

Complexity of Gottesman-Kitaev-Preskill States

Lukas Brenner^{1,2,*} Libor Caha^{1,2,†} Xavier Coiteux-Roy^{3,4,‡} and Robert Koenig^{1,2,§}

¹*School of Computation, Information and Technology, Technical University of Munich, Munich, Germany*

²*Munich Center for Quantum Science and Technology, Munich, Germany*

³*Department of Computer Science, University of Calgary, Calgary, Canada*

⁴*Department of Physics and Astronomy, University of Calgary, Calgary, Canada*



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We initiate the study of state complexity for continuous-variable quantum systems. Concretely, we consider a setup with bosonic modes and auxiliary qubits, where available operations include Gaussian one- and two-mode operations and single- and two-qubit operations as well as qubit-controlled phase-space displacements. We define the (approximate) complexity of a bosonic state by the minimum size of a circuit that prepares an approximation to the state in trace distance. We propose a new circuit which prepares an approximate Gottesman-Kitaev-Preskill (GKP) state $|\text{GKP}_{\kappa,\Delta}\rangle$. Here, κ^{-2} is the variance of the envelope, and Δ^2 is the variance of the individual peaks. We show that the circuit accepts with constant probability and—conditioned on acceptance—the output state is polynomially close in (κ, Δ) to the state $|\text{GKP}_{\kappa,\Delta}\rangle$. The size of our circuit is linear in $(\log 1/\kappa, \log 1/\Delta)$. To our knowledge, this is the first protocol for GKP-state preparation with fidelity guarantees for the prepared state. We also show converse bounds, establishing that the linear circuit-size dependence of our construction is optimal. This fully characterizes the complexity of GKP states.

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I. GKP STATES AND THEIR USE

Gottesman-Kitaev-Preskill (GKP) states are a key resource in continuous-variable (CV) quantum information processing [1]. Originally introduced as basis states of an error-correcting code protecting against phase-space displacement noise, their primary use has traditionally been in the context of quantum fault-tolerance. While the original paper [1] proposes the use of such states to encode individual qubits into oscillators, a number of subsequent works make essential use of GKP states as building blocks in other fault-tolerance constructions: Examples include toric- and surface-GKP codes (obtained by concatenating a qubit code with the GKP code) [2–5] and oscillator-to-oscillator codes where GKP states are used in auxiliary oscillators to define encoding isometries [6]. Beyond the design of robust quantum memories, several GKP-based schemes for achieving universal fault-tolerant quantum

computation in CV systems subject to random displacement errors have been proposed, starting from the original work [1]; see, e.g., Ref. [4]. Fault-tolerant CV measurement-based computation schemes based on GKP codes were proposed in Ref. [7]. In this context, GKP states not only act as a substrate for protecting quantum information, but also provide computational power: As shown in Ref. [8], universal fault-tolerant quantum computation can be achieved using only Gaussian (linear optics) operations when a supply of GKP states is provided.

Applications of GKP states outside the area of error correction include schemes for achieving maximal violations of Clauser-Horne-Shimony-Holt (CHSH)-type Bell inequalities [9,10] and procedures for process tomography of small displacements [11] (themselves applicable to quantum fault-tolerance). Distributed sensing protocols relying on GKP codes were proposed in Refs. [12,13]. Finally, an algorithmic application of GKP states was proposed in Ref. [14], where the inherent lattice structure of GKP states was used for integer factorization algorithm with constantly many modes and a single qubit.

II. PRIOR WORK ON GKP-STATE PREPARATION

The versatility and central role of GKP states in these applications directly motivates the design and analysis of corresponding preparation procedures. Numerous proposals have been made in the past. We refer to Ref. [15] for a

*Contact author: lukas.brenner@tum.de

†Contact author: cahalibor@me.com

‡Contact author: xavier.coiteuxroy@ucalgary.ca

§Contact author: robert.koenig@tum.de

recent review, which includes a discussion of concrete physical setups.

The primary distinction between different proposals is the type of non-Gaussian operation involved. In the seminal work [1], the use of a (unitary) cubic coupling between two oscillators [with a Hamiltonian of the form $Q_1(Q_2^2 + P_2^2)$] was suggested. Protocols such as Refs. [16–18] use a single auxiliary qubit and a qubit-oscillator coupling (generated by a Hamiltonian of the form $\sigma_z P$) allowing for qubit-controlled displacements. Such qubit-controlled displacements were also used in Ref. [19], which provides a remarkable proof-of-principle experimental demonstration of GKP codes and associated fault-tolerance operations.

A different approach to creating GKP states is that of engineering a “GKP Hamiltonian” whose ground states are (approximate) GKP code states, an approach already suggested in the seminal work [1]; see, e.g., Refs. [20–22]. Closely related to this are proposals to realize such Hamiltonians as effective evolution operators, e.g., by dynamical decoupling [23] or as effective (Floquet) Hamiltonians in periodically driven systems; see, e.g., Refs. [24,25]. We refer to Ref. [15] for a more complete discussion of these different proposals as well as experimental demonstrations.

III. COMPLEXITY OF BOSONIC STATES

We initiate the study of the complexity of CV states, with a special focus on (approximate) GKP states.

To define the notion of the complexity of a CV quantum state formally, let us briefly review the established definition of complexity for an n -qubit “target” state $|\Psi_{\text{target}}\rangle \in (\mathbb{C}^2)^{\otimes n}$; see, e.g., Ref. [26]. For $\varepsilon \geq 0$, the (unitary state) complexity $C_\varepsilon(|\Psi_{\text{target}}\rangle)$ of $|\Psi_{\text{target}}\rangle$ is defined as the minimum size (i.e., number of operations) of a circuit U over a gate set $\mathcal{G}$ which turns a product state $|0\rangle^{\otimes(n+m)}$ consisting of n “system” qubits and m auxiliary qubits into an ε -approximate version of $|\Psi_{\text{target}}\rangle$ on the system qubits. Closeness is measured in terms of the L^1 norm; i.e., the reduced density operator of the state $U|0\rangle^{\otimes(n+m)}$ after tracing out the m auxiliary qubits should satisfy

$$\left\| \text{tr}_m U|0\rangle\langle 0|^{\otimes(n+m)} U^\dagger - |\Psi_{\text{target}}\rangle\langle \Psi_{\text{target}}| \right\|_1 \leq \varepsilon.$$

Here, $\mathcal{G}$ is a set of allowed (unitary) operations that is computationally universal. For example, it could be chosen as the set of all single-qubit Hadamard and T gates, and all two-qubit CNOT gates, but the exact choice of gate set $\mathcal{G}$ is typically irrelevant for complexity-theoretic considerations. This is because particular choices affect only overall constants and do not affect the (asymptotic) scaling.

Beyond unitary state complexity, similar natural notions of the complexity of a state can be defined by considering preparation protocols involving measurements and

adaptivity (meaning that the applied gates depend on previously obtained measurement results) as well as post-selection (heralding). Our definitions for the bosonic case mirror these concepts of complexity of an n -qubit state.

IV. ALLOWED OPERATIONS FOR HYBRID QUBIT-BOSON SYSTEMS

In the following, we consider single-mode bosonic states $|\Psi_{\text{target}}\rangle \in L^2(\mathbb{R})$ for simplicity (but these considerations immediately generalize to multimode systems). To define a notion of complexity of such a state, we need to agree on the available resources, i.e., the auxiliary systems and operations used. Our notion of the complexity $C^*(|\Psi\rangle)$ of a state $|\Psi\rangle$ should be such that the function $n \mapsto C^*(|\Psi_n\rangle)$ [respectively, $(\kappa, \Delta) \mapsto C^*(|\Psi_{\kappa, \Delta}\rangle)$] quantifies the number of operations needed to prepare these states for different elements in the family yet with the same set of operations. In particular, the considered set of operations should consist of only “physically reasonable” operations. For example, this means that only bounded-strength operations should be allowed, implying that highly squeezed states necessarily have high complexity as their preparation requires application of a large number of (squeezing) unitaries from such a gate set. Our choice of operations will satisfy this requirement.

In our work, we are particularly interested in qubit-oscillator couplings as an experimental building block for designing GKP-preparation procedures, similar to Refs. [16–18]. In addition to such qubit-oscillator operations, we allow arbitrary single- and two-qubit operations and (limited) bosonic Gaussian operations. In more detail, the systems we consider consist of one “system” oscillator (boson), m auxiliary oscillators (bosons), and m' auxiliary qubits. That is, it is described by the Hilbert space $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})^{\otimes m} \otimes (\mathbb{C}^2)^{\otimes m'}$. The set of operations we consider consists of the following.

- (i) Preparation of the single-qubit computational basis state $|0\rangle$ on a qubit and of the vacuum state $|\text{vac}\rangle$ on any mode.
- (ii) Arbitrary single- and two-qubit unitaries acting on any qubit or any pair of qubits.
- (iii) Gaussian one- and two-mode unitaries generated by Hamiltonians, which are quadratic in the mode operators and have bounded strength. In more detail, for an n -mode bosonic system (here, $n = m + 1$), we consider Hamiltonians of the form

$$H(A) = \frac{1}{2} \sum_{j,k=1}^{2n} A_{j,k} R_j R_k =: \frac{1}{2} R^T A R,$$

where $R = (Q_1, P_1, \dots, Q_n, P_n)$ denotes the vector of quadrature operators and where $A = A^T \in \text{Mat}_{2n \times 2n}(\mathbb{R})$ is a symmetric matrix with real entries. We say that the unitary $U(A) = e^{iH(A)}$ has

bounded strength if the operator norm $\|A\|$ is bounded by a constant, which can be arbitrary but fixed (its choice does not affect the scaling of the resulting notion of complexity). Concretely, we say that $U(A)$ is bounded if

$$\|A\| \leq 2\pi. \quad (1)$$

- (iv) One-mode phase-space displacements of constant strength. For an n -mode (here, $n = m + 1$) bosonic system with (vector of) quadrature operators $R = (Q_1, P_1, \dots, Q_n, P_n)$, every vector $d \in \mathbb{R}^{2n}$ defines a unitary

$$D(d) = e^{i \sum_{j=1}^{2n} d_j R_j}$$

called the (Weyl) phase-space displacement by d . Its action on mode operators is given by

$$D(d)^\dagger R_j D(d) = R_j + d_j I$$

for a $j \in [2n]$. Correspondingly, the vector d is often referred to as a displacement vector. The unitary $D(d)$ is a single-mode displacement (acting on a mode $j \in [n]$) if all entries of d other than d_{2j-1} (associated with Q_j) and d_{2j} (associated with P_j) vanish. It is of bounded strength if the Euclidean norm $\|d\| = \sqrt{\sum_{j=1}^{2n} d_j^2}$ of d is bounded by a constant. In the following, we again arbitrarily choose this constant to be 2π ; i.e., we say that $D(d)$ is a bounded-strength phase-space displacement if and only if

$$\|d\| \leq 2\pi. \quad (2)$$

- (v) Qubit-controlled single-mode displacements of bounded strength. We say that $\text{ctrl}_j D(d)$ is single mode and of bounded strength if this is the case for the displacement operator $D(d)$.
- (vi) Homodyne (Q)-quadrature measurements of any bosonic mode and computational basis measurements of any qubit.

Each of these building blocks can be physically implemented using current technology within suitable experimental platforms: Operations from linear optics are well established, and qubit-oscillator interactions—typically described by Jaynes-Cummings-type Hamiltonians—are becoming standard tools across various systems. Notably, in superconducting circuits, qubit-controlled displacements, both under resonant drive [27] and in the dispersive regime [19], as well as qubit-controlled phase-space rotations in the dispersive regime [28] have been experimentally demonstrated. We refer the interested readers to Ref. [29] for a comprehensive and up-to-date review of the state of the art,

including detailed discussions on the physical realization of the operations used in this work.

In the following, we refer to the operations from (i)–(vi) as elementary operations. Moreover, we define the set of all unitaries of the form (ii), (iii), (iv), and (v) as $\mathcal{G}$.

We consider the use of each of these elementary operations as contributing one unit of complexity to any protocol. Since we further assume that any bosonic mode or qubit is initialized before further manipulations, this means that the overall complexity of any protocol will be lower bounded by the total number of bosonic modes and qubits involved. (The latter quantity is sometimes referred to as the width of a circuit or protocol.)

In fact, the protocols we propose are expressed entirely in terms of beam splitters, phase-space rotations, and single-mode squeezing of constant strength [which generate unitaries from (iii)] as well as single-qubit Clifford unitaries and the qubit-boson operation (v).

The constant-strength restriction in (iii), (iv), and (v) is motivated by the fact that coupling strengths derived from physical interactions are typically constant (i.e., independent of the system size). In particular, this means that the evolution time required to realize such a unitary depends (typically linearly) on the involved parameter. A physically reasonable notion of complexity should take this into account. We achieve this by restricting our parameters to fixed finite intervals. We note that the choice of constants in Eqs. (1) and (2) is arbitrary and does not affect the overall scaling we obtain in asymptotic settings.

A. Unitary state complexity and circuit complexity

Let us first consider the problem of preparing a target state $|\Psi_{\text{target}}\rangle \in L^2(\mathbb{R})$ by means of unitary operations. Concretely, we consider protocols of the following form on a system with Hilbert space $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})^{\otimes m} \otimes (\mathbb{C}^2)^{\otimes m'}$, i.e., $m + 1$ oscillators and m' qubits.

- Each qubit is initialized in the computational basis state $|0\rangle$, and each bosonic mode is initialized in the vacuum state $|\text{vac}\rangle$. After this step, the system is in the state $|\Psi\rangle = |\text{vac}\rangle \otimes |\text{vac}\rangle^{\otimes m} \otimes |0\rangle^{\otimes m'}$.
- A sequence of $T \in \mathbb{N}$ unitaries $U_1, \dots, U_T \in \mathcal{G}$ is applied. Denoting by $U = U_T \dots U_1$ the corresponding unitary, the state after this step is the state $U|\Psi\rangle$.
- The output of the protocol is the reduced density operator of the first mode, i.e., $\rho = \text{tr}_{m,m'} U|\Psi\rangle\langle\Psi|U^\dagger$, where $\text{tr}_{m,m'}$ denotes the partial trace over all m auxiliary modes and m' auxiliary qubits. The output state ρ is supposed to be a good approximation to the target state $|\Psi_{\text{target}}\rangle\langle\Psi_{\text{target}}|$ (as quantified below).

A qubit-boson protocol for preparing a target state $|\Psi_{\text{target}}\rangle$ of this form is referred to a unitary state preparation protocol. Observe that the total number of operations from the set (i)–(vi) of allowed operations is equal to $T + (m + 1) + m'$, since $m + 1$ vacuum states and m'

qubits are prepared initially. We say that the protocol achieves error $\varepsilon \geq 0$ if the output state is ε -close in L^1 distance to the desired target state $|\Psi_{\text{target}}\rangle$, i.e., if

$$\|\text{tr}_{m,m'} U |\Psi\rangle \langle \Psi| U^\dagger - |\Psi_{\text{target}}\rangle \langle \Psi_{\text{target}}|\|_1 \leq \varepsilon.$$

We say that a state $|\Psi_{\text{target}}\rangle \in L^2(\mathbb{R})$ has unitary circuit complexity $\mathcal{C}_\varepsilon(|\Psi_{\text{target}}\rangle)$ for $\varepsilon \geq 0$ if there is a unitary state preparation protocol achieving error $\varepsilon \geq 0$ such that

$$\mathcal{C}_\varepsilon(|\Psi_{\text{target}}\rangle) = T + (m + 1) + m',$$

and the rhs is minimal among all unitary state preparation protocols with this property. In other words, the reduced density matrix on the first mode of $U(|\text{vac}\rangle \otimes |\text{vac}\rangle^{\otimes m} \otimes |0\rangle^{\otimes m'})$ is ε -close in L^1 distance to the target state $|\Psi_{\text{target}}\rangle$, and the protocol is resource optimal in the sense that the number $T + (m + 1) + m'$ of operations used is minimal.

To study this notion of (unitary) complexity of a state $|\Psi_{\text{target}}\rangle$, we often decompose a given unitary $U \in L^2(\mathbb{R})$ into a product $U = U_T \dots U_1$ of unitaries $U_1, \dots, U_T$ belonging to our allowed gate set $\mathcal{G}$ (we consider only cases where this kind of factorization exists and is exact). We denote by

$$\mathcal{C}_\mathcal{G}(U) = \min\{T \in \mathbb{N} | U = U_T \dots U_1, U_i \in \mathcal{G}\} \quad (3)$$

the minimum number of unitaries needed in such a factorization. Note that this is closely connected to the unitary state complexity of a state: If $\|U|\text{vac}\rangle \langle \text{vac}| U^\dagger - |\Psi_{\text{target}}\rangle \langle \Psi_{\text{target}}|\|_1 \leq \varepsilon$, then the unitary state complexity is bounded by $\mathcal{C}_\varepsilon(|\Psi_{\text{target}}\rangle) \leq \mathcal{C}_\mathcal{G}(U) + 1$. This is because the factorization of the unitary U into unitaries from $\mathcal{G}$ provides a protocol for preparing $|\Psi\rangle$, and this protocol involves only the preparation of one vacuum state and application of $\mathcal{C}_\mathcal{G}(|\Psi_{\text{target}}\rangle)$ gates from $\mathcal{G}$.

1. The zero-error unitary state complexity of a coherent state

As a simple warm-up example, we consider the (unitary) complexity of coherent states. These states can be prepared exactly using the set of operations we consider (i.e., preparation of $|\text{vac}\rangle$ on bosonic modes, $|0\rangle$ on qubits, and unitaries belonging to the set $\mathcal{G}$; see Sec. IV). Motivated by this, we consider their zero-error complexity only.

To fix notation, let $D(d) = e^{i(d_Q Q - d_P P)}$ be the (Weyl) phase-space displacement operator associated with $d = (d_Q, d_P) \in \mathbb{R}^2$. Throughout this section, we consider the coherent state

$$|d\rangle = D(d)|\text{vac}\rangle \quad (4)$$

with $d = (d_Q, d_P) \in \mathbb{R}^2$. To illustrate how the notion of the complexity $\mathcal{C}(|d\rangle)$ of the state $|d\rangle$ depends on the set of

operations we allow, first, suppose we allow only preparation of a bosonic mode in the state $|\text{vac}\rangle$ and application of displacements $\{D(f)\}_f$ of constant strength, i.e., $\|f\| \leq 2\pi$. In this case, the complexity of $\mathcal{C}(|d\rangle)$ is upper bounded by $\mathcal{C}(|d\rangle) \leq \|d\| + 1$, since $|d\rangle = D(d/\|d\|)^{\|d\|} |\text{vac}\rangle$ (if $\|d\| \notin \mathbb{N}$, rounding should be applied). Indeed, preparing one vacuum state and $\|d\|$ applications of unitaries from our gate set $\{D(f)\}_{\|f\| \leq 2\pi}$ results in the target state $|d\rangle$.

If we assume that the available set of operations includes also allows unitaries of the form (iii), i.e., those generated by quadratic Hamiltonians, the complexity $\mathcal{C}(|d\rangle)$ of the coherent state $|d\rangle$ is upper bounded by $\mathcal{C}(|d\rangle) \leq 2 \log \|d\| + 3$. A corresponding protocol goes as follows. First, we apply the displacement $D(0, \|d\|)$ to the vacuum state $|\text{vac}\rangle$ using single-mode squeezing operators $S(z) = e^{iz(QP+PQ)/2}$. This is based on the equality

$$D[(0, c)] = S(\log 1/c) D[(0, 1)] S(\log c)$$

for all $c > 0$. Second, we rotate in phase space by applying the unitary $R(\theta) = e^{-i\theta(Q^2+P^2)/2}$, where $\theta = \pi - \arg d \in [0, 2\pi)$. Decomposing the squeezing and rotation unitaries into the respective constant-strength quantities shows the claim.

In fact, we can show that this protocol is optimal up to constant factors, establishing that the (unitary) complexity of the coherent state $|d\rangle$ is given by $\mathcal{C}_0(|d\rangle) = \Theta(\log \|d\|)$ as $\|d\| \rightarrow \infty$. To argue that every protocol needs $\Omega(\log \|d\|)$ elementary operations, we use moment bounds; see, e.g., Refs. [30,31]. We exploit the fact that every bounded-strength Gaussian unitary is moment-limited; i.e., the change in energy of the (harmonic oscillator) Hamiltonian $H = Q^2 + P^2$ under application of such a unitary is upper bounded by an affine-linear function [i.e., $\langle U\Psi, H U\Psi \rangle \leq a \langle \Psi, H \Psi \rangle + b$ for constants (a, b)]. In particular, starting from the vacuum state $|\text{vac}\rangle$, a circuit $U = U_T \dots U_1$ composed of T elementary operations can create only a state whose energy is bounded by a function exponential in T . Because the coherent state has energy $\langle d, H d \rangle = \|d\|^2 + 1$, its preparation needs $\Omega(\log \|d\|)$ operations. Remarkably, this matches (up to a constant) the number of operations in the preparation protocol described above. We can summarize this in the following statement.

Lemma 4.1 (zero-error complexity of coherent states). For $d \in \mathbb{R}^2$, let $|d\rangle \in L^2(\mathbb{R})$ be the coherent state defined by Eq. (4). Then, we have

$$\mathcal{C}_0(|d\rangle) = \Theta(\log \|d\|) \quad \text{for } \|d\| \rightarrow \infty.$$

We refer to Appendixes B 1 and B 2 for a rigorous proof of the statement. The derivation of Lemma 4.1 illustrates a few of the building blocks for our main result, namely, using an explicit state-preparation protocol for the upper bound and moment limits for the lower bounds.

However, the protocol and its analysis are somewhat more involved and prepare an approximation only to the state $|\text{GKP}_{\kappa,\Delta}\rangle$ [i.e., we consider the unitary and heralded complexities $\mathcal{C}_\varepsilon^*(|\text{GKP}_{\kappa,\Delta}\rangle)$ and $\mathcal{C}_{p,\varepsilon}^{\text{her}}(|\text{GKP}_{\kappa,\Delta}\rangle)$ with non-zero error $\varepsilon > 0$].

B. Heralded state complexity

Going beyond unitary preparation, we also consider heralded state generation. Here, the protocol additionally produces an output flag $F \in \{\text{acc}, \text{rej}\}$ (i.e., a classical output bit) indicating whether the protocol accepts (succeeds) or rejects (in the case of failure) the output. In more detail, we consider protocols of the following form.

- (i) Each mode is initialized in the vacuum state $|\text{vac}\rangle$ and each qubit in the computational state $|0\rangle$; i.e., the initial state is $|\Psi\rangle = |\text{vac}\rangle \otimes |\text{vac}\rangle^{\otimes m} \otimes |0\rangle^{m'}$.
- (ii) Subsequently, a unitary U as described before is applied; i.e., U consists of gates from the set $\mathcal{G}$. We denote by T_1 the number of such gates.
- (iii) A measurement is then applied to the auxiliary modes and the qubits of the prepared state $U|\Psi\rangle$. More precisely, a homodyne position measurement is applied to every auxiliary bosonic mode and a computational basis measurement to every auxiliary qubit. Let us denote the corresponding positive operator-valued measure (POVM) by $\{E_\alpha\}_{\alpha \in \mathcal{M}}$, where $\mathcal{M} = \mathbb{R}^m \times \{0, 1\}^{m'}$ denotes the set of measurement outcomes.
- (iv) Depending on the measurement outcome α , a flag $F(\alpha) \in \{\text{acc}, \text{rej}\}$ is computed by means of an efficiently computable function $F: \mathcal{M} \rightarrow \{\text{acc}, \text{rej}\}$.
- (v) If $F(\alpha) = \text{acc}$, a vector $d(\alpha) = [d_Q(\alpha), d_P(\alpha)] \in \mathbb{R}^2$ is computed from the measurement outcome by means of an efficiently computable function $d: \mathcal{M} \rightarrow \mathbb{R}^2$. The postmeasurement state on the first mode is then displaced by application of the unitary $D[d(\alpha)] = e^{i[d_Q(\alpha)Q - d_P(\alpha)P]}$, i.e., a phase-space displacement by the vector $d(\alpha)$.

We note that this kind of “correction” operation by translation is commonly considered, e.g., in the context of Steane-type code state preparation for GKP codes [1].

- (vi) The output of the protocol is the output flag $F(\alpha)$ and, assuming that $F(\alpha) = \text{acc}$, the reduced density operator on the first mode. Conditioned on acceptance, we want the state of the first mode to be close to the target state $|\Psi_{\text{target}}\rangle$.

We call a protocol for state preparation of this form a heralded state-preparation protocol. The definition implies that the (average) output state of the protocol conditioned on acceptance is given by

$$\rho_{\text{acc}} = \frac{1}{p_{\text{acc}}} \int_{F^{-1}(\{\text{acc}\})} p(\alpha) D[d(\alpha)] \rho^{(\alpha)} D[d(\alpha)]^\dagger d\alpha,$$

where we introduce normalized states $\rho^{(\alpha)}$ as

$$p(\alpha) \rho^{(\alpha)} = \text{tr}_{m,m'}((I \otimes E_\alpha) U |\Psi\rangle \langle \Psi| U^\dagger)$$

with $p(\alpha) = \langle \Psi | U^\dagger (I \otimes E_\alpha) U | \Psi \rangle$ and where

$$p_{\text{acc}} = \int_{F^{-1}(\{\text{acc}\})} p(\alpha) d\alpha \quad (5)$$

is the probability that the protocol accepts [i.e., that $F(\alpha) = \text{acc}$]. We say that the protocol produces an ε approximation of the state $|\Psi_{\text{target}}\rangle$ with acceptance probability p_{acc} [see Eq. (5)] if

$$\|\rho_{\text{acc}} - |\Psi_{\text{target}}\rangle \langle \Psi_{\text{target}}|\|_1 \leq \varepsilon.$$

The number of operations from the set (i)–(vi) applied in this protocol is determined by the number T_1 of unitaries applied to implement U [see (ii)] and the unitaries applied to implement the “correction” $D[d(\alpha)]$ after obtaining the measurement result α . We consider the worst case; i.e., we use the minimal number of elementary unitaries from $\mathcal{G}$ required to implement $D[d(\alpha)]$, maximized over all possible shifts $d(\alpha)$ associated with different measurement outcomes $\alpha \in F^{-1}(\{\text{acc}\})$. That is, we use the quantity [cf. Eq. (3)]

$$T_2 = \sup_{\alpha \in F^{-1}(\{\text{acc}\})} \mathcal{C}_{\mathcal{G}}\{D[d(\alpha)]\}.$$

Taking into account the $(m+1) + m'$ single-mode and single-qubit preparations, the maximal number $T_1 + T_2$ of unitary operations, and the $m + m'$ single-mode and single-qubit measurements, the maximal (i.e., worst-case over measurement outcomes) number of operations from (i)–(vi) in such a protocol is, therefore,

$$T_1 + T_2 + (2m+1) + 2m'. \quad (6)$$

(Here, we do not consider the complexity of the classical computation used to evaluate the functions F and d but simply assume that this is efficient. Indeed, this is the case for the protocols we consider.)

We say that a state $|\Psi_{\text{target}}\rangle \in L^2(\mathbb{R})$ has heralded state complexity $\mathcal{C}_{p,\varepsilon}^*(|\Psi_{\text{target}}\rangle)$ for $\varepsilon \geq 0$ and $p > 0$ if there is a heralded state-preparation protocol which prepares an ε approximation to the state $|\Psi_{\text{target}}\rangle$ with probability at least $p_{\text{acc}} \geq p$, whose number of operations [see Eq. (6)] is equal to

$$\mathcal{C}_{p,\varepsilon}^*(|\Psi_{\text{target}}\rangle) = T_1 + T_2 + (2m+1) + 2m'$$

and has the property that the rhs of this equation is minimal among all heralded state preparation protocols with this property.

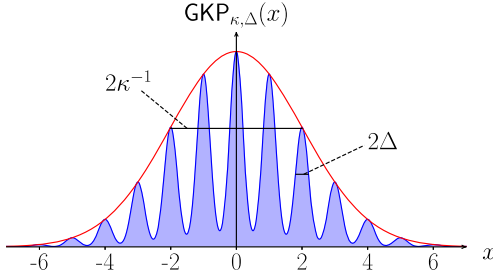


FIG. 1. The approximate GKP state $|\text{GKP}_{\kappa, \Delta}\rangle$ in position space. The red line represents the envelope $\eta_{\kappa}(x) \propto e^{-\kappa^2 x^2/2}$ of the state, a Gaussian with variance κ^{-2} . The GKP wave function is illustrated in blue (the shading is for visual emphasis only). According to our convention, this function has Gaussian peaks of variance Δ^2 at all integers.

V. COMPLEXITY BOUNDS FOR GKP STATES

We are interested in characterizing the complexity of states in the two-parameter family $\{|\text{GKP}_{\kappa, \Delta}\rangle\}_{\kappa, \Delta > 0} \subset L^2(\mathbb{R})$ of states defined by

$$\text{GKP}_{\kappa, \Delta}(x) = \frac{C_{\kappa, \Delta} \sqrt{\kappa}}{\sqrt{\pi} \Delta} \sum_{z \in \mathbb{Z}} e^{-\kappa^2 z^2/2} e^{-(x-z)^2/(2\Delta^2)} \quad (7)$$

for $x \in \mathbb{R}$. Here, $C_{\kappa, \Delta} > 0$ is a constant such that the vector $|\text{GKP}_{\kappa, \Delta}\rangle$ is normalized. For each pair $(\kappa, \Delta) \in [0, \infty) \times [0, \infty)$, Eq. (7) defines an approximate (finitely squeezed) GKP state $|\text{GKP}_{\kappa, \Delta}\rangle$ with peaks of width Δ localized around each integer (constituting a “grid” following common terminology in the GKP literature) and a Gaussian envelope of variance $1/\kappa^2$; see Fig. 1. We note that our conventions ensures an integer spacing of the peaks. This convention is slightly different from the one typically used in the literature. More precisely, this difference is accounted for by an application of a constant-strength squeezing operator, which amounts to rescaling the parameters (κ, Δ) by (irrelevant) constant factors. The main result of this work is the following characterization of the complexity of GKP states.

Theorem 5.1. There is a polynomial $r(\kappa, \Delta)$ with $r(0, 0) = 0$ such that, for all functions $p(\kappa, \Delta)$ and $\varepsilon(\kappa, \Delta)$ satisfying $r(\kappa, \Delta) \leq \varepsilon(\kappa, \Delta) \leq p(\kappa, \Delta) \leq 1/10$ for sufficiently small (κ, Δ) , the heralded state complexity satisfies in the limit $(\kappa, \Delta) \rightarrow (0, 0)$

$$\mathcal{C}_{p(\kappa, \Delta), \varepsilon(\kappa, \Delta)}^{*, \text{her}}(|\text{GKP}_{\kappa, \Delta}\rangle) = \Theta(\log 1/\kappa + \log 1/\Delta).$$

This result implies, in particular, that the approximation error is lower bounded by a constant [independent of (κ, Δ)] for any protocol that has constant acceptance probability p_{acc} and uses a number of elementary operations that is sublinear in $(\log 1/\kappa, \log 1/\Delta)$.

Protocol 1. Approximate GKP-state-preparation protocol

Input: Parameters $\kappa \in (0, 1/4)$ and $\Delta \in (0, 1/4)$.

Output: Either **accept** or **reject**; and in the case of acceptance, a state of a single mode that is close to the state $|\text{GKP}_{\kappa, \Delta}\rangle$ state, see Theorem 5.2.

- 1: Apply the comb-state-preparation Protocol 2 with input Δ and $n = \lfloor \frac{4}{3} \log_2 1/\kappa \rfloor$. This results in a state close to the approximate comb state $|\text{III}_{L, \Delta}\rangle \langle \text{III}_{L, \Delta}|$, where $L = 2^n$.
- 2: Apply the envelope-shaping Protocol 3 with input state τ and with parameters κ and L . If Protocol 3 accepts, **return accept** and the single-mode state that it produced.
- 3: **return reject** otherwise.

A. Efficient preparation of GKP states

The proof of Theorem 5.1 actually shows a stronger statement: An approximation to the state $|\text{GKP}_{\kappa, \Delta}\rangle$ can be obtained probabilistically using a system consisting of only two bosonic modes and a single qubit.

Protocol 1 provides an efficient GKP-state-preparation scheme in the sense that the required number of operations to realize it scales logarithmically in the GKP-state parameters (κ, Δ) while the achieved L^1 distance is bounded polynomially in (κ, Δ) . More precisely, we show that the following holds.

Theorem 5.2. Given sufficiently small κ and Δ , the output state of Protocol 1 conditioned on acceptance is close to $|\text{GKP}_{\kappa, \Delta}\rangle$. The protocol accepts with probability at least $\Pr[\text{acc}] \geq 1/10$. The output state $\rho \in \mathcal{B}[L^2(\mathbb{R})]$ conditioned on acceptance satisfies

$$\|\rho - |\text{GKP}_{\kappa, \Delta}\rangle \langle \text{GKP}_{\kappa, \Delta}|\|_1 \leq O(\sqrt{\Delta}) + O(\kappa^{1/3}).$$

Furthermore, the protocol requires $O(\log 1/\kappa + \log 1/\Delta)$ elementary operations (see Sec. IV).

In the following, we elaborate on the two key ingredients Protocol 1 relies on, which are also illustrated in Fig. 2. For a detailed analysis of this protocol and the leveraged subprotocols, we refer to Appendixes C–E.

1. Comb-state preparation

Using an auxiliary qubit enables us to apply non-Gaussian transformations on the initial vacuum state. We use this in order imprint a discrete structure into a CV state, namely, (finitely many) equidistant peaks (local maxima). We denote these states comb states, defined pointwise as

$$\text{III}_{L, \Delta}(x) \propto \sum_{z=-L/2}^{L/2-1} e^{-(x-z)^2/(2\Delta^2)},$$

where L is number of peaks and determines $\Delta > 0$ the width of each individual peak. Protocol 2 provides an efficient way to prepare the comb state. More precisely, we

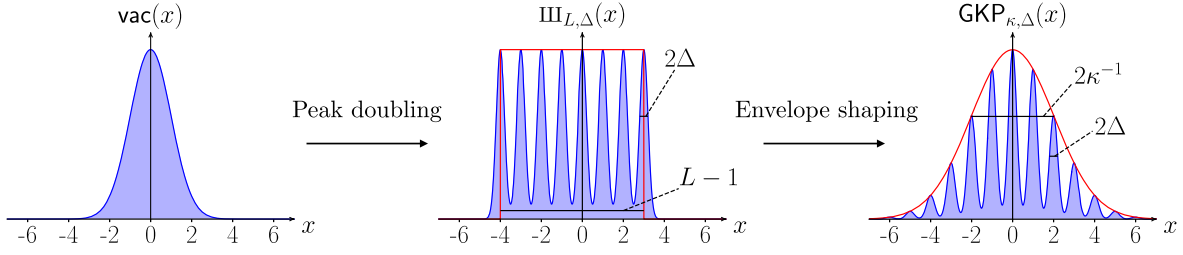
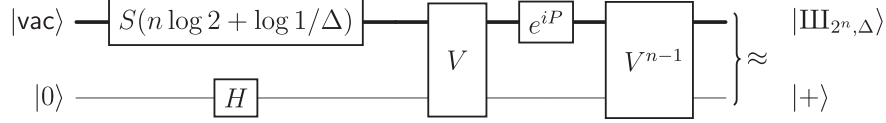


FIG. 2. The conceptual steps of the state-preparation protocol.

FIG. 3. Circuit diagram of Protocol 2. It uses the V gate described in Fig. 4. The exponent of V indicates the number of applications. The squeezing unitary is realized by composing single-mode squeezing operations from the set $\mathcal{G}$ (see Sec. IV).

can approximately generate a comb state with $L = 2^n$ in n rounds, each of which doubles the previous number of peaks by applying constant-depth qubit-boson unitary V (see Fig. 4 for a circuit). To give some intuition on the repeated action of the unitary V , let us consider its action on a single-peak state of the form $|x\rangle \otimes |+\rangle$, where $|x\rangle$ is the (unnormalized) position eigenstate associated with an integer eigenvalue $x \in \mathbb{Z}$ of the position operator Q . Applying the unitary V splits the peak in two:

$$\begin{aligned}
 |x\rangle \otimes |+\rangle &\xrightarrow{S(-\log 2) \otimes I} |2x\rangle \otimes |+\rangle \\
 &\xrightarrow{\text{ctrl} e^{-iP}} |2x\rangle \otimes |0\rangle + |2x+1\rangle \otimes |1\rangle \\
 &\xrightarrow{I \otimes H} |2x\rangle \otimes |+\rangle + |2x+1\rangle \otimes |-\rangle \\
 &\xrightarrow{\text{ctrl} e^{i\pi Q}} (|2x\rangle + |2x+1\rangle) \otimes |+\rangle,
 \end{aligned}$$

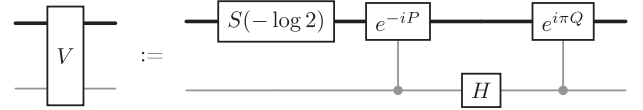
that is,

$$V(|x\rangle \otimes |+\rangle) \propto (|2x\rangle + |2x+1\rangle) \otimes |+\rangle$$

for $x \in \mathbb{Z}$. After n iterations of V , we thus obtain 2^n peaks from the initial single peak; i.e., we have

$$\begin{aligned}
 V^n(|x\rangle \otimes |+\rangle) \\
 \propto (|2^n x\rangle + |2^n x+1\rangle + \dots + |2^n x+2^n-1\rangle) \otimes |+\rangle
 \end{aligned}$$

for $x \in \mathbb{Z}$. In physically realistic scenarios, we have to replace position eigenstates by their normalizable approximation, i.e., squeezed vacuum states. In order to stay sufficiently located around integers, we have to first squeeze the initial vacuum state by application of the squeezing unitary $S(n \log 2 + \log 1/\Delta)$, which, when decomposed into a sequence of constant-strength squeezing unitaries, requires $n + O(\log 1/\Delta)$ elementary

FIG. 4. Circuit implementing the unitary V used in the comb-state-preparation protocol. It uses two qubit-controlled displacements: by 1 in the Q direction and by π in the P direction.

operations to implement. The unitary circuit implementing the comb-state preparation in Protocol 2 is illustrated in Fig. 3. It uses $O(n + \log 1/\Delta)$ elementary operations in order to achieve an L^1 -norm distance of $\text{poly}(\Delta)$ from the state $|\text{III}_{2^n,\Delta}\rangle$. Importantly, it requires only one auxiliary qubit (i.e., $m = 0$ and $m' = 1$).

In the context of GKP-state preparation (cf. Protocol 1), a comb state with $L = \Theta[1/\text{poly}(\kappa)]$ peaks is generated in $n = O(\log 1/\kappa)$ rounds of the peak-doubling scheme. We refer to Appendix C 2 for a rigorous statement about the output state of Protocol 2.

Protocol 2. Comb-state preparation

Input: A squeezing parameter $\Delta \in (0, 1/4)$ and a number of rounds $n \geq 1$.

Output: A state close to the state $|\text{III}_{2^n,\Delta}\rangle$.

- 1: Prepare the squeezed vacuum state $|\Psi_{2^{-n}\Delta}\rangle = S(n \log 2 + \log 1/\Delta)|\text{vac}\rangle$.
- 2: Prepare the qubit state $|+\rangle$. Denote the resulting state by $|\Phi^{(0)}\rangle \leftarrow |\Psi_{2^{-n}\Delta}\rangle \otimes |+\rangle$.
- 3: Apply $(e^{iP} \otimes I)V$ to the state $|\Phi^{(0)}\rangle$ yielding $|\Phi^{(1)}\rangle \leftarrow (e^{iP} \otimes I)V|\Phi^{(0)}\rangle$.
- 4: **for** $k \in \{2, \dots, n\}$ **do**
- 5: Apply V to the state $|\Phi^{(k-1)}\rangle$, yielding $|\Phi^{(k)}\rangle \leftarrow V|\Phi^{(k-1)}\rangle$.
- 6: **return** the first register of the state $|\Phi^{(n)}\rangle$, i.e., the state $\text{tr}_{\text{qubit}}|\Phi^{(n)}\rangle\langle\Phi^{(n)}|$.

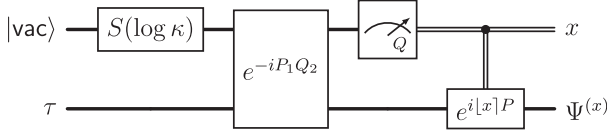


FIG. 5. Circuit used in the envelope-Gaussification protocol (Protocol 3). For measurement outcomes $x \in \mathbb{R}$ with $x \notin [-L/8 - 1/2, L/8 + 1/2]$, the protocol returns **reject**, and there is no output state in this case. For values $x \in [-L/8 - 1/2, L/8 + 1/2]$ (a case illustrated in the figure), the unitary $e^{i[x]P}$ is applied to the second mode. This is a classically controlled displacement gate; i.e., it involves the parameter $[x] \in \mathbb{Z}$ which is computed classically. Note that both the squeezing gate $S(\log \kappa)$ and $e^{i[x]P}$ need to be decomposed in terms of constant-strength squeezing and displacements gates, respectively, to obtain operations from the set $\mathcal{G}$ (see Sec. IV).

2. Envelope shaping

In the next step, the comb state $|\text{III}_{L,\Delta}\rangle$ is equipped with the desired Gaussian envelope. Let us briefly motivate the importance of the Gaussian envelope. One reason to impose a Gaussian envelope stems partly from the fact that it ensures that the description of GKP states in position and momentum space are equivalent in the sense that the Fourier transform essentially interchanges the roles of the parameters κ and Δ . This property is crucial in the analysis of the factoring algorithm presented in Ref. [14]. Another argument for a Gaussian envelope is based on energy considerations; see Ref. [7].

The squeezed vacuum state $S(\log 1/\kappa)|\text{vac}\rangle$, previously prepared in an auxiliary mode, is “added” via the bosonic gate $e^{-iP_1 Q_2}$ (realizable in constant depth using constant-strength single-mode squeezers and a beam splitter). In the next step, we perform a homodyne position measurement of one of the modes, yielding a measurement outcome x . Intuitively speaking, the postmeasurement state corresponds to a comb state multiplied by a Gaussian envelope which is centered around the measurement outcome x . If $|x|$ is not too large, we can correct for this by a phase-space displacement of the postmeasurement state while increasing closeness to the desired GKP state. To be more precise, depending on the measurement outcome being contained in an acceptance region Ω_L , Protocol 3 either accepts and performs a centering shift by $[x] = [x + 1/2]$, or it rejects and discards. We show that the acceptance probability is lower bounded by a constant. Furthermore, we prove that the (average) output state conditioned on acceptance is close to the state $|\text{GKP}_{\kappa,\Delta}\rangle$.

Protocol 3 is implemented by the adaptive circuit in Fig. 5. This circuit is adaptive in the sense that it involves a unitary (displacement) that is classically controlled by (a function of) the measurement result. We refer to Appendix D 2 for a trace distance bound for comb states as input to Protocol 3.

Protocol 3. Envelope-Gaussification protocol

-
- Input:** A state $\tau \in \mathcal{B}[L^2(\mathbb{R})]$, a parameter $\kappa \in (0, 1/4)$, and a parameter $L \in 8\mathbb{N}$.
- Output:** Either **accept** or **reject** and in the case of acceptance a state of a single mode. Conditioned on acceptance, this output state is close to $|\text{GKP}_{\kappa,\Delta}\rangle$.
- 1: Prepare the squeezed vacuum state $|\eta_\kappa\rangle = S(\log \kappa)|\text{vac}\rangle$ in the first register.
 - 2: Apply the unitary $e^{-iP_1 Q_2}$.
 - 3: Perform a homodyne position measurement on the first mode, resulting in an outcome $x \in \mathbb{R}$ and a postmeasurement state of the second mode.
 - 4: **if** $x \in \Omega_L = [-L/8 - 1/2, L/8 + 1/2]$, **then**
 - 5: Round the result x to the nearest integer, yielding $[x] \in \mathbb{Z}$.
 - 6: Apply the classically controlled correction unitary $e^{i[x]P}$ on the second mode.
 - 7: **return accept** and the state of the second mode.
 - 8: **else**
 - 9: **return reject**.
-

B. Lower bounds on the GKP-state complexity

Complementary to the upper bound, we establish the following complexity lower bounds by using the fact that the state $|\text{GKP}_{\kappa,\Delta}\rangle$ has a mean energy scaling as a function of (κ, Δ) and that the operations (iii)–(v) are moment-limited: Starting from the initial state $|\Psi\rangle = |\text{vac}\rangle^{\otimes(m+1)} \otimes |0\rangle^{\otimes m'}$ and applying a sequence of T unitaries $U_T \dots U_1$, the energy can at most grow exponentially. In more detail, we have

$$\langle \Psi, U^\dagger H U \Psi \rangle \leq e^{\mu T} (m + 2) =: E_m(T),$$

where $H = \sum_{j=1}^{m+1} (Q_j^2 + P_j^2)$ and for some constant $\mu > 0$ depending on the restrictions (1) and (2). Using a Markov-type argument, this directly relates to the concentration of the output states ρ of both unitary and heralding protocols in the sense that the projection on an interval $[-R, R]$ in position space $\text{tr}(\Pi_{[-R,R]}\rho)$ as well as the corresponding projection in momentum space $\text{tr}(\hat{\Pi}_{[-R,R]}\rho)$ is lower bounded by $1 - A[E_m(T)/R^2]$, where the constant A depends on the type of protocol, i.e., whether it is unitary or heralding. On the other hand, the corresponding projection of the target state $|\text{GKP}_{\kappa,\Delta}\rangle$ can be bounded as $\langle \text{GKP}_{\kappa,\Delta}, \Pi_{[R,R]} \text{GKP}_{\kappa,\Delta} \rangle \leq b\kappa R + c(\kappa, \Delta)$ and $\langle \text{GKP}_{\kappa,\Delta}, \hat{\Pi}_{[R,R]} \text{GKP}_{\kappa,\Delta} \rangle \leq \hat{b}\Delta R + c(\kappa, \Delta)$ for some constants $b, \hat{b} > 0$ and a polynomial c in (κ, Δ) with $c(0, 0) = 0$. Combining the concentration bounds yields a restriction of the L^1 -norm distance $\|\rho - |\text{GKP}_{\kappa,\Delta}\rangle\langle \text{GKP}_{\kappa,\Delta}|\|_1$ achievable with a protocol that uses at most a given number of elementary operations. These arguments culminate in the following result.

