

Eigenstate Thermalization in Thermal First-Order Phase Transitions

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(Received 27 January 2026; revised 11 May 2026; accepted 9 July 2026; published 18 August 2026)

The eigenstate thermalization hypothesis (ETH) posits how isolated quantum many-body systems thermalize, assuming that individual eigenstates at the same energy density have identical expectation values of local observables in the limit of large systems. While the ETH apparently holds across a wide range of interacting quantum systems, in this work, we show that it may require generalization in the presence of thermal first-order phase transitions. We introduce a class of all-to-all spin models, featuring first-order thermal phase transitions that stem from two distinct local maxima of entropy (two mean-field solutions that we dub “branches”) that exchange dominance in the many-body density of states as the energy is varied. We argue that, for energies in the vicinity of the thermal phase transition, eigenstate expectation values do not need to converge to the same thermal value. The system has a regime with coexistence of two classes of eigenstates corresponding to the two branches with distinct expectation values at the same energy density and another regime with Schrödinger-cat-like eigenstates that are interbranch superpositions; these two regimes are separated by an eigenstate phase transition. We propose a more general form of the ETH *Ansatz*, support our results by semiclassical calculations and an exact diagonalization study of a microscopic spin model, and argue that the structure of eigenstates in the vicinity of thermal first-order phase transitions can be experimentally probed via nonequilibrium dynamics.

DOI: [10.1103/4zs8-7kf4](https://doi.org/10.1103/4zs8-7kf4)

Subject Areas: Computational Physics,
Quantum Physics, Statistical Physics

I. INTRODUCTION

The eigenstate thermalization hypothesis (ETH) [1–3] provides a central framework for understanding thermalization in isolated quantum many-body systems and has been extensively tested across a wide range of models [4]. For a generic interacting quantum model, where the ETH is expected to hold, quantum dynamics initiated from typical nonequilibrium states leads to relaxation to locally near-thermal states and the emergence of an effective hydrodynamic description [5,6]. From this perspective, systems in which the ETH is violated or requires modification represent candidates for qualitatively different dynamics and merit special attention. Recent progress in studying such systems includes many-body localization [7,8] and quantum many-body scars [9–11], both of which feature

distinct dynamical behavior. These examples naturally raise the question of whether there exist other generic situations where ETH must be generalized or modified.

In this work, we demonstrate that thermal first-order phase transitions provide a setting in which ETH may need to be generalized, and we establish this concretely in a class of all-to-all spin models. While continuous phase transitions are more commonly studied, due to their emergent universal description, here, we focus on *first-order* phase transitions that are more generic [12]. One main result of our work is that thermal first-order transitions force a generalization of ETH: Eigenstates can organize into multiple stable or metastable ergodic components, characterized by distinct expectation values of local observables at the same energy density. Before discussing this further, we first review the description of first-order thermal phase transitions.

In the canonical ensemble perspective, where temperature is the control parameter, a first-order thermal phase transition is a discontinuous change in the system’s energy as a function of temperature. Crucially, to study the ETH, a natural perspective is provided by the *microcanonical* ensemble that uses the energy of the system as the control parameter. When equivalence between canonical and

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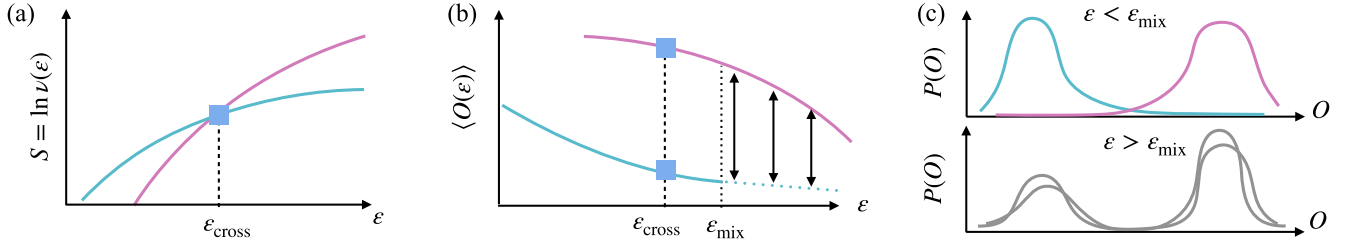


FIG. 1. Schematic picture of the first-order thermal phase transition that we study. (a) Microcanonical mean-field behavior with two distinct solutions, with one solution dominating the many-body density of states $\nu(\epsilon)$ for energy densities $\epsilon < \epsilon_{\text{cross}}$ and the second solution becoming dominant for larger energies. This results in a nonconvex region in the entropy. (b) Expectations for the ETH, that has two different branches and absence of complete ergodicity for $\epsilon < \epsilon_{\text{mix}}$. The ergodicity is restored for $\epsilon > \epsilon_{\text{mix}}$, leading to the disappearance of the lower branch due to a combination of classical and possibly quantum tunneling shown schematically by black arrows. The relative ordering of ϵ_{cross} and ϵ_{mix} is not fixed: the sketch illustrates $\epsilon_{\text{cross}} < \epsilon_{\text{mix}}$, while $\epsilon_{\text{cross}} \geq \epsilon_{\text{mix}}$ is also possible. (c) The distribution of local observables in eigenstates for energy densities below ϵ_{mix} has two species of eigenstates. In contrast, in the mixing regime, the eigenstates have bimodal distribution of local observables.

microcanonical ensembles breaks down, as it does at thermal first-order phase transitions, we understand ETH as the equivalence between quantum expectation values in eigenstates and those in the corresponding *microcanonical* ensemble. (We keep using the term “ETH” although, strictly speaking, the reference to temperature in eigenstate *thermalization* hypothesis should perhaps be omitted when ensemble equivalence breaks down.)

In the microcanonical description, the characteristic discontinuity in energy as a function of temperature is absent. Instead, the first-order phase transition is manifested as a nonconvex region in the microcanonical entropy (the logarithm of the density of states) calculated as a function of energy; see Fig. 1(a) for a schematic of the behavior in the class of models that we study here. To have a model where this nonconvexity in the entropy density survives in the limit of an infinite system [12], we study systems with long-range interactions. In models with short-range interactions, the system removes the nonconvex entropy density in the energy range of the transition by spatial phase separation. By contrast, for systems with sufficiently long-range interactions, phase separation does not occur, the entropy density can remain nonconvex, and the inequivalence of the canonical and microcanonical ensembles allows the temperature as a function of the energy density to be nonmonotonic in the microcanonical ensemble [13,14].

In this work, we introduce an analytically tractable route to studying the first-order thermal phase transition in certain spin models: We consider systems whose equilibrium thermodynamic properties follow a mean-field description and have two distinct mean-field solutions coexisting at the same energy in the energy range near the microcanonical first-order transition. This is achieved by introducing a generalized Lipkin-Meshkov-Glick (LMG) model [15] describing a collection of spins $1/2$ with all-to-all interactions. The LMG model has symmetry under permutation of the spins which allows a mean-field

description, makes the magnitude of the total spin conserved, and reduces the dynamics to that of a single large spin. Then, we perturb this model with additional weak random terms that break the permutation symmetry, allow the magnitude of the total spin to fluctuate, and make the system have true many-body dynamics, but without modifying the system’s equilibrium thermodynamics. This approach allows both analytical and numerical study of the model’s ETH behavior.

The thermal first-order phase transition is naturally obtained within the mean-field description when microcanonical entropies of two distinct mean-field solutions overtake each other with energy density; see Fig. 1(a). The two distinct mean-field solutions (local entropy maxima) provide specific realization of two distinct ergodic components (assuming a generic perturbation that leads to ergodicity within each component) that are called “branches” in what follows. These two branches are generally not related to each other by any symmetry and, hence, have macroscopically different expectation values of local observables as shown in Fig. 1(b). The focus of the present work is to understand the fate of expectation values of local observables in individual eigenstates, therefore providing insight into ETH behavior in systems with a first-order phase transition. Note that, in general, mean-field solutions with thermal first-order phase transitions need not have two branches with a crossing like this. Instead, there may be only one branch with the entropy as a function of the energy being nonconvex. Thus, what we consider here is a subset of the thermal first-order transitions that can occur in mean-field theory.

One key result of our work is the existence of the two possibilities summarized in Figs. 1(b) and 1(c). For some energies, the system may feature eigenstates that can be separated into two distinct classes according to their mean-field branch origin, the regime that we call *multibranch* ETH, where the eigenstates are each localized to a single branch. In this regime, if one does not distinguish branches,

the correspondence between eigenstate expectation values and the microcanonical ensemble implied by ETH breaks down, and ergodicity *between* different ergodic components (branches) is broken even at infinite times. For higher energy density, we expect the quantum tunneling and thermal fluctuations that couple the branches to become sufficient to cause the eigenstates to be hybridized (delocalized) between branches. We call this regime *mixed-branch* ETH, and here the equivalence between eigenstate expectation values and the microcanonical ensemble is restored. In other words, ergodicity between different ergodic components is restored, albeit at long times; hence, they are only *metastable*. This regime is still qualitatively different from conventional ETH, since the probability distributions of local observables for individual eigenstates are bimodal, therefore featuring anomalously large fluctuations; see the illustration in Fig. 1(c). Hence, in some aspects, the mixed-branch ETH may be more unusual compared to the multibranch regime. In Sec. II E, we formalize this picture as a branch-resolved generalization of the ETH *Ansatz* [1–3], in which eigenstates can carry an additional discrete branch label.

The existence of two similar regimes was also proposed earlier [16–19], however in the context of systems with $\mathbb{Z}_2$ symmetry and a continuous phase transition, where in the multibranch regime all eigenstates come in near-degenerate symmetry-related pairs. Also, a similar transition was discussed in quantum p -spin models [20], where the number of ergodic components is increasing with the number of spins. However, Ref. [20] focused on analytical calculations, and it remains unclear if this behavior can be studied numerically in finite systems.

The remainder of this paper is organized as follows. Section II introduces the all-to-all model of spin 1/2 and the resulting mean-field phase diagram and also provides a semiclassical calculation of the tunneling that determines the fate of the ETH. This information is used in Sec. III to guide exact diagonalization numerics and demonstrate regimes of multibranch and mixed-branch ETH. We conclude our work with Sec. IV, discussing the potential implications of our results for more physical models with power-law or local interactions and outlining potential dynamical probes of the two distinct ETH regimes identified in our work.

