangle + \partial\langle 123 \rangle) - \theta(U_{014}, \partial\langle 012 \rangle + \partial\langle 123 \rangle) \right) \\
 & + \left(\theta(U_{023}, \partial\langle 024 \rangle + \partial\langle 124 \rangle + \partial\langle 123 \rangle) + \theta(U_{014}, \partial\langle 024 \rangle + \partial\langle 023 \rangle + \partial\langle 124 \rangle + \partial\langle 123 \rangle) \right. \\
 & \quad \left. - \theta(U_{023}, \partial\langle 024 \rangle + \partial\langle 014 \rangle + \partial\langle 124 \rangle + \partial\langle 123 \rangle) - \theta(U_{014}, \partial\langle 024 \rangle + \partial\langle 124 \rangle + \partial\langle 123 \rangle) \right) \\
 & + \left(\theta(U_{023}, \partial\langle 034 \rangle + \partial\langle 134 \rangle) + \theta(U_{014}, \partial\langle 034 \rangle + \partial\langle 023 \rangle + \partial\langle 134 \rangle) \right. \\
 & \quad \left. - \theta(U_{023}, \partial\langle 034 \rangle + \partial\langle 014 \rangle + \partial\langle 134 \rangle) - \theta(U_{014}, \partial\langle 034 \rangle + \partial\langle 134 \rangle) \right) \\
 & + \left(\theta(U_{023}, \partial\langle 013 \rangle) + \theta(U_{014}, \partial\langle 013 \rangle + \partial\langle 023 \rangle) - \theta(U_{023}, \partial\langle 013 \rangle + \partial\langle 014 \rangle) - \theta(U_{014}, \partial\langle 013 \rangle) \right) \\
 & = 0 \pmod{2\pi},
 \end{aligned} \tag{C16}$$

where the terms with matched colors are completely canceled out in the last equality. Thus, we have demonstrated that Eq. (C14) indeed represents a valid locality identity.

Note that for the locality identity of each color or symbol, we have the freedom to multiply the overall minus sign; e.g., the locality identity on the yellow plaquettes in Eq. (C13) can be modified as

$$\begin{aligned}
 0 &= -\theta([U_{234}, [U_{012}, U_{013}], \partial\langle 023 \rangle) \\
 &= -\theta(U_{012}^{-1}U_{013}^{-1}U_{012}U_{013}, \partial\langle 023 \rangle) \\
 &\quad - \theta(U_{013}^{-1}U_{012}^{-1}U_{013}U_{012}, \partial\langle 023 \rangle + \partial\langle 234 \rangle) \\
 &= \theta(U_{013}^{-1}U_{012}^{-1}U_{013}U_{012}, \partial\langle 023 \rangle) \\
 &\quad + \theta(U_{012}^{-1}U_{013}^{-1}U_{012}U_{013}, \partial\langle 023 \rangle + \partial\langle 234 \rangle). \tag{C17}
 \end{aligned}$$

The physical interpretation of this choice of locality identity is that it reverses the orientation of the yellow plaquettes in Fig. 8(b). This same argument applies to each symbol: We can simultaneously reverse the orientations of four plaquettes with the same symbol. We can derive different unitary sequences by altering the orientation of each color or symbol. However, all these sequences are equivalent, as they differ only by locality identities. Specifically, changing the orientation corresponds to adding (-2) times the locality identity, and the unitary sequence is the sum of all locality identities divided by 2. For example, by selecting a different orientation, we obtain the 36-step unitary sequence shown in Fig. 9, which exactly matches the sequence proposed in Ref. [45].

Finally, we used computers to enumerate all possible orientations for each locality identity. There are $2^{15} = 32768$ different configurations corresponding to the three symbols and 12 colors. Our computations verified that the minimal number of steps required is 24. Therefore, the unitary sequence presented in Fig. 8 is optimal, meaning there is no shorter sequence capable of detecting the loop statistics.

3. Particle fusion process in $(1+1)\text{D}$

In this section, we prove that the $(1+1)\text{D}$ $\mathbb{Z}_2$ particle fusion statistics (14) is an invariant in E_{inv} . More precisely, on the complex

$$\begin{array}{c}
 2 \\
 \swarrow \quad \searrow \\
 U_{02} \quad U_{12} \\
 \searrow \quad \swarrow \\
 0 \quad \quad 1 \\
 \quad \quad U_{01}
 \end{array}, \tag{C18}$$

we show that twice the phase of the $(1+1)\text{D}$ particle fusion process can be written as a combination of locality identities.

One such identity is

$$\langle \triangleleft | [U_{12}, [U_{02}, U_{01}]] | \triangleleft \rangle = 1, \tag{C19}$$

which implies that

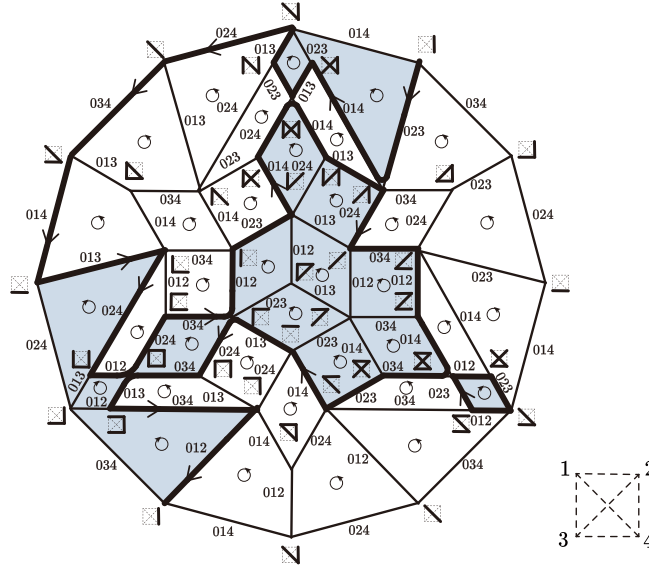


FIG. 9. The 36-step unitary sequence corresponding to $\mathbb{Z}_2$ loop statistics in $(3+1)\text{D}$. Compared to Fig. 8(b), the orientations of certain locality identities have been reversed. Any choice of orientation for each locality identity yields a valid sequence for detecting the loop statistics. This particular orientation corresponds to the 36-step process proposed in Ref. [45], where a different computational method was used to derive the sequence.

$$\begin{aligned}
 & \theta([U_{02}, U_{01}], \triangle) + \theta(U_{12}, \triangle) + \theta([U_{02}, U_{01}]^{-1}, \triangle) + \theta(U_{12}^{-1}, \triangle) \\
 &= \theta([U_{02}, U_{01}], \triangle) + \theta([U_{02}, U_{01}]^{-1}, \triangle) \\
 &= \theta(U_{01}, \triangle) + \theta(U_{02}, \triangle) + \theta(U_{01}^{-1}, \triangle) + \theta(U_{02}^{-1}, \triangle) \\
 &\quad + \theta(U_{02}, \triangle) + \theta(U_{01}, \triangle) + \theta(U_{02}^{-1}, \triangle) + \theta(U_{01}^{-1}, \triangle) \\
 &= 0 \pmod{2\pi}.
 \end{aligned} \tag{C20}$$

In configuration space, these phases can be visualized as

$$\begin{aligned}
 & \begin{array}{c} \triangle \\ \uparrow U_{01}^{-1} \\ \triangle \end{array} \begin{array}{c} \triangle \\ \leftarrow U_{01}^{-1} \\ \triangle \end{array} \\
 & \begin{array}{c} \triangle \\ \downarrow U_{02}^{-1} \\ \triangle \end{array} \begin{array}{c} \triangle \\ \rightarrow U_{02} \\ \triangle \end{array} \\
 & + \begin{array}{c} \triangle \\ \uparrow U_{01}^{-1} \\ \triangle \end{array} \begin{array}{c} \triangle \\ \rightarrow U_{01}^{-1} \\ \triangle \end{array} \\
 & \begin{array}{c} \triangle \\ \downarrow U_{02}^{-1} \\ \triangle \end{array} \begin{array}{c} \triangle \\ \leftarrow U_{02} \\ \triangle \end{array} = 0 \pmod{2\pi}.
 \end{aligned} \tag{C21}$$