Theorem 5.3. There exists a polynomial $s(\kappa, \Delta)$ with $s(0, 0) = 0$ such that, for all functions $p(\kappa, \Delta)$ and $\varepsilon(\kappa, \Delta)$ satisfying $0 \leq \varepsilon(\kappa, \Delta) \leq p(\kappa, \Delta)$ and $s(\kappa, \Delta) \leq p(\kappa, \Delta) \leq 1$ for sufficiently small (κ, Δ) , we have

$$\mathcal{C}_{p(\kappa, \Delta), \varepsilon(\kappa, \Delta)}^{*, \text{her}}(|\text{GKP}_{\kappa, \Delta}\rangle) \geq \Omega(\log 1/\kappa + \log 1/\Delta)$$

for $(\kappa, \Delta) \rightarrow (0, 0)$.

A rigorous proof of this statement can be found in Appendix F 5.

VI. ALTERNATIVE FIGURES OF MERIT FOR GKP-STATE PREPARATION

Our notion of state complexity quantifies the accuracy of the prepared states using the L^1 -norm distance. To the best of our knowledge, this is the first time such a stringent quality measure has been considered for (approximate) GKP states. This notion is especially important for the algorithmic use that typically requires an outcome distribution corresponding to a measurement close in L^1 distance to a target distribution. This is exemplified in Ref. [14] in an algorithmic application to the integer factorization problem.

In the quantum fault-tolerance literature, the figure of merit that is typically considered is motivated by the fact that an ideal GKP state $|\text{GKP}\rangle \propto \sum_{z \in \mathbb{Z}} |z\rangle$ is the simultaneous $+1$ eigenstates of two commuting phase-space displacement operators S_P and S_Q (with respect to our conventions, these are $S_P = e^{-iP}$ and $S_Q = e^{2\pi i Q}$). Correspondingly, effective squeezing parameters $\Delta_P(\rho)$ and $\Delta_Q(\rho)$ of a density operator $\rho \in \mathcal{B}[L^2(\mathbb{R})]$ are introduced in Ref. [11] as

$$\begin{aligned} \Delta_P(\rho) &:= \sqrt{\log 1/|\text{tr}(S_P \rho)|^2} \quad \text{and} \\ \Delta_Q(\rho) &:= \sqrt{\log 1/|\text{tr}(S_Q \rho)|^2}. \end{aligned} \quad (8)$$

These (formally) vanish for the ideal GKP state $|\text{GKP}\rangle$, and, correspondingly, upper bounds on these quantities are used as a quality measure for the prepared state; see, e.g., Refs. [18, 32]. In Appendix H, we show upper bounds on these quantities for states ρ produced by our protocol.

We emphasize, however, that upper bounds on $\Delta_P(\rho)$ and $\Delta_Q(\rho)$ only are not sufficient to conclude that ρ is close to a state of the form $|\text{GKP}_{\kappa, \Delta}\rangle$. For example, although such upper bounds have been established for the output state ρ of a certain protocol (similar to ours) in Ref. [32], the state ρ obtained in that reference is far (in L^1 distance) from an approximate GKP state. Nevertheless, the use of the effective squeezing parameters given in Eq. (8) is suitable and typically sufficient for fault-tolerance applications, where the key property of the prepared states is an approximate phase-space translation invariance with

respect to a lattice of phase-space translation vectors. Upper bounds on the quantities (8) provide a quantitative expression of this translation invariance.

VII. DIFFERENT APPROACHES TO PREPARING GKP STATES

There are a number of existing protocols for approximate GKP-state preparation based on the defining property of ideal GKP states. For example, the protocols proposed in Refs. [16–18] rely on the idea of gradually projecting into an eigenspace of the operator S_P with eigenvalue $e^{i\theta}$ for some $\theta \in [0, 2\pi)$ while extracting information about θ . This is achieved via standard phase estimation procedures for (controlled) unitaries. Finally, the outcome state is corrected by a shift that depends on an estimate $\hat{\theta}$ of θ , thereby approximately preparing a $+1$ eigenstate of S_P .

In more detail, the protocol first introduced in Ref. [16] and further modified in Ref. [17] uses an auxiliary qubit to perform n rounds of phase estimation by repetition; that is, intermediate qubit resets are required. Starting from a squeezed vacuum state, each round adds two new equidistant peaks, with amplitudes following a binomial distribution postselected on all relevant qubit measurements yielding zero. Asymptotically, the resulting wave function can be approximated by a Gaussian distribution with an envelope parameter scaling as $\kappa = O(1/n)$, as implied by the central limit theorem. This realizes an approximate projection onto the $+1$ eigenspace of S_P . For different (nonzero) qubit measurement outcomes, the corresponding eigenvalue can be estimated, and a suitable (constant-strength) correction is applied. Because this protocol adds only two peaks at a time, the run-time is linear in the number L of peaks generated. In contrast, our protocol has a logarithmic run-time.

The protocol in Ref. [18] takes a different form but can also be interpreted as performing a gradual projection onto the $+1$ eigenspace of S_P . It employs n rounds and 2^n squeezed cat states as input to create an approximate GKP state with $L = 2^n$ peaks. In the k th round, two states with 2^k peaks are combined using a beam splitter to produce a state with 2^{k+1} peaks, while a P -quadrature measurement is performed on one of the output modes. Combining the outcomes of all homodyne measurements allows estimation of the eigenvalue of S_P , followed by a (constant-strength) corrective shift. The corresponding circuit, thus, has a logarithmic depth in the number L of peaks of the final approximate GKP state, which matches our protocol. However, our protocol uses only two modes instead of a number of modes which are linear in L .

Two preparation protocols tailored to the neutral atom platform are presented in Ref. [33]. The first starts from a squeezed vacuum state and employs postselection based on midcircuit measurements of an auxiliary qubit, adding two peaks per round via constant-strength qubit-controlled

displacements. The second protocol uses qubit-controlled displacements with varying strengths and requires mid-circuit resets of the auxiliary qubit. Both have linear complexity in the number L of peaks generated.

A somewhat different approach is a unitary circuit proposed in Ref. [32]. This protocol has a superficial resemblance to ours, since it proceeds in n rounds, where each round doubles the number of peaks (meaning that the number of rounds is logarithmic in L). However, the protocol applies qubit-controlled displacements of strength $2^{\Theta(n-k)}$ in the k th round, meaning that this protocol uses gates of unbounded strength. Decomposing these gates into bounded-strength operations leads to a polylogarithmic complexity in the number L of peaks generated.

We emphasize that these existing works have a number of shortcomings in terms of the analysis provided: In several instances, only the case of a finite number (e.g., $L \leq 8$) of peaks is studied. Furthermore, some of these protocols depend on parameters which are nonexplicit: They are solutions to certain optimization problems without proven feasibility guarantees for large L . Most importantly, only effective squeezing parameters [see Eq. (8)] are used to quantify the quality of the prepared state.

In contrast, our protocol is fully explicit, uses only two modes and a qubit, has logarithmic complexity in the number of peaks generated, and comes with an accuracy guarantee in terms of the trace norm, the strongest desirable measure of closeness for quantum states. We also note that our protocol first prepares a comb state and imposes a Gaussian envelope only in a subsequent step. For some applications, these comb states may be of independent interest.

VIII. CONCLUSIONS

We demonstrated that the complexities of coherent states and approximate GKP states are essentially determined by their energies. The scaling of their complexities can, thus, be determined by straightforward back-of-the-envelope calculations. Based on these two qualitatively very different examples, it is natural to ask whether the observed relationship between complexity and energy can be generalized to arbitrary single-mode bosonic states. More specifically, can every single-mode bosonic state be (approximately) prepared with a number of elementary operations that scales logarithmically with the energy of the state?

While our state-preparation protocols achieve optimal complexity under ideal, noiseless conditions, a compelling open question is whether efficient fault-tolerant implementations can be devised with similar fidelity guarantees. This question is particularly relevant for approximate GKP states, as these are instrumental in universal fault-tolerant quantum computation with continuous-variable systems. Existing work [17] in this direction provides first examples of such protocols but covers only highly restricted error models. We note that, in its current form,

the comb-state-preparation part of our GKP-state-preparation protocol is particularly susceptible to noise, because errors get amplified over the course of the protocol by the repeated squeezing operations. On the other hand, the Gaussification subprotocol has some inherent robustness to small displacement errors in its execution: The final state after a noisy execution will differ from the desired target state by only a comparatively small displacement.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

No data were created or analyzed in this study.

These appendixes are structured as follows. In Appendix A, we introduce the notion of complexity of bosonic states and the basic structure of unitary as well as heralding state preparation. In Appendix B, we study the complexity of coherent states. In Appendix C, we introduce comb states as a special kind of grid state featuring a rectangular envelope. We provide a unitary quantum circuit that prepares these states efficiently in the number of desired peaks and with high fidelity. In Appendix D, we present a heralding protocol that imprints a Gaussian envelope on a comb state. Consequently, in Appendix E, we combine the results from Appendixes C and D, yielding a protocol that prepares approximate GKP states. In Appendix F, we prove a converse bound for the heralded state complexity of GKP states.

APPENDIX A: COMPLEXITY OF BOSONIC STATES

In this section, we detail the allowed operations and restate the basic notions of complexity of bosonic states from the main text.

1. Allowed operations for hybrid qubit-boson systems

The systems we consider consist of one distinct oscillator (boson) that carries the output state along with m auxiliary oscillators (bosons) and m' auxiliary qubits. Hence, the corresponding Hilbert space is given by $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})^{\otimes m} \otimes (\mathbb{C}^2)^{\otimes m'}$. We allow the following operations.

- (i) Preparation of the single-qubit computational basis state $|0\rangle$ on a qubit and of the vacuum state $|\text{vac}\rangle$ on

any mode. [Here, $|\text{vac}\rangle \in L^2(\mathbb{R})$ is the ground state of the harmonic oscillator Hamiltonian, i.e., a centered Gaussian state saturating Heisenberg's uncertainty relation for the position and momentum quadratures.]

- (ii) Arbitrary single- and two-qubit unitaries acting on any qubit or any pair of qubits.
- (iii) Gaussian one- and two-mode unitaries generated by Hamiltonians, which are quadratic in the mode operators and have bounded strength. In more detail, for an n -mode bosonic system (here, $n = m + 1$), we consider Hamiltonians of the form

$$H(A) = \frac{1}{2} \sum_{j,k=1}^{2n} A_{j,k} R_j R_k =: \frac{1}{2} R^T A R,$$

where $R = (Q_1, P_1, \dots, Q_n, P_n)$ denotes the vector of quadrature operators and where $A = A^T \in \text{Mat}_{2n \times 2n}(\mathbb{R})$ is a symmetric matrix with real entries. Denoting by $J = -J^T \in \text{Mat}_{2n \times 2n}(\mathbb{R})$ the symplectic form defined by the canonical commutation relations $[R_j, R_k] = iJ_{j,k}I$ (where I denotes the identity on the Hilbert space), the action of the associated Gaussian unitary $U(A) = e^{iH(A)}$ on $L^2(\mathbb{R})^{\otimes n}$ can be described by the symplectic group element

$$\begin{aligned} S(A) &= e^{AJ} \in \text{Sp}(2n) \\ &= \{S \in \text{Mat}_{2n \times 2n}(\mathbb{R}) | SJS^T = J\}. \end{aligned}$$

That is, the (Gaussian) unitary $U(A)$ acts as

$$\begin{aligned} U(A)^\dagger R_j U(A) &= \sum_{k=1}^{2n} S(A)_{j,k} R_k \text{ for } j \in [2n] := \{1, \dots, 2n\}. \end{aligned} \quad (\text{A1})$$

The unitary $U(A)$ is a one-mode unitary acting on a mode $j \in [n]$ if $H(A)$ is a linear combination of monomials containing the position and momentum operators Q_j and P_j associated with the j th mode only. Similarly, $U(A)$ is a two-mode unitary acting on two modes (j, k) , $j \neq k$, if and only if $H(A)$ involves only monomials that are products of the operators $\{Q_j, P_j, Q_k, P_k\}$. We say that $U(A)$ has bounded strength if the operator norm $\|A\|$ is bounded by a constant, which can be arbitrary but fixed. In the following, we say that $U(A)$ is bounded if

$$\|A\| \leq 2\pi.$$

- (iv) One-mode phase-space displacements of constant strength. For an n -mode (here, $n = m + 1$) bosonic

system with (vector of) quadrature operators $R = (Q_1, P_1, \dots, Q_n, P_n)$, every vector $d \in \mathbb{R}^{2n}$ defines a unitary

$$D(d) = e^{i \sum_{j=1}^{2n} d_j R_j}$$

called the (Weyl) phase-space displacement by d . Its action on mode operators is given by

$$D(d)^\dagger R_j D(d) = R_j + d_j I \text{ for } j \in [2n].$$

Correspondingly, the vector d is often referred to as a displacement vector. The unitary $D(d)$ is a single-mode displacement (acting on a mode $j \in [n]$) if all entries of d other than d_{2j-1} (associated with Q_j) and d_{2j} (associated with P_j) vanish. It is of bounded strength if the Euclidean norm $\|d\| = \sqrt{\sum_{j=1}^{2n} d_j^2}$ of d is bounded by a constant. In the following, we again arbitrarily choose this constant to be 2π ; i.e., we say that $D(d)$ is a bounded-strength phase-space displacement if and only if

$$\|d\| \leq 2\pi.$$

- (v) Qubit-controlled single-mode displacements of bounded strength. For a system of n bosonic modes (here, $n = m + 1$) and m' qubits, this is defined as follows. Given an index $r \in [m']$ of a qubit and a vector $d \in \mathbb{R}^{2n}$, the qubit- r -controlled phase-space displacement by d is the unitary

$$\text{ctrl}_r D(d) = I_{L^2(\mathbb{R})^{\otimes n}} \otimes |0\rangle\langle 0|_r \otimes + D(d) \otimes |1\rangle\langle 1|_r.$$

Here, we write $|0\rangle\langle 0|_r$ for the m' -qubit operator $I^{\otimes r-1} \otimes |0\rangle\langle 0| \otimes I^{\otimes m'-r}$ and similarly for $|1\rangle\langle 1|_r$. We say that $\text{ctrl}_r D(d)$ is single mode and of bounded strength if this is the case for the displacement operator $D(d)$.

- (vi) Homodyne (Q -)quadrature measurements of any bosonic mode and computational basis measurements on any qubit. The former is defined in terms of the spectral decomposition of the quadrature operator, i.e., measuring the position (for Q) or momentum (for P). We assume that these measurements are destructive; i.e., we do not consider postmeasurement states on the measured modes.

Recall that we denote the operations from (i)–(vi) as elementary operations and we define the set of all unitaries of the form (ii)–(v) as $\mathcal{G}$. We attribute to each of these elementary operations one unit of complexity when used in any state-preparation protocol.

Let us comment on the choice of the unitaries in the above list. We note that, by using the Solovay-Kitaev algorithm (see, e.g., Ref. [34]), any single-qubit or two-

qubit unitary U can be approximated to arbitrary precision by a product of generators from a finite universal gate set such as $\{H, T, \text{CNOT}\}$. For convenience, we include the set of all single- and two-qubit unitaries in our set of operations; see (ii) above. As a consequence, our complexity measure involves the number of one- and two-qubit operations (but not the number of individual gates from a finite gate set).

Similar observations can be made about our bosonic operations: The bosonic and the qubit-boson unitaries (iii)–(v) can be generated by the following set of generators.

- (a) One- and two-mode Gaussian unitary operations of constant strength. Specifically, we use single-mode displacements of the form

$$e^{iaQ}, e^{iaP} \quad \text{with} \quad a \in [-2\pi, 2\pi],$$

single-mode squeezing operations of the form

$$S(z) = e^{i(z/2)(QP+PQ)} \quad \text{with} \quad z \in [-2\pi, 2\pi], \quad (\text{A2})$$

single-mode phase-space rotations (phase shifts), i.e.,

$$P(\phi) = e^{i(\phi/2)(Q^2+P^2)} \quad \text{with} \quad \phi \in [-2\pi, 2\pi],$$

and two-mode beam splitters acting on two modes j and k , that is,

$$B_{j,k}(\omega) = e^{i\omega(Q_j Q_k + P_j P_k)} \quad \text{with} \quad \omega \in [-2\pi, 2\pi].$$

Note that these gates generate the group of Gaussian unitaries on the bosonic modes. Indeed, the group of passive (i.e., number-preserving) Gaussian unitaries (corresponding to the orthogonal symplectic group) is generated by beam splitters and phase shifters only; see Ref. [35]. Note that, by the Euler decomposition (which factorizes every symplectic matrix as $S = O_1 Z O_2$ with O_1 and O_2 orthogonal symplectic and Z diagonal symplectic; see Ref. [36]), every Gaussian unitary is a product of passive Gaussian unitaries and single-mode squeezers. Since passive Gaussian unitaries can be compiled into a sequence of phase shifters and beam splitters (see Ref. [35] for a description of a corresponding algorithm), it follows that every Gaussian unitary can be realized by beam splitters, phase shifters, and single-mode-squeezing operations.

- (b) Qubit-controlled phase-space displacements of constant strength. These are unitaries of the form

$$\text{ctrl}e^{i\theta P} := |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \exp(i\theta P), \quad \theta \in [-2\pi, 2\pi],$$

$$\text{ctrl}e^{i\theta Q} := |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \exp(i\theta Q), \quad \theta \in [-2\pi, 2\pi].$$

In fact, the protocols we propose are expressed entirely in terms of the generators (a) and (b) as well as single-qubit Clifford unitaries and the qubit-boson operation (v).

2. Unitary state complexity and circuit complexity

In the setting of unitary state preparation, given a target state $|\Psi_{\text{target}}\rangle \in L^2(\mathbb{R})$, we consider protocols of the following form on a system with Hilbert space $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})^{\otimes m} \otimes (\mathbb{C}^2)^{\otimes m'}$.

- Each qubit is initialized in the computational basis state $|0\rangle$, and each bosonic mode is initialized in the vacuum state $|\text{vac}\rangle$. After this step, the system is in the state $|\Psi\rangle = |\text{vac}\rangle \otimes |\text{vac}\rangle^{\otimes m} \otimes |0\rangle^{\otimes m'}$.
- A sequence of $T \in \mathbb{N}$ unitaries $U_1, \dots, U_T \in \mathcal{G}$ is applied. Denoting by $U = U_T \dots U_1$ the corresponding unitary, the state after this step is the state $U|\Psi\rangle$.
- The output of the protocol is the reduced density operator of the first mode, i.e., $\rho = \text{tr}_{m,m'} U|\Psi\rangle\langle\Psi|U^\dagger$, where $\text{tr}_{m,m'}$ denotes the partial trace over all m auxiliary modes and m' auxiliary qubits. The output state ρ is supposed to be a good approximation to the target state $|\Psi_{\text{target}}\rangle\langle\Psi_{\text{target}}|$ (as quantified below).

Observe that the total number of operations from the set (i)–(vi) of allowed operations is equal to $T + (m + 1) + m'$, since $m + 1$ vacuum states and m' qubits are prepared initially. We say that the protocol achieves error $\varepsilon \geq 0$ if the output state is ε -close in L^1 distance to the desired target state $|\Psi_{\text{target}}\rangle$, i.e., if

$$\|\text{tr}_{m,m'} U|\Psi\rangle\langle\Psi|U^\dagger - |\Psi_{\text{target}}\rangle\langle\Psi_{\text{target}}|\|_1 \leq \varepsilon.$$

We say that a state $|\Psi_{\text{target}}\rangle \in L^2(\mathbb{R})$ has unitary circuit complexity $\mathcal{C}_\varepsilon^*(|\Psi_{\text{target}}\rangle)$ for $\varepsilon \geq 0$ if there is a unitary state-preparation protocol achieving error $\varepsilon \geq 0$ such that

$$\mathcal{C}_\varepsilon^*(|\Psi_{\text{target}}\rangle) = T + (m + 1) + m',$$

and the rhs is minimal among all unitary state preparation protocols with this property. In other words, the reduced density matrix on the first mode of $U(|\text{vac}\rangle \otimes |\text{vac}\rangle^{\otimes m} \otimes |0\rangle^{\otimes m'})$ is ε -close in L^1 distance to the target state $|\Psi_{\text{target}}\rangle$, and the protocol is resource optimal in the sense that the number $T + (m + 1) + m'$ of operations used is minimal.

To study this notion of (unitary) complexity of a state $|\Psi_{\text{target}}\rangle$, we often decompose a given unitary $U \in L^2(\mathbb{R})$ into a product $U = U_T \dots U_1$ of unitaries $U_1, \dots, U_T$ belonging to our allowed gate set $\mathcal{G}$ (we consider only cases where this kind of factorization exists and is exact). We denote by

$$\mathcal{C}_\mathcal{G}(U) = \min\{T \in \mathbb{N} \mid \exists U_1, \dots, U_T \in \mathcal{G} \text{ such that } U = U_T \dots U_1\} \quad (\text{A3})$$

the minimum number of unitaries needed in such a factorization. Note that this is closely connected to the unitary state complexity of a state: If $\|U|\text{vac}\rangle\langle\text{vac}|U^\dagger - |\Psi_{\text{target}}\rangle\langle\Psi_{\text{target}}|\|_1 \leq \varepsilon$, then the unitary state complexity is bounded by $\mathcal{C}_\varepsilon(|\Psi_{\text{target}}\rangle) \leq \mathcal{C}_\mathcal{G}(U) + 1$. This is because the factorization of the unitary U into unitaries from $\mathcal{G}$ provides a protocol for preparing $|\Psi\rangle$, and this protocol involves only the preparation of one vacuum state and application of $\mathcal{C}_\mathcal{G}(|\Psi_{\text{target}}\rangle)$ gates from $\mathcal{G}$.

3. Heralded state complexity

The second type of state-preparation protocols we consider is based on heralding state generation. Here, the protocol additionally produces an output flag $F \in \{\text{acc}, \text{rej}\}$ (i.e., a classical output bit) indicating whether the protocol accepts (succeeds) or rejects (in the case of failure) the output. In more detail, we consider protocols of the following form.

- (i) Each mode is initialized in the vacuum state $|\text{vac}\rangle$ and each qubit in the computational state $|0\rangle$; i.e., the initial state is $|\Psi\rangle = |\text{vac}\rangle \otimes |\text{vac}\rangle^{\otimes m} \otimes |0\rangle^{m'}$.
- (ii) Subsequently, a unitary U as described before is applied; i.e., U consists of gates from the set $\mathcal{G}$. We denote by T_1 the number of such gates.
- (iii) A measurement is then applied to the auxiliary modes and the qubits of the prepared state $U|\Psi\rangle$. More precisely, a homodyne position measurement is applied to every auxiliary bosonic mode and a computational basis measurement to every auxiliary qubit. Let us denote the corresponding POVM by $\{E_\alpha\}_{\alpha \in \mathcal{M}}$, where $\mathcal{M} = \mathbb{R}^m \times \{0, 1\}^{m'}$ denotes the set of measurement outcomes.
- (iv) Depending on the measurement outcome α , a flag $F(\alpha) \in \{\text{acc}, \text{rej}\}$ is computed by means of an efficiently computable function $F: \mathcal{M} \rightarrow \{\text{acc}, \text{rej}\}$.
- (v) If $F(\alpha) = \text{acc}$, a vector $d(\alpha) = [d_Q(\alpha), d_P(\alpha)] \in \mathbb{R}^2$ is computed from the measurement outcome by means of an efficiently computable function $d: \mathcal{M} \rightarrow \mathbb{R}^2$. The postmeasurement state on the first mode is then displaced by application of the unitary $D[d(\alpha)] = e^{i[d_Q(\alpha)Q - d_P(\alpha)P]}$, i.e., a phase-space displacement by the vector $d(\alpha)$.
- (vi) The output of the protocol is the output flag $F(\alpha)$ and, assuming that $F(\alpha) = \text{acc}$, the reduced density operator on the first mode. Conditioned on acceptance, we want the state of the first mode to be close to the target state $|\Psi_{\text{target}}\rangle$.

We call a protocol for state preparation of this form a heralded state preparation protocol. The definition implies that the (average) output state of the protocol conditioned on acceptance is given by

$$\rho_{\text{acc}} = \frac{1}{p_{\text{acc}}} \int_{F^{-1}(\{\text{acc}\})} p(\alpha) D[d(\alpha)] \rho^{(\alpha)} D[d(\alpha)]^\dagger d\alpha, \quad (\text{A4})$$

where we introduce normalized states $\rho^{(\alpha)}$ as

$$p(\alpha) \rho^{(\alpha)} = \text{tr}_{m, m'}((I \otimes E_\alpha) U |\Psi\rangle\langle\Psi| U^\dagger) \quad \text{with} \\ p(\alpha) = \langle\Psi| U^\dagger (I \otimes E_\alpha) U |\Psi\rangle$$

and where

$$p_{\text{acc}} = \int_{F^{-1}(\{\text{acc}\})} p(\alpha) d\alpha \quad (\text{A5})$$

is the probability that the protocol accepts [i.e., that $F(\alpha) = \text{acc}$]. We say that the protocol produces an ε approximation of the state $|\Psi_{\text{target}}\rangle$ with acceptance probability p_{acc} [see Eq. (A5)] if

$$\|\rho_{\text{acc}} - |\Psi_{\text{target}}\rangle\langle\Psi_{\text{target}}|\|_1 \leq \varepsilon.$$

The number of operations from the set (i)–(vi) applied in this protocol is determined by the number T_1 of unitaries applied to implement U [see (ii)] and the unitaries applied to implement the “correction” $D[d(\alpha)]$ after obtaining the measurement result α . We consider the worst case; i.e., we use the minimal number of elementary unitaries from $\mathcal{G}$ required to implement $D[d(\alpha)]$, maximized over all possible shifts $d(\alpha)$ associated with different measurement outcomes $\alpha \in F^{-1}(\{\text{acc}\})$. That is, we use the quantity [cf. Eq. (A3)]

$$T_2 = \sup_{\alpha \in F^{-1}(\{\text{acc}\})} \mathcal{C}_\mathcal{G}\{D[d(\alpha)]\}. \quad (\text{A6})$$

We establish upper and lower bounds on the quantity $\mathcal{C}_\mathcal{G}[D(d)]$ for arbitrary $d \in \mathbb{R}^2$ in Appendix B.

Taking into account the $(m+1) + m'$ single-mode and single-qubit preparations, the maximal number $T_1 + T_2$ of unitary operations, and the $m + m'$ single-mode and single-qubit measurements, the maximal (i.e., worst-case over measurement outcomes) number of operations from (i)–(vi) in such a protocol is, therefore,

$$T_1 + T_2 + (2m+1) + 2m'. \quad (\text{A7})$$

(Here, we do not consider the complexity of the classical computation used to evaluate the functions F and d but simply assume that this is efficient. Indeed, this is the case for the protocols we consider.)

We say that a state $|\Psi_{\text{target}}\rangle \in L^2(\mathbb{R})$ has heralded state complexity $\mathcal{C}_{p, \varepsilon}^*(|\Psi_{\text{target}}\rangle)$ for $\varepsilon \geq 0$ and $p > 0$ if there is a heralded state preparation protocol which prepares an ε approximation to the state $|\Psi_{\text{target}}\rangle$ with probability at least $p_{\text{acc}} \geq p$, whose number of operations [see Eq. (A7)] is equal to

$$\mathcal{C}_{p, \varepsilon}^*(|\Psi_{\text{target}}\rangle) = T_1 + T_2 + (2m+1) + 2m'$$

and has the property that the rhs of this equation is minimal among all heralded state preparation protocols with this property.

APPENDIX B: THE (ZERO-ERROR) UNITARY COMPLEXITY OF COHERENT STATES

In this section, we consider the (unitary) complexity of coherent states. These states can be prepared exactly using the set of operations we consider (i.e., preparation of $|\text{vac}\rangle$ on bosonic modes, $|0\rangle$ on qubits, and unitaries belonging to the set $\mathcal{G}$; see Appendix A 1). Motivated by this, we consider their zero-error complexity only. This simple problem serves as a warm-up and illustrates some key concepts.

To fix notation, let $D(d) = e^{i(d_Q Q - d_P P)}$ be the (Weyl) phase-space displacement operator associated with $d = (d_Q, d_P) \in \mathbb{R}^2$. Throughout this section, we consider the coherent state

$$|d\rangle = D(d)|\text{vac}\rangle \quad \text{with} \quad d = (d_Q, d_P) \in \mathbb{R}^2. \quad (\text{B1})$$

1. Protocols for preparing coherent states

Let us first consider protocols for generating the state (B1).

Clearly, one way of generating the coherent state $|d\rangle$ using only a single bosonic mode, no auxiliary modes (i.e., $m = 0$), and no qubits (i.e., $m' = 0$) is to simply realize the displacement $D(d)$ by a sequence of displacements:

$$D(d) \propto U(\theta)D[(0, c)]U(-\theta) \quad \text{with}$$

It is easy to verify that

$$D[(0, c)] = e^{-icP} = S(-\log c)e^{-iP}S(\log c) \quad \text{for all } c > 0,$$

where $S(z)$ for $z \in \mathbb{R}$ denotes single-mode squeezing; see Eq. (A2). Since $S(z_1)S(z_2) = S(z_1 + z_2)$ for all $z_1, z_2 \in \mathbb{R}$, we have

$$S(z) = S(z/\lceil |z| \rceil)^{\lceil |z| \rceil} \quad \text{for any } z \in \mathbb{R}. \quad (\text{B5})$$

Combining Eqs. (B4)–(B5), we obtain the factorization

$$D(d) \propto U(\theta_d)S(-z_d)^{N_d}e^{-iP}S(z_d)^{N_d}U(-\theta_d) \quad \text{with} \quad \begin{aligned} z_d &= \frac{\log \|d\|}{\lceil \log \|d\| \rceil}, \\ N_d &= \lceil \log \|d\| \rceil, \\ \theta_d &= \pi - \arg d \bmod (2\pi). \end{aligned} \quad (\text{B6})$$

We have $z_d \in [-1, 1]$; hence, $S(z_d), S(-z_d) \in \mathcal{G}$ are constant-strength Gaussian unitaries. Since we also have $U(\theta) \in \mathcal{G}$ for any $\theta \in [-2\pi, 2\pi]$ and $e^{-iP} \in \mathcal{G}$, it follows from Eq. (B6) that $D(d)$ can be realized by a sequence of $T = 2N_d + 2$ gates from $\mathcal{G}$. [The first phase-space rotation $U(-\theta_d)$ does not need to be applied, because the state $|\text{vac}\rangle$ is invariant under such rotations.] In particular, using only a single mode and T gates from $\mathcal{G}$, the coherent state $|d\rangle$ can be prepared from the vacuum state $|\text{vac}\rangle$. We have, thus, shown the following.

Because $D(d_1)D(d_2) \propto D(d_1 + d_2)$ by the Weyl relations, the state $|d\rangle$ is proportional to

$$|d\rangle \propto D(d/T)^T|\text{vac}\rangle \quad \text{for any } T \in \mathbb{N}. \quad (\text{B2})$$

Choosing $T = \lceil \|d\| \rceil$, where $\|d\| = \sqrt{d_Q^2 + d_P^2}$ denotes the Euclidean norm of $d = (d_Q, d_P) \in \mathbb{R}^2$, ensures that each unitary $D(d/T)$ is a displacement of constant strength, i.e., belongs to the gate set $\mathcal{G}$. Since Eq. (B2) shows that the state $|d\rangle$ can be created exactly (i.e., with error $\varepsilon = 0$) with $T = \lceil \|d\| \rceil$ gates from $\mathcal{G}$ and 1 bosonic mode, the (zero-error) unitary state complexity of $|d\rangle$ is upper bounded by $C_0^*(|d\rangle) \leq \lceil \|d\| \rceil + 1$. We note that the factorization

$$D(d) \propto D(d/\lceil \|d\| \rceil)^{\lceil \|d\| \rceil} \quad (\text{B3})$$

used here also implies that the circuit complexity of the displacement operator $D(d)$ is bounded by $\mathcal{C}_{\mathcal{G}}[D(d)] \leq \lceil \|d\| \rceil$.

This naïve approach to preparing the coherent state $|d\rangle$ expressed by Eq. (B2) [respectively, Eq. (B3)] is, however, far from optimal: In fact, a number of gates from $\mathcal{G}$ that is only logarithmic in $\|d\|$ suffices. To see this, recall that the (passive) unitary $U(\theta) = e^{-i\theta(Q^2 + P^2)/2}$ realizes a rotation in phase space, i.e.,

$$\begin{aligned} U(\theta)QU(\theta)^\dagger &= (\cos \theta)Q + (\sin \theta)P \\ U(\theta)PU(\theta)^\dagger &= -(\sin \theta)Q + (\cos \theta)P \end{aligned} \quad \text{for } \theta \in [0, 2\pi),$$

which implies that

$$\begin{aligned} c &= \|d\|, \\ \theta &= \pi - \arg d \bmod (2\pi) \in [0, 2\pi). \end{aligned} \quad (\text{B4})$$

Lemma B.1. Let $d \in \mathbb{R}^2$ be arbitrary. The circuit complexity of the displacement operator $D(d)$ is bounded by

$$\mathcal{C}_G[D(d)] \leq 2\lceil \log \|d\| \rceil + 3.$$

In particular, the (zero-error) complexity of the coherent state $|d\rangle$ is bounded by

$$\mathcal{C}_0^*(|d\rangle) \leq 2\lceil \log \|d\| \rceil + 3.$$

2. Lower bounds on the complexity of coherent states

Let us now turn to lower bounds on the complexity of a coherent state $|d\rangle$, $d \in \mathbb{R}^2$. Such lower bounds can be obtained by considering the (harmonic oscillator) Hamiltonian

$$H = Q^2 + P^2.$$

Recalling that $\langle \text{vac} | H | \text{vac} \rangle = 1$ and $\langle \text{vac} | Q | \text{vac} \rangle = \langle \text{vac} | P | \text{vac} \rangle = 0$, it is easy to check that the energy of the coherent state $|d\rangle$ is equal to

$$\langle d | H | d \rangle = \|d\|^2 + 1. \quad (\text{B7})$$

Now consider any protocol that starts from the state $|\Psi\rangle = |\text{vac}\rangle \otimes |\text{vac}\rangle^{\otimes m} \otimes |0\rangle^{\otimes m'}$ on $m+1$ bosonic modes and m' auxiliary qubits, applies a unitary U consisting of T gates from the set $\mathcal{G}$, and subsequently outputs the state $\rho = \text{tr}_{m,m'} U |\Psi\rangle \langle \Psi| U^\dagger$ on the first mode. By using moment limits, we show below (see Lemma F.6) that the energy of the output state ρ is bounded: We have

$$\text{tr}(H\rho) \leq e^{8\pi T}(m+2) \quad \text{for any } T \in \mathbb{N}, \quad m, m' \in \mathbb{N} \cup \{0\}. \quad (\text{B8})$$

Now suppose that the considered protocol prepares the coherent state $|d\rangle$ exactly, i.e., $\rho = |d\rangle \langle d|$. Comparing Eqs. (B7) and (B8), we then conclude that we must have $\|d\|^2 + 1 \leq e^{8\pi T}(m+2)$, i.e.,

$$\begin{aligned} T &\geq \frac{1}{8\pi} \log \left(\frac{\|d\|^2 + 1}{(m+2)} \right) \\ &= \frac{1}{8\pi} (2 \log \|d\| + \log(1 + 1/\|d\|^2) - \log(m+2)). \end{aligned} \quad (\text{B9})$$

Using that

$$m+1 - \frac{1}{8\pi} \log(m+2) \geq 0 \quad \text{for all } m \in \mathbb{N} \cup \{0\},$$

we conclude that Eq. (B9) implies

$$\begin{aligned} T + (m+1) + m' &\geq \frac{1}{8\pi} (2 \log \|d\| + \log(1 + 1/\|d\|^2)) \\ &=: f(\|d\|). \end{aligned}$$

Since this inequality is satisfied for any protocol preparing the state $|d\rangle$, we conclude that the (zero-error) unitary state complexity of the coherent state $|d\rangle$ is at least $\mathcal{C}_0^*(|d\rangle) \geq f(\|d\|)$.

Analogous reasoning applies to the circuit complexity of the displacement operator $D(d)$: If $U = D(d)$ for a circuit U consisting of T gates, then T must satisfy Eq. (B9) (with $m = m' = 0$), because U gives rise to a protocol preparing $|d\rangle = U|\text{vac}\rangle$ from a single vacuum state. In particular, we have

$$\mathcal{C}_G[D(d)] \geq f(\|d\|) - 1 \geq \frac{1}{4\pi} \log \|d\| - 1. \quad (\text{B10})$$

In summary, we have, thus, shown the following.

Lemma B.2. Let $d \in \mathbb{R}^2$ be arbitrary. Then,

$$\mathcal{C}_0^*(|d\rangle) = \Omega(\log \|d\|) \quad \text{for } \|d\| \rightarrow \infty$$

and

$$\mathcal{C}_G[D(d)] = \Omega(\log \|d\|) \quad \text{for } \|d\| \rightarrow \infty.$$

3. The zero-error unitary state complexity of a coherent state

Combining the preparation procedure (characterized by Lemma B.1) with Lemma B.2, we obtain the following scaling of the zero-error complexity of a coherent state.

Corollary B.2 (zero-error complexity of coherent states). For $d \in \mathbb{R}^2$, let $|d\rangle \in L^2(\mathbb{R})$ be the coherent state defined by Eq. (B1). Then, we have

$$\mathcal{C}_0^*(|d\rangle) = \Theta(\log \|d\|) \quad \text{for } \|d\| \rightarrow \infty.$$

The proof of Corollary B.3 illustrates a few of the building blocks for our main result. For example, in the GKP-state-preparation protocol we propose, we also use the factorization (B6) of a phase-space displacement operator $D(d)$. However, the protocol and its analysis are somewhat more involved and prepare only an approximation to the state $|\text{GKP}_{\kappa,\Delta}\rangle$ [i.e., we consider the unitary and heralded complexities $\mathcal{C}_e^*(|\text{GKP}_{\kappa,\Delta}\rangle)$ and $\mathcal{C}_{p,e}^{*,\text{her}}(|\text{GKP}_{\kappa,\Delta}\rangle)$ with nonzero error $\varepsilon > 0$].

We also derive corresponding lower bounds, again using moment (energy) limits. We note that, in principle, identical arguments can be used to derive lower bounds on $\mathcal{C}_e^*(|d\rangle)$ and the heralded complexity $\mathcal{C}_{p,e}^{*,\text{her}}(|d\rangle)$ of a coherent state $|d\rangle$. Since our focus is on GKP states, we omit the details here.

APPENDIX C: COMB-STATE PREPARATION

In this section, we consider the problem of preparing comb states. We give the formal definition of these states in Appendix C 1. In Appendix C 2, we give a protocol for their preparation. In Appendix C 3, we explain the underlying ideas of the protocol. Finally, in Appendix C 4, we prove that our protocol indeed prepares (approximate) comb states.

As we argue in Appendix D, comb states can be converted to GKP states rather easily. In other words, the comb-state-preparation protocol presented here is a key building block for our GKP-state-preparation protocol.

1. Definition of comb states

Comb states are GKP-like wave functions that have support on an interval of length $L > 0$ and a rectangular envelope given by

$$\square_L(z) = \frac{1}{\sqrt{L}} \delta_{z \in [-L/2, L/2]}.$$

That is, for a squeezing parameter $\Delta > 0$ (associated with the width of each peak), we define the states

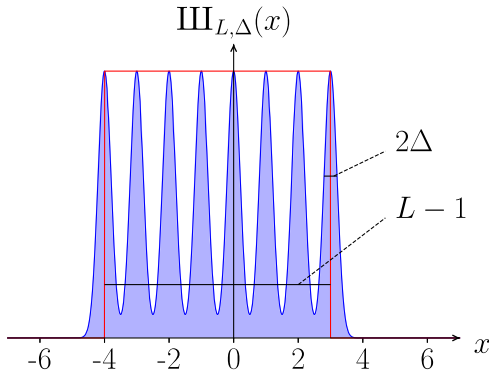


FIG. 6. An illustration of the comb state $|\text{III}_{L,\Delta}\rangle$ with $L = 8$ (i.e., with eight local maxima). For any large even integer $L \in 2\mathbb{N}$, the state $|\text{III}_{L,\Delta}\rangle$ is “almost centered”: Its peaks lie at positions $\mathcal{L}_L := \{-L/2, \dots, -1, 0, \dots, L/2 - 1\}$.

$$\begin{aligned} |\text{III}_{L,\Delta}\rangle &= D_{L,\Delta} \sum_{z \in \mathbb{Z}} \square_L(z) |\chi_\Delta(z)\rangle \\ &= \frac{D_{L,\Delta}}{\sqrt{L}} \sum_{z=-L/2}^{L/2-1} |\chi_\Delta(z)\rangle. \end{aligned} \quad (\text{C1})$$

Here, $D_{L,\Delta}$ is the normalization factor, and χ_Δ are translated Gaussians:

$$\begin{aligned} [\chi_\Delta(z)](x) &= \Psi_\Delta(x - z), \quad \text{where} \\ \Psi_\Delta(x) &= \frac{1}{(\pi\Delta^2)^{1/4}} e^{-x^2/(2\Delta^2)}. \end{aligned} \quad (\text{C2})$$

Note that $|\Psi_1\rangle = |\text{vac}\rangle$ is the vacuum state, i.e., the ground state of the harmonic oscillator Hamiltonian.

We call a state $|\text{III}_{L,\Delta}\rangle$ a comb state. It is illustrated in Fig. 6.

2. Unitary comb-state preparation

We give a unitary circuit for comb-state preparation in Protocol 2. It is illustrated in Fig. 7. It uses one auxiliary qubit (i.e., $m = 0$ and $m' = 1$).

Its core is the repeated use of the unitary V defined as

$$\begin{aligned} V &:= V^{(4)} V^{(3)} V^{(2)} V^{(1)}, \quad \text{where} \\ V^{(1)} &= S(-\log 2) \otimes I, \\ V^{(2)} &= \text{ctrl} e^{-iP}, \\ V^{(3)} &= I \otimes H, \\ V^{(4)} &= \text{ctrl} e^{i\pi Q}. \end{aligned} \quad (\text{C3})$$

The circuit diagram for this unitary is given in Fig. 8. The main result of this section is the following.

Theorem C.1. Given a squeezing parameter $\Delta \in (0, 1/4)$ and a number of rounds $n > 0$, Protocol 2 returns a quantum state $\rho \in \mathcal{B}(L^2(\mathbb{R}))$ close in L^1 distance to the comb state $|\text{III}_{2^n,\Delta}\rangle$; i.e., the output state ρ satisfies

$$\|\rho - |\text{III}_{2^n,\Delta}\rangle\langle\text{III}_{2^n,\Delta}|\|_1 \leq 17\sqrt{\Delta}. \quad (\text{C4})$$

The protocol can be realized by a circuit using $5n + \lceil \log 1/\Delta \rceil + 4$ elementary operations.

Since Protocol 2 uses only one auxiliary qubit in addition to the bosonic system mode, Theorem C.1 implies that the

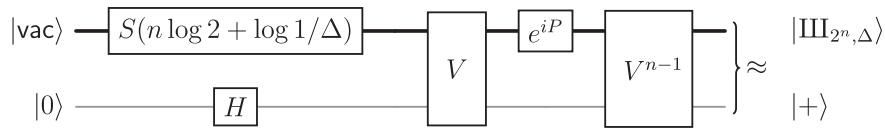


FIG. 7. Circuit diagram of Protocol 2. It uses the V gate described in Fig. 8. The exponent of V indicates the number of applications. The squeezing unitary is realized by composing single-mode squeezing operations from the set $\mathcal{G}$ (see Appendix A 2); see the factorization given in Eq. (C5).

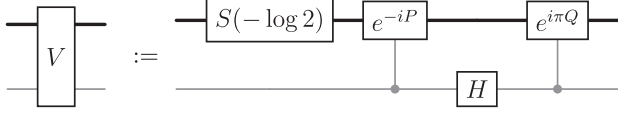


FIG. 8. Circuit implementing the unitary V used in the comb-state-preparation protocol. It uses two qubit-controlled displacements: by 1 in the Q direction and by π in the P direction.

unitary state complexity of the comb state with $L = 2^n$ peaks is upper bounded by

$$C_{17\sqrt{\Delta}}^*(|\mathbb{III}_{2^n, \Delta}\rangle) \leq 5n + \lceil \log 1/\Delta \rceil + 4.$$

Before proving Eq. (C4), let us verify that the stated number of elementary operations (cf. Appendix A 1) is correct. The circuit in Fig. 7 uses gates from the set $\mathcal{G}$ (used in the definition of V) only, except for the single-mode squeezing operator $S(n \log 2 + \log 1/\Delta)$, which is used to prepare the squeezed vacuum state $|\Psi_{2^{-n}\Delta}\rangle$ from the vacuum state $|\text{vac}\rangle$ (see step 1 in Protocol 2). We can decompose this unitary as

$$S(n \log 2 + \log 1/\Delta) = S[\log(2)]^n S(z_\Delta)^{\lceil \log 1/\Delta \rceil}, \quad \text{where} \\ z_\Delta = \frac{\log 1/\Delta}{\lceil \log 1/\Delta \rceil} \in (0, 1]. \quad (\text{C5})$$

Observe that the rhs involves only single-mode squeezing operators with squeezing parameters $z \in [-2\pi, 2\pi]$ in accordance with the definition of the set $\mathcal{G}$; i.e., it is a sequence of $n + \lceil \log 1/\Delta \rceil$ gates belonging to the gate set $\mathcal{G}$.

Since the circuit involves n applications of V (each composed of four gates from $\mathcal{G}$), one application of e^{iP} , and application of the Hadamard gate H as well as the squeezing unitary $S(n \log 2 + \log 1/\Delta)$, the total number of gates is $5n + \lceil \log 1/\Delta \rceil + 2$. Adding the initialization of $|\text{vac}\rangle$ and $|0\rangle$ implies the claim.

3. Underlying ideas for the comb-state-preparation circuit

Given a copy of the truncated comb state $|\mathbb{III}_{L, \Delta}\rangle$ and a qubit in the state $|+\rangle$, the unitary V generates an approximate instance of the comb state $|\mathbb{III}_{2L, 2\Delta}\rangle$; see Fig. 9 for an illustration. We give a quantitative statement in Lemma C.3 below. In other words, the unitary V effectively doubles the number L of peaks while also doubling the squeezing parameter Δ .

To give some intuition on the repeated action of the unitary V , let us consider its action on a single-peak state of the form $|x\rangle \otimes |+\rangle$, where $|x\rangle$ is the (unnormalized) position eigenstate associated with an integer eigenvalue $x \in \mathbb{Z}$ of the position operator Q . Applying the unitary V splits the peak in two:

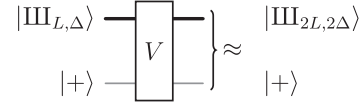


FIG. 9. Key to the protocol construction is that the unitary V essentially doubles the number of peaks when the qubit is in the state $|+\rangle$. The output represented on the right-hand side is approximate (see Lemma C.3 for details). Note that the qubit approximately acts as a catalyst.

$$\begin{aligned} |x\rangle \otimes |+\rangle &\xrightarrow{S(-\log 2) \otimes I} |2x\rangle \otimes |+\rangle \\ &\xrightarrow{\text{ctrl } e^{-iP}} |2x\rangle \otimes |0\rangle + |2x+1\rangle \otimes |1\rangle \\ &\xrightarrow{I \otimes H} |2x\rangle \otimes |+\rangle + |2x+1\rangle \otimes |-\rangle \\ &= (|2x\rangle + |2x+1\rangle) \otimes |0\rangle + (|2x\rangle - |2x+1\rangle) \otimes |1\rangle \\ &\xrightarrow{\text{ctrl } e^{i\pi Q}} (|2x\rangle + |2x+1\rangle) \otimes |+\rangle; \end{aligned}$$

that is,

$$V(|x\rangle \otimes |+\rangle) \propto (|2x\rangle + |2x+1\rangle) \otimes |+\rangle \quad \text{for } x \in \mathbb{Z}.$$

After n iterations of V , we thus obtain 2^n peaks from the initial single peak; i.e., we have

$$\begin{aligned} V^n(|x\rangle \otimes |+\rangle) &\propto (|2^n x\rangle + |2^n x+1\rangle + \dots + |2^n x+2^n-1\rangle) \\ &\quad \otimes |+\rangle \quad \text{for } x \in \mathbb{Z}. \end{aligned} \quad (\text{C6})$$

This brief calculation illustrates the effect of repeatedly applying V .

We note that—according to Eq. (C6)—repeated application of V to $|0\rangle \otimes |+\rangle$, where $|0\rangle$ is the position eigenstate with eigenvalue $x = 0$, results in a comblike state shifted to the right. An approximately centered state can be obtained by application of an (extensive) phase-shift unitary of the form $e^{i2^{n-1}P}$. We use an alternative approach, applying a phase shift e^{iP} after the first application of V ; see step 3 of the protocol.

The detailed analysis of Protocol 2 follows similar reasoning and is presented in the following section. The protocol uses a squeezed vacuum state $|\Psi_{2^{-n}\Delta}\rangle$ in place of the position eigenstate $|0\rangle$. Here, a comb state results from the repeated application of the unitary V . The corresponding process is illustrated in Fig. 10.

4. Proof of Theorem C.1

The proof of Theorem C.1 uses approximate comb states which we introduce in Appendix C 4 a. In Appendix C 4 b, we analyze a single application of the unitary V . In Appendix C 4 c, we complete the proof of Theorem C.1.

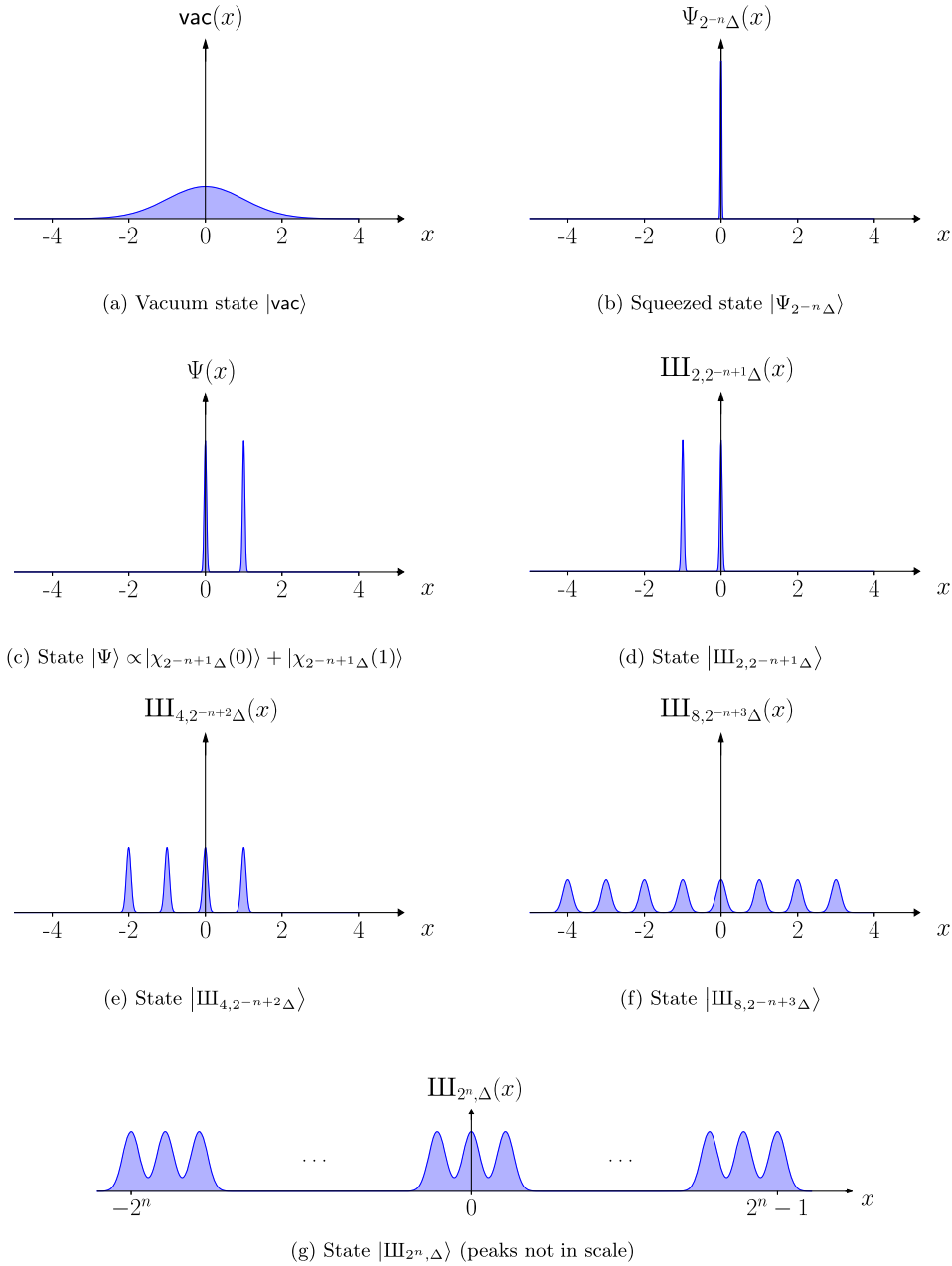


FIG. 10. Representation of the sequence of states generated in the comb-state-preparation protocol (Protocol 2).

a. Definition of approximate comb states

Our analysis involves truncated comb states. A truncated comb state is obtained by taking a comb state $|\text{III}_{L,\Delta}\rangle$ and truncating each Gaussian peak to have support only on an interval of width 2ε . Concretely, for $\varepsilon \in (0, 1/2)$, we define

$$\text{III}_{L,\Delta}^\varepsilon = \frac{\Pi_{\mathbb{Z}(\varepsilon)} \text{III}_{L,\Delta}}{\|\Pi_{\mathbb{Z}(\varepsilon)} \text{III}_{L,\Delta}\|}, \quad (\text{C7})$$

where $\mathbb{Z}(\varepsilon) = \mathbb{Z} + [-\varepsilon, \varepsilon]$ (where addition is understood as Minkowski sum) and where, for a subset $S \subseteq \mathbb{R}$, we denote by Π_S the orthogonal projection onto functions having support contained in S .

The state $|\text{III}_{L,\Delta}^\varepsilon\rangle$ illustrated in Fig. 11. Observe that, for an even integer $L \in 2\mathbb{N}$, this state has the form

$$|\text{III}_{L,\Delta}^\varepsilon\rangle = \frac{1}{\sqrt{L}} \sum_{z=-L/2}^{L/2-1} |\chi_\Delta^\varepsilon(z)\rangle, \quad (\text{C8})$$

where we use

$$[\chi_\Delta^\varepsilon(z)](x) = \Psi_\Delta^\varepsilon(x-z) \quad \text{and} \quad \Psi_\Delta^\varepsilon = \frac{\Pi_{[-\varepsilon,\varepsilon]} \Psi_\Delta}{\|\Pi_{[-\varepsilon,\varepsilon]} \Psi_\Delta\|}. \quad (\text{C9})$$

Here, $\Psi_\Delta \in L^2(\mathbb{R})$ is the squeezed vacuum state defined in Eq. (C2). Clearly, for suitably chosen (ε, Δ) , the truncated state $|\Psi_\Delta^\varepsilon\rangle$ is close to $|\Psi_\Delta\rangle$; see Lemma G.2 in Appendix G for a quantitative statement.

Similarly, the state $|\text{III}_{L,\Delta}^\varepsilon\rangle$ is close to the state $|\text{III}_{L,\Delta}\rangle$ for suitably chosen parameters (L, Δ, ε) . We refer to Lemma G.6 for a quantitative statement.

b. Analysis of the peak-doubling unitary V

Our analysis of Protocol 2 starts with a quantitative description of the effect of a single application of the unitary V introduced in Eq. (C3) (and illustrated in Fig. 8).

We first consider the application of V to a single (truncated) squeezed vacuum state $|\Psi_\Delta^\varepsilon\rangle$.

Lemma C.2. Let $\varepsilon \in (0, 1/2)$ and $\Delta > 0$. Then,

$$\|e^{-iP}|\text{III}_{2,2\Delta}^{2\varepsilon}\rangle\langle\text{III}_{2,2\Delta}^{2\varepsilon}|e^{iP} \otimes |+\rangle\langle+| - V(|\Psi_\Delta^\varepsilon\rangle\langle\Psi_\Delta^\varepsilon| \otimes |+\rangle\langle+|)V^\dagger\|_1 \leq 9\varepsilon.$$

Proof. We use the factorization of $V = V^{(4)}V^{(3)}V^{(2)}V^{(1)}$ of the unitary V from Eq. (C3). Since the squeezing operator $S(z)$ acts on an element $\Psi \in L^2(\mathbb{R})$ as $[S(z)\Psi](x) = e^{z/2}\Psi(e^zx)$, we have $S(-\log 2)|\Psi_\Delta^\varepsilon\rangle = |\Psi_{2\Delta}^{2\varepsilon}\rangle$. It follows that

$$V^{(1)}(|\Psi_\Delta^\varepsilon\rangle \otimes |+\rangle) = |\Psi_{2\Delta}^{2\varepsilon}\rangle \otimes |+\rangle.$$

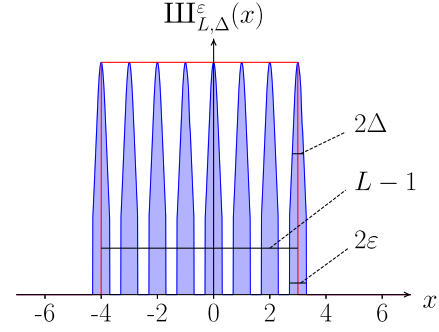


FIG. 11. An illustration of the truncated comb state $|\text{III}_{L,\Delta}^\varepsilon\rangle$ with $L = 8$ and $\varepsilon = 0.3$.

Writing $|\Psi_{2\Delta}^{2\varepsilon}\rangle = |\chi_{2\Delta}^{2\varepsilon}(0)\rangle$ and using that $e^{-iP}|\Psi_{2\Delta}^{2\varepsilon}\rangle = |\chi_{2\Delta}^{2\varepsilon}(1)\rangle$, we obtain

$$\begin{aligned} V^{(2)}V^{(1)}(|\Psi_\Delta^\varepsilon\rangle \otimes |+\rangle) \\ = \frac{1}{\sqrt{2}}[|\chi_{2\Delta}^{2\varepsilon}(0)\rangle \otimes |0\rangle + |\chi_{2\Delta}^{2\varepsilon}(1)\rangle \otimes |1\rangle]. \end{aligned}$$

In particular, applying a Hadamard gate to the qubit results in the state

$$\begin{aligned} V^{(3)}V^{(2)}V^{(1)}(|\Psi_\Delta^\varepsilon\rangle \otimes |+\rangle) &= \frac{1}{\sqrt{2}}[|\chi_{2\Delta}^{2\varepsilon}(0)\rangle \otimes |+\rangle + |\chi_{2\Delta}^{2\varepsilon}(1)\rangle \otimes |-\rangle] \\ &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}[|\chi_{2\Delta}^{2\varepsilon}(0)\rangle + |\chi_{2\Delta}^{2\varepsilon}(1)\rangle] \otimes |0\rangle + \frac{1}{\sqrt{2}}[|\chi_{2\Delta}^{2\varepsilon}(0)\rangle - |\chi_{2\Delta}^{2\varepsilon}(1)\rangle] \otimes |1\rangle\right). \end{aligned}$$

The final unitary $V^{(4)} = \text{ctrl}e^{i\pi Q}$ has the effect of approximately eliminating the phase (-1) in the second term. In more detail, the final state is

$$\begin{aligned} V(|\Psi_\Delta^\varepsilon\rangle \otimes |+\rangle) &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}[|\chi_{2\Delta}^{2\varepsilon}(0)\rangle + |\chi_{2\Delta}^{2\varepsilon}(1)\rangle] \otimes |0\rangle + \frac{1}{\sqrt{2}}[e^{i\pi Q}|\chi_{2\Delta}^{2\varepsilon}(0)\rangle - e^{i\pi Q}|\chi_{2\Delta}^{2\varepsilon}(1)\rangle] \otimes |1\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(\frac{1}{2}(I + e^{i\pi Q})|\chi_{2\Delta}^{2\varepsilon}(0)\rangle + \frac{1}{2}(I - e^{i\pi Q})|\chi_{2\Delta}^{2\varepsilon}(1)\rangle\right) \otimes |+\rangle \\ &\quad + \frac{1}{\sqrt{2}}\left(\frac{1}{2}(I - e^{i\pi Q})|\chi_{2\Delta}^{2\varepsilon}(0)\rangle + \frac{1}{2}(I + e^{i\pi Q})|\chi_{2\Delta}^{2\varepsilon}(1)\rangle\right) \otimes |-\rangle. \end{aligned} \tag{C10}$$

Now consider the state

$$\begin{aligned} e^{-iP}|\text{III}_{2,2\Delta}^{2\varepsilon}\rangle &= \frac{1}{\sqrt{2}}e^{-iP}(|\chi_{2\Delta}^{2\varepsilon}(-1)\rangle + |\chi_{2\Delta}^{2\varepsilon}(0)\rangle) \\ &= \frac{1}{\sqrt{2}}(|\chi_{2\Delta}^{2\varepsilon}(0)\rangle + |\chi_{2\Delta}^{2\varepsilon}(1)\rangle). \end{aligned} \tag{C11}$$

The overlap of Eqs. (C10) and (C11) is

$$\left\langle e^{-iP} |\mathbb{III}_{2,2\Delta}^{2\varepsilon}\rangle \otimes |+\rangle, V(|\Psi_{\Delta}^{\varepsilon}\rangle \otimes |+\rangle) \right\rangle = \frac{1}{2} \cdot \left\langle \chi_{2\Delta}^{2\varepsilon}(0), \frac{1}{2}(I + e^{i\pi Q}) \chi_{2\Delta}^{2\varepsilon}(0) \right\rangle + \frac{1}{2} \cdot \left\langle \chi_{2\Delta}^{2\varepsilon}(1), \frac{1}{2}(I - e^{i\pi Q}) \chi_{2\Delta}^{2\varepsilon}(1) \right\rangle,$$

where we use that $|\chi_{2\Delta}^{2\varepsilon}(0)\rangle$ and $|\chi_{2\Delta}^{2\varepsilon}(1)\rangle$ [and, thus, also $e^{i\pi Q}|\chi_{2\Delta}^{2\varepsilon}(0)\rangle$ and $|\chi_{2\Delta}^{2\varepsilon}(1)\rangle$, etc.] have nonoverlapping support (because $\varepsilon < 1/2$) and are, thus, orthogonal, i.e.,

$$\begin{aligned} \langle \chi_{2\Delta}^{2\varepsilon}(0), \chi_{2\Delta}^{2\varepsilon}(1) \rangle &= \langle e^{i\pi Q} \chi_{2\Delta}^{2\varepsilon}(0), \chi_{2\Delta}^{2\varepsilon}(1) \rangle \\ &= \langle \chi_{2\Delta}^{2\varepsilon}(0), e^{i\pi Q} \chi_{2\Delta}^{2\varepsilon}(1) \rangle = 0. \end{aligned}$$

Using that

$$\langle \chi_{\Delta}^{\varepsilon}(z), e^{-i\pi z} e^{i\pi Q} \chi_{\Delta}^{\varepsilon}(z) \rangle \geq 1 - 5\varepsilon^2 \quad \text{for every } z \in \mathbb{R}$$

(see Lemma G.5), we obtain (for $z \in \{0, 1\}$, and for 2ε instead of ε)

$$\langle \chi_{2\Delta}^{2\varepsilon}(z), \frac{1}{2}[I + (-1)^z e^{i\pi Q}] \chi_{2\Delta}^{2\varepsilon}(z) \rangle \geq 1 - 10\varepsilon^2 \quad \text{for } z \in \{0, 1\}$$

and, thus,

$$\left\langle e^{-iP} |\mathbb{III}_{2,2\Delta}^{2\varepsilon}\rangle \otimes |+\rangle, V(|\Psi_{\Delta}^{\varepsilon}\rangle \otimes |+\rangle) \right\rangle \geq 1 - 10\varepsilon^2.$$

In particular,

$$\left| \left\langle e^{-iP} |\mathbb{III}_{2,2\Delta}^{2\varepsilon}\rangle \otimes |+\rangle, V(|\Psi_{\Delta}^{\varepsilon}\rangle \otimes |+\rangle) \right\rangle \right|^2 \geq 1 - 20\varepsilon^2,$$

and the claim follows from the identity

$$\| |\Psi\rangle\langle\Psi| - |\Phi\rangle\langle\Phi| \|_1 = 2\sqrt{1 - |\langle\Psi, \Phi\rangle|^2} \quad (\text{C12})$$

relating the trace distance and the overlap for two pure states $|\Phi\rangle$ and $|\Psi\rangle$ and the inequality $2\sqrt{20} \leq 9$. ■

Given the state $|\mathbb{III}_{L,\Delta}^{\varepsilon}\rangle$, the unitary V generates an approximation of the state $|\mathbb{III}_{2L,2\Delta}^{2\varepsilon}\rangle$ when applied to a product state with the qubit in the state $|+\rangle$. (The qubit approximately acts as a catalyst; i.e., the state of the qubit after application of V is approximately equal to $|+\rangle$.) A detailed description of this “peak-doubling” effect is the following.

Lemma C.3. Let $\varepsilon \in (0, 1/2)$, $\Delta > 0$, and $L \in 2\mathbb{N}$. Then,

$$\| |\mathbb{III}_{2L,2\Delta}^{2\varepsilon}\rangle \langle \mathbb{III}_{2L,2\Delta}^{2\varepsilon}| \otimes |+\rangle \langle +| - V(|\mathbb{III}_{L,\Delta}^{\varepsilon}\rangle \langle \mathbb{III}_{L,\Delta}^{\varepsilon}| \otimes |+\rangle \langle +|) V^{\dagger} \|_1 \leq 9\varepsilon.$$

Proof. Let us analyze the action of V on the state $|\mathbb{III}_{L,\Delta}^{\varepsilon}\rangle \otimes |+\rangle$. To lighten the notation, we first define the states

$$\Psi_z^{\text{even}} = |\chi_{2\Delta}^{2\varepsilon}(2z)\rangle \quad \text{and} \quad \Psi_z^{\text{odd}} = |\chi_{2\Delta}^{2\varepsilon}(2z+1)\rangle \quad \text{for } z \in \mathbb{Z}.$$

Since $\varepsilon < 1/2$, these states are pairwise orthogonal; i.e., we have

$$\begin{aligned} \langle \Psi_z^{\text{odd}}, \Psi_{z'}^{\text{even}} \rangle &= 0 \\ \langle \Psi_z^{\text{even}}, \Psi_{z'}^{\text{even}} \rangle &= \langle \Psi_z^{\text{odd}}, \Psi_{z'}^{\text{odd}} \rangle = \delta_{z,z'} \end{aligned} \quad \text{for } z, z' \in \mathbb{Z}. \quad (\text{C13})$$

For later reference, we note that—since these wave functions have pairwise orthogonal support (in the position basis)—these orthogonality relations also hold when an additional phase (in the position basis) is introduced. In particular, we have

$$\begin{aligned} \langle \Psi_z^{\text{odd}}, e^{i\pi Q} \Psi_{z'}^{\text{even}} \rangle &= 0 \\ \langle \Psi_z^{\text{even}}, e^{i\pi Q} \Psi_{z'}^{\text{even}} \rangle &= \langle \Psi_z^{\text{odd}}, e^{i\pi Q} \Psi_{z'}^{\text{odd}} \rangle = 0 \end{aligned} \quad \text{for } z \neq z' \in \mathbb{Z}. \quad (\text{C14})$$

With the factorization $V = V^{(4)} V^{(3)} V^{(2)} V^{(1)}$ of the unitary V from Eq. (C3), we can analyze the action of V as follows. It is easy to check that

$$V^{(1)}(|\mathbb{III}_{L,\Delta}^{\varepsilon}\rangle \otimes |+\rangle) = \frac{1}{\sqrt{L}} \sum_{z=-L/2}^{L/2-1} \Psi_z^{\text{even}} |+\rangle.$$

Hence, we obtain

$$V^{(2)}V^{(1)}(|\mathbb{W}_{L,\Delta}^\epsilon\rangle \otimes |+\rangle) = \frac{1}{\sqrt{L}} \sum_{z=-L/2}^{L/2-1} \frac{1}{\sqrt{2}} (\Psi_z^{\text{even}} \otimes |0\rangle + \Psi_z^{\text{odd}} \otimes |1\rangle)$$

using $\text{ctrl}e^{-iP}|\Psi_{z'}^{\text{even}}\rangle = |\Psi_{z'}^{\text{odd}}\rangle$. Applying the single-qubit Hadamard gate $V^{(3)} = I \otimes H$ to this state yields

$$\begin{aligned} V^{(3)}V^{(2)}V^{(1)}(|\mathbb{W}_{L,\Delta}^\epsilon\rangle \otimes |+\rangle) &= \frac{1}{\sqrt{L}} \sum_{z=-L/2}^{L/2-1} \frac{1}{\sqrt{2}} (|\Psi_z^{\text{even}}\rangle \otimes |+\rangle + |\Psi_z^{\text{odd}}\rangle \otimes |-\rangle) \\ &= \frac{1}{2\sqrt{L}} \sum_{z=-L/2}^{L/2-1} \left((|\Psi_z^{\text{even}}\rangle + |\Psi_z^{\text{odd}}\rangle) \otimes |0\rangle + (|\Psi_z^{\text{even}}\rangle - |\Psi_z^{\text{odd}}\rangle) \otimes |1\rangle \right). \end{aligned}$$

The final state after application of $V^{(4)} = \text{ctrl}e^{i\pi Q}$ is, thus,

$$\begin{aligned} V(|\mathbb{W}_{L,\Delta}^\epsilon\rangle \otimes |+\rangle) &= \frac{1}{2\sqrt{L}} \sum_{z=-L/2}^{L/2-1} \left((|\Psi_z^{\text{even}}\rangle + |\Psi_z^{\text{odd}}\rangle) \otimes |0\rangle + e^{i\pi Q} (|\Psi_z^{\text{even}}\rangle - |\Psi_z^{\text{odd}}\rangle) \otimes |1\rangle \right) \\ &= \frac{1}{2\sqrt{2L}} \sum_{z=-L/2}^{L/2-1} \left(((I + e^{i\pi Q})|\Psi_z^{\text{even}}\rangle + (I - e^{i\pi Q})|\Psi_z^{\text{odd}}\rangle) \otimes |+\rangle \right. \\ &\quad \left. + ((I - e^{i\pi Q})|\Psi_z^{\text{even}}\rangle + (I + e^{i\pi Q})|\Psi_z^{\text{odd}}\rangle) \otimes |-\rangle \right). \end{aligned}$$

We compute the overlap between the target state $|\mathbb{W}_{2L,2\Delta}^{2\epsilon}\rangle \otimes |+\rangle$ and the state $V(|\mathbb{W}_{L,\Delta}^\epsilon\rangle \otimes |+\rangle)$ prepared by the protocol. For convenience, we rewrite the target state in a form that resembles the form of the latter. We have

$$|\mathbb{W}_{2L,2\Delta}^{2\epsilon}\rangle = \frac{1}{\sqrt{2L}} \sum_{z=-L}^{L-1} |\chi_{2\Delta}^{2\epsilon}(z)\rangle = \frac{1}{\sqrt{2L}} \sum_{z=-L/2}^{L/2-1} (|\Psi_z^{\text{even}}\rangle + |\Psi_z^{\text{odd}}\rangle).$$

Therefore, the overlap between $|\mathbb{W}_{2L,2\Delta}^{2\epsilon}\rangle \otimes |+\rangle$ and $V(|\mathbb{W}_{L,\Delta}^\epsilon\rangle \otimes |+\rangle)$ is

$$\begin{aligned} (\langle \mathbb{W}_{2L,2\Delta}^{2\epsilon} | \otimes \langle + |) V(|\mathbb{W}_{L,\Delta}^\epsilon\rangle \otimes |+\rangle) &= \frac{1}{4L} \sum_{z=-L/2}^{L/2-1} \sum_{z'=-L/2}^{L/2-1} \langle \Psi_{z'}^{\text{even}} + \Psi_{z'}^{\text{odd}}, (I + e^{i\pi Q})\Psi_z^{\text{even}} + (I - e^{i\pi Q})\Psi_z^{\text{odd}} \rangle \\ &= \frac{1}{4L} \sum_{z=-L/2}^{L/2-1} \left(2 + \langle \Psi_z^{\text{even}}, e^{i\pi Q}\Psi_z^{\text{even}} \rangle - \langle \Psi_z^{\text{odd}}, e^{i\pi Q}\Psi_z^{\text{odd}} \rangle \right), \end{aligned}$$

where we use the orthogonality relations (C13) and (C14). Using that $e^{-i\pi(2z)} = 1$ and $e^{-i\pi(2z+1)} = -1$ for every integer $z \in \mathbb{Z}$, this can be rewritten as

$$\begin{aligned} (\langle \mathbb{W}_{2L,2\Delta}^{2\epsilon} | \otimes \langle + |) V(|\mathbb{W}_{L,\Delta}^\epsilon\rangle \otimes |+\rangle) &= \frac{1}{4L} \sum_{z=-L/2}^{L/2-1} \left(2 + \langle \Psi_z^{\text{even}}, e^{-i\pi(2z)} e^{i\pi Q} \Psi_z^{\text{even}} \rangle + \langle \Psi_z^{\text{odd}}, e^{-i\pi(2z+1)} e^{i\pi Q} \Psi_z^{\text{odd}} \rangle \right) \\ &\geq \frac{1}{4L} \sum_{z=-L/2}^{L/2-1} \left(2 + 2[1 - 5(2\epsilon)^2] \right) \\ &= 1 - 10\epsilon^2. \end{aligned} \tag{C15}$$

Here, we use the fact (see Lemma G.5) that $\langle \chi_\Delta^\epsilon(z), e^{-i\pi z} e^{i\pi Q} \chi_\Delta^\epsilon(z) \rangle \geq 1 - 5\epsilon^2$ to obtain the last inequality. We prove this fact in the appendix. Since Eq. (C15) implies

$$|(\langle \mathbb{I}_{2L,2\Delta}^{2\varepsilon} | \otimes \langle + |) V (| \mathbb{I}_{L,\Delta}^\varepsilon \rangle \otimes | + \rangle)|^2 \geq 1 - 20\varepsilon^2.$$

By using the relation between the overlap of two states and their trace distance [cf. Eq. (C12)], we conclude that

$$\| | \mathbb{I}_{2L,2\Delta}^{2\varepsilon} \rangle \langle \mathbb{I}_{2L,2\Delta}^{2\varepsilon} | \otimes | + \rangle \langle + | - V (| \mathbb{I}_{L,\Delta}^\varepsilon \rangle \langle \mathbb{I}_{L,\Delta}^\varepsilon | \otimes | + \rangle \langle + |) V^\dagger \|_1 \leq 2\sqrt{20}\varepsilon.$$

This implies the claim, since $2\sqrt{20} \leq 9$. ■

c. Completing the proof of Theorem C.1

In this section, we complete the proof of Theorem C.1. We have already argued that Protocol 2 can be realized using at most $5n + \lceil \log 1/\Delta \rceil + 4$ allowed elementary operations; see the discussion following the statement of the theorem. It thus remains to show Eq. (C4), i.e., that the output state ρ of the protocol is close to the state $| \mathbb{I}_{2^n,\Delta} \rangle$.

To do so, let us consider the repeated action of V . The following combines Lemmas C.2 and C.3.

Lemma C.4. For $n \in \mathbb{N}$, let us define the unitary

$$U_n := V^{n-1} (e^{iP} \otimes I) V.$$

Suppose $\varepsilon \in (0, 2^{-(n+1)})$ and $\Delta > 0$. Then,

$$\left\| | \mathbb{I}_{2^n,2^n\Delta}^{2^n\varepsilon} \rangle \langle \mathbb{I}_{2^n,2^n\Delta}^{2^n\varepsilon} | \otimes | + \rangle \langle + | - U_n (| \Psi_\Delta^\varepsilon \rangle \langle \Psi_\Delta^\varepsilon | \otimes | + \rangle \langle + |) U_n^\dagger \right\|_1 \leq 9\varepsilon \cdot (2^n - 1).$$

Proof. Define

$$\begin{aligned} |\Phi^{(0)}\rangle &= |\Psi_\Delta^\varepsilon\rangle \otimes |+\rangle, \\ |\Phi^{(1)}\rangle &= (e^{iP} \otimes I) V |\Phi^{(0)}\rangle, \\ |\Phi^{(k)}\rangle &= V |\Phi^{(k-1)}\rangle \quad \text{for } k \in \{2, \dots, n\}. \end{aligned}$$

We show inductively that

$$\left\| |\Phi^{(k)}\rangle \langle \Phi^{(k)}| - | \mathbb{I}_{2^k,2^k\Delta}^{2^k\varepsilon} \rangle \langle \mathbb{I}_{2^k,2^k\Delta}^{2^k\varepsilon} | \otimes | + \rangle \langle + | \right\|_1 \leq 9\varepsilon \cdot (2^k - 1) \quad \text{for every } k \in \{1, \dots, n\}. \quad (\text{C16})$$

By Lemma C.2 and the invariance of the norm $\|\cdot\|_1$ under unitaries, we have

$$\left\| |\Phi^{(1)}\rangle \langle \Phi^{(1)}| - | \mathbb{I}_{2,2\Delta}^{2\varepsilon} \rangle \langle \mathbb{I}_{2,2\Delta}^{2\varepsilon} | \otimes | + \rangle \langle + | \right\|_1 \leq 9\varepsilon.$$

This establishes Eq. (C16) for $k = 1$.

Suppose that we have shown the claim (C16) for $k - 1$. Then, we have by definition and by the triangle inequality that

$$\begin{aligned} & \left\| |\Phi^{(k)}\rangle \langle \Phi^{(k)}| - | \mathbb{I}_{2^k,2^k\Delta}^{2^k\varepsilon} \rangle \langle \mathbb{I}_{2^k,2^k\Delta}^{2^k\varepsilon} | \otimes | + \rangle \langle + | \right\|_1 \\ &= \left\| V |\Phi^{(k-1)}\rangle \langle \Phi^{(k-1)}| V^\dagger - | \mathbb{I}_{2^k,2^k\Delta}^{2^k\varepsilon} \rangle \langle \mathbb{I}_{2^k,2^k\Delta}^{2^k\varepsilon} | \otimes | + \rangle \langle + | \right\|_1 \\ &\leq \left\| V |\Phi^{(k-1)}\rangle \langle \Phi^{(k-1)}| V^\dagger - V \left(| \mathbb{I}_{2^{k-1},2^{k-1}\Delta}^{2^{k-1}\varepsilon} \rangle \langle \mathbb{I}_{2^{k-1},2^{k-1}\Delta}^{2^{k-1}\varepsilon} | \otimes | + \rangle \langle + | \right) V^\dagger \right\|_1 \\ &+ \left\| V \left(| \mathbb{I}_{2^{k-1},2^{k-1}\Delta}^{2^{k-1}\varepsilon} \rangle \langle \mathbb{I}_{2^{k-1},2^{k-1}\Delta}^{2^{k-1}\varepsilon} | \otimes | + \rangle \langle + | \right) V^\dagger - | \mathbb{I}_{2^k,2^k\Delta}^{2^k\varepsilon} \rangle \langle \mathbb{I}_{2^k,2^k\Delta}^{2^k\varepsilon} | \otimes | + \rangle \langle + | \right\|_1. \end{aligned}$$

By the invariance of the norm under unitaries and by the induction hypothesis, we have

$$\left\| V |\Phi^{(k-1)}\rangle \langle \Phi^{(k-1)}| V^\dagger - V \left(| \mathbb{I}_{2^{k-1},2^{k-1}\Delta}^{2^{k-1}\varepsilon} \rangle \langle \mathbb{I}_{2^{k-1},2^{k-1}\Delta}^{2^{k-1}\varepsilon} | \right) V^\dagger \right\|_1 \leq 9\varepsilon \cdot (2^{k-1} - 1).$$

Furthermore, we have

$$\left\| V \left(|\mathbb{I}_{2^{k-1}, 2^{k-1}\Delta}^{2^{k-1}\varepsilon}\rangle \langle \mathbb{I}_{2^{k-1}, 2^{k-1}\Delta}^{2^{k-1}\varepsilon}| \otimes |+\rangle \langle +| \right) V^\dagger - |\mathbb{I}_{2^k, 2^k\Delta}^{2^k\varepsilon}\rangle \langle \mathbb{I}_{2^k, 2^k\Delta}^{2^k\varepsilon}| \otimes |+\rangle \langle +| \right\|_1 \leq 9 \cdot 2^{k-1}\varepsilon$$

by Lemma C.3. The latter can be applied since $2^k\varepsilon \in (0, 1/2)$ by the assumption $\varepsilon \in (0, 2^{-(n+1)})$. Since $2^{k-1} - 1 + 2^{k-1} = 2^k - 1$, this implies Eq. (C16) for k .

Because $|\Phi^{(n)}\rangle = U_n(|\Psi_\Delta^\varepsilon\rangle \otimes |+\rangle)$ by definition, Eq. (C16) with $k = n$ implies the claim. ■

With Lemma C.4, we can complete the proof of Theorem C.1 as follows. Let $\Delta \in (0, 1/4)$ and $\varepsilon \in (0, 1/2)$. Then, Lemma C.4 [with $(2^{-n}\varepsilon, 2^{-n}\Delta)$ in place of (ε, Δ)] implies that

$$\left\| |\mathbb{I}_{2^n, \Delta}^\varepsilon\rangle \langle \mathbb{I}_{2^n, \Delta}^\varepsilon| \otimes |+\rangle \langle +| - U_n \left(|\Psi_{2^{-n}\Delta}^{2^{-n}\varepsilon}\rangle \langle \Psi_{2^{-n}\Delta}^{2^{-n}\varepsilon}| \otimes |+\rangle \langle +| \right) U_n^\dagger \right\|_1 \leq 9(2^{-n}\varepsilon) \cdot (2^n - 1) \leq 9\varepsilon. \quad (\text{C17})$$

By Corollary G.3, we have that for any $\varepsilon \in (\sqrt{\Delta}, 1/2)$ the truncated squeezed vacuum state $|\Psi_{2^{-n}\Delta}^{2^{-n}\varepsilon}\rangle$ is close to the squeezed vacuum state $|\Psi_{2^{-n}\Delta}\rangle$, i.e.,

$$\left\| |\Psi_{2^{-n}\Delta}\rangle \langle \Psi_{2^{-n}\Delta}| - |\Psi_{2^{-n}\Delta}^{2^{-n}\varepsilon}\rangle \langle \Psi_{2^{-n}\Delta}^{2^{-n}\varepsilon}| \right\|_1 \leq 3\sqrt{\Delta}. \quad (\text{C18})$$

Combining Eqs. (C17) and (C18) with the triangle inequality and using the invariance of the norm under unitaries, we conclude that

$$\left\| |\mathbb{I}_{2^n, \Delta}^\varepsilon\rangle \langle \mathbb{I}_{2^n, \Delta}^\varepsilon| \otimes |+\rangle \langle +| - U_n(|\Psi_{2^{-n}\Delta}\rangle \langle \Psi_{2^{-n}\Delta}| \otimes |+\rangle \langle +|) U_n^\dagger \right\|_1 \leq 3\sqrt{\Delta} + 9\varepsilon.$$

Corollary G.7 states that, for $\Delta \in (0, 1/4)$ and any $\varepsilon \in [\sqrt{\Delta}, 1/2)$, we have

$$\left\| |\mathbb{I}_{2^n, \Delta}\rangle \langle \mathbb{I}_{2^n, \Delta}| - |\mathbb{I}_{2^n, \Delta}^\varepsilon\rangle \langle \mathbb{I}_{2^n, \Delta}^\varepsilon| \right\|_1 \leq 5\sqrt{\Delta};$$

hence, we obtain with the choice $\varepsilon = \sqrt{\Delta}$

$$\left\| |\mathbb{I}_{2^n, \Delta}\rangle \langle \mathbb{I}_{2^n, \Delta}| \otimes |+\rangle \langle +| - U_n(|\Psi_{2^{-n}\Delta}\rangle \langle \Psi_{2^{-n}\Delta}| \otimes |+\rangle \langle +|) U_n^\dagger \right\|_1 \leq 3\sqrt{\Delta} + 9\varepsilon + 5\sqrt{\Delta} = 17\sqrt{\Delta}$$

by the triangle inequality. Using the fact that L^1 norm is contractive under completely positive trace-preserving (CPTP) maps (in particular, under tracing out the qubit system), the claim follows, since the output state of the protocol is

$$\rho = \text{tr}_{\text{qubit}} U_n(|\Psi_{2^{-n}\Delta}\rangle \langle \Psi_{2^{-n}\Delta}| \otimes |+\rangle \langle +|) U_n^\dagger$$

by definition.

APPENDIX D: THE ENVELOPE-GAUSSIFICATION PROTOCOL

In the following, we explain how to turn a comb state (a state with a rectangular envelope) into a state with a Gaussian envelope. In Appendix D 1, we introduce an alternative notion of approximate GKP states (where the envelope is defined differently). This proves helpful for our analysis. In Appendix D 2, we introduce our heralded envelope-Gaussification protocol and establish its main properties.

Given an input comb state $|\mathbb{I}_{L, \Delta}\rangle$ and a parameter κ (specifying the width of the desired Gaussian envelope), the protocol either rejects or accepts. Conditioned on acceptance, the output state is a quantum state close to the approximate GKP state $|\text{GKP}_{\kappa, \Delta}\rangle$. Finally, in Appendix D 4, we present the proof to the main result of this section: We show that applying the Gaussification protocol to a comb state produces a state with a Gaussian envelope.

1. An alternative type of approximate GKP state

The protocol considered in this section (Protocol 3) takes a comb state and applies a Gaussian envelope to it. The protocol does not produce approximate GKP state with “peakwise” Gaussian envelope $|\text{GKP}_{\kappa, \Delta}\rangle$ but (states close to) approximate GKP states with “pointwise” Gaussian envelope $|\text{gkp}_{\kappa, \Delta}\rangle$ that we define here.

To define the state $|\text{gkp}_{\kappa, \Delta}\rangle$ and to highlight the difference to the peakwise GKP state $|\text{GKP}_{\kappa, \Delta}\rangle$ [cf. Eq. (7)], let us rewrite the latter wave function as

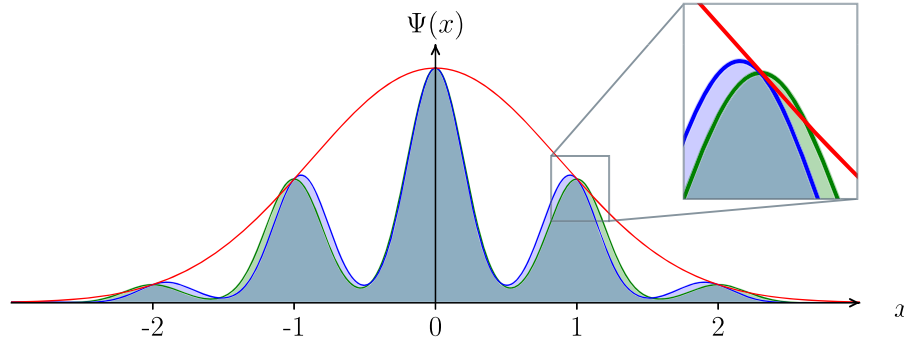


FIG. 12. Comparison of the pointwise and peakwise envelope models. The pointwise GKP state $|\text{gkp}_{\kappa,\Delta}\rangle$ is depicted in blue, and the peakwise GKP state $|\text{GKP}_{\kappa,\Delta}\rangle$ is depicted in green.

$$\text{GKP}_{\kappa,\Delta}(x) := C_{\kappa,\Delta} \sum_{z \in \mathbb{Z}} \eta_{\kappa}(z) \chi_{\Delta}(z)(x), \quad (\text{D1})$$

where $\eta_{\kappa} \in L^2(\mathbb{R})$ is the Gaussian envelope

$$\eta_{\kappa}(z) = \frac{\sqrt{\kappa}}{\pi^{1/4}} e^{-\kappa^2 z^2/2} \quad (\text{D2})$$

with parameter $\kappa > 0$ and $\chi_{\Delta}(z) \in L^2(\mathbb{R})$ is a Gaussian with variance Δ^2 centered at $z \in \mathbb{R}$; see Eq. (C2).

We define a pointwise GKP state $|\text{gkp}_{\kappa,\Delta}\rangle \in L^2(\mathbb{R})$ by

$$\text{gkp}_{\kappa,\Delta}(x) := D_{\kappa,\Delta} \sum_{z \in \mathbb{Z}} \eta_{\kappa}(x) \chi_{\Delta}(z)(x), \quad (\text{D3})$$

where $D_{\kappa,\Delta}$ is normalization factor. We illustrate the difference between the states $|\text{gkp}_{\kappa,\Delta}\rangle$ and $|\text{GKP}_{\kappa,\Delta}\rangle$ in Fig. 12.

We note that peakwise and pointwise approximate GKP states [with an appropriate choice of parameters (κ, Δ)] are close to each other in L^1 distance (see Corollary G.17 for further details).

2. Envelope shaping by an adaptive “measure-then-correct” protocol

Here, we describe our envelope-Gaussification protocol (cf. Protocol 3). The protocol takes as input (a state close to) a comb state $|\text{III}_{L,\Delta}\rangle$, as well as a parameter $\kappa > 0$ specifying the targeted Gaussian envelope η_{κ} . It either accepts or rejects and outputs a one-mode state when it accepts. We show that the acceptance probability is lower bounded by a constant. Furthermore, we prove that the (average) output state conditioned on acceptance is close to the state $|\text{GKP}_{\kappa,\Delta}\rangle$.