II. MEAN-FIELD AND SEMICLASSICAL TUNNELING

In this section, we introduce a generalization of the LMG model that will be studied throughout this work, discuss its mean-field description, and calculate the semiclassical tunneling amplitude between different mean-field solutions. Finally, we provide the mean-field phase diagrams and determine promising choices of parameters for exact diagonalization studies.

A. Generalization of the LMG model

To explore how the ETH behaves in the presence of a thermal first-order phase transition, we consider a generalized LMG model. The standard LMG model [15,21] describes pairwise interacting spins with infinite-range couplings and provides one of the simplest microscopic realizations of mean-field physics in a fully quantum setting. Specifically, it is defined on a system of N spin-1/2 degrees of freedom and contains single-spin terms in the form of longitudinal and transverse field that can be expressed via collective spin operators:

$$X = \sum_{i=1}^N \sigma_i^x, \quad Z = \sum_{i=1}^N \sigma_i^z, \quad (1)$$

where $\sigma_i^{x,z}$ are the standard Pauli matrices acting on the i th spin. Pairwise all-to-all interactions in the standard LMG model can be expressed as a properly normalized square of the total collective spin operator Z^2 , leading to the Hamiltonian $H_{\text{LMG}} = (1/N)Z^2 + h_z Z + h_x X$. This model, studied in numerous contexts, in particular, for $h_z = 0$ allows one to have spontaneous symmetry breaking and study the continuous phase transition [16,22]. However, the standard LMG model does not have a first-order *thermal* phase transition. The $\mathbb{Z}_2$ symmetry pins the line of first-order phase transitions to the $h_z = 0$ axis, and it cannot be crossed by increasing only the energy density or temperature.

In order to realize a thermal first-order phase transition, we use a generalized three-spin interacting LMG model, denoted as LMG-3 in what follows. This model was studied in the context of annealing [23–27]. The Hamiltonian

$$H_{\text{LMG-3}} = \frac{1}{N^2} Z^3 + h_z Z + h_x X \quad (2)$$

depends on two coupling parameters, $h_{z,x}$ fields, as we fixed the coupling in the three-spin interaction term. We omit the two-spin interaction for simplicity. Including that term would not qualitatively change the result, although it breaks the chiral symmetry. This model strongly breaks $\mathbb{Z}_2$ symmetry also at the level of the interaction term and, as we show below, has a thermal first-order phase transition for a broad region of parameters h_x and h_z . This Hamiltonian $H_{\text{LMG-3}}$ also has a chiral symmetry generated by the operator $\mathcal{C} = \prod_{i=1}^N \sigma_i^y$ that flips x and z directions of all spins. The operator $\mathcal{C}$ anticommutes with the Hamiltonian (2), resulting in the energy spectrum being symmetric around zero energy. Because of this symmetry, we focus on only the regime of negative energies, $E \leq 0$, as the behavior at positive energies can be obtained by application of the operator $\mathcal{C}$.

In addition to the chiral symmetry, this LMG-3 model also features the standard spin permutation symmetry characteristic of models with uniform all-to-all interactions. This symmetry means the magnitude of the total spin is conserved. However, in what follows, we add a thermodynamically irrelevant random perturbation that breaks this permutation symmetry and allows the system to thermalize between different total spin sectors. Therefore, below, we use the mean-field treatment to first understand the thermodynamic phase diagram of the LMG-3 model before considering its dynamics in the presence of a weak random perturbation. As we discuss below, we expect similar physics to occur in LMG models with p -spin interactions for $p \geq 4$ [23–27]. However, we focus on the simplest possible case beyond familiar two-spin Ising interaction.

B. Density matrix mean field

The mean-field treatment of the LMG-3 model relies on the factorizable *Ansatz* for the many-body spin density matrix:

$$\rho = \otimes_i \rho_i, \quad \rho_i = \frac{1}{2} + \frac{1}{2} m (\cos \alpha \sigma_i^z + \sin \alpha \sigma_i^x), \quad (3)$$

parametrized by globally uniform parameters $\alpha \in [0, 2\pi]$ and $m \in [0, 1]$. These parameters specify the direction and magnitude of the magnetization per spin, that reads

$$\vec{m} = \text{tr} \rho \vec{\sigma} = m (\cos \alpha \hat{z} + \sin \alpha \hat{x}). \quad (4)$$

Note that we are using only one angle α to specify the direction of the magnetization, since the LMG-3 model's Hamiltonian (2) is real in the usual σ^z basis, which means the equilibrium magnetization is in the x - z plane. The entropy per spin of this mean-field state (3) is [16]

$$s(m) = \frac{1}{2} [2 \ln 2 - (1-m) \ln(1-m) - (1+m) \ln(1+m)], \quad (5)$$

which monotonically increases as the magnetization m is reduced from 1 (maximal total spin, entropy density $s = 0$) to 0 (vanishing total spin, entropy density $s = \ln 2$).

Equipped with expressions for spin expectation values and entropy, we set up a *microcanonical* mean-field calculation. Specifically, at each fixed value of energy density ε , given by

$$\varepsilon = \frac{H_{\text{LMG-3}}}{N} = m^3 \cos^3 \alpha + h_z m \cos \alpha + h_x m \sin \alpha, \quad (6)$$

we seek to *maximize* the entropy or, equivalently, *minimize* the magnetization m . The extrema of m , corresponding to vanishing derivative, are attained at points where

$dm/d\alpha|_\varepsilon = 0$. Solving for this condition, we obtain the following expression of m as a function of α :

$$m = \sqrt{\frac{h_x \cos \alpha - h_z \sin \alpha}{3 \cos^2 \alpha \sin \alpha}}, \quad (7)$$

subject to the condition of m being in the physically allowed interval, $m \in [0, 1]$. Plugging this expression into Eqs. (5) and (6) gives the values of entropy density and energy density at the extremal point as a function of α . Generally, the entropy density is a nonmonotonic function of α at each fixed energy, corresponding to the existence of several extremal points. Among those extremal points, each local minimum of m (local maximum of the entropy) is locally a microcanonically stable mean-field solution.

In order to illustrate the solutions of the microcanonical mean-field equations described above in the regime that we focus on, we fix parameters of the LMG-3 model to $h_z = -0.2$ and $h_x = 0.1$. With this choice of parameters, we plot the contours of constant energy density ε in the plane of parameters (α, m) in Fig. 2(a). For strongly negative energy densities, there is a unique local minimum of m , with $m^z < 0$, giving rise to the low-energy branch of mean-field solutions that we call the $m^z < 0$ branch. This branch has entropy monotonically increasing with energy density, as shown in Fig. 2(b), and is characterized by a negative expectation value of m^z ; see Fig. 2(c). Notably, upon increasing the energy density, the constant energy contours show the appearance of another local minimum of m . When it first appears, we have $m = 1$ at this new minimum; hence, another branch of mean-field solutions enters in Fig. 2(b), initially appearing at zero entropy density near $\varepsilon = -0.13$. This branch has positive magnetization and is labeled as $m^z > 0$; see Fig. 2(c). Given the chiral symmetry mentioned above, there is also another $m^z > 0$ branch that is mostly at positive energy that is equivalent under this symmetry to the $m^z < 0$ branch; this branch first appears very close to $\varepsilon = 0$ and is slightly visible in Figs. 2(b) and 2(c). We focus only on the first two branches that appear at negative energy, so we mostly ignore this third, higher-energy branch.

When the $m^z > 0$ branch first appears, it has lower entropy than the $m^z < 0$ branch. But as the energy is increased, the entropies of these two branches in Fig. 2(b) cross each other at energy density $\varepsilon_{\text{cross}}$. So the LMG-3 model in this regime of parameters realizes the branch crossing scenario sketched in Figs. 1(a) and 1(b). These branches are characterized by macroscopically different expectation values of local observables, such as the magnetization density. The branches exchange dominance at the energy $\varepsilon_{\text{cross}}$, leading to discontinuities in microcanonical expectation values as a function of the energy. Hence, our model realizes a strongly first-order phase transition, even in the microcanonical ensemble. Note that the all-to-all nature of the interactions does not allow the

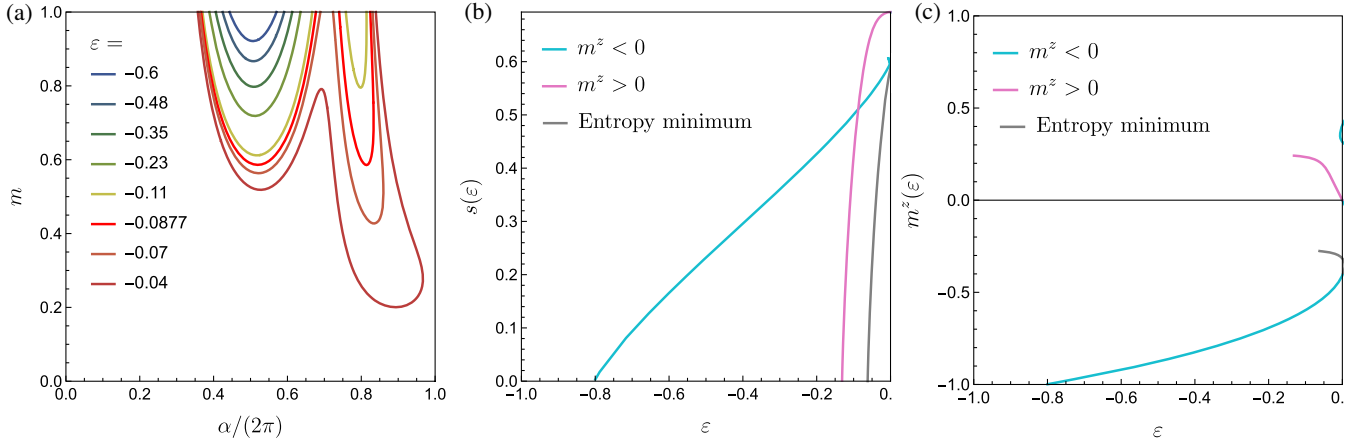


FIG. 2. (a) Contours of constant energy density ε have a unique local minimum value of m for very negative ε . This minimum gives rise to the branch of mean-field solutions that we label as $m^z < 0$. Upon increasing the energy density, a second local minimum of m emerges. This second minimum, labeled as the $m^z > 0$ branch, upon further increasing the energy attains a lower value of m compared to the $m^z < 0$ branch. (b) Entropy densities of the two branches as a function of ε have a crossing at energy density $\varepsilon_{\text{cross}} \approx -0.0877$. (c) Expectation value of z magnetization per spin, m^z , shows that the mean-field solution appearing at lower energies has negative z magnetization, while the second branch that appears upon increasing the energy has positive expectation value of m^z . The gray line corresponds to the local entropy minimum (local maximum of m) that eventually merges with the $m^z < 0$ branch at a small positive value of ε (not shown). Parameters of the LMG-3 model shown here are $h_z = -0.2$ and $h_x = 0.1$.

spatial phase separation that would occur in this model in microcanonical states if the interactions were instead only short-range.