Another identity is

$$\langle \triangle | [U_{12}, [U_{02}^{-1}, U_{01}]] | \triangle \rangle = 1, \tag{C22}$$

which implies that

$$\begin{aligned}
& \theta([U_{02}^{-1}, U_{01}], \triangle) + \theta(U_{12}^{-1}, \triangle) + \theta([U_{02}^{-1}, U_{01}]^{-1}, \triangle) + \theta(U_{12}, \triangle) \\
&= \theta([U_{02}^{-1}, U_{01}], \triangle) + \theta([U_{02}^{-1}, U_{01}]^{-1}, \triangle) \\
&= \theta(U_{01}, \triangle) + \theta(U_{02}^{-1}, \triangle) + \theta(U_{01}^{-1}, \triangle) + \theta(U_{02}, \triangle) \\
&\quad + \theta(U_{02}^{-1}, \triangle) + \theta(U_{01}, \triangle) + \theta(U_{02}, \triangle) + \theta(U_{01}^{-1}, \triangle) \\
&= 0 \pmod{2\pi}.
\end{aligned} \tag{C23}$$

In configuration space, these phases can be represented as

$$\begin{aligned}
& \begin{array}{c} \triangle \\ \uparrow U_{02} \\ \square \\ \downarrow U_{01} \\ \triangle \end{array} \quad \begin{array}{c} \triangle \\ \leftarrow U_{01}^{-1} \\ \square \\ \rightarrow U_{02}^{-1} \\ \triangle \end{array} + \begin{array}{c} \triangle \\ \uparrow U_{02} \\ \square \\ \downarrow U_{01} \\ \triangle \end{array} \quad \begin{array}{c} \triangle \\ \leftarrow U_{01}^{-1} \\ \square \\ \rightarrow U_{02}^{-1} \\ \triangle \end{array} = 0 \pmod{2\pi}.
\end{aligned} \tag{C24}$$

From the definition of U^{-1} in Eq. (73), we obtain

$$\begin{aligned}
& \begin{array}{c} \triangle \\ \downarrow U_{02}^{-1} \\ \triangle \end{array} = - \begin{array}{c} \triangle \\ \uparrow U_{02} \\ \triangle \end{array}, \quad \begin{array}{c} \triangle \\ \uparrow U_{02}^{-1} \\ \triangle \end{array} = - \begin{array}{c} \triangle \\ \downarrow U_{02} \\ \triangle \end{array}.
\end{aligned} \tag{C25}$$

By adding Eqs. (C21) and (C24) and applying Eq. (C25), we find

$$2 \times \begin{array}{c} \triangle \\ \leftarrow U_{01}^{-1} \\ \square \\ \rightarrow U_{01} \\ \triangle \end{array} = 0 \pmod{2\pi}, \tag{C26}$$

which implies that $[U_{02}, U_{01}^2]^2 = 1$. This calculation produces the fusion statistics in Eq. (14).

4. Loop-membrane mutual statistics in (3+1)D

In this section, we prove that the loop-membrane statistic introduced in Sec. IID 2 is an invariant belonging to E_{inv} . For the setup in Eq. (38), we introduce a new tetrahedron t_3 adjacent to t_1 and t_2 such that $t_1 \cap t_2 \cap t_3 = v_1 v_2$, the edge connecting v_1 and v_2 . We then define t_4 as the complement of $t_1 \cup t_2 \cup t_3$, which satisfies $\partial t_4 = \partial t_1 + \partial t_2 + \partial t_3$. In this configuration, we have $f \cap t_1 \cap t_2 \cap t_4 = \emptyset$. Moreover, since f can be replaced by its exterior without affecting the commutator, the higher commutators among U_f , U_{t_1} , U_{t_2} , and U_{t_3} are also trivial. Under these conditions, the following four locality identities hold:

- (1) $\langle \emptyset | [U_f, [U_{t_3}, [U_{t_2}, U_{t_1}]]] | \emptyset \rangle = 1$
- (2) $\langle \emptyset | [U_f, [U_{t_3}, [U_{t_2}^{-1}, U_{t_1}]]] | \emptyset \rangle = 1$
- (3) $\langle \partial U_{t_3} | [U_f, [U_{t_4}, [U_{t_2}, U_{t_1}]]] | \partial U_{t_3} \rangle = 1$
- (4) $\langle \partial U_{t_3} | [U_f, [U_{t_4}, [U_{t_2}^{-1}, U_{t_1}]]] | \partial U_{t_3} \rangle = 1$

By summing these locality identities, we obtain

$$\begin{aligned}
& \theta([U_f, [U_{t_3}, [U_{t_2}, U_{t_1}]]], \emptyset) + \theta([U_f, [U_{t_3}, [U_{t_2}^{-1}, U_{t_1}]]], \emptyset) + \theta([U_f, [U_{t_4}, [U_{t_2}, U_{t_1}]]], \partial U_{t_3}) + \theta([U_f, [U_{t_4}, [U_{t_2}^{-1}, U_{t_1}]]], \partial U_{t_3}) \\
& = \theta([U_{t_3}, [U_{t_2}, U_{t_1}]], \emptyset) + \theta([U_{t_3}, [U_{t_2}, U_{t_1}]^{-1}], \partial U_f) \\
& \quad + \theta([U_{t_3}, [U_{t_2}^{-1}, U_{t_1}]], \emptyset) + \theta([U_{t_3}, [U_{t_2}^{-1}, U_{t_1}]^{-1}], \partial U_f) \\
& \quad + \theta([U_{t_4}, [U_{t_2}, U_{t_1}]], \partial U_{t_3}) + \theta([U_{t_4}, [U_{t_2}, U_{t_1}]^{-1}], \partial U_f + \partial U_{t_3}) + \theta([U_{t_4}, [U_{t_2}^{-1}, U_{t_1}]], \partial U_{t_3}) \\
& \quad + \theta([U_{t_4}, [U_{t_2}^{-1}, U_{t_1}]^{-1}], \partial U_f + \partial U_{t_3}) \\
& = \theta([U_{t_3}, [U_{t_2}, U_{t_1}]], \emptyset) + \theta([U_{t_3}, [U_{t_2}^{-1}, U_{t_1}]], \emptyset) + \theta([U_{t_4}, [U_{t_2}, U_{t_1}]], \partial U_{t_3}) + \theta([U_{t_4}, [U_{t_2}^{-1}, U_{t_1}]], \partial U_{t_3}) \\
& \quad + (\theta([U_{t_3}, [U_{t_2}, U_{t_1}]^{-1}], \partial U_f) + \theta([U_{t_3}, [U_{t_2}^{-1}, U_{t_1}]]^{-1}], \partial U_f) + \theta([U_{t_4}, [U_{t_2}, U_{t_1}]^{-1}], \partial U_f + \partial U_{t_3}) \\
& \quad + \theta([U_{t_4}, [U_{t_2}^{-1}, U_{t_1}]^{-1}], \partial U_f + \partial U_{t_3})) = 2\theta([U_{t_2}, U_{t_1}^2], \emptyset) + 2\theta([U_{t_2}, U_{t_1}^2]^{-1}], \partial U_f) \\
& = 2\theta([U_f, [U_{t_2}, U_{t_1}^2]], \emptyset) = \theta([U_f, [U_{t_2}, U_{t_1}^2]^2], \emptyset) = 0 \pmod{2\pi}. \tag{C27}
\end{aligned}$$

Therefore, the loop-membrane statistical process $[U_f, [U_{t_2}, U_{t_1}^2]]$ follows directly from the locality identities.

APPENDIX D: DETAILED ANALYSIS OF THE STRUCTURE OF E_{id}

In this appendix, we show the following statement: Suppose we have an invariant Θ in E_{inv} for an Abelian symmetry group G expressed as

$$e^{i\Theta} = \langle a | U(s_n)^\pm, \dots, U(s_1)^\pm | a \rangle. \tag{D1}$$

Let us pick any element $s_0 \in \mathcal{S}$. Then, the following ratio is an element of E_{id} :

$$e^{i\Phi} = \frac{\langle a | U(s_n)^\pm, \dots, U(s_1)^\pm | a \rangle}{\langle a + \partial s_0 | U(s_n)^\pm, \dots, U(s_1)^\pm | a + \partial s_0 \rangle}. \tag{D2}$$

In other words, we show that the phase $e^{i\Phi}$ can be expressed as the product of higher commutators of the form

$$\begin{aligned}
& \langle a' | [U(s'_m), [\dots, [U(s'_2), U(s'_1)]]]] | a' \rangle = 1, \\
& a' \in \mathcal{A}, s'_1, s'_2, \dots, s'_m \in \mathcal{S} | s'_1 \cap s'_2 \cap \dots \cap s'_m = \emptyset, \tag{D3}
\end{aligned}$$

which we refer to as the locality identity in the main text. This property holds when X is a combinatorial manifold (Theorems VI.3 and VI.10 of Ref. [108]), but it fails for a generic simplicial complex X . A crucial step in the proof is the introduction of localization for excitation models. For completeness, we present a less abstract argument here.

We first rewrite the phase Φ in the following fashion:

$$\begin{aligned}
\Phi &= \sum_{j=0}^{n-1} \theta(U(s_{j+1})^\pm, a + t_j) - \theta(U(s_{j+1})^\pm, a + \partial s_0 + t_j) \\
&= \sum_{j=0}^{n-1} \theta(U(s_{j+1})^\pm, a + t_j) - \theta(U(s_{j+1})^\pm, a \\
& \quad + \partial s_0 + t_j) + \theta(U(s_0), a + t_{j+1}) - \theta(U(s_0), a + t_j), \tag{D4}
\end{aligned}$$

where $t_j = \sum_{k=1}^j \pm \partial s_k$. In the second equation, we used $t_n = t_0 = 0$. Noting that

$$\begin{aligned}
& \arg(\langle a | [U(s_2), U(s_1)] | a \rangle) \\
&= \theta(s_1, a) + \theta(s_2, a + \partial s_1) - \theta(s_1, a + \partial s_2) - \theta(s_2, a), \tag{D5}
\end{aligned}$$

and $\theta(U(s), a) = -\theta(U(s)^{-1}, a + \partial s)$, we get

$$\begin{aligned}
\Phi &= \sum_{\substack{j=0 \\ +\text{sign}}}^{n-1} \langle a + t_j | [U(s_0), U(s_{j+1})] | a + t_j \rangle \\
& \quad + \sum_{\substack{j=0 \\ -\text{sign}}}^{n-1} \langle a + t_{j+1} | [U(s_0), U(s_{j+1})]^{-1} | a + t_{j+1} \rangle, \tag{D6}
\end{aligned}$$

where the sum breaks into the two parts depending on the sign on $U(s_j)^\pm$. Let us fix an element $s \in \mathcal{S}$. In the expression (D6), let us collect the commutators involving $U(s_j)^\pm$ with $s_j = s$. Focusing on these commutators, the sum is schematically written as

$$\Phi = \sum_{a'} \langle a' | [U(s_0), U(s)]^\pm | a' \rangle + \dots, \quad (\text{D7})$$

where ... denotes the commutators involving $s_j \neq s$. If $\text{supp}(s_0) \cap \text{supp}(s) = \emptyset$, the commutator can be regarded as the locality identity. Hence, we assume $\text{supp}(s_0) \cap \text{supp}(s)$ is nontrivial.

Let us first study the case with $\text{supp}(s_0) \neq \text{supp}(s)$. For $a', b' \in \mathcal{A}$, if $a' = b'$ at $\text{supp}(s_0) \cap \text{supp}(s')$, there exists $s_{ab} \in \mathcal{S}$ satisfying $U(s_{ab})|a'\rangle = |b'\rangle$ and $\text{supp}(s_0) \cap \text{supp}(s') \cap \text{supp}(s_{ab}) = \emptyset$. In that case, one can convert a' into b' by the locality identity:

$$\frac{\langle a' | [U(s_0), U(s)]^\pm | a' \rangle}{\langle b' | [U(s_0), U(s)]^\pm | b' \rangle} = \langle [U(s_{ab}), [U(s_0), U(s)]^\pm] \rangle = 1. \quad (\text{D8})$$

This implies that the phase Φ in Eq. (D7) depends on a' only through the configuration of a' in the vicinity of the mutual support $\text{supp}(s_0) \cap \text{supp}(s)$. Concretely, the configuration of a' near the mutual support of the unitaries can be specified as follows. The mutual support $\text{supp}(s_0) \cap \text{supp}(s)$ generally becomes a p -simplex σ_p . Pick $(p+1)$ vertices $v_0, \dots, v_p$ of this p -simplex σ_p . Then, a' in the expression (D7) can be represented by a $(p+1)$ -tuple of the restricted configurations $(a'|_{v_0}, \dots, a'|_{v_p})$. In other words, we can replace $|a'\rangle$ with a canonical representative of the state $|(a'|_{v_0}, \dots, a'|_{v_p})\rangle$ in $\mathcal{A}$ whose restriction to vertices becomes the fixed ones. This can be done by using the locality identity and leaves the value of Φ invariant.

Let us write the set of restricted configurations $a|_{v_j}$ as $\mathcal{A}_{v_j}$. We also introduce a shorthand notation $a|_{v_j} = \alpha_j$. We can express Φ as

$$\Phi = \sum_{\oplus_j \mathcal{A}_{v_j}} \epsilon_+(s, \{\alpha_j\}) \langle \{\alpha_j\} | [U(s_0), U(s)] | \{\alpha_j\} \rangle + \sum_{\oplus_j \mathcal{A}_{v_j}} \epsilon_-(s, \{\alpha_j\}) \langle \{\alpha_j\} | [U(s_0), U(s)]^{-1} | \{\alpha_j\} \rangle, \quad (\text{D9})$$

where $\epsilon_\pm$ is the positive integer coefficient counting the number of the commutator $\langle \{\alpha_j\} | [U(s_0), U(s)]^\pm | \{\alpha_j\} \rangle$ appears in the sum of Φ . Let us define ϵ as

$$\epsilon(s, \{\alpha_j\}) = \epsilon_+(s, \{\alpha_j\}) - \epsilon_-(s, \{\alpha_j\}). \quad (\text{D10})$$

Then, due to the condition of E_{inv} shown in Eq. (55) satisfied by the invariant Θ , ϵ satisfies

$$\sum_{\oplus_{j \neq k} \mathcal{A}_{v_j}} \epsilon(s, (\alpha_0, \dots, \alpha_p)) = 0 \quad (\text{D11})$$

for any $\alpha_k \in \mathcal{A}_{v_k}$ and any $0 \leq k \leq p$. The sum is over $\alpha_j \in \mathcal{A}_{v_j}$ with $j \neq k$ while α_k is fixed. Further, once we fix a

choice of α_{v_k} , it fixes the configuration of the excitation at the p -simplex σ_p which we write a_{σ_p} . Then, ϵ is nonzero if and only if $\alpha_0, \dots, \alpha_p$ all specify the same a_{σ_p} at σ_p . This allows us to write a refined version of Eq. (D11) as

$$\sum_{\oplus_{j \neq k} \mathcal{A}_{v_j}(a_{\sigma_p})} \epsilon(s, (\alpha_0, \dots, \alpha_p)) = 0, \quad (\text{D12})$$

where we defined $\mathcal{A}_{v_j}(a_{\sigma_p}) \subset \mathcal{A}_{v_j}$ as a set of α_j , which has the fixed configuration a_{σ_p} at the simplex σ_p .

Equation (D12) gives a number of constraints on $\epsilon(s, \{\alpha_j\})$. One can show that the solution to Eq. (D12) is generated by the following (overcomplete) basis: Let us pick a pair of elements $\alpha_j^{(0)}, \alpha_j^{(1)}$ from each $\mathcal{A}_{v_j}(a_{\sigma_p})$. Then, the basis is given by

$$\epsilon(s, \{\alpha_j^{(n_j)}\}) = (-1)^{\sum_j n_j}, \quad (\text{D13})$$

with $n_j = 0, 1$, with the other ϵ zero.

Let us show that the basis (D13) corresponds to the trivial phase due to the locality identity. To see this, take a unitary $U(s_{\alpha_k})$ that shifts $\alpha_k^{(0)}$ to $\alpha_k^{(1)}$, but leaves the other α_j with $j \neq k$ invariant. Note that the mutual support of the unitaries $\text{supp}(s_0) \cap \text{supp}(s) \cap \text{supp}(s_{\alpha_k})$ becomes a 0-simplex v_k . In particular, this implies that

$$\left(\bigcap_{j=0}^p \text{supp}(s_{\alpha_j}) \right) \cap \text{supp}(s_0) \cap \text{supp}(s) = \emptyset. \quad (\text{D14})$$

Now, the basis (D13) corresponds to the invariant given by the following higher commutator:

$$\langle \{\alpha_j^{(0)}\} | [U(s_{\alpha_0}), [\dots, [U(s_{\alpha_p}), [U(s_0), U(s)]]]] | \{\alpha_j^{(0)}\} \rangle, \quad (\text{D15})$$

so each basis corresponds to the locality identity. Noticing this for each choice of a_{σ_p} at σ_p completes the proof for $\text{supp}(s_0) \neq \text{supp}(s)$.

When $\text{supp}(s_0) = \text{supp}(s)$, the proof is done in a similar logic: The difference is that the support of $[U(s_0), U(s)]$ is given by $\partial(\text{supp}(s))$, so we need to take $p+2$ vertices of a $(p+1)$ -simplex s to fix the configuration of excitations in the vicinity of the operator support. Let us write these vertices as $v_0, \dots, v_{p+1}$. The boundary $p+1$ -simplexes of s are written as $\sigma_0, \dots, \sigma_{p+1}$, where σ_j does not contain the vertex v_j . We again write the set of restricted configurations $a|_{v_j} = \alpha_j$ as $\mathcal{A}_{v_j}$, with $0 \leq j \leq p+1$. One can again express Φ as

$$\begin{aligned} \Phi = & \sum_{\oplus_j \mathcal{A}_{v_j}} \epsilon_+(s, \{\alpha_j\}) \langle \{\alpha_j\} | [U(s_0), U(s)] | \{\alpha_j\} \rangle \\ & + \sum_{\oplus_j \mathcal{A}_{v_j}} \epsilon_-(s, \{\alpha_j\}) \langle \{\alpha_j\} | [U(s_0), U(s)]^{-1} | \{\alpha_j\} \rangle, \end{aligned} \quad (\text{D16})$$

where $\epsilon_{\pm}$ is the positive integer coefficient counting the number of the given commutator appearing in the sum of Φ . We again define ϵ as

$$\epsilon(\{\alpha_j\}) = \epsilon_+(s, \{\alpha_j\}) - \epsilon_-(s, \{\alpha_j\}), \quad (\text{D17})$$

where we suppressed the dependence on s . Then, due to the condition of E_{inv} in Eq. (55) satisfied by the invariant Θ , ϵ satisfies

$$\sum_{\oplus_{j \neq k} \mathcal{A}_{v_j}} \epsilon(\alpha_0, \dots, \alpha_{p+1}) = 0 \quad (\text{D18})$$

for any $\alpha_k \in \mathcal{A}_{v_k}$ and any $0 \leq k \leq p+1$. The sum is over $\alpha_j \in \mathcal{A}_{v_j}$ with $j \neq k$ while α_k is fixed. Once we fix the restricted configuration α_k , it fixes the configuration of the excitation a at p -simplexes $\{\sigma_j\}$ except for σ_k .

Let us define the set $\mathcal{A}_{v_k}(\{a_{\sigma_j}\}) \subset \mathcal{A}_{v_k}$ as a set of α_k , which has the fixed configurations of excitations $\{a_{\sigma_j}\}$ at the p -simplexes $\sigma_0, \dots, \sigma_{p+1}$ of s . Note that this set does not depend on a_{σ_k} .

One can show that the solution to Eq. (D18) is generated by the following basis:

- (1) Let us fix a configuration of excitations $\{a_{\sigma_j}\}$ at the p -simplexes of s , and consider $\mathcal{A}_{v_k}(\{a_{\sigma_j}\})$. Let us pick two elements $\alpha_k^{(0)}, \alpha_k^{(1)} \in \mathcal{A}_{v_k}(\{a_{\sigma_j}\})$ for each $0 \leq k \leq p+1$. Then, the basis is given by

$$\epsilon(\{\alpha_j^{(n_j)}\}) = (-1)^{\sum_j n_j}, \quad (\text{D19})$$

with $n_j = 0, 1$, with the other ϵ zero.

- (2) Let us fix a choice of the vertex v_k . Take two configurations of the excitations $\{a_{\sigma_j}\}, \{a'_{\sigma_j}\}$, where $a_{\sigma_j} = a'_{\sigma_j}$ for $j \neq k$. Pick two elements $\alpha_k^{(0)}, \alpha_k^{(1)} \in \mathcal{A}_{v_k}(\{a_{\sigma_j}\})$. Note that $\alpha_k^{(0)}, \alpha_k^{(1)} \in \mathcal{A}_{v_k}(\{a'_{\sigma_j}\})$. For $l \neq k$, let us pick a set of elements $\{\alpha_l\}_{l \neq k}, \{\alpha'_l\}_{l \neq k}$ from $\{\mathcal{A}_{v_l}(\{a_{\sigma_j}\})\}_{l \neq k}, \{\mathcal{A}_{v_l}(\{a'_{\sigma_j}\})\}_{l \neq k}$. Then, the basis is given by

$$\begin{aligned} \epsilon(\alpha_k^{(0)}, \{\alpha_l\}_{l \neq k}) &= 1, \\ \epsilon(\alpha_k^{(1)}, \{\alpha_l\}_{l \neq k}) &= -1, \\ \epsilon(\alpha_k^{(0)}, \{\alpha'_l\}_{l \neq k}) &= -1, \\ \epsilon(\alpha_k^{(1)}, \{\alpha'_l\}_{l \neq k}) &= 1, \end{aligned} \quad (\text{D20})$$

with the other ϵ zero.

- (3) For each p -simplex σ_j , let us take a pair of configurations of excitation $a_{\sigma_j}^{(0)}, a_{\sigma_j}^{(1)}$. For each configuration $\{a_{\sigma_j}^{(n_j)}\}$ at p -simplexes, we choose a single excitation configuration $\{\alpha_j\}_{\{n_j\}} \in \mathcal{A}(\{a_{\sigma_j}^{(n_j)}\})$. These excitation configurations are chosen so that when $n_j = n'_j$ for $j \neq k$, we have $(\alpha_k)_{\{n_j\}} = (\alpha_k)_{\{n'_j\}}$. Then, the basis is given by

$$\epsilon(\{\alpha_j\}_{\{n_j\}}) = (-1)^{\sum_j n_j}. \quad (\text{D21})$$

Each type of the basis (D19) corresponds to the higher commutators:

- (1) For the first type of the basis, let us take a unitary $U(s_{\alpha_k})$ that transforms $\alpha_k^{(0)}$ into $\alpha_k^{(1)}$, leaving the other $\{\alpha_j\}$ invariant. Note that the mutual support of the unitaries $\text{supp}(s) \cap \text{supp}(s_{\alpha_k})$ becomes a 0-simplex v_k . In particular, this implies that

$$\left(\bigcap_{j=0}^{p+1} \text{supp}(s_{\alpha_j}) \right) \cap \text{supp}(s) = \emptyset. \quad (\text{D22})$$

Now, the basis (D19) corresponds to the invariant given by the following higher commutator:

$$\langle \{\alpha_j^{(0)}\} | [U(s_{\alpha_0}), [\dots, [U(s_{\alpha_{p+1}}), [U(s_0), U(s)]]]] | \{\alpha_j^{(0)}\} \rangle. \quad (\text{D23})$$

- (2) For the second type of the basis (D20), let us take a unitary $U(s_{\alpha_k})$ that transforms $\alpha_k^{(0)}$ into $\alpha_k^{(1)}$, leaving the other $\{\alpha_j\}$ invariant. Let us also take a unitary $U(s_{\hat{k}})$ that transforms $\{\alpha_l\}_{l \neq k}$ into $\{\alpha'_l\}_{l \neq k}$, leaving α_k invariant. The mutual support of the unitaries becomes $\text{supp}(s_{\alpha_k}) \cap \text{supp}(s) = v_k$, $\text{supp}(s_{\hat{k}}) \cap \text{supp}(s) = \sigma_{\hat{k}}$. We then have

$$\text{supp}(s_{\alpha_k}) \cap \text{supp}(s_{\hat{k}}) \cap \text{supp}(s) = \emptyset. \quad (\text{D24})$$

The basis (D20) corresponds to the invariant given by the following higher commutator:

$$\langle \alpha_k^{(0)}, \{\alpha_l\}_{l \neq k} | [U(s_{\alpha_k}), [U(s_{\hat{k}}), [U(s_0), U(s)]]] | \alpha_k^{(0)}, \{\alpha_l\}_{l \neq k} \rangle. \quad (\text{D25})$$

- (3) For the third type of the basis (D21), let us take a unitary $U(s_{\hat{k}})$ that transforms $\{\alpha_j\}_{\{n_j\}}$ into $\{\alpha_j\}_{\{n'_j\}}$, with $n'_k = n_k + 1 \bmod 2$, and $n_j = n'_j$ with $j \neq k$. Note that the mutual support $\text{supp}(s) \cap \text{supp}(s_{\hat{k}})$ becomes a p -simplex $\sigma_{\hat{k}}$. This implies that

$$\left(\bigcap_{j=0}^{p+1} \text{supp}(s_{\hat{j}}) \right) \cap \text{supp}(s) = \emptyset. \quad (\text{D26})$$

The basis (D21) corresponds to the invariant given by the following higher commutator:

$$\langle \{\alpha_j\}_{\{n_j=0\}} | [U(s_{\hat{0}}), [\dots, [U(s_{\widehat{p+1}}), [U(s_0), U(s)]]]] | \{\alpha_j\}_{\{n_j=0\}} \rangle. \quad (\text{D27})$$

This completes the proof that Φ is given by linear combinations of the locality identities.

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- [1] S. N. Bose, *Plancks gesetz und lichtquantenhypothese*, *Z. Phys.* **26**, 178 (1924).
- [2] A. Einstein, *Quantentheorie des einatomigen idealen gases*, Albert Einstein: Akademie-Vorträge (2005), pp. 237–244.
- [3] L. Landau, *Theory of the superfluidity of helium II*, *Phys. Rev.* **60**, 356 (1941).
- [4] L. Onsager, *Statistical hydrodynamics*, *Il Nuovo Cimento* (1943-1954) **6**, 279 (1949).
- [5] V. L. Ginzburg and L. D. Landau, *On the theory of superconductivity*, *Zh. Eksp. Teor. Fiz.* **20**, 1064 (1950).
- [6] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Theory of superconductivity*, *Phys. Rev.* **108**, 1175 (1957).
- [7] Zheng-Cheng Gu and Xiao-Gang Wen, *Symmetry-protected topological orders for interacting fermions: Fermionic topological nonlinear σ models and a special group supercohomology theory*, *Phys. Rev. B* **90**, 115141 (2014).
- [8] Davide Gaiotto and Anton Kapustin, *Spin TQFTs and fermionic phases of matter*, *Int. J. Mod. Phys. A* **31**, 1645044 (2016).
- [9] Lakshya Bhardwaj, Davide Gaiotto, and Anton Kapustin, *State sum constructions of spin-TFTs and string net constructions of fermionic phases of matter*, *J. High Energy Phys.* **04** (2017) 096.
- [10] Maissam Barkeshli, Yu-An Chen, Po-Shen Hsin, and Naren Manjunath, *Classification of $(2+1)$ D invertible fermionic topological phases with symmetry*, *Phys. Rev. B* **105**, 235143 (2022).
- [11] Frank Wilczek, *Quantum mechanics of fractional-spin particles*, *Phys. Rev. Lett.* **49**, 957 (1982).
- [12] R. B. Laughlin, *Anomalous quantum Hall effect: An incompressible quantum fluid with fractionally charged excitations*, *Phys. Rev. Lett.* **50**, 1395 (1983).
- [13] Horst L. Stormer, *Nobel lecture: The fractional quantum Hall effect*, *Rev. Mod. Phys.* **71**, 875 (1999).
- [14] Gregory Moore and Nicholas Read, *Non-Abelions in the fractional quantum Hall effect*, *Nucl. Phys.* **B360**, 362 (1991).
- [15] Eric Dennis, Alexei Kitaev, Andrew Landahl, and John Preskill, *Topological quantum memory*, *J. Math. Phys.* (N.Y.) **43**, 4452 (2002).
- [16] Michael H. Freedman, Michael Larsen, and Zhenghan Wang, *A modular functor which is universal for quantum computation*, *Commun. Math. Phys.* **227**, 605 (2002).
- [17] Michael Freedman, Alexei Kitaev, Michael Larsen, and Zhenghan Wang, *Topological quantum computation*, *Bull. Am. Math. Soc.* **40**, 31 (2003).
- [18] A. Yu. Kitaev, *Fault-tolerant quantum computation by anyons*, *Ann. Phys. (Amsterdam)* **303**, 2 (2003).
- [19] Chetan Nayak, Steven H. Simon, Ady Stern, Michael Freedman, and Sankar Das Sarma, *Non-Abelian anyons and topological quantum computation*, *Rev. Mod. Phys.* **80**, 1083 (2008).
- [20] Zijian Liang, Ke Liu, Hao Song, and Yu-An Chen, *Generalized toric codes on twisted tori for quantum error correction*, *PRX Quantum* **6**, 020357 (2025).
- [21] Keyang Chen, Yuanling Liu, Yiming Zhang, Zijian Liang, Yu-An Chen, Ke Liu, and Hao Song, *Anyon theory and topological frustration of high-efficiency quantum low-density parity-check codes*, *Phys. Rev. Lett.* **135**, 076603 (2025).
- [22] Alexei Kitaev, *Anyons in an exactly solved model and beyond*, *Ann. Phys. (Amsterdam)* **321**, 2 (2006).
- [23] Michael A. Levin and Xiao-Gang Wen, *String-net condensation: A physical mechanism for topological phases*, *Phys. Rev. B* **71**, 045110 (2005).
- [24] Eric C. Rowell, *From quantum groups to unitary modular tensor categories*, *arXiv:math/0503226*.
- [25] Alexei Kitaev and Liang Kong, *Models for gapped boundaries and domain walls*, *Commun. Math. Phys.* **313**, 351 (2012).
- [26] Liang Kong, *Anyon condensation and tensor categories*, *Nucl. Phys.* **B886**, 436 (2014).
- [27] Chenjie Wang and Michael Levin, *Braiding statistics of loop excitations in three dimensions*, *Phys. Rev. Lett.* **113**, 080403 (2014).

- [28] Shenghan Jiang, Andrej Mesaros, and Ying Ran, *Generalized modular transformations in $(3+1)$ D topologically ordered phases and triple linking invariant of loop braiding*, *Phys. Rev. X* **4**, 031048 (2014).
- [29] Chao-Ming Jian and Xiao-Liang Qi, *Layer construction of 3D topological states and string braiding statistics*, *Phys. Rev. X* **4**, 041043 (2014).
- [30] Juven C. Wang, Zheng-Cheng Gu, and Xiao-Gang Wen, *Field-theory representation of gauge-gravity symmetry-protected topological invariants, group cohomology, and beyond*, *Phys. Rev. Lett.* **114**, 031601 (2015).
- [31] Chenjie Wang, Chien-Hung Lin, and Michael Levin, *Bulk-boundary correspondence for three-dimensional symmetry-protected topological phases*, *Phys. Rev. X* **6**, 021015 (2016).
- [32] Peng Ye and Zheng-Cheng Gu, *Topological quantum field theory of three-dimensional bosonic Abelian-symmetry-protected topological phases*, *Phys. Rev. B* **93**, 205157 (2016).
- [33] Pavel Putrov, Juven Wang, and Shing-Tung Yau, *Braiding statistics and link invariants of bosonic/fermionic topological quantum matter in $2+1$ and $3+1$ dimensions*, *Ann. Phys. (Amsterdam)* **384**, 254 (2017).
- [34] AtMa P. O. Chan, Peng Ye, and Shinsei Ryu, *Braiding with borromean rings in $(3+1)$ -dimensional spacetime*, *Phys. Rev. Lett.* **121**, 061601 (2018).
- [35] Qing-Rui Wang, Meng Cheng, Chenjie Wang, and Zheng-Cheng Gu, *Topological quantum field theory for Abelian topological phases and loop braiding statistics in $(3+1)$ -dimensions*, *Phys. Rev. B* **99**, 235137 (2019).
- [36] Zhi-Feng Zhang and Peng Ye, *Compatible braidings with hopf links, multiloop, and borromean rings in $(3+1)$ -dimensional spacetime*, *Phys. Rev. Res.* **3**, 023132 (2021).
- [37] Zhi-Feng Zhang and Peng Ye, *Topological orders, braiding statistics, and mixture of two types of twisted BF theories in five dimensions*, *J. High Energy Phys.* **04** (2022) 138.
- [38] Maissam Barkeshli, Yu-An Chen, Sheng-Jie Huang, Ryohei Kobayashi, Nathanan Tantivasadakarn, and Guanyu Zhu, *Codimension-2 defects and higher symmetries in $(3+1)$ D topological phases*, *SciPost Phys.* **14**, 065 (2023).
- [39] Maissam Barkeshli, Yu-An Chen, Po-Shen Hsin, and Ryohei Kobayashi, *Higher-group symmetry in finite gauge theory and stabilizer codes*, *SciPost Phys.* **16**, 089 (2024).
- [40] Ryan Thorngren, *Framed Wilson operators, fermionic strings, and gravitational anomaly in 4D*, *J. High Energy Phys.* **02** (2015) 152.
- [41] Ryohei Kobayashi, Kantaro Ohmori, and Yuji Tachikawa, *On gapped boundaries for SPT phases beyond group cohomology*, *J. High Energy Phys.* **11** (2019) 131.
- [42] Theo Johnson-Freyd, *$(3+1)$ D topological orders with only a $\mathbb{Z}_2$ -charged particle*, *arXiv:2011.11165*.
- [43] Po-Shen Hsin, Wenjie Ji, and Chao-Ming Jian, *Exotic invertible phases with higher-group symmetries*, *arXiv:2105.09454*.
- [44] Yu-An Chen and Po-Shen Hsin, *Exactly solvable lattice Hamiltonians and gravitational anomalies*, *SciPost Phys.* **14**, 089 (2023).
- [45] Lukasz Fidkowski, Jeongwan Haah, and Matthew B. Hastings, *Gravitational anomaly of $(3+1)$ -dimensional $\mathbb{Z}_2$ toric code with fermionic charges and fermionic loop self-statistics*, *Phys. Rev. B* **106**, 165135 (2022).
- [46] Srivatsa Tata, Ryohei Kobayashi, Daniel Bulmash, and Maissam Barkeshli, *Anomalies in $(2+1)$ D fermionic topological phases and $(3+1)$ D path integral state sums for fermionic spts*, *Commun. Math. Phys.* **397**, 199 (2022).
- [47] Jeongwan Haah, Lukasz Fidkowski, and Matthew B. Hastings, *Nontrivial quantum cellular automata in higher dimensions*, *Commun. Math. Phys.* **398**, 469 (2023).
- [48] Yu-An Chen and Sri Tata, *Higher cup products on hypercubic lattices: Application to lattice models of topological phases*, *J. Math. Phys. (N.Y.)* **64**, 091902 (2023).
- [49] Lukasz Fidkowski, Jeongwan Haah, and Matthew B. Hastings, *A QCA for every SPT*, *arXiv:2407.07951*.
- [50] Po-Shen Hsin, Ho Tat Lam, and Nathan Seiberg, *Comments on one-form global symmetries and their gauging in 3D and 4D*, *SciPost Phys.* **6**, 039 (2019).
- [51] Liang Wang and Zhenghan Wang, *In and around Abelian anyon models*, *J. Phys. A* **53**, 505203 (2020).
- [52] Maissam Barkeshli, Hong-Chen Jiang, Ronny Thomale, and Xiao-Liang Qi, *Generalized Kitaev models and extrinsic non-Abelian twist defects*, *Phys. Rev. Lett.* **114**, 026401 (2015).
- [53] Yu-An Chen, Anton Kapustin, and Djordje Radicevic, *Exact bosonization in two spatial dimensions and a new class of lattice gauge theories*, *Ann. Phys. (Amsterdam)* **393**, 234 (2018).
- [54] Yu-An Chen and Anton Kapustin, *Bosonization in three spatial dimensions and a 2-form gauge theory*, *Phys. Rev. B* **100**, 245127 (2019).
- [55] Yu-An Chen, *Exact bosonization in arbitrary dimensions*, *Phys. Rev. Res.* **2**, 033527 (2020).
- [56] Yu-Xin Yang, Meng Cheng, and Ji-Yao Chen, *Chiral spin liquid in a generalized kitaev honeycomb model with z_4 1-form symmetry*, *arXiv:2408.02046*.
- [57] More precisely, an anomalous higher-form symmetry S cannot preserve any short-range-entangled state; that is, there is no state $|\Psi\rangle$ obtained from a product state by a finite-depth quantum circuit that satisfies $S|\Psi\rangle \propto |\Psi\rangle$.
- [58] Michael Levin and Zheng-Cheng Gu, *Braiding statistics approach to symmetry-protected topological phases*, *Phys. Rev. B* **86**, 115109 (2012).
- [59] Tyler D. Ellison, Yu-An Chen, Arpit Dua, Wilbur Shirley, Nathanan Tantivasadakarn, and Dominic J. Williamson, *Pauli stabilizer models of twisted quantum doubles*, *PRX Quantum* **3**, 010353 (2022).
- [60] Tyler D. Ellison, Yu-An Chen, Arpit Dua, Wilbur Shirley, Nathanan Tantivasadakarn, and Dominic J. Williamson, *Pauli topological subsystem codes from Abelian anyon theories*, *Quantum* **7**, 1137 (2023).
- [61] Wilbur Shirley, Yu-An Chen, Arpit Dua, Tyler D. Ellison, Nathanan Tantivasadakarn, and Dominic J. Williamson, *Three-dimensional quantum cellular automata from chiral semion surface topological order and beyond*, *PRX Quantum* **3**, 030326 (2022).

- [62] Li-Mei Chen, Tyler D. Ellison, Meng Cheng, Peng Ye, and Ji-Yao Chen, *Chiral spin liquid in a F_3 Kitaev model*, *Phys. Rev. B* **109**, 155161 (2024).
- [63] Blazej Ruba and Bowen Yang, *Homological invariants of Pauli stabilizer codes*, *Commun. Math. Phys.* **405**, 126 (2024).
- [64] Zijian Liang, Yijia Xu, Joseph T. Iosue, and Yu-An Chen, *Extracting topological orders of generalized Pauli stabilizer codes in two dimensions*, *PRX Quantum* **5**, 030328 (2024).
- [65] Zijian Liang, Bowen Yang, Joseph T. Iosue, and Yu-An Chen, *Operator algebra and algorithmic construction of boundaries and defects in $(2+1)$ D topological Pauli stabilizer codes*, *arXiv:2410.11942*.
- [66] Zijian Liang, Jens Niklas Eberhardt, and Yu-An Chen, *Planar quantum low-density parity-check codes with open boundaries*, *PRX Quantum* **6**, 040330 (2025).
- [67] Michael Levin and Xiao-Gang Wen, *Fermions, strings, and gauge fields in lattice spin models*, *Phys. Rev. B* **67**, 245316 (2003).
- [68] Kyle Kawagoe and Michael Levin, *Microscopic definitions of anyon data*, *Phys. Rev. B* **101**, 115113 (2020).
- [69] In the literature, this unitary operator is sometimes referred to as a shallow-depth circuit, meaning that the circuit depth is small compared to the system size under consideration.
- [70] In condensed matter physics, a particle is called deconfined if it can be separated from its antiparticle at an arbitrarily large distances without incurring an energy cost that grows with the distance. In contrast, a confined particle has an energy cost proportional to the separation between excitations, since the corresponding string operator $U(s)$ violates Hamiltonian terms all along its length, not just near the end points.
- [71] Anton Kapustin and Ryan Thorngren, *Higher symmetry and gapped phases of gauge theories*, *Progress Math.* **324**, 177 (2017).
- [72] Francesco Benini, Clay Córdova, and Po-Shen Hsin, *On 2-group global symmetries and their anomalies*, *J. High Energy Phys.* **03** (2019) 118.
- [73] Qing-Rui Wang and Meng Cheng (unpublished).
- [74] Elliott Lieb, Theodore Schultz, and Daniel Mattis, *Two soluble models of an antiferromagnetic chain*, *Ann. Phys. (N.Y.)* **16**, 407 (1961).
- [75] Masaki Oshikawa, *Commensurability, excitation gap, and topology in quantum many-particle systems on a periodic lattice*, *Phys. Rev. Lett.* **84**, 1535 (2000).
- [76] M. B. Hastings, *Lieb-schultz-mattis in higher dimensions*, *Phys. Rev. B* **69**, 104431 (2004).
- [77] Ryohei Kobayashi, Ken Shiozaki, Yuta Kikuchi, and Shinsei Ryu, *Lieb-Schultz-Mattis type theorem with higher-form symmetry and the quantum dimer models*, *Phys. Rev. B* **99**, 014402 (2019).
- [78] Dominic V. Else and Ryan Thorngren, *Topological theory of lieb-schultz-mattis theorems in quantum spin systems*, *Phys. Rev. B* **101**, 224437 (2020).
- [79] Meng Cheng and Nathan Seiberg, *Lieb-Schultz-Mattis, Luttinger, and 't Hooft—anomaly matching in lattice systems*, *SciPost Phys.* **15**, 051 (2023).
- [80] Sahand Seifnashri, *Lieb-Schultz-Mattis anomalies as obstructions to gauging (non-on-site) symmetries*, *SciPost Phys.* **16**, 098 (2024).
- [81] Robbert Dijkgraaf and Edward Witten, *Topological gauge theories and group cohomology*, *Commun. Math. Phys.* **129**, 393 (1990).
- [82] Xie Chen, Zheng-Cheng Gu, Zheng-Xin Liu, and Xiao-Gang Wen, *Symmetry protected topological orders and the group cohomology of their symmetry group*, *Phys. Rev. B* **87**, 155114 (2013).
- [83] Dominic V. Else and Chetan Nayak, *Classifying symmetry-protected topological phases through the anomalous action of the symmetry on the edge*, *Phys. Rev. B* **90**, 235137 (2014).
- [84] Chenjie Wang and Michael Levin, *Topological invariants for gauge theories and symmetry-protected topological phases*, *Phys. Rev. B* **91**, 165119 (2015).
- [85] Nathanan Tantivasadakarn, *Dimensional reduction and topological invariants of symmetry-protected topological phases*, *Phys. Rev. B* **96**, 195101 (2017).
- [86] Carolyn Zhang, *Topological invariants for symmetry-protected topological phase entanglers*, *Phys. Rev. B* **107**, 235104 (2023).
- [87] Xie Chen, Zheng-Xin Liu, and Xiao-Gang Wen, *Two-dimensional symmetry-protected topological orders and their protected gapless edge excitations*, *Phys. Rev. B* **84**, 235141 (2011).
- [88] Wenjie Ji and Xiao-Gang Wen, *Categorical symmetry and noninvertible anomaly in symmetry-breaking and topological phase transitions*, *Phys. Rev. Res.* **2**, 033417 (2020).
- [89] Arkya Chatterjee and Xiao-Gang Wen, *Symmetry as a shadow of topological order and a derivation of topological holographic principle*, *Phys. Rev. B* **107**, 155136 (2023).
- [90] Tyler D. Ellison and Lukasz Fidkowski, *Disentangling interacting symmetry-protected phases of fermions in two dimensions*, *Phys. Rev. X* **9**, 011016 (2019).
- [91] Yu-An Chen, Tyler D. Ellison, and Nathanan Tantivasadakarn, *Disentangling supercohomology symmetry-protected topological phases in three spatial dimensions*, *Phys. Rev. Res.* **3**, 013056 (2021).
- [92] Yu-An Chen and Yijia Xu, *Equivalence between fermion-to-qubit mappings in two spatial dimensions*, *PRX Quantum* **4**, 010326 (2023).
- [93] For $N = 2$, the statistics can be simplified as $Z_4^I(a, b) = [U(a)_C, [U(a)_B, [U(a)_A, U(b)_{A+B+C+D}]]]$.
- [94] A different boundary theory (“anomalous projective semion states”) of this SPT phase was studied in Ref. [95].
- [95] Zitao Wang and Xie Chen, *Twisted gauge theories in three-dimensional walker-wang models*, *Phys. Rev. B* **95**, 115142 (2017).
- [96] In the case of loop excitations, the membrane operators U_{0ij} and U_{0ji} represent the same operator.
- [97] For $N = 2$, a simpler expression is $Z_5(g) := [U(g)_D, [U(g)_C, [U(g)_B, U(g)_A^2]]]$.
- [98] Michael H. Freedman and Matthew B. Hastings, *Double semions in arbitrary dimension*, *Commun. Math. Phys.* **347**, 389 (2016).

- [99] Lukasz Fidkowski, Jeongwan Haah, Matthew B. Hastings, and Nathanan Tantivasadakarn, *Disentangling the generalized double semion model*, *Commun. Math. Phys.* **380**, 1151 (2020).
- [100] Lukasz Fidkowski, Jeongwan Haah, and Matthew B. Hastings, *Exactly solvable model for a 4 + 1D beyond-cohomology symmetry-protected topological phase*, *Phys. Rev. B* **101**, 155124 (2020).
- [101] For simplicity, we typically assume that M is a sphere, ensuring that the ground state has no topological degeneracy. This assumption is valid since generalized statistics are local properties and are insensitive to the global topology.
- [102] One can extend Eq. (57) to a more generic setup
- $$\begin{aligned} & \langle a | [U(s_n), [\dots, [U(s_2), U(s_1)]]] | a \rangle \\ &= \langle b | [U(s_n), [\dots, [U(s_2), U(s_1)]]] | b \rangle, \end{aligned} \quad (\text{D28})$$
- for all states $|a\rangle, |b\rangle$ that are identical at the mutual support of unitaries $\text{supp}(s_1) \cap \dots \cap \text{supp}(s_n)$. Note that this is equivalent to Eq. (57) when $X \setminus \{\text{supp}(s_1) \cap \dots \cap \text{supp}(s_n)\}$ is connected, allowing us to find an additional operator that connects states $|a\rangle$ and $|b\rangle$ without overlapping with the mutual support of unitaries. Therefore, when X is a triangulation of a closed manifold, such as a sphere S^d , it is sufficient to consider Eq. (57). This paper will focus on this case.
- [103] This property follows from the commutator decomposition $[AB, C] = B^{-1}[A, C]B[B, C]$, which expresses the commutator of a product as a product of commutators up to conjugation. Therefore, if all higher commutators involving each $U(s)$ are trivial, then the higher commutators of any product $V = \prod_{s \in S} U(s)^{n_s}$ with $n_s \in \mathbb{Z}$ are also trivial, provided that the intersections of their supports vanish.
- [104] Clay Córdova, Daniel S. Freed, and Constantin Teleman, *Gapped theories have torsion anomalies*, [arXiv:2408.15148](#).
- [105] Ph Mullhaupt, *Review of “SHEAFHOM: Software for sparse integer matrices”*, M. McConnell, *Pure Appl. Math. Q.* **3** (2007), no. 1, part 3, 307–322 (2009).
- [106] The minimal triangulation of S^d is formed by $d + 1$ d -simplices, which are identified as the boundary of a single $(d + 1)$ -simplex. Its vertices and edges form a complete graph K_{d+2} .
- [107] To be precise, T behaves more similar to the homology of Eilenberg-MacLane space: $T = H_{d+2}(B^{d-p}G, \mathbb{Z})$ (but they are isomorphic) [108].
- [108] Hanyu Xue, *Statistics of invertible topological excitations*, [arXiv:2412.07653](#).
- [109] James R. Munkres, *Topology: A First Course* (Prentice-Hall, Inc., New York, 1975), Sec. 7-9.
- [110] Jean-Louis Loday, *Cyclic Homology*, 2nd ed., Grundlehren der mathematischen Wissenschaften (Springer, Berlin, Heidelberg, 1997), pp. XIX, 516, Chap. 13, Sec. II.
- [111] Xinping Yang and Meng Cheng, *Gapped boundary of (4 + 1)d beyond-cohomology bosonic SPT phase*, *Phys. Rev. B* **110**, 045137 (2024).
- [112] Yasurō Tomonaga, *Some properties of Pontryagin Classes Mod 3*, *Trans. Am. Math. Soc.* **117**, 37 (1965).
- [113] S. Bravyi, M. B. Hastings, and F. Verstraete, *Lieb-Robinson bounds and the generation of correlations and topological quantum order*, *Phys. Rev. Lett.* **97**, 050401 (2006).
- [114] Dorit Aharonov and Yonathan Touati, *Quantum circuit depth lower bounds for homological codes*, [arXiv:1810.03912](#).
- [115] Zhi Li, Dongjin Lee, and Beni Yoshida, *How much entanglement is needed for emergent anyons and fermions?*, [arXiv:2405.07970](#).
- [116] To be precise, Eq. (109) states only that the state $|jk\rangle$ has the form of $|jk\rangle = |\psi\rangle_{jk} \otimes |0\rangle_{\overline{jk}}$, where $|\psi\rangle_{jk}$ is a possibly entangled state localized at two vertices j, k . Since a pair of zero-dimensional excitations is disconnected, we need extra work to show that $|\psi\rangle_{jk}$ is factorized as $|\psi\rangle_{jk} = |j\rangle \otimes |k\rangle$. To see this, we consider a state of four Abelian anyons $|jklm\rangle$, which is expressed as either $|jklm\rangle = |\psi\rangle_{jk} \otimes |\psi\rangle_{lm} \otimes |0\rangle_{\overline{jklm}}$ or $|jklm\rangle = |\psi\rangle_{jm} \otimes |\psi\rangle_{kl} \otimes |0\rangle_{\overline{jklm}}$, depending on whether we use $U_{jk}U_{lm}$ or $U_{jm}U_{kl}$ to create them. We then have $|\psi\rangle_{jk} \otimes |\psi\rangle_{lm} = |\psi\rangle_{jm} \otimes |\psi\rangle_{kl}$, which implies that $|\psi\rangle_{jk}$ does not have entanglement between j and k , hence, $|\psi\rangle_{jk} = |j\rangle \otimes |k\rangle$. See Ref. [115] for this discussion on general Abelian anyons.
- [117] To be precise, when we define the phases such as $\theta(U_{j; jkl}^{(0)}, a), \theta(\tilde{U}_{jkl}, a)$, we are fixing the choice of the intermediate states, e.g., $\tilde{U}_{jkl}|a\rangle_A$ in terms of the patch of MPS. This amounts to fixing the MPS at the intermediate edges expressed as the green lines in Fig. 6(b).
- [118] Anton Kapustin and Ryan Thorngren, *Fermionic SPT phases in higher dimensions and bosonization*, *J. High Energy Phys.* **10** (2017) 080.
- [119] Maissam Barkeshli, Po-Shen Hsin, and Ryohei Kobayashi, *Higher-group symmetry of (3 + 1)D fermionic $\mathbb{Z}_2$ gauge theory: Logical CCZ, CS, and T gates from higher symmetry*, *SciPost Phys.* **16**, 122 (2024).
- [120] Ryan Thorngren and Yifan Wang, *Fusion category symmetry I: Anomaly in-flow and gapped phases*, [arXiv:1912.02817](#).
- [121] Carolyn Zhang and Clay Córdova, *Anomalies of (1 + 1)D categorical symmetries*, [arXiv:2304.01262](#).
- [122] Clay Cordova, Po-Shen Hsin, and Carolyn Zhang, *Anomalies of non-invertible symmetries in (3 + 1)D*, [arXiv:2308.11706](#).
- [123] Po-Shen Hsin, Ryohei Kobayashi, and Carolyn Zhang, *Fractionalization of coset non-invertible symmetry and exotic Hall conductance*, *SciPost Phys.* **17**, 095 (2024).
- [124] Hoi Chun Po, Lukasz Fidkowski, Takahiro Morimoto, Andrew C. Potter, and Ashvin Vishwanath, *Chiral Floquet phases of many-body localized bosons*, *Phys. Rev. X* **6**, 041070 (2016).
- [125] D. Gross, V. Nesme, H. Vogts, and R. F. Werner, *Index theory of one dimensional quantum walks and cellular automata*, *Commun. Math. Phys.* **310**, 419 (2012).
- [126] Jie-Yu Zhang, Meng-Yuan Li, and Peng Ye, *Higher-order cellular automata generated symmetry-protected*

- topological phases and detection through multi point strange correlators*, [PRX Quantum](#) **5**, 030342 (2024).
- [127] Lukasz Fidkowski and Matthew B. Hastings, *Pumping chirality in three dimensions*, [Phys. Rev. B](#) **109**, 235142 (2024).
- [128] A key fact is that $\gamma_{14}/a_{14} \pmod{2}$ remains invariant under $\prod_v X_v$, which shifts each $a_i \rightarrow a_i + 1$.
- [129] Here, the vertex 1 is regarded as a zero-dimensional face.
- [130] A. Hatcher, *Algebraic Topology* (Cambridge University Press, Cambridge, England, 2002).
- Correction:* The equal contribution statement for the first three authors has been moved to the first page as a byline footnote.