Protocol 3 is implemented by the adaptive circuit in Fig. 13. This circuit is adaptive in the sense that it involves a unitary (displacement) that is classically controlled by a function of the measurement result.

The main result of this section is the following.

Theorem D.1. There are constants $b_1, b_2 > 0$ such that the following holds. Assume $\xi > 0$, $\kappa \in (0, 1/4)$, $\Delta \in (0, 1/4)$, and $L \in 8\mathbb{N}$. Let $\tau \in \mathcal{B}[L^2(\mathbb{R})]$ be a state close to $|\text{III}_{L,\Delta}\rangle$, i.e.,

$$\|\tau - |\text{III}_{L,\Delta}\rangle\langle\text{III}_{L,\Delta}|\|_1 \leq \xi.$$

Given the comb state parameter L , the squeezing parameter κ (specifying a Gaussian envelope) and the input state τ , Protocol 3 accepts with probability at least

$$\Pr[\text{Protocol 3 accepts } |\tau|] \geq \frac{1}{8}(1 - 2e^{-\kappa^2 L^2/256}) - \frac{5}{2}\sqrt{\Delta} - \frac{\xi}{2}. \quad (\text{D4})$$

Conditioned on acceptance, the output state $\tau_{\text{acc}} \in \mathcal{B}[L^2(\mathbb{R})]$ on the second mode is close to the state $|\text{GKP}_{\kappa,\Delta}\rangle$, i.e.,

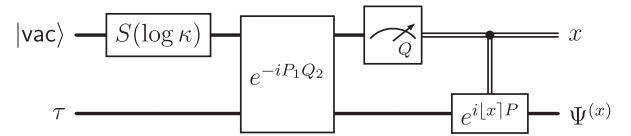


FIG. 13. Circuit used in the envelope-Gaussification protocol (Protocol 3). For measurement outcomes $x \in \mathbb{R}$ with $x \notin [-L/8 - 1/2, L/8 + 1/2]$, the protocol returns **reject**, and there is no output state in this case. For values $x \in [-L/8 - 1/2, L/8 + 1/2]$ (a case illustrated in the figure), the unitary $e^{i[x]P}$ is applied to the second mode. This is a classically controlled displacement gate; i.e., it involves the parameter $[x] \in \mathbb{Z}$ which is computed classically. Note that both the squeezing gate $S(\log \kappa)$ and $e^{i[x]P}$ need to be decomposed in terms of constant-strength squeezing and displacement gates, respectively, to obtain operations from the set $\mathcal{G}$ (see Appendix A 1).

$$\begin{aligned} & \|\tau_{\text{acc}} - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 \\ & \leq \frac{5\sqrt{\Delta} + \xi}{\frac{1}{4}(1 - 2e^{-\kappa^2 L^2/16})} + 6\sqrt{\Delta} + 6\kappa\sqrt{L} + 7e^{-\kappa^2 L^2/128}. \end{aligned} \quad (\text{D5})$$

The protocol can be realized using at most $b_1 \log L + b_2 \log 1/\kappa$ elementary operations.

Before proving Eqs. (D4) and (D5), let us compute the number of elementary operations that are needed to implement Protocol 3.

The protocol first prepares the Gaussian state $|\eta_\kappa\rangle = S(\log \kappa)|\text{vac}\rangle$. As we allow for only bounded-strength operations, we decompose the unitary $S(\log \kappa)$ into consecutive bounded-strength squeezing operators $S(z)$ with $z \in [-2\pi, 2\pi]$. An analysis similar to Eq. (C5) shows that the unitary $S(\log \kappa)$ can be realized by $\lceil \log 1/\kappa \rceil$ gates of this form.

Subsequently, the unitary $e^{-iP_1 Q_2}$ is applied. This is a Gaussian unitary of constant strength; hence, it can be written as a product of a constant number of Gaussian unitaries from the set $\mathcal{G}$ (see the discussion in Appendix A 1).

Then, the protocol performs a homodyne measurement of the first register that results in an outcome $x \in \mathbb{R}$. The classical outcome x is rounded to the next integer $\lfloor x \rfloor \in \mathbb{Z}$. Then, the protocol applies a classically controlled shift (displacement) unitary $e^{i\lfloor x \rfloor P_2}$ depending on $\lfloor x \rfloor$. This operation again does not have bounded strength, in general (we have $|x| \leq L/8 + 1/2$, meaning that $\lfloor x \rfloor$ can scale with L), and needs to be decomposed into gates from $\mathcal{G}$. From Lemma B.1, we know that we can decompose the unitary $e^{i\lfloor x \rfloor P_2}$ using at most $2\lceil \log \lfloor x \rfloor \rceil + 3$ unitaries from $\mathcal{G}$. As the acceptance region is $\Omega_L = [-L/8 - 1/2, L/8 + 1/2]$, we conclude that the shift correction after acceptance needs at most $2\lceil \log(L/8 + 1/2) \rceil + 3$ gates from $\mathcal{G}$ to be implemented. As the state initialization of the vacuum $|\text{vac}\rangle$ and the homodyne measurement require a constant amount (two) elementary operations, choosing $b_1, b_2 > 0$ large enough shows the claim.

3. Analysis of envelope-Gaussification with $|\text{III}_L^\epsilon\rangle$ as input

In this section, we analyze Protocol 3 in the case where the input state is $|\text{III}_L^\epsilon\rangle$, i.e., a truncated comb state.

Specifically, we proceed as follows: In Appendix D 3 a, we translate the circuit defined by Protocol 3 to an equivalent circuit that is nonadaptive (to simplify the analysis). In Appendix D 3 b, we then show that, upon acceptance, the output state is indeed a state close the desired approximate GKP state. In Appendix D 3 c, we show that the acceptance probability is lower bounded by a constant independent of the envelope parameters.

These results are subsequently used in Appendix D 4 to extend the analysis to input states that are close to a comb state.

a. A nonadaptive description of Protocol 3

We note that Protocol 3 (cf. Fig. 13) is adaptive; i.e., it involves a unitary (the unitary $e^{i\lfloor x \rfloor P_2}$) whose parameter is classically controlled and determined by the measurement result $x \in \mathbb{R}$. To analyze Protocol 3, it will be convenient to consider a nonadaptive version of the circuit depicted in Fig. 13. The nonadaptive circuit contains a unitary that is outside our allowed gate set $\mathcal{G}$: This is the (non-Gaussian) unitary $e^{i\lfloor Q \rfloor P_2}$. Here, the operator $\lfloor Q \rfloor$ acts in position space on functions Ψ as a multiplication operator, i.e.,

$$(\lfloor Q \rfloor \Psi)(x) = \lfloor x \rfloor \Psi(x) \quad \text{for } x \in \mathbb{R}.$$

That is, we consider the nonadaptive circuit in Fig. 14(b), which is equivalent to the circuit in Fig. 14(a).

We use the notion of a quantum instrument to describe the homodyne position measurement and the associated postmeasurement state of the circuit in Fig. 14(b). Quantum instruments are based on completely positive trace-non-increasing maps (CPTNIM) $\mathcal{K}: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}')$ defined on two Hilbert spaces $\mathcal{H}, \mathcal{H}'$; see, e.g., Ref. [37] for further details. An instrument is a CPTNIM-valued measure on a suitable measure space. In the case of homodyne position measurement, the measure space is given by the Borel- σ

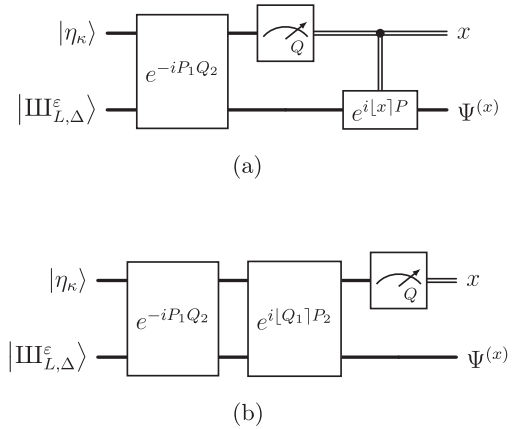


FIG. 14. (a) Adaptive circuit implementing Protocol 3. The last gate in the circuit is classically controlled by the parameter $\lfloor x \rfloor \in \mathbb{Z}$, a function of the measurement result $x \in \mathbb{R}$. (b) Nonadaptive circuit implementing Protocol 3. Here, the unitary before the measurement does not belong to the set of operations $\mathcal{G}$ as it is non-Gaussian. All unitaries are nonadaptive (i.e., they are not controlled by measurement results). Two equivalent circuits realizing the envelope-Gaussification protocol (Protocol 3) on input $|\text{III}_{L,\Delta}^\epsilon\rangle$. The first one is identical to that shown in Fig. 13, up to the fact that the squeezed vacuum state $|\eta_\kappa\rangle = S(\log \kappa)|\text{vac}\rangle$ is drawn as an input. We use the second circuit to analyze the behavior of the first.

algebra of $\mathbb{R}$. If the position of the first mode of a bipartite system of two oscillators is measured, then $\mathcal{H} \cong L^2(\mathbb{R})^{\otimes 2} =: \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{H}' \cong L^2(\mathbb{R})$, and the instrument $\mathcal{K}$ associated with homodyne position measurement is defined by

$$\begin{aligned} \mathcal{K}[A]: \mathcal{B}[L^2(\mathbb{R})^{\otimes 2}] &\rightarrow \mathcal{B}[L^2(\mathbb{R})] \\ \rho &\mapsto \mathcal{K}[A](\rho) = \text{tr}_{\mathcal{H}_1}((\Pi_A \otimes \text{id})\rho), \end{aligned}$$

for any Borel set $A \subseteq \mathbb{R}$. In this expression, Π_A denotes the orthogonal projection onto the subspace $L^2(\mathbb{R})$ of functions Ψ having support contained in A . The interpretation is as follows: For a bipartite state $\rho \in \mathcal{B}[L^2(\mathbb{R})^{\otimes 2}]$, the measurement outcome X is a random variable satisfying

$$\Pr[X \in A] = \text{tr} \mathcal{K}[A](\rho).$$

Furthermore, if this expression is nonzero, then the conditional postmeasurement state $\rho|_A$ conditioned on the event $X \in A$ is given by the expression

$$\rho|_A = \frac{\mathcal{K}[A](\rho)}{\text{tr} \mathcal{K}[A](\rho)}.$$

The following lemma gives an expression for the overlap of this conditional state when measuring the first mode of a state of the form $e^{i\lfloor Q_1 \rfloor P_2} e^{-iP_1 Q_2} (\Psi_1 \otimes \Psi_2)$. This will be useful to analyze the circuit in Fig. 14(b).

Lemma D.2. Let $\Psi_1, \Psi_2 \in L^2(\mathbb{R})$ be two states. Assume that Ψ_1 is even, i.e.,

$$\Psi_1(x) = \Psi_1(-x) \quad \text{for all } x \in \mathbb{R}.$$

Define

$$\begin{aligned} \Phi(y) &= \frac{1}{\sqrt{p(0)}} \Psi_1(y) \Psi_2(y) \quad \text{for all } y \in \mathbb{R}, \quad \text{where} \\ p(0) &= \int_{\mathbb{R}} |\Psi_1(z) \Psi_2(z)|^2 dz. \end{aligned}$$

[That is, $\Phi = M_{\Psi_1} \Psi_2 / \|M_{\Psi_1} \Psi_2\|$, where $M_{\Psi_1}: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is the multiplication operator which outputs the pointwise product of Ψ_1 and the function it is applied to.] Suppose we apply a position measurement to the first mode of the state

$$|\Psi\rangle = e^{i\lfloor Q_1 \rfloor P_2} e^{-iP_1 Q_2} (|\Psi_1\rangle \otimes |\Psi_2\rangle).$$

For $x \in \mathbb{R}$ and its associated noninteger part $\delta(x) = x - \lfloor x \rfloor$, we define

$$\begin{aligned} p(x) &= \int_{\mathbb{R}} |\Psi_1(x-y)|^2 |\Psi_2(y)|^2 dy, \\ m(x) &= \int_{\mathbb{R}} \overline{\Psi_1[y - \delta(x)]} \Psi_1(y) \overline{\Psi_2(\lfloor x \rfloor + y)} \Psi_2(y) dy. \end{aligned}$$

Let $A \subseteq \mathbb{R}$ be a Borel set. Assume that $p(A) = \Pr[X \in A] > 0$ for the measurement outcome X . Let $\rho|_A$ denote the corresponding postmeasurement state on the second mode conditioned on the event $X \in A$ that the measurement outcome belongs to A .

Then, the probability of the event $X \in A$ is equal to

$$p(A) = \int_A p(x) dx,$$

and the conditional state $\rho|_A$ conditioned on the event $X \in A$ satisfies

$$\langle \Phi | \rho|_A | \Phi \rangle = \frac{1}{p(0)p(A)} \cdot \int_A |m(x)|^2 dx. \quad (\text{D6})$$

Proof. Let $|\Psi'\rangle = e^{-iP_1 Q_2} (|\Psi_1\rangle \otimes |\Psi_2\rangle)$. Then, we have

$$\Psi'(x, y) = (e^{-iyP_1} \Psi_1)(x) \Psi_2(y) = \Psi_1(x-y) \Psi_2(y) \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

Since $|\Psi\rangle = e^{i\lfloor Q_1 \rfloor P_2} |\Psi'\rangle$, it follows that

$$\Psi(x, y) = \Psi'(x, y + \lfloor x \rfloor) = \Psi_1[x - (y + \lfloor x \rfloor)] \Psi_2(x + y) = \Psi_1[y - \delta(x)] \Psi_2(\lfloor x \rfloor + y),$$

for all $(x, y) \in \mathbb{R}^2$, where we use that Ψ_1 is even in the last step. By definition of the conditional postmeasurement state $\rho|_A = p(A)^{-1} \mathcal{K}[A](\rho)$, we have

$$\begin{aligned}
\langle \Phi | \rho_A | \Phi \rangle &= \frac{1}{p(A)} \langle \Psi | (\Pi_A \otimes |\Phi\rangle\langle\Phi|) | \Psi \rangle \\
&= \frac{1}{p(A)} \int_A dx \int_{\mathbb{R}^2} dy dy' \overline{\Psi(x, y)} \Phi(y) \overline{\Phi(y')} \Psi(x, y') \\
&= \frac{1}{p(A)} \int_A dx \int_{\mathbb{R}^2} dy dy' \overline{\Psi_1[y - \delta(x)] \Psi_2(\lfloor x \rfloor + y)} \Phi(y) \cdot \overline{\Phi(y')} \Psi_1(y' - \delta) \Psi_2(\lfloor x \rfloor + y') \\
&= \frac{1}{p(0)p(A)} \int_A dx \int_{\mathbb{R}} dy \overline{\Psi_1[y - \delta(x)] \Psi_1(y) \Psi_2(\lfloor x \rfloor + y)} \Psi_2(y) \cdot \int_{\mathbb{R}} dy' \overline{\Psi_1(y') \Psi_1(y' - \delta(x))} \overline{\Psi_2(y')} \Psi_2(\lfloor x \rfloor + y') \\
&= \frac{1}{p(0)p(A)} \int_A dx \left| \int_{\mathbb{R}} dy \overline{\Psi_1[y - \delta(x)] \Psi_1(y) \Psi_2(\lfloor x \rfloor + y)} \Psi_2(y) \right|^2.
\end{aligned}$$

The claim follows from this.

b. Implications for approximate comb states

We use Lemma D.2 to analyze the circuit in Fig. 14(b). To do so, we apply it to the state $|\Psi_1\rangle = |\eta_\kappa\rangle$ and the state $|\Psi_2\rangle = |\text{III}_{L,\Delta}^\varepsilon\rangle$. In this case, the state $|\Phi\rangle$ of Lemma D.2 is equal to

$$\Phi(y) = \frac{1}{\sqrt{p(0)}} \eta_\kappa(y) \text{III}_{L,\Delta}^\varepsilon(y).$$

That is, the state $|\Phi\rangle = |\text{gkp}_{L,\kappa,\Delta}^\varepsilon\rangle$, where

$$\text{gkp}_{L,\kappa,\Delta}^\varepsilon(x) = \frac{1}{\sqrt{p(0)}} \frac{1}{\sqrt{L}} \sum_{z=-L/2}^{L/2-1} \eta_\kappa(x) \chi_\Delta^\varepsilon(z)(x), \quad (\text{D7})$$

is a truncated version of the state $|\text{gkp}_{L,\kappa,\Delta}\rangle$ defined by Eq. (D3). For later reference, we note that the norm of this state is equal to

$$1 = \|\text{gkp}_{L,\kappa,\Delta}^\varepsilon\|^2 = \frac{1}{Lp(0)} \sum_{k=-L/2}^{L/2-1} I_k(0), \quad (\text{D8})$$

where we use the expression

$$I_k(\delta) = \int \chi_\Delta^\varepsilon(k)(y)^2 \eta_\kappa(y - \delta)^2 dy. \quad (\text{D9})$$

[Equation (D8) follows because the functions $\{\chi_\Delta^\varepsilon(z)\}_{z \in \{-L/2, \dots, L/2-1\}}$ have pairwise-disjoint supports for $\varepsilon < 1/2$.]

The corresponding probability density functions of outcomes are equal to

$$\begin{aligned}
p(x) &= \int \eta_\kappa(x - y)^2 |\text{III}_{L,\Delta}^\varepsilon(y)|^2 dy \\
&= \int \eta_\kappa[y - \delta(x)]^2 |\text{III}_{L,\Delta}^\varepsilon(y + \lfloor x \rfloor)|^2 dy,
\end{aligned}$$

■ where we use that $x = \lfloor x \rfloor + \delta(x)$ and that $\eta_\kappa(z) = \eta_\kappa(-z)$ for $z \in \mathbb{R}$ and where we substitute $y - \lfloor x \rfloor$ for y . Inserting the definition of $\text{III}_{L,\Delta}^\varepsilon$ into the above, we obtain

$$\begin{aligned}
p(x) &= \frac{1}{L} \int \eta_\kappa[y - \delta(x)]^2 \left(\sum_{z=-L/2}^{L/2-1} \chi_\Delta^\varepsilon(z)(y + \lfloor x \rfloor) \right)^2 dy \\
&= \frac{1}{L} \int \eta_\kappa[y - \delta(x)]^2 \left(\sum_{z=-L/2}^{L/2-1} \chi_\Delta^\varepsilon(z - \lfloor x \rfloor)(y) \right)^2 dy \\
&= \frac{1}{L} \int \eta_\kappa[y - \delta(x)]^2 \left(\sum_{k=-L/2-\lfloor x \rfloor}^{L/2-\lfloor x \rfloor-1} \chi_\Delta^\varepsilon(k)(y) \right)^2 dy,
\end{aligned}$$

where we first use that

$$\begin{aligned}
\chi_\Delta^\varepsilon(z)(y + \lfloor x \rfloor) &= \Psi_\Delta^\varepsilon(y + \lfloor x \rfloor - z) \\
&= \Psi_\Delta^\varepsilon[y - (z - \lfloor x \rfloor)] = \chi_\Delta^\varepsilon(z - \lfloor x \rfloor)(y)
\end{aligned} \quad (\text{D10})$$

by the symmetry of the truncated centered Gaussian Ψ_Δ^ε and then change the index of summation. We can write Eq. (D10) as

$$p(x) = \frac{1}{L} \sum_{k_1=-L/2-\lfloor x \rfloor}^{L/2-\lfloor x \rfloor-1} \sum_{k_2=-L/2-\lfloor x \rfloor}^{L/2-\lfloor x \rfloor-1} M_{k_1,k_2}(x), \quad (\text{D11})$$

where

$$M_{k_1,k_2}(x) := \int \eta_\kappa[y - \delta(x)]^2 \chi_\Delta^\varepsilon(k_1)(y) \chi_\Delta^\varepsilon(k_2)(y) dy.$$

Because $\varepsilon < 1/2$, the expression $\chi_\Delta^\varepsilon(k_1)(y) \chi_\Delta^\varepsilon(k_2)(y)$ can be nonzero only if $k_1 = k_2$. Thus, the integral $M_{k_1,k_2}(x)$ vanishes unless $k_1 = k_2$, and we have $M_{k,k}(x) = I_k[\delta(x)]$, where $I_k(\delta)$ is defined by Eq. (D9). We conclude that

$$\begin{aligned}
p(x) &= \frac{1}{L} \sum_{k=-L/2-\lfloor x \rfloor}^{L/2-1-\lfloor x \rfloor} I_k[\delta(x)] \\
&= \frac{1}{L} \sum_{k=-L/2}^{L/2-1} I_{k-\lfloor x \rfloor}[\delta(x)] \\
&= \frac{1}{L} \sum_{k=-L/2}^{L/2-1} I_k[\lfloor x \rfloor + \delta(x)] \\
&= \frac{1}{L} \sum_{k=-L/2}^{L/2-1} I_k(x), \quad \text{since } x = \lfloor x \rfloor + \delta(x), \quad (\text{D12})
\end{aligned}$$

where we relabel the index of the sum in the second step and use that $I_{k-\lfloor x \rfloor}[\delta(x)] = I_k[\lfloor x \rfloor + \delta(x)]$ in the third line [cf. Eq. (G55) for a proof of this relation]. For later reference, we observe that we can rephrase Eq. (D12) as follows. For an integer $m \in \mathbb{Z}$ and $\delta \in [-1/2, 1/2)$, the probability density function $p(m + \delta)$ is equal to the following two expressions:

$$p(m + \delta) = \frac{1}{L} \sum_{k=-L/2-m}^{L/2-m-1} I_k(\delta) = \frac{1}{L} \sum_{k=-L/2}^{L/2-1} I_k(m + \delta). \quad (\text{D13})$$

We show that the conditional postmeasurement state $\rho_{|\Omega_L}$ in the second register, which is conditioned on the measurement result X in the first register being in the acceptance region

$$\Omega_L = [-L/8 - 1/2, L/8 + 1/2] \quad (\text{D14})$$

of Protocol 3 (line 4), i.e., conditioned on the event $X \in \Omega_L$, is close to the state $|\text{gkp}_{L,\kappa,\Delta}^\varepsilon\rangle$.

Lemma D.3. Let $L \in 8\mathbb{N}$ be an integer multiple of 8, let $\Delta > 0$, let $\kappa > 0$, let $\varepsilon \in (0, 1/2)$, and let $\Omega_L \subset \mathbb{R}$ be the set (D14) of measurement results $x \in \mathbb{R}$ for which Protocol 3 accepts. Let us denote by $\rho_{|\Omega_L}(\varepsilon, \kappa, \Delta, L) \in \mathcal{B}[L^2(\mathbb{R})]$ the output state conditioned on the event that the protocol

accepts on input $(|\text{III}_{L,\Delta}^\varepsilon\rangle, \kappa, L)$. Then,

$$\begin{aligned}
&\langle \text{gkp}_{L,\kappa,\Delta}^\varepsilon | \rho_{|\Omega_L}(\varepsilon, \kappa, \Delta, L) | \text{gkp}_{L,\kappa,\Delta}^\varepsilon \rangle \\
&\geq 1 - 3\kappa^2 L/2 - 4e^{-\kappa^2 L^2/32}.
\end{aligned}$$

Proof. Because the state $|\Phi\rangle$ defined in Lemma D.2 is equal to $|\Phi\rangle = |\text{gkp}_{L,\kappa,\Delta}^\varepsilon\rangle$, the claim follows from Eq. (D6), which can be rewritten as

$$\begin{aligned}
&\langle \text{gkp}_{L,\kappa,\Delta}^\varepsilon | \rho_{|\Omega_L}(\varepsilon, \kappa, \Delta, L) | \text{gkp}_{L,\kappa,\Delta}^\varepsilon \rangle \\
&= \int_{\Omega_L} \frac{p(x)}{p(\Omega_L)} \cdot \frac{|m(x)|^2}{p(0)p(x)} dx,
\end{aligned}$$

because $p(x) > 0$ for all $x \in \Omega_L$ [in fact, we even have $p(x) > 0$ for all $x \in \mathbb{R}$]. Observing that $p(x)/p(\Omega_L) = p(x|\Omega_L)$ is the conditional distribution given that the measurement result satisfies $X \in \Omega_L$, it follows that

$$\begin{aligned}
&\langle \text{gkp}_{L,\kappa,\Delta}^\varepsilon | \rho_{|\Omega_L}(\varepsilon, \kappa, \Delta, L) | \text{gkp}_{L,\kappa,\Delta}^\varepsilon \rangle \\
&= \int_{\Omega_L} p(x|\Omega_L) \frac{|m(x)|^2}{p(0)p(x)} dx \geq \inf_{x \in \Omega_L} \frac{m(x)^2}{p(0)p(x)}.
\end{aligned}$$

Here, we use that $m(x) \geq 0$ for all $x \in \mathbb{R}$. The claim is a consequence of this inequality, the definition (D14) of Ω_L , and Lemma D.4 below. ■

Lemma D.4. Suppose that $\kappa > 0$, $\Delta > 0$, $\varepsilon \in (0, 1/2)$, and $L \in 8\mathbb{N}$. Let $x = m + \delta$ with $m = \lfloor x \rfloor$, $|m| \leq L/8$, and $|\delta| \leq 1/2$. Consider the quantities $p(x) \in \mathbb{R}$ and $m(x) \in \mathbb{C}$ defined by Lemma D.2 applied to $\Psi_1 = \eta_\kappa$ and $\Psi_2 = \text{III}_{L,\Delta}^\varepsilon$. Then,

$$\frac{m(x)^2}{p(0)p(x)} \geq 1 - 3\kappa^2 L/2 - 4e^{-\kappa^2 L^2/32}.$$

Proof. By definition of $x = m + \delta$, we have $\lfloor x \rfloor = m$ and $\delta = x - \lfloor x \rfloor$ and, thus,

$$\begin{aligned}
m(x) &= \int_{\mathbb{R}} \eta_\kappa(y - \delta) \eta_\kappa(y) \text{III}_{L,\Delta}^\varepsilon(y) \text{III}_{L,\Delta}^\varepsilon(m + y) dy \\
&= \frac{1}{L} \sum_{k_1=-L/2}^{L/2-1} \sum_{k_2=-L/2}^{L/2-1} \int \eta_\kappa(y - \delta) \eta_\kappa(y) \chi_\Delta^\varepsilon(k_1)(y) \chi_\Delta^\varepsilon(k_2)(m + y) dy.
\end{aligned}$$

Similarly as before [see Eq. (D10)], we can use that $\chi_\Delta^\varepsilon(k_2)(m + y) = \chi_\Delta^\varepsilon(k_2 - m)(y)$ to obtain

$$\begin{aligned}
m(x) &= \frac{1}{L} \sum_{k_1=-L/2}^{L/2-1} \sum_{k_2=-L/2}^{L/2-1} \int \eta_\kappa(y-\delta) \eta_\kappa(y) \chi_\Delta^\varepsilon(k_1)(y) \chi_\Delta^\varepsilon(k_2-m)(y) dy \\
&= \frac{1}{L} \sum_{k_1=-L/2}^{L/2-1} \sum_{k_2=-L/2-m}^{L/2-m-1} \int \eta_\kappa(y-\delta) \eta_\kappa(y) \chi_\Delta^\varepsilon(k_1)(y) \chi_\Delta^\varepsilon(k_2)(y) dy \\
&= \frac{1}{L} \sum_{k_1=-L/2}^{L/2-1} \sum_{k_2=-L/2-m}^{L/2-m-1} M'_{k_1, k_2}(\delta),
\end{aligned}$$

where we shift the summation index k_2 and introduce the scalars

$$M'_{k_1, k_2}(\delta) = \int \eta_\kappa(y-\delta) \eta_\kappa(y) \chi_\Delta^\varepsilon(k_1)(y) \chi_\Delta^\varepsilon(k_2)(y) dy.$$

Because the expression $\chi_\Delta^\varepsilon(k_1)(y) \chi_\Delta^\varepsilon(k_2)(y)$ can be non-zero only when $k_1 = k_2$ (since $\varepsilon < 1/2$), the integral $M'_{k_1, k_2}(\delta)$ vanishes unless $k_1 = k_2$. We conclude that

$$m(x) = \begin{cases} \frac{1}{L} \sum_{k=-L/2}^{L/2-m-1} M'_k(\delta) & \text{if } m \geq 0, \\ \frac{1}{L} \sum_{k=-L/2+|m|}^{L/2-1} M'_k(\delta) & \text{if } m < 0, \end{cases}$$

where we set $M'_k = M'_{k, k}$. Recalling that we are considering $m \in \mathbb{Z}$ with $|m| \leq L/8$ and using that each term

$$M'_k(\delta) = \int \eta_\kappa(y-\delta) \eta_\kappa(y) \chi_\Delta^\varepsilon(k)(y)^2 dy \quad (\text{D15})$$

is non-negative, we obtain the lower bound

$$\begin{aligned}
m(x) &\geq \frac{1}{L} \sum_{k=-L/2+L/8}^{L/2-L/8-1} M'_k(\delta) \\
&\geq \frac{1}{L} \sum_{k=-L/4}^{L/4-1} M'_k(\delta), \quad (\text{D16})
\end{aligned}$$

where we use that $L/2 - L/8 - 1 \geq L/4 - 1$ and $-L/2 + L/8 < -L/4$ since $L \geq 8$ by the assumption that L is an integer multiple of 8.

In the expression (D15) defining $M'_k(\delta)$, we can restrict the domain of integration to $[k - \varepsilon, k + \varepsilon]$ as the expression $\chi_\Delta^\varepsilon(k)(y)$ vanishes for $y \in \mathbb{R}$ outside this interval; i.e., we have

$$M'_k(\delta) = \int_{k-\varepsilon}^{k+\varepsilon} \eta_\kappa(y-\delta) \eta_\kappa(y) \chi_\Delta^\varepsilon(k)(y)^2 dy. \quad (\text{D17})$$

We show that

$$M'_k(\delta) \geq e^{-\kappa^2 L/4} \int_{k-\varepsilon}^{k+\varepsilon} \eta_\kappa(y)^2 \chi_\Delta^\varepsilon(k)(y)^2 dy \quad \text{for all } k \in \{-L/4, \dots, L/4 - 1\}. \quad (\text{D18})$$

We note that, by using the definition (D9) of the integral $I_k(\delta)$, Eq. (D18) can be expressed as

$$M'_k(\delta) \geq e^{-\kappa^2 L/4} I_k(0) \quad \text{for all } k \in \{-L/4, \dots, L/4 - 1\}. \quad (\text{D19})$$

We use the identity

$$\begin{aligned}
\eta_\kappa(y-\delta) &= \frac{\sqrt{\kappa}}{\pi^{1/4}} e^{-\kappa^2(y-\delta)^2/2} \\
&= \frac{\sqrt{\kappa}}{\pi^{1/4}} e^{-\kappa^2 y^2/2} \cdot e^{\kappa^2 y \delta} \cdot e^{-\kappa^2 \delta^2/2} \\
&= \eta_\kappa(y) \cdot e^{\kappa^2 y \delta} \cdot e^{-\kappa^2 \delta^2/2},
\end{aligned}$$

which implies that

$$\eta_\kappa(y-\delta) \geq \eta_\kappa(y) \cdot e^{-\kappa^2(|y| \cdot |\delta| + \delta^2/2)}.$$

For $y \in [k - \varepsilon, k + \varepsilon]$, we have $|y| \leq |k| + \varepsilon \leq L/4 + \varepsilon$ by the assumption $k \in \{-L/4, \dots, L/4 - 1\}$; hence, we have for $\delta \in [-1/2, 1/2]$

$$\begin{aligned}
\eta_\kappa(y-\delta) &\geq \eta_\kappa(y) \cdot e^{-\kappa^2((L/4+\varepsilon)/2+1/8)} \\
&\geq \eta_\kappa(y) \cdot e^{-\kappa^2 L/4} \quad \text{for } y \in [k - \varepsilon, k + \varepsilon]. \quad (\text{D20})
\end{aligned}$$

Here, we use that $(L/4 + \varepsilon)/2 + 1/8 \leq L/4$ for $L \in 8\mathbb{N}$ and $\varepsilon < 1/2$. Bounding the term $\eta_\kappa(y-\delta)$ in Eq. (D17) using Eq. (D20) and using the monotonicity of the integral implies the claim in Eq. (D18).

Applying Eq. (D19) to each term in Eq. (D16), we obtain the bound

$$\begin{aligned} \frac{m(x)}{\sqrt{p(x)p(0)}} &\geq \frac{1}{L\sqrt{p(x)p(0)}} \sum_{k=-L/4}^{L/4-1} M'_k(\delta) \\ &\geq e^{-\kappa^2 L/4} \frac{1}{L\sqrt{p(x)p(0)}} \sum_{k=-L/4}^{L/4-1} I_k(0) \end{aligned} \quad (\text{D21})$$

on the quantity of interest. Dividing Eq. (D21) by the norm (D8) gives the expression

$$\begin{aligned} \frac{m(x)}{\sqrt{p(x)p(0)}} &\geq e^{-\kappa^2 L/4} \left(\frac{p(0)}{p(x)} \right)^{1/2} \cdot \frac{\sum_{k=-L/4}^{L/4-1} I_k(0)}{\sum_{k=-L/2}^{L/2-1} I_k(0)} \\ &= e^{-\kappa^2 L/4} \left(\frac{\sum_{k=-L/2}^{L/2-1} I_k(0)}{\sum_{k=-L/2-m}^{L/2-1-m} I_k(\delta)} \right)^{1/2} \\ &\quad \cdot \frac{\sum_{k=-L/4}^{L/4-1} I_k(0)}{\sum_{k=-L/2}^{L/2-1} I_k(0)}, \end{aligned} \quad (\text{D22})$$

where, in the last step, we use Eq. (D13) and the assumption $x = m + \delta$.

We lower bound each factor in Eq. (D22) separately. By Lemma G.26 and the assumption $|m| \leq L/8$, we have

$$\begin{aligned} \sum_{k=-L/2-m}^{L/2-1-m} I_k(\delta) &\leq e^{\kappa^2 L} \sum_{k=-L/2-m}^{L/2-1-m} I_k(0) \\ &\leq e^{\kappa^2 L} \sum_{k=-L}^{L-1} I_k(0), \end{aligned} \quad (\text{D23})$$

where we use that each term $I_k(0)$ is non-negative in the second inequality. Equation (D23) implies that

$$\begin{aligned} \frac{\sum_{k=-L/2}^{L/2-1} I_k(0)}{\sum_{k=-L/2-m}^{L/2-1-m} I_k(\delta)} &\geq e^{-\kappa^2 L} \frac{\sum_{k=-L/2}^{L/2-1} I_k(0)}{\sum_{k=-L}^{L-1} I_k(0)} \\ &\geq e^{-\kappa^2 L} \cdot (1 - e^{-\kappa^2 L^2/8}) \end{aligned} \quad (\text{D24})$$

by Lemma G.24. Similarly, Lemma G.24 applied with $L/2$ instead of L gives the lower bound

$$\frac{\sum_{k=-L/4}^{L/4-1} I_k(0)}{\sum_{k=-L/2}^{L/2-1} I_k(0)} \geq 1 - e^{-\kappa^2 L^2/32}. \quad (\text{D25})$$

Combining Eqs. (D24) and (D25) with Eq. (D22) gives the lower bound

$$\begin{aligned} \frac{m(x)}{\sqrt{p(x)p(0)}} &\geq e^{-\kappa^2 L/4} e^{-\kappa^2 L/2} (1 - e^{-\kappa^2 L^2/8})^{1/2} (1 - e^{-\kappa^2 L^2/32}) \\ &\geq e^{-\kappa^2 L/4} e^{-\kappa^2 L/2} (1 - e^{-\kappa^2 L^2/8}) (1 - e^{-\kappa^2 L^2/32}) \\ &\geq e^{-3\kappa^2 L/4} (1 - e^{-\kappa^2 L^2/8} - e^{-\kappa^2 L^2/32}) \\ &\geq (1 - 3\kappa^2 L/4) (1 - 2e^{-\kappa^2 L^2/32}) \\ &\geq 1 - 3\kappa^2 L/4 - 2e^{-\kappa^2 L^2/32}, \end{aligned}$$

where we use inequality $\sqrt{1-x} \geq 1-x$ for $x \in (0, 1)$ in the second step, the inequality $(1-x)(1-y) \geq 1-x-y$ for $x, y \geq 0$ in the third and last step, and the inequality $e^{-x} \geq 1-x$ in the fourth step. The claim follows by inequality $(1-x)^2 \geq 1-2x$ for $x \in \mathbb{R}$. ■

We show that the conditional output state $\rho_{|\Omega_L}(\varepsilon, \kappa, \Delta, L) \in \mathcal{B}(L^2(\mathbb{R}))$ of the second mode—conditioned on the measurement result x (when applying a position measurement to the first mode) belonging to the “acceptance region” Ω_L (cf. Protocol 3, line 4)—is close to an approximate GKP state [for suitably chosen parameters (κ, Δ) and L].

Lemma D.5. Assume $\kappa \in (0, 1/4)$, $\Delta \in (0, 1/4)$, $\varepsilon \in [\sqrt{\Delta}, 1/2)$, and $L \in \mathbb{N}$. Let $\rho_{|\Omega_L}(\varepsilon, \kappa, \Delta, L) \in L^2(\mathbb{R})$ be the output state of Protocol 3 conditioned on acceptance (see Lemma D.2), on input $(|\text{III}_{L,\Delta}^\varepsilon\rangle, \kappa, L)$. Then,

$$\begin{aligned} \|\rho_{|\Omega_L}(\varepsilon, \kappa, \Delta, L) - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 \\ \leq 6\kappa\sqrt{L} + 6\sqrt{\Delta} + 7e^{-\kappa^2 L^2/64}. \end{aligned}$$

We note that Lemma D.5 implies that in the limit $(\kappa, \Delta) \rightarrow (0, 0)$ a choice of L scaling, e.g., as $L = \Theta[(1/\kappa)^{4/3}]$ ensures that the approximation error scales polynomially in (κ, Δ) .

Proof. Lemma D.3 together with relation $\|\rho - |\Psi\rangle\langle\Psi|\|_1 = 2\sqrt{1 - \langle\Psi, \rho\Psi\rangle}$ between the overlap and the L^1 distance for a state $\rho \in \mathcal{B}[L^2(\mathbb{R})]$ and a pure state $|\Psi\rangle \in L^2(\mathbb{R})$ implies

$$\begin{aligned} \|\text{gkp}_{L,\kappa,\Delta}^\varepsilon\rangle\langle\text{gkp}_{L,\kappa,\Delta}^\varepsilon| - \rho_{|\Omega_L}(\varepsilon, \kappa, \Delta, L)\|_1 \\ \leq 2\sqrt{3\kappa^2 L/2 + 4e^{-\kappa^2 L^2/32}} \\ \leq 3\kappa\sqrt{L} + 4e^{-\kappa^2 L^2/64}, \end{aligned} \quad (\text{D26})$$

where we use the inequality $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ for all $x, y \geq 0$ on the last line. By assumption, we have $\varepsilon \in [\sqrt{\Delta}, 1/2)$; thus, Corollary G.17 yields

$$\begin{aligned} \|\text{gkp}_{L,\kappa,\Delta}^\varepsilon\rangle\langle\text{gkp}_{L,\kappa,\Delta}^\varepsilon| - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 \\ \leq 3\kappa\sqrt{L} + 6\sqrt{\Delta} + 3e^{-\kappa^2 L^2/8}. \end{aligned} \quad (\text{D27})$$

Equations (D26) and (D27) combined with the triangle inequality imply the claim. ■

c. Bounding the acceptance probability of Protocol 3

We prove a lower bound on the acceptance probability of Protocol 3 given the input state $|\mathbb{III}_{L,\Delta}^\varepsilon\rangle$ and the input parameters (κ, L) .

Lemma D.6. Assume $\kappa > 0$, $\Delta > 0$, $\varepsilon \in (0, 1/2)$, and $L \in 16\mathbb{N}$. Given as input the state $|\mathbb{III}_{L,\Delta}^\varepsilon\rangle$ and the parameters κ and L , Protocol 3 accepts with probability at least

$$\Pr[\text{Protocol 3 accepts } |\mathbb{III}_{L,\Delta}^\varepsilon\rangle] \geq \frac{1}{8}(1 - 2e^{-\kappa^2 L^2/256}).$$

Proof. The probability that Protocol 3 accepts is the probability of obtaining a measurement outcome x belonging to the acceptance region (cf. line 4 in Protocol 3)

$$\Omega_L = [-L/8 - 1/2, L/8 + 1/2].$$

That is,

$$\Pr[\text{Protocol 3 accepts } |\mathbb{III}_{L,\Delta}^\varepsilon\rangle] = p(\Omega_L) = \int_{\Omega_L} p(x) dx,$$

where $p(x)$ is the probability density of obtaining outcome x [defined in Eq. (D11)]. By Eq. (D12), it satisfies

$$\begin{aligned} p(\Omega_L) &= \int_{-L/8-1/2}^{L/8+1/2} p(x) dx \\ &= \frac{1}{L} \sum_{k=-L/2}^{L/2-1} \int_{-L/8-1/2}^{L/8+1/2} I_k(x) dx, \end{aligned}$$

where we recall [cf. Eq. (D9)] that

$$I_k(\delta) := \int \eta_\kappa(u - \delta)^2 \chi_\Delta^\varepsilon(k)(u)^2 du.$$

By using Fubini's theorem, we obtain

$$\begin{aligned} p(\Omega_L) &= \frac{1}{L} \sum_{k=-L/2}^{L/2-1} \int \left(\int_{-L/8-1/2}^{L/8+1/2} \eta_\kappa(u-x)^2 dx \right) \chi_\Delta^\varepsilon(k)(u)^2 du \\ &= \frac{1}{L} \sum_{k=-L/2}^{L/2-1} \int \left(\int_{-L/8-1/2-u}^{L/8+1/2-u} \eta_\kappa(z)^2 dz \right) \chi_\Delta^\varepsilon(k)(u)^2 du, \end{aligned}$$

where we substitute $z = u - x$. Inserting the definition of $\chi_\Delta^\varepsilon(k)$ and defining the inner integral as $\Theta(u)$, i.e.,

$$\Theta(u) = \int_{-L/8-1/2-u}^{L/8+1/2-u} \eta_\kappa(z)^2 dz,$$

we can write

$$\begin{aligned} p(\Omega_L) &= \frac{1}{L} \sum_{k=-L/2}^{L/2-1} \int \Theta(u) \Psi_\Delta^\varepsilon(u-k)^2 du \\ &= \frac{1}{L} \sum_{k=-L/2}^{L/2-1} \int \Theta(v+k) \Psi_\Delta^\varepsilon(v)^2 dv, \end{aligned}$$

where we substitute $v = u - k$. Moreover, as $\text{supp}(\Psi_\Delta^\varepsilon) \subseteq [-\varepsilon, \varepsilon]$, we can restrict us to $|v| \leq \varepsilon < 1/2$. Because of the positivity of the integrand, we have

$$\begin{aligned} \Theta(v+k) &= \int_{-L/8-1/2-v-k}^{L/8+1/2-v-k} \eta_\kappa(z)^2 dz \\ &\geq \int_{-L/8-k}^{L/8-k} \eta_\kappa(z)^2 dz \quad \text{for } -L/2 \leq k \leq L/2 - 1. \end{aligned}$$

Using that $\|\Psi_\Delta^\varepsilon\| = 1$, we can bound

$$\begin{aligned} p(\Omega_L) &\geq \frac{1}{L} \sum_{k=-L/2}^{L/2-1} \int \left(\int_{-L/8-k}^{L/8-k} \eta_\kappa(z)^2 dz \right) \Psi_\Delta^\varepsilon(v)^2 dv \\ &= \frac{1}{L} \sum_{k=-L/2}^{L/2-1} \int_{-L/8-k}^{L/8-k} \eta_\kappa(z)^2 dz \int \Psi_\Delta^\varepsilon(v)^2 dv \\ &= \frac{1}{L} \sum_{k=-L/2}^{L/2-1} \int_{-L/8-k}^{L/8-k} \eta_\kappa(z)^2 dz. \end{aligned} \quad (\text{D28})$$

Using that the integrand is non-negative and that the interval $[-L/8 - k, L/8 - k]$ contains the interval $[-L/16, L/16]$ for all $k \in \{-L/16, \dots, L/16\}$ and that there are at least $L/8$ such values of $k \in \{-L/2, \dots, L/2 - 1\}$ (by the assumption $L \in 8\mathbb{N}$), we obtain the lower bound

$$p(\Omega_L) \geq \frac{1}{8} \int_{-L/16}^{L/16} \eta_\kappa(z)^2 dz.$$

Inserting this into Eq. (D28) yields

$$p(\Omega_L) \geq \frac{1}{8} \int_{-L/16}^{L/16} \eta_\kappa(z)^2 dz. \quad (\text{D29})$$

Since $\eta_\kappa(\cdot)^2$ is the probability density function of a centered normal random variable $X \sim \mathcal{N}[0, 1/(2\kappa^2)]$, we obtain by the Chernoff bound (see Ref. [38])

$$\begin{aligned} \Pr[|X| \geq L/16] &\leq 2e^{-(L/16)^2 \kappa^2} \\ &= 2e^{-L^2 \kappa^2/256}. \end{aligned}$$

Inserting this tail bound into Eq. (D29) yields the claim. ■

4. Proof of Theorem D.1

In this section, we prove Theorem D.1. We rely on the analysis of Protocol 3 in the case where the input is a

truncated comb state $|\text{III}_{L,\Delta}^\varepsilon\rangle$; see Appendix D 3. To extend to arbitrary input states that are close to $|\text{III}_{L,\Delta}\rangle$, we first analyze the effect of heralding; see Appendix D 4 a. We show that—assuming a constant lower bound on the acceptance probability—a heralding channel is stable under deviations in the input state (quantified in terms of the L^1 distance). Building on this stability result, we then complete the proof of Theorem D.1 in Appendix D 4 b.

a. Heralding channels and approximate input states

In this section, we examine what conclusions can be drawn if we know that a heralding channel prepares, with known probability, an output that is close to a target state. More formally, we consider a quantum channel, i.e., a completely positive map

$$\mathcal{E}: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}') \otimes \mathbb{C}^2$$

$$\rho \mapsto \mathcal{E}_{\text{acc}}(\rho) \otimes |\text{acc}\rangle\langle\text{acc}| + \mathcal{E}_{\text{rej}}(\rho) \otimes |\text{rej}\rangle\langle\text{rej}|,$$

where $\mathcal{E}_{\text{acc}}$ and $\mathcal{E}_{\text{rej}}$ are CPTNIMs, $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert spaces, and the second register is a qubit representing classical information spanned by the two orthonormal basis states $\{|\text{acc}\rangle, |\text{rej}\rangle\}$.

For any input state ρ , let us define the heralded states

$$\rho_{\text{acc}} = \frac{1}{\text{Pr}[\text{acc}|\rho]} \mathcal{E}_{\text{acc}}(\rho) \quad \text{and} \quad \rho_{\text{rej}} = \frac{1}{\text{Pr}[\text{rej}|\rho]} \mathcal{E}_{\text{rej}}(\rho),$$

where

$$\text{Pr}[\text{acc}|\rho] = \text{tr} \mathcal{E}_{\text{acc}}(\rho) \quad \text{and} \quad \text{Pr}[\text{rej}|\rho] = \text{tr} \mathcal{E}_{\text{rej}}(\rho)$$

are, respectively, the success and failure probabilities of the heralding. We note that the acceptance probability can be written as

$$\text{Pr}[\text{acc}|\rho] = \text{tr}(\Pi\rho), \quad \text{where } \Pi = \mathcal{E}_{\text{acc}}^\dagger(I) \quad (\text{D30})$$

and $\mathcal{E}_{\text{acc}}^\dagger$ denotes the adjoint map of $\mathcal{E}_{\text{acc}}$. The operator Π satisfies $0 \leq \Pi \leq I$. We call an operator Π with this property a POVM element in the following. It corresponds to the binary-outcome POVM $\{\Pi, I - \Pi\}$.

The main result of this section is the following statement expressed by Lemma D.7. Assume that, for an input state ρ , application of $\mathcal{E}$ results in acceptance with probability close to 1 and that the heralded state ρ_{acc} is close to a desired target state σ_{targ} . Then the same is true for a different input state τ , assuming that τ is close to ρ .

Lemma D.7. Let $\delta, \gamma > 0$ and $\rho, \tau, \sigma_{\text{targ}} \in \mathcal{B}(\mathcal{H})$ be states. Consider the heralding quantum channel $\mathcal{E}: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}') \otimes \mathbb{C}^2$ as introduced above and assume the following conditions are satisfied:

- (i) $\|\rho - \tau\|_1 \leq \delta$,
- (ii) $\|\rho_{\text{acc}} - \sigma_{\text{targ}}\|_1 \leq \gamma$, and

(iii) $\text{Pr}[\text{acc}|\rho] \neq 0$.

Then, the probability of acceptance on input τ is at least

$$\text{Pr}[\text{acc}|\tau] \geq \text{Pr}[\text{acc}|\rho] - \delta/2, \quad (\text{D31})$$

and upon acceptance the heralded state τ_{acc} and the target state σ_{targ} satisfy

$$\|\tau_{\text{acc}} - \sigma_{\text{targ}}\|_1 \leq \frac{\delta}{\text{Pr}[\text{acc}|\rho]} + \gamma. \quad (\text{D32})$$

Proof. Our proof heavily exploits that the trace distance between two states $\rho, \tau \in \mathcal{B}(\mathcal{H})$ can be written in variational form as

$$\|\rho - \tau\|_1 = 2 \max_{0 \leq \Pi \leq I} \text{tr}(\Pi(\rho - \tau)), \quad (\text{D33})$$

where $\Pi \in \mathcal{B}(\mathcal{H})$ satisfies $0 \leq \Pi \leq I$. The assumption (i) and the specialization of the expression (D33) to $\Pi = \mathcal{E}^\dagger(I)$ [see Eq. (D30)] give us the bound

$$\begin{aligned} \text{Pr}[\text{acc}|\rho] - \text{Pr}[\text{acc}|\tau] &\leq \max_{0 \leq \Pi \leq I} \text{tr}(\Pi(\rho - \tau)) \\ &= \frac{1}{2} \|\rho - \tau\|_1 \leq \frac{\delta}{2}; \end{aligned}$$

cf. Eq. (D30). This implies the claimed lower bound (D31) on $\text{Pr}[\text{acc}|\tau]$.

We now prove that, on input τ and conditioned on acceptance, the output τ_{acc} is close to the target state σ_{targ} [see Eq. (D32)]. Because of the contractivity of the trace distance under quantum channels, we have by the assumption (i) that

$$\|\mathcal{E}(\rho) - \mathcal{E}(\tau)\|_1 \leq \|\rho - \tau\|_1 \leq \delta. \quad (\text{D34})$$

For simplicity, we abbreviate the difference in acceptance probability as

$$\Delta_{\text{acc}} := \text{Pr}[\text{acc}|\rho] - \text{Pr}[\text{acc}|\tau]$$

and use the compact notation

$$\begin{aligned} \alpha_{\text{acc}} &:= (\text{Pr}[\text{acc}|\rho](\rho_{\text{acc}} - \tau_{\text{acc}}) + \Delta_{\text{acc}}\tau_{\text{acc}}) \otimes |\text{acc}\rangle\langle\text{acc}|, \\ \alpha_{\text{rej}} &:= (\text{Pr}[\text{rej}|\rho]\rho_{\text{rej}} - \text{Pr}[\text{rej}|\tau]\tau_{\text{rej}}) \otimes |\text{rej}\rangle\langle\text{rej}|, \end{aligned}$$

such that $\mathcal{E}(\rho) - \mathcal{E}(\tau) = \alpha_{\text{acc}} + \alpha_{\text{rej}}$. By expressing the norm $\|\mathcal{E}(\rho) - \mathcal{E}(\tau)\|_1$ in Eq. (D34) in variational form [cf. Eq. (D33)] and using that $\| - A \|_1 = \| A \|_1$, we have

$$\begin{aligned} \delta &\geq \|(-1)^\sigma(\alpha_{\text{acc}} + \alpha_{\text{rej}})\|_1 \\ &= 2 \max_{0 \leq \Pi \leq I} \text{tr}(\Pi(-1)^\sigma(\alpha_{\text{acc}} + \alpha_{\text{rej}})) \quad \text{for any } \sigma \in \{0, 1\}. \end{aligned}$$

Let us restrict to POVM elements of the form $\Pi = \Pi' \otimes |\text{acc}\rangle\langle\text{acc}| + \Pi'' \otimes |\text{rej}\rangle\langle\text{rej}|$, where $\Pi', \Pi'' \in \mathcal{B}(\mathcal{H}')$ are

arbitrary POVM elements. Then,

$$\begin{aligned} & 2 \max_{0 \leq \Pi \leq I} \text{tr}(\Pi(-1)^\sigma(\alpha_{\text{acc}} + \alpha_{\text{rej}})) \\ & \geq 2 \max_{\substack{0 \leq \Pi' \leq I \\ 0 \leq \Pi'' \leq I}} \text{tr} \left((\Pi' \otimes |\text{acc}\rangle\langle\text{acc}| \right. \\ & \quad \left. + \Pi'' \otimes |\text{rej}\rangle\langle\text{rej}|)(-1)^\sigma(\alpha_{\text{acc}} + \alpha_{\text{rej}}) \right) \\ & = 2 \max_{\substack{0 \leq \Pi' \leq I \\ 0 \leq \Pi'' \leq I}} (-1)^\sigma \text{tr}((\Pi' \otimes I)\alpha_{\text{acc}} + (\Pi'' \otimes I)\alpha_{\text{rej}}) \quad (\text{D35}) \end{aligned}$$

$$\geq 2 \max_{0 \leq \Pi' \leq I} (-1)^\sigma \text{tr}((\Pi' \otimes I)\alpha_{\text{acc}}) \quad (\text{D36})$$

$$= 2 \max_{0 \leq \Pi' \leq I} (-1)^\sigma \text{tr} \left(\Pi' (\text{Pr}[\text{acc}|\rho](\rho_{\text{acc}} - \tau_{\text{acc}}) + \Delta_{\text{acc}}\tau_{\text{acc}}) \right), \quad (\text{D37})$$

where Eq. (D35) is a consequence of the orthogonality of the classical states $|\text{acc}\rangle\langle\text{acc}|$ and $|\text{rej}\rangle\langle\text{rej}|$, Eq. (D36) can be seen setting $\Pi'' = 0$, and Eq. (D37) is obtained by tracing out the classical register.

Choose $\sigma \in \{0, 1\}$ such that $(-1)^\sigma \Delta_{\text{acc}} \geq 0$, i.e., $(-1)^\sigma$ is the sign of Δ_{acc} . Then, positivity of $\text{tr}(\Pi'\tau_{\text{acc}})$ and Eq. (D37) implies that

$$\begin{aligned} \delta & \geq 2 \max_{0 \leq \Pi' \leq I} (-1)^\sigma \text{tr} \left(\text{Pr}[\text{acc}|\rho] \cdot \Pi'(\rho_{\text{acc}} - \tau_{\text{acc}}) \right) \\ & = 2 \text{Pr}[\text{acc}|\rho] \cdot \max_{0 \leq \Pi' \leq I} \text{tr} \left(\Pi'(-1)^\sigma(\rho_{\text{acc}} - \tau_{\text{acc}}) \right) \\ & = \text{Pr}[\text{acc}|\rho] \cdot \|(-1)^\sigma(\rho_{\text{acc}} - \tau_{\text{acc}})\|_1 \\ & = \text{Pr}[\text{acc}|\rho] \cdot \|\rho_{\text{acc}} - \tau_{\text{acc}}\|_1, \end{aligned}$$

where we use the linearity of the trace in the first identity, the variational form of the norm $\|\cdot\|_1$ to obtain the second identity, and the fact that $\| -A \|_1 = \|A\|_1$ to reach the last identity.

By the triangle inequality and by the assumption (ii), we conclude that

$$\begin{aligned} \|\tau_{\text{acc}} - \sigma_{\text{targ}}\|_1 & \leq \|\tau_{\text{acc}} - \rho_{\text{acc}}\|_1 + \|\rho_{\text{acc}} - \sigma_{\text{targ}}\|_1 \\ & \leq \frac{\delta}{\text{Pr}[\text{acc}|\rho]} + \gamma, \end{aligned}$$

as claimed. $\blacksquare$

b. Completing the proof of Theorem D.1

We now combine the results on Protocol 3 with the truncated comb state $|\text{III}_L^\varepsilon\rangle$ as input (obtained in Appendix D3) with Lemma D.7 from Appendix D4a about heralding channels. This gives our main result, Theorem D.1, establishing that the envelope-shaping

protocol also produces an approximate GKP state if the input state is only close to a comb state.

Proof of Theorem D.1. We consider Protocol 3 with input state $\rho = |\text{III}_{L,\Delta}^\varepsilon\rangle\langle\text{III}_{L,\Delta}^\varepsilon|$. Let us denote the output state conditioned on acceptance by $\rho_{\text{acc}} = \rho|_{\Omega_L}(\varepsilon, \kappa, \Delta, L)$. By Lemma D.5 applied with $\varepsilon = \sqrt{\Delta}$, this output state is close to the target state $\sigma_{\text{targ}} = |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|$; that is, we have

$$\|\rho_{\text{acc}} - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 \leq 6\kappa\sqrt{L} + 6\sqrt{\Delta} + 7e^{-\kappa^2 L^2/64}.$$

Moreover, by Lemma D.6, we have

$$\text{Pr}[\text{Protocol 3 accepts } |\rho\rangle] \geq \frac{1}{8}(1 - 2e^{-\kappa^2 L^2/256}).$$

By assumption, $\tau \in \mathcal{B}[L^2(\mathbb{R})]$ is a state that is ξ -close to $|\text{III}_{L,\Delta}\rangle\langle\text{III}_{L,\Delta}|$. By Corollary G.7, we have

$$\| |\text{III}_{L,\Delta}\rangle\langle\text{III}_{L,\Delta}| - |\text{III}_{L,\Delta}^\varepsilon\rangle\langle\text{III}_{L,\Delta}^\varepsilon| \|_1 \leq 5\sqrt{\Delta},$$

and, hence, the triangle inequality implies that

$$\|\tau - |\text{III}_{L,\Delta}^\varepsilon\rangle\langle\text{III}_{L,\Delta}^\varepsilon|\|_1 \leq 5\sqrt{\Delta} + \xi.$$

We apply Lemma D.7 with parameters

$$\delta = 5\sqrt{\Delta} + \xi \quad [\text{cf. Eq. (D39)}],$$

$$\gamma = 6\kappa\sqrt{L} + 6\sqrt{\Delta} + 7e^{-\kappa^2 L^2/64} \quad [\text{cf. Eq. (D39)}].$$

We conclude that, on input τ , Protocol 3 accepts with probability at least

$$\text{Pr}[\text{Protocol 3 accepts } |\tau\rangle] \geq \frac{1}{8}(1 - 2e^{-\kappa^2 L^2/256}) - \frac{5}{2}\sqrt{\Delta} - \frac{\xi}{2}.$$

Furthermore, conditioned on acceptance, the output state τ_{acc} is close to the target state $|\text{III}_{L,\Delta}\rangle\langle\text{III}_{L,\Delta}|$, i.e.,

$$\begin{aligned} \|\tau_{\text{acc}} - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 & \leq \frac{5\sqrt{\Delta} + \xi}{\frac{1}{8}(1 - 2e^{-\kappa^2 L^2/256})} + 6\kappa\sqrt{L} + 6\sqrt{\Delta} + 7e^{-\kappa^2 L^2/64}. \end{aligned}$$

This is the claim. (We have already discussed the complexity, i.e., number of operators from the set $\mathcal{G}$ of the circuit implementing this protocol.) $\blacksquare$

APPENDIX E: APPROXIMATE GKP-STATE PREPARATION

In this section, we present our main protocol for preparing approximate GKP states. We show that, given parameters (κ, Δ) , this protocol accepts with probability at least

$$\Pr[\text{acc}] \geq \frac{1}{10},$$

in which case it prepares a quantum state $\tau_{\text{acc}} \in \mathcal{B}[L^2(\mathbb{R})]$ satisfying

$$\|\tau_{\text{acc}} - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 \leq O(\sqrt{\Delta}) + O(\kappa^{1/3});$$

or it rejects. The protocol is efficient—it uses only a linear number of operations in $(\log 1/\kappa, 1/\Delta)$ from the set $\mathcal{G}$ [cf. Appendix A (a) to (b)].

The protocol works in two stages. First, it creates a comb state (using the comb-state-preparation protocol in Appendix C). Second, it shapes the prepared comb state by a Gaussian envelope (using the envelope-Gaussification protocol in Appendix D). We refer to Fig. 15 for the circuit of the preparation protocol.

Theorem E.1. There are constants $c_1, c_2 > 0$ such that the following holds. Given inputs $\kappa, \Delta \in (0, 10^{-6})$, the output state of Protocol 1 conditioned on acceptance is close to $|\text{GKP}_{\kappa,\Delta}\rangle$. The protocol accepts with probability at least

$$\Pr[\text{acc}] \geq \frac{1}{10}.$$

The output state $\rho \in \mathcal{B}[L^2(\mathbb{R})]$ conditioned on acceptance

satisfies

$$\|\rho - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 \leq 190\sqrt{\Delta} + 24\kappa^{1/3}.$$

Furthermore, the protocol requires fewer than $c_1 \log 1/\kappa + c_2 \log 1/\Delta$ elementary operations (see Appendix A 1).

Proof. As described in Protocol 1, we choose parameters

$$n = \left\lceil \frac{4}{3} \log_2 1/\kappa \right\rceil \quad \text{and} \quad L = 2^n. \quad (\text{E1})$$

First, Protocol 1 prepares a state close to the comb state $|\text{III}_{L,\Delta}\rangle$ using the comb-state-preparation protocol (Protocol 2). The comb-state-preparation protocol is run with input parameters Δ and number of rounds $n = \log_2 L$. By Theorem C.1, this protocol (deterministically) prepares a quantum state $\tau \in \mathcal{B}[L^2(\mathbb{R})]$ such that

$$\|\tau - |\text{III}_{L,\Delta}\rangle\langle\text{III}_{L,\Delta}|\|_1 \leq 17\sqrt{\Delta}.$$

Subsequently, Protocol 1 applies the envelope-Gaussification protocol (Protocol 3) with parameters κ and L to the state τ . By Theorem D.1 (setting $\xi = 17\sqrt{\Delta}$), the corresponding output state τ_{acc} conditioned on acceptance in Protocol 3 satisfies

$$\|\tau_{\text{acc}} - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 \leq \left(\frac{22}{\frac{1}{8}(1 - 2e^{-\kappa^2 L^2/256})} + 6 \right) \sqrt{\Delta} + 6\kappa\sqrt{L} + 7e^{-\kappa^2 L^2/64}. \quad (\text{E2})$$

With the chosen parameters, we have $\kappa L \geq \kappa \cdot 2^{(4/3)(\log_2 1/\kappa)-1} = \kappa/(2\kappa^{4/3}) = \kappa^{-1/3}/2$, and, by the assumption $0 < \kappa < 10^{-6}$, we have

$$\frac{1}{8}(1 - 2e^{-\kappa^2 L^2/256}) \geq \frac{12}{100}. \quad (\text{E3})$$

Therefore, using this bound in Eq. (E2) gives

$$\begin{aligned} \|\tau_{\text{acc}} - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 &\leq 190\sqrt{\Delta} + 6\kappa\sqrt{L} + 7e^{-\kappa^2 L^2/64} \\ &\leq 190\sqrt{\Delta} + 6\kappa^{1/3} + 7e^{-\kappa^2 L^2/64} \quad \text{by } \kappa\sqrt{L} \leq \kappa^{1/3} \\ &\leq 190\sqrt{\Delta} + 6\kappa^{1/3} + 7e^{-\kappa^{-2/3}/256} \quad \text{by } \kappa L \geq \kappa^{-1/3}/2 \\ &\leq 190\sqrt{\Delta} + 6\kappa^{1/3} + 1792\kappa^{2/3} \quad \text{by } e^{-x} \leq x^{-1} \text{ for } x > 0 \\ &\leq 190\sqrt{\Delta} + 6\kappa^{1/3} + 18\kappa^{1/3} \quad \text{by the assumption } \kappa < 10^{-6} \\ &= 190\sqrt{\Delta} + 24\kappa^{1/3}. \end{aligned}$$

Furthermore, Protocol 3 accepts with probability at least

$$\Pr[\text{Protocol 3 accepts } |\tau|] \geq \frac{1}{8}(1 - 2e^{-\kappa^2 L^2/256}) - 11\sqrt{\Delta}. \quad (\text{E4})$$

By inserting the bound (E3) into Eq. (E4) and using the assumption $0 < \Delta < 10^{-6}$, we obtain

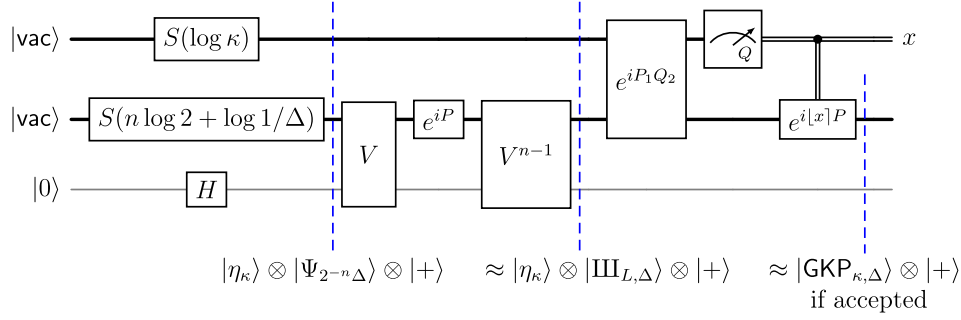


FIG. 15. Circuit for heralded preparation of an approximate GKP state. The unitary V is defined in Fig. 8. The exponent above the gate V indicates the number of repetitions. Again, both the squeezing operations and the classically controlled displacements are implemented as a product of their constant-strength equivalents; see the discussion surrounding Eq. (C5).

$$\Pr[\text{Protocol 3 accepts } |\tau| \geq \frac{12}{100} - 11\sqrt{\Delta} \geq \frac{1}{10}.$$

Finally, we analyze the circuit complexity of Protocol 1. Since this protocol is simply the composition of Protocols 2 and 3, the number of operations used is the sum of the corresponding numbers for these protocols. By Theorem C.1, we have that Protocol 2 uses $5n + \lceil \log 1/\Delta \rceil + 5$ elementary operations, and by Theorem D.1, we have that Protocol 3 uses at most $b_1 \log L + b_2 \log 1/\kappa$ operations for some constants $b_1, b_2 > 0$. Since the parameters in Protocol 1 are fixed as in Eq. (E1), i.e., $n = \lceil (4 \log_2 \kappa)/3 \rceil$ and $\log L = 2^n$, we can find constants c_1 and c_2 such that in Protocol 1 the total number of elementary operations used is upper bounded by $c_1 \log 1/\kappa + c_2 \log 1/\Delta$. ■

Theorem E.1 directly implies the following asymptotic statement about the heralded complexity of approximate GKP states.

Corollary E.2. There is a polynomial $q(\kappa, \Delta)$ with $q(0, 0) = 0$ such that, for all functions $\varepsilon(\kappa, \Delta)$ and $p(\kappa, \Delta)$ satisfying $p(\kappa, \Delta) \in [0, 1/10]$ and $\varepsilon(\kappa, \Delta) \geq q(\kappa, \Delta)$ for sufficiently small (κ, Δ) , we have

$$C_{p(\kappa, \Delta), \varepsilon(\kappa, \Delta)}^{*, \text{her}}(|\text{GKP}_{\kappa, \Delta}\rangle) \leq O(\log 1/\kappa + \log 1/\Delta) \text{ for } (\kappa, \Delta) \rightarrow (0, 0).$$

It is useful to phrase Theorem E.1 in terms of a single parameter.

Corollary E.3. Let $N \in \mathbb{N}$ be sufficiently large and assume $\kappa = \text{poly}(1/N)$ and $\Delta = \text{poly}(1/N)$. Then, Protocol 1 prepares a state $\rho \in \mathcal{B}[L^2(\mathbb{R})]$ such that

$$\|\rho - |\text{GKP}_{\kappa, \Delta}\rangle\langle \text{GKP}_{\kappa, \Delta}| \|_1 \leq \text{poly}(1/N),$$

with probability at least $1/10$. The protocol uses $O(\log N)$ elementary operations.

APPENDIX F: LOWER BOUNDS ON THE COMPLEXITY OF APPROXIMATE GKP STATES

In this section, we establish lower bounds on the unitary and heralded complexity of the state $|\text{GKP}_{\kappa, \Delta}\rangle$. We proceed as follows. In Appendix F.2, we establish an upper bound on the energy of a state $U|\Psi\rangle$ produced by applying a circuit $U = U_T \dots U_1$ with a limited number T of gates from the set $\mathcal{G}$. Here, $|\Psi\rangle = |\text{vac}\rangle \otimes |\text{vac}\rangle^{\otimes m} \otimes |0\rangle^{\otimes m'}$. Concretely, we show that

$$\langle U\Psi, HU\Psi \rangle \leq e^{8\pi T} (m+2) =: E_m(T), \quad (\text{F1})$$

where $H = \sum_{k=1}^{m+1} (Q_k^2 + P_k^2)$; see Lemma F.6. This is an immediate consequence of the fact that the unitary operations constituting the set $\mathcal{G}$ are moment-limited.

Equation (F1) implies that the reduced density operator $\rho = \text{tr}_{m, m'} U|\Psi\rangle\langle \Psi|U^\dagger$ on the first mode has most of its support on the subspace spanned by functions with support on a bounded interval $[-R, R]$ in position space; i.e., we have

$$\text{tr}(\Pi_{[-R, R]}\rho) \geq 1 - E_m(T)/R^2 \text{ for } R > 0. \quad (\text{F2})$$

Here, $\Pi_{[-R, R]}$ denotes the orthogonal projection onto the subspace of $L^2(\mathbb{R})$ of functions whose support is contained in the interval $[-R, R]$. Equation (F2) is a direct consequence of Markov's inequality; see Lemma F.7. A bound analogous to Eq. (F2) applies to the orthogonal projection $\hat{\Pi}_{[-R, R]}$ onto the subspace of functions whose Fourier transform has support on $[-R, R]$.

Finally, we show that a bound of the form (F2) immediately implies a lower bound on the distance of ρ to the state $|\text{GKP}_{\kappa, \Delta}\rangle\langle \text{GKP}_{\kappa, \Delta}|$; see Lemma F.9. This is because the norm $\|\Pi_{[-R, R]}|\text{GKP}_{\kappa, \Delta}\rangle\|$ of the projected state is bounded (for suitably chosen R), and the same holds for the projection $\hat{\Pi}_{[-R, R]}$ in momentum space. We establish corresponding tail bounds in Appendix F.3. Combined, these results yield a lower bound on the unitary complexity of $|\text{GKP}_{\kappa, \Delta}\rangle$; see Appendix F.4. We then extend these

arguments to the heralded state complexity in Appendix F 5.

1. Moment limits on the gate set $\mathcal{G}$

Here, we argue that each unitary $U \in \mathcal{G}$ in our gate set $\mathcal{G}$ is moment-limited; i.e., it cannot significantly increase the norm of the displacement vector (i.e., the first moments)

$$\begin{aligned} \text{tr}(He^{-iaR_i}\rho e^{iaR_i}) &\leq \text{tr}(H\rho) + (2\pi)^2 + 4\pi\|s(\rho)\| \quad \text{for all } a \in [-2\pi, 2\pi], \\ \text{tr}(HP_j(\phi)^\dagger \rho P_j(\phi)) &= \text{tr}(H\rho) \quad \text{for all } \phi \in [-2\pi, 2\pi], \\ \text{tr}(HB_{j,k}(\omega)^\dagger \rho B_{j,k}(\omega)) &= \text{tr}(H\rho) \quad \text{for all } \omega \in [-2\pi, 2\pi], \\ \text{tr}(HS_j(z)^\dagger \rho S_j(z)) &\leq e^{4\pi}\text{tr}(H\rho) \quad \text{for all } z \in [-2\pi, 2\pi] \end{aligned}$$

when applied to a state $\rho \in \mathcal{B}[L^2(\mathbb{R})^{\otimes n}]$ with finite first and second moments. Here, $s(\rho) \in \mathbb{R}^{2n}$ is defined by its entries $s(\rho)_j = \text{tr}(R_j\rho)$.

For completeness, we establish analogous bounds for our gate set $\mathcal{G}$, whose unitaries act on a Hilbert space of the form $L^2(\mathbb{R})^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes n'}$.

Lemma F.1 (moment limit on phase-space displacements). Consider an n -mode bosonic system with (vector of) mode operators $R = (Q_1, P_1, \dots, Q_n, P_n)$. For $d = (d^Q, d^P) \in \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$, let $D(d) = e^{i \sum_{j=1}^n (d_j^Q Q_j - d_j^P P_j)}$ be the displacement operator in the direction d . Let $\rho \in \mathcal{B}[L^2(\mathbb{R})]$ be a state with finite first and second moments. Then,

$$\text{tr}[HD(d)\rho D(d)^\dagger] \leq \text{tr}(H\rho) + 2\|d\| \cdot \|s(\rho)\| + \|d\|^2, \quad (\text{F3})$$

where $\|d\| = \sqrt{\sum_{j=1}^{2n} d_j^2}$ denotes the Euclidean norm of d and where $s(\rho) \in \mathbb{R}^{2n}$ is the displacement vector of ρ defined by its entries $s(\rho)_j = \text{tr}(R_j\rho)$. Furthermore, the Euclidean norm of the displacement of the resulting state is bounded as

$$\|s[D(d)\rho D(d)^\dagger]\| \leq \|s(\rho)\| + \|d\|. \quad (\text{F4})$$

By definition of $\mathcal{G}$, Lemma F.1 implies that, for any single-mode displacement $D(d) \in \mathcal{G}$, we have

$$\begin{aligned} \|s[D(d)\rho D(d)^\dagger]\| &\leq \|s(\rho)\| + 2\pi, \\ \text{tr}[HD(d)\rho D(d)^\dagger] &\leq \text{tr}(H\rho) + 4\pi\|s(\rho)\| + (2\pi)^2. \end{aligned} \quad (\text{F5})$$

Proof. Since the displacement operator $D(d)$ acts on mode operators as

$$\begin{aligned} D(d)^\dagger Q_j D(d) &= Q_j + d_j^Q I \\ D(d)^\dagger P_j D(d) &= P_j + d_j^P I \end{aligned} \quad \text{for all } j \in [n], \quad (\text{F6})$$

we conclude that the displacement vector of the state $D(d)\rho D(d)^\dagger$ is equal to

of a state nor the energy (a sum of second moments). Such moment bounds are well known [30,31] and widely used. For example, denoting the Hamiltonian of n independent harmonic oscillators by $H = \sum_{j=1}^{2n} R_j^2$, where $R = (Q_1, P_1, \dots, Q_n, P_n)$, phase-space displacements, rotations, beam splitters, and single-mode squeezers satisfy (see, e.g., p. 30 in Ref. [39])

$$s[D(d)\rho D(d)^\dagger] = s(\rho) + d.$$

Equation (F4) immediately follows from this using the triangle inequality for the Euclidean norm.

Again using Eq. (F6), we also have

$$\begin{aligned} D(d)^\dagger H D(d) &= \sum_{j=1}^{2n} (R_j + d_j I)^2 \\ &= \sum_{j=1}^{2n} R_j^2 + 2 \sum_{j=1}^{2n} d_j R_j + \left(\sum_{j=1}^{2n} d_j^2 \right) \cdot I \\ &= H + 2 \sum_{j=1}^{2n} d_j R_j + \|d\|^2 \cdot I. \end{aligned}$$

This implies that

$$\begin{aligned} \text{tr}[HD(d)\rho D(d)^\dagger] &= \text{tr}[D(d)^\dagger H D(d)\rho] \\ &= \text{tr}(H\rho) + 2s(\rho)^T d + \|d\|^2 \\ &\leq \text{tr}(H\rho) + 2\|s(\rho)\| \cdot \|d\| + \|d\|^2 \end{aligned}$$

by the Cauchy-Schwarz inequality. This is the claim in Eq. (F3). ■

Lemma F.2 (moment limit on quadratic Gaussian unitaries). Consider an n -mode bosonic system with (vector of) mode operators $R = (Q_1, P_1, \dots, Q_n, P_n)$. Let $H = \sum_{j=1}^n (Q_j^2 + P_j^2) = \sum_{k=1}^{2n} R_k^2$. Let $A = A^T \in \text{Mat}_{2n \times 2n}(\mathbb{R})$ be a symmetric matrix, and let $U(A) = e^{iH(A)}$ be the Gaussian unitary defined in terms of the Hamiltonian $H(A) = \frac{1}{2} R^T A R$. Let $\rho \in \mathcal{B}[L^2(\mathbb{R})]$ be a state with finite first and second moments. Then, the norms of the displacement vectors $s(\rho), s[U(A)\rho U(A)^\dagger] \in \mathbb{R}^{2n}$ are related by

$$\|s[U(A)\rho U(A)^\dagger]\| \leq e^{\|A\|} \cdot \|s(\rho)\|. \quad (\text{F7})$$

Furthermore, we have

$$\text{tr}[HU(A)\rho U(A)^\dagger] \leq e^{2\|A\|} \text{tr}(H\rho). \quad (\text{F8})$$

By definition of $\mathcal{G}$, Lemma F.2 means that, for any Gaussian unitary $U = U(A) \in \mathcal{G}$, we have the inequalities

$$\begin{aligned} \|s[U(A)\rho U(A)^\dagger]\| &\leq e^{2\pi} \|s(\rho)\|, \\ \text{tr}[HU(A)\rho U(A)^\dagger] &\leq e^{4\pi} \text{tr}(H\rho). \end{aligned} \quad (\text{F9})$$

Proof. Let $J \in \text{Mat}_{2n \times 2n}(\mathbb{R})$ be the symplectic form associated with the n -mode system, and let $S(A) = e^{AJ} \in \text{Sp}(2n)$ be the symplectic matrix describing the action of $U(A)$; see Eq. (A1). Since

$$U(A)^\dagger R_j U(A) = \sum_{k=1}^{2n} S(A)_{j,k} R_k,$$

the displacement vector of the state $U(A)\rho U(A)^\dagger$ is given by matrix-vector multiplication; i.e.,

$$s[U(A)\rho U(A)^\dagger] = S(A)s(\rho).$$

It follows that

$$\|s[U(A)\rho U(A)^\dagger]\| \leq \|S(A)\| \cdot \|s(\rho)\|, \quad (\text{F10})$$

where $\|S(A)\|$ denotes the operator norm of $S(A)$. By submultiplicativity of the operator norm, we have

$$\|S(A)\| = \|e^{AJ}\| \leq e^{\|AJ\|} \leq e^{\|A\|},$$

where we use the fact that $\|J\| = 1$. It follows from Eq. (F10) that

$$\|s[U(A)\rho U(A)^\dagger]\| \leq e^{\|A\|} \|s(\rho)\|,$$

as claimed in Eq. (F7).

In the following, we show Eq. (F8). Without loss of generality, we assume that $\rho = |\Psi\rangle\langle\Psi|$ is a pure state. The general case follows by spectrally decomposing ρ . For any symmetric matrix $M = M^T \in \text{Mat}_{2n \times 2n}(\mathbb{R})$, we also have

$$U(A)^\dagger H(M) U(A) = H[S(A)^T M S(A)]$$

by Eq. (A1). In particular, because $H = 2H(I_{2n \times 2n})$, this means that

$$U(A)^\dagger H U(A) = 2H[S(A)^T S(A)].$$

Observe that, for any symmetric matrix $M = M^T \in \text{Mat}_{2n \times 2n}(\mathbb{R})$, we have

$$\langle\Psi, H(M)\Psi\rangle = \frac{1}{2} \sum_{j,k=1}^{2n} M_{j,k} G_{j,k} = \frac{1}{2} \text{tr}(MG),$$

where the Hermitian matrix $G = G^\dagger \in \text{Mat}_{2n \times 2n}(\mathbb{C})$ is defined by its entries

$$G_{j,k} = \langle R_j \Psi, R_k \Psi \rangle \quad \text{for } j, k \in [2n]$$

and where we use the symmetry of M in the second identity.

Applied to the expression of interest with $H = 2H(I_{2n \times 2n})$, we have

$$\text{tr}(H\rho) = \text{tr}(G) \quad (\text{F11})$$

and

$$\text{tr}[HU(A)\rho U(A)^\dagger] = \text{tr}[U(A)^\dagger H U(A)\rho] = \text{tr}[S(A)^T S(A)G]. \quad (\text{F12})$$

Since the operator norm is submultiplicative, we have

$$\|S(A)^T S(A)\| = \|(e^{AJ})^T e^{AJ}\| \leq \|(e^{AJ})^T\| \cdot \|e^{AJ}\| \leq e^{\|(AJ)^T\|} \cdot e^{\|AJ\|} = e^{2\|AJ\|} \leq e^{2\|A\|},$$

where we use that $\|AJ\| \leq \|A\| \cdot \|J\| = \|A\|$ since $\|J\| = 1$ for any symmetric matrix $A = A^T \in \text{Mat}_{2n \times 2n}(\mathbb{R})$. In particular, this means that we have the operator inequality

$$S(A)^T S(A) \leq e^{2\|A\|} I_{2n \times 2n} \quad \text{for any symmetric matrix } A = A^T \in \text{Mat}_{2n \times 2n}(\mathbb{R}).$$

Inserting this into Eq. (F12) and combining with Eq. (F11) implies the claim (F8), because G is a Gram matrix and, thus, positive semidefinite. ■

Lemma F.3 (moment limit on qubit-controlled displacements). Consider a system with Hilbert space $L^2(\mathbb{R})^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes n'}$. Consider the result $[\text{ctrl}_j D(d)]\rho[\text{ctrl}_j D(d)]^\dagger$ of applying a qubit- j -controlled displacement $D(d)$, $d \in \mathbb{R}^{2n}$ to a state $\rho \in \mathcal{B}[L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes n'}]$ with finite first moments. The Euclidean norm of the displacement vector of this state is bounded by

$$\|s([\text{ctrl}_j D(d)]\rho[\text{ctrl}_j D(d)]^\dagger)\| \leq \|s(\rho)\| + \|d\|. \quad (\text{F13})$$

Furthermore, for any state $\rho \in \mathcal{B}[L^2(\mathbb{R})^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes n'}]$ with finite first and second moments, we have

$$\text{tr}(H[\text{ctrl}_j D(d)]\rho[\text{ctrl}_j D(d)]^\dagger) \leq 2\text{tr}(H\rho) + 2\|d\| \cdot \|s(\rho)\| + \|d\|^2.$$

Proof. We give the proof for $j = n' = 1$ (the general case is analogous). Let U be an arbitrary unitary on $L^2(\mathbb{R})^{\otimes n}$, i.e., on the bosonic modes. Then, we have

$$(\text{ctrl}U)^\dagger R_k (\text{ctrl}U) = |0\rangle\langle 0| \otimes R_k + |1\rangle\langle 1| \otimes U^\dagger R_k U \quad \text{for } k \in [2n]. \quad (\text{F14})$$

In the case where $U = D(d)$ is a displacement, Eq. (F14) specializes to

$$[\text{ctrl}D(d)]^\dagger R_k [\text{ctrl}D(d)] = |0\rangle\langle 0| \otimes R_k + |1\rangle\langle 1| \otimes (R_k + d_k I) = I \otimes R_k + d_k I \otimes I,$$

and we conclude that

$$s([\text{ctrl}_j D(d)]\rho[\text{ctrl}_j D(d)]^\dagger) = s(\rho) + d.$$

The claim (F13) follows from the triangle inequality. The second claim follows immediately from Lemma F.4, which follows, applied to the unitary $U = D(d)$, together with the moment bound (F3) for the unitary $D(d)$. ■

Lemma F.4 (second moment limit on controlled unitaries). Suppose U is a unitary on $L^2(\mathbb{R})^{\otimes n}$ with the following property: There are constants $a, b, c > 0$ such that we have

$$\text{tr}(HU\rho U^\dagger) \leq a\text{tr}(H\rho) + b\|s(\rho)\| + c \quad (\text{F15})$$

for any state $\rho \in \mathcal{B}[L^2(\mathbb{R})^{\otimes n}]$ with finite first and second moments. Consider a system with Hilbert space $L^2(\mathbb{R})^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes n'}$ and the controlled unitary

$$\text{ctrl}_j U = |0\rangle\langle 0|_j \otimes I_{L^2(\mathbb{R})^{\otimes n}} + |1\rangle\langle 1|_j \otimes U,$$

where the bosonic unitary U is controlled by the j th qubit, with $j \in [n']$. (Here, we write $|0\rangle\langle 0|_j$ for the n' -qubit operator $I^{\otimes j-1} \otimes |0\rangle\langle 0| \otimes I^{\otimes n-j}$ and similarly for $|1\rangle\langle 1|_j$). Then, we have

$$\text{tr}(H(\text{ctrl}_j U)\rho(\text{ctrl}_j U)^\dagger) \leq (a+1)\text{tr}(H\rho) + b\|s(\rho)\| + c$$

for any state $\rho \in \mathcal{B}[L^2(\mathbb{R})^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes n'}]$ with finite first and second moments.

Proof. We give the proof for $j = n' = 1$ (the general case is analogous). Then, we have

$$\begin{aligned} (\text{ctrl}U)^\dagger H(\text{ctrl}U) &= |0\rangle\langle 0| \otimes H + |1\rangle\langle 1| \otimes U^\dagger H U \\ &\leq I \otimes H + I \otimes U^\dagger H U, \end{aligned}$$

where we use the operator inequalities $|0\rangle\langle 0| \leq I$ and $|1\rangle\langle 1| \leq I$ for a single qubit and the fact that $H \geq 0$. It follows that

$$\begin{aligned} \text{tr}(H(\text{ctrl}U)\rho(\text{ctrl}U)^\dagger) &\leq \text{tr}(H\rho') + \text{tr}(U^\dagger H U \rho') \\ &= \text{tr}(H\rho') + \text{tr}(H U \rho' U^\dagger), \end{aligned}$$

where $\rho' = \text{tr}_n \rho$ denotes the reduced density operator of the bosonic modes after tracing out the qubits. The claim now follows from the assumption (F15) applied to the state ρ' . ■

Combining Eqs. (F5) and (F9) and Lemma F.3 gives the following corollary.

Corollary F.5 (moment limits on gates from $\mathcal{G}$). Let $\rho \in \mathcal{B}[L^2(\mathbb{R})^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes n'}]$ be a state of n bosons and n' qubits. Assume that ρ has finite first and second moments. Let $s(\rho) \in \mathbb{R}^{2n}$ denote its displacement vector. Then,

$$\begin{aligned} \|s(U\rho U^\dagger)\| &\leq e^{2\pi}\|s(\rho)\| + 2\pi \\ \text{tr}(H U \rho U^\dagger) &\leq e^{4\pi}\text{tr}(H\rho) + 4\pi\|s(\rho)\| + (2\pi)^2 \end{aligned} \quad \text{for every } U \in \mathcal{G}.$$

2. Moment limits on low-complexity states

Here, we argue that a state produced by a small circuit with gates from $\mathcal{G}$ has small energy.

Lemma F.6. Consider a circuit $U = U_T \dots U_1$ composed of T unitaries $U_1, \dots, U_T$ from the set $\mathcal{G}$ acting on the initial state $|\Psi\rangle = |\text{vac}\rangle \otimes |\text{vac}\rangle^{\otimes m} \otimes |0\rangle^{\otimes m'}$. We define the Hamiltonian

$$H = \sum_{k=1}^{m+1} (Q_k^2 + P_k^2).$$

Then,

$$\langle \Psi | U^\dagger H U | \Psi \rangle \leq e^{8\pi T} (m+2) := E_m(T).$$

Proof. Let $R = (R_1, \dots, R_{2(m+1)}) := (Q_1, P_1, \dots, Q_{(m+1)}, P_{(m+1)})$ be the vector of mode operators, and let

$$s(\rho) = \text{tr}(R\rho) \in \mathbb{R}^{2(m+1)}$$

be the displacement vector of a state ρ on $L^2(\mathbb{R})^{\otimes(m+1)} \otimes (\mathbb{C}^2)^{\otimes m'}$. Denoting by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^{2m}$, we have by Corollary F.5 that

$$\|s(U_t \rho U_t^\dagger)\| \leq \gamma \|s(\rho)\| + \delta \quad (\text{F16})$$

with $\gamma = e^{2\pi}$ and $\delta = 2\pi$ for any unitary $U_t \in \mathcal{G}$ belonging to our gate set $\mathcal{G}$ and any state ρ . Setting $\rho^{(0)} = |\Psi\rangle\langle\Psi|$ and $\rho^{(t)} = U_t \rho^{(t-1)} U_t^\dagger$ for $t \geq 1$, we can deduce from Eq. (F16) that

$$\|s(\rho^{(t)})\| \leq f[\|s(\rho^{(t-1)})\|],$$

where $f(s) = \gamma s + \delta$. In particular, defining $u_0 := \|s(\rho^{(0)})\| = 0$ and $u_t := f(u_{t-1})$ for $t \geq 1$, we get the upper bound

$$\begin{aligned} \|s(\rho^{(t)})\| &\leq u_t = f^{\circ t}(u_0) \\ &= \gamma^t u_0 + \delta(1 + \gamma + \dots + \gamma^{t-1}) \\ &= \delta \cdot \frac{\gamma^t - 1}{\gamma - 1}, \quad \text{since } u_0 = 0, \\ &\leq \delta \cdot \gamma^t, \quad \text{since } \gamma > 2, \\ &= 2\pi \cdot e^{2\pi t}. \end{aligned} \quad (\text{F17})$$

Again using Corollary F.5, we have for $1 \leq t \leq T$ that

$$\begin{aligned} \text{tr}(H\rho^{(t)}) &\leq e^{4\pi} \text{tr}(H\rho^{(t-1)}) + 4\pi \|s(\rho^{(t)})\| + (2\pi)^2 \\ &\leq e^{4\pi} \text{tr}(H\rho^{(t-1)}) + 8\pi^2 \cdot e^{2\pi t} + (2\pi)^2 \\ &\leq e^{4\pi} \text{tr}(H\rho^{(t-1)}) + 8\pi^2 \cdot e^{2\pi T} + (2\pi)^2. \end{aligned}$$

Therefore, setting $A = e^{4\pi}$, $B = 8\pi^2 \cdot e^{2\pi T} + (2\pi)^2$, and $x_t = \text{tr}(H\rho^{(t)})$, we can rephrase the previous bound as

$$x_t \leq A x_{t-1} + B.$$

Proceeding similarly as in Eq. (F17), we have

$$\begin{aligned} x_t &\leq A^t x_0 + B(1 + A + \dots + A^{t-1}) \\ &\leq A^t x_0 + B A^t \quad \text{as } A > 2. \end{aligned}$$

With $t = T$, we have

$$\begin{aligned} \text{tr}(H\rho^{(T)}) &\leq e^{4\pi T} \text{tr}(H\rho^{(0)}) + [8\pi^2 \cdot e^{2\pi T} + (2\pi)^2] \cdot e^{4\pi T} \\ &\leq e^{8\pi T} (\text{tr}(H\rho^{(0)}) + 1), \end{aligned} \quad (\text{F18})$$

where we use that $[8\pi^2 \cdot e^{2\pi T} + (2\pi)^2] \leq e^{4\pi T}$ for $T \geq 1$. Obviously, the bound (F18) is also valid in the case $T = 0$. The claim follows from $\langle \text{vac} | (Q^2 + P^2) | \text{vac} \rangle = 1$, which gives $\text{tr}(H\rho^{(0)}) = m + 1$. ■

The following lemma shows that an energy-limited state has most of its support on a subspace spanned by functions with support in a bounded interval, in both position space and momentum space. Given a set $I \subseteq \mathbb{R}$, we define the operator Π_I as the projection onto the subspace of $L^2(\mathbb{R})$ of functions with support contained in I in position space. In the following, we define the Fourier transformation as the operator

$$\begin{aligned} \mathcal{F} L^2(\mathbb{R}) &\rightarrow L^2(\mathbb{R}), \\ f &\mapsto \mathcal{F}(f) \end{aligned}$$

acting on a function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ as

$$\mathcal{F}(f)(p) = \hat{f}(p) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ipx} dx. \quad (\text{F19})$$

Note that we have $\|f\|_2 = \|\hat{f}\|_2$. Hence, the Fourier transformation is isometric on a dense subspace $[L^1(\mathbb{R}) \cap L^2(\mathbb{R})]$, and, thus, it uniquely extends to a unitary operator on $L^2(\mathbb{R})$. In this context, we define the operator $\hat{\Pi}_I := \mathcal{F}^\dagger \Pi_I \mathcal{F}$, which is just the orthogonal projection onto the subspace of $L^2(\mathbb{R})$ of functions f whose Fourier transform $\mathcal{F}(f)$ has support contained in I .