Finally, we argue that the multibranch structure and first-order thermal phase transitions are generic for LMG models with p -spin interactions for $p \geq 3$. Such models were originally introduced to study first-order phase transitions in the low-lying part of the spectrum relevant for quantum annealing [23–26], while our focus is on their thermodynamics at finite energy density. In any all-to-all spin-1/2 model with p -body interactions, the energy density $\varepsilon(m, \alpha)$ is a polynomial of degree p in the magnetization m , while the entropy density $s(m)$ in Eq. (5) depends only on $|m|$. The microcanonical mean-field problem of maximizing $s(m)$ at fixed ε reduces to finding local minima of m on the constant-energy contours in the (α, m) plane, as in Fig. 2(a). For $p = 2$, these contours admit at most two such minima, related by $\mathbb{Z}_2$ symmetry, so a thermal first-order transition is forbidden away from the symmetric line. For $p \geq 3$, as we demonstrated above for $p = 3$, the constant-energy contours generically admit two or more inequivalent local minima of m (branches) not related by symmetry, and crossings of their entropies as a function of ε are generic rather than fine-tuned.

C. Semiclassical tunneling

Next, we consider the stability of these two coexisting mean-field solutions in the LMG-3 model with respect to tunneling processes and fluctuations. In the LMG-3 model with exact permutation symmetry, the total spin is conserved and so does not fluctuate. Thus, now we add a weak

additional term to the Hamiltonian to break this symmetry but weak enough to not affect the thermodynamics, as discussed in more detail below. This now allows fluctuations in m within each eigenstate. For simplicity, we mostly focus on the value of the energy density $\varepsilon_{\text{cross}}$, where the two mean-field solutions have the same $m = m_{\text{cross}}$ and, thus, the same entropy; see the inset in Fig. 3 for an illustration of this constant energy line in the plane of (α, m) parameters. These two extremal points may be coupled via two processes: quantum tunneling at fixed m

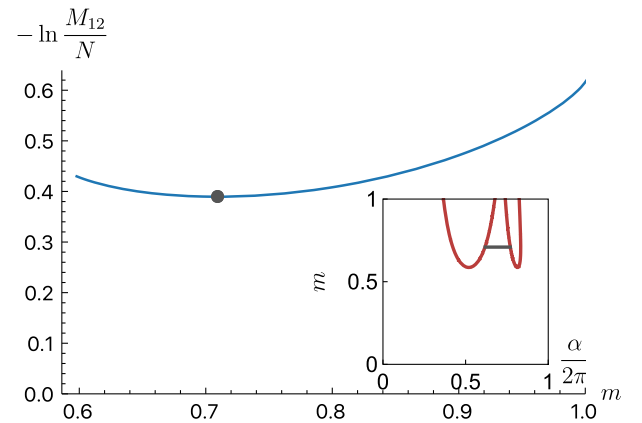


FIG. 3. Tunneling at the energy density $\varepsilon_{\text{cross}} = -0.0877$ and $m_{\text{cross}} = 0.586$ corresponding to the branch crossing for $h_z = -0.2$, $h_x = 0.1$. The maximal matrix element is attained at a value of $m_t \approx 0.709 > m_{\text{cross}}$ shown by the gray dot. The inset shows the constant energy ($\varepsilon = \varepsilon_{\text{cross}}$) contour in the α, m plane, with the dark gray line indicating the value of m_t and range of angles where optimal tunneling occurs.

through intermediate states at different energies and thermal fluctuations of m and α at fixed energy. The former process occurs within the LMG-3 model with full permutation symmetry, while the latter occurs only due to the weak breaking of that symmetry.

During quantum tunneling in this all-to-all model, the total energy of the system goes extensively “off shell,” so the matrix element of such processes is suppressed exponentially with the number of spins, N . Alternatively, the system can use thermal fluctuations to decrease the entropy (increase m) and go part way up or even fully over the entropy barrier staying at the same energy, a process also featuring a probability exponentially suppressed with N due to the extensively lower entropy during this process. Notably, going “over” the entropy barrier is possible only in cases where the two entropy maxima are separated by a local entropy minimum that is within the physically allowed $m \leq 1$ [see the gray line in Figs. 2(b) and 2(c)]; for the case shown in the inset in Fig. 3, this process is not possible. Generally, we expect that a certain *optimal* combination, characterized by the largest net transition rate, of thermal fluctuations to a larger m and quantum tunneling at that m (or no tunneling if the optimum is to go over the entropy barrier), will be the dominant process coupling the two branches in the limit of a large system. In order to find this optimum, we assume eigenstate thermalization within each branch, use the entropy to obtain the probability of the system being thermally excited to m , and use a semiclassical calculation of the quantum tunneling matrix element at m .

A semiclassical calculation of the tunneling matrix element, outlined in Appendix A, gives the exponent of the quantum tunneling matrix element between the two branches at m , $M_{12}^{(q)} \propto e^{-NA_q(m)}$, where we ignore the multiplicative factor in front of the exponential that contains polynomial dependence on N . The exponential suppression of the matrix element is calculated up to the leading order in N via the integral over the instanton trajectory connecting the two branches at m . The resulting expression contains an integral over variable z , defined as a rescaled z projection of the total spin:

$$A_q(m) = \frac{m}{2} \int_{z_1}^{z_2} \frac{dz}{\sinh \varphi} \sqrt{\frac{1-z}{1+z}} \times \left[\frac{6m^2 z^2 + h_z}{h_x} - \frac{z \cosh \varphi}{\sqrt{1-z^2}} \right], \quad (8)$$

where the angle φ is defined via the relation $\cosh \varphi = (\epsilon_{\text{cross}} - m^3 z^3 - h_z m z) / (m h_x \sqrt{1-z^2})$, with ϵ_{cross} being the energy density and $m \in [m_{\text{cross}}, 1]$. The integral is calculated between the closest points $z_{1,2}$ on the two branches, where $\cosh \varphi = 1$; see Appendix A.

Assuming that the full matrix element for a transition between branches is the probability of thermal excitation to

m times the matrix element for tunneling at m [16], we obtain the following expression for the full matrix element:

$$M_{12}(m) \propto \exp\left(-NA_q(m) - N[s(m_{\text{cross}}) - s(m)]\right), \quad (9)$$

that depends on the magnitude of the spin, $m \geq m_{\text{cross}}$, where the quantum tunneling process happens. The term in the exponent in Eq. (9) has to be minimized over the values of m in the physically allowed interval, $m \in [m_{\text{cross}}, 1]$. An example of such minimization is illustrated in Fig. 3, where the gray dot indicates the optimal value of m , denoted in what follows as m_t , that generally is inside the allowed interval, $m_{\text{cross}} < m_t < 1$.

The resulting matrix element calculated at $m = m_t$ has to be compared to the many-body level spacing δ that is equal between branches at the crossing point:

$$\delta \propto e^{-Ns(m_{\text{cross}})} \quad (10)$$

and that similarly features exponential suppression with the number of spins, N . The competition between level spacing and matrix element, both exponentially suppressed with N , leads to two distinct regimes separated by an eigenstate phase transition, as we discuss in the following.

D. Multi- and mixed-branch ETH regimes

In the case when the optimal matrix element is much smaller than the level spacing, $M_{12} \ll \delta$, or, equivalently, $-\ln M_{12}(m_t)/N > -(\ln \delta)/N$, leading to condition:

$$A_q(m_t) > s(m_t), \quad (11)$$

the tunneling process is unable to effectively couple the eigenstates from one branch to the other. Moreover, the ratio between matrix element and level spacing is exponentially suppressed with the number of spins, N . In this *multibranch* regime, each branch is dynamically stable with respect to quantum tunneling and thermal fluctuation processes. The eigenstates are each localized on one of the branches, so they can reliably be classified according to the branch they belong to, and any interbranch hybridization is suppressed upon increasing the system size N . Since we are considering eigenstates at the energy where the two branches have the same entropy, this localization of the eigenstates to a single branch is a weak violation of the ETH. We have assumed that the eigenstates do obey ETH within each branch, but in this regime they do not obey ETH once both branches are considered.

In the opposite, *mixed-branch* regime, $-\ln M_{12}(m_t)/N < -(\ln \delta)/N$, the ratio between matrix element and level spacing is exponentially increasing with N . This leads to an effective mixing of the two macroscopically distinct mean-field branches. As a result, the eigenstates in this mixed-branch regime are delocalized between the branches.

Because of the macroscopically distinct character of the two mean-field branches, the eigenstates in this regime are Schrödinger-cat-like, featuring anomalous bimodal distributions of local observables. In this regime, interbranch thermal equilibration does happen in the limit of long times but at a rate that decreases exponentially with increasing N , due to the entropy barrier that has to be either surmounted or tunneled through.

Until now, we have restricted our attention to the point of the crossing between two branches, where the branches have equal entropy. Away from the crossing, the entropies differ extensively between branches, so the many-body level spacings differ, $\delta_1 \gg \delta_2$, where we have labeled the lower-entropy branch 1. The multibranch regime, with eigenstates localized to both branches, is then $\delta_1 \gg \delta_2 \gg M_{12}$. The mixed-branch regime, with full thermalization between branches, is $M_{12} \gg \delta_1 \gg \delta_2$. There is also an intermediate regime, $\delta_1 \gg M_{12} \gg \delta_2$, where no eigenstates are localized to the lower-entropy branch, but the eigenstates are not fully thermalized between branches because the eigenstates vary in how close they are to resonant with a state within the lower-entropy branch [28]. There are then two eigenstate phase transitions, but the transition out of the multibranch regime (where $\delta_1 \gg M_{12} \sim \delta_2$) is more apparent, since this where eigenstates localized to the lower-entropy branch disappear and interbranch relaxation onsets.

The multibranch regime occurs when the tunneling matrix element M_{12} is small, so one way to leave this regime and go through the eigenstate phase transition is to change model parameters to increase the quantum fluctuations. In the LMG-3 model, this can be done by increasing the transverse field h_x in the parameter range we study numerically below. One can also go through the transition by changing the energy and, thus, the entropies of the two branches. In the LMG-3 model, as the entropies are increased, the entropy barrier also decreases, as can be seen in Fig. 2(a). Thus, by increasing the entropies and, thus, decreasing the δ 's, we also increase M_{12} , so both effects go in the direction of moving across the eigenstate phase transition.

E. Generalized ETH Ansatz

The multibranch ETH regime introduced above suggests a generalization of the ETH Ansatz. The standard ETH Ansatz [1–3], which should apply in the mixed branch regime, imposes the following structure of matrix elements of local operators: $\langle E|O|E'\rangle = \mathcal{O}(\varepsilon)\delta_{EE'} + e^{-S(\varepsilon)/2}f_O(\varepsilon, \omega)R_{EE'}$, where $\varepsilon = (E+E')/(2N)$ and $\omega = E - E'$. The smooth function $\mathcal{O}(\varepsilon)$ controls the diagonal expectation values, $f_O(\varepsilon, \omega)$ controls the off-diagonal matrix elements and, thus, the dynamics, the prefactor $e^{-S(\varepsilon)/2}$ encodes the suppression by the density of states, and $R_{EE'}$ is a pseudorandom matrix element of order unity. In the multibranch regime, the standard

Ansatz fails: At the same energy density ε , there are two distinct expectation values $\mathcal{O}_1(\varepsilon) \neq \mathcal{O}_2(\varepsilon)$, so $\mathcal{O}(\varepsilon)$ is not single valued [see Fig. 5(a)]. The natural generalization extends the eigenstate label by a discrete *branch* index $b = 1, 2$ and posits within-branch ETH with branch-resolved smooth functions:

$$\langle E, b|O|E', b'\rangle = \mathcal{O}_b(\varepsilon)\delta_{EE'}\delta_{bb'} + e^{-S_b(\varepsilon)/2}f_{O,bb'}(\varepsilon, \omega)R_{EE'}^{bb'}, \quad (12)$$

with branch-resolved entropies $S_b(\varepsilon)$ and spectral functions $f_{O,bb'}(\varepsilon, \omega)$. The branch index labels which local entropy maximum (mean-field solution) the eigenstate is associated with; in our chaotic LMG-3 model, it coincides with the sign of $\langle m^z \rangle$.

The two regimes identified in Sec. II D correspond to these two distinct versions of ETH. In the multibranch regime, $f_{O,12}$ inherits an additional exponential suppression from the semiclassical tunneling matrix element [Eq. (9)], on top of the standard $e^{-S_b/2}$ prefactor. This suppression is strong enough in this regime to suppress interbranch eigenstate hybridization of typical eigenstates, and the branch number is, thus, an emergent quantum number. If these exponentially suppressed interbranch matrix elements are ignored, this *Ansatz* (12) reduces to two independent within-branch ETH *Ansätze*. In the mixed-branch regime, in contrast, the eigenstates are coherent superpositions of branches, so do not each have a branch label. If this regime applies at the crossing, the eigenstates retain Schrödinger-cat-like bimodal Born distributions $p(m^z)$; see Fig. 5(b).

The different structure of eigenstates in the multi- vs mixed-branch regimes can be probed via dynamics or, equivalently, via the spectral functions $f_O(\varepsilon, \omega)$ and $f_{O,bb'}(\varepsilon, \omega)$ entering Eq. (12). In the multibranch regime, initial states with local observables that agree with a particular branch will retain memory of their branch to infinite time. Specifically, in our chaotic LMG-3 model, the z magnetization initialized to be negative may never thermally equilibrate, despite not corresponding to a globally conserved quantity. In contrast, in the mixed-branch regime, expectation values of observables will relax to their microcanonical expectation value over all eigenstates at the corresponding energy density. There, even if the magnetization is initialized to be negative, it may melt to near-zero or even positive values, depending on the relative entropy of the two mean-field branches. As we discussed above, these two distinct ETH regimes are separated by an eigenstate phase transition.

In Sec. III below, we provide numerical evidence for both regimes and the transition between them: Figure 5(a) tests the branch-resolved diagonal $\mathcal{O}_b(\varepsilon)$, Figs. 5(b) and 5(c) probe the relative weights of branches in the mixed-regime, and Fig. 8 checks the structure of off-diagonal matrix elements, with the observed ω^{-2} tail being a direct consequence of perturbative interbranch mixing.

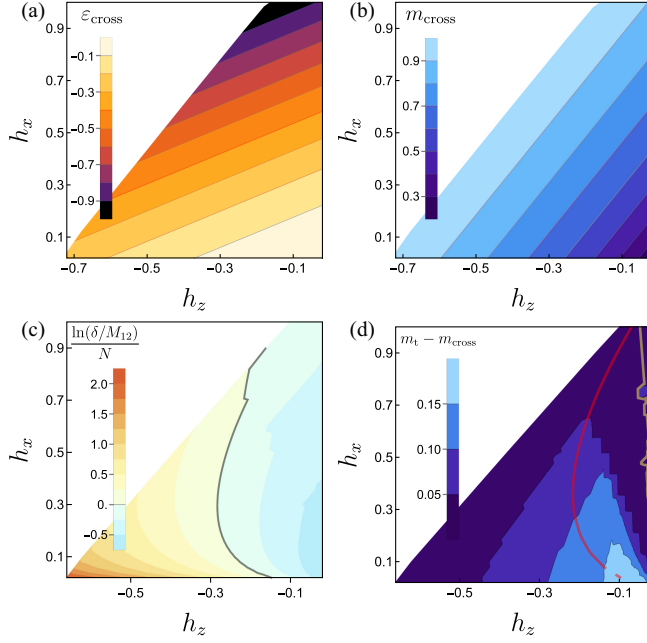


FIG. 4. Phase diagrams (a) value of energy density $\varepsilon_{\text{cross}}$ and (b) spin size m_{cross} at the branch crossing. White regions correspond to parameters where the crossing does not occur. (c) Positive and negative values of the logarithm of the ratio between level spacing and matrix element correspond to localized and hybridizing phases, respectively. (d) Difference between optimal m_t where tunneling happens and m_{cross} at the crossing shows that the role of within-branch fluctuations is larger at small values of $h_{x,z}$. The red line shows the contour where local spin maximum at $\varepsilon_{\text{cross}}$ is nearly outside of the physically allowed range ($m_{\text{max}} = 0.98$); to the left of the red line, only a combination of quantum and classical tunneling is allowed. To the right of the yellow line, tunneling becomes fully classical.

F. Phase diagrams

Finally, we combine our results to systematically study the phase diagram of the LMG-3 model as a function of the couplings $h_{z,x}$. Our aim is to identify the most promising couplings for numerical finite-size studies of multibranch and mixed-branch ETH regimes separated by an eigenstate phase transition in the LMG-3 model. To this end, we perform a systematic mean-field calculation for a dense grid of points in the space of parameters $h_{z,x}$, with $h_z \in [-0.8, -0.02]$ and $h_x \in [0.02, 1]$ and grid spacing $\Delta h_{x,z} = 0.02$. For simplicity, we focus on the energy density corresponding to the branch crossing point, $\varepsilon_{\text{cross}}$. First, in Figs. 4(a) and 4(b), we show the energy density $\varepsilon_{\text{cross}}$ and spin size m_{cross} as a function of the couplings. The white region in these plots correspond to the regime where the crossing is absent. Moreover, Figs. 4(a) and 4(b) show that the regime with $\varepsilon_{\text{cross}}$ close to zero and small m_{cross} , where the density of states is large and many such eigenstates, therefore, appear in finite-size systems, occurs at small couplings $h_{z,x}$. We, therefore, focus our numerical studies on this region.

Next, we consider the relation between the optimal matrix element and the level spacing at the branch crossing point; see Fig. 4(c). Warm colors in this panel correspond to the region where the crossing point is in the multibranch regime (localized), while blue colors mark the mixed-branch (delocalized) regime. This implies that in the warm region the $\varepsilon_{\text{cross}} < \varepsilon_{\text{mix}}$ holds, matching the situation sketched in Fig. 1(b). In the region with cold colors in Fig. 4(c), the opposite condition is true, and the boundary between two regions shown by the line gives an estimate of parameters where these two energy densities match, $\varepsilon_{\text{cross}} \approx \varepsilon_{\text{mix}}$. The accompanying Fig. 4(d) shows the difference between m_t and spin magnitude at the crossing. We observe that in a broad range of parameters the quantum tunneling is essential, since two local minima corresponding to different branches do not have a maximum of m between them within physical range, similar to the case shown in the inset in Fig. 3.

The data in Fig. 4(c) reveal that, for a fixed value of the h_z field near $h_z = -0.20$, increasing the transverse field h_x leads to enhanced coupling between branches and eventual delocalization. This expected trend is clear for small h_x , although at larger values of h_x the data in Fig. 4(c) suggest the possibility of reentrant behavior. Provided that we would like the crossing point to be at small energy density and yet potentially realize the transition between multibranch and mixed-branch ETH regimes, Fig. 4 suggests the value of $h_z = -0.2$ to be near optimal, and, for this fixed value of h_z , changing h_x in the interval $[0.05, 0.15]$ is expected to shift the crossing point from multibranch to mixed-branch ETH regime. In the next section, we proceed with an exact diagonalization study of the deformed LMG-3 model that confirms the presence of two distinct ETH regimes separated by an eigenstate transition.

III. NUMERICAL STUDY OF ETH AND DYNAMICAL TRANSITIONS

In this section, we present an exact diagonalization study of eigenstate properties of LMG-3 models with up to 16 spins. To this end, we start with the discussion of the weak perturbation that we add to the LMG-3 model that is thermodynamically irrelevant but breaks the permutation symmetry and allows for numerical study of the system's ETH properties in a fully many-body regime of its dynamics. After this, we demonstrate numerical signatures of the multibranch and mixed-branch ETH regimes and the expected transition between them. Finally, in Sec. III D, we discuss the behavior of spectral functions of local observables and potential signatures of the dynamical phase transition separating the different ETH regimes. While our numerical results are relatively well converged in the multibranch and mixed-branch ETH regimes, convincing numerical characterization of the eigenstate phase transition appears to require system sizes beyond what we are able to access here.

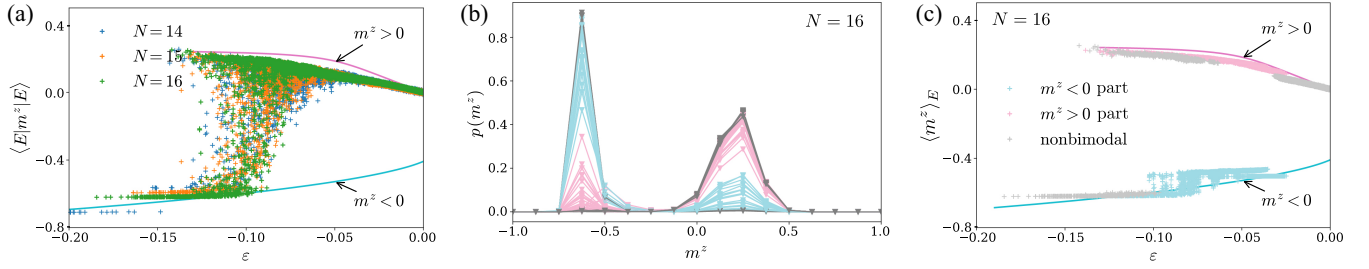


FIG. 5. (a) Expectation value of z magnetization for individual eigenstates for different $N = 14, 15, 16$. The mean-field branches are also shown. (b) Born-rule distribution of m_z measurement outcomes for $N = 16$ in the 80 eigenstates closest to energy density $\epsilon = -0.1$, with each curve corresponding to a single eigenstate. The spread of shapes shows the presence of nonbimodal distributions, as well as bimodal distributions with peaks corresponding to the two branches, demonstrating coexistence of qualitatively different eigenstates within a narrow energy window. (c) After separating the bimodal eigenstates each into two parts, as described in the main text, the expectation values of the z magnetization in each part, denoted as $\langle m^z \rangle_E$ clearly show the presence of the two branches for $N = 16$. Values of magnetic fields: $h_x = 0.1$ and $h_z = -0.2$.

A. Model and parameter choice

In order to break the permutation symmetry present in the uniform all-to-all LMG-3 Hamiltonian (2), we perturb it with a random all-to-all three-spin term

$$H = H_{\text{LMG-3}} + H_{\text{rand}}, \quad H_{\text{rand}} = \frac{\lambda}{N^p} \sum_{i>j>k} J_{ijk} \sigma_i^z \sigma_j^z \sigma_k^z, \quad (13)$$

where the couplings J_{ijk} are independently sampled from the Gaussian distribution with vanishing average and unit variance. This random term, averaged over the full Hilbert space, scales as $\langle H_{\text{rand}}^2 \rangle \sim N^{3-2p}$. Thus, for $1 < p < 3/2$ this term is subextensive and does not influence thermodynamic properties but does grow in magnitude upon increasing N . In what follows, we fix $p = 5/4$ to be in the middle of this physically justified interval and fix $\lambda = 0.2$.

B. Multi- and mixed-branch ETH regimes

After fixing parameters, we run numerical full diagonalization on the Hamiltonian H from Eq. (13) for three different system sizes $N = 14, 15$, and 16 , fixed value of z -magnetic field $h_z = -0.2$ (see Sec. II F), and a range of values of x -magnetic field h_x . Unless explicitly specified, we perform no disorder averaging but use a single realization of disorder. Using numerically obtained eigenenergies and eigenstates, we study the fate of ETH in this model, focusing on expectation values of local magnetization, defined as

$$m^z = \frac{1}{N} \sum_{i=1}^N \sigma_i^z, \quad (14)$$

consistent with Eq. (4) used in the mean-field treatment.

The expectation value of m^z across all eigenstates with negative energy density is shown in Fig. 5(a) for $h_x = 0.10$,

where we focus on the range of energy densities $\epsilon \in [-0.2, 0]$, since most of the negative-energy eigenstates are concentrated in this region. The data for $-0.15 < \epsilon < -0.10$ reveal the presence of two branches, indicating that some of these eigenstates are localized within the branches. However, as the energy density is increased past $\epsilon > -0.1$, the lower branch of eigenstates with $\langle m^z \rangle < 0$ starts to disappear. This indicates that eigenstates do not remain localized to this branch in this higher energy range, even though this lower branch does extend in Fig. 2(c) all the way to zero energy density.

In order to get insight into the disappearance of the negative magnetization branch, we study the distribution of m^z defined in Eq. (14) in individual eigenstates near energy density $\epsilon = -0.1$. These distributions, shown in Fig. 5(b) for 80 different eigenstates, reveal the presence of many mixed-branch eigenstates with a bimodal distribution of m^z , as well as other eigenstates where the distribution is nonbimodal, indicating that those eigenstates are localized within one branch. This supports the interpretation that the disappearance of the $\langle m^z \rangle < 0$ branch for $\epsilon > -0.10$ in Fig. 5(a) is due to interbranch delocalization of all eigenstates in this energy range, since in this range the $m^z > 0$ branch has the higher entropy and, thus, dominates in these m^z distributions.

To further explore the transition from multibranch to mixed-branch regimes as ϵ is increased toward zero, we demonstrate that it is possible to numerically remove the branch mixing in Fig. 5(c). To this end, any eigenstate that has a bimodal distribution of m^z is separated into “ $m^z < 0$ ” and “ $m^z > 0$ ” parts by selecting the left or right peak in the distribution of m^z (see Appendix B for details of the procedure). After proper renormalization, we plot the resulting expectation values of magnetization in Fig. 5(c). Notably, this figure shows that such numerical removal of interbranch mixing restores a rough agreement between mean-field and exact diagonalization results for all energy densities. Thus, the deformed LMG-3 model for the

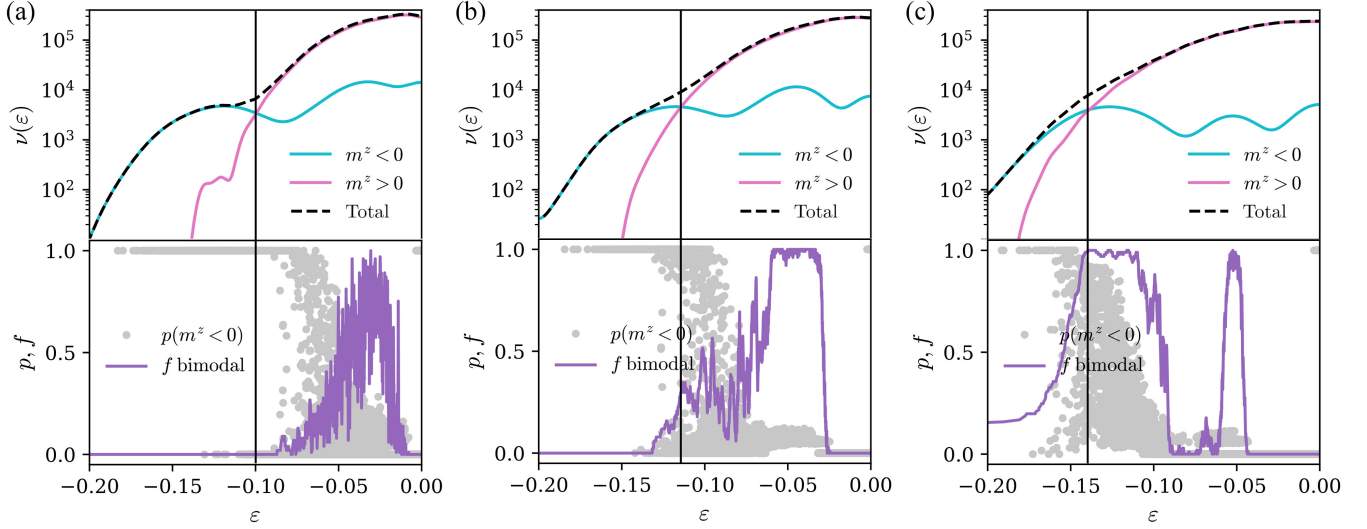


FIG. 6. (a)–(c) Top: density of states in the two branches for three different values of transverse field, $h_x = 0.05, 0.1$, and 0.15 ; the crossing moves to lower energy densities with increasing h_x . Bottom: weights of the negative magnetization peak p for all eigenstates as gray dots. The magenta line gives the fraction of bimodal eigenstates f (defined according to criterion $p \in [0.05, 0.95]$; see Appendix B 1 for details). Bottom plots illustrate that with increasing h_x the onset of hybridization (of bimodal eigenstates) shifts to lower energy densities. All data are for $N = 16$ and $h_z = -0.2$.

particular choice of parameters $h_z = -0.2$ and $h_x = 0.1$ as guided by the mean-field results is capable of realizing two distinct regimes of ETH, multibranch and mixed-branch. Below, we proceed with more study of these two regimes.

C. Transition between multi- and mixed-branch regimes

To systematically study the extent of the two different ETH regimes as a function of transverse field magnitude h_x , we consider the branch-resolved density of states (DOS). To split the total DOS into contributions from each branch, we use our classification of eigenstates into nonbimodal and bimodal (see Appendix B 1). Nonbimodal eigenstates get classified according to the sign of the expectation value of m^z and contribute to the DOS in only one of the branches. In contrast, bimodal eigenstates are separated into two parts, with the probability weight assigned to each branch accordingly. The resulting per-branch DOS, as well as total DOS are shown in top in Figs. 6(a)–6(c) for three different values of h_x . These figures reveal that the crossing point is shifting upon increasing h_x toward more negative-energy densities, consistent with the trend predicted by the mean-field results [see Fig. 4(a)]. The systematic study of the crossing point with system size and h_x is shown in Appendix B.

In order to see the onset of the branch mixing, in the bottom in Figs. 6(a)–6(c), we plot the averaged fraction of bimodal eigenstates present at each energy density, $f(\epsilon)$ (see Appendix B for criterion and details). The onset of branch mixing is where this fraction rises from near zero to near one. We see that with increasing h_x the onset of branch hybridization happens at energies above the crossing point

for $h_x = 0.05$, therefore corresponding to the situation $\epsilon_{\text{cross}} < \epsilon_{\text{mix}}$ sketched in Fig. 1(b). For $h_x = 0.15$, the mixing occurs before the crossing point, $\epsilon_{\text{cross}} > \epsilon_{\text{mix}}$, which also qualitatively agrees with the mean-field predictions in Fig. 4(c). Finally, for $h_x = 0.1$, we find that ϵ_{mix} is quite close to ϵ_{cross} .

Finally, we illustrate the behavior of the distribution of the expectation value of $\langle E | m^z | E \rangle$ in different eigenstates in Fig. 7 near the energy density corresponding to the DOS crossing point. At the small value of $h_x = 0.05$, this distribution stays clearly bimodal for all considered system sizes. In contrast, at the larger value of $h_x = 0.15$, this distribution flows toward nonbimodal upon increasing the number of spins, N . Note that, at the crossing point, nonbimodal within-single-eigenstate distributions $P(m^z)$ produce the bimodal distribution seen in the top in Fig. 7, while bimodal within-single-eigenstate distributions produce the nonbimodal distribution seen for $N = 16$ in the bottom of that figure. The presence of an eigenstate phase transition implies that the changes in this distribution and in related diagnostics should sharpen with increasing system size. However, the limited system sizes accessible to exact diagonalization can at best provide supporting evidence for such a transition and are, by themselves, inconclusive without the mean-field and semiclassical tunneling analyses presented in Sec. II.

D. Spectral function signatures of eigenstate phase transition

To explore consequences of the branch mixing transition in dynamics, we study the behavior of off-diagonal matrix elements of the magnetization operator defined in Eq. (14).

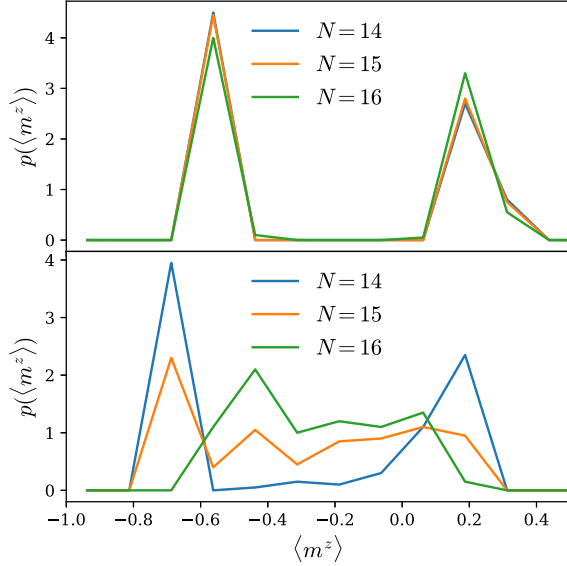


FIG. 7. Distribution of eigenstate magnetization expectation values at the crossing for two values of the transverse field. For weak $h_x = 0.05$ (top), this distribution is bimodal, indicating we are in the multibranch regime, whereas for $h_x = 0.15$ (bottom) it flows toward nonbimodal with increasing system size, as expected for the mixed-branch regime.

We define the spectral function via rescaled off-diagonal matrix elements averaged over a narrow window of eigenstates around a target energy density:

$$f^2(\omega) = \frac{N}{\delta} \langle |\langle E | m^z | E' \rangle|^2 \delta(E - E' - \omega) \rangle_{E/N \in [\varepsilon - \Delta, \varepsilon + \Delta]}. \quad (15)$$

The rescaling is by a factor of N/δ , where the inverse level spacing $1/\delta$ is the density of states at this energy density ε and the factor N appears because m^z is the magnetization density.

The spectral function defined in this way coincides with the ETH ansatz for off-diagonal matrix elements [4,29]. Moreover, it is related to the Fourier transform of the real-time autocorrelation function of the magnetization density m^z [4,29]. Therefore, long-time behavior of magnetization dynamics is mapped to the low-frequency behavior of $f^2(\omega)$. In particular, the time on which dynamics of magnetization saturates, T_{sat} , sets the Thouless energy as $E_{\text{Th}} = 1/T_{\text{sat}}$ where $f^2(\omega \lesssim E_{\text{Th}}) \sim \text{const.}$ This flat region in $f^2(\omega)$ is known as the Thouless plateau in the literature [30–32], and the corresponding energy scale is expected to be much larger than the level spacing $E_{\text{Th}} \gg \delta$ for systems that thermalize well. Cases where E_{Th} is less than or near the level spacing, as is the case in strongly disordered systems [30,32–34] and also in some models featuring quantum many-body scars [35], are typically associated with anomalous or nonergodic dynamics.

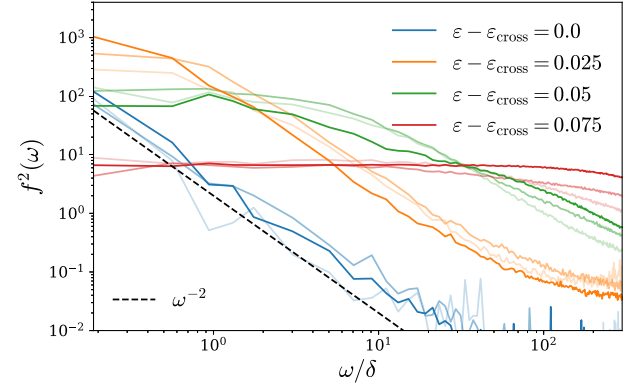


FIG. 8. Rescaled spectral function of local magnetization density shows a qualitative change in behavior as a function of energy density at a fixed value of $h_x = 0.1$ and $h_z = -0.2$. Its value at frequency of the order of the level spacing first goes up and then decreases, simultaneously showing the formation of broad Thouless plateau. Different color intensities label system sizes: 14 (fainter), 15, and 16 (bolder). Energy density is plotted relative to the crossing energy density at $N = 14$, and the window used for averaging is set by $\Delta = 0.025$. For $\varepsilon - \varepsilon_{\text{cross}} < 0$, the spectral function displays qualitatively similar power-law-like behavior to that at the crossing; however, the decreasing density of states below the crossing reduces the available data and leads to increased noise.

Examples of the spectral function (15) calculated for the perturbed LMG-3 Hamiltonian are shown in Fig. 8 for different energy densities. For energy densities close to zero (large values of $\varepsilon - \varepsilon_{\text{cross}}$), associated with mixed-branch ETH regime, the spectral function shows a broad Thouless plateau much larger than level spacing. Thus, in the mixed-branch regime, the z magnetization relaxes to its microcanonical expectation value at a rate much faster than inverse level spacing. Upon decreasing energy density, as the system is approaching the eigenstate phase transition, we observe in Fig. 8 that the extent of the plateau E_{Th} becomes of the order of the level spacing. This is consistent with the slowdown in the dynamics of magnetization relaxation at the eigenstate phase transition, ε_{mix} , where the matrix element coupling eigenstates on two different branches becomes of the order of the level spacing. This corresponds to the relaxation time becoming of the order of inverse level spacing and Thouless energy $E_{\text{Th}} \sim \delta$.

For even smaller energy densities $\varepsilon < \varepsilon_{\text{mix}}$, when the system is in the multibranch ETH regime, the magnetization does not relax to its microcanonical expectation value but retains memory of the initial state. Such dynamics is not visible in the spectral function that is averaged over all eigenstates within a certain energy window but can potentially be detected by distributions of off-diagonal matrix elements. The decrease in the density of states as ε becomes more negative prevents reliable averaging of the spectral function deep in the multibranch regime, and we do not show data for $\varepsilon \lesssim \varepsilon_{\text{cross}}$.

Additional support to the onset of the multibranch ETH is given by the characteristic power-law dependence of $f^2(\omega) \sim \omega^{-2}$, shown by the dashed line in Fig. 8. This dependence follows from the assumption of small matrix elements between eigenstates on different branches, $M_{12} \ll \omega \lesssim \delta_{1,2}$. The small matrix elements M_{12} weakly mix the eigenstates on the two branches, with mixing amplitude estimated as $M_{12}/\omega \ll 1$ at the first order in M_{12} . This produces a matrix element for the magnetization that is of the order of the mixing, leading to dependence of the spectral function $f^2(\omega) \propto \omega^{-2}$, as observed in Fig. 8 for small values of $\varepsilon - \varepsilon_{\text{cross}}$. Similar power-law dependence is present also for other values of $h_x \in [0.05, 0.15]$. The power-law behavior of the spectral function was observed in the context of many-body localization [30,32] and interpreted as a signature of resonant avoided level crossings at different scales [33,34].

To conclude this section, we discussed the expectations for the dynamics of magnetization as the energy density is tuned between multibranch and mixed-branch regimes via the eigenstate phase transition. We supported our conclusion by numerically studying the structure of average off-diagonal matrix elements of magnetization. Crucially, the exponential density of states on each of the branches enables the preparation of physical (short-range entangled) initial states within a narrow energy window that can be localized on individual branches. Therefore, dynamics of magnetization in different ETH regimes can be experimentally probed with quantum quenches.

IV. DISCUSSION

In summary, we demonstrated the existence of two distinct ETH regimes, dubbed multibranch and mixed-branch, in certain quantum systems with a thermal first-order phase transition and all-to-all interactions. Within the energy range of the first-order phase transition, such systems have inequivalence between the canonical and microcanonical ensembles, and eigenstate “thermalization” then corresponds to agreement with microcanonical distributions and not with canonical distributions. (The statement in this sentence appears to apply to thermal first-order transitions quite generally.) In the multibranch regime, the system is characterized by incomplete ergodicity, manifesting in the existence within the same energy range of two sets of eigenstates with macroscopically distinct observable properties, that correspond to the two phases separated by the first-order transition. In this regime, each of these two equilibrium phases is dynamically stable to infinite time even in a finite system, since almost all eigenstates are each localized primarily in one of the two phases.

In the mixed-branch regime, on the other hand, the quantum dynamics does produce slow relaxation between these macroscopically different states via a combination of quantum tunneling and thermal fluctuations, leading to a

long-time restoration of full microcanonical ergodicity in large finite systems. In this regime, the eigenstates are Schrödinger-cat-like states with the two phases that are separated by the first-order phase transition coexisting within each eigenstate with weights that match the microcanonical distribution in the limit of large systems. The relaxation between these two “coexisting” phases happens on a timescale that is exponentially long in the size of the system, so in this regime the branches are only metastable. The multibranch and mixed-branch regimes are separated by an eigenstate (and dynamical) phase transition that can be probed via this exponentially slow nonequilibrium dynamics of observables.

Our work makes a step in extending our understanding of ETH in quantum systems with thermal first-order phase transitions. We established the mean-field language using the model with all-to-all interactions and a specific arrangement of two distinct mean-field local entropy maxima (branches). We focused on this particular regime that shows a sharp first-order phase transition as a function of energy in the microcanonical ensemble. However, qualitatively different branch arrangements can and do occur. For example, the false vacuum decay scenario that has recently attracted significant interest [36–40] may also be realized in all-to-all models as two branches that never cross each other. In this scenario, the eigenstates on the subdominant branch in the multibranch regime may be interpreted as quantum many-body scarred eigenstates [9], and they will be destroyed by the onset of interbranch mixing. A systematic exploration of alternative branch structures and their implications for ETH is an interesting direction for future work.

Another promising direction is to understand the existence and fate of distinct ETH regimes in models with shorter-range interactions. Two distinct classes of such models include power-law interacting models as well as fully local models with only short-range interactions. Spin models with power-law interactions are implementable in trapped ion quantum simulators [41], whereas local two-dimensional models may be realized in neutral atom or superconducting qubit quantum simulators [42]. These models allow for phase separation in the vicinity of the thermal first-order phase transition, whose effects on ETH behavior remain to be understood. Generally, we expect that unusual eigenstates and dynamics may emerge in the vicinity of thermal first-order phase transitions. The branch-resolved *Ansatz* Eq. (12) is the natural extension of ETH that we propose; understanding its fate in phase-separated short-range systems is left to future work. Preliminary exact diagonalization results for a spin-1 chain with power-law interactions, which breaks the all-to-all permutation symmetry and is, thus, not in the mean-field class, show the same two-branch structure of eigenstate expectation values of the order parameter; a systematic study of thermal first-order phase transitions in spin-1

models, including finite-size scaling, will be reported elsewhere [43].

Another avenue for future work is understanding the structure of off-diagonal matrix elements as well as their higher-order correlation functions [44] related to out-of-time-ordered correlation functions. To this end, it is important to find specific microscopic models realizing thermal phase transitions and amenable to numerical studies in finite-size systems. General insight provided in this direction by our work is these models should typically include more than two-spin interactions in order to strongly break the Ising $\mathbb{Z}_2$ symmetry at the level of interactions. Alternatively, considering lattice models with three or more local Hilbert space dimensions may also be a fruitful avenue [43].

Finally, our findings may be also viewed in the context of local minima of the energy (metastability) in quantum systems, that were recently discussed for pure quantum states in the Hilbert space [45] and also for finite temperature Gibbs states [46]. There, a local energy minimum (metastable) state or density matrix for a particular model is one whose energy cannot be lowered by the action of local operators or quantum channels. In the context of our work, one needs rather an entropy-based definition: A density matrix is a local entropy maximum if its entropy cannot be increased by a local quantum channel without changing the energy. Within this definition, systems with a thermal first-order phase transition, such as considered here, have two such local entropy maxima in both the mixed-branch regime and multibranch regimes. The difference between the two regimes is in the closed system's unitary dynamics: The lifetime of the lower-entropy branch can be finite [i.e., the state is metastable (mixed-branch)] or infinite (multibranch) in large finite systems.

In conclusion, we expect that future theoretical and experimental studies of quantum models with thermal first-order phase transitions will extend our understanding of possible ETH regimes and modifications in physical systems. Recalling that mean-field-inspired “branches” correspond to distinct ergodic components, we expect such metastable ergodic components to exist also in short-range models in dimension above one. In contrast, multiple distinct fully stable ergodic components at the same energy (“multibranch regime”) appear to require extensive entropy or energy barriers, which are possible only in long-range interacting systems or for models on expander graphs. Studies of metastable ergodic components in short-range models may reveal generic and robust mechanisms for anomalously slow dynamics and unusual eigenstates at high energy densities.

ACKNOWLEDGMENTS

A. A. acknowledges discussions and prior collaboration on related topics with Anatoly Dymarsky. M. S. acknowledges Ashwin Vishwanath for introducing him to the idea

of thermal first-order phase transitions in quantum systems. This research was supported in part by Grant No. NSF PHY-2309135 to the Kavli Institute for Theoretical Physics (KITP) and by the Erwin Schrödinger International Institute for Mathematics and Physics (ESI). O. K. D. acknowledges support from the NSF through a grant for ITAMP at Harvard University. D. A. H. was supported in part by NSF QLCI Grant No. OMA-2120757.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: DETAILS OF SEMICLASSICAL CALCULATION

In this appendix, we calculate the leading terms in the semiclassical tunneling action. First, we reproduce the known result for the standard LMG model and then derive generalized expressions for the LMG-3 Hamiltonian.

1. Tunneling in LMG model

We start with reproducing the known results for the semiclassical tunneling amplitude in the conventional LMG model. The tunneling action calculation is known to have subtleties related to the formulation of coherent state path integrals for spins [47–50]. However, since we focus on leading-order terms in N , these subtleties do not affect the present calculation.

We denote collective spin operators $s^{x,z}$ which differ from X and Z introduced in Eq. (1) by an additional factor of $1/2$. In these notations, the Hamiltonian of standard LMG model may be written as $H_{\text{LMG}} = (g/2)(s_z^2 - s_x^2)$. Translating this into Euclidean-time action, we obtain

$$L = i(p - s)\dot{\alpha} - H_{\text{LMG}}(p, \alpha), \quad (\text{A1})$$

$$H_{\text{LMG}} = \frac{g}{2}[(p^2 - s^2)\cos^2\alpha + p^2], \quad (\text{A2})$$

where we introduced notations

$$p = s^z \quad (\text{A3})$$

and α for canonically conjugate variables and remaining spin component can be expressed via them as

$$s^x = \sqrt{s^2 - p^2} \cos \alpha, \quad s = \frac{Nm}{2}. \quad (\text{A4})$$

The energy in Eq. (A2) has two minima confined to the $x - y$ plane of the Bloch sphere with the energy $E_0 = -s^2g/2$. We are interested in calculating the tunneling action between these minima as

$$A = \int_{-\infty}^{\infty} d\tau i(p-s)\dot{\alpha}, \quad (\text{A5})$$

using equations of motion $i\dot{\alpha} = (\partial H/\partial p)$ and $i\dot{p} = -(\partial H/\partial \alpha)$. These equations of motion can be integrated into explicit instanton trajectory

$$p = \pm \frac{is}{\cosh(gs\sqrt{2}\tau)}, \quad \cos^2 \alpha = \frac{\cosh^2(gs\sqrt{2}\tau) - 1}{\cosh^2(gs\sqrt{2}\tau) + 1}, \quad (\text{A6})$$

leading to a trajectory that has $p = 0$ at $\tau = \pm\infty$ and purely imaginary otherwise and connects points $\alpha = 0$ and $\alpha = \pi$. Explicit integration over τ gives

$$A_{\text{LMG}} = -s[i\pi + 2\ln(\sqrt{2} + 1)]. \quad (\text{A7})$$

The imaginary part corresponds to Berry's phase that would lead to cancellation between two different instanton trajectories for half-integer s . The real part of the action is correct to the leading order in s and has nontrivial $O(1)$ corrections. A more direct way of calculating the action is to switch the integration variable into α and use equations of motion with energy conservation, leading to

$$A_{\text{LMG}} = is \int_0^\pi d\alpha \left[\sqrt{\frac{\cos^2 \alpha - 1}{\cos^2 \alpha + 1}} - 1 \right], \quad (\text{A8})$$

that reproduces the same answer.

2. Generalization to LMG-3 model

Here, we calculate the tunneling action at the energy where two minima are degenerate in energy. In this case, the energy where the tunneling happens will be given by $E_{\min} = N\epsilon_{\text{cross}}$, where ϵ_{cross} is the energy density where two mean-field branches have identical value of $m = m_{\text{cross}}$. After this, we discuss extension of the calculation to the case when tunneling happens at the higher value of m , corresponding to the lower entropy, but the energy and energy density are kept fixed.

Writing the generalized LMG-3 model using notations from Eqs. (A3) and (A4), we obtain

$$H_{\text{LMG-3}} = \frac{8}{N^2} p^3 + 2h_z p + 2h_x \sqrt{s^2 - p^2} \cos \alpha, \quad (\text{A9})$$

where now the two distinct minima will be located on the big meridian of the Bloch sphere corresponding to $\cos \alpha = -1$. In the present case, we expect the instanton trajectory to have imaginary value of α in the tunneling process and the value of p to stay real. Thus, we can directly change the integration coordinate to p using conservation of energy. We solve for $\cos \alpha$ using condition $H_{\text{LMG-3}} = N\epsilon_{\text{cross}}$ with Hamiltonian from Eq. (A9) as

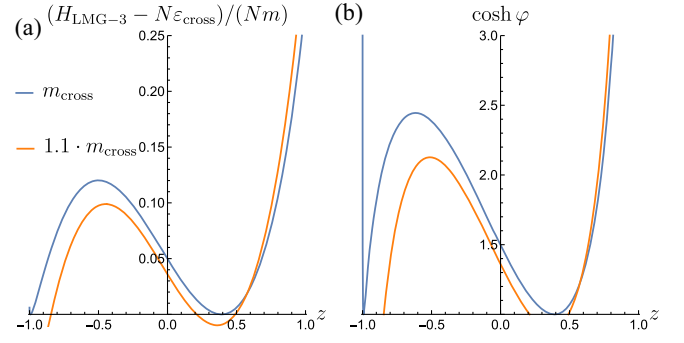


FIG. 9. (a) Effective energy potential as a function of z for fixed $\cos \alpha = -1$ has two minima at zero when $m = m_{\text{cross}}$. Tunneling happens between points $z_1 = 0.9926$ and $z_2 = 0.3909$. For larger values of m , for instance, $m = 1.1m_{\text{cross}}$, tunneling happens between closest roots of effective potential that move closer together. (b) Value of $\cosh \alpha$ as a function of z on the instanton trajectory is consistently larger than 1 between tunneling points, indicating that the value of original angle α is imaginary under the tunneling barrier.

$$\cos \alpha = \frac{\epsilon_{\text{cross}} - 8p^3/N^3 - 2h_z p/N}{2h_x \sqrt{(s/N)^2 - (p/N)^2}}. \quad (\text{A10})$$

Under the energy barrier $|\cos \alpha| > 1$, hence corresponding to imaginary α . Denoting $\alpha = i\varphi$, we get

$$A_{\text{LMG-3}} = - \int_{-\infty}^{\infty} d\tau (p-s)\dot{\varphi} = \int_{p_1}^{p_2} dp (s-p) \frac{\dot{\varphi}}{\dot{p}}, \quad (\text{A11})$$

where we change the integration variable from time to p and integration limits are given by two solutions of $\cos \alpha = -1$ of Eq. (A10). The ratio $\dot{\varphi}/\dot{p}$ can be simplified using equations of motion, which after introducing rescaled variable

$$z = \frac{p}{s} = \frac{2p}{Nm}, \quad (\text{A12})$$

rescaling an overall prefactor of $-N$, and substituting $m = m_{\text{cross}}$, gives

$$A_q = -\frac{A_{\text{LMG3}}}{N} = \frac{m_{\text{cross}}}{2} \int_{z_1}^{z_2} \frac{dz}{\sinh \varphi} \sqrt{\frac{1-z}{1+z}} \times \left[\frac{6m_{\text{cross}}^2 z^2 + h_z}{h_x} - \frac{z \cosh \varphi}{\sqrt{1-z^2}} \right], \quad (\text{A13})$$

where $\cosh \varphi$ and $\sinh \varphi$ are functions of z given by

$$\cosh \varphi = \frac{\epsilon_{\text{cross}} - m_{\text{cross}}^3 z^3 - h_z m_{\text{cross}} z}{m_{\text{cross}} h_x \sqrt{1-z^2}}. \quad (\text{A14})$$

Figure 9 shows the example of the potential and plots the function $\cosh \alpha$ obtained from equations of motion for specific values of $h_x = 0.1$ and $h_z = -0.2$ used in the main

text. The numerical calculation of the integral in Eq. (A13) gives the value $A_q = 0.446$.

However, tunneling is possible not only from the bottom of the potential well shown in Fig. 9(a). As we discuss in the main text, the system can use classical fluctuations to decrease entropy (increase the value of m). This incurs additional suppression from classical tunneling but, however, decreases quantum tunneling action. Thus, we generalize the expression for the action when the tunneling process occurs at $m > m_{\text{cross}}$. The resulting expression [also given in the main text in Eq. (8)] is identical to Eq. (A13), however, with the value of m_{cross} replaced by m that is allowed to take values in the interval $m \in [m_{\text{cross}}, 1]$. We emphasize that we keep the value of energy density constant, as it is fixed within the microcanonical ensemble. Nevertheless, increasing the value of $m > m_{\text{cross}}$ effectively lowers the energy barrier as shown in Fig. 9(a), where for $m > m_{\text{cross}}$ now tunneling happens between two values of z that are closest to each other. Inside this interval of values of z , the expression of $\cosh \varphi$ becomes larger than one [see Fig. 9(b)], indicating that angle α is imaginary on the instanton trajectory.

Finally, we note that spin lives on the Bloch sphere, and, hence, the tunneling between two minima may, in principle, be also possible using instanton trajectory going in the other direction on the big meridian of the Bloch sphere. However, in contrast to the conventional LMG model that has $\mathbb{Z}_2$ symmetry, in the present case we have no symmetry that relates the action of two tunneling trajectories. Hence, the chosen instanton trajectory is expected to give the dominant contribution to tunneling.

APPENDIX B: ADDITIONAL NUMERICAL DATA

In this appendix, we present details of the numerical procedures used in the main text as well as additional numerical data supporting our conclusions.

1. Determination of bimodal eigenstates

Here, we discuss the algorithm and criteria used to determine the bimodal eigenstates (see Fig. 5) and also to plot their fraction (Fig. 6). In order to analyze the character of eigenstates, we first calculate the distribution of magnetization m^z for each individual eigenstate. Magnetization m^z can take a small discrete set of $N + 1$ values for N spin-1/2 degrees of freedom, $m^z = -1, -1 + 2/N, -1 + 4/N, \dots, 1$. The resulting distribution of magnetization is smoothed using function `gaussian_filter1d` from the `scipy.ndimage` package. Afterward, we apply peak finding function `find_peaks` from `scipy.signal` with parameters `prominence = 0.01`, and `distance = 2`. Provided this function returns two peaks, the eigenstate is denoted as bimodal, and we also extract the location of minimum separating two peaks, used below for obtaining per-branch DOS.

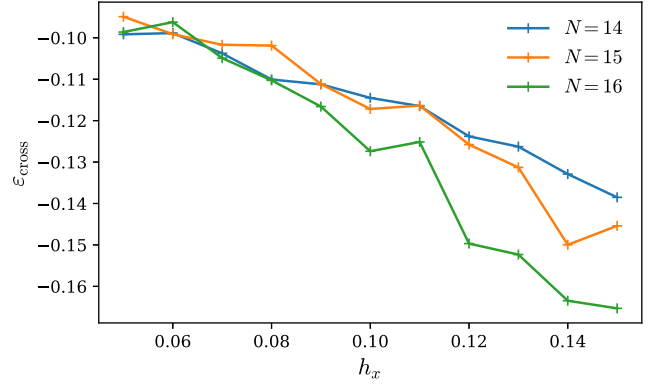


FIG. 10. Energy density where two mean-field branches cross each other as a function of h_x for three different system sizes.

The minimum separating two peaks for bimodal eigenstates is used to extract the weight in the left peak, denoted as p . This weight is used to plot the fraction of bimodal eigenstates, defined as those where $p \in [0.05, 0.95]$, in the bottom in Fig. 6 after sliding window average with the window fixed to 101 eigenstates.

2. Crossing point in DOS

Here, we systematically study the behavior of the crossing point in DOS of two branches. The branch-resolved DOS is obtained using the algorithm for determining bimodal eigenstates described above. Eigenstates not identified as bimodal are fully assigned to one of the branches depending on the expectation value of magnetization. For the bimodal eigenstates the distribution of m^z has two clearly defined peaks separated by the minimum, see Fig. 5. We use the location of the minimum to separate the contribution of the two peaks: The weight to the left of the minimum including the minimum point is assigned to the $m^z < 0$ branch that is denoted as p , and the weight to the right is attributed to the $m^z > 0$ branch, being equal to $1 - p$ by normalization. These two weights by definition sum to one; hence, the sum of two DOS for individual branches equals to the total DOS.

In order to determine crossing point, the resulting discrete dataset of energies of individual eigenstates and their per-branch weights is smoothed using the `gaussian_kde` from `scipy` package using bandwidth method with bandwidth set to 0.1. The crossing point of two DOS is extracted numerically from this data (see Fig. 6 as an example) and plotted in Fig. 10. The location of the crossing point systematically shifts to lower energy densities upon increasing h_x , consistent with the mean field phase diagram; see Fig. 4(a). In addition, the crossing point also shifts with N , indicating presence of finite-size effects and displays weak fluctuations with different realizations of disorder not shown here.

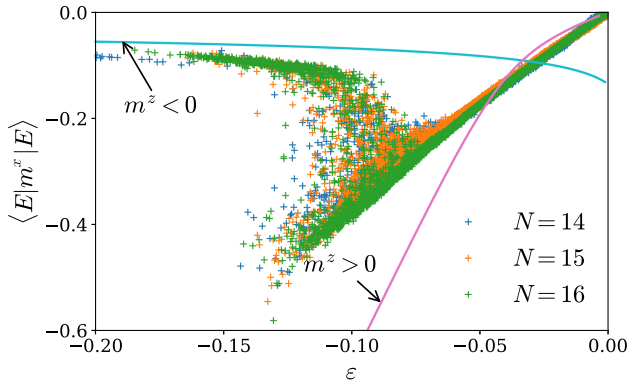


FIG. 11. Expectation value of m^x in eigenstates for $h_x = 0.1$ and $h_z = -0.2$ for three different system sizes. Lines correspond to mean-field prediction, with the branch labeled as $m^z < 0$ featuring small negative m^x values and the branch with $m^z > 0$ having considerably larger magnetization in the x direction.

3. Diagonal matrix elements of m^x

In the main text, we focused on studying matrix elements of the m^z operator and observed two distinct behaviors denoted as multibranch and mixed-branch regimes. In order to illustrate that this behavior holds for other general operators, we study expectation values of m^x magnetization in eigenstates. Figure 11 shows these expectation values along with mean-field predictions in the same interval of energy densities as Fig. 5 in the main text. While we observe a larger discrepancy between mean-field predictions and expectation values in eigenstates, the overall trend is similar: At lower energy densities, we see two distinct branches that start considerably hybridizing around $\epsilon \approx 0.1$ and for higher energy densities only the branch that dominates the DOS (labeled as $m^z > 0$) survives.

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