

Floquet Thermalization via Instantons near Dynamical Freezing

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Periodically driven Floquet quantum many-body systems have revealed new insights into the rich interplay of thermalization and growth of entanglement. The phenomenology of *dynamical freezing*, whereby a translationally invariant many-body system exhibits emergent conservation laws and a slow growth of entanglement entropy at certain fixed ratios of a drive amplitude and frequency, presents a novel paradigm for retaining memory of an initial state up to late times. Previous studies of dynamical freezing have largely been restricted to a high-frequency Floquet-Magnus expansion and numerical exact diagonalization. Both techniques are unable to capture the slow approach to thermalization, or lack thereof, in a systematic fashion. By employing Floquet *flow renormalization*, where the time-dependent part of the Hamiltonian is gradually decoupled from the effective Hamiltonian using a sequence of unitary transformations, we unveil the universal approach to dynamical freezing and beyond, at asymptotically late times. We analyze the *fixed-point* behavior associated with the flow renormalization at and near freezing using both exact-diagonalization and tensor-network-based methods and contrast the results with the conventional prethermal phenomenon. For a generic nonintegrable spin Hamiltonian with a periodic cosine wave drive, the flow approaches an unstable fixed point with an approximate emergent symmetry. We observe that at freezing the thermalization timescales are delayed compared to away from freezing, and the flow trajectory undergoes a series of *instanton* events. Our numerical results are supported by analytical solutions to the flow equations.

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I. INTRODUCTION

A cornerstone of statistical mechanics of closed and chaotic systems is the ergodic hypothesis, whereby in the late-time limit the entire many-body phase space is explored subject to the macroscopic constraints. This hypothesis underpins the equal probability assumption associated with the microcanonical ensemble and ensures that the late-time average of observables reflect the microcanonical predictions. The generalization to quantum statistical mechanics, based on the eigenstate thermalization hypothesis, provides a framework for thermalization in closed quantum systems evolving under their own unitary many-body dynamics [1–5]. As per the eigenstate thermalization hypothesis, the eigenstates of generic quantum

Hamiltonians exhibit “typical” statistical properties, aligning closely with predictions from the microcanonical ensemble. Extending these concepts to periodically driven (i.e., Floquet) nonintegrable quantum many-body problems without energy conservation, the natural fate is thermalization to an “infinite-temperature” (i.e., featureless) state [6,7]. A number of recent advances have revealed exciting mechanisms to evade the process of Floquet thermalization. One such possibility arises in strongly disordered Floquet many-body localized (MBL) systems [8–14], building on progress in the study of MBL [15–19], whose stability in the thermodynamic limit [20–23] at asymptotically late times remains a topic of active research. Another possibility are quantum scars [24–26], which represent a unique but relatively rare class of excited states in the many-body spectrum that preserve memory of initial conditions. Finally, yet another mechanism invokes the phenomenon of Hilbert-space shattering [27,28] for breaking ergodicity, which requires a set of conservation laws that fragment the Hilbert space into exponentially many dynamically isolated subsectors, revealing dynamical behavior akin to both MBL and quantum scars [29]. At high drive frequencies, heating and thermalization can be significantly suppressed, leading to the emergence of prethermal plateaus [30–38].

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In parallel, a number of studies based primarily on numerical exact diagonalization (ED) and supporting Floquet-Magnus expansion have reported the emergence of approximate conservation laws and slow dynamics in generic translationally invariant, periodically driven spin models at special ratios of the drive amplitude and frequency. This distinct route to avoiding Floquet thermalization has been denoted “dynamical freezing” [39–53]. Recent works by some of us have extended this phenomenology to a strongly interacting fermionic setup analyzed using Keldysh field-theoretic methods [54] and interacting bosonic models with a large local Hilbert-space dimension where a semiclassical correspondence reveals suppression of the chaotic Lyapunov exponent at the frozen points [55]. Unfortunately, a vast majority of analytical attempts to study dynamical freezing have relied on the *uncontrolled* and *perturbative* Floquet-Magnus expansion in a rotating frame, which does not produce a convergent series for generic interacting systems and potentially misses important *nonperturbative* effects relevant for addressing fundamental questions related to thermalization (or lack thereof) at freezing. This has left a number of basic questions about dynamical freezing for more generic drives unanswered, which are addressed definitively in this work. Additionally, the numerical studies have largely been restricted to ED on relatively modest system sizes up to finite times. The only notable exception is the recent numerical study for an infinite system size but carried out for *only* a square-wave drive protocol [53]. In contrast, the focus of this manuscript is on the fate of a Floquet many-body system with a sinusoidal drive—which is known to exhibit distinct behavior even within the perturbative Floquet-Magnus expansion [55]—but now obtained using a number of complementary numerical and analytical nonperturbative methods.

First, it is natural to question what determines the location of the dynamically frozen points. For spin models with low-dimensional Hilbert spaces, the location typically correlates with the values of ratio of drive amplitude and frequency where the *leading-order* Magnus correction vanishes. However, some of us have previously demonstrated a quantitative shift in these locations due to the higher-order Magnus corrections in models with a larger on-site Hilbert-space dimension [55]. Finding an alternative method to determine the freezing points, without relying on a combination of numerics and the Floquet-Magnus expansion, is a desirable goal. Second, and more importantly, whether dynamical freezing is *exact* and *robust* in the thermodynamic system-size and late-time limit, and if not, what controls the possible decay away from this special behavior has not been determined beyond the same limitations imposed by exact diagonalization and Magnus expansion for *generic* drives.

In this work, we improve on these limitations of prior studies and provide a universal description of dynamical

freezing based on a Floquet flow-renormalization group (fRG) framework [56–64]. Fundamentally, the formalism generates a sequence of infinitesimal unitary transformations to systematically eliminate the time-dependent part of the driven Hamiltonian, resulting in a flow equation for the different coupling constants in the effective model. The method in the equilibrium setting is a generalization of the more widely known Schrieffer-Wolff transformations [65]; see Appendix A for a brief review of this connection and Appendix B for details on its numerical implementation. As we demonstrate in this paper by focusing on a generic, nonintegrable one-dimensional spin model with a “cosine-wave” drive, the flow trajectory along the fRG time at the *putative* freezing point organizes into a sequence of *unstable* fixed points, except the possibly inevitable thermal (T) fixed point that appears at asymptotically late times [for finite systems, this final fixed point appears “thermal” only when the drive frequency $\Omega < \Omega_*(L)$, where $\Omega_*(L)$ is a system-size- (L -) dependent threshold frequency; see Secs. II and V for more details]; see Fig. 1 for a schematic depiction. Along the flow trajectory, the approach to the first fixed point starting from the bare Hamiltonian is of special interest. It is an *unstable* prethermal (P) fixed point, associated with the emergence of an *approximately* conserved effective Hamiltonian, valid up to an exponentially long timescale in the drive frequency, Ω . Subsequently, the system flows via a complex sequence of intermediate fixed points $I_1, I_2, \dots$. The flow in between these

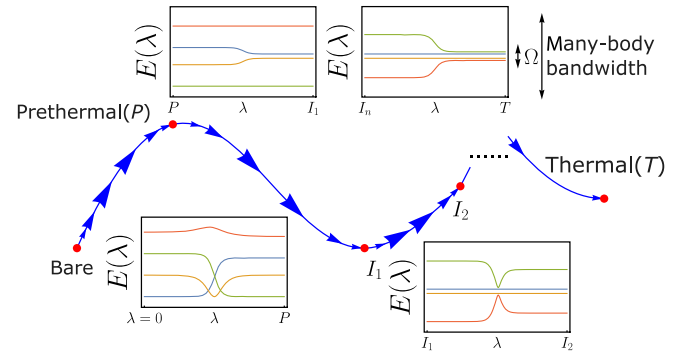


FIG. 1. A schematic depiction of the Floquet flow-renormalization trajectory (blue) along the fRG time λ . The insets show the evolution of many-body energy levels between the different fixed points, e.g., from $\lambda = 0$ to the first (unstable) prethermal fixed point (“P”), followed by a sequence of instanton events that connect intermediate fixed points ($I_1, I_2, \dots$), and final approach to a possibly thermal (“T”) fixed point, depending on a nontrivial interplay of the drive frequency and finite system size. The flow starts with the bare Hamiltonian (many-body bandwidth W) and a Floquet drive with frequency, Ω . The flow to P leads to a renormalization of the many-body spectrum. The energy levels associated with the effective Hamiltonian at the thermal fixed point T are folded into an interval smaller than Ω . See Table I for a relationship between the fRG time λ and real time t along with a summary of the different regimes encountered during the flow *vis-à-vis* these timescales.

intermediate fixed points show characteristics of instanton events. However, as we discuss below, it is important to note at the outset that capturing the flow to the thermal fixed point numerically is exceptionally challenging given the exponential timescales involved and remains an exciting arena for future research.