Lemma F.7. Let $\rho \in \mathcal{B}[L^2(\mathbb{R})]$ be a state. Then,

$$\min \{ \text{tr}(\Pi_{[-R,R]}\rho), \text{tr}(\hat{\Pi}_{[-R,R]}\rho) \} \geq 1 - \frac{\text{tr}((Q^2 + P^2)\rho)}{R^2}$$

for any $R > 0$.

Proof. Markov's inequality in the form

$$\Pr[Q^2 > R^2] \leq \frac{\mathbb{E}[Q^2]}{R^2}$$

implies that

$$\text{tr}((I - \Pi_{[-R,R]})\rho) \leq \frac{\text{tr}(Q^2\rho)}{R^2} \leq \frac{\text{tr}((Q^2 + P^2)\rho)}{R^2};$$

that is,

$$\text{tr}(\Pi_{[-R,R]}\rho) \geq 1 - \frac{\text{tr}((Q^2 + P^2)\rho)}{R^2}.$$

Interchanging the roles of P and Q yields the analogous bound for $\text{tr}(\hat{\Pi}_{[-R,R]}\rho)$. ■

3. Tail bounds for approximate GKP states

In this section, we focus on the decay properties of approximate GKP states $|\text{GKP}_{\kappa,\Delta}\rangle$ in position and momentum space. We repeatedly use the variational characterization

$$\|\rho - \sigma\|_1 = 2 \sup_{\Pi} \text{tr}(\Pi(\rho - \sigma)) \quad (\text{F20})$$

of the L^1 distance of any two states $\rho, \sigma \in \mathcal{B}[L^2(\mathbb{R})]$, where the supremum is taken over all orthogonal projections Π acting on $L^2(\mathbb{R})$.

Lemma F.8. Let $\kappa \in (0, 1/4)$, $\Delta \in (0, 1/100)$, and $R > 0$. Then,

$$\|\Pi_{[-R,R]}|\text{GKP}_{\kappa,\Delta}\rangle\|^2 \leq 4\kappa R + 5\sqrt{\kappa} + 7\sqrt{\Delta}, \quad (\text{F21})$$

$$\left\| \hat{\Pi}_{[-R,R]}|\text{GKP}_{\kappa,\Delta}\rangle \right\|^2 \leq 2\Delta R + 5\sqrt{\kappa} + 7\sqrt{\Delta}. \quad (\text{F22})$$

Proof. In the following, we fix $\varepsilon = \sqrt{\Delta}$. Note that $\varepsilon \in (0, 1/2)$ by the assumption on Δ .

We first prove claim (F21). We have

$$\begin{aligned} \|\Pi_{[-R,R]}|\text{GKP}_{\kappa,\Delta}\rangle\|^2 &= \langle \text{GKP}_{\kappa,\Delta} | \Pi_{[-R,R]} | \text{GKP}_{\kappa,\Delta} \rangle \\ &\leq \langle \text{GKP}_{\kappa,\Delta}^\varepsilon | \Pi_{[-R,R]} | \text{GKP}_{\kappa,\Delta}^\varepsilon \rangle \\ &\quad + \frac{1}{2} \left\| |\text{GKP}_{\kappa,\Delta}\rangle \langle \text{GKP}_{\kappa,\Delta}| \right. \\ &\quad \left. - |\text{GKP}_{\kappa,\Delta}^\varepsilon\rangle \langle \text{GKP}_{\kappa,\Delta}^\varepsilon| \right\|_1, \end{aligned} \quad (\text{F23})$$

where we use Eq. (F20). By Lemma G.15, we can bound

$$\begin{aligned} \left\| \hat{\Pi}_{[-R,R]}|\text{GKP}_{\kappa,\Delta}\rangle \right\|^2 &= \langle \text{GKP}_{\kappa,\Delta} | \hat{\Pi}_{[-R,R]} | \text{GKP}_{\kappa,\Delta} \rangle \\ &\leq \langle \text{GKP}_{\kappa,\Delta}^\varepsilon | \hat{\Pi}_{[-R,R]} | \text{GKP}_{\kappa,\Delta}^\varepsilon \rangle + \frac{1}{2} \left\| |\text{GKP}_{\kappa,\Delta}\rangle \langle \text{GKP}_{\kappa,\Delta}| - |\text{GKP}_{\kappa,\Delta}^\varepsilon\rangle \langle \text{GKP}_{\kappa,\Delta}^\varepsilon| \right\|_1. \end{aligned} \quad (\text{F25})$$

We bound separately the two terms on the rhs of Eq. (F25), starting with the first. By Lemma G.20, we have

$$\langle \text{GKP}_{\kappa,\Delta}^\varepsilon | \hat{\Pi}_{[-R,R]} | \text{GKP}_{\kappa,\Delta}^\varepsilon \rangle \leq 2\Delta R + 12\Delta. \quad (\text{F26})$$

Moreover, from Corollary G.22, we know that

$$\left\| |\text{GKP}_{\kappa,\Delta}\rangle \langle \text{GKP}_{\kappa,\Delta}| - |\text{GKP}_{\kappa,\Delta}^\varepsilon\rangle \langle \text{GKP}_{\kappa,\Delta}^\varepsilon| \right\|_1 \leq 4\sqrt{\kappa} + 13\sqrt{\Delta}. \quad (\text{F27})$$

Combining Eq. (F26) with Eq. (F27) and using that by assumption $\sqrt{\Delta} \leq 1/10$, we have $\Delta \leq \sqrt{\Delta}/10$, which yields

$$\left\| \hat{\Pi}_{[-R,R]}|\text{GKP}_{\kappa,\Delta}\rangle \right\|^2 \leq 2\Delta R + 12\Delta + 2\sqrt{\kappa} + 13\sqrt{\Delta}/2 \leq 2\Delta R + 2\sqrt{\kappa} + 7\sqrt{\Delta} \leq 2\Delta R + 5\sqrt{\kappa} + 7\sqrt{\Delta}.$$

This completes the proof. ■

the first term as

$$\langle \text{GKP}_{\kappa,\Delta}^\varepsilon | \Pi_{[-R,R]} | \text{GKP}_{\kappa,\Delta}^\varepsilon \rangle \leq 4\kappa R + 10\kappa.$$

Using Corollary G.10, the second term in Eq. (F23) (by the assumption $\varepsilon = \sqrt{\Delta}$) is bounded by

$$\left\| |\text{GKP}_{\kappa,\Delta}\rangle \langle \text{GKP}_{\kappa,\Delta}| - |\text{GKP}_{\kappa,\Delta}^\varepsilon\rangle \langle \text{GKP}_{\kappa,\Delta}^\varepsilon| \right\|_1 \leq 6\sqrt{\Delta}.$$

It follows that

$$\begin{aligned} \|\Pi_{[-R,R]}|\text{GKP}_{\kappa,\Delta}\rangle\|^2 &\leq 4\kappa R + 10\kappa + 3\sqrt{\Delta} \\ &\leq 4\kappa R + 5\sqrt{\kappa} + 3\sqrt{\Delta} \\ &\leq 4\kappa R + 5\sqrt{\kappa} + 7\sqrt{\Delta}, \end{aligned}$$

where we use that $\kappa < 1/4$ to obtain the second inequality.

To show the claim (F22), we define

$$|\text{GKP}_{\kappa,\Delta}^\varepsilon\rangle := \frac{\hat{\Pi}_{(2\pi\mathbb{Z})(\varepsilon)}|\text{GKP}_{\kappa,\Delta}\rangle}{\|\hat{\Pi}_{(2\pi\mathbb{Z})(\varepsilon)}|\text{GKP}_{\kappa,\Delta}\rangle\|}, \quad (\text{F24})$$

where we define $(2\pi\mathbb{Z})(\varepsilon) := 2\pi\mathbb{Z} + [-\varepsilon, \varepsilon]$. This is to say that, in momentum space, we have

$$\text{GKP}_{\kappa,\Delta}^\varepsilon(p) = D_{\kappa,\Delta}^\varepsilon \sum_{z \in \mathbb{Z}} \eta_\Delta(p) \chi_\kappa^\varepsilon(2\pi z)(p) \quad \text{for } p \in \mathbb{R},$$

where $D_{\kappa,\Delta}^\varepsilon$ is a normalization constant such that $\|\text{GKP}_{\kappa,\Delta}^\varepsilon\| = 1$. We conclude with Eq. (F20) that

Given the established tail bounds for the state $|\text{GKP}_{\kappa,\Delta}\rangle$, we can use the projections $\Pi_{[-R,R]}$ and $\hat{\Pi}_{[-R,R]}$ to establish lower bounds on the distance of a state ρ to $|\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|$, as follows.

Lemma F.9. Let $\rho \in \mathcal{B}[L^2(\mathbb{R})]$ be a state, $\kappa \in (0, 1/4)$, and $\Delta \in (0, 1/100)$. Then,

$$\begin{aligned} \|\rho - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 &\geq 2\left(\text{tr}(\Pi_{[-R,R]}\rho) - (4\kappa R + 5\sqrt{\kappa} + 7\sqrt{\Delta})\right) \\ &\quad \text{for all } R > 0. \\ \|\rho - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 &\geq 2\left(\text{tr}(\hat{\Pi}_{[-R,R]}\rho) - (2\Delta R + 5\sqrt{\kappa} + 7\sqrt{\Delta})\right) \end{aligned}$$

Proof. The claim follows from Eq. (F20) with the choice $\Pi = \Pi_{[-R,R]}$ (respectively, $\Pi = \hat{\Pi}_{[-R,R]}$) and Lemma F.8 giving an upper bound on the norms of the projected states $\Pi_{[-R,R]}|\text{GKP}_{\kappa,\Delta}\rangle$ ($\hat{\Pi}_{[-R,R]}|\text{GKP}_{\kappa,\Delta}\rangle$). ■

4. A lower bound on the unitary complexity of $|\text{GKP}_{\kappa,\Delta}\rangle$

In this section, we establish a lower bound on the approximate unitary state complexity $\mathcal{C}_\varepsilon^*(|\text{GKP}_{\kappa,\Delta}\rangle)$ of the state $|\text{GKP}_{\kappa,\Delta}\rangle$ as introduced in Appendix A 2. We prove the following.

Theorem F.10. Let $\kappa, \Delta > 0$ be such that

$$20\sqrt{\kappa} + 28\sqrt{\Delta} \leq 1. \quad (\text{F28})$$

Then, we have

$$\mathcal{C}_1^*(|\text{GKP}_{\kappa,\Delta}\rangle) \geq \frac{1}{8\pi}(\log 1/\kappa + \log 1/\Delta) - 1.$$

In particular, we infer the following from Theorem F.10.
Corollary F.11.

$$\mathcal{C}_1^*(|\text{GKP}_{\kappa,\Delta}\rangle) = \Omega(\log 1/\kappa + \log 1/\Delta) \quad \text{for } (\kappa, \Delta) \rightarrow (0, 0).$$

Proof of Theorem F.10. Consider a circuit U preparing a distance-1 approximation to $|\text{GKP}_{\kappa,\Delta}\rangle$ with minimal complexity $\mathcal{C}_1^*(|\text{GKP}_{\kappa,\Delta}\rangle)$. In other words, the circuit U uses $m+1$ bosonic modes and m' qubits and is composed of T unitaries from the set $\mathcal{G}$. These parameters satisfy

$$\mathcal{C}_1^*(|\text{GKP}_{\kappa,\Delta}\rangle) = T + (m+1) + m'. \quad (\text{F29})$$

Furthermore, denoting the state of the first mode after application of U to the initial state $|\Psi\rangle = |\text{vac}\rangle \otimes |\text{vac}\rangle^{\otimes m} \otimes |0\rangle^{\otimes m'}$ by $\rho := \text{tr}_{m,m'} U|\Psi\rangle\langle\Psi|U^\dagger$, we have

$$\|\rho - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 \leq 1.$$

We proceed in two steps. First, the assumption (F28) implies that $5\sqrt{\kappa} + 7\sqrt{\Delta} < 1/4$ and also that $\kappa < 1/4$ and $\Delta < 1/100$. Therefore, by the first inequality of Lemma F.7 and by Lemma F.9, we have

$$\begin{aligned} \|\rho - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 &\geq 2\left(1 - \left(\frac{E_m(T)}{R^2} + 4\kappa R\right) - 1/4\right) \quad \text{for all } R > 0, \end{aligned}$$

which implies

$$E_m(T) \geq (1/4 - 4\kappa R)R^2 \quad \text{for } R > 0.$$

We maximize the rhs for $R > 0$. The global maximum of the function $g(R) = (1/4 - 4\kappa R)R^2$ restricted to $R > 0$ is at $R_0 = 1/(24\kappa)$ with $g(R_0) = 1/(12 \cdot 24^2 \kappa^2)$.

Inserting the definition of $E_m(T)$ from Lemma F.6 into the above gives the lower bound

$$8\pi T + \log(m+2) \geq \log 1/\kappa^2 - \log(12 \cdot 24^2);$$

hence, using that $(\log(m+2))/(8\pi) \leq m+1$ for all $m \geq 0$ and that $\log(12 \cdot 24^2)/(8\pi) < 1$, we obtain

$$T + (m+1) + m' \geq \frac{1}{4\pi} \cdot \log 1/\kappa - 1.$$

Inserting this into Eq. (F29) gives the first lower bound

$$\mathcal{C}_1^*(|\text{GKP}_{\kappa,\Delta}\rangle) \geq \frac{1}{4\pi} \cdot \log 1/\kappa - 1 \quad (\text{F30})$$

on the complexity $\mathcal{C}_1^*(|\text{GKP}_{\kappa,\Delta}\rangle)$.

Second, by a similar calculation using the bound involving $\hat{\Pi}_{[-R,R]}$ in Lemma F.7, we have [by the assumption (F28), which is $5\sqrt{\kappa} + 7\sqrt{\Delta} < 1/4$]

$$\begin{aligned} \|\rho - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 &\geq 2\left(1 - \left(\frac{E_m(T)}{R^2} + 2\Delta R\right) - 1/4\right) \quad \text{for all } R > 0. \end{aligned}$$

This implies

$$E_m(T) \geq (1/4 - 2\Delta R)R^2 \quad \text{for all } R > 0.$$

Proceeding as we did in the calculation involving $\Pi_{[-R,R]}$, we find

$$E_m(T) \geq 1/(6 \cdot 12^2)1/\Delta^2.$$

Again inserting $E_m(T)$ from Lemma F.6 yields

$$8\pi T + \log(m+2) \geq \log 1/\Delta^2 - \log(6 \cdot 12^2);$$

hence, by the same argument as before, along with $[\log(6 \cdot 12^2)]/(8\pi) < 1$, we get

$$T + (m+1) + m' \geq \frac{1}{4\pi} \cdot \log 1/\Delta - 1.$$

Therefore, this second analysis shows that the unitary complexity is bounded as

$$C_1^*(|\text{GKP}_{\kappa,\Delta}\rangle) \geq \frac{1}{4\pi} \cdot \log 1/\Delta - 1. \quad (\text{F31})$$

Combining the bounds from Eqs. (F30) and (F31), we deduce

$$C_1^*(|\text{GKP}_{\kappa,\Delta}\rangle) \geq \frac{1}{8\pi} (\log 1/\kappa + \log 1/\Delta) - 1.$$

The claim follows. $\blacksquare$

5. A lower bound on the heralded complexity of $|\text{GKP}_{\kappa,\Delta}\rangle$

In this section, we prove a lower bound on the complexity of preparing the state $|\text{GKP}_{\kappa,\Delta}\rangle$ with heralding protocols as introduced in Appendix A 3. We argue that, as long as the number of operations (cf. Appendix A 3) is bounded, the resulting states will be far from a GKP state except with low probability. This immediately implies a lower bound on the heralded complexity of $|\text{GKP}_{\kappa,\Delta}\rangle$ states. The procedure is similar to bounding the unitary complexity. We first need to lower bound the terms of the form $\text{tr}(\Pi_{[-R,R]}\rho_{\text{acc}})$ and $\text{tr}(\hat{\Pi}_{[-R,R]}\rho_{\text{acc}})$, where ρ_{acc} is the output state conditioned on acceptance. We show the following.

Lemma F.12. Consider a heralding protocol described by the unitary U on $L^2(\mathbb{R})^{\otimes(m+1)} \otimes (\mathbb{C}^2)^{\otimes m'}$ composed of T_1 gates from $\mathcal{G}$ (before the measurements) and acceptance probability $p_{\text{acc}} \in (0, 1]$. Let $R > 0$ and assume that the maximal norm of a displacement vector associated with a displacement operator applied after measurements is bounded by

$$d_{\text{acc}} := \sup_{\alpha \in F^{-1}(\{\text{acc}\})} \|d(\alpha)\| \leq R/2.$$

Then,

$$\min \{ \text{tr}(\Pi_{[-R,R]}\rho_{\text{acc}}), \text{tr}(\hat{\Pi}_{[-R,R]}\rho_{\text{acc}}) \} \geq p_{\text{acc}} - \frac{4E(T_1)}{R^2},$$

with the function E_m defined in Lemma F.6.

Proof. We show only

$$\text{tr}(\Pi_{[-R,R]}\rho_{\text{acc}}) \geq p_{\text{acc}} - \frac{4E(T_1)}{R^2};$$

the inequality for $\hat{\Pi}_{[-R,R]}$ follows by analogous arguments. As $|d_p(\alpha)| \leq \|d(\alpha)\| \leq R/2$ for all $\alpha \in F^{-1}(\{\text{acc}\})$, we have

$$[-R/2, R/2] \subseteq [-R + d_p(\alpha), R + d_p(\alpha)].$$

Moreover, we note that

$$D[d(\alpha)]^\dagger \Pi_{[-R,R]} D[d(\alpha)] = \Pi_{[-R+d_p(\alpha), R+d_p(\alpha)]}.$$

Hence, we conclude that

$$\begin{aligned} \text{tr}(\Pi_{[-R,R]}\rho_{\text{acc}}) &= \int_{F^{-1}(\{\text{acc}\})} \text{tr}((\Pi_{[-R,R]} \otimes I) \{D[d(\alpha)] \otimes E_\alpha\} U|\Psi\rangle\langle\Psi|U^\dagger \{D[d(\alpha)]^\dagger \otimes I\}) d\alpha \\ &= \int_{F^{-1}(\{\text{acc}\})} \text{tr}(\{D[d(\alpha)]^\dagger \Pi_{[-R,R]} D[d(\alpha)] \otimes E_\alpha\} U|\Psi\rangle\langle\Psi|U^\dagger) d\alpha \\ &= \int_{F^{-1}(\{\text{acc}\})} \text{tr}(\Pi_{[-R+d_p(\alpha), R+d_p(\alpha)]} (I \otimes E_\alpha) U|\Psi\rangle\langle\Psi|U^\dagger) d\alpha \\ &\geq \int_{F^{-1}(\{\text{acc}\})} \text{tr}(\Pi_{[-R/2, R/2]} \otimes E_\alpha) U|\Psi\rangle\langle\Psi|U^\dagger d\alpha \\ &= \langle U\Psi, (\Pi_{[-R/2, R/2]} \otimes E_{\text{acc}}) U\Psi \rangle, \end{aligned} \quad (\text{F32})$$

where we define $E_{\text{acc}} := \int_{F^{-1}(\{\text{acc}\})} E_\alpha d\alpha$. For later reference, we also define the POVM element associated with the output flag rej by $E_{\text{rej}} := \int_{F^{-1}(\{\text{rej}\})} E_\alpha d\alpha$. Let us consider the postmeasurement state without shift corrections $D[d(\alpha)]$, which we can decompose according to the flag outputs acc and rej as

$$\text{tr}_{m,m'} U|\Psi\rangle\langle\Psi|U^\dagger = p_{\text{acc}} \rho'_{\text{acc}} + (1 - p_{\text{acc}}) \rho'_{\text{rej}},$$

where

$$\rho'_{\text{acc}} = \frac{1}{p_{\text{acc}}} \int_{F^{-1}(\{\text{acc}\})} p(\alpha) \rho^{(\alpha)} d\alpha \quad \text{and} \quad \rho'_{\text{rej}} = \frac{1}{1 - p_{\text{acc}}} \int_{F^{-1}(\{\text{rej}\})} p(\alpha) \rho^{(\alpha)} d\alpha$$

and where $p(\alpha) \rho^{(\alpha)} = \langle \Psi | U^\dagger (I \otimes E_\alpha) U | \Psi \rangle$. Clearly, we have

$$\langle U\Psi, (\Pi_{[-R/2, R/2]} \otimes I) U\Psi \rangle = \langle U\Psi, (\Pi_{[-R/2, R/2]} \otimes E_{\text{acc}}) U\Psi \rangle + \langle U\Psi, (\Pi_{[-R/2, R/2]} \otimes E_{\text{rej}}) U\Psi \rangle \quad (\text{F33})$$

and

$$\langle U\Psi, (\Pi_{[-R/2, R/2]} \otimes E_{\text{rej}}) U\Psi \rangle \leq \langle U\Psi, (I \otimes E_{\text{rej}}) U\Psi \rangle = 1 - p_{\text{acc}}.$$

Combining Eqs. (F32) and (F33), we obtain

$$\begin{aligned} \text{tr}(\Pi_{[-R, R]} \rho_{\text{acc}}) &\geq \langle U\Psi, (\Pi_{[-R/2, R/2]} \otimes E_{\text{acc}}) U\Psi \rangle \\ &= \langle U\Psi, (\Pi_{[-R/2, R/2]} \otimes I) U\Psi \rangle - \langle U\Psi, (\Pi_{[-R/2, R/2]} \otimes E_{\text{rej}}) U\Psi \rangle \\ &\geq \langle U\Psi, (\Pi_{[-R/2, R/2]} \otimes I) U\Psi \rangle - (1 - p_{\text{acc}}) \quad \text{by Eq. (F34)} \\ &= p_{\text{acc}} - (1 - \langle U\Psi, (\Pi_{[-R/2, R/2]} \otimes I) U\Psi \rangle). \end{aligned} \quad (\text{F34})$$

Moreover, we can bound

$$\begin{aligned} \langle U\Psi, (\Pi_{[-R/2, R/2]} \otimes I) U\Psi \rangle &= \text{tr}(\Pi_{[-R/2, R/2]} \text{tr}_{m, m'} U | \Psi \rangle \langle \Psi | U^\dagger) \geq 1 - \frac{\text{tr}((Q^2 + P^2) \text{tr}_{m, m'} U | \Psi \rangle \langle \Psi | U^\dagger)}{(R/2)^2} \\ &= 1 - \frac{\langle \Psi | U^\dagger (Q_1^2 + P_1^2) U | \Psi \rangle}{(R/2)^2} \geq 1 - \frac{\langle \Psi | U^\dagger (\sum_{i=1}^{m+1} Q_i^2 + P_i^2) U | \Psi \rangle}{(R/2)^2}, \end{aligned} \quad (\text{F35})$$

where we use Lemma F.7 applied to the reduced state $\text{tr}_{m, m'} U | \Psi \rangle \langle \Psi | U^\dagger$ to obtain the first inequality. The claim follows from combining Eqs. (F34) and (F35) with the upper bound on the energy E_m stated in Lemma F.6. ■

Theorem F.13. Let $\kappa, \Delta > 0$. Then, for all $p \in (0, 1]$ and $\varepsilon > 0$ such that

$$20\sqrt{\kappa} + 28\sqrt{\Delta} \leq p \quad \text{and} \quad \varepsilon \leq p, \quad (\text{F36})$$

we have

$$\mathcal{C}_{p, \varepsilon}^{\text{her},*}(|\text{GKP}_{\kappa, \Delta}\rangle) \geq \frac{1}{200} (\log 1/\kappa + \log 1/\Delta) - 1.$$

Theorem F.13 implies the following corollary.

Corollary F.14. There exists a polynomial $s(\kappa, \Delta)$ with $s(0, 0) = 0$ such that, for all functions $p(\kappa, \Delta)$ and $\varepsilon(\kappa, \Delta)$ satisfying $0 \leq \varepsilon(\kappa, \Delta) \leq p(\kappa, \Delta)$ and $s(\kappa, \Delta) \leq p(\kappa, \Delta) \leq 1$ for sufficiently small (κ, Δ) , we have

$$\begin{aligned} \mathcal{C}_{p(\kappa, \Delta), \varepsilon(\kappa, \Delta)}^{\text{her},*}(|\text{GKP}_{\kappa, \Delta}\rangle) \\ \geq \Omega(\log 1/\kappa + \log 1/\Delta) \quad \text{for } (\kappa, \Delta) \rightarrow (0, 0). \end{aligned}$$

Proof of Theorem F.13. Consider a heralding protocol, i.e., a unitary U composed of T_1 gates from $\mathcal{G}$, a POVM

$\{E_\alpha\}$, a map F computing a flag, and shift corrections $\{D[d(\alpha)]\}$, preparing with probability at least p a distance- ε approximation to $|\text{GKP}_{\kappa, \Delta}\rangle$ with minimal complexity $\mathcal{C}_{p, \varepsilon}^{\text{her},*}(|\text{GKP}_{\kappa, \Delta}\rangle)$. In other words, the protocol uses $m + 1$ bosonic modes and m' qubits and requires (including shift corrections) $T_1 + T_2$ gates from the set $\mathcal{G}$, where T_2 accounts for the complexity of shift corrections [cf. the definition in Eq. (A6)]. These parameters satisfy

$$\mathcal{C}_{p, \varepsilon}^{\text{her},*}(|\text{GKP}_{\kappa, \Delta}\rangle) = T_1 + T_2 + (2m + 1) + 2m'. \quad (\text{F37})$$

In addition, the output state conditioned on acceptance ρ_{acc} [cf. Eq. (A4)] satisfies

$$\|\rho_{\text{acc}} - |\text{GKP}_{\kappa, \Delta}\rangle \langle \text{GKP}_{\kappa, \Delta}|\|_1 \leq \varepsilon.$$

In the following, we argue similarly as in the proof of Theorem F.10.

For convenience, we define

$$d_{\text{acc}} := \sup_{\alpha \in F^{-1}(\{\text{acc}\})} \|d(\alpha)\| \leq R/2.$$

By Lemmas F.9 and F.12 (using the assumption $p_{\text{acc}} \geq p$) and considering the projector $\Pi_{[-R, R]}$, we have

$$\begin{aligned} & \|\rho_{\text{acc}} - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 \\ & \geq 2 \left(p - \left(\frac{4E_m(T_1)}{R^2} + 4\kappa R \right) - (7\sqrt{\kappa} + 5\sqrt{\Delta}) \right) \end{aligned}$$

for any $R \geq 2d_{\text{acc}}$, where d_{acc} is defined in Lemma F.12. By the assumption (F36), we have $5\sqrt{\kappa} + 7\sqrt{\Delta} < p/4$. Thus, solving for $E_m(T_1)$, we find

$$\begin{aligned} E_m(T_1) & \geq \frac{1}{4}((3p/4 - \varepsilon/2) - 4\kappa R)R^2 \\ & \geq \frac{1}{4}(p/4 - 4\kappa R)R^2 =: f(R) \quad \text{for all } R \geq 2d_{\text{acc}}, \quad (\text{F38}) \end{aligned}$$

where we use the assumption $\varepsilon \leq p$. Since this bound is valid for any $R \geq 2d_{\text{acc}}$, we can maximize the rhs of this inequality. That is, we choose

$$R_{\text{max}} = \arg \max_{R \geq 2d_{\text{acc}}} f(R).$$

Note that, for $A, B > 0$, the function $x \mapsto (A - Bx)x^2$ restricted to $x \geq 0$ has its global maximum at $x_0 = 2A/(3B)$ and is monotonically decreasing for $x \geq x_0$. Hence, the global maximum of $f(R)$ on $[0, \infty)$ is at

$$R_0 = \frac{p}{24\kappa}.$$

We have to distinguish two cases, depending on whether this global maximum is contained in $[2d_{\text{acc}}, \infty)$ or not.

- (a) If $R_0 \geq 2d_{\text{acc}}$, we conclude that the maximum of $f(R)$ restricted to $[2d_{\text{acc}}, \infty)$ is equal to $R_{\text{max}} = R_0$, and so, inserting this into Eq. (F38), we find

$$E_m(T_1) \geq \frac{p^3}{432 \cdot 4^3} \cdot 1/\kappa^2.$$

Inserting the definition of E_m from Lemma F.6 and solving for T_1 , we obtain

$$8\pi T_1 + \log(m+2) \geq \log 1/\kappa^2 + 3 \log p - \log(432 \cdot 4^3).$$

Since $T_1 \leq T_1 + T_2$, $[\log(m+2)]/(8\pi) \leq m+1$ for all $m \geq 0$, and $[\log(432 \cdot 4^3)]/(8\pi) < 1$, this implies

$$\begin{aligned} & T_1 + T_2 + (2m+1) + 2m' \\ & \geq \frac{1}{4\pi} \log 1/\kappa + \frac{3}{8\pi} \log p - 1; \end{aligned}$$

hence, with Eq. (F37),

$$C_{p,\varepsilon}^{*,\text{her}}(|\text{GKP}_{\kappa,\Delta}\rangle) \geq \frac{1}{4\pi} \log 1/\kappa + \frac{3}{8\pi} \log p - 1.$$

- (b) If $R_0 < 2d_{\text{acc}}$, we have

$$\begin{aligned} T_2 & = \max_{\alpha \in F^{-1}(\{\text{acc}\})} C_G\{D[d(\alpha)]\} \\ & \geq \max_{\alpha \in F^{-1}(\{\text{acc}\})} \frac{1}{4\pi} (\log \|d(\alpha)\|) - 1 \\ & = \frac{1}{4\pi} (\log d_{\text{acc}}) - 1, \end{aligned}$$

where we use the lower bound (B10) on the complexity of a displacement operator $D(d)$. We conclude that

$$\begin{aligned} & T_1 + T_2 + (2m+1) + 2m' \\ & \geq T_2 + 1 \geq \frac{1}{4\pi} \log d_{\text{acc}} \geq \frac{1}{4\pi} \log R_0/2 \\ & \geq \frac{1}{4\pi} \log 1/\kappa + \frac{1}{4\pi} \log(p/48). \end{aligned}$$

Hence, in this case using $\log(48)/(4\pi) < 1$, we have

$$\begin{aligned} C_{p,\varepsilon}^{*,\text{her}}(|\text{GKP}_{\kappa,\Delta}\rangle) & \geq \frac{1}{4\pi} \log 1/\kappa + \frac{1}{4\pi} \log(p/48) \\ & \geq \frac{1}{4\pi} \log 1/\kappa + \frac{1}{4\pi} \log p - 1. \end{aligned}$$

In both cases, we have shown that

$$C_{p,\varepsilon}^{*,\text{her}}(|\text{GKP}_{\kappa,\Delta}\rangle) \geq \frac{1}{4\pi} \log 1/\kappa + \frac{3}{8\pi} \log p - 1.$$

Because of the assumption (F36), we have $p \geq \sqrt{\kappa}$. This implies that

$$\begin{aligned} C_{p,\varepsilon}^{*,\text{her}}(|\text{GKP}_{\kappa,\Delta}\rangle) & \geq \left(\frac{1}{4\pi} - \frac{3}{16\pi} \right) \log 1/\kappa - 1 \\ & \geq \frac{1}{200} \log 1/\kappa - 1. \end{aligned} \quad (\text{F39})$$

Analogously, we can derive bounds using $\hat{\Pi}_{[-R,R]}$ instead of $\Pi_{[-R,R]}$. Again by Lemmas F.9 and F.12, the assumption (F36), and $p_{\text{acc}} \geq p$, we have

$$\begin{aligned} & \|\rho_{\text{acc}} - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 \\ & \geq 2 \left(3p/4 - \left(\frac{4E_m(T_1)}{R^2} + 2\Delta R \right) \right) \end{aligned}$$

for any $R \geq 2d_{\text{acc}}$. Solving for $E_m(T_1)$ and using that $\varepsilon \leq p$, we find

$$E_m(T_1) \geq \frac{1}{4}(p/4 - 2\Delta R)R^2 \quad \text{for all } R \geq 2d_{\text{acc}}.$$

Maximizing the rhs for $R \geq 2d_{\text{acc}}$ and proceeding as before with a case distinction, we arrive again at

$$\mathcal{C}_{p,\varepsilon}^{*,\text{her}}(|\text{GKP}_{\kappa,\Delta}\rangle) \geq \frac{1}{4\pi} \log 1/\Delta + \frac{3}{8\pi} (\log p) - 1.$$

By the same argument as in Eq. (F39), we infer

$$\mathcal{C}_{p,\varepsilon}^{*,\text{her}}(|\text{GKP}_{\kappa,\Delta}\rangle) \geq \frac{1}{100} \log 1/\Delta - 1. \quad (\text{F40})$$

Combining Eqs. (F39) and (F40), it follows that

$$\mathcal{C}_{p,\varepsilon}^{*,\text{her}}(|\text{GKP}_{\kappa,\Delta}\rangle) \geq \frac{1}{200} (\log 1/\kappa + \log 1/\Delta) - 1.$$

In particular, Theorem F.13 covers all heralding protocols whose acceptance probability p_{acc} is lower bounded by a constant for sufficiently small κ and Δ , as it is the case in Protocol 1 (cf. Theorem E.1). We are in the position to combine the upper bound from Corollary E.2 and the upper bound on the heralded state complexity stated in Corollary F.14 to get the following.

Corollary F.15. There is a polynomial $r(\kappa, \Delta)$ with $r(0, 0) = 0$ such that, for all functions $p(\kappa, \Delta)$ and $\varepsilon(\kappa, \Delta)$ satisfying $r(\kappa, \Delta) \leq \varepsilon(\kappa, \Delta) \leq p(\kappa, \Delta) \leq 1/10$ for sufficiently small (κ, Δ) , we have

$$\begin{aligned} \mathcal{C}_{p(\kappa,\Delta),\varepsilon(\kappa,\Delta)}^{*,\text{her}}(|\text{GKP}_{\kappa,\Delta}\rangle) \\ = \Theta(\log 1/\kappa + \log 1/\Delta) \quad \text{for } (\kappa, \Delta) \rightarrow (0, 0). \end{aligned}$$

APPENDIX G: BASIC PROPERTIES OF APPROXIMATE GKP STATES

In this section, we prove closeness of various types of approximate GKP states that we use.

1. Bounds on sums of Gaussians

In this section, we establish several useful bounds on sums of Gaussians.

Lemma G.1. Let $c > 0$. Then,

$$\sqrt{\frac{\pi}{c}} - 1 \leq \sum_{z \in \mathbb{Z}} e^{-cz^2} \leq \sqrt{\frac{\pi}{c}} + 1. \quad (\text{G1})$$

In particular,

$$\sum_{z \in \mathbb{Z} \setminus \{0\}} e^{-cz^2} \leq \sqrt{\frac{\pi}{c}}. \quad (\text{G2})$$

Moreover, we can bound

$$\sum_{z \in \mathbb{Z}} e^{-c(z-1/2)^2} \geq \sqrt{\frac{\pi}{c}} - 1. \quad (\text{G3})$$

We also have for $\varepsilon \in (0, 1/2)$ that

$$\sum_{z \in \mathbb{Z}} e^{-c(|z|+\varepsilon)^2} \geq \sqrt{\frac{\pi}{c}} - 2(1 + \varepsilon), \quad (\text{G4})$$

$$\sum_{z \in \mathbb{Z} \setminus \{0\}} e^{-c(|z|-\varepsilon)^2} \leq \sqrt{\frac{\pi}{c}} + 2. \quad (\text{G5})$$

Proof. We obtain the bounds in Eqs. (G1) and (G2) as follows. Notice that $x \mapsto e^{-cx^2}$ is a monotonically decreasing function for $c > 0$ and $x \geq 0$. Thus,

$$\begin{aligned} e^{-cz^2} &\leq \int_{z-1}^z e^{-cx^2} dx \quad \text{for any } z \geq 1, \\ e^{-cz^2} &\geq \int_z^{z+1} e^{-cx^2} dx \quad \text{for any } z \geq 0. \end{aligned} \quad (\text{G6})$$

Thus, we obtain the upper bound in Eq. (G1):

$$\sum_{z \in \mathbb{Z}} e^{-cz^2} = 1 + \sum_{z \in \mathbb{Z} \setminus \{0\}} e^{-cz^2} \leq 1 + \int e^{-cx^2} dx = \sqrt{\frac{\pi}{c}} + 1.$$

This also implies the upper bound (G2). By Eq. (G6), we have

$$\sum_{z \in \mathbb{Z}} e^{-cz^2} \geq 2 \int_1^\infty e^{-cx^2} dx + 1. \quad (\text{G7})$$

Since $e^{-cx^2} \leq 1$ for all $x \in \mathbb{R}$, we infer $\int_0^1 e^{-cx^2} dx \leq 1$. This with Eq. (G7) gives the lower bound in Eq. (G1):

$$\sum_{z \in \mathbb{Z}} e^{-cz^2} \geq \int e^{-cx^2} dx - 1 = \sqrt{\frac{\pi}{c}} - 1.$$

To show the bound (G3), we note that

$$\begin{aligned} \sum_{z \in \mathbb{Z}} e^{-c(z-1/2)^2} &= 2 \sum_{z \in \mathbb{N}} e^{-c(z-1/2)^2} \geq 2 \int_{1/2}^\infty e^{-cx^2} dx \\ &\geq \int e^{-cx^2} dx - 1 = \sqrt{\frac{\pi}{c}} - 1, \end{aligned}$$

where we use Eq. (G6) to obtain the first inequality and the fact that $e^{-cx^2} \leq 1$ for all x in the second inequality.

Next, we show Eq. (G4). Observe that

$$\sum_{z \in \mathbb{Z}} e^{-c(|z|+\varepsilon)^2} = \sum_{z>0} e^{-c(z+\varepsilon)^2} + \sum_{z \in \mathbb{N}} e^{-c(z+\varepsilon)^2} \geq 2 \sum_{z \in \mathbb{N}} e^{-c(z+\varepsilon)^2}. \quad (\text{G8})$$

By Eq. (G6) and by the assumption $\varepsilon \in (0, 1/2)$, we have

$$\begin{aligned} 2 \sum_{z \in \mathbb{N}} e^{-c(z+\varepsilon)^2} &\geq 2 \int_{1+\varepsilon}^{\infty} e^{-cx^2} dx \\ &\geq \int e^{-cx^2} dx - 2(1+\varepsilon), \end{aligned}$$

where we use that $e^{-cx^2} \leq 1$ for all $x > 0$ and $c > 0$. Combining this with Eq. (G8) implies the claim in Eq. (G4):

$$\sum_{z \in \mathbb{Z}} e^{-c(|z|+\varepsilon)^2} \geq \sqrt{\frac{\pi}{c}} - 2(1+\varepsilon).$$

Finally, we show Eq. (G5). By $e^{-cx^2} \leq 1$ for all $x \in \mathbb{R}$ and $c > 0$, we have

$$\sum_{z \in \mathbb{Z} \setminus \{0\}} e^{-c(|z|-\varepsilon)^2} = 2 \sum_{z \in \mathbb{N}} e^{-c(z-\varepsilon)^2} \leq 2 + 2 \sum_{z \geq 2} e^{-c(z-\varepsilon)^2}.$$

This, together with the upper bound from Eq. (G6), implies the claim

$$\begin{aligned} \sum_{z \in \mathbb{Z} \setminus \{0\}} e^{-c(|z|-\varepsilon)^2} &\leq 2 + 2 \int_{1-\varepsilon}^{\infty} e^{-cx^2} dx \\ &\leq 2 + \int e^{-cx^2} dx \\ &= 2 + \sqrt{\frac{\pi}{c}}. \end{aligned}$$

2. Distance bounds between approximate comb and GKP states

In this section, we prove several bounds on the closeness of Gaussian states, approximate comb states, and approximate GKP states.

a. Bounds on Gaussian states

Let us recall the definition of Gaussian state with parameter Δ [cf. Eq. (C2)]:

$$\Psi_{\Delta}(x) = \frac{1}{(\pi\Delta^2)^{1/4}} e^{-x^2/(2\Delta^2)} \quad (\text{G9})$$

and its truncated variant [cf. Eq. (C9)]:

$$\Psi_{\Delta}^{\varepsilon} = \frac{\Pi_{[-\varepsilon, \varepsilon]} \Psi_{\Delta}}{\|\Pi_{[-\varepsilon, \varepsilon]} \Psi_{\Delta}\|}, \quad (\text{G10})$$

where $\|\cdot\|$ is the Euclidean norm.

We show that, for a certain choice of parameters, these states are close to each other.

Lemma G.2. Let $\Delta > 0$ and $\varepsilon \in (0, 1/2)$. Then,

$$|\langle \Psi_{\Delta}, \Psi_{\Delta}^{\varepsilon} \rangle|^2 \geq 1 - 2e^{-(\varepsilon/\Delta)^2}. \quad (\text{G11})$$

Furthermore, if $\varepsilon \in [\sqrt{\Delta}, 1/2)$, we have

$$|\langle \Psi_{\Delta}, \Psi_{\Delta}^{\varepsilon} \rangle|^2 \geq 1 - 2e^{-1/\Delta} \geq 1 - 2\Delta. \quad (\text{G12})$$

Proof. By definition of $\Psi_{\Delta}^{\varepsilon}$ [cf. Eq. (G10)], we have

$$\begin{aligned} \langle \Psi_{\Delta}, \Psi_{\Delta}^{\varepsilon} \rangle &= \|\Pi_{[-\varepsilon, \varepsilon]} \Psi_{\Delta}\|^{-1} \langle \Psi_{\Delta}, \Pi_{[-\varepsilon, \varepsilon]} \Psi_{\Delta} \rangle \\ &= \|\Pi_{[-\varepsilon, \varepsilon]} \Psi_{\Delta}\|. \end{aligned} \quad (\text{G13})$$

By definition of Ψ_{Δ} [cf. Eq. (G9)], we have

$$\|\Pi_{[-\varepsilon, \varepsilon]} \Psi_{\Delta}\|^2 = \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{\pi}\Delta} e^{-(x/\Delta)^2} dx = \Pr[|X| \leq \varepsilon],$$

where we use that the integrand $x \mapsto 1/(\sqrt{\pi}\Delta) e^{-(x/\Delta)^2}$ is the probability density function of the random variable X sampled from Gaussian distribution with mean 0 and variance $\Delta^2/2$, i.e., $X \sim \mathcal{N}(0, \Delta^2/2)$. Thus, we can use the Chernoff bound (see, e.g., Ref. [38]) to obtain

$$\begin{aligned} \|\Pi_{[-\varepsilon, \varepsilon]} \Psi_{\Delta}\|^2 &= 1 - 2\Pr[X \geq \varepsilon] \\ &\geq 1 - 2e^{-(\varepsilon/\Delta)^2}. \end{aligned}$$

Inserting this into the square of Eq. (G13) implies Eq. (G11).

Equation (G12) follows from Eq. (G11), from the assumption $\varepsilon \in [\sqrt{\Delta}, 1/2)$ and from the inequality $e^{-x} \leq 1/x$ for all $x \geq 0$. We have

$$\begin{aligned} |\langle \Psi_{\Delta}, \Psi_{\Delta}^{\varepsilon} \rangle|^2 &\geq 1 - 2e^{-(\varepsilon/\Delta)^2} \\ &\geq 1 - 2e^{-1/\Delta} \\ &\geq 1 - 2\Delta. \end{aligned}$$

Corollary G.3. Assume $\Delta > 0$ and $\varepsilon \in [\sqrt{\Delta}, 1/2)$. Then,

$$\| |\Psi_{\Delta}\rangle \langle \Psi_{\Delta}| - |\Psi_{\Delta}^{\varepsilon}\rangle \langle \Psi_{\Delta}^{\varepsilon}| \|_1 \leq 3\sqrt{\Delta}.$$

Proof. By Lemma G.2 and by the relation between the trace distance and the overlap [cf. Eq. (C12)], we have

$$\| |\Psi_{\Delta}\rangle \langle \Psi_{\Delta}| - |\Psi_{\Delta}^{\varepsilon}\rangle \langle \Psi_{\Delta}^{\varepsilon}| \|_1 \leq 2\sqrt{2\Delta} \leq 3\sqrt{\Delta}.$$

Let us recall the definition of translated Gaussian states [Eq. (C2)] and translated truncated Gaussian states [Eq. (C9)]:

$$[\chi_\Delta(z)](x) := \Psi_\Delta(x-z) \quad \text{and} \quad [\chi_\Delta^\varepsilon(z)](x) := \Psi_\Delta^\varepsilon(x-z).$$

They satisfy the following statements.

Lemma G.4. Let $z, z' \in \mathbb{Z}$ and $\Delta \in (0, 1/4)$. We have

$$\langle \chi_\Delta(z), \chi_\Delta(z') \rangle = e^{-(z-z')^2/(4\Delta^2)}.$$

Proof. We have by definition that

$$\begin{aligned} \langle \chi_\Delta(z), \chi_\Delta(z') \rangle &= \frac{1}{\sqrt{\pi}\Delta} \int_{-\infty}^{\infty} e^{-(x-z)^2/(2\Delta^2)} e^{-(x-z')^2/(2\Delta^2)} dx \\ &= \frac{1}{\sqrt{\pi}\Delta} \int_{-\infty}^{\infty} e^{-[(x-z)^2 + (x-z')^2]/(2\Delta^2)} dx. \end{aligned}$$

Note that

$$(x-z)^2 + (x-z')^2 = 2[x - (z+z')/2]^2 + (z-z')^2/2.$$

Therefore,

$$\begin{aligned} \langle \chi_\Delta(z), \chi_\Delta(z') \rangle &= \frac{1}{\sqrt{\pi}\Delta} \left(\int_{-\infty}^{\infty} e^{-[x - (z+z')/2]^2/(2\Delta^2)} dx \right) e^{-(z-z')^2/4\Delta^2} \\ &= e^{-(z-z')^2/(4\Delta^2)}, \end{aligned}$$

where we use that $\|\Psi_\Delta\| = 1$.

Lemma G.5. Let $\varepsilon \in (0, 1/2)$ and $\Delta > 0$. Then,

$$\begin{aligned} \delta_{z,z'}(1 - 5\varepsilon^2) &\leq \langle \chi_\Delta^\varepsilon(z'), e^{-i\pi z} e^{i\pi Q} \chi_\Delta^\varepsilon(z) \rangle \\ &\leq \delta_{z,z'} \quad \text{for all } z, z' \in \mathbb{Z}. \end{aligned}$$

In particular, the overlap $\langle \chi_\Delta^\varepsilon(z'), e^{-i\pi z} e^{i\pi Q} \chi_\Delta^\varepsilon(z) \rangle$ is real.

Proof. First, we consider the case $z \neq z'$. By definition, the unitary $e^{i\pi Q}$ acts as a multiplication operator on $L^2(\mathbb{R})$. As the functions $\chi_\Delta^\varepsilon(z)$ and $\chi_\Delta^\varepsilon(z')$ have disjoint support, we conclude that

$$\langle \chi_\Delta^\varepsilon(z), e^{-i\pi z} e^{i\pi Q} \chi_\Delta^\varepsilon(z) \rangle = 0 \quad \text{if } z \neq z'. \quad (\text{G14})$$

Next, we consider the case $z = z'$. By definition of the unitary $e^{i\pi Q}$, we have

$$\begin{aligned} \langle \chi_\Delta^\varepsilon(z), e^{-i\pi z} e^{i\pi Q} \chi_\Delta^\varepsilon(z) \rangle &= \int_{\mathbb{R}} \overline{\chi_\Delta^\varepsilon(z)(x)} e^{-i\pi z} e^{i\pi x} \chi_\Delta^\varepsilon(z)(x) dx \\ &= \int_{z-\varepsilon}^{z+\varepsilon} |\chi_\Delta^\varepsilon(z)(x)|^2 e^{-i\pi z} e^{i\pi x} dx \\ &= \int_{z-\varepsilon}^{z+\varepsilon} |\Psi_\Delta^\varepsilon(x-z)|^2 e^{i\pi(x-z)} dx \end{aligned} \quad (\text{G15})$$

$$= \int_{z-\varepsilon}^{z+\varepsilon} |\Psi_\Delta^\varepsilon(x-z)|^2 e^{i\pi(x-z)} dx \quad (\text{G16})$$

$$= \int_{-\varepsilon}^{\varepsilon} |\Psi_\Delta^\varepsilon(x)|^2 \cos(\pi x) dx. \quad (\text{G17})$$

We obtain Eq. (G15) from the fact that the support of $\chi_\Delta^\varepsilon(z)$ is contained in the interval $[z-\varepsilon, z+\varepsilon]$. Equation (G17) follows from a variable substitution and using that the support of Ψ_Δ^ε is contained in $[-\varepsilon, \varepsilon]$. Equation (G17) follows from the fact that $|\Psi_\Delta^\varepsilon|^2$ is even and cosine is odd. We can bound Eq. (G17) as follows:

$$\begin{aligned} \int_{-\varepsilon}^{\varepsilon} |\Psi_\Delta^\varepsilon(x)|^2 \cos(\pi x) dx &\geq \cos(\pi \varepsilon) \int_{-\varepsilon}^{\varepsilon} |\Psi_\Delta^\varepsilon(x)|^2 dx \\ &= \cos(\pi \varepsilon) \end{aligned} \quad (\text{G18})$$

$$\geq 1 - 5\varepsilon^2. \quad (\text{G19})$$

Inequality (G18) follows from the fact that cosine is an even function monotonically decreasing on the interval $[0, \pi/2]$. Equation (G19) is a consequence of the bound $\cos x \geq 1 - x^2/2$ for all $x \in \mathbb{R}$. Finally, from Eqs. (G14) and (G17), we conclude that $\langle \chi_\Delta^\varepsilon(z), e^{-i\pi z} e^{i\pi Q} \chi_\Delta^\varepsilon(z) \rangle \in \mathbb{R}$ for all $z, z' \in \mathbb{Z}$. ■

b. Bounds on approximate comb states

Let us recall the definitions of the approximate comb states [cf. Eq. (C1)] and of the truncated approximate comb states [cf. Eqs. (C7) and (C8)]. Those are

$$|\mathbb{I}_{L,\Delta}\rangle = \frac{D_{L,\Delta}}{\sqrt{L}} \sum_{z=-L/2}^{L/2-1} |\chi_\Delta(z)\rangle, \quad (\text{G20})$$

where $D_{L,\Delta}$ is a normalization factor and χ_Δ are translated Gaussians as in Eq. (G9), and

$$|\mathbb{I}_{L,\Delta}^\varepsilon\rangle = \frac{1}{\sqrt{L}} \sum_{z=-L/2}^{L/2-1} |\chi_\Delta^\varepsilon(z)\rangle, \quad (\text{G21})$$

where χ_Δ are truncated translated Gaussians as in Eq. (G10).

Lemma G.6. Let $\varepsilon \in (0, 1/2)$, $\Delta \in (0, 1/4)$, and $L \in 2\mathbb{N}$. Then,

$$|\langle \mathbb{I}_{L,\Delta}, \mathbb{I}_{L,\Delta}^\varepsilon \rangle|^2 \geq 1 - 16\Delta^2 - 2e^{-(\varepsilon/\Delta)^2}.$$

Proof. By Eqs. (G20) and (G21), we have

$$\begin{aligned} \langle \mathbb{I}_{L,\Delta}, \mathbb{I}_{L,\Delta}^\varepsilon \rangle &= \frac{D_{L,\Delta}}{L} \sum_{z=-L/2}^{L/2-1} \sum_{z'=-L/2}^{L/2-1} \langle \chi_\Delta(z), \chi_\Delta^\varepsilon(z') \rangle \\ &\geq \frac{D_{L,\Delta}}{L} \sum_{z=-L/2}^{L/2-1} \langle \Psi_\Delta, \Psi_\Delta^\varepsilon \rangle \end{aligned} \quad (\text{G22})$$

$$\begin{aligned}
&= D_{L,\Delta} \langle \Psi_\Delta, \Psi_\Delta^\varepsilon \rangle \\
&\geq D_{L,\Delta} \left(1 - 2e^{-(\varepsilon/\Delta)^2} \right)^{1/2}, \tag{G23}
\end{aligned}$$

where inequality (G22) follows from the non-negativity of $\chi_\Delta(z)(\cdot)$ and $\chi_\Delta^\varepsilon(z)(\cdot)$ and the equality $\langle \chi_\Delta(z), \chi_\Delta^\varepsilon(z) \rangle = \langle \Psi_\Delta, \Psi_\Delta^\varepsilon \rangle$ for all $z \in \mathbb{Z}$. The inequality (G23) follows from Lemma G.2.

We have

$$\begin{aligned}
D_{L,\Delta}^{-2} &= \frac{1}{L} \sum_{z=-L/2}^{L/2-1} \sum_{z'=-L/2}^{L/2-1} \langle \chi_\Delta(z), \chi_\Delta(z') \rangle \\
&= \frac{1}{L} \sum_{z=-L/2}^{L/2-1} \sum_{z'=-L/2}^{L/2-1} e^{-(z-z')^2/4\Delta^2}, \tag{G24}
\end{aligned}$$

where we use that $\langle \chi_\Delta(z), \chi_\Delta(z') \rangle = e^{-(z-z')^2/4\Delta^2}$ by Lemma G.4. We split the double sum into diagonal and off-diagonal parts, S and J , respectively. This gives

$$\sum_{z=-L/2}^{L/2-1} \sum_{z'=-L/2}^{L/2-1} e^{-(z-z')^2/4\Delta^2} = S + J,$$

where

$$\begin{aligned}
S &= \sum_{z=-L/2}^{L/2-1} e^{-(z-z)^2/4\Delta^2} = \sum_{z=-L/2}^{L/2-1} 1 = L, \\
J &= \sum_{\substack{z, z' \in \{-L/2, \dots, L/2-1\} \\ z \neq z'}} e^{-(z-z')^2/4\Delta^2}. \tag{G25}
\end{aligned}$$

Let us rewrite J as

$$\begin{aligned}
J &= 2 \sum_{z=-L/2}^{L/2-1} \sum_{z'=z+1}^{L/2-1} e^{-(z-z')^2/4\Delta^2} \\
&= 2 \sum_{z=-L/2}^{L/2-1} \sum_{y=1}^{L/2-z-1} e^{-y^2/4\Delta^2} \\
&\leq 2L \sum_{y=1}^{L-1} e^{-y^2/4\Delta^2}.
\end{aligned}$$

We use that $y \leq y^2$ for $y \geq 1$ and bound J as follows:

$$\begin{aligned}
J &\leq 2L \sum_{y=1}^{L-1} \left(e^{-1/4\Delta^2} \right)^y \\
&= 2L e^{-1/4\Delta^2} \frac{1 - e^{-(L-1)/4\Delta^2}}{1 - e^{-1/4\Delta^2}} \text{ by sum of geometric series} \\
&\leq 2L \frac{e^{-1/4\Delta^2}}{1 - e^{-1/4\Delta^2}}.
\end{aligned}$$

Notice that $e^{-1/(4\Delta^2)} \leq 1/4$ because $0 < \Delta < 1/4$. Therefore, we can use the inequality $x/(1-x) \leq 2x$ that holds for all $0 \leq x \leq 1/2$ and obtain

$$\begin{aligned}
J &\leq 2L \cdot 2e^{-1/4\Delta^2} \\
&\leq 16L\Delta^2 \text{ by } e^{-x} \leq x^{-1} \text{ for all } x \geq 0.
\end{aligned}$$

Combining this bound with Eqs. (G24)–(G25) gives

$$D_{L,\Delta}^{-2} \leq 1 + 16\Delta^2.$$

Finally, Eq. (G23) together with the bound on $D_{L,\Delta}^{-2}$ gives

$$\begin{aligned}
|\langle \mathbb{I}_{L,\Delta}, \mathbb{I}_{L,\Delta}^\varepsilon \rangle|^2 &\geq \frac{1}{1 + 16\Delta^2} \left(1 - 2e^{-(\varepsilon/\Delta)^2} \right) \\
&\geq (1 - 16\Delta^2) \left(1 - 2e^{-(\varepsilon/\Delta)^2} \right) \\
&\geq 1 - 16\Delta^2 - 2e^{-(\varepsilon/\Delta)^2},
\end{aligned}$$

where we use $(1+x)^{-1} \geq 1-x$ for all $x \geq 0$ on the first line and $(1-x)(1-y) \geq 1-x-y$ for all $x, y \geq 0$ on the last line. ■

Corollary G.7. Let $\Delta \in (0, 1/4)$, $\varepsilon \in [\sqrt{\Delta}, 1/2)$, and $L \in 2\mathbb{N}$. Then,

$$\| |\mathbb{I}_{L,\Delta} \rangle \langle \mathbb{I}_{L,\Delta}| - |\mathbb{I}_{L,\Delta}^\varepsilon \rangle \langle \mathbb{I}_{L,\Delta}^\varepsilon| \|_1 \leq 5\sqrt{\Delta}.$$

Proof. By Lemma G.6, we have

$$\begin{aligned}
|\langle \mathbb{I}_{L,\Delta}, \mathbb{I}_{L,\Delta}^\varepsilon \rangle|^2 &\geq 1 - 16\Delta^2 - 2e^{-(\varepsilon/\Delta)^2} \\
&\geq 1 - 16\Delta^2 - 2e^{-1/\Delta} \text{ by } \varepsilon \geq \sqrt{\Delta} \\
&\geq 1 - 16\Delta^2 - 2\Delta \text{ by } e^{-x} \leq x^{-1} \text{ for } x > 0 \\
&\geq 1 - 6\Delta \text{ by } \Delta < 1/4.
\end{aligned}$$

Using the relation between the trace distance and the overlap [cf. Eq. (C12)] gives the claim

$$\| |\mathbb{I}_{L,\Delta} \rangle \langle \mathbb{I}_{L,\Delta}| - |\mathbb{I}_{L,\Delta}^\varepsilon \rangle \langle \mathbb{I}_{L,\Delta}^\varepsilon| \|_1 \leq 2\sqrt{6\Delta} \leq 5\sqrt{\Delta}.$$

■

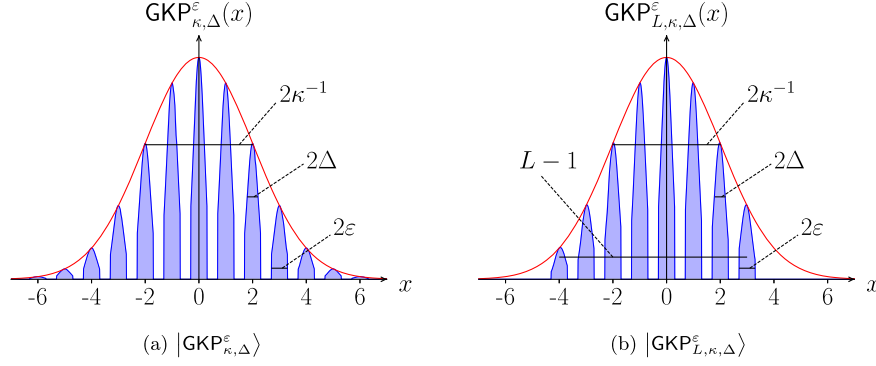


FIG. 16. Illustration of GKP states with truncated peaks $|\text{GKP}_{\kappa, \Delta}^\epsilon\rangle$ and with truncated peaks and compactly supported envelope $|\text{GKP}_{L, \kappa, \Delta}^\epsilon\rangle$.

c. Bounds on approximate GKP states

Let us recall the definition of the approximate $|\text{GKP}_{\kappa, \Delta}\rangle$ state with parameters κ and Δ [cf. Eqs. (D1) and (7)]:

$$|\text{GKP}_{\kappa, \Delta}\rangle := C_{\kappa, \Delta} \sum_{z \in \mathbb{Z}} \eta_{\kappa}(z) |\chi_{\Delta}(z)\rangle, \quad (\text{G26})$$

where $C_{\kappa, \Delta}$ is the normalization factor.

It is convenient to define approximate GKP states with truncated peaks $|\text{GKP}_{\kappa, \Delta}^\epsilon\rangle$ and approximate GKP states with truncated peaks and compactly supported envelope $|\text{GKP}_{L, \kappa, \Delta}^\epsilon\rangle$. We define these [for $\epsilon \in (0, 1/2)$ and $\kappa, \Delta > 0$] as

$$|\text{GKP}_{\kappa, \Delta}^\epsilon\rangle := C_{\kappa} \sum_{z \in \mathbb{Z}} \eta_{\kappa}(z) |\chi_{\Delta}^\epsilon(z)\rangle,$$

$$|\text{GKP}_{L, \kappa, \Delta}^\epsilon\rangle := C_{L, \kappa} \sum_{z=-L/2}^{L/2-1} \eta_{\kappa}(z) |\chi_{\Delta}^\epsilon(z)\rangle,$$

where C_{κ} and $C_{L, \kappa}$ are the respective normalization factors and where we use

$$[\chi_{\Delta}^\epsilon(z)](x) := \Psi_{\Delta}^\epsilon(x - z) \quad \text{and} \quad \Psi_{\Delta}^\epsilon := \frac{\Pi_{[-\epsilon, \epsilon]} \Psi_{\Delta}}{\|\Pi_{[-\epsilon, \epsilon]} \Psi_{\Delta}\|}$$

as in Eq. (C9).

We display these approximate GKP states in Fig. 16 and show by the following lemmas and subsequent corollaries that they are close to state $\text{GKP}_{\kappa, \Delta}$ for suitably chosen ϵ and L .

For future reference, observe that [by the orthogonality of the states $\{|\chi_{\Delta}^\epsilon(z)\rangle\}_{z \in \mathbb{Z}}$] the normalization constants satisfy

$$C_{\kappa}^{-2} = \sum_{k \in \mathbb{Z}} \eta_{\kappa}(k)^2, \quad (\text{G27})$$

$$C_{L, \kappa}^{-2} = \sum_{k=-L/2}^{L/2-1} \eta_{\kappa}(k)^2. \quad (\text{G28})$$

We can bound the quantities C_{κ}^{-2} , $C_{\kappa, \Delta}^{-2}$, and $C_{L, \kappa}^{-2}$ related to normalization constants by the following technical lemma, which we use repeatedly in our proofs.

Lemma G.8. Let $\kappa, \Delta > 0$. Then,

$$1 - \frac{\kappa}{\sqrt{\pi}} \leq C_{\kappa}^{-2} \leq 1 + \frac{\kappa}{\sqrt{\pi}}, \quad (\text{G29})$$

$$C_{\kappa, \Delta}^{-2} \leq 1 + \frac{\kappa}{\sqrt{\pi}} + 2(\sqrt{2\pi} + \kappa)\Delta, \quad (\text{G30})$$

$$\left(1 - 2e^{-(\kappa L/2)^2}\right) \cdot \left(1 + \frac{\kappa}{\sqrt{\pi}}\right) \leq C_{L, \kappa}^{-2}. \quad (\text{G31})$$

Furthermore, we have

$$1 - \frac{2(\sqrt{2\pi} + \kappa)}{1 - \kappa/\sqrt{\pi}} \Delta \leq \frac{C_{\kappa, \Delta}^2}{C_{\kappa}^2} \leq 1. \quad (\text{G32})$$

Proof. We obtain the lower bound in Eq. (G29) by using Lemma G.1 (with $c = \kappa^2$):

$$\begin{aligned} C_{\kappa}^{-2} &= \sum_{z \in \mathbb{Z}} \eta_{\kappa}(z)^2 \\ &= \frac{\kappa}{\sqrt{\pi}} \sum_{z \in \mathbb{Z}} e^{-\kappa^2 z^2} \\ &\geq \frac{\kappa}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{\kappa} - 1 \right) \\ &= 1 - \frac{\kappa}{\sqrt{\pi}}. \end{aligned}$$

We obtain the upper bound in Eq. (G29) analogously [by using Lemma G.1 (with $c = \kappa^2$)]:

$$C_{\kappa}^{-2} \leq 1 + \frac{\kappa}{\sqrt{\pi}}.$$

We get the upper bound on $C_{\kappa, \Delta}^{-2}$ in Eq. (G30) by first noting that

$$C_{\kappa,\Delta}^{-2} = \sum_{z,z' \in \mathbb{Z}} \eta_{\kappa}(z) \eta_{\kappa}(z') \langle \chi_{\Delta}(z), \chi_{\Delta}(z') \rangle = \sum_{z \in \mathbb{Z}} \eta_{\kappa}(z)^2 + \sum_{z \in \mathbb{Z}} \sum_{\substack{z' \in \mathbb{Z} \\ z' \neq z}} \eta_{\kappa}(z) \eta_{\kappa}(z') \langle \chi_{\Delta}(z), \chi_{\Delta}(z') \rangle = C_{\kappa}^{-2} + J, \quad (\text{G33})$$

where we define J as the second term in Eq. (G33). Because we have $J \geq 0$ [because each function $\chi_{\Delta}(z)$ is non-negative], the second inequality in Eq. (G32) follows.

Since $\langle \chi_{\Delta}(z), \chi_{\Delta}(z') \rangle = e^{-(z-z')^2/4\Delta^2}$, we have

$$J = \sum_{z \in \mathbb{Z}} \eta_{\kappa}(z) \sum_{k \in \mathbb{Z} \setminus \{0\}} \eta_{\kappa}(z-k) e^{-k^2/(4\Delta^2)}$$

by Lemma G.4. Since $\eta_{\kappa}(z') \leq \sqrt{\kappa}/\pi^{1/4}$ for any $z' \in \mathbb{Z}$, we obtain

$$\begin{aligned} J &\leq \sum_{z \in \mathbb{Z}} \eta_{\kappa}(z) \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\sqrt{\kappa}}{\pi^{1/4}} e^{-k^2/(2\Delta)^2} \\ &\leq \left(\sum_{z \in \mathbb{Z}} \eta_{\kappa}(z) \right) \left(\frac{\sqrt{\kappa}}{\pi^{1/4}} 2\sqrt{\pi}\Delta \right) \quad \text{by Lemma G.1 with } c = 1/(2\Delta)^2 \\ &\leq \frac{\kappa}{\sqrt{\pi}} \left(\frac{\sqrt{2\pi}}{\kappa} + 1 \right) 2\sqrt{\pi}\Delta \quad \text{by Lemma G.1 with } c = \kappa^2/2 \\ &= 2(\sqrt{2\pi} + \kappa)\Delta. \end{aligned}$$

We obtain the claim (G30) by inserting this and Eq. (G28) in Eq. (G33). This further gives

$$\frac{C_{\kappa,\Delta}^2}{C_{\kappa}^2} = \frac{C_{\kappa}^{-2}}{C_{\kappa,\Delta}^{-2}} = \frac{C_{\kappa}^{-2}}{C_{\kappa}^{-2} + J} = (1 + J/C_{\kappa}^2)^{-1} \geq 1 - J/C_{\kappa}^{-2}$$

by inequality $(1+x)^{-1} \geq 1-x$ for all $x > 0$. Thus,

$$\frac{C_{\kappa,\Delta}^2}{C_{\kappa}^2} \geq 1 - \frac{2(\sqrt{2\pi} + \kappa)\Delta}{1 - \kappa/\sqrt{\pi}}.$$

We get the upper bound on $C_{L,\kappa}^{-2}$ in Eq. (G31) as follows. By Eqs. (G27) and (G28), we can write

$$C_{L,\kappa}^{-2} = C_{\kappa}^{-2} - K, \quad (\text{G34})$$

where

$$K = \sum_{k=L/2+1}^{\infty} \eta_{\kappa}(-k)^2 + \sum_{k=L/2}^{\infty} \eta_{\kappa}(k)^2.$$

We have

$$\begin{aligned} K &\leq 2 \sum_{k=L/2}^{\infty} \eta_{\kappa}(k)^2 \quad \text{using } \eta_{\kappa}(-k) = \eta_{\kappa}(k) = \frac{2\kappa}{\sqrt{\pi}} \sum_{k=L/2}^{\infty} e^{-\kappa^2 k^2} \\ &= \frac{2\kappa}{\sqrt{\pi}} e^{-(\kappa L/2)^2} \sum_{k=L/2}^{\infty} e^{-\kappa^2 [k^2 - (L/2)^2]} \leq \frac{2\kappa}{\sqrt{\pi}} e^{-(\kappa L/2)^2} \sum_{k=L/2}^{\infty} e^{-\kappa^2 (k-L/2)^2} \quad \text{since } (k-a)^2 \leq k^2 - a^2 \quad \text{for } 0 < a \leq k \\ &= \frac{2\kappa}{\sqrt{\pi}} e^{-(\kappa L/2)^2} \sum_{r=0}^{\infty} e^{-\kappa^2 r^2}. \end{aligned} \quad (\text{G35})$$

By Lemma G.1 with $c = \kappa^2$, we have

$$\sum_{r=0}^{\infty} e^{-\kappa^2 r^2} = \frac{1}{2} \left(1 + \sum_{r=-\infty}^{\infty} e^{-\kappa^2 r^2} \right) \leq \left(1 + \frac{\sqrt{\pi}}{2\kappa} \right).$$

By inserting this into Eq. (G35), we obtain

$$K \leq \frac{2\kappa}{\sqrt{\pi}} e^{-(\kappa L/2)^2} \left(1 + \frac{\sqrt{\pi}}{2\kappa} \right) = e^{-(\kappa L/2)^2} \left(1 + \frac{2\kappa}{\sqrt{\pi}} \right) \leq 2e^{-(\kappa L/2)^2} \left(1 + \frac{\kappa}{\sqrt{\pi}} \right).$$

Inserting this bound and the bound from Eq. (G29) into Eq. (G34) gives the claim (G31). ■

Lemma G.9. Let $\kappa \in (0, 1/4)$, $\Delta > 0$, and $\varepsilon \in (0, 1/2)$. Then,

$$|\langle \text{GKP}_{\kappa, \Delta}, \text{GKP}_{\kappa, \Delta}^\varepsilon \rangle|^2 \geq 1 - 7\Delta - 2e^{-(\varepsilon/\Delta)^2}.$$

Proof. We have

$$\begin{aligned} \langle \text{GKP}_{\kappa, \Delta}, \text{GKP}_{\kappa, \Delta}^\varepsilon \rangle &= C_{\kappa, \Delta} C_\kappa \sum_{z, z' \in \mathbb{Z}} \eta_\kappa(z) \eta_\kappa(z') \langle \chi_\Delta(z), \chi_\Delta^\varepsilon(z') \rangle \\ &\geq C_{\kappa, \Delta} C_\kappa \sum_{z \in \mathbb{Z}} \eta_\kappa(z)^2 \langle \Psi_\Delta, \Psi_\Delta^\varepsilon \rangle \\ &= \frac{C_{\kappa, \Delta}}{C_\kappa} \langle \Psi_\Delta, \Psi_\Delta^\varepsilon \rangle \\ &\geq \frac{C_{\kappa, \Delta}}{C_\kappa} \left(1 - 2e^{-(\varepsilon/\Delta)^2} \right)^{1/2}, \end{aligned} \quad (\text{G36})$$

where the first inequality follows from non-negativity of $\chi_\Delta(z)(\cdot)$ and $\chi_\Delta^\varepsilon(z)(\cdot)$ and $\langle \chi_\Delta(z), \chi_\Delta^\varepsilon(z) \rangle = \langle \Psi_\Delta, \Psi_\Delta^\varepsilon \rangle$ for all $z \in \mathbb{Z}$. The last inequality is from Lemma G.2.

We use Lemma G.8 to obtain

$$\frac{C_{\kappa, \Delta}^2}{C_\kappa^2} \geq 1 - \frac{2(\sqrt{2\pi} + \kappa)}{1 - \kappa/\sqrt{\pi}} \Delta.$$

Thus, by the assumption $\kappa < 1/4$, we get

$$\frac{C_{\kappa, \Delta}^2}{C_\kappa^2} \geq 1 - 7\Delta.$$

Inserting this into the square of Eq. (G36) and using the inequality $(1-x)(1-y) \geq 1-x-y$ for $x, y \geq 0$ implies the claim. ■

Corollary G.10. Let $\kappa \in (0, 1/4)$, $\Delta > 0$, and $\varepsilon \in [\sqrt{\Delta}, 1/2)$. Then,

$$||| \text{GKP}_{\kappa, \Delta} \rangle \langle \text{GKP}_{\kappa, \Delta} | - | \text{GKP}_{\kappa, \Delta}^\varepsilon \rangle \langle \text{GKP}_{\kappa, \Delta}^\varepsilon | |||_1 \leq 6\sqrt{\Delta}.$$

Proof. By Lemma G.9 and the assumption $\varepsilon \geq \sqrt{\Delta}$, we have

$$\begin{aligned} |\langle \text{GKP}_{\kappa, \Delta}, \text{GKP}_{\kappa, \Delta}^\varepsilon \rangle|^2 &\geq 1 - 7\Delta - 2e^{-(\varepsilon/\Delta)^2} \\ &\geq 1 - 7\Delta - 2e^{-1/\Delta} \\ &\geq 1 - 9\Delta, \end{aligned}$$

where the last inequality follows from the inequality $e^{-x} \leq x^{-1}$ for $x > 0$. Using the relation between the trace distance and the overlap [cf. Eq. (C12)] gives the claim

$$\begin{aligned} ||| \text{GKP}_{\kappa, \Delta} \rangle \langle \text{GKP}_{\kappa, \Delta} | - | \text{GKP}_{\kappa, \Delta}^\varepsilon \rangle \langle \text{GKP}_{\kappa, \Delta}^\varepsilon | |||_1 &\leq 2\sqrt{9\Delta} \\ &= 6\sqrt{\Delta}. \end{aligned}$$

Lemma G.11. Let $\kappa \in (0, 1/4)$, $\Delta > 0$, $\varepsilon \in (0, 1/2)$, and $L \in 2\mathbb{N}$. Then,

$$|\langle \text{GKP}_{L, \kappa, \Delta}^\varepsilon, \text{GKP}_{\kappa, \Delta}^\varepsilon \rangle|^2 \geq 1 - 2e^{-\kappa^2 L^2/4}.$$

Proof. By the orthogonality of the states $\{|\chi_\Delta^\varepsilon(z)\rangle\}_{z \in \mathbb{Z}}$ for $\varepsilon < 1/2$ and by Eq. (G28), we have

$$\langle \text{GKP}_{L, \kappa, \Delta}^\varepsilon, \text{GKP}_{\kappa, \Delta}^\varepsilon \rangle = C_\kappa C_{L, \kappa} \sum_{k=-L/2}^{L/2-1} \eta_\kappa(k)^2 = \frac{C_\kappa}{C_{L, \kappa}}.$$

Thus, Lemma G.8 gives the claim

$$\begin{aligned} \frac{C_\kappa^2}{C_{L, \kappa}^2} &= \frac{C_{L, \kappa}^{-2}}{C_\kappa^{-2}} \geq \frac{\left(1 - 2e^{-(\kappa L/2)^2} \right) \cdot (1 + \kappa/\sqrt{\pi})}{1 + \kappa/\sqrt{\pi}} \\ &= 1 - 2e^{-(\kappa L/2)^2}. \end{aligned}$$

■
Corollary G.12. Assume $\kappa \in (0, 1/4)$ and $L \in 2\mathbb{N}$. We have

$$||| \text{GKP}_{L, \kappa, \Delta}^\varepsilon \rangle \langle \text{GKP}_{L, \kappa, \Delta}^\varepsilon | - | \text{GKP}_{\kappa, \Delta}^\varepsilon \rangle \langle \text{GKP}_{\kappa, \Delta}^\varepsilon | |||_1 \leq 3e^{-\kappa^2 L^2/8}.$$

Proof. By Lemma G.11 and the relation between the trace distance and the overlap [cf. Eq. (C12)], we have

$$\begin{aligned} ||| \text{GKP}_{L, \kappa, \Delta}^\varepsilon \rangle \langle \text{GKP}_{L, \kappa, \Delta}^\varepsilon | - | \text{GKP}_{\kappa, \Delta}^\varepsilon \rangle \langle \text{GKP}_{\kappa, \Delta}^\varepsilon | |||_1 \\ \leq 2\sqrt{2e^{-\kappa^2 L^2/4}} \leq 3e^{-\kappa^2 L^2/8}. \end{aligned}$$

■
Lemma G.13. Let $\kappa \in (0, 1/4)$, $\Delta > 0$, and $\varepsilon \in (0, 1/2)$. Then,

$$|\langle \text{GKP}_{\kappa, \Delta}^\varepsilon, e^{-iP} \text{GKP}_{\kappa, \Delta}^\varepsilon \rangle|^2 \geq 1 - 4\kappa.$$

Proof. From pairwise orthogonality of the states $\{|\chi_\Delta^\varepsilon(z)\rangle\}_{z \in \mathbb{Z}}$ and $e^{-iP}|\chi_\Delta^\varepsilon(z)\rangle = |\chi_\Delta^\varepsilon(z-1)\rangle$, we have

$$\begin{aligned} \langle \text{GKP}_{\kappa, \Delta}^\varepsilon, e^{-iP} \text{GKP}_{\kappa, \Delta}^\varepsilon \rangle &= C_\kappa^2 \sum_{z, z' \in \mathbb{Z}} \eta_\kappa(z) \eta_\kappa(z') \langle \chi_\Delta^\varepsilon(z), \chi_\Delta^\varepsilon(z'-1) \rangle \\ &= C_\kappa^2 \sum_{z \in \mathbb{Z}} \eta_\kappa(z) \eta_\kappa(z-1) \\ &= C_\kappa^2 e^{-\kappa^2/4} \sum_{z \in \mathbb{Z}} \eta_\kappa(z-1/2)^2, \end{aligned} \quad (\text{G37})$$

where the last step is by the definition of η_κ [cf. Eq. (D2)] that implies

$$\begin{aligned}
\eta_\kappa(z)\eta_\kappa(z-1) &= \frac{\kappa}{\sqrt{\pi}} e^{-\kappa^2 z^2/2} e^{-\kappa^2 (z-1)^2/2} \\
&= \frac{\kappa}{\sqrt{\pi}} e^{-\frac{\kappa^2}{2}(2(z-1/2)^2+1/2)} \\
&= e^{-\kappa^2/4} \eta_\kappa(z-1/2)^2.
\end{aligned}$$

By definition of η_κ and by using Lemma G.1 (with $c = \kappa^2$), we have

$$\sum_{z \in \mathbb{Z}} \eta_\kappa(z-1/2)^2 \geq \frac{\kappa}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{\kappa} - 1 \right) = 1 - \frac{\kappa}{\sqrt{\pi}}.$$

Combining Eq. (G37), the upper bound on C_κ^{-2} from Lemma G.8 gives

$$\begin{aligned}
\langle \text{GKP}_{\kappa,\Delta}^\varepsilon, e^{-iP} \text{GKP}_{\kappa,\Delta}^\varepsilon \rangle &\geq e^{-\kappa^2/4} \frac{1 - \kappa/\sqrt{\pi}}{1 + \kappa/\sqrt{\pi}} \\
&\geq e^{-\kappa^2/4} \left(1 - 2 \frac{\kappa}{\sqrt{\pi}} \right) \quad \text{by } \frac{1-x}{1+x} \geq 1-2x \quad \text{for } x > 0 \\
&\geq \left(1 - \frac{1}{4} \kappa^2 \right) \left(1 - \frac{2}{\sqrt{\pi}} \kappa \right) \quad \text{by } e^{-x} \geq 1-x \\
&\geq 1 - \frac{1}{4} \kappa^2 - \frac{2}{\sqrt{\pi}} \kappa \\
&\geq 1 - 2\kappa,
\end{aligned}$$

where we use the inequality $(1-x)(1-y) \geq 1-x-y$ for $x, y \geq 0$ on the penultimate line and the assumption $\kappa \leq 1/4$ on the last line. The claim follows by $(1-x)^2 \geq 1-2x$ for all $x \in \mathbb{R}$. ■

Lemma G.14. Let $\varepsilon \in (0, 1/2)$ and $\kappa > 0$. Then,

$$|\langle \text{GKP}_{\kappa,\Delta}^\varepsilon, e^{2\pi i Q} \text{GKP}_{\kappa,\Delta}^\varepsilon \rangle|^2 \geq 1 - 40\varepsilon^2.$$

Proof. Since for $\varepsilon < 1/2$ and $z, z' \in \mathbb{Z}$, $\chi_\Delta^\varepsilon(z)$ and $\chi_\Delta^\varepsilon(z')$ [respectively, $e^{2\pi i Q} \chi_\Delta^\varepsilon(z')$] have disjoint support for $z \neq z'$, we have

$$\begin{aligned}
\langle \chi_\Delta^\varepsilon(z'), e^{2\pi i Q} \chi_\Delta^\varepsilon(z) \rangle &= \delta_{z,z'} \int_{z-\varepsilon}^{z+\varepsilon} \overline{\chi_\Delta^\varepsilon(z)(x)} e^{2\pi i x} \chi_\Delta^\varepsilon(z)(x) dx \\
&= \delta_{z,z'} \int_{z-\varepsilon}^{z+\varepsilon} \overline{\Psi_\Delta^\varepsilon(x-z)} e^{2\pi i x} \Psi_\Delta^\varepsilon(x-z) dx \\
&= \delta_{z,z'} \int_{-\varepsilon}^{\varepsilon} |\Psi_\Delta^\varepsilon(x)|^2 e^{2\pi i x} dx \\
&= \delta_{z,z'} \int_{-\varepsilon}^{\varepsilon} |\Psi_\Delta^\varepsilon(x)|^2 \cos(2\pi x) dx,
\end{aligned} \tag{G38}$$

where Eq. (G38) follows from $\Psi_\Delta^\varepsilon(-x) = \Psi_\Delta^\varepsilon(x)$ and the fact that sinus is an odd function. Moreover, since cosine is an even function monotonically increasing on the interval $[-\pi, 0]$ and monotonically decreasing on the interval $[0, \pi]$,

we get

$$\begin{aligned}
\int_{-\varepsilon}^{\varepsilon} |\Psi_\Delta^\varepsilon(x)|^2 \cos(2\pi x) dx &\geq \cos(2\pi \varepsilon) \int_{-\varepsilon}^{\varepsilon} |\Psi_\Delta^\varepsilon(x)|^2 dx \\
&= \cos(2\pi \varepsilon) \\
&\geq 1 - 20\varepsilon^2,
\end{aligned}$$

where we use the bound $\cos x \geq 1 - x^2/2$ for all $x \in \mathbb{R}$ to obtain the last inequality. Therefore, we have

$$\begin{aligned}
&\langle \text{GKP}_{\kappa,\Delta}^\varepsilon, e^{2\pi i Q} \text{GKP}_{\kappa,\Delta}^\varepsilon \rangle \\
&= C_\kappa^2 \sum_{z,z' \in \mathbb{Z}} \eta_\kappa(z') \eta_\kappa(z) \langle \chi_\Delta^\varepsilon(z'), e^{2\pi i Q} \chi_\Delta^\varepsilon(z) \rangle \\
&\geq C_\kappa^2 \sum_{z \in \mathbb{Z}} \eta_\kappa(z)^2 (1 - 20\varepsilon^2) \\
&= C_\kappa^2 C_\kappa^{-2} (1 - 20\varepsilon^2) \\
&= 1 - 20\varepsilon^2,
\end{aligned}$$

where we use the identity for C_κ^{-2} from Eq. (G27) on the penultimate line. The claim follows using $(1-x)^2 \geq 1-2x$ for $x \in \mathbb{R}$. ■

Lemma G.15. Let $\kappa \in (0, 1/4)$, $\Delta \in (0, 1/4)$, and $\varepsilon \in (0, 1/2)$. Then,

$$\langle \text{GKP}_{\kappa,\Delta}^\varepsilon | \Pi_{[-R,R]} | \text{GKP}_{\kappa,\Delta}^\varepsilon \rangle \leq 4\kappa R + 10\kappa \quad \text{for any } R > 0.$$

Proof. Recall that the support of $\chi_\Delta^\varepsilon(z)$ is contained in the interval $[z - \varepsilon, z + \varepsilon]$ for all $z \in \mathbb{Z}$. Therefore, by definition of the state $|\text{GKP}_{\kappa,\Delta}^\varepsilon\rangle$, we have

$$\begin{aligned}
&\langle \text{GKP}_{\kappa,\Delta}^\varepsilon | \Pi_{[-R,R]} | \text{GKP}_{\kappa,\Delta}^\varepsilon \rangle \\
&= C_\kappa^2 \sum_{z \in \mathbb{Z}} \eta_\kappa(z)^2 \int_{-R}^R \chi_\Delta^\varepsilon(z)^2(x) dx \\
&\leq C_\kappa^2 \sum_{z=-\lceil R+1/2 \rceil}^{\lceil R+1/2 \rceil} \eta_\kappa(z)^2 \\
&\leq C_\kappa^2 \sum_{z=-\lceil R+1/2 \rceil}^{\lceil R+1/2 \rceil} \kappa \quad \text{since } \eta_\kappa(z)^2 \leq \kappa/\sqrt{\pi} < \kappa \\
&\leq C_\kappa^2 (2\kappa R + 5\kappa),
\end{aligned} \tag{G39}$$

where we use that $\lceil R+1/2 \rceil \leq R+2$ to obtain the last inequality. Combining Eq. (G39) and the lower bound on C_κ^{-2} from Lemma (G.2.3), we have

$$\begin{aligned}
\langle \text{GKP}_{\kappa,\Delta}^\varepsilon | \Pi_{[-R,R]} | \text{GKP}_{\kappa,\Delta}^\varepsilon \rangle &\leq \left(1 - \frac{\kappa}{\sqrt{\pi}} \right)^{-1} \cdot (2\kappa R + 5\kappa) \\
&\leq 3\kappa R + 6\kappa,
\end{aligned}$$

where we use $\kappa < 1/4$ in the last step. ■

d. Bounds on pointwise approximate GKP states

Recall the pointwise GKP state [cf. Eq. (D7)]:

$$\text{gkp}_{L,\kappa,\Delta}^\varepsilon(x) = D_{L,\kappa,\Delta}^\varepsilon \sum_{z=-L/2}^{L/2-1} \eta_\kappa(x) \chi_\Delta^\varepsilon(z)(x),$$

where $D_{L,\kappa,\Delta}^\varepsilon$ is a normalization factor.

It is convenient to define

$$I_k(y) := \int \eta_\kappa(x-y)^2 \chi_\Delta^\varepsilon(k)(x)^2 dx, \quad (\text{G40})$$

$$I'_k := \int \eta_\kappa(x) \eta_\kappa(k) \chi_\Delta^\varepsilon(k)(x)^2 dx \quad (\text{G41})$$

for $k \in \mathbb{N}$ and $y \in \mathbb{R}$.

We prove the following relations between the pointwise GKP states and the peakwise GKP states.

Lemma G.16. Let $\kappa > 0$, $\Delta > 0$, and $\varepsilon \in (0, 1/2)$. Then,

$$|\langle \text{gkp}_{L,\kappa,\Delta}^\varepsilon, \text{GKP}_{L,\kappa,\Delta}^\varepsilon \rangle|^2 \geq 1 - 4\kappa^2 \varepsilon L.$$

Proof. We note that the normalization constant $D_{L,\kappa,\Delta}^\varepsilon$ for $|\text{gkp}_{L,\kappa,\Delta}^\varepsilon\rangle$ is

$$D_{L,\kappa,\Delta}^\varepsilon = \left(\sum_{k_1=-L/2}^{L/2-1} \sum_{k_2=-L/2}^{L/2-1} M_{k_1,k_2} \right)^{-1/2},$$

where

$$M_{k_1,k_2} := \int \eta_\kappa(x)^2 \chi_\Delta^\varepsilon(k_1)(x) \chi_\Delta^\varepsilon(k_2)(x) dx.$$

Because of the factor $\chi_\Delta^\varepsilon(k_1)(x) \chi_\Delta^\varepsilon(k_2)$, k_1 and k_2 being integers and $\varepsilon < 1/2$, we have $M_{k_1,k_2} = 0$ unless $k_1 = k_2$. Furthermore, we have $M_{k,k} = I_k(0)$; thus,

$$D_{L,\kappa,\Delta}^\varepsilon = \left(\sum_{k=-L/2}^{L/2-1} I_k(0) \right)^{-1/2}. \quad (\text{G42})$$

We have

$$\langle \text{gkp}_{L,\kappa,\Delta}^\varepsilon, \text{GKP}_{L,\kappa,\Delta}^\varepsilon \rangle = C_{L,\kappa} D_{L,\kappa,\Delta}^\varepsilon \sum_{k_1=-L/2}^{L/2-1} \sum_{k_2=-L/2}^{L/2-1} I'_{k_1,k_2},$$

where

$$I'_{k_1,k_2} = \int \eta_\kappa(x) \eta_\kappa(k_2) \chi_\Delta^\varepsilon(k_1)(x) \chi_\Delta^\varepsilon(k_2)(x) dx.$$

The integral I'_{k_1,k_2} vanishes unless $k_1 = k_2$ because of the term $\chi_\Delta^\varepsilon(k_1)(x) \chi_\Delta^\varepsilon(k_2)(x)$, and it follows that

$$\langle \text{gkp}_{L,\kappa,\Delta}^\varepsilon, \text{GKP}_{L,\kappa,\Delta}^\varepsilon \rangle = C_{L,\kappa} D_{L,\kappa,\Delta}^\varepsilon \sum_{k=-L/2}^{L/2-1} I'_k, \quad (\text{G43})$$

where $I'_k = I'_{k,k}$ [see the definition (G41)].

Combining Eq. (G43) with Eqs. (G42) and (G27) gives

$$\begin{aligned} \langle \text{gkp}_{L,\kappa,\Delta}^\varepsilon, \text{GKP}_{L,\kappa,\Delta}^\varepsilon \rangle &= \left(\sum_{k=-L/2}^{L/2-1} \eta_\kappa(k)^2 \right)^{-1/2} \cdot \left(\sum_{k=-L/2}^{L/2-1} I_k(0) \right)^{-1/2} \cdot \sum_{k=-L/2}^{L/2-1} I'_k \\ &= \left(\frac{\sum_{k=-L/2}^{L/2-1} I_k(0)}{\sum_{k=-L/2}^{L/2-1} \eta_\kappa(k)^2} \right)^{1/2} \left(\frac{\sum_{k=-L/2}^{L/2-1} I'_k}{\sum_{k=-L/2}^{L/2-1} I_k(0)} \right). \end{aligned}$$

By using Lemmas G.23 and G.25, we obtain

$$\langle \text{gkp}_{L,\kappa,\Delta}^\varepsilon, \text{GKP}_{L,\kappa,\Delta}^\varepsilon \rangle \geq (1 - 2\kappa^2 \varepsilon L)^{1/2} (1 - \kappa^2 \varepsilon L/2) \geq 1 - 2\kappa^2 \varepsilon L,$$

where we use the inequality $(1 - 2x)^{1/2}(1 - x/2) \geq 1 - 2x$ for $0 \leq x \leq 1/2$. The claim follows using $(1 - x)^2 \geq 1 - 2x$ for all $x \in \mathbb{R}$. ■

Corollary G.17. Let $\kappa \in (0, 1/4)$, $\Delta \in (0, 1/4)$, and $\varepsilon \in (0, 1/2)$. Then,

$$\begin{aligned} &\| |\text{gkp}_{L,\kappa,\Delta}^\varepsilon\rangle \langle \text{gkp}_{L,\kappa,\Delta}^\varepsilon| - |\text{GKP}_{L,\kappa,\Delta}^\varepsilon\rangle \langle \text{GKP}_{L,\kappa,\Delta}^\varepsilon| \|_1 \\ &\leq 4\kappa\sqrt{\varepsilon L} \quad \text{and} \end{aligned} \quad (\text{G44})$$

$$\begin{aligned} &\| |\text{gkp}_{L,\kappa,\Delta}^\varepsilon\rangle \langle \text{gkp}_{L,\kappa,\Delta}^\varepsilon| - |\text{GKP}_{\kappa,\Delta}^\varepsilon\rangle \langle \text{GKP}_{\kappa,\Delta}^\varepsilon| \|_1 \\ &\leq 4\kappa\sqrt{\varepsilon L} + 3e^{-\kappa^2 L^2/8}. \end{aligned} \quad (\text{G45})$$

Furthermore, if $\varepsilon \in [\sqrt{\Delta}, 1/2)$, we have

$$\begin{aligned} &\| |\text{gkp}_{L,\kappa,\Delta}^\varepsilon\rangle \langle \text{gkp}_{L,\kappa,\Delta}^\varepsilon| - |\text{GKP}_{\kappa,\Delta}^\varepsilon\rangle \langle \text{GKP}_{\kappa,\Delta}^\varepsilon| \|_1 \\ &\leq 3\kappa\sqrt{L} + 6\sqrt{\Delta} + 3e^{-\kappa^2 L^2/8}. \end{aligned} \quad (\text{G46})$$

Proof. Equation (G44) is an immediate corollary of Lemma G.16 and the relation between the trace distance and the overlap [cf. Eq. (C12)].

Equation (G45) follows by application of the triangle inequality to Eq. (G44) and the bound in Corollary G.12.

Equation (G46) is obtained by application of the triangle inequality to Eq. (G45) and the bound in Corollary G.10. We have

$$\begin{aligned} & \| |\text{gkp}_{L,\kappa,\Delta}^\varepsilon\rangle \langle \text{gkp}_{L,\kappa,\Delta}^\varepsilon| - |\text{GKP}_{\kappa,\Delta}\rangle \langle \text{GKP}_{\kappa,\Delta}| \|_1 \\ & \leq 4\kappa\sqrt{\varepsilon L} + 6\sqrt{\Delta} + 4e^{-\kappa^2 L^2/8} \\ & \leq 3\kappa\sqrt{L} + 6\sqrt{\Delta} + 4e^{-\kappa^2 L^2/8}, \end{aligned}$$

where we use the assumption $\varepsilon < 1/2$ to obtain the last inequality. ■

e. Bounds on approximate GKP states in momentum space

In this section, we are concerned with the Fourier transform (cf. Appendix F2) of the wave function $|\text{GKP}_{\kappa,\Delta}\rangle \in L^2(\mathbb{R})$.

Lemma G.18. Let $\kappa > 0$ and $\Delta > 0$. Then, we have

$$\widehat{\text{GKP}}_{\kappa,\Delta}(p) = \sqrt{2\pi} C_{\kappa,\Delta} \sum_{z \in \mathbb{Z}} \eta_\Delta(p) \Psi_\kappa(p - 2\pi z).$$

Proof. We apply the Fourier transformation [cf. Eq. (F19)] to the definition of approximate GKP states in Eq. (G26). By linearity, we have

$$\widehat{\text{GKP}}_{\kappa,\Delta}(p) = C_{\kappa,\Delta} \hat{\Psi}_\Delta(p) \sum_{z \in \mathbb{Z}} \eta_\kappa(z) e^{-izp}.$$

We use the Poisson summation formula (see, e.g., Ref. [40]), which states that the following holds for a Schwarz function f :

$$\sum_{z \in \mathbb{Z}} f(z) = \sqrt{2\pi} \sum_{z \in \mathbb{Z}} \hat{f}(2\pi z).$$

By applying it with $f(z) = \eta_\kappa(z) e^{-ipz}$, we get

$$\begin{aligned} & \widehat{\text{GKP}}_{\kappa,\Delta}(p) \\ & = C_{\kappa,\Delta} \hat{\Psi}_\Delta(p) \left(\sqrt{2\pi} \sum_{z \in \mathbb{Z}} \hat{\eta}_\kappa(p + 2\pi z) \right) \text{ for } p \in \mathbb{R}. \end{aligned}$$

From the definition of Ψ_Δ and η_κ [see Eqs. (C2) and (D2)] along with

$$\frac{1}{\sqrt{2\pi}} \int e^{-cx^2/2} e^{-ipx} dx = \frac{1}{\sqrt{a}} e^{-p^2/(2a)},$$

we obtain (by setting a as $1/\Delta^2$ and κ^2 , respectively) that

$$\begin{aligned} \widehat{\text{GKP}}_{\kappa,\Delta}(p) & = \sqrt{2\pi} C_{\kappa,\Delta} \frac{\sqrt{\Delta}}{\sqrt{\pi\kappa}} e^{-\Delta^2 p^2/2} \sum_{z \in \mathbb{Z}} e^{-(p+2\pi z)^2/(2\kappa^2)} \\ & = \sqrt{2\pi} C_{\kappa,\Delta} \sum_{z \in \mathbb{Z}} \eta_\Delta(p) \Psi_\kappa(p - 2\pi z). \end{aligned}$$

■

From Lemma G.18, we see that, in momentum space, the roles of κ and Δ are interchanged; the spacing of the peaks is altered from 1 to 2π ; and the envelope is acting pointwise, similar to the wave function of the $|\text{gkp}_{\kappa,\Delta}\rangle$ state in position space.

In the remaining part of this section, we show properties of the truncated GKP state in momentum space $|\text{GKP}_{\kappa,\Delta}^\varepsilon\rangle$ [cf. Eq. (F24)], which we recall to be defined pointwise in momentum space as

$$\text{GKP}_{\kappa,\Delta}^\varepsilon(p) = D_{\kappa,\Delta}^\varepsilon \sum_{z \in \mathbb{Z}} \eta_\Delta(p) \chi_\kappa^\varepsilon(2\pi z)(p) \text{ for } p \in \mathbb{R}.$$

It is convenient to first bound the quantity $(D_{\kappa,\Delta}^\varepsilon)^{-2}$.

Lemma G.19. Let $\varepsilon \in (0, 1/2)$. Then,

$$\frac{1}{2\pi} (1 + 6\sqrt{\pi}\Delta) \geq (D_{\kappa,\Delta}^\varepsilon)^{-2} \geq \frac{1}{2\pi} (1 - 6\sqrt{\pi}\Delta). \quad (\text{G47})$$

Proof. We have that

$$\begin{aligned} (D_{\kappa,\Delta}^\varepsilon)^{-2} & = \sum_{z, z' \in \mathbb{Z}} \int_{-\infty}^{\infty} \eta_\Delta(p)^2 \chi_\kappa^\varepsilon(2\pi z)(p) \chi_\kappa^\varepsilon(2\pi z')(p) dp \\ & = \sum_{z \in \mathbb{Z}} \int_{2\pi z - \varepsilon}^{2\pi z + \varepsilon} \eta_\Delta(p)^2 \chi_\kappa^\varepsilon(2\pi z)(p)^2 dp, \end{aligned} \quad (\text{G48})$$

where we use that the functions $\chi_\kappa^\varepsilon(2\pi z)$ and $\chi_\kappa^\varepsilon(2\pi z')$ with $\varepsilon \in (0, 1/2)$ have disjoint support for $z \neq z'$. Note that

$$\eta_\Delta(p) \geq \eta_\Delta(2\pi|z| + \varepsilon) \text{ for } p \in [2\pi z - \varepsilon, 2\pi z + \varepsilon]. \quad (\text{G49})$$

By this and the expression $\eta_\Delta(x)^2 = \Delta/\sqrt{\pi} \cdot e^{-\Delta^2 x^2}$, we have

$$(D_{\kappa,\Delta}^\varepsilon)^{-2} \geq \sum_{z \in \mathbb{Z}} \eta_\Delta(2\pi|z| + \varepsilon)^2 = \frac{1}{2\pi} \sum_{z \in \mathbb{Z}} \eta_{2\pi\Delta} [|z| + \varepsilon/(2\pi)]^2.$$

Lemma G.1 [with $c = (2\pi\Delta)^2$] and the assumption $\varepsilon \in (0, 1/2)$ imply the lower bound of the claim (G47):

$$(D_{\kappa,\Delta}^\varepsilon)^{-2} \geq \frac{1}{2\pi} \frac{2\pi\Delta}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2\pi\Delta} - 3 \right) = \frac{1}{2\pi} (1 - 6\sqrt{\pi}\Delta).$$

We obtain the upper bound in the claim (G47) analogously as follows. Similarly to Eq. (G49), we have

$$\eta_\Delta(p) \leq \eta_\Delta(2\pi|z| - \varepsilon) \text{ for } p \in [2\pi z - \varepsilon, 2\pi z + \varepsilon], \quad z \in \mathbb{Z} \setminus \{0\}.$$

By this, by Eq. (G48), by the fact that $\eta_\Delta(p) \leq \eta_\Delta(0)$ for all p , and by the definition of $\eta_\Delta(\cdot)^2$, we have

$$\begin{aligned} (D_{\kappa,\Delta}^\varepsilon)^{-2} &\leq \eta_\Delta(0)^2 + \sum_{z \in \mathbb{Z} \setminus \{0\}} \eta_\Delta(2\pi|z| - \varepsilon)^2 \\ &= \eta_\Delta(0)^2 + \frac{1}{2\pi} \sum_{z \in \mathbb{Z} \setminus \{0\}} \eta_{2\pi\Delta} [|z| - \varepsilon/(2\pi)]^2. \end{aligned}$$

Therefore, by the bound on this sum from Lemma G.1 [with $c = (2\pi\Delta)^2$] and by the definition of $\eta_\Delta(\cdot)^2$, we have

$$\begin{aligned} (D_{\kappa,\Delta}^\varepsilon)^{-2} &\leq \eta_\Delta(0)^2 + \frac{1}{2\pi} \frac{2\pi\Delta}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2\pi\Delta} + 2 \right) \\ &= \frac{1}{2\pi} (1 + 6\sqrt{\pi}\Delta). \end{aligned}$$

Lemma G.20. Let $\kappa \in (0, 1/4)$, $\Delta \in (0, 1/100)$, and $\varepsilon \in (0, 1/2)$. Then,

$$\langle \text{GKP}_{\kappa,\Delta}^{\hat{\varepsilon}} | \hat{\Pi}_{[-R,R]} | \text{GKP}_{\kappa,\Delta}^{\hat{\varepsilon}} \rangle \leq 2\Delta R + 12\Delta$$

for any $R > 0$.

Proof. The support of $\chi_\kappa^\varepsilon(2\pi z)$ is contained in the interval $[2\pi z - \varepsilon, 2\pi z + \varepsilon]$ for all $z \in \mathbb{Z}$. Hence, by definition of the state $|\text{GKP}_{\kappa,\Delta}^{\hat{\varepsilon}}\rangle$, we have

$$\begin{aligned} &\langle \text{GKP}_{\kappa,\Delta}^{\hat{\varepsilon}} | \hat{\Pi}_{[-R,R]} | \text{GKP}_{\kappa,\Delta}^{\hat{\varepsilon}} \rangle \\ &= (D_{\kappa,\Delta}^\varepsilon)^2 \sum_{z \in \mathbb{Z}} \int_{-R}^R \eta_\Delta(p)^2 \chi_\kappa^\varepsilon(2\pi z)(p)^2 dp \\ &\leq (D_{\kappa,\Delta}^\varepsilon)^2 \sum_{z \in \mathbb{Z}} \int_{-2\pi[R/(2\pi)]-\varepsilon}^{2\pi[R/(2\pi)]+\varepsilon} \eta_\Delta(p)^2 \chi_\kappa^\varepsilon(2\pi z)(p)^2 dp \\ &= (D_{\kappa,\Delta}^\varepsilon)^2 \sum_{z=-[R/(2\pi)]}^{[R/(2\pi)]} \int_{2\pi z-\varepsilon}^{2\pi z+\varepsilon} \eta_\Delta(p)^2 \chi_\kappa^\varepsilon(2\pi z)(p)^2 dp. \end{aligned}$$

Notice that η_Δ is an even function monotonically decreasing on the interval $[0, \infty)$; it holds for $p \in [2\pi z - \varepsilon, 2\pi z + \varepsilon]$ and $z \in \mathbb{Z} \setminus \{0\}$ that

$$\eta_\Delta(p)^2 \leq \eta_\Delta(2\pi|z| - \varepsilon)^2 \leq \eta_\Delta[2\pi(|z| - 1)]^2.$$

Therefore,

$$\begin{aligned} \langle \text{GKP}_{\kappa,\Delta}^{\hat{\varepsilon}} | \hat{\Pi}_{[-R,R]} | \text{GKP}_{\kappa,\Delta}^{\hat{\varepsilon}} \rangle &\leq (D_{\kappa,\Delta}^\varepsilon)^2 \left(\eta_\Delta(0)^2 + 2 \sum_{z=1}^{[R/(2\pi)]} \eta_\Delta[2\pi(z-1)]^2 \int_{2\pi z-\varepsilon}^{2\pi z+\varepsilon} \chi_\kappa^\varepsilon(2\pi z)(p)^2 dp \right) \\ &= (D_{\kappa,\Delta}^\varepsilon)^2 \left(\eta_\Delta(0)^2 + 2 \sum_{z=1}^{[R/(2\pi)]} \eta_\Delta(2\pi(z-1))^2 \right) \\ &= (D_{\kappa,\Delta}^\varepsilon)^2 \left(\eta_\Delta(0)^2 + 2 \sum_{z=0}^{[R/(2\pi)]-1} \eta_\Delta(2\pi z)^2 \right), \end{aligned} \tag{G50}$$

where we use that $\chi_\kappa^\varepsilon(2\pi z)$ are normalized. Clearly, we have

$$\eta_\Delta(p)^2 \leq \eta_\Delta(0)^2 = \frac{\Delta}{\sqrt{\pi}} \quad \text{for all } p \in \mathbb{R}.$$

By using this bound on each term in the sum in Eq. (G50), we obtain

$$\langle \text{GKP}_{\kappa,\Delta}^{\hat{\varepsilon}} | \hat{\Pi}_{[-R,R]} | \text{GKP}_{\kappa,\Delta}^{\hat{\varepsilon}} \rangle \leq (D_{\kappa,\Delta}^\varepsilon)^2 \frac{\Delta}{\sqrt{\pi}} (1 + 2[R/(2\pi)]). \tag{G51}$$

Combining Eq. (G51), the fact that $[R/(2\pi)] \leq R/(2\pi) + 1$, and the bound from Lemma G.19 gives the claim

$$\langle \text{GKP}_{\kappa,\Delta}^{\hat{\varepsilon}} | \hat{\Pi}_{[-R+s,R+s]} | \text{GKP}_{\kappa,\Delta}^{\hat{\varepsilon}} \rangle \leq 2\pi(1 - 6\sqrt{\pi}\Delta)^{-1} \cdot \left(\frac{\Delta R}{\pi^{3/2}} + \frac{3\Delta}{\sqrt{\pi}} \right) \leq 2\Delta R + 12\Delta,$$

where we use $\Delta < 1/100$ in the last step. ■

Lemma G.21. Let $\kappa \in (0, 1/4)$, $\Delta \in (0, 1/4)$, and $\varepsilon \in [\sqrt{\Delta}, 1/2)$. Then,

$$\left| \langle \text{GKP}_{\kappa, \Delta}, \text{GKP}_{\kappa, \Delta}^{\hat{\varepsilon}} \rangle \right|^2 \geq 1 - 3\kappa - 39\Delta.$$

Proof. By Lemma G.18 and by the fact that the Fourier transformation is unitary, we get

$$\begin{aligned} \langle \text{GKP}_{\kappa, \Delta}, \text{GKP}_{\kappa, \Delta}^{\hat{\varepsilon}} \rangle &= \sqrt{2\pi} C_{\Delta, \kappa} D_{\kappa, \Delta}^{\varepsilon} \sum_{z, z' \in \mathbb{Z}} \int \eta_{\Delta}(p)^2 \chi_{\kappa}(2\pi z)(p) \chi_{\kappa}^{\varepsilon}(2\pi z')(p) dp \\ &\geq \sqrt{2\pi} C_{\Delta, \kappa} D_{\kappa, \Delta}^{\varepsilon} \sum_{z \in \mathbb{Z}} \int \eta_{\Delta}(p)^2 \chi_{\kappa}(2\pi z)(p) \chi_{\kappa}^{\varepsilon}(2\pi z)(p) dp, \end{aligned}$$

where we use the non-negativity of the integrands. We have

$$\begin{aligned} \int \eta_{\Delta}(p)^2 \chi_{\kappa}(2\pi z)(p) \chi_{\kappa}^{\varepsilon}(2\pi z)(p) dp &= \int \eta_{\Delta}(p)^2 \Psi_{\kappa}(p - 2\pi z) \Psi_{\kappa}^{\varepsilon}(p - 2\pi z) dp \\ &= \int_{-\varepsilon}^{\varepsilon} \eta_{\Delta}(u + 2\pi z)^2 \Psi_{\kappa}(u) \Psi_{\kappa}^{\varepsilon}(u) du, \end{aligned}$$

where we use that the support of $\Psi_{\kappa}^{\varepsilon}$ is contained in $[-\varepsilon, \varepsilon]$ and substituted $u := p - 2\pi z$ in the last step. Because of the symmetry and monotonicity of η_{Δ} , we infer that

$$\eta_{\Delta}(u + 2\pi z) \geq \eta_{\Delta}(2\pi|z| + \varepsilon) \quad \text{for } u \in [-\varepsilon, \varepsilon], \quad z \in \mathbb{Z}.$$

Therefore, we can bound

$$\begin{aligned} \langle \text{GKP}_{\kappa, \Delta}, \text{GKP}_{\kappa, \Delta}^{\hat{\varepsilon}} \rangle &\geq \sqrt{2\pi} C_{\kappa, \Delta} D_{\kappa, \Delta}^{\varepsilon} \sum_{z \in \mathbb{Z}} \eta_{\Delta}(2\pi|z| + \varepsilon)^2 \int_{-\varepsilon}^{\varepsilon} \Psi_{\kappa}(p) \Psi_{\kappa}^{\varepsilon}(p) dp \\ &= \sqrt{2\pi} C_{\kappa, \Delta} D_{\kappa, \Delta}^{\varepsilon} \sum_{z \in \mathbb{Z}} \eta_{\Delta}(2\pi|z| + \varepsilon)^2 \langle \Psi_{\kappa}, \Psi_{\kappa}^{\varepsilon} \rangle. \end{aligned} \quad (\text{G52})$$

We bound the squares of the four factors in Eq. (G52) separately. By Lemma G.8 and by inequality $(1+x)^{-1} \geq 1-x$ for $x > 0$, we have

$$C_{\kappa, \Delta}^2 \geq 1 - \frac{\kappa}{\sqrt{\pi}} - 2(\sqrt{2\pi} + \kappa)\Delta \geq 1 - \kappa - 6\Delta, \quad (\text{G53})$$

where we use the assumption $\kappa < 1/4$ in the last inequality. By Lemma G.19, we have

$$(D_{\kappa, \Delta}^{\varepsilon})^2 \geq 2\pi(1 + 6\sqrt{\pi}\Delta)^{-1} \geq 2\pi(1 - 6\sqrt{\pi}\Delta) \geq 2\pi(1 - 11\Delta),$$

where we use $(1+x)^{-1} \geq 1-x$ for all $x > 0$. By the definition of $\eta_{\Delta}(\cdot)$, $\varepsilon \in (0, 1/2)$, and by Lemma G.1 (with $c = 2\pi\Delta$), we have

$$\begin{aligned} \sum_{z \in \mathbb{Z}} \eta_{\Delta}(2\pi|z| + \varepsilon)^2 &= \frac{1}{2\pi} \sum_{z \in \mathbb{Z}} \eta_{2\pi\Delta}(|z| + \varepsilon/(2\pi))^2 \geq \frac{1}{2\pi} \frac{2\pi\Delta}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2\pi\Delta} - 3 \right) \\ &= \frac{1}{2\pi} (1 - 6\sqrt{\pi}\Delta) \geq \frac{1}{2\pi} (1 - 11\Delta). \end{aligned}$$

Thus,

$$\left(\sum_{z \in \mathbb{Z}} \eta_{\Delta}(2\pi|z| + \varepsilon)^2 \right)^2 \geq \frac{1}{(2\pi)^2} (1 - 12\sqrt{\pi}\Delta) \geq \frac{1}{(2\pi)^2} (1 - 22\Delta).$$

Finally, by Lemma G.2, we have for $\varepsilon \in [\sqrt{\kappa}, 1/2)$ that

$$|\langle \Psi_{\kappa}, \Psi_{\kappa}^{\varepsilon} \rangle|^2 \geq 1 - 2\kappa. \quad (\text{G54})$$

Combining the bounds from Eq. (G52) with Eqs. (G53)–(G54) and repeatedly using $(1-x)(1-y) \geq 1-x-y$ for $x, y \geq 0$ gives

$$\left| \langle \text{GKP}_{\kappa, \Delta}, \text{GKP}_{\kappa, \Delta}^{\hat{\varepsilon}} \rangle \right|^2 \geq (1 - \kappa - 6\Delta)(1 - 11\Delta)(1 - 22\Delta)(1 - 2\kappa) \geq 1 - 3\kappa - 39\Delta.$$

This is the claim. ■

Corollary G.22. Let $\kappa \in (0, 1/4)$, $\Delta \in (0, 1/4)$, and $\varepsilon \in [\sqrt{\Delta}, 1/2)$. Then,

$$\left\| |\text{GKP}_{\kappa, \Delta}\rangle \langle \text{GKP}_{\kappa, \Delta}| - |\text{GKP}_{\kappa, \Delta}^{\hat{\varepsilon}}\rangle \langle \text{GKP}_{\kappa, \Delta}^{\hat{\varepsilon}}| \right\|_1 \leq 4\sqrt{\kappa} + 13\sqrt{\Delta}.$$

Proof. From Lemma G.21 and from the relation between the trace distance and the overlap [cf. Eq. (C12)], we have

$$\left\| |\text{GKP}_{\kappa, \Delta}\rangle \langle \text{GKP}_{\kappa, \Delta}| - |\text{GKP}_{\kappa, \Delta}^{\hat{\varepsilon}}\rangle \langle \text{GKP}_{\kappa, \Delta}^{\hat{\varepsilon}}| \right\|_1 \leq 2\sqrt{3\kappa} + 2\sqrt{39\Delta} \leq 4\sqrt{\kappa} + 13\sqrt{\Delta},$$

where we use $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ for $x, y \geq 0$. ■

3. Bounds on convolutions

In this section, we prove bounds on and relations between convolutions of Gaussians with summands (indexed by $k \in \mathbb{Z}$) of the following form [cf. Eqs. (G40) and (G41), where we use the definition of translated Gaussians (C9)]:

$$I_k(y) = \int \eta_{\kappa}(x-y)^2 \Psi_{\Delta}^{\varepsilon}(x-k)^2 dx, \quad \text{where } y \in \mathbb{R}, \text{ and}$$

$$I'_k = \int \eta_{\kappa}(x) \eta_{\kappa}(k) \Psi_{\Delta}^{\varepsilon}(x-k)^2 dx.$$

For future reference, we observe that, for any integer $m \in \mathbb{Z}$ and $\delta \in \mathbb{R}$, it holds by the variable substitution $z := x - m$ that

$$\begin{aligned} I_k(m + \delta) &= \int \eta_{\kappa}[x - (m + \delta)]^2 \Psi_{\Delta}^{\varepsilon}(x-k)^2 dx \\ &= \int \eta_{\kappa}(z - \delta)^2 \Psi_{\Delta}^{\varepsilon}[z - (k - m)]^2 dz \\ &= I_{k-m}(\delta). \end{aligned} \quad (\text{G55})$$

Furthermore, by the variable substitution $z := x - k$, we obtain the identities

$$I_k(s) = \int_{-\varepsilon}^{\varepsilon} \eta_{\kappa}[(k+z) - s]^2 \Psi_{\Delta}^{\varepsilon}(z)^2 dz, \quad (\text{G56})$$

$$I'_k = \int_{-\varepsilon}^{\varepsilon} \eta_{\kappa}(k+z) \eta_{\kappa}(k) \Psi_{\Delta}^{\varepsilon}(z)^2 dz. \quad (\text{G57})$$

In our proofs, we frequently use the following two identities regarding the product of shifted Gaussians:

$$\begin{aligned} \eta_{\kappa}(k+z)^2 &= \frac{\kappa}{\sqrt{\pi}} e^{-\kappa^2(k+z)^2} \\ &= \frac{\kappa}{\sqrt{\pi}} e^{-\kappa^2 k^2} \cdot e^{-2\kappa^2 kz} \cdot e^{-\kappa^2 z^2} \\ &= \eta_{\kappa}(k)^2 \cdot e^{-2\kappa^2 kz} \cdot e^{-\kappa^2 z^2} \end{aligned} \quad (\text{G58})$$

and

$$\begin{aligned} \eta_{\kappa}(k+z) \eta_{\kappa}(k) &= \frac{\kappa}{\sqrt{\pi}} e^{-\kappa^2(k+z)^2/2} \cdot e^{-\kappa^2 k^2/2} \\ &= \frac{\kappa}{\sqrt{\pi}} e^{-\kappa^2(k+z)^2} \cdot e^{\kappa^2(k+z)^2/2 - \kappa^2 k^2/2} \\ &= \frac{\kappa}{\sqrt{\pi}} e^{-\kappa^2(k+z)^2} \cdot e^{\kappa^2 kz + \kappa^2 z^2/2} \\ &= \eta_{\kappa}(k+z)^2 \cdot e^{\kappa^2 kz} \cdot e^{\kappa^2 z^2/2}. \end{aligned} \quad (\text{G59})$$

We prove the following statements.

Lemma G.23. Let $I_k(0)$ be as in Eq. (G40) with $\kappa > 0$, $\Delta > 0$, and $\varepsilon \in (0, 1/2)$. Let $L \in 2\mathbb{N}$. Then,

$$\sum_{k=-L/2}^{L/2-1} I_k(0) \geq (1 - 2\kappa^2 \varepsilon L) \cdot \sum_{k=-L/2}^{L/2-1} \eta_{\kappa}(k)^2.$$

Proof. We prove the claim by bounding integrands in $I_k(0)$ [cf. identity (G56) with $s = 0$]. We use that

$$e^{-2\kappa^2 kz} \cdot e^{-\kappa^2 z^2} \geq e^{-\kappa^2 L \varepsilon} \cdot e^{-\kappa^2 \varepsilon^2}$$

for $z \in (-\varepsilon, \varepsilon)$ and $k \in \{-L/2, \dots, L/2 - 1\}$ together with Eq. (G58) to obtain

$$\eta_{\kappa}(k+z)^2 \geq \eta_{\kappa}(k)^2 \cdot e^{-\kappa^2 \varepsilon(L+\varepsilon)}.$$

By inserting this lower bound into the identity for $I_k(y)$ in Eq. (G56), we obtain (for $k \in \{-L/2, \dots, L/2 - 1\}$)

$$\begin{aligned} I_k(0) &\geq e^{-\kappa^2 \varepsilon(L+\varepsilon)} \eta_\kappa(k)^2 \int_{-\varepsilon}^{\varepsilon} \Psi_\Delta^\varepsilon(x)^2 dz \\ &= e^{-\kappa^2 \varepsilon(L+\varepsilon)} \eta_\kappa(k)^2, \end{aligned}$$

where we use the fact that $\Psi_\Delta^\varepsilon \in L^2(\mathbb{R})$ is normalized and has support on $(-\varepsilon, \varepsilon)$. Summing over $k \in \{-L/2, \dots, L/2 - 1\}$ yields

$$\sum_{k=-L/2}^{L/2-1} I_k(0) \geq e^{-\kappa^2 \varepsilon(L+\varepsilon)} \sum_{k=-L/2}^{L/2-1} \eta_\kappa(k)^2. \quad (\text{G60})$$

By the inequality $e^{-x} \geq 1 - x$ for $x \geq 0$ and by the assumption on ε , we obtain

$$\begin{aligned} e^{-\kappa^2 \varepsilon(L+\varepsilon)} &\geq 1 - \kappa^2 \varepsilon(L + \varepsilon) \\ &\geq 1 - 2\kappa^2 \varepsilon L, \end{aligned}$$

which together with Eq. (G60) proves the claim. $\blacksquare$

Lemma G.24. Let $I_k(0)$ be as defined in Eq. (G40) with $\kappa > 0$, $\Delta > 0$, and $\varepsilon \in (0, 1/2)$. Let $L \in 2\mathbb{N}$ such that $L \geq 4$. We have

$$\sum_{k=-L/2}^{L/2-1} I_k(0) \geq (1 - e^{-\kappa^2 L^2/8}) \sum_{k=-L}^{L-1} I_k(0).$$

Proof. We prove the claim by comparing integrands in $I_k(0)$ [cf. identity (G56)]. Since both $\eta_\kappa(\cdot)$ and $\Psi_\Delta^\varepsilon(\cdot)$ are even functions, we have from Eq. (G40) by the variable substitution $z' = -z$ that $I_k(0)$ is even in k . Indeed,

$$\begin{aligned} I_k(0) &= \int_{-\varepsilon}^{\varepsilon} \eta_\kappa(k+z)^2 \Psi_\Delta^\varepsilon(z)^2 dz \\ &= \int_{-\varepsilon}^{\varepsilon} \eta_\kappa(-k+z')^2 \Psi_\Delta^\varepsilon(z')^2 dz' \\ &= I_{-k}(0). \end{aligned} \quad (\text{G61})$$

By Eq. (G58) (with $m+z$ in place of k and $L/2$ in place of z), we have for $m \geq 0$ and $z \in (-\varepsilon, \varepsilon)$ that

$$\begin{aligned} \eta_\kappa(L/2 + m + z)^2 &= \eta_\kappa(m+z)^2 \cdot e^{-\kappa^2 m L/2} \cdot e^{-\kappa^2 z L/2} \cdot e^{-\kappa^2 L^2/4} \\ &\leq \eta_\kappa(m+z)^2 \cdot e^{\kappa^2 \varepsilon L/2} \cdot e^{-\kappa^2 L^2/4} \\ &\leq \eta_\kappa(m+z)^2 \cdot e^{-\kappa^2 (L^2/4 - \varepsilon L/2)} \\ &\leq \eta_\kappa(m+z)^2 \cdot e^{-\kappa^2 L^2/8}, \end{aligned} \quad (\text{G62})$$

where we use that $\kappa^2 m L/2 \geq 0$ and $z \geq -\varepsilon$ on the second line and the following inequality $\varepsilon L/2 \leq L/2 \leq (L/2) \cdot$

$(L/4) = L^2/8$ for $L \geq 4$ on the last line. Inserting the bound from Eq. (G62) into the identity for $I_k(y)$ from Eq. (G56) gives

$$I_{L/2+m}(0) \leq e^{-\kappa^2 L^2/8} I_m(0) \quad \text{for } m \geq 0.$$

Since $I_k(0)$ is even in k [cf. Eq. (G61)], we also have

$$I_{-L/2-m}(0) \leq e^{-\kappa^2 L^2/8} I_{-m}(0) \quad \text{for } m \geq 0.$$

In particular, this implies

$$\begin{aligned} \sum_{k=-L}^{L-1} I_k(0) &= \sum_{k=-L/2}^{L/2-1} I_k(0) + \sum_{k=L/2+1}^{L-1} I_{-k}(0) + \sum_{k=L/2}^{L-1} I_k(0) \\ &\leq \sum_{k=-L/2}^{L/2} I_k(0) + e^{-\kappa^2 L^2/8} \sum_{m=1}^{L/2-1} I_{-m}(0) \\ &\quad + e^{-\kappa^2 L^2/8} \cdot \sum_{m=0}^{L/2-1} I_m(0) \\ &= \sum_{k=-L/2}^{L/2} I_k(0) + e^{-\kappa^2 L^2/8} \sum_{k=-L/2+1}^{L/2-1} I_k(0) \\ &\leq \left(1 + e^{-\kappa^2 L^2/16}\right) \cdot \sum_{k=-L/2}^{L/2} I_k(0). \end{aligned}$$

In the last inequality, we use that $I_k(0) \geq 0$ for every $k \in \mathbb{Z}$. Rewriting this as

$$\left(1 + e^{-\kappa^2 L^2/8}\right)^{-1} \sum_{k=-L}^{L-1} I_k(0) \leq \sum_{k=-L/2}^{L/2} I_k(0)$$

and using that $1 - x \leq (1 + x)^{-1}$ for $x \in (0, 1)$ gives the claim. $\blacksquare$

Lemma G.25. Let $I_k(0)$ be as defined in Eq. (G40) with $\kappa > 0$, $\Delta > 0$, and $\varepsilon \in (0, 1/2)$. Let $L \geq 0$.

$$\sum_{k=-L/2}^{L/2-1} I'_k \geq (1 - \kappa^2 \varepsilon L/2) \sum_{k=-L/2}^{L/2-1} I_k(0).$$

Proof. We prove the claim by comparing integrands in I'_k and $I_k(0)$ [cf. identities (G57) and (G56)]. By the identity (G59), we have

$$\begin{aligned} \eta_\kappa(k+z) \eta_\kappa(k) &= \eta_\kappa(k+z)^2 \cdot e^{\kappa^2 k z} \cdot e^{\kappa^2 z^2/2} \\ &\geq \eta_\kappa(k+z)^2 \cdot e^{\kappa^2 k z}, \end{aligned}$$

where we use that $e^{\kappa^2 z^2/2} \geq 1$ that holds for $\kappa > 0$ and $z \in \mathbb{R}$. Furthermore, for $k \in \{-L/2, \dots, L/2 - 1\}$ and $z \in (-\varepsilon, \varepsilon)$, we have

$$e^{\kappa^2 k z} \geq e^{-\kappa^2 L \varepsilon / 2}.$$

It follows that (for this choice of z and k)

$$\eta_\kappa(k+z)\eta_\kappa(k) \geq \eta_\kappa(k+z)^2 e^{-\kappa^2 L \varepsilon / 2}. \quad (\text{G63})$$

Inserting Eq. (G63) into Eq. (G57), we get (for $k \in \{-L/2, \dots, L/2-1\}$)

$$\begin{aligned} I'_k &\geq e^{-\kappa^2 L \varepsilon / 2} \int_{-\varepsilon}^{\varepsilon} \Psi_\Delta^\varepsilon(z)^2 \eta_\kappa(k+z)^2 dz \\ &= e^{-\kappa^2 L \varepsilon / 2} I_k(0), \end{aligned}$$

where we use the identity (G56) on the last line. Summing over $k \in \{-L/2, \dots, L/2-1\}$ and using the inequality $e^{-x} \geq 1-x$ for $x \geq 0$ gives the claim. ■

Lemma G.26. Let $\kappa \in \mathbb{R}$, $\varepsilon \in [0, 1/2)$, and $L \in 2\mathbb{N}$. Let $m \in \{-L/4, \dots, L/4\}$ and $s \in (-1/2, 1/2]$. Then,

$$\sum_{k=-L/2-m}^{L/2-1-m} I_k(s) \leq e^{\kappa^2 L} \sum_{k=-L/2-m}^{L/2-1-m} I_k(0).$$

Proof. We prove the claim by comparing integrands in $I_k(s)$ [cf. identity (G56)]. By Eq. (G58), we have

$$\begin{aligned} \eta_\kappa[(k+z)-s]^2 &= \eta_\kappa(k+z)^2 \cdot e^{2\kappa^2(k+z)s} \cdot e^{-\kappa^2 s^2} \\ &\leq \eta_\kappa(k+z)^2 \cdot e^{2\kappa^2 k s} \cdot e^{2\kappa^2 z s}, \end{aligned} \quad (\text{G64})$$

where we use $e^{-\kappa^2 s^2} \leq 1$ (for $s, \kappa \in \mathbb{R}$) on the last line. Since for $s \in (-1/2, 1/2]$ we have

$$\max_{k \in \{-L/2-m, \dots, L/2-1-m\}} e^{2\kappa^2 k s} \leq e^{\kappa^2 (L/2+|m|)} \quad \text{for any } m \in \mathbb{Z},$$

we obtain

$$e^{2\kappa^2 k s} \leq e^{3\kappa^2 L/4}$$

for any $m \in \{-L/4, \dots, L/4\}$, $k \in \{-L/2-m, \dots, L/2-1-m\}$. Furthermore, we have

$$e^{2\kappa^2 z s} \leq e^{\kappa^2 \varepsilon} \quad \text{for any } z \in (-\varepsilon, \varepsilon) \text{ and } s \in (-1/2, 1/2].$$

Inserting the previous two inequalities into Eq. (G64), we conclude that, for such m, k, z , and s , we have

$$\eta_\kappa[(k+z)-s]^2 \leq \eta_\kappa(k+z)^2 \cdot e^{3\kappa^2 L/4} \cdot e^{\kappa^2 \varepsilon}. \quad (\text{G65})$$

Inserting Eq. (G65) into Eq. (G56), we obtain

$$\begin{aligned} I_k(s) &\leq e^{3\kappa^2 L/4 + \kappa^2 \varepsilon} \cdot \int_{-\varepsilon}^{\varepsilon} \eta_\kappa(k+z)^2 \Psi_\Delta^\varepsilon(z)^2 dz \\ &= e^{\kappa^2 (3L/4 + \varepsilon)} \cdot I_k(0) \\ &\leq e^{\kappa^2 L} I_k(0), \end{aligned} \quad (\text{G66})$$

whenever $k \in \{-L/2-m, \dots, L/2-1-m\}$ with $m \in \{-L/4, \dots, L/4\}$. The claim follows by summing Eq. (G66) over $k \in \{-L/2-m, \dots, L/2-1-m\}$. ■

APPENDIX H: EFFECTIVE SQUEEZING PARAMETERS

In this section, we bound the effective squeezing parameter, a metric to quantify the quality of GKP states (see Refs. [18,32]). Let us define the unitaries $S_P = e^{-iP}$ and $S_Q = e^{2\pi i Q}$. We want to quantify how invariant a state is under these unitaries. To do so, the effective squeezing parameters Δ_P and Δ_Q are introduced, respectively, as

$$\begin{aligned} \Delta_P(\rho) &:= \sqrt{\log(1/|\langle S_P \rangle_\rho|^2)} \quad \text{and} \\ \Delta_Q(\rho) &:= \sqrt{\log(1/|\langle S_Q \rangle_\rho|^2)}, \end{aligned}$$

where we define $\langle U \rangle_\rho := \text{tr}(U\rho)$ for a unitary U . The motivation of these quantities is the case of the ideal GKP state

$$|\text{GKP}\rangle \propto \sum_{z \in \mathbb{Z}} |z\rangle,$$

which has stabilizers S_P and S_Q and for which we, thus, have $\Delta_P(\text{GKP}) = \Delta_Q(\text{GKP}) = 0$. We show the following.

Lemma H.1. Let $\rho \in \mathcal{B}[L^2(\mathbb{R})]$ be a quantum state prepared by Protocol 1 (cf. Theorem E.1). For κ, Δ sufficiently small, it holds that

$$\begin{aligned} \Delta_P &\leq 39\Delta^{1/4} + 16\kappa^{1/6}, \\ \Delta_Q &\leq 41\Delta^{1/4} + 16\kappa^{1/6}. \end{aligned}$$

Proof. Note that for a unitary U and two quantum states ρ and σ , we have

$$|\text{tr}[U\rho]| \geq |\text{tr}[U\sigma]| - |\text{tr}[U(\rho - \sigma)]| \geq |\text{tr}[U\sigma]| - \|\rho - \sigma\|_1.$$

The first inequality uses the triangle inequality; the second uses Hölder's inequality, implying that, for any trace class operator X and any unitary U , we have $|\text{tr}[U X]| \leq \|X\|_1$. In particular, we conclude

$$|\langle U \rangle_\rho|^2 \geq |\langle U \rangle_\sigma|^2 - 2\|\rho - \sigma\|_1. \quad (\text{H1})$$

By definition, $S_P = e^{-iP}$ and $S_Q = e^{2\pi i Q}$. Because of Lemma G.13, we have

$$|\langle \text{GKP}_{\kappa,\Delta}^\varepsilon, e^{-iP} \text{GKP}_{\kappa,\Delta}^\varepsilon \rangle|^2 \geq 1 - 4\kappa.$$

Moreover, setting $\varepsilon = \sqrt{\Delta}$, we infer from Corollary G.10 that

$$||| \text{GKP}_{\kappa,\Delta}^\varepsilon \rangle \langle \text{GKP}_{\kappa,\Delta}^\varepsilon | - | \text{GKP}_{\kappa,\Delta} \rangle \langle \text{GKP}_{\kappa,\Delta} | |||_1 \leq 6\sqrt{\Delta}. \quad (\text{H2})$$

Finally, we have by Theorem E.1 that

$$||\rho - | \text{GKP}_{\kappa,\Delta} \rangle \langle \text{GKP}_{\kappa,\Delta} |||_1 \leq 190\sqrt{\Delta} + 24\kappa^{1/3}. \quad (\text{H3})$$

Thus, by the triangle inequality applied to Eqs. (H2) and (H3), we infer

$$||\rho - | \text{GKP}_{\kappa,\Delta}^\varepsilon \rangle \langle \text{GKP}_{\kappa,\Delta}^\varepsilon |||_1 \leq 190\sqrt{\Delta} + 30\kappa^{1/3}. \quad (\text{H4})$$

Hence, by Eq. (H1) along with Eq. (H4),

$$|\langle S_P \rangle_\rho|^2 \geq 1 - 380\sqrt{\Delta} - 64\kappa^{1/3}.$$

As $\log[(1-x)^{-1}] = -\log(1-x) \leq 2x$ for $0 \leq x \leq 1/2$, we deduce for small enough Δ and κ that

$$\Delta_P(\rho) \leq 2\sqrt{380\sqrt{\Delta} + 64\kappa^{1/3}} \leq 39\Delta^{1/4} + 16\kappa^{1/6}.$$

Similarly, by Lemma G.14, we have that

$$|\langle \text{GKP}_{\kappa,\Delta}^\varepsilon, e^{2\pi i Q} \text{GKP}_{\kappa,\Delta}^\varepsilon \rangle|^2 \geq 1 - 40\varepsilon^2 = 1 - 40\Delta.$$

Hence, by Eq. (H1) along with Eq. (H4) and using $\sqrt{x} \geq x$ for $x \in [0, 1]$, we have that

$$\begin{aligned} |\langle S_Q \rangle_\rho|^2 &\geq 1 - 40\Delta - 380\sqrt{\Delta} - 60\kappa^{1/3} \\ &\geq 1 - 420\sqrt{\Delta} - 60\kappa^{1/3}. \end{aligned}$$

It follows that

$$\Delta_Q(\rho) \leq 2\sqrt{420\sqrt{\Delta} + 60\kappa^{1/3}} \leq 41\Delta^{1/4} + 16\kappa^{1/6},$$

where we use $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ for $x, y \geq 0$. ■

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