

Experimentally Probing Entropy Reduction via Iterative Quantum Information Transfer

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Thermodynamic principles governing energy and information are important tools for a deeper understanding and better control of quantum systems. In this work, we experimentally investigate the interplay of the thermodynamic costs and information flow in a quantum system undergoing iterative quantum measurement and feedback. Our study employs a state stabilization protocol involving repeated measurement and feedback on an electronic spin qubit associated with a silicon-vacancy center in diamond, which is strongly coupled to a diamond nanocavity. This setup allows us to verify the fundamental laws of nonequilibrium quantum thermodynamics, including the second law and the fluctuation theorem, both of which incorporate measures of quantum information flow induced by iterative measurement and feedback. We further assess the reducible entropy based on the feedback's causal structure and quantitatively demonstrate the thermodynamic advantages of non-Markovian feedback over Markovian feedback. For that purpose, we extend the theoretical framework of quantum thermodynamics to include the causal structure of the applied feedback protocol. Our work lays the foundation for investigating the entropic and energetic costs of real-time quantum control in various quantum systems.

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I. INTRODUCTION

Recent advances in nonequilibrium thermodynamics [1–6] have revealed the fundamental relationship between information and thermodynamics [7,8]. The reduction of entropy in a system through feedback control by the “Maxwell demon” is fundamentally bounded by the information accessible to the demon, which is characterized by mutual information and information flow. This is reflected in generalizations of the second law of thermodynamics (SL) and the fluctuation theorem (FT) [9–20], in which information and thermodynamic quantities are treated on an

equal footing even in far from equilibrium dynamics. In the classical regime, the generalized thermodynamic principles have been experimentally verified, clarifying the role of the continuous flow of classical information [21–23], beyond single measurement-feedback protocols [24–32]. In the quantum regime [33–38], however, the thermodynamic role of information flow induced by iterative quantum feedback has not been probed experimentally.

Exploring the impact of information flow on quantum thermodynamics is critical to characterize the dynamical behavior of a system under multiple rounds of quantum measurement and feedback, which is practically relevant to various tasks in quantum experiments. Iterative or continuous feedback control can be used to create specific quantum resource states [39,40] and to execute quantum error correction protocols [41–43]. To perform these tasks effectively, it is crucial to find the optimal feedback protocol that fully exploits the available information [44,45]. However, the thermodynamic description of the

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causal structure of quantum control remains to be investigated, unlike the well-explored classical case [11–15,46–49].

In this work, we experimentally explore the role of information flow in quantum thermodynamics using successive quantum feedback on a solid-state spin qubit. Specifically, our quantum system consists of a silicon-vacancy (SiV) defect center coupled to a diamond nanocavity [50,51]. Using our platform, we first experimentally validate the fundamental thermodynamic laws, the SL and FT, which incorporate a measure of quantum information flow, as theoretically predicted in Ref. [18]. These results confirm that the measure of information flow bounds the fundamental capability of quantum feedback control, i.e., the amount of entropy reduction. Moreover, we experimentally explore the role of the causal structure of the feedback protocol, which is responsible for how efficiently the obtained information is used. By turning on and off the dependence of the feedback on the past measurement outcomes, we generate non-Markovian and Markovian control sequences as different feedback causal structures. To properly describe the effect of non-Markovian feedback protocols, we expand the theoretical framework [18] and derive the generalized SL and FT by quantifying the unavailable information due to the constraints on feedback design.

Based on the theoretical framework developed in this study, we experimentally show the increased accessible information and reducible entropy of non-Markovian feedback over Markovian feedback, thus not only highlighting but also quantifying the benefit of non-Markovianity in thermodynamic terms. Our framework offers a new tool set for understanding the role of information flow, optimizing feedback to control quantum systems, learning the energy cost of quantum information processing, and designing efficient quantum heat engines under feedback control in the real-time domain.

II. KEY IDEA: THERMODYNAMICS UNDER FEEDBACK

Originally developed for macroscopic systems, thermodynamics has since been extended to microscopic systems where thermal and quantum fluctuations play a significant role [2,3]. Because of these fluctuations, physical systems follow different trajectories for individual trials, and stochastic thermodynamic quantities can be defined for each trajectory. For example, in an ensemble of orthogonal basis $\{|x\rangle\}_x$ with probability $p(x)$, the stochastic entropy for $|x\rangle$ can be defined as $s \equiv -\ln p(x)$, whose average denoted by $\langle \cdot \rangle$ becomes the von Neumann entropy of $\rho \equiv \sum_x p(x)|x\rangle\langle x|$ [i.e., $\langle s \rangle = S(\rho) \equiv -\text{Tr}[\rho \ln \rho]$].

The SL and FT are the fundamental thermodynamic principles of such stochastic systems. The SL is denoted as $\langle \sigma \rangle \geq 0$ with the entropy production defined as $\sigma \equiv s_f - s_i - \beta Q$ [52–54], where s_f and s_i represent the stochastic entropy of the final and initial states, β is the

inverse temperature of the heat bath, and Q denotes the input heat from the bath. The entropy production quantifies the thermodynamic cost required for quantum dynamics [6]. While the SL concerns only the average value of the entropy production $\langle \sigma \rangle$, the FT, defined as $\langle e^{-\sigma} \rangle = 1$, also includes its higher-order moments, thus describing the system's thermodynamic behavior at the level of fluctuations. The FT plays a central role in nonequilibrium thermodynamics since it leads to the derivation of many thermodynamic laws, including the SL and the fluctuation-dissipation theorems [55,56].

In the presence of feedback by the Maxwell demon, the entropy production can be reduced by the amount of the demon's information gain i , resulting in the generalization of the SL as $\langle \sigma \rangle \geq -\langle i \rangle$ and the FT as $\langle e^{-\sigma-i} \rangle = 1$. For classical systems, the information i is typically characterized by the transfer entropy [12–14]. It is a well-established measure of classical information flow [57], widely used in time series analysis across various fields, including finance [58], neuroscience [59], and biochemistry [60].

For a quantum system under iterative measurement and feedback, shown in Fig. 1(a), the relevant measure of information flow is the quantum-classical-transfer (QC-transfer) entropy [18], denoted as i_{QCT} , and the SL and the FT are generalized as follows (see Appendix B for details):

$$\langle \sigma \rangle \geq -\langle i_{\text{QCT}} \rangle, \quad (1)$$

$$\langle e^{-\sigma-i_{\text{QCT}}} \rangle = 1. \quad (2)$$

The QC-transfer entropy i_{QCT} is defined as the accumulation of newly obtained information in each measurement, with its increment in the n th measurement process given by

$$\Delta i_{\text{QCT}} = -\ln p(b_n|Y_{n-1}) + \ln p(a_n|Y_n), \quad (3)$$

$$\Delta \langle i_{\text{QCT}} \rangle = S(\rho_n|Y_{n-1}) - S(\tau_n|Y_n). \quad (4)$$

Here, the n th measurement outcome is expressed as y_n , and all the outcomes until y_n are expressed as $Y_n \equiv (y_1, y_2, \dots, y_n)$. The conditional probabilities $p(b_n|Y_{n-1})$ and $p(a_n|Y_n)$ correspond to the populations of the conditional density operators $\rho_n^{Y_{n-1}}$ and $\tau_n^{Y_n}$ in their respective diagonal bases. These operators represent the system's quantum states immediately before and after the n th measurement, respectively. The indices b_n and a_n label the eigenstates of those diagonalizations, realized in each quantum trajectory. $\Delta \langle i_{\text{QCT}} \rangle$ quantifies how much the conditional entropy $S(\rho|y) \equiv \sum_y p_y S(\rho^y)$ is reduced through the n th measurement process.

Our experiments explore the dynamics of a single quantum spin system under repeated measurement and feedback [Fig. 1(b)]. We extract thermodynamic quantities such as entropy and information gained from measurements to verify and study the thermodynamic laws.

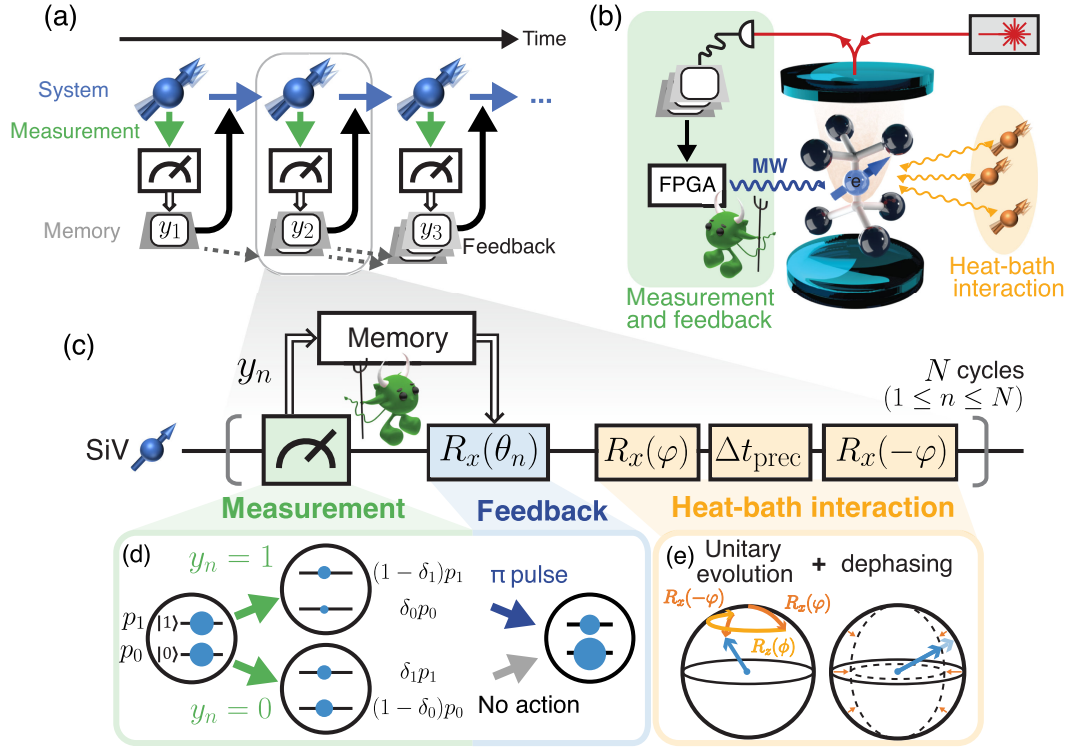


FIG. 1. The silicon-vacancy center under iterative measurement and feedback. (a) Schematic of a quantum system under repeated measurement and feedback. The operation in the n th feedback process is determined by all past measurement outcomes $(y_1, y_2, \dots, y_n)$. (b) Experimental setup. The SiV coupled to a nanophotonic optical cavity in diamond is placed in a cryostat at a 200 mK temperature. A readout laser at ~ 737 nm is sent to the SiV-cavity system through an optical fiber, and the reflected light intensity is monitored with a superconducting nanowire single-photon detector to determine the SiV electron spin state. The measurement results are stored and sent to an FPGA that plays the role of the Maxwell demon and can apply quantum gates on the SiV electronic spin qubit through the MW application. The decoherence mechanism is established through SiV's interaction with a heat bath consisting of spins in the diamond lattice. (c) A detailed experimental sequence of a step shown in (a). The SiV electronic spin qubit is measured, and the result y_n is sent to the Maxwell demon's memory, which then applies the feedback unitary rotation $\hat{R}_x(\theta_n)$ based on the past result(s). The spin then evolves for time Δt_{prec} in a tilted basis set by the prerotation and postrotation gates $\hat{R}_x(\varphi)$ and $\hat{R}_x(-\varphi)$. (d) Details of the measurement and feedback with a specific Markovian feedback. The population change by the measurement and feedback is illustrated separately for $y_n = 1$ and $y_n = 0$, where coherence between $|0\rangle$ and $|1\rangle$ is ignored for simplicity in this figure. (e) Bloch sphere representation of the unitary evolution and dephasing.

Applying weak measurements and implementing both coherent and incoherent quantum evolution in between successive measurement steps allows us to probe regimes of nontrivial evolution of thermodynamic quantities. The outcomes allow us to explore the differences between quantum and classical dynamics and properties of the information flow that depend on the causal structure of implemented feedback protocols.

III. EXPERIMENTAL SYSTEM

We treat the electron spin degree of freedom of the SiV center in a diamond nanocavity in our experimental setup as a two-level system or a qubit with the ground $|0\rangle$ (corresponding to $|\downarrow\rangle$) and excited $|1\rangle$ (corresponding to $|\uparrow\rangle$) states split by an externally applied magnetic field. We iteratively measure the state of the spin qubit in the $\hat{\sigma}_z$ basis. The measurement outcomes are then fed back into the

system through the application of microwave (MW) pulses near resonant ($\Omega_\omega \gg \Delta_\omega$, with Ω_ω and Δ_ω the pulse Rabi frequency and detuning, respectively) with the spin qubit MW transition to stabilize either the ground or excited state. The qubit is then left to evolve for time Δt_{prec} under a constant Hamiltonian while also experiencing dephasing noise through the interaction with the ^{13}C nuclear spin bath in diamond. The ^{13}C nuclear spins are modeled as a heat bath whose inverse temperature is $\beta = 0$. For all experiments, we initialize the qubit in a maximally mixed state so that the success of the target state stabilization implies the entropy reduction $\langle \sigma \rangle = \langle s_f \rangle - \langle s_i \rangle < 0$.

The experimental protocol shown in Fig. 1(c) consists of successive rounds of measurement, feedback, and heat-bath interactions. We measure the qubit by shining a readout laser near the SiV electron resonance and monitoring the reflected photon counts to determine the qubit state. This measurement process can be described by the

positive operator-valued measure (POVM) with operators $E_{y_n=0} \equiv (1 - \delta_0)|0\rangle\langle 0| + \delta_1|1\rangle\langle 1|$ and $E_{y_n=1} \equiv \delta_0|0\rangle\langle 0| + (1 - \delta_1)|1\rangle\langle 1|$, where the parameter δ_0 (δ_1) represents the error probability that $y_n = 1$ ($y_n = 0$) is read out, even though the system is in the $|0\rangle$ ($|1\rangle$) state. The values of δ_0 and δ_1 are adjusted by setting the readout time and tuning the laser frequency with respect to the SiV optical transition, which enables us to explore different information transfer regimes. Details of the optical readout, including the definition and tuning of δ_0 and δ_1 , are provided in Appendix A.

The measurement outcomes are immediately processed by a field programmable gate array (FPGA), and fed back in 50 ns, which is much shorter than the qubit coherence time. In each feedback process, a θ_n rotation around the x axis $\hat{R}_x(\theta_n) \equiv e^{-i\theta_n\hat{\sigma}_x}$ is applied with a MW pulse, where θ_n depends on past measurement outcome(s) through a pre-programmed feedback function.

The system then experiences noise against the stabilization of the $\hat{\sigma}_z$ eigenstates [Fig. 1(e)] through dephasing ($T_2^* \sim 4 \mu\text{s}$) as well as coherent evolution under the Hamiltonian $\hat{H} = \Delta_\omega \hat{\sigma}_z$ for duration Δt_{prec} , where Δ_ω is the detuning between the MW drive frequency and the qubit MW transition frequency. As $T_1 > 1$ s for the SiV, the effect of T_1 -dependent depolarization is not significant and can be ignored. Since both the dephasing and coherent evolution processes leave the $\hat{\sigma}_z$ eigenstates unaffected, we apply rotations $\hat{R}_x(\phi)$ and $\hat{R}_x(-\phi)$ before and after the evolution time Δt_{prec} to effectively implement off-axis (from the measurement basis) evolution of the system. The coherent evolution can then be described as the application of the unitary gate $\hat{V} \equiv \hat{R}_x(\phi)\hat{R}_z(\phi)\hat{R}_x(-\phi)$, where $\phi = \Delta_\omega \Delta t_{\text{prec}}$. The unitary gate $\hat{V}$ introduces the quantum nature into the system's dynamics by generating quantum coherence between $|0\rangle$ and $|1\rangle$. Without $\hat{V}$, the dynamics of the system would be completely classical as a mixture of $|0\rangle$ and $|1\rangle$ states, and thus not suitable for our exploration of thermodynamics in the quantum regime. Additional details about experimental parameters and analysis can be found in Sec. S1 of Supplemental Material [61].

IV. EXPERIMENTAL RESULTS

A. Verification of the generalized thermodynamic laws under Markovian feedback

We begin by experimentally verifying the generalized SL [Eq. (1)] and FT [Eq. (2)] in our system by applying a Markovian feedback protocol. The applied sequence, shown in Fig. 1(d), is Markovian because it does not utilize the memory of past measurements. The resulting feedback pulse angle θ_n depends on the measurement result y_n only: $\theta_n = \pi$ if $y_n = 1$ and $\theta_n = 0$ if $y_n = 0$. This effectively stabilizes the ground state $|0\rangle$. The applied

sequence contains 10 successive cycles of measurement and feedback. To obtain the thermodynamic quantities shown in Fig. 2, we repeated the experimental trial 7000 times and computed the ensemble averages over the sampled trajectories.

Figure 2(a) shows a reduction of the entropy of the system (i.e., $\langle \sigma \rangle < 0$) due to the ground-state stabilization effect of the feedback sequence. At the same time, the entropy is lower bounded by the negative QC-transfer entropy $-\langle i_{\text{QCT}} \rangle$ in accordance with the generalized SL [Eq. (1)]. The observed gap between $\langle \sigma \rangle$ and $-\langle i_{\text{QCT}} \rangle$ reflects the fact that not all of the extracted information is used for entropy reduction, primarily due to the Markovian nature of the feedback protocol, which ignores the past outcomes. We also observe that the classical transfer entropy i_{TE} , while a

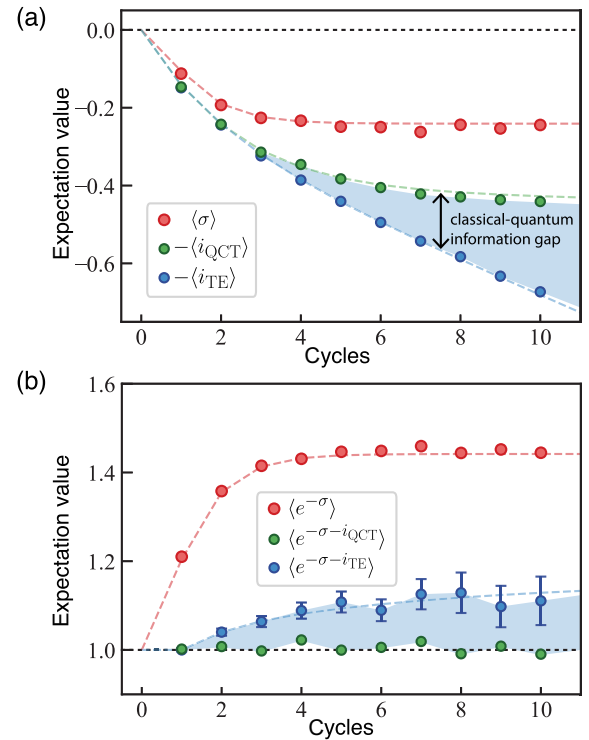


FIG. 2. Evolution of thermodynamic quantities under Markovian feedback. (a) Experimentally obtained entropy production $\langle \sigma \rangle$, negative transfer entropy $-\langle i_{\text{TE}} \rangle$, and negative QC-transfer entropy $-\langle i_{\text{QCT}} \rangle$ against the number of feedback cycles. The blue shaded region illustrates the information loss due to measurement backaction, indicating the gap between $-\langle i_{\text{TE}} \rangle$ and $-\langle i_{\text{QCT}} \rangle$. (b) Statistical averages $\langle e^{-\sigma} \rangle$, $\langle e^{-\sigma - i_{\text{TE}}} \rangle$, and $\langle e^{-\sigma - i_{\text{QCT}}} \rangle$ from experimental data are plotted against the number of feedback cycles. The blue shaded region highlights the gap between the ensemble averages with classical information measure $\langle e^{-\sigma - i_{\text{TE}}} \rangle$ and quantum information measure $\langle e^{-\sigma - i_{\text{QCT}}} \rangle$. In both (a) and (b), error bars, which represent the standard error of the mean, are smaller than the marker sizes except for $\langle e^{-\sigma - i_{\text{TE}}} \rangle$. Dotted lines represent theoretical predictions for the same colored plots, obtained from numerical simulations using the experimental parameters.

well-established measure of classical information flow [57], is not a suitable measure in the quantum regime. The increment of i_{TE} in the n th measurement process, given by $\Delta i_{\text{TE}} = -\ln p(b_n|Y_{n-1}) + \ln p(b_n|Y_n)$, provides a looser bound on the entropy reduction than its quantum counterpart i_{QCT} . This is because i_{TE} does not capture the state change induced by measurement backaction, whereas i_{QCT} does. Further details on this distinction are provided in Appendix D.

Figure 2(b) shows that the generalized FT with QC-transfer entropy Eq. (2) holds, with $\langle e^{-\sigma-i_{\text{QCT}}} \rangle \sim 1$ throughout all measurement and feedback cycles. We observe that the uncorrected statistical average $\langle e^{-\sigma} \rangle$ deviates from 1 due to the presence of feedback control, mirroring the negative entropy production shown in Fig. 2(a). We can also observe that the classical transfer entropy i_{TE} does not accurately describe the information flow in this experiment as the generalized FT with classical transfer entropy i_{TE} clearly does not hold, and $\langle e^{-\sigma-i_{\text{TE}}} \rangle$ significantly diverges from 1. We note that since i_{QCT} reduces to i_{TE} under fully classical dynamics, these divergences can also be interpreted as indicating the quantum nature of the system's dynamics (see Appendix D for more information on the relationship between quantum and classical information measures). In both Figs. 2(a) and 2(b), experimental measurements of the thermodynamic quantities are in good agreement with the theoretical predictions. Our results confirm that the QC-transfer entropy is a relevant information measure under iterative quantum feedback, which characterizes thermodynamic quantities both at the ensemble and individual trajectory levels.

B. Non-Markovian feedback and the role of causal structure of feedback in quantum thermodynamics

While the above section treated a memoryless (that is, Markovian) feedback protocol, many feedback protocols contain a more general causal structure incorporating multiple past measurements stored in memory to make an optimum feedback decision. Systems under such non-Markovian feedback are best described by non-Markovian dynamics. In the following, we first extend the theoretical framework to quantify how much information is actually used for the entropy reduction, depending on the causal structure (or Markovianity) of feedback. Based on that, we experimentally demonstrate and measure the thermodynamic advantage of non-Markovian feedback over Markovian feedback.

In order to formulate the causal structure of feedback, we introduce the notion of k th-order Markovian feedback, in which only the last k measurement outcomes are utilized in designing the feedback protocol. The entropy reduction achieved with k th-order Markovian feedback can be characterized in a manner that reflects the restriction imposed on the feedback design. To this end, we introduce the information measure i_{FB}^k , which quantifies the net

amount of information that can be fed back to the system if a k th-order Markovian protocol is employed (see Appendix C for the precise definition). Since measurement outcomes other than the most recent k are not used in such a protocol, the inequality $\langle i_{\text{FB}}^k \rangle \leq \langle i_{\text{QCT}} \rangle$ holds, with $\langle i_{\text{QCT}} \rangle$ being the total information gain available from all measurement outcomes. Another notable property is the inequality $\langle i_{\text{FB}}^k \rangle \leq \langle i_{\text{FB}}^{k'} \rangle$ when $k < k'$, which reflects that higher-order Markovian feedback is less restrictive. We note that the information not fed back to system, $\langle i_{\text{QCT}} \rangle - \langle i_{\text{FB}}^k \rangle$, was introduced and named backward transfer entropy [49] in the classical regime.

Now, a refinement of the SL Eq. (1) under k th-order Markovian feedback is obtained as

$$\langle \sigma \rangle \geq -\langle i_{\text{FB}}^k \rangle. \quad (5)$$

This provides a tighter bound for the entropy reduction by accounting for the Markovianity of feedback, while the inequality (1) serves as the universal bound regardless of the feedback protocol details. The FT can also be generalized by incorporating i_{FB}^k as

$$\langle e^{-\sigma-i_{\text{FB}}^k} \rangle = 1, \quad (6)$$

which reveals the fundamental relationship between σ and i_{FB}^k at the trajectory level.

We apply this theoretical framework to a series of experiments where k th-order Markovian feedback for $k = 1, 2, 3, 4$ is implemented. The experiment still follows the general sequence shown in Fig. 1 but now with the feedback sequences as shown in Figs. 3(a) and 3(b) for $k = 1$ and $k = 2$, respectively. Here, we perform stabilization of the excited state $|1\rangle$ instead of the ground state $|0\rangle$: $\theta_n = \pi$ when all of the last k outcomes are 0 [i.e., $(y_{n-k+1}, \dots, y_n) = (0, \dots, 0)$], and $\theta_n = 0$ otherwise. Measurement results before the last k are erased and overwritten on the FPGA performing the feedback. Note that at the beginning of an experimental trial, when the number of feedback cycles n is smaller than the non-Markovianity k , we set $\theta_n = \pi$ if all of the last n outcomes are 0, and $\theta_n = 0$ otherwise. Again, 10 successive cycles of measurement and feedback are applied for one experimental trial, which is repeated 10000 times.

The data in Fig. 3(c) show an improvement in the excited-state stabilization with an increase in non-Markovianity, or memory length k . The more measurements are taken into consideration, the more accurately the system's state can be determined. The advantage of non-Markovian feedback can also be observed in terms of the information-thermodynamic efficiency η [24,27,35], which quantifies the proportion of total information gain that can be converted to the entropy reduction. This efficiency is defined as

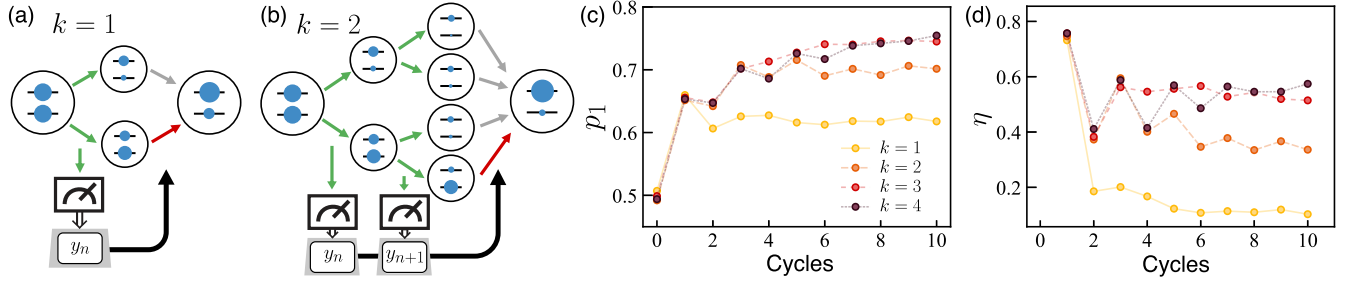


FIG. 3. k th-order Markovian feedback protocols and their performance in the experiments. (a),(b) Schematics for Markovian and second-order Markovian feedback protocols stabilizing the excited state $|1\rangle$. The free evolution between the measurement of y_n and y_{n+1} is not shown in (b) for simplicity. (c) Experimentally measured excited-state populations p_1 under k th-order Markovian feedback ($k = 1, 2, 3, 4$) as a function of the number of feedback cycles. (d) Experimentally measured conversion efficiencies for the same k th-order Markovian feedback conditions ($k = 1, 2, 3, 4$) plotted against the number of feedback cycles. As shown, the non-Markovianity enhances the information-thermodynamic efficiency of the feedback protocol.

$$\eta \equiv \frac{-\langle \sigma \rangle}{\langle i_{\text{QCT}} \rangle}. \quad (7)$$

As shown in Fig. 3(d), η has the same value for all k when the number of feedback cycles n is less than k , while it

increases with k for a larger number of cycles. However, we note that this efficiency does not continue to increase indefinitely with increasing k : The improvement from $k = 3$ to $k = 4$ is less significant than that from $k = 1$ to $k = 2$. A more detailed discussion of this behavior is provided in Supplemental Material [61].

Figure 4(a) shows the entropy reduction $\langle \sigma \rangle < 0$ as a result of the excited-state stabilization. $\langle \sigma \rangle$ is always lower bounded by $-\langle i_{\text{FB}}^k \rangle$ in accordance with Eq. (5). However, it is not lower bounded by the Markovian feedback information value $-\langle i_{\text{FB}}^1 \rangle$ when the applied sequence is non-Markovian, e.g., with $k \geq 2$. This is highlighted with the shaded purple region in Fig. 4(a), explicitly representing the thermodynamic advantage of non-Markovian feedback over Markovian feedback. We note that the difference in $-\langle i_{\text{FB}}^1 \rangle$ between $k = 1$ and $k = 4$ in Fig. 4(a) is due to slight run-to-run variations in experimental parameters (see Supplemental Material Sec. S1 for more details [61]). Figure 4(b) shows the generalized FT [Eq. (6)] incorporating feedback-induced information flow i_{FB}^k . The modified FT is satisfied for all k with $\langle e^{-\sigma - i_{\text{FB}}^k} \rangle$ staying near unity throughout the experiment. This demonstrates the relevance of i_{FB}^k to thermodynamic benefit at a single trajectory level.

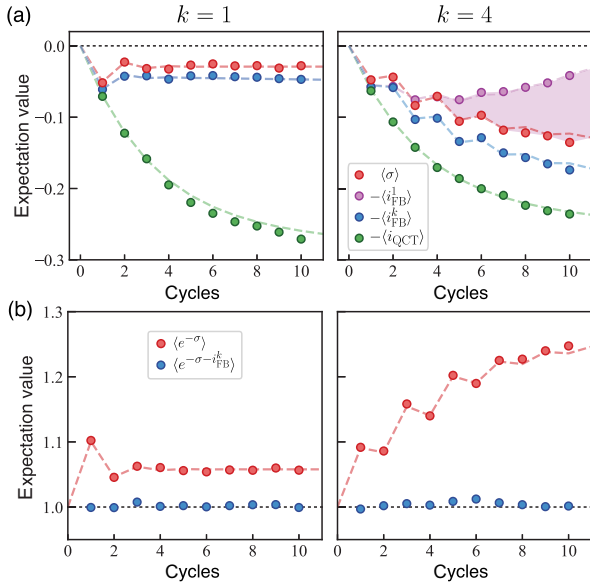


FIG. 4. Generalized second law and fluctuation theorem under k th-order Markovian feedback. (a) Experimentally measured entropy production $\langle \sigma \rangle$, lower bound for the entropy production under Markovian feedback $-\langle i_{\text{FB}}^1 \rangle$, negative values of the net information fed back through k th-order Markovian feedback $-\langle i_{\text{FB}}^k \rangle$, and negative QC-transfer entropy $-\langle i_{\text{QCT}} \rangle$ are plotted against the number of feedback cycles for $k = 1, 4$. The purple shaded region indicates the entropy reduction beyond the Markovian bound $\langle \sigma \rangle < -\langle i_{\text{FB}}^1 \rangle$. (b) Experimentally measured $\langle e^{-\sigma} \rangle$ and $\langle e^{-\sigma - i_{\text{FB}}^k} \rangle$ are plotted against the number of feedback cycles. In both (a) and (b), dotted lines represent theoretical predictions corresponding to the same colored plots. Error bars, which represent the standard error of the mean, are smaller than the marker sizes.

V. CONCLUSION

In this work, we experimentally verified the generalized second law of thermodynamics and fluctuation theorem, which incorporate measures of quantum information flow, using an SiV center coupled to a diamond nanocavity. Through a series of experiments varying the Markovianity and memory length of feedback protocols, we quantified the thermodynamic advantage of non-Markovian over Markovian feedback. To enable the analysis of the experimental protocols, we have extended the theoretical framework to describe the thermodynamic benefits resulting from the causal structure of the feedback. Our work directly probes the fundamental relationship between

thermodynamic benefit and information flow in quantum systems, and explicitly reveals the thermodynamic advantage of non-Markovianity in thermodynamic terms.

While we focus on a simple quantum system in this experiment, our work provides a framework that could be extended to more general and complex quantum systems. For example, it would be interesting to experimentally investigate finite temperature regime $\beta > 0$, where the energetic contribution also plays a significant role. Studies in these systems could yield new insights for improved quantum control. The applications include efficient quantum feedback design for quantum state preparation and stabilization [45,62] and quantum error correction [63,64], which can be seen as processes controllably removing entropy from a quantum system. Furthermore, our work would pave the way for exploring the thermodynamic impact of real-time state estimation such as the quantum Kalman filter [44], which will play an essential role in achieving optimal control.

Since the thermodynamic information measures used in our work put strict bounds on the entropy production of a feedback and measurement process, they tell us the achievable state purity and fidelity. This paves the way for more optimal quantum measurement and feedback strategies. In particular, when detailed prior knowledge of the evolution and measurement implemented in the experiment is available, the information-thermodynamic efficiency can serve as a more appropriate benchmark for evaluating a feedback process than, for example, state fidelity or state purity. This is due to its ability to quantify how close the entropy reduction is to the information extracted from the system through the measurement instead of quantifying how close the system is to a desired target state. In other words, the information-thermodynamic efficiency is saturated when a feedback protocol completely uses the information that is extracted from the system through the measurement, whereas the state fidelity or purity is saturated only when the system is perfectly stabilized. Furthermore, since in the classical regime measures of classical information flow such as the transfer entropy are relevant in feedback efficiency evaluation [48], we expect their quantum counterparts covered in our work to play an important role in evaluating and improving quantum feedback protocols.

Other quantum systems such as quantum heat engines and batteries, proposed quantum analogs of classical heat engines [65,66], can also benefit from our nonequilibrium thermodynamic framework, including successive quantum feedback control, for example, for the robustness to external noise.

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APPENDIX A: OPTICAL READOUT AND δ_0, δ_1 TUNING

The Purcell-enhanced frequency-dependent reflectivity of the SiV-cavity system exhibits large spin state-dependent reflection intensity contrast around the spinlike optical transitions. The contrast value is set by the SiV-cavity cooperativity, as well as the detuning of the SiV and cavity transitions. The cavity transition frequency is tuned so as to achieve the highest contrast when on resonance with the SiV transition. The SiV spin state is read out in the $\hat{\sigma}_z$ basis by shining light resonant with the spinlike transition for one of the spin states and monitoring the reflected photons. The resulting reflected photon distribution obeys Poissonian statistics with the average photon number dependent on the spin state. The photon distribution is then thresholded with a chosen value n to determine the spin state (e.g., less than n photons corresponds to $|\downarrow\rangle$, n photons or more to $|\uparrow\rangle$). Here, up-spin and down-spin state correspond to the excited state $|1\rangle$ and ground state $|0\rangle$. Every readout consists first of a short readout (y_n), the result of which is stored by a fast FPGA for feedforward decision making, and a longer readout with the result sent to a time tagger for data analysis (z_n). The Kraus operators for this measurement become

$$M_{y_n, z_n} \equiv \begin{cases} \sqrt{1 - \delta_0}|0\rangle\langle 0| & (y_n = 0, z_n = 0) \\ \sqrt{\delta_1}|1\rangle\langle 1| & (y_n = 0, z_n = 1) \\ \sqrt{\delta_0}|0\rangle\langle 0| & (y_n = 1, z_n = 0) \\ \sqrt{1 - \delta_1}|1\rangle\langle 1| & (y_n = 1, z_n = 1). \end{cases} \quad (\text{A1})$$

The corresponding POVM operator for the outcome y_n is defined as $E_{y_n} \equiv \sum_{z_n} M_{y_n, z_n}^\dagger M_{y_n, z_n}$.

The strength of the read—and thus the values of δ_0, δ_1 —can be tuned in two ways. The first way is by changing the number of photons sent to the SiV-cavity system [Figs. 5(e) and 5(f)], with a higher number of incident photons

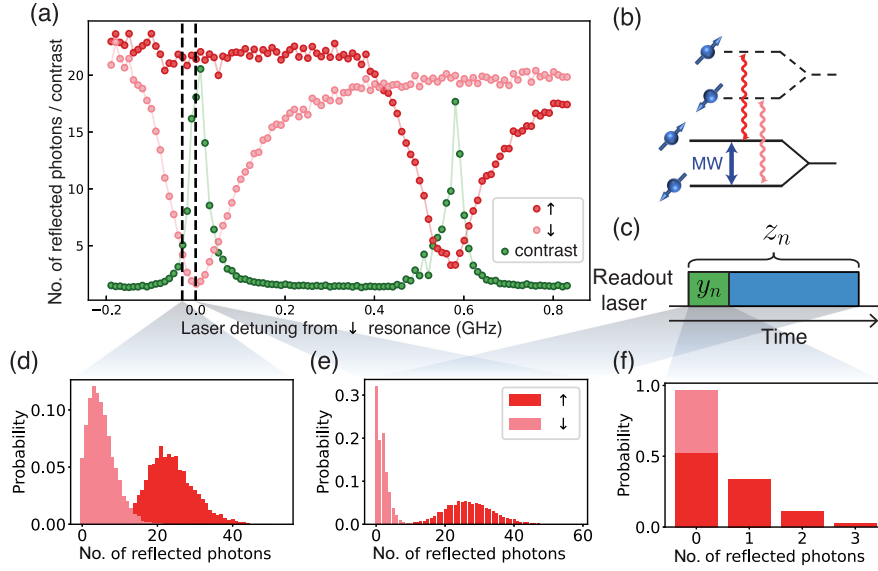


FIG. 5. Optical readout of the SiV electron spin state. (a) Spin state-dependent number of reflected photons and optical contrast in function of readout laser frequency detuning. (b) The energy level structure of the SiV, showing optical transitions in pink and red, and showing microwave-accessible transitions in dark blue. (c) Spin readout sequence, with the shorter readout (y_n) used for feedback and the longer readout (z_n) used for state tomography. Reflected photon count distribution for (d) off-resonant readout [corresponding to laser readout frequency of the left-hand vertical dashed line in (a)], (e) on-resonant readout [corresponding to laser readout frequency of the right-hand vertical dashed line in (a)], and (f) shorter readout.

resulting in an increase of the splitting between the Poisson distributions and thus a stronger read (the splitting increases linearly with average photon number, while the width of the distribution increases with the square root of the average photon number). The second way is by detuning the readout laser from the SiV resonance frequency [Figs. 5(d) and 5(e)], so that the optical contrast is reduced (see Fig. 5) and therefore the splitting between the Poisson distribution is reduced. The parameters δ_0 , δ_1 then obey the equations

$$\delta_0 = \sum_{i=n}^{\infty} \frac{\mu_{\downarrow}^i e^{-\mu_{\downarrow}}}{i!},$$

$$\delta_1 = \sum_{i=0}^{n-1} \frac{\mu_{\uparrow}^i e^{-\mu_{\uparrow}}}{i!},$$

where $\mu_{\downarrow/\uparrow}$ is the average reflected photon number for the $|\downarrow/\uparrow\rangle$ state, and n is the threshold value to distinguish the SiV state based on the number of reflected photons. Experimentally, the parameters δ_0 , δ_1 are explicitly calculated from the sampled data as

$$\delta_0 \equiv \frac{\text{NUM}(y=1, z=0)}{\text{NUM}(y=0, z=0) + \text{NUM}(y=1, z=0)},$$

$$\delta_1 \equiv \frac{\text{NUM}(y=0, z=1)}{\text{NUM}(y=0, z=1) + \text{NUM}(y=1, z=1)},$$

where $\text{NUM}(y, z)$ represents the number that the measurement outcome set (y, z) is sampled in the experiment.

APPENDIX B: THERMODYNAMICS UNDER ITERATIVE QUANTUM FEEDBACK

We review quantum thermodynamics under iterative measurement and feedback and detail the SL and FT generalized with the QC-transfer entropy, given by Eqs. (1) and (2) in the main text. While these generalized thermodynamic laws have already been introduced in Ref. [18], we restate them here in a slightly more general setup, for the completeness of the paper. We note that the generalized SL [Eq. (5)] and FT [Eq. (6)] incorporating i_{FB}^k , which are newly derived in this paper, are discussed in Appendix C.

1. Setup

In this work, we consider the quantum system under iterative measurement and feedback interacting with the heat bath at inverse temperature β , as shown in Fig. 6. This includes not only the situation of sequential discrete measurement and feedback but also the continuous feedback situation where the dynamics is described with the stochastic master equation [44,67]. We describe the outcome of n th measurement as y_n , and all the outcomes until y_n as $Y_n \equiv (y_1, \dots, y_n)$. The total number of feedback cycles is denoted as N , and $\rho_{n-1}^{Y_n}$ and $\tau_n^{Y_n}$ represent the conditional density operators right before and after the n th measurement process, respectively, for $1 \leq n \leq N$.

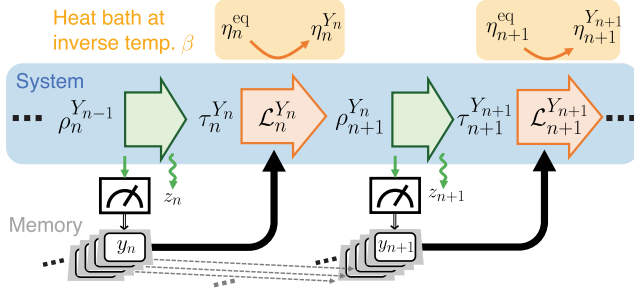


FIG. 6. Schematic of the quantum system under iterative measurement and feedback interacting with a heat bath. Here, y_n denotes the n th measurement outcome and z_n denotes the label that is not stored in memory nor utilized in feedback. The conditional density operators $\rho_{n-1}^{Y_{n-1}}$ and $\tau_n^{Y_n}$ denote the state before and after the n th measurement, respectively, and $\mathcal{L}_n^{Y_n}$ represents the n th feedback process. η_n and $\eta_n^{Y_n}$ are states of the heat bath before and after the n th feedback process, respectively.

The n th measurement is described with the set of Kraus operators $\{M_{y_n, z_n}\}_{y_n, z_n}$, where the label y_n is measured and utilized in the feedback while the label z_n is lost. Therefore, conditional density operator after the n th measurement becomes $P[y_n|Y_{n-1}]\tau_{n-1}^{Y_{n-1}, y_n} = \sum_{z_n} M_{y_n, z_n} \rho_{n-1}^{Y_{n-1}} M_{y_n, z_n}^\dagger$. The n th feedback process and heat-bath interaction are described as $\mathcal{L}_n^{Y_n}(\tau_n^{Y_n}) \equiv \text{Tr}_B[U^{Y_n}(\tau_n^{Y_n} \otimes \eta_n^{\text{eq}})U^{Y_n\dagger}]$, where U^{Y_n} is the unitary operation to the composite system (quantum system and heat bath) depending on the measurement outcomes Y_n , and $\text{Tr}_B[\cdot]$ denotes tracing out the heat-bath degree of freedom. $\eta_n^{\text{eq}} \equiv (e^{-\beta H_{B,n}}/Z_\beta)$ denotes the thermal state of heat bath with Hamiltonian $H_{B,n}$ at inverse temperature β , where $Z_\beta \equiv \text{Tr}[e^{-\beta H_{B,n}}]$ is the partition function. The set of Kraus operators for $\mathcal{L}_n^{Y_n}$, denoted as $\{L_{d_n}^{Y_n}\}_{d_n}$, is represented as $L_{(m,l)}^{Y_n} \equiv \langle E_l|U^{Y_n}|E_m\rangle \sqrt{(e^{-\beta E_m}/Z_\beta)}$, where $|E_l\rangle$ and $|E_m\rangle$ are the energy eigenbases of the heat bath. The operator $L_{(m,l)}^{Y_n}$ represents the transition from $|E_m\rangle$ to $|E_l\rangle$, and the subscript d_n represents the labels (m, l) . We note that Hamiltonian $H_{B,n}$, thermal state η_n^{eq} , and energy eigenbasis $\{|E_l\rangle\}_l$ can also depend on Y_n , while its dependence is abbreviated here for simplicity.

Additionally, we employ a so-called two-time projective measurement scheme, where projective measurements (PMs) are performed before and after the N feedback cycles. The bases for the initial and final PMs are denoted as $\{|b_1\rangle\}$ and $\{|b_{N+1}\rangle\}$. In this case, the initial state ρ_1 must be diagonal in the basis $\{|b_1\rangle\}$, and the final state, which we denote by ρ_{N+1} , becomes diagonal in $\{|b_{N+1}\rangle\}$. The final state ρ_{N+1} is defined as $\rho_{N+1} \equiv \sum_{Y_N} P[Y_N] \rho_{N+1}^{Y_N}$, where the conditional state $\rho_{N+1}^{Y_N}$ is described as $\rho_{N+1}^{Y_N} \equiv \sum_{b_{N+1}} \Pi_{b_{N+1}} \mathcal{L}_N^{Y_N}(\tau_N^{Y_N}) \Pi_{b_{N+1}}$ with the rank-one projector $\Pi_{b_{N+1}} \equiv |b_{N+1}\rangle\langle b_{N+1}|$. These PMs enable us to quantify

the entropy change at the trajectory level, as discussed later in Appendix C. It is the standard scheme of quantum thermodynamics [5,6,68], and has also been employed in some experiments [33,35,36].

Finally, we note that in a series of previous works [19,20,69–72], a thermodynamic framework for general stochastic quantum dynamics has been established, which includes the setup of iterative quantum feedback. However, these works differ from our framework significantly, both in their underlying conceptual approach and in the thermodynamic laws derived. Specifically, their framework aims at establishing thermodynamic consistency by explicitly including ancilla systems (such as memory) and considering thermodynamics at the level of composite system (system and ancilla). In contrast, our framework focuses on how measurement and feedback operations modify the thermodynamic behavior of the system itself. A more detailed discussion of these differences is provided in Supplemental Material [61].

2. Generalized second law

The average heat transferred from the heat bath until the N th feedback process is defined as [5,68]

$$\langle Q \rangle \equiv \sum_{n=1}^N \sum_{Y_n} P[Y_n] \text{Tr}[H_{B,n}(\eta_n^{\text{eq}} - \eta_n^{Y_n})], \quad (\text{B1})$$

where $\eta_n^{Y_n} \equiv \text{Tr}_S[U^{Y_n}(\tau_n^{Y_n} \otimes \eta_n^{\text{eq}})U^{Y_n\dagger}]$ represents the state of n th heat bath after n th feedback process. We also define the heat income when the measurement outcomes are Y_n as $Q_n^{Y_n} \equiv \text{Tr}[H_{B,n}(\eta_n^{\text{eq}} - \eta_n^{Y_n})]$. The averaged entropy production is defined as

$$\langle \sigma \rangle \equiv S(\rho_{N+1}) - S(\rho_1) - \beta \langle Q \rangle, \quad (\text{B2})$$

where $S(\rho) \equiv -\text{Tr}[\rho \ln \rho]$ denotes the von Neumann entropy. The entropy of the initial and final states $S(\rho_1)$ and $S(\rho_{N+1})$ correspond to $\langle s_i \rangle$ and $\langle s_f \rangle$ in the main text. The QC-transfer entropy, which is a measure of total information gain by iterative measurements, is defined as [18]

$$\langle i_{\text{QCT}} \rangle = \sum_{n=1}^N S(\rho_n|Y_{n-1}) - S(\tau_n|Y_n), \quad (\text{B3})$$

where $S(\rho|y) \equiv \sum_y P[y] S(\rho^y)$ represents the conditional von Neumann entropy. Equation (4), which represents the increment of $\langle i_{\text{QCT}} \rangle$ by the n th measurement, is consistent to the definition of Eq. (B3). We will show later in Appendix D 2 that the QC-transfer entropy can be interpreted as the quantum counterpart of the (classical) transfer entropy [57], which is a widely used measure of classical information flow.

With these thermodynamic quantities and information terms, we can derive the generalized SL Eq. (1):

$$\langle \sigma \rangle \geq -\langle i_{\text{QCT}} \rangle.$$

The derivation of this inequality is already performed in Ref. [18]. Also, we will provide the derivation of the tighter second law Eq. (5) in Appendix C 2.

3. Generalized fluctuation theorem

We introduce the definitions of the stochastic entropy production σ and the stochastic QC-transfer entropy i_{QCT} , with which the generalized FT $\langle e^{-\sigma - i_{\text{QCT}}} \rangle = 1$, given by Eq. (6), is derived. For that purpose, we need to properly decompose the averaged dynamics into trajectories by utilizing *fine unraveling* introduced in Ref. [18].

The essence of the fine unraveling is to insert the nondemolition projective measurements before and after the measurement of y_n , in addition to the original iterative measurement and the heat-bath interaction, as shown in Fig. 7. The bases for these inserted PMs are determined by the spectral decompositions of the conditional density operators $\rho_n^{Y_{n-1}}$ and $\tau_n^{Y_n}$. Specifically, denoting their diagonalizations as

$$\rho_n^{Y_{n-1}} \equiv \sum_{b_n} p(b_n | Y_{n-1}) |b_n\rangle \langle b_n|^{Y_{n-1}},$$

$$\tau_n^{Y_n} \equiv \sum_{a_n} p(a_n | Y_n) |a_n\rangle \langle a_n|^{Y_n},$$

the bases for inserted PMs before and after the measurement of y_n are given by the orthonormal bases $\{|b_n\rangle^{Y_{n-1}}\}$ and $\{|a_n\rangle^{Y_n}\}$, respectively. The dependence of the diagonal bases $|b_n\rangle$ and $|a_n\rangle$ on the measurement outcomes Y_{n-1} and Y_n are abbreviated when it is clear from the context. It is noteworthy that their insertions do not destroy the original dynamics at the average level (i.e., the original dynamics is

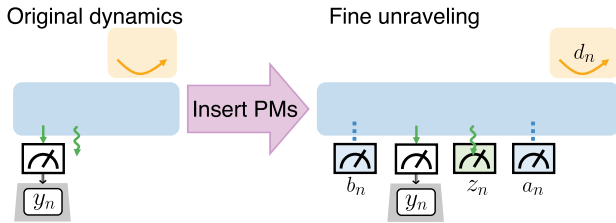


FIG. 7. Unraveling of the system dynamics. In the original dynamics at the ensemble level, only the measurement outcome y_n is read out in the n th cycle. On the other hand, under fine unraveling, the trajectories are designated not only by y_n but also by the other labels b_n , z_n , a_n , and d_n . Here, b_n and a_n denote the outcomes of the inserted PMs, which are performed on the diagonal bases of $\rho_n^{Y_{n-1}}$ and $\tau_n^{Y_n}$, respectively. The label z_n denotes the outcome of the n th measurement that is not stored in memory, and d_n represents the label for the energy change of the heat bath.

recovered by taking the ensemble average over the trajectory), since the bases of these PMs are diagonal with respect to the conditional density operators. We also note that the inserted PM with the diagonal basis of ρ_1 , $\{|b_1\rangle\}$, duplicates with the two-time PMs and thus can be abbreviated.

The fine-unraveled trajectory is designated with the outcomes of iterative measurements $\{y_n\}_{n=1}^N$ and $\{z_n\}_{n=1}^N$, heat-bath monitoring $\{d_n\}_{n=1}^N$, two-time PMs b_1, b_{N+1} , and the inserted PMs $\{b_n\}_{n=2}^N$ and $\{a_n\}_{n=1}^N$. We introduce the notation to denote all these labels together as

$$\psi_N \equiv (\{b_n\}_{n=1}^{N+1}, \{y_n\}_{n=1}^N, \{z_n\}_{n=1}^N, \{a_n\}_{n=1}^N, \{d_n\}_{n=1}^N).$$

The set of Kraus operators for the fine unraveling is described as

$$\left\{ \Pi_{b_{N+1}} \left(\prod_{n=1}^N L_{d_n}^{Y_n} \Pi_{a_n}^{Y_n} M_{y_n, z_n} \Pi_{b_n}^{Y_{n-1}} \right) \right\}_{\psi_N}, \quad (\text{B4})$$

where $\Pi_{b_n}^{Y_{n-1}} \equiv |b_n\rangle \langle b_n|^{Y_{n-1}}$ and $\Pi_{a_n}^{Y_n} \equiv |a_n\rangle \langle a_n|^{Y_n}$ for $1 \leq n \leq N$ represent the rank-one projectors of inserted PMs, and $\Pi_{b_{N+1}} \equiv |b_{N+1}\rangle \langle b_{N+1}|$ is the rank-one projector for the final PM. By utilizing this Kraus operator representation, the probability for the trajectory ψ_N is calculated as

$$P_f[\psi_N] = |\langle b_{N+1} | L_{d_N}^{Y_N} \Pi_{a_N}^{Y_N} M_{y_N, z_N} | b_N \rangle^{Y_{N-1}}|^2$$

$$\times \left\{ \prod_{n=1}^{N-1} |Y_n \langle b_{n+1} | L_{d_n}^{Y_n} \Pi_{a_n}^{Y_n} M_{y_n, z_n} | b_n \rangle^{Y_{n-1}}|^2 \right\} p(b_1), \quad (\text{B5})$$

where the subscript f of $P_f[\psi_N]$ represents the fine unraveling.

Under the fine unraveling, both the stochastic entropy production σ and the stochastic QC-transfer entropy i_{QCT} can be defined to each trajectory ψ_N . The stochastic entropy production is defined as [52–54]

$$\sigma[\psi_N] \equiv -\ln p(b_{N+1}) + \ln p(b_1) - \beta Q[\psi_N], \quad (\text{B6})$$

where the stochastic entropies of the final and initial states are $s_f \equiv -\ln p(b_{N+1})$ and $s_i \equiv -\ln p(b_1)$, respectively, and the stochastic heat input is defined as

$$Q[\psi_N] \equiv -\sum_{n=1}^N \Delta_{d_n}^{Y_n}. \quad (\text{B7})$$

Here, $\Delta_{(m,l)}^{Y_n} \equiv E_l^{Y_n} - E_m^{Y_n}$ represents the energy change of the heat bath due to the interaction with the system, under the condition that the past measurement outcomes are Y_n . The stochastic QC-transfer entropy is defined as [18]

$$i_{\text{QCT}}[\psi_N] \equiv \sum_{n=1}^N -\ln p(b_n|Y_{n-1}) + \ln p(a_n|Y_n), \quad (\text{B8})$$

where $p(b_n|Y_{n-1})$ and $p(a_n|Y_n)$ are diagonal probability distribution for $\rho_{n-1}^{Y_{n-1}}$ and $\tau_n^{Y_n}$, respectively. These stochastic quantities Eqs. (B6), (B7), (B8) are reduced to Eqs. (B2), (B1), and (B3), respectively, by taking the ensemble averages with respect to ψ_N , which means that the definitions of the averaged and stochastic quantities are consistent.

With these stochastic thermodynamic quantities and information term, we can derive the generalized FT [Eq. (2)]:

$$\langle e^{-\sigma - i_{\text{QCT}}} \rangle = 1.$$

The derivation of this equality is provided in Ref. [18].

APPENDIX C: QUANTUM THERMODYNAMICS DEPENDING ON THE CAUSAL STRUCTURE OF FEEDBACK

We here introduce the measures of information transfer under k th-order Markovian feedback, and elaborate on the generalized SL and FT, given by Eqs. (5) and (6) in the main text.

1. Information transfer under k th-order Markovian feedback

To formulate the causal structure, we use the notion of k th-order Markovian feedback, in which only the last k measurement outcomes are utilized in feedback, as shown in Fig. 8. We introduce the notation $Y_n^k \equiv (y_{n-k+1}, \dots, y_{n-1}, y_n)$ to describe the last k measurement outcomes when $n \geq k$, while Y_n^k is reduced to Y_n when $n < k$. The density operators conditioned only on the last k outcomes are described as $\tau_n^{Y_n^k}$ and $\rho_{n+1}^{Y_n^k}$. These operators satisfy the relationships like $P[Y_n^k|\rho_{n+1}^{Y_n^k}] \equiv \sum_{Y_{n-k}} P[Y_n|\rho_{n+1}^{Y_n^k}]$, where $P[Y_n]$ and $P[Y_n^k]$ are the probabilities for the outcomes Y_n and Y_n^k , respectively. Under k th-order Markovian feedback, the n th feedback process and heat-bath interaction depends only on Y_n^k , and therefore the corresponding completely positive and trace-preserving map (CPTP map) can be described as $\mathcal{L}_n^{Y_n^k}(\tau_n^{Y_n^k}) \equiv \text{Tr}_B[U^{Y_n^k}(\tau_n^{Y_n^k} \otimes \eta_n^{\text{eq}})U^{Y_n^k\dagger}]$.

To derive the generalized SL under k th-order Markovian feedback, we first introduce the net information that can be fed back to the system $\langle i_{\text{FB}}^k \rangle$. It is defined as

$$\langle i_{\text{FB}}^k \rangle \equiv \sum_{n=1}^N \{ \chi(\tau_n : Y_n^k) - \chi(\rho_{n+1} : Y_n^k) - \Delta S_n^{\text{BA}} \}, \quad (\text{C1})$$

where $\Delta S_n^{\text{BA}} \equiv S(\tau_n) - S(\rho_n)$ represents the entropy increase due to the backaction of n th measurement, and $\chi(\rho : y) \equiv S(\rho) - \sum_y p_y \rho^y$ denotes the Holevo information [73],

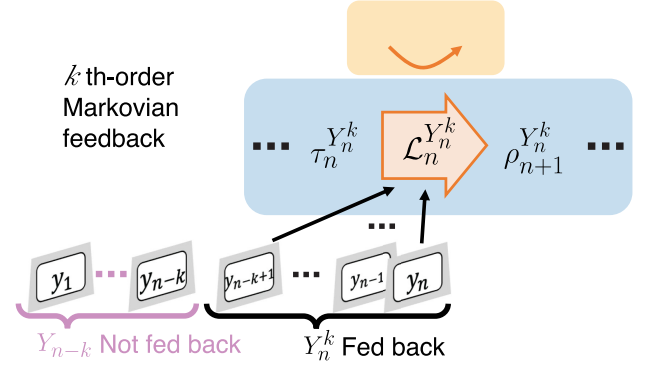


FIG. 8. Schematic of the k th-order Markovian feedback setup. While the last k measurement outcomes Y_n^k are utilized to determine the feedback protocol $\mathcal{L}_n^{Y_n^k}$, it does not depend on the earlier outcomes Y_{n-k} .

a well-known information measure to quantify the correlation between quantum system and classical variable. Considering the quantum-classical state $\sum_y P[y] \rho^y \otimes |y\rangle\langle y|$ by explicitly introducing the classical degree of freedom y as the register system with orthogonal basis $\{|y\rangle\}_y$, the Holevo information is interpreted as the quantum mutual information between quantum system and classical register. We can see from Eq. (C1) that $\langle i_{\text{FB}}^k \rangle$ quantifies the net information fed back, which is decomposed to two contributions: the correlation consumed during the n th feedback process, $\chi(\tau_n : Y_n^k) - \chi(\rho_{n+1} : Y_n^k)$, and the information lost due to the n th measurement backaction ΔS_n^{BA} .

Under k th-order Markovian feedback, $\langle i_{\text{FB}}^k \rangle$ satisfies the inequality

$$\langle i_{\text{FB}}^k \rangle \leq \langle i_{\text{FB}}^{k'} \rangle \leq \langle i_{\text{QCT}} \rangle, \quad \text{if } k' > k. \quad (\text{C2})$$

This inequality shows that the information fed back into the system is always smaller than the total information gain, and it decreases with the non-Markovianity of feedback k . The first inequality of Eq. (C2) is shown as

$$\begin{aligned} \langle i_{\text{FB}}^{k'} \rangle - \langle i_{\text{FB}}^k \rangle &= \sum_{n=1}^N \sum_{Y_n^{k'}} P[Y_n^{k'}] \{ S(\tau_n^{Y_n^{k'}} \| \tau_n^{Y_n^k}) - S(\rho_{n+1}^{Y_n^{k'}} \| \rho_{n+1}^{Y_n^k}) \} \\ &\geq 0, \end{aligned}$$

where $S(\rho \| \tau) \equiv \text{Tr}[\rho(\ln \rho - \ln \tau)]$ denotes the quantum relative entropy. We here utilize the relationships $\rho_{n+1}^{Y_n^{k'}} = \mathcal{L}_n^{Y_n^{k'}}(\tau_n^{Y_n^{k'}})$ and $\rho_{n+1}^{Y_n^k} = \mathcal{L}_n^{Y_n^k}(\tau_n^{Y_n^k})$ and the monotonicity of relative entropy under CPTP map $\mathcal{L}_n^{Y_n^k}$ [74]. The second inequality in Eq. (C2) can also be derived as

$$\begin{aligned} \langle i_{\text{QCT}} \rangle - \langle i_{\text{FB}}^k \rangle &= \sum_{n=1}^N \sum_{Y_n} P[Y_n] \{ S(\tau_n^{Y_n} \| \tau_n^{Y_n^k}) - S(\rho_{n+1}^{Y_n} \| \rho_{n+1}^{Y_n^k}) \} \\ &\quad + \chi(\rho_{N+1} : Y_N) \geq 0, \end{aligned}$$

where we use the relationships $\rho_{n+1}^{Y_n} = \mathcal{L}_n^{Y_n}(\tau_n^{Y_n})$ and $\rho_{n+1}^{Y_n'} = \mathcal{L}_n^{Y_n'}(\tau_n^{Y_n'})$, monotonicity of relative entropy under CPTP map $\mathcal{L}_n^{Y_n}$ [74], and the positivity of Holevo information $\chi(\rho_{N+1}: Y_N) \geq 0$ for the derivation.

Here, we further introduce the information measure named backward QC-transfer entropy, which is wasted due to the restriction in feedback structure. As shown in Fig. 8, while a portion of information gain $\langle i_{\text{QCT}} \rangle$ is fed back to system, another portion is wasted without being utilized in the feedback protocol. Under k th-order Markovian feedback, it is defined as

$$I_{\text{BQC}}^k \equiv \langle i_{\text{QCT}} \rangle - \langle i_{\text{FB}}^k \rangle$$

$$= \left\{ \sum_{n=k+1}^N \chi(\tau_n: Y_{n-k} | Y_n^k) - \chi(\rho_{n+1}: Y_{n-k} | Y_n^k) \right\}$$

$$+ \chi(\rho_{N+1}: Y_N). \quad (\text{C3})$$

The conditional Holevo information is defined as $\chi(\rho: y_2 | y_1) \equiv \sum_{y_1} P[y_1] S(\rho^{y_1}) - \sum_{y_1, y_2} P[y_1, y_2] S(\rho^{y_1, y_2})$ for the ensemble of the quantum state ρ^{y_1, y_2} with the probability $P[y_1, y_2]$. Here, $P[y_1] \equiv \sum_{y_2} P[y_1, y_2]$ is the marginal probability, $\rho^{y_1} \equiv \sum_{y_2} \{ (P[y_1, y_2]) / (P[y_1]) \} \rho^{y_1, y_2}$ is the quantum state conditioned only on y_1 , and $\rho \equiv \sum_{y_1, y_2} P[y_1, y_2] \rho^{y_1, y_2}$ is the averaged quantum state. Considering the quantum-classical-classical state $\sum_{y_1, y_2} P[y_1, y_2] \rho^{y_1, y_2} \otimes |y_1\rangle\langle y_1| \otimes |y_2\rangle\langle y_2|$ with the classical registers with orthogonal basis $\{|y_1\rangle\}_{y_1}$ and $\{|y_2\rangle\}_{y_2}$, the conditional Holevo information is equivalent to the conditional quantum mutual information which quantifies the correlation between quantum system and y_2 under the condition of y_1 . Therefore, the term $\chi(\tau_n: Y_{n-k} | Y_n^k)$ quantifies the system-memory correlation which is not utilized in the feedback process from the n th time onward, as shown in Fig. 8, and $\chi(\tau_n: Y_{n-k} | Y_n^k) - \chi(\rho_{n+1}: Y_{n-k} | Y_n^k)$ means the amount of unusable correlation that is lost during the feedback process $\mathcal{L}_n^{Y_n^k}$. Therefore, we can see from Eq. (C3) that I_{BQC}^k quantifies the wasted correlations between the system and memory as the summation of the two contributions: the correlations lost during the dynamics (i.e., the terms in curly brackets) and the correlations remained at the end [i.e., $\chi(\rho_{N+1}: Y_N)$]. We can also prove that when the dynamics is fully classical, I_{BQC}^k is reduced to the (classical) backward transfer entropy [49], which is the measure of wasted classical correlation due to the restriction in the causal structure of feedback (see Appendix D 3 for the proof).

2. Generalized second law under k th-order Markovian feedback

We derive the SL generalized with the net information fed back $\langle i_{\text{FB}}^k \rangle$, described as

$$\langle \sigma \rangle \geq -\langle i_{\text{FB}}^k \rangle \geq -\langle i_{\text{QCT}} \rangle. \quad (\text{C4})$$

The second inequality has already been shown in Appendix C 1 as Eq. (C2). The first inequality of Eq. (C4) can also be derived by utilizing the positivity of the relative entropy, as follows:

$$\langle \sigma \rangle + \langle i_{\text{FB}}^k \rangle$$

$$= \sum_{n=1}^N \sum_{Y_n^k} P[Y_n^k] \{ S(\rho_{n+1}^{Y_n^k}) - S(\tau_n^{Y_n^k}) + \beta \text{Tr}[H_{\text{B},n}(\eta_n^{Y_n^k} - \eta_n^{\text{eq}})] \}$$

$$= \sum_{n=1}^N \sum_{Y_n^k} P[Y_n^k] S(U^{Y_n^k}(\tau_n^{Y_n^k} \otimes \eta_n^{\text{eq}}) U^{Y_n^k \dagger} \| \rho_{n+1}^{Y_n^k} \otimes \eta_n^{\text{eq}}) \geq 0,$$

where $\eta_n^{Y_n^k} \equiv \text{Tr}_S[U^{Y_n^k}(\tau_n^{Y_n^k} \otimes \eta_n^{\text{eq}}) U^{Y_n^k \dagger}]$ represents the state of n th heat bath after the n th feedback process under the condition that the last k measurement outcomes are Y_n^k .

3. Generalized fluctuation theorem under k th-order Markovian feedback

To derive the generalized FT given by Eq. (6), we need to introduce the unraveling such that both the stochastic entropy production σ and stochastic information flow i_{FB}^k are properly defined to each trajectory. Since i_{FB}^k cannot be defined to the fine-unraveled trajectories discussed in Appendix B 3, we here introduce another unraveling named k th-order coarse unraveling, where i_{FB}^k is well defined to each trajectory.

The essential idea of k th-order coarse unraveling is similar to that of fine unraveling, where each trajectory is designated with the labels of iterative measurement, heat-bath interaction, and inserted PMs, as shown in Fig. 7. The only difference from the fine unraveling is that the basis of inserted PMs are the diagonal basis of $\rho_{n-1}^{Y_n^k}$ and $\tau_n^{Y_n^k}$, not $\rho_{n-1}^{Y_n}$ and $\tau_n^{Y_n}$. The density operators $\rho_{n-1}^{Y_n^k}$ and $\tau_n^{Y_n^k}$ are diagonalized as

$$\rho_{n-1}^{Y_n^k} \equiv \sum_{b_n} p(b_n | Y_{n-1}^k) |b_n\rangle\langle b_n|^{Y_{n-1}^k},$$

$$\tau_n^{Y_n^k} \equiv \sum_{a_n} p(a_n | Y_n^k) |a_n\rangle\langle a_n|^{Y_n^k},$$

where the dependence of the diagonal bases $|b_n\rangle$ and $|a_n\rangle$ on the measurement outcomes Y_{n-1}^k and Y_n^k is abbreviated when it is clear from the context. The set of Kraus operators for this unraveling is described as

$$\left\{ \Pi_{b_{N+1}} \left(\prod_{n=1}^N L_{d_n}^{Y_n^k} \Pi_{a_n}^{Y_n^k} M_{y_n, z_n} \Pi_{b_n}^{Y_{n-1}^k} \right) \right\}_{\psi_N}, \quad (\text{C5})$$

where $\Pi_{b_{N+1}}^{Y_N^k} \equiv |b_N\rangle\langle b_N|^{Y_N^k}$ and $\Pi_{a_n}^{Y_n^k} \equiv |a_n\rangle\langle a_n|^{Y_n^k}$ denote the projectors of inserted PMs. Therefore, the probability

for the trajectory ψ_N under k th-order coarse unraveling can be calculated as

$$P_{k,c}[\psi_N] = |\langle b_{N+1} | L_{d_N}^{Y_N} \Pi_{a_N}^{Y_N} M_{y_N, z_N} | b_N \rangle^{Y_{N-1}}|^2 \times \left\{ \prod_{n=1}^{N-1} |\langle b_{n+1} | L_{d_n}^{Y_n} \Pi_{a_n}^{Y_n} M_{y_n, z_n} | b_n \rangle^{Y_{n-1}}|^2 \right\} P(b_1). \quad (\text{C6})$$

Because of the choice of the PM bases, we can recover the original dynamics of density operators $\rho_n^{Y_{n-1}}$ and $\tau_n^{Y_n}$ by taking the ensemble average, if the feedback protocol is k th-order Markovian. We note that as can be seen from Eqs. (B5) and (C6), the probability for the trajectories with the same label ψ_N is different depending on the types of the unraveling.

Under k th-order coarse unraveling, the stochastic entropy production can be defined in the same way as the fine unraveling, as described in Eq. (B6). Furthermore, the stochastic information transfer i_{FB}^k is also well defined:

$$i_{\text{FB}}^k[\psi_N] \equiv \left\{ \sum_{n=1}^N \ln p(a_n | Y_n^k) - \ln p(b_{n+1} | Y_n^k) \right\} + \ln p(b_{N+1}) - \ln p(b_1). \quad (\text{C7})$$

We can confirm that this definition of i_{FB}^k is consistent to the definition of $\langle i_{\text{FB}}^k \rangle$ given by Eq. (C1), by taking the ensemble average. With these stochastic quantities, we can derive the generalized FT described as

$$\langle e^{-\sigma - i_{\text{FB}}^k} \rangle = 1.$$

The derivation of this equality is basically the same as that of Eq. (2) discussed in Ref. [18]. The detailed derivation of Eq. (6) is provided in Supplemental Material [61].

APPENDIX D: RELATIONSHIP BETWEEN QUANTUM AND CLASSICAL INFORMATION MEASURES

When the dynamics are fully classical (i.e., can be described with a classical stochastic process), the quantum information measures are reduced to the corresponding classical measures. Specifically, we show that the QC-transfer entropy $\langle i_{\text{QCT}} \rangle$ and the backward QC-transfer entropy $I_{\text{BQC}}^k \equiv \langle i_{\text{QCT}} \rangle - \langle i_{\text{FB}}^k \rangle$ are reduced to transfer entropy $\langle i_{\text{TE}} \rangle$ [57] and backward transfer entropy I_{BTE}^k [49], respectively.

1. Fully classical dynamics

Our setup of iterative feedback shown in Fig. 6 generally exhibits quantum nature, such as the measurement back-action. We here consider the special case of this setup

where the dynamics are effectively reduced to the classical stochastic process. Specifically, we consider the situation where the measurement operators are described as

$$M_{y_n, z_n} = \sum_i \sqrt{q(y_n, z_n | i)} |i\rangle \langle i|, \quad (\text{D1})$$

and the feedback process satisfies

$$\mathcal{L}_n^{Y_n}(|i\rangle \langle i|) = \sum_j w(j|i, Y_n^k) |j\rangle \langle j|, \quad (\text{D2})$$

where $\{|i\rangle\}_i$ is a set of fixed orthogonal basis. Under these conditions, the system is always described as a classical mixture of $\{|i\rangle\}$ (i.e., both $\{|a_n\rangle^{Y_n^k}\}$ and $\{|b_n\rangle^{Y_{n-1}^k}\}$ are reduced to $\{|i\rangle\}$), and there is no measurement backaction even at the trajectory level (i.e., $a_n = b_n$ for all n). The stochastic process of $\{b_n\}_{n=1}^N$ and $\{y_n\}_{n=1}^N$ in such fully classical situation is described in Fig. 9 with causal networks (see, e.g., Ref. [75] for a review of causal network representation).

In such case, conditional density operators $\rho_n^{Y_{n-1}}$ and $\tau_n^{Y_n}$ can be represented as

$$\rho_n^{Y_{n-1}} = \sum_i p(b_n = i | Y_{n-1}) |i\rangle \langle i|, \\ \tau_n^{Y_n} = \sum_i p(a_n = i | Y_n) |i\rangle \langle i|.$$

Since $\sum_{z_n} M_{y_n, z_n} \rho_n^{Y_{n-1}} M_{y_n, z_n}^\dagger = P[y_n | Y_{n-1}] \tau_n^{Y_n}$ holds, we have the equality

$$p(a_n = i | Y_n) = \frac{\sum_{z_n} q(y_n, z_n | i) p(b_n = i | Y_{n-1})}{P[y_n | Y_{n-1}]}, \quad (\text{D3})$$

which relates the diagonal probability distributions.

2. Transfer entropy

In the setup of iterative quantum feedback, (classical) transfer entropy is defined as follows:

$$i_{\text{TE}} = \sum_{n=1}^N -\ln p(b_n | Y_{n-1}) + \ln p(b_n | Y_n), \quad (\text{D4})$$

$$\langle i_{\text{TE}} \rangle = \sum_{n=1}^N I(b_n : y_n | Y_{n-1}). \quad (\text{D5})$$

Here, conditional probability $p(b_n | Y_n)$ is defined as

$$p(b_n | Y_n) \equiv \frac{p(y_n | b_n, Y_{n-1}) p(b_n | Y_{n-1})}{P[y_n | Y_{n-1}]}, \quad (\text{D6})$$

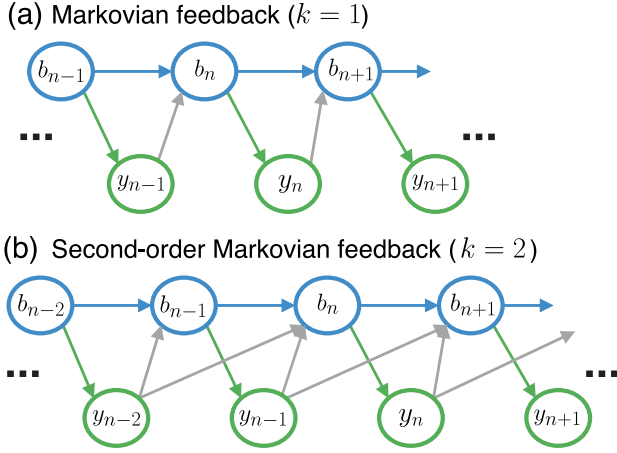


FIG. 9. Causal network representation of the classical stochastic processes under (a) Markovian and (b) second-order Markovian feedback. These fully classical dynamics are considered as special cases of our setup of quantum iterative feedback. Here, $\{b_n\}_{n=1}^N$ are the set of variables representing the system's dynamics, and $\{y_n\}_{n=1}^N$ denotes the set of variables for the measurement outcomes.

with $p(y_n|b_n, Y_{n-1}) \equiv \sum_{z_n} \|M_{y_n, z_n} |b_n\rangle^{Y_{n-1}}\|^2$ and $P[y_n|Y_{n-1}] \equiv \sum_{z_n} \text{Tr}[M_{y_n, z_n} \rho_n^{Y_{n-1}} M_{y_n, z_n}^\dagger]$. The conditional mutual information $I(b_n : y_n | Y_{n-1})$ quantifies the correlation newly generated between the system state $|b_n\rangle^{Y_{n-1}}$ and the memory, in the n th measurement.

When the dynamics is fully classical, we can show i_{TE} is reduced to the QC-transfer entropy, defined as

$$i_{\text{QCT}} \equiv \sum_{n=1}^N -\ln p(b_n|Y_{n-1}) + \ln p(a_n|Y_n).$$

The reduction follows from the fact that $p(a_n|Y_n) = p(b_n|Y_n)$, which implies the absence of measurement backaction. This equality can be derived from Eqs. (D3) and (D6). This reduction implies that if i_{QCT} and i_{TE} take different values, the system dynamics are not fully classical.

We now illustrate the difference between the averaged transfer entropy and the QC-transfer entropy through two simple examples that highlight the role of measurement backaction. To this end, we consider a two-level system and focus on the first measurement ($n=1$). We adopt the measurement operators used in our experiment [Eq. (A1)] and examine the following two cases: (i) $\delta_0 = \delta_1 = 0$ (i.e., projective measurement in the computational basis $\{|0\rangle, |1\rangle\}$) and (ii) $\delta_0 = \delta_1 = \frac{1}{2}$ (i.e., dephasing in the same basis). The initial state ρ_1 is expressed via its spectral decomposition as $\rho_1 = \sum_{b_1=0,1} q(b_1) |b_1\rangle\langle b_1|$. We define the overlaps between the eigenbasis $\{|b_1=0\rangle, |b_1=1\rangle\}$ of ρ_1 and the computational basis as $|\langle 0|b_1=0\rangle|^2 = |\langle 1|b_1=1\rangle|^2 = \alpha$, with $0 < \alpha < 1$. This setup ensures that

the two basis sets are not identical. In this setting, we compare the quantum and classical information measures:

$$\begin{aligned} \langle i_{\text{QCT}} \rangle &= S(\rho_1) - \sum_{y_1} P(y_1) S(\tau_1^{y_1}), \\ \langle i_{\text{TE}} \rangle &= I(b_1 : y_1). \end{aligned}$$

First, in case (i) with $\delta_0 = \delta_1 = 0$, the QC-transfer entropy becomes $\langle i_{\text{QCT}} \rangle = S(\rho_1)$, since each conditional state $\tau_1^{y_1=0,1}$ becomes a pure state. The classical transfer entropy in this case is given by $\langle i_{\text{TE}} \rangle = S(\rho_1) - h(\alpha)$, where $h(p) = -p \ln p - (1-p) \ln(1-p)$ is the binary entropy function. This follows from the joint probability $p(b_1 = i, y_1) = q(b_1 = i) |\langle b_1 = i | y_1 \rangle|^2$. Thus, we obtain the inequality $\langle i_{\text{QCT}} \rangle > \langle i_{\text{TE}} \rangle$. In case (ii) with $\delta_0 = \delta_1 = \frac{1}{2}$, the classical transfer entropy becomes $\langle i_{\text{TE}} \rangle = 0$, since the joint probability becomes $p(b_1, y_1) = \frac{1}{2} q(b_1)$, indicating statistical independence of b_1 and y_1 . The QC-transfer entropy is given by $\langle i_{\text{QCT}} \rangle = S(\rho_1) - S(\mathcal{D}(\rho_1))$, where $\mathcal{D}(\rho) = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|$ represents the dephasing channel. If $q(0) \neq q(1)$, then $S(\mathcal{D}(\rho_1)) > S(\rho_1)$, implying $\langle i_{\text{QCT}} \rangle < 0$, which leads to $\langle i_{\text{QCT}} \rangle < \langle i_{\text{TE}} \rangle$. These two examples demonstrate that the QC-transfer entropy deviates from the classical transfer entropy due to measurement backaction and that there is no fixed ordering between them in general.

3. Backward transfer entropy

We show that the backward QC-transfer entropy I_{BQC}^k , introduced in Eq. (C3), is reduced to the (classical) backward transfer entropy when the dynamics is fully classical. Under the classical dynamics, I_{BQC}^k is represented as

$$\begin{aligned} I_{\text{BQC}}^k &= \left\{ \sum_{n=k+1}^N \chi(\tau_n : Y_{n-k} | Y_n^k) - \chi(\rho_{n+1} : Y_{n-k} | Y_n^k) \right\} \\ &\quad + \chi(\rho_{N+1} : Y_N) \\ &= \left\{ \sum_{n=k+1}^N I(b_n : Y_{n-k} | Y_n^k) - I(b_{n+1} : Y_{n-k} | Y_n^k) \right\} \\ &\quad + I(b_{N+1} : Y_N), \end{aligned}$$

where the Holevo information is reduced to the classical mutual information. In the following, we show that in fully classical dynamics under k th-order Markovian feedback, I_{BQC}^k is equivalent to the backward transfer entropy I_{BTE}^k , defined as [49,76]

$$I_{\text{BTE}}^k \equiv \sum_{n=k+1}^N I(b_n : y_{n-k} | Y_N^{N-(n-k)}) + I(b_{N+1} : Y_N^k). \quad (\text{D7})$$

The equality $I_{\text{BQC}}^k = I_{\text{BTE}}^k$ can be derived as follows:

$$\begin{aligned}
I_{\text{BQC}}^k &= \left\langle \sum_{n=k+1}^N I(b_n : Y_{n-k} | Y_n^k) - I(b_n : Y_{n-k-1} | Y_{n-1}^k) \right\rangle \\
&\quad + I(b_{N+1} : Y_N^k) \\
&= \left\langle \ln \prod_{n=k+1}^N \frac{p(Y_{n-k} | b_n, Y_n^k)}{P[Y_{n-k} | Y_n^k]} \frac{P[Y_{n-k-1} | Y_{n-1}^k]}{p(Y_{n-k-1} | b_n, Y_{n-1}^k)} \right\rangle \\
&\quad + I(b_{N+1} : Y_N^k) \\
&= \left\langle \ln \prod_{n=k+1}^N \frac{p(Y_{n-k} | b_n, Y_n^k)}{p(Y_{n-k-1} | b_n, Y_n^{k+1})} \frac{P[Y_{n-k-1} | Y_n^k]}{P[Y_{n-k} | Y_n^k]} \right\rangle \\
&\quad + I(b_{N+1} : Y_N^k) \\
&= \left\langle \ln \left(\prod_{n=k+1}^N \frac{p(b_n, Y_n^{k+1})}{p(b_n, Y_n^k)} \right) \frac{P[Y_N^k]}{P[Y_N]} \right\rangle + I(b_{N+1} : Y_N^k) \\
&= \left\langle \ln \prod_{n=k+1}^N \frac{p(y_{n-k} | b_n, Y_n^k)}{P[y_{n-k} | Y_N^{N-(n-k)}]} \right\rangle + I(b_{N+1} : Y_N^k) \\
&= \left\langle \ln \prod_{n=k+1}^N \frac{p(y_{n-k} | b_n, Y_N^{N-(n-k)})}{P[y_{n-k} | Y_N^{N-(n-k)}]} \right\rangle + I(b_{N+1} : Y_N^k) \\
&= \sum_{n=k+1}^N I(b_n : y_{n-k} | Y_N^{N-(n-k)}) + I(b_{N+1} : Y_N^k) \\
&= I_{\text{BTE}}^k,
\end{aligned}$$

where $\langle \cdot \rangle$ represents the ensemble average. In the derivation of the third equality, we utilize the relationship $p(Y_{n-k-1} | b_n, Y_n^{k+1}) = p(Y_{n-k-1} | b_n, Y_{n-1}^k)$, which is equivalent to $p(y_{n-k} | b_n, Y_{n-1}) = p(y_{n-k} | b_n, Y_{n-1}^k)$. This equality follows from the chain rule $Y_{n-k-1} \rightarrow (b_n, Y_{n-1}^k) \rightarrow y_n$. In the derivation of the sixth equality, we use the relationship $p(y_{n-k} | b_n, Y_N^{N-(n-k)}) = p(y_{n-k} | b_n, Y_n^k)$, which follows from the chain rule $y_{n-k} \rightarrow (b_n, Y_n^k) \rightarrow Y_N^{N-n}$ representing the k th-order Markovianity of feedback. We can check these chain rules with the causal networks shown in Fig. 9.

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