We demonstrate explicitly that dynamical freezing manifests as an additional emergent conservation law associated with the prethermal fixed point. *Exact* dynamical freezing would correspond to a *vanishing* operator norm for the commutator between the emergent conserved quantity and the undriven part of the Hamiltonian near the prethermal fixed point. Generically, we find that the above commutator norm does *not* vanish exactly at freezing but exhibits a nontrivial behavior as a function of the drive frequency, system size, and fRG time. Specifically, the emergent symmetry becomes more pronounced as the high-frequency limit $\Omega \rightarrow \infty$ is approached. From that perspective, dynamical freezing is *asymptotic* in the high-frequency limit.

In this work, we have also analyzed carefully the flow away from the unstable prethermal fixed point by combining powerful numerical and analytical techniques. Interestingly, we find that the system flows via a sequence of “fixed points,” where the intermediate evolution in fRG time shows characteristics of instantonlike events. The renormalized effective Hamiltonian in the vicinity of these intermediate fixed points is approximately constant, and the associated flow is exponentially slow in $\exp(-\Omega/J)$, where J is the typical single-particle energy scale of the system. The transition between two fixed points occurs rapidly in a “short” window of fRG time, which is on the order of J^{-1} . We associate these “tunneling”-type events as instantons (see Fig. 1), which are fundamentally nonperturbative and lie beyond the reach of more conventional perturbative methods, which includes the Floquet-Dyson perturbation theory [45] or the Floquet-Magnus expansion [66], but are readily accessible via the flow formalism. These perturbative methods are expected to describe the flow from the bare Hamiltonian to the prethermal fixed point. In contrast, the instantons signal a departure from the prethermal fixed point and the breakdown of perturbative expansions, which ultimately must capture the *universal* flow to the thermal fixed point. We note that capturing and diagnosing the full flow toward this “latest” time thermalization for any appreciable system size is an exceedingly slow process, given the exponentially long timescales involved, and is, therefore, well beyond the scope of any numerical method. However, as we discuss below, our results associated with the sequence of instantonlike events suggest that the ultimate fate and fixed-point behavior is tied to this slow thermalization for drive frequencies that lie below a system-size-dependent threshold.

The rest of the paper is organized as follows: In Sec. II, we review the Floquet-flow formalism and the flow

equation. We also introduce the characteristic features associated with dynamical freezing, as manifest in the flow formalism. In Sec. III, we use a simple example of a driven harmonic oscillator to illustrate the flow toward putative freezing using the flow formalism. In Sec. IV, we turn to the solutions of the flow equations for a non-integrable spin chain driven by a cosine wave first using both exact diagonalization and matrix-product operator-based methods. In Sec. V, we present results for a number of numerical diagnostics that characterize the approach to thermalization near dynamical freezing as a function of system size and drive frequency. In Sec. VI, we turn to a general analytical treatment of the flow equation that applies to models beyond the specific spin-chain Hamiltonian, which can describe the early-time approach to the unstable prethermal fixed point, as well as the late-time thermalization due to instantons. We end with an outlook in Sec. VII. The Appendixes contain a number of technical details on the methods used in this paper. In Appendix A, we review the connection between Schrieffer-Wolff transformations and the flow formalism. In Appendix B, we describe some additional numerical details for simulating the flow equations. In Appendix C, we present some additional numerical results supporting conclusions in the main text. In Appendix D, we present details of analytical analysis of the flow equation in the early-time regime, which behaves similarly as the Floquet-Magnus expansion. In Appendix E, we estimate the timescales associated with thermalization in the flow formalism. In Appendix F, we review the Floquet-Magnus expansion of the spin-chain Hamiltonian we used.

II. FLOQUET-FLOW RENORMALIZATION AND DYNAMICAL FREEZING

We review the flow formalism, developed originally by Wegner, Wilson, and others [67,68] and adapted to a variety of strongly correlated problems [56] (including problems beyond the equilibrium setting [57–60,62–64,69–71]). The discussion here closely resembles the one introduced in Ref. [63], which is reviewed in greater detail in Appendix A. We are interested in time-dependent Hamiltonians expressed as

$$H(t) = H_0 + H_1 e^{i\Omega t} + H_{-1} e^{-i\Omega t}, \quad (1)$$

where $H_{-1} = H_1^\dagger$. Treating Eq. (1) as the initial state of a renormalization process, the flow generates a sequence of infinitesimal unitary transformations, preserving the spectrum of the stroboscopic unitary evolution operator [72]:

$$U(T + t_0, t_0) = \mathcal{T} e^{-i \int_{t_0}^{T+t_0} d\tau H(\tau)}, \quad (2)$$

where $\mathcal{T}$ denotes time ordering. Our goal here is to realize a unitary transformation that systematically decouples the

TABLE I. Glossary of flow renormalization group (λ) and real (t) timescales that describe the Floquet dynamics based on the Wegner-Wilson flow framework described in this work.

fRG time (λ)	Physical significance	Real time (t)	Physical significance
Ramp-up regime: $\mathcal{O}(1) \times \Omega^{-1}$	RG timescale beyond which the effective Hamiltonian $H_0(\lambda)$ approaches the prethermal fixed point	$t \ll t_{\text{th}}$ (defined below)	Real timescale where the Floquet-Magnus expansion is a good approximation of the dynamics
Prethermal regime: $\lambda_{\text{min}} \sim (1/J) \ln(\Omega/J)$	RG timescale where $\ H_1(\lambda)\ $ attains its minimal value after flowing near the prethermal fixed point	$t_{\text{eff}}(\lambda) \sim \ H_1(\lambda)\ ^{-1} \ \hat{1}\ $	Real timescale up to which $H_0(\lambda)$ serves as a reliable approximation for dynamics
Thermalization: $\lambda_{\text{th}} \gtrsim \lambda_{\text{min}}$	RG timescale where the system flows away from the prethermal fixed point	Thermalization: $t_{\text{th}} = t_{\text{eff}}(\lambda_{\text{min}}) \sim (\Omega/J)^{\Omega/J}$	Real timescale beyond which the prethermal Hamiltonian $H_0(\lambda_{\text{min}})$ is no longer reliable

time-dependent driving component H_1 from the effective theory while at the same time to preserve the spectrum of $U(T + t_0, t_0)$. Building on earlier work, the appropriate flow equations are

$$\partial_\lambda H_0(\lambda) = 2[H_1(\lambda), H_1^\dagger(\lambda)], \quad (3a)$$

$$\partial_\lambda H_1(\lambda) = -\Omega H_1(\lambda) - [H_0(\lambda), H_1(\lambda)], \quad (3b)$$

where λ represents the “fRG time” along the flow trajectory. Note that λ enters here dimensionally as real time with dimension $\sim [\text{time}]$ in contrast to the conventional Wegner-Wilson picture [67,68], where $\lambda \sim [\text{time}]^2$. See Table I for a glossary of the different timescales that describe the dynamics within the flow RG framework.

We solve the coupled matrix equations, Eqs. (3a) and (3b), using two approaches. First, for small system sizes we employ exact diagonalization by representing the operators explicitly in the many-body Hilbert space and solving the equations exactly using the fourth-order Runge-Kutta (RK4) method [57]. In Supplemental Material [73], we provide numerical evidences that Eq. (3) does preserve the eigenvalues of the Floquet Hamiltonian. The second approach is tailored for larger system sizes and utilizes matrix-product operators (MPOs) [63]. MPOs encode general many-body operators as products of operator-valued matrices, with each matrix acting locally at a single site. Once the operators $H_0(\lambda)$ and $H_1(\lambda)$ are expressed in the MPO form, we again solve the equations using the Runge-Kutta method. MPOs facilitate operations such as addition, evaluation, and computation of commutators, yielding exponentially localized MPOs [74,75]. MPOs have been applied to solve flow equations for many-body localized Hamiltonians [60,76,77]. All ED results are obtained using the QuSpin [78,79] PYTHON package, while the MPO-based computations are performed using the JULIA package ITensor [80–82]. See Appendix B for additional details.

We can view the nonoscillatory component $H_0(\lambda)$ as an effective Hamiltonian responsible for generating the unitary evolution:

$$U_{\text{eff}}(\lambda, t, t') = e^{-iH_0(\lambda)(t-t')}. \quad (4)$$

The distance between $U_{\text{eff}}(\lambda, t_0 + nT, t_0)$ and $U(t_0 + nT, t_0)$ [Eq. (2)], evaluated after n cycles of the Floquet system and averaged over the starting time t_0 within one full cycle, is claimed to be bounded by the Frobenius norm of $nT \times H_1(\lambda)$ [63]. This implies that U_{eff} serves as a reliable approximation for U up to a time

$$t_{\text{eff}}(\lambda) \sim \|H_1(\lambda)\|^{-1} \|\hat{1}\|, \quad (5)$$

which provides useful estimates on the (pre)thermalization timescales for our driven problem, as we describe below. Here, $\|M\|^2 = \text{Tr}(M^\dagger M)$ denotes the Frobenius norm of the operator, and Eq. (5) is normalized by the norm of the identity operator $\hat{1}$ on the many-body Hilbert space.

The typical behavior of the flow equation (3) can be divided into three regimes (see also Table I).

- For fRG time, $0 \leq \lambda \lesssim \Omega^{-1}$, the flow is in a ramp-up regime where $H_0(\lambda)$ is rapidly renormalized by the drive. In the high-frequency limit, this ramp-up flow can be described perturbatively using an expansion similar to the Floquet-Magnus expansion (see Sec. VI).
- Following this ramp-up regime, the system enters a prethermal regime where $\Omega^{-1} \lesssim \lambda \lesssim \lambda_{\text{min}}$. Here, the static part $H_0(\lambda)$ is approximately independent of λ , which leads to the interpretation that the system is close to the prethermal fixed point. The oscillatory part $H_1(\lambda)$ decays exponentially $\sim \exp(-\Omega\lambda)$ until $\lambda = \lambda_{\text{min}}$. Combining this with Eqs. (4) and (5) implies that in the prethermal regime the system shows approximate conservation of the Hamiltonian $H_0(\lambda_{\text{min}})$ up to an exponential long real time

$t_{\text{pre}} \propto \exp(\Omega \lambda_{\text{min}})$. Here, the scale λ_{min} is determined primarily by the local excitation energy scale (such as a single spin flip in a spin chain) of the system, up to logarithmic corrections in Ω . The finiteness of λ_{min} implies that the prethermal fixed point is unstable.

Dynamical freezing manifests as an approximate emergent symmetry of $H_0(\lambda)$ in the prethermal regime, for all $\Omega^{-1} \lesssim \lambda \lesssim \lambda_{\text{min}}$. Consider the emergent symmetry associated with the bare drive component in the Hamiltonian, $H_1(0) = H_{-1}(0) = A\mathcal{Q}$, where $\mathcal{Q}$ is the “charge” associated with the emergent symmetry and A is the drive amplitude. The freezing phenomenon occurs in the fast and strong drive limit (i.e., large A , Ω), where the ratio A/Ω takes discrete special values. At the freezing point, the prethermal Hamiltonian $H_0(\lambda)$ shows an emergent approximate symmetry, characterized by the commutator $[H_0(\lambda), \mathcal{Q}] \sim \mathcal{O}(1/\Omega^2)$, which vanishes in the $\Omega \rightarrow \infty$ limit. Therefore, freezing is only approximate at any finite Ω and, importantly, does *not* survive when the system flows out of the prethermal regime.

- (c) When $\lambda \gtrsim \lambda_{\text{min}}$, the system enters the thermalization regime, where $H_0(\lambda)$ flows to a more nonlocal Hamiltonian. When the drive frequency Ω is smaller than a system-size-dependent threshold $\Omega_*(L)$, the system eventually flows to a “thermal” fixed point, where the operator-entanglement entropy is close to that of a random matrix [83]. Based on our numerical results on small system sizes (see Sec. V), we find that $\Omega_*(L)$ weakly increases with L . Whether the threshold diverges in the thermodynamic limit or saturates to a finite value will determine the eventual fate of freezing, which remains an exciting direction to analyze further via new numerical developments in future work. From the explicit integration of the flow equations, we find that this complex process occurs through a series of instanton events, where $H_1(\lambda)$ becomes sizable in a narrow window of fRG time, and in the same window a pair of eigenvalues of $H_0(\lambda)$ approach each other. As a result of the instanton events, eigenvalues of $H_0(\lambda)$ will ultimately be pushed into an interval of width Ω , which means $H_0(\lambda \rightarrow \infty)$ is now interpreted as a Floquet Hamiltonian. A more detailed technical discussion of these timescales is presented in Sec. VI and Appendix E.

We end this section by noting that, in our numerical computations, we can readily access the ramp-up and prethermal regimes, respectively. In order to access the thermalization regime, we have to restrict ourselves to relatively small drive frequencies due to the limitations associated with numerical floating-point precision.

III. PERIODICALLY DRIVEN HARMONIC OSCILLATOR

To illustrate some of the key ideas used to analyze the flow toward dynamical freezing using the above method,

we consider a simple (but illustrative) example of a driven noninteracting Hamiltonian:

$$H(t) = \omega_0 \hat{a}^\dagger \hat{a} + \omega_1 (\hat{a} + \hat{a}^\dagger) + A \hat{a}^\dagger \hat{a} \cos(\Omega t), \quad (6)$$

where $\hat{a}^\dagger(\hat{a})$ represents a bosonic creation (annihilation) operator and A and Ω represent the drive amplitude and frequency, respectively. Rewriting the above in terms of the decomposition in Eq. (1),

$$H_0(\lambda) = A_0(\lambda) \hat{a}^\dagger \hat{a} + B_0(\lambda) \hat{a}^\dagger + B_0^*(\lambda) \hat{a}, \quad (7a)$$

$$H_1(\lambda) = A_1(\lambda) \hat{a}^\dagger \hat{a} + B_1(\lambda) \hat{a}^\dagger + C_1^*(\lambda) \hat{a}. \quad (7b)$$

Here, $A_0(\lambda)$ and $A_1(\lambda)$ are real functions, while $B_0(\lambda)$, $B_1(\lambda)$, and $C_1(\lambda)$ can be complex, and we have dropped the parts proportional to the identity operator. The two flow equations (3a) and (3b) become

$$\frac{\partial A_0}{\partial \lambda} = 0, \quad (8a)$$

$$\frac{\partial A_1}{\partial \lambda} = -\Omega A_1, \quad (8b)$$

$$\frac{\partial B_0}{\partial \lambda} = 2A_1(C_1 - B_1), \quad (8c)$$

$$\frac{\partial B_1}{\partial \lambda} = -\Omega B_1 - A_0 B_1 + A_1 B_0, \quad (8d)$$

$$\frac{\partial C_1}{\partial \lambda} = -\Omega C_1 + A_0 C_1 - A_1 B_0, \quad (8e)$$

and the initial conditions at $\lambda = 0$ are $A_0 = \omega_0$, $A_1 = A/2$, $B_0 = \omega_1$, and $B_1 = C_1 = 0$. The flow equations reduce to first-order differential equations for $A_{0,1}$, $B_{0,1}$, and C_1 , respectively. Solutions for $A_0(\lambda) = A_0$ (constant) and $A_1(\lambda) = (A/2)e^{-\Omega\lambda}$, while others can be solved numerically. Figure 2 depicts flow trajectories in the (B_0, B_1) plane for varying ratios of A/Ω and ω_0/Ω , respectively. Because of integrability of Eq. (6), the system directly flows to the prethermal fixed point, with $\lambda_{\text{min}} = \infty$ and $\|H_1(\lambda_{\text{min}})\| = 0$.

In this example, dynamical freezing is associated with an emergent U(1) symmetry generated by $\mathcal{Q} = \hat{a}^\dagger \hat{a}$, with the distance between $H_0(\lambda)$ and $\mathcal{H}_{\text{sym}}$ —the space of U(1)-symmetric Hamiltonians—measured by $|B_0(\lambda)|$. The flow in Fig. 2 follows a spiral trajectory in the (B_0, B_1) plane. Since, at the end of the flow, the oscillatory part of the Hamiltonian is completely removed, the flow will end on the $B_1 = 0$ line, the space of prethermal Hamiltonians $\mathcal{H}_{\text{pre}}$. For a nonfreezing point, the flow ends on $\mathcal{H}_{\text{sym}}$ with

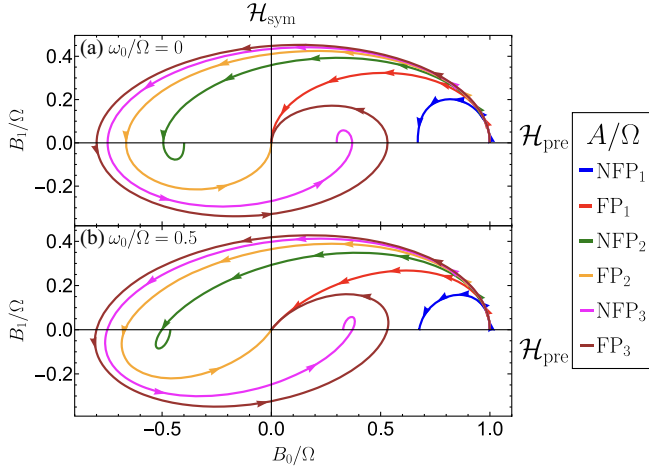


FIG. 2. Flow trajectories of a driven harmonic oscillator given by Eq. (8) for $\omega_1 = \Omega$, (a) $\omega_0/\Omega = 0$, and (b) $\omega_0/\Omega = 0.5$ in the (B_0, B_1) plane, depicted for varying A/Ω . The n th freezing point, denoted FP_n , corresponds to distinct values of A/Ω in (a) and (b), while NFP_n stands for nonfreezing point. $\mathcal{H}_{\text{pre}}$ denotes the subspace of prethermal Hamiltonians (locus $B_1 = 0$), and $\mathcal{H}_{\text{sym}}$ denotes subspace of $U(1)$ symmetric Hamiltonians (locus $B_0 = 0$). Dynamical freezing corresponds to the flow landing exactly on the intersection of $\mathcal{H}_{\text{pre}}$ and $\mathcal{H}_{\text{sym}}$, as shown by the curves associated with FP_1 (red line), FP_2 (orange line), and FP_3 (brown line).

$B_0(\lambda \rightarrow \infty) \neq 0$. This corresponds to the presence of a $U(1)$ -breaking term in the renormalized Hamiltonian. In contrast, freezing occurs for $B_0(\lambda \rightarrow \infty) = 0$ (equivalently, $[H_0(\lambda \rightarrow \infty), \mathcal{Q}] = 0$), linked to certain initial values of A/Ω , which generally depend on ω_0 . In the limit $\omega_0/\Omega \rightarrow 0$, these values align with the Magnus expansion, namely, the zeros of the Bessel function $J_0(\dots)$ (see Sec. VI and Appendix D for more details). Since $H_0(\lambda \rightarrow \infty)$ depends on A/Ω continuously, the order of the freezing point can be related to the geometry of the flow trajectory. In particular, every time $H_0(\lambda)$ crosses the axis $B_0 = 0$, the order of the freezing point increases by one; see Fig. 2.

As a caveat, some of the features of the simple example shown above *do not* generalize to a generic many-body Hamiltonian. First, here the system is fully integrable and does not “thermalize” in the conventional sense, meaning that the flow explores only the ramp-up and prethermal regimes, respectively. Second, since there is only one set of terms $(\hat{a}, \hat{a}^\dagger)$ that violates the $U(1)$ symmetry, we can realize freezing for any oscillator frequency ω_0/Ω by tuning the parameter A/Ω . In a generic many-body system, there can be multiple symmetry-violating operators in $H_0(\lambda)$, and $[H_0(\lambda), \mathcal{Q}]$ need not vanish by tuning a single parameter A/Ω . However, as we show later in Sec. IV, the minimum value decays as a power law in $1/\Omega$. Moreover, many of the qualitative features discovered here via the flow trajectories generalize to varying degrees in the more generic setting, as we highlight below.

IV. FLOQUET SPIN CHAIN: NUMERICAL RESULTS

We now turn to the main example of interest in this paper and demonstrate that the key features discussed above can be generalized to a many-body setting using the same fRG. We have analyzed the flow trajectories for a one-dimensional, nonintegrable Ising spin chain with a Floquet cosine-wave drive coupled to the transverse field. The results are sufficiently universal, even in the presence of additional perturbations to the Hamiltonian, that we believe our results are representative of the general phenomenology introduced above. The Hamiltonian is given by

$$H(t) = \sum_i \left[-JS_i^z S_{i+1}^z - J_2 S_i^z S_{i+2}^z + 2AS_i^x \cos(\Omega t) \right], \quad (9)$$

where the second-neighbor coupling J_2 is responsible for the nonintegrable nature; for $J_2 = 0$, freezing in the integrable (free) Hamiltonian has been analyzed previously [39]. Note that, in the absence of the drive term, the above Hamiltonian is entirely classical. Hence, we have also analyzed the flow RG in the vicinity of freezing points as a function of a *static* transverse magnetic field in the x direction, B_x , for the modified Hamiltonian

$$H(t) \rightarrow H(t) + \sum_i B_x S_i^x. \quad (10)$$

We can split the above Hamiltonian into two parts:

$$H_0(0) = \sum_i \left[-JS_i^z S_{i+1}^z - J_2 S_i^z S_{i+2}^z + B_x S_i^x \right], \quad (11a)$$

$$H_1(0) = A \sum_i S_i^x, \quad (11b)$$

which represent the initial condition for the flow associated with $H_0(\lambda)$ and $H_{\pm 1}(\lambda)$, respectively. Our numerical results are presented in units where $J = 1$. Since the phenomenology remains qualitatively similar with the inclusion of B_x , we first begin by setting $B_x = 0$ and then study the problem as a function of increasing (but static) B_x . Nevertheless, there are interesting quantitative modifications to the freezing phenomenology with the inclusion of $B_x \neq 0$, which we discuss briefly in Sec. IV B below and elaborate further in greater detail in Appendix C.

At the dynamically frozen point, we expect the operator $\mathcal{O} = \sum_i S_i^x (\equiv \mathcal{Q})$ to exhibit an approximate emergent conservation law. One of the principle quantities of interest will be

$$P(\lambda) = \frac{\| [H_0(\lambda), \mathcal{O}] \|}{\| H_0(\lambda) \|}, \quad (12)$$

where $\|M\| = \sqrt{\text{Tr}(MM^\dagger)}$ is the Frobenius norm. A small $P(\lambda)$ implies approximate emergent symmetry generated by $\mathcal{O}$. To compare across different system size, we have

normalized $P(\lambda)$ to be an intensive quantity. It is also useful to define the intensive ratio of the norms of the driven part of the Hamiltonian with respect to the static part:

$$Q(\lambda) = \frac{\|H_1(\lambda)\|}{\|H_0(\lambda)\|}, \quad (13)$$

which quantifies the extent to which the oscillatory component has been removed during the flow. As discussed in the discussion surrounding Eq. (5), a smaller $Q(\lambda)$ implies that $H_0(\lambda)$ is a better approximation for the effective Floquet Hamiltonian up to later times.

A. Approach to prethermal regime at freezing point

To diagnose dynamical freezing numerically, we evolve Eqs. (3) (i.e., let them flow) to a fRG time $\lambda = \lambda_c$, such that $Q(\lambda_c) \ll 1$ and $P(\lambda)$ exhibits a λ -independent plateau, as introduced in regime (b) in Sec. II; see also Table I. In other words, λ_c is then in the initial prethermal regime. In this subsection, we choose $\lambda_c \lesssim \lambda_{\min}$, the fRG time where $H_1(\lambda)$ minimizes. For all of the numerical computations in this section, we find that a $\lambda_c = 1 \times J^{-1}$ satisfies these numerical criteria. In general, we find that λ_c should be set to the inverse intrinsic single-particle energy scales associated with the local thermalization dynamics.

In Fig. 3(a), we show explicit results for $P(\lambda)$ for $B_x = 0$ at and away from the freezing points; clearly, this requires us to identify the location for the freezing points. Thus, we first performed a scan of $P(\lambda = \lambda_c)$ over a fine interval of A/Ω (including for a $L = 40$ chain using MPO), which clearly reveals the sharp features associated with freezing. These are encoded in the striking drop of $P(\lambda_c)$ at specific values of A/Ω . Note that λ_c is chosen such that $P(\lambda)$ is already in a plateaulike regime, as explained above and shown in Fig. 3(b). Interestingly, there is an “asymptotic” plateau associated with $P(\lambda)$ as a function λ , both at and away from freezing (we show in Appendix C that the plateau feature at $\Omega = 10J$ can extend to much longer

fRG time). However, the flow toward the plateau at the first, second, and third freezing points is accompanied by a rich structure, reminiscent of some aspects of the flow trajectories for the driven harmonic oscillator in Fig. 2, which we describe in more detail below. We note here that the data in Fig. 3(a) are obtained for a fixed Ω and varying A . The numerical results remain identical, if instead A is fixed and Ω is varied (see Appendix C). However, the only requirement remains that both A and Ω are large compared to the single-particle energy scales in $H(t)$, with $A/\Omega \sim O(1)$ for FP_n with $n \sim O(1)$.

The locations of the freezing points are roughly consistent with the expectations based on Floquet-Magnus expansion, namely, $J_0(A/4\Omega) = 0$, where J_0 is the Bessel function (see Appendix F for derivation). As we demonstrate in Appendix C, the position of the freezing point is renormalized away from the leading-order Magnus result, with a stronger renormalization for smaller Ω .

The flow initially shows a ramp-up behavior where $H_0(\lambda)$ [and, relatedly, $P(\lambda)$] varies rapidly with the fRG time λ . This is followed by the prethermal regime where $H_0(\lambda)$ flows slowly with λ , and, as a result, $P(\lambda)$ reaches a plateau. As noted before, the hallmark of “dynamical freezing” is the strong reduction in the value associated with the plateau in $P(\lambda)$. It is worth highlighting that this plateau value at freezing, while at least an order of magnitude smaller than the initial value $P(0)$, does not vanish exactly. This leads to only an approximate conservation of $\mathcal{O}$. In Sec. IV B, we return to this point when $B_x \neq 0$.

We note two additional features associated with $P(\lambda)$ at freezing. First, they develop strong “dips” in the ramp-up regime [marked by arrows in Fig. 3(b)], and the number of such dips is given by $(n - 1)$, where n represents the order of freezing point. This is reminiscent of the driven harmonic oscillator example in Fig. 2, where, with varying A/Ω , the flow trajectory can cross the $\mathcal{H}_{\text{sym}}$ plane, associated with the subspace of symmetric Hamiltonians. In the many-body situation, the crossing is not exact.

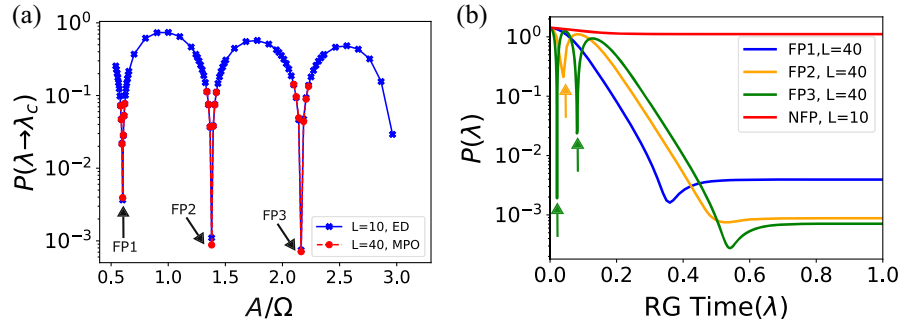


FIG. 3. (a) $P(\lambda)$ defined in Eq. (12) as a function of A/Ω for $\lambda \rightarrow \lambda_c$. The sharp drop in $P(\lambda_c)$ marks the freezing points, FP_1 , FP_2 , and FP_3 , in the considered range of A/Ω . The results for $P(\lambda)$ are obtained for system size $L = 10$ using exact diagonalization and for $L = 40$ from MPO calculations. (b) $P(\lambda)$ as a function of fRG time (λ) for the different freezing points shows a saturation to the plateau at “late” times. Results for all freezing points are shown for $L = 40$ from MPO computations and for nonfreezing point (NFP) for $L = 10$ using ED. Other parameters: $J = 1$, $J_2 = 0.2$, and $\Omega = 10$. We show the results for $P(\lambda)$ for $L = 10$ until $\lambda = 10$ in Appendix C.

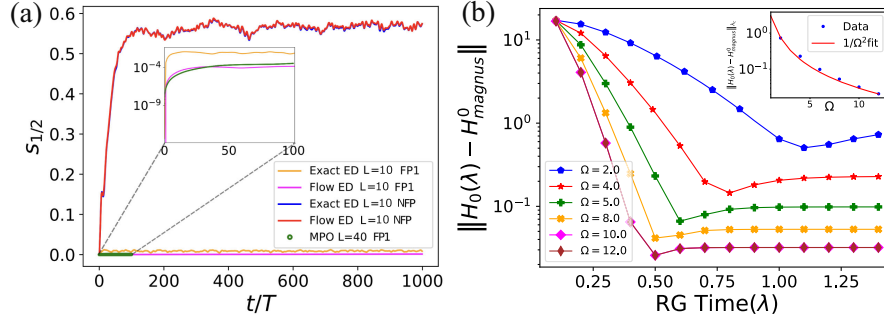


FIG. 4. (a) Half-chain entanglement entropy (EE) density [i.e., $s_{1/2} = S_{1/2}/(L/2)$] plotted stroboscopically as a function of real time starting from a fully polarized state $|\psi\rangle = |x; +\rangle^{\otimes L}$. The results are shown for $L = 10$ (using both exact and flow ED) and $L = 40$ (using flow MPO and MPS), at and away from freezing. Inset: early-time evolution of $s_{1/2}$ at FP1. Other parameters are set to $J = 1$, $J_2 = 0.2$, and $\Omega = 10$. (b) Frobenius norm of $H_0(\lambda) - H_{\text{magnus}}^0$ as a function of fRG time for varying Ω . H_{magnus}^0 is computed analytically in Appendix F. Inset: saturation value at late fRG time falls off as $1/\Omega^2$ at FP1 ($L = 10$).

Nevertheless, associated with each such near crossing, the distance (in the space of Hamiltonians) between the effective Hamiltonian and $\mathcal{H}_{\text{sym}}$, which is given by $P(\lambda)$, develops a minimum. This corresponds to the dips. Second, for higher drive amplitudes A/Ω , the flow takes a longer fRG time to reach the plateau. This can be intuitively understood as follows: The flow effectively integrates away the oscillatory part $H_1(\lambda)$ at a rate proportional to the drive frequency Ω , such that a stronger drive spends a longer time in the ramp-up regime. This feature is also consistent with our analytical results in Sec. VI.

It is worth emphasizing that the finite value of $P(\lambda)$ at freezing is in sharp contrast to the harmonic oscillator example in Fig. 2 presented earlier, where $P(\lambda)$ vanishes exactly. The difference can be traced back to our concluding comments in Sec. III, where we pointed out that, in a many-body system, a variety of operators that violate the emergent conservation can be generated along the flow, which cannot all be suppressed by fine-tuning a single parameter, A/Ω . However, these additional symmetry-violating operators can be universally suppressed in the high-frequency $\Omega \rightarrow \infty$ limit. We find $P(\lambda_c)$ scales with the frequency as $1/\Omega^2$ at the freezing points (see Appendix C for more detail).

Let us end by relating our findings based on the flow formalism to the real-time dynamics of the driven system. As discussed in Sec. II, the static part of the Hamiltonian $H_0(\lambda)$ is to be interpreted as an approximation for the actual Floquet Hamiltonian H_F with an error bounded by $\|H_1\|$. As discussed in Appendix C, $\|H_1(\lambda_c)\|$ decays exponentially with Ω , implying that $H_0(\lambda_c)$ should be an excellent approximation to H_F in the large- Ω limit. To test this, we have computed the growth of the half-chain entanglement entropy density of the system, $s_{1/2} = S_{1/2}/(L/2)$, starting from the polarized state $|\psi\rangle = |x; +\rangle^{\otimes L}$, using two different methods (see Appendix B for additional numerical details):

- We use the exact real-time evolution based on exact diagonalization to evaluate $s_{1/2}$.
- We treat the problem as an effective static system, but with Hamiltonian $H_0(\lambda_c)$, obtained from the solutions to the flow equations using exact diagonalization, as well as MPO-based evolution using the time-dependent-variational principle (TDVP) [81,82,84–87], respectively.

The results for $s_{1/2}$ obtained using both sets of methods are plotted in Fig. 4(a), which show excellent numerical agreement at and away from freezing. This provides further evidence for our Floquet flow renormalization being able to capture the essential universal aspects of the problem, including the growth of entanglement entropy in real time. We also note that $H_0(\lambda)$ in the prethermal regime accurately captures the real-time dynamics up to an exponentially long time. It is worth noting that the growth of entanglement entropy in Fig. 4(a) shows clear differences between freezing points (FP1) and nonfreezing points (NFP). Clearly, at early to intermediate times, the freezing phenomenology is associated with much slower dynamics; however, this does not imply that this difference would survive up to much later times. The growth of entanglement is exponentially slow in the drive frequency as well, which can be seen quite clearly by focusing on smaller system sizes [73].

Finally, we directly compare the “distance” between $H_0(\lambda)$ and H_{magnus}^0 (quantified by Frobenius norm) in Fig. 4(b), the leading-order result of the Floquet-Magnus expansion (see Appendix F for detail), which is expected to capture the prethermal dynamics in the $\Omega \rightarrow \infty$ limit. Indeed, we see that this distance decreases along the flow trajectory λ and reaches a plateau in the prethermal regime; the plateau value decays as $1/\Omega^2$ in the high-frequency limit, which is consistent with our analytical result in Sec. VI. Furthermore, $H_0(\lambda_c)$ is approximately isospectral to the actual Floquet Hamiltonian H_F [73].

B. Effect of a static transverse magnetic field

As noted previously, H_0 is purely classical when $B_x = 0$; see Eq. (11a). Here, we analyze the effect of introducing an additional static transverse field along the x direction. Specifically, we study the norm of $P(\lambda)$ as in Eq. (12) near the dynamical freezing point. In Fig. 5(a), we present results for $P(\lambda \rightarrow \lambda_c)$ as a function of (A/Ω) near FP₁ for increasing values of B_x . These results demonstrate that the location of the freezing point associated with the first (prethermal) fixed point is unaffected by increasing B_x , as indicated by the nearly identical locations of the minimum in $P(\lambda_c)$ for all B_x .

However, the robustness (i.e., “quality”) of freezing depends on the value of B_x , as we now demonstrate by investigating the Floquet eigenstates. To analyze how the nonzero transverse field affects the Floquet eigenstates, we compute their localization properties in the many-body Hilbert space. Specifically, we calculate the inverse participation ratio (IPR) of a Floquet eigenstate, $|m\rangle$, in the eigenbasis ($|i_x\rangle$) of $\hat{S}_x$:

$$\text{IPR} = \sum_{i_x=1}^{D_H} |\langle i_x | m \rangle|^4, \quad (14)$$

where $D_H = 2^L$ is the dimension associated with the many-body Hilbert space. When $|m\rangle$ is localized over only a few [i.e., $O(1)$] of the eigenstates $|i_x\rangle$, we expect $\text{IPR} \sim O(1)$. In contrast, when $|m\rangle$ is delocalized over all of the eigenstates in the eigenbasis of any local operator (e.g., $\hat{S}_x$), the IPR falls off rapidly with increasing system size. In the present setting of a driven many-body system, where the drive couples to S_i^x , unbounded heating would lead to increasing delocalization and decreasing IPR. In Fig. 5(b), we observe that, at the freezing point, the IPR increases and approaches an $O(1)$ value with increasing B_x , indicating enhanced localization in the many-body Hilbert space. Interestingly, by analyzing the IPR for the Floquet eigenstates of $H_0(\lambda_c)$ in the x basis both at freezing (FP1) and away from freezing (NFP), as well as with and without a finite B_x , we find a higher tendency toward localization when $B_x \neq 0$. However, the Floquet eigenstates also develop an increased tendency toward delocalization in the eigenbasis of any local operator, as evaluated based on the IPR for the Floquet eigenstates of $H_0(\lambda_c)$. At the freezing point, the effective Floquet Hamiltonian to leading order is proportional to the $YY + ZZ$ terms; see Appendix F. This structure gives rise to an emergent $\mathbb{Z}_2$ symmetry, resulting in degenerate Floquet eigenstates [equivalently, eigenstates of $H_0(\lambda_c)$], and, consequently, the expectation values of S_x are nearly zero for all states. Introducing a small nonzero B_x lifts the degeneracy, making the Floquet eigenstates to align with the eigenstates of S_x . We have further analyzed the effect of B_x on the expectation values of the magnetization in Appendix C.

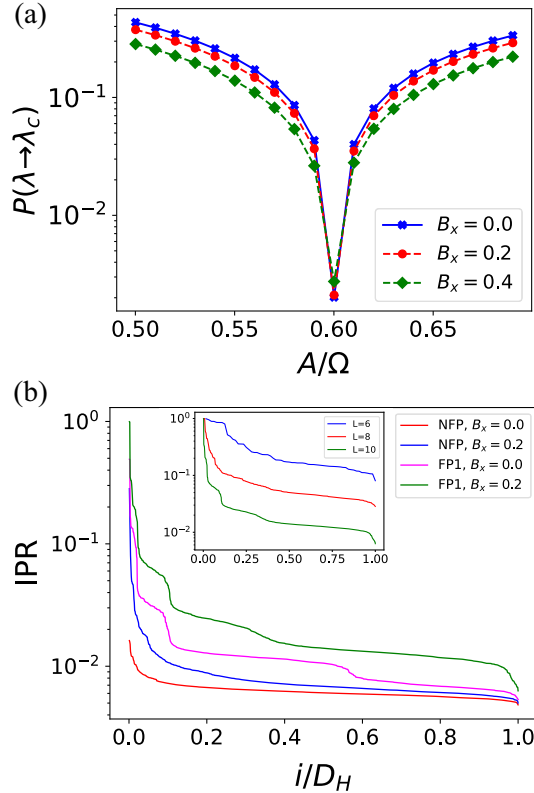


FIG. 5. (a) $P(\lambda \rightarrow \lambda_c)$ obtained from Eq. (12) as a function of (A/Ω) near FP₁ with increasing B_x for $L = 10$. All other parameters are the same as in Fig. 3. (b) IPR defined via Eq. (14) for eigenstates of $H_0(\lambda_c)$ in the $\hat{S}_x$ basis plotted in decreasing order for $L = 10$, comparing the freezing point (FP1) and the nonfreezing point (NFP), both with and without the magnetic field term in the x direction. Inset: IPR vs i/D_H for increasing system size for FP₁ with $B_x \neq 0$. Other parameters are $J = 1$, $J_2 = 0.5$, $\Omega = 5$, and $\lambda_c = 5.0$.

C. Thermalization via instantons

Let us now address the flow away from the prethermal fixed point. For that matter, we study $\|H_1(\lambda)\|$ for different values of the next-nearest-neighbor coupling J_2 , at and away from a freezing point, up to intermediate fRG time λ_c . In this subsection, we set $\lambda_c \gg \lambda_{\min}$. We note that $J_2 \neq 0$ breaks free-fermion integrability of the model. Therefore, we expect that $H_1(\lambda)$ ceases to decay and shows an upturn at intermediate fRG times when $J_2 \neq 0$. Our access to this regime is partly constrained by two factors: First, the initial exponential decay of $H_1(\lambda)$ is limited by numerical floating-point precision. The prethermal cutoff fRG time $\lambda_{\min}$ and the drive frequency Ω are limited by $\Omega\lambda_{\min} < \ln(1/\epsilon_{\text{fp}})$, where $\epsilon_{\text{fp}} \sim 10^{-16}$ is the floating-point precision. Beyond that, $H_1(\lambda)$ will decay below floating-point precision before reaching the upturn. Second, the growth of $\|H_1(\lambda)\|$ is associated with a nontrivial spreading of operators in space, causing the bond dimension of the MPO to grow rapidly. Because of these constraints, we choose to study this regime using the ED method with a

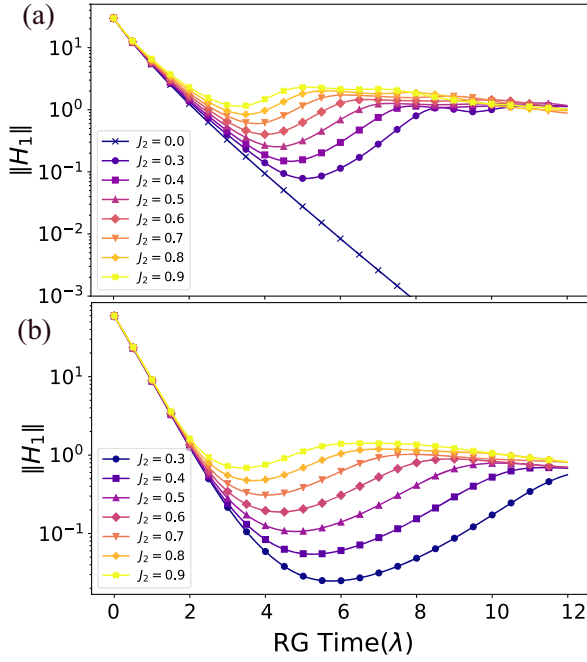


FIG. 6. The Frobenius norm of (a),(b) H_1 as a function of fRG time for increasing J_2 for $L = 10$. (a) Away from freezing, and for $J_2 = 0$, the system flows toward an integrable fixed point. For $J_2 \neq 0$, at small fRG times the system first flows toward a prethermal fixed point and eventually flows to a thermal fixed point. (b) At freezing, the system flows toward a prethermal fixed point for small and intermediate fRG times, controlled by the emergent approximate integrability, with the subsequent flow to the thermal fixed point delayed relative to (a). Other parameters are set to $\Omega = 2J$, $J = 1$, $A/\Omega = 0.3$ away from freezing, and $A/\Omega = 0.601$ at FP_1 .

smaller value of Ω , as shown in Fig. 6. The latter issue can be addressed using more sophisticated MPO compression in future work [88], but we expect that the qualitative considerations are likely to be similar to what we describe below.

As noted before, for a small fRG time λ in the ramp-up regime, $H_1(\lambda)$ decays exponentially with λ , as shown in Fig. 6. However, for intermediate values of λ , $\|H_1(\lambda)\|$ reaches a minimum and starts to grow at $\lambda = \lambda_{\min}$. Concurrently, the static component $H_0(\lambda)$ is no longer in the plateau regime and starts to flow as well. Using Eq. (5), we can interpret the minimum of $\|H_1(\lambda)\|$ as the termination of the prethermal regime, and the associated time-scale $t_{\text{eff}} \sim \|\hat{1}\|/\|H_1(\lambda_{\min})\|$ marks the maximal (real) time of validity of the prethermal Hamiltonian $H_0(\lambda_{\min})$. In the fRG language, this implies that the prethermal fixed point is unstable in generic nonintegrable systems, and the flow turns away from the fixed point at $\lambda = \lambda_{\min}$.

Clearly, as the system flows away from the prethermal fixed point, we expect the emergent symmetry associated with dynamical freezing to also be lost. However, we point

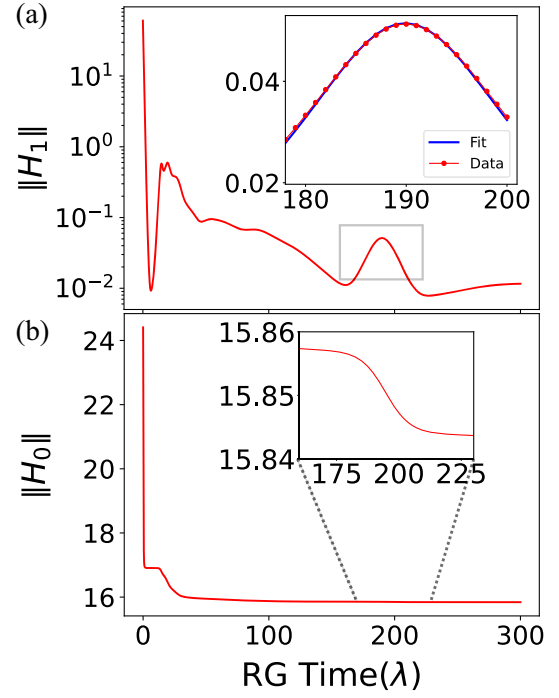


FIG. 7. (a) Frobenius norm of $H_1(\lambda)$ as a function of fRG time for $L = 10$. Inset: isolated peak at late fRG time along with theoretical fit $[(a/2)\text{sech}[a(\lambda-b)]]$. Parameters: $J = 1$, $J_2 = 0.2$, and $\Omega = 2.0$ at FP_1 . (b) Frobenius norm of $H_0(\lambda)$, with a steplike feature correlated with the peak in $H_1(\lambda)$ in (a). Additional results for $L = 6$ are shown in Appendix C.

out that dynamical freezing does provide additional mitigation against thermalization. As we see in Fig. 6, for a many-body system driven at the freezing point, the minimum fRG time $\lambda_{\min}$ of $\|H_1(\lambda)\|$ is indeed *delayed* compared to the same system away from freezing. At the same time, $\|H_1(\lambda_{\min})\|$ is smaller, and $H_0(\lambda)$ shows a longer plateau-like interval [see Ref. [73] for the corresponding figure of $H_0(\lambda)$]. This implies that the dynamically frozen regime (for the cosine-wave drive) thermalizes slower compared to a generic driven (unfrozen) system, in the sense of both fRG time and real time.

Next, we address the flow dynamics beyond the prethermal fixed point. We run the flow equation (3) using the ED method up to a fRG time $\lambda \gg \lambda_{\min}$, and the result is shown in Fig. 7. After the minimum associated with the termination of the prethermal fixed point, $\|H_1(\lambda)\|$ shows additional peaks, separated by regions of exponential decay. To interpret the behavior of $\|H_1(\lambda)\|$, we generalize the lesson we learn from the prethermal fixed point: The flow equation (3) shows that the flow rate of $H_0(\lambda)$ is determined by the magnitude of $H_1(\lambda)$. When $\|H_1(\lambda)\|$ decays exponentially, it implies that the system is close to an unstable fixed point and that the flow of $H_0(\lambda)$ runs slowly. In contrast, when $\|H_1(\lambda)\|$ grows and attains a peak, it is interpreted as a rapid flow from one fixed point to the

other. As we show in Fig. 7, when $H_1(\lambda)$ shows a peak, there is a concurrent steplike feature in $H_0(\lambda)$. Therefore, the trajectory at large fRG times can be interpreted as the flow between a series of unstable fixed points. In particular, most of the fRG time is spent near a fixed point, and the transition between fixed points takes only a short interval of fRG time. Therefore, it is appropriate to associate these transitions with instantons [89].

As we discuss in Sec. VI, an instanton event is associated with the reorganization of the spectrum of $H_0(\lambda)$. When $\lambda \gg \lambda_{\min}$, most instantons are elementary in the sense that they involve only a pair of eigenvalues of $H_0(\lambda)$ and a single matrix element of $H_1(\lambda)$, which is amenable to an analytical description. As a result, $\|H_1(\lambda)\|$ shows a simple “single-peak” structure that agrees with the analytical solution in Eq. (26) below. However, we note there are also instances where multiple instantons are close to each other, and there can be nontrivial interactions between them. As shown in Fig. 7, the first growth regime of $\|H_1(\lambda)\|$ after $\lambda = \lambda_{\min}$ is one of such examples, where two instantons are close to one another resulting in a “double-peak” feature in $\|H_1(\lambda)\|$. In Appendix C, we present results for a smaller system size ($L = 6$), where we were able to simulate the flow for significantly longer fRG times and access more instanton events. We found examples of well-isolated instantons, which continue to be well described by the theoretical results in Sec. VI. Finally, we remark that the analytical treatment in Sec. VI relates the width of the instanton with its height, but the exact location and the height of the instanton is determined by the microscopic details of the Hamiltonian. Nevertheless, we expect the first instanton to happen at a RG timescale λ that is parametrically comparable to $\lambda_{\min}$.

V. NUMERICAL DIAGNOSIS OF THERMALIZATION

Our analysis thus far has established the emergence of prethermal regimes and instanton-mediated dynamics near dynamical freezing. However, we have not addressed the fundamentally important question of the extent to which the system thermalizes and loses memory of its initial state in the late-time limit. To address this question, we now present results for two complementary numerical diagnostics: (i) the operator entanglement entropy (OPEE) [83,90] associated with $H_0(\lambda)$, the flowing effective Hamiltonian, which probes the buildup of operator nonlocality and complexity, and (ii) the diagonal ensemble (DE) average of local observables, which captures the infinite-time limit of their expectation value after dephasing in the Floquet basis. The OPEE serves as a global measure that is sensitive to random-matrix-like behavior, while the DE average tests for the loss of memory of the initial state. Together, these quantities allow us to assess the extent of thermalization at and near freezing and to identify regimes where prethermal behavior persists up to late times.

A. Operator entanglement entropy

The OPEE quantifies the degree of operator nonlocality by treating it as a state in a doubled Hilbert space. Given an operator O acting on a Hilbert space of L spin- $\frac{1}{2}$ degrees of freedom (i.e., of dimension 2^L), we begin by *vectorizing* it into a pure quantum state in a doubled Hilbert space $\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}$, each of dimension 2^L . This is done by flattening O into a column vector using row-major order: $|O\rangle = [1/(\|O\|_{\text{Fro}})\text{vec}(O)]$. This normalization ensures that $\langle O|O\rangle = 1$, allowing $|O\rangle$ to be interpreted as a state in $\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}} \cong \mathbb{C}^{2^L} \otimes \mathbb{C}^{2^L}$. We reshape the vector $|O\rangle$ into a rank- $2L$ tensor with dimensions $(2, 2, \dots, 2)$, corresponding to the physical spin degrees of freedom. We then bipartition the system into two subsystems, A and B , and the tensor is transposed to group all indices in A and B and reshaped into a matrix M_{ab} , where multi-indices a and b correspond to the configurations of subsystems A and B , respectively. We finally perform a singular value decomposition on the reshaped matrix: $M = U\Sigma V^\dagger$. The singular values $\{\lambda_i\}$ in Σ correspond to the Schmidt coefficients of the bipartition $A|B$. The reduced density matrix ρ_A has eigenvalues λ_i^2 , and the von Neumann entanglement entropy [91] is computed as

$$S[O] = -\sum_i \lambda_i^2 \log \lambda_i^2. \quad (15)$$

Our numerical results for OPEE are shown in Fig. 8. In Figs. 8(a) and 8(b), we show the evolution of the OPEE as a function of the RG time λ , for low- $\Omega \lesssim \Omega_*$ and high-frequency $\Omega \gtrsim \Omega_*$, where Ω_* is a system-size-dependent “threshold” scale. In Fig. 8(c), we show the asymptotic saturation value of the OPEE as $\lambda \rightarrow \infty$ as a function of Ω .

As shown in Fig. 8(a), when $\Omega \lesssim \Omega_*$ the OPEE appears to be insensitive to the difference between freezing and nonfreezing points. Furthermore, the saturation value for OPEE agrees with that prescribed by random-matrix theory, with volume-law scaling [83]. On the other hand, for higher frequencies, the OPEE shows difference between freezing and nonfreezing points as can be seen in Fig. 8(b). In particular, the OPEE at the freezing point saturates to a lower value compared to that at the nonfreezing point. In Fig. 8(c), we plot the saturation value of OPEE as a function of Ω across different system sizes. Moreover, this asymptotic saturation value deviates from the random-matrix prediction and exhibits a sub-volume-law scaling. This leads us to conjecture that the eventual thermalization behavior of the driven system, even at freezing, depends on how the drive frequency compares to a system-size-dependent threshold, $\Omega_*(L)$.

To further investigate the thermalization properties in the thermodynamic limit $L \rightarrow \infty$, we need to analyze the L dependence of Ω_* . Based on the previous observations, if $\Omega_*(L)$ diverges with L in the thermodynamic limit, it suggests that an infinite system driven at a finite frequency

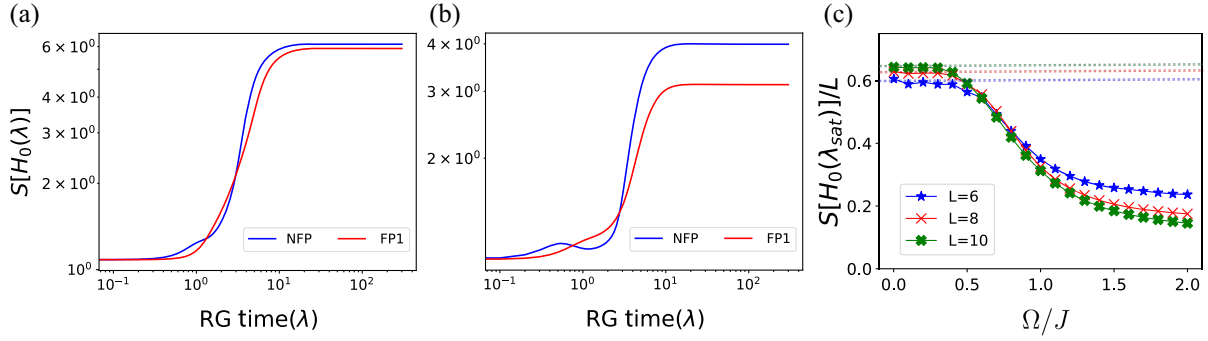


FIG. 8. (a) The operator entanglement entropy (OPEE) for the operator $H_0(\lambda)$ is plotted as a function of RG time, both at and away from the freezing point ($L = 10$, $\Omega/J = 0.5$). Both of the curves approach the RMT value. (b) For $\Omega/J = 1.0$, at the freezing point, the OPEE is significantly suppressed compared to the nonfreezing point. (c) The saturation value of the OPEE is shown as a function of the driving frequency at the first freezing point, across different system sizes. As the frequency increases, the OPEE saturation value approaches subvolume law. At lower frequencies, the OPEE scales as $\mathcal{O}(L)$. The RMT value is shown as dashed lines. Other parameters: $J = 1$, $J_2 = 0.9$, and $B_x = 0.1$.

always thermalizes. On the other hand, if $\Omega_*(L \rightarrow \infty)$ is finite, it suggests that even an infinite system can avoid thermalization if driven at a sufficiently high but finite frequency. While our current numerical results are limited in size by ED, finite-size scaling suggests that Ω_* increases slowly with L , and a more systematic future analysis is required to settle this question definitively.

We note here that a quantum state at freezing can exhibit volume-law scaling of the *state* entanglement entropy [45], which has been attributed to Hilbert space fragmentation related to additional conserved quantities [53]. The OPEE that is studied here is a distinct concept from the state entanglement, which quantifies the spread of operator in the Liouville space. Saturation to the full random matrix theory (RMT) value we observed here for $\Omega < \Omega_*(L)$ is a significantly stronger diagnostic than the observation of a volume law alone: It requires unrestricted mixing across the entire operator Hilbert space, implying that $H_0(\lambda)$ explores the full Liouville space without constraint.

Our study of the OPEE has also revealed two additional features tied to freezing (see Ref. [73] for details), which include (i) the OPEE for higher-order freezing points display an additional oscillatory behavior for short RG time λ in the ramp-up regime, and the number of oscillations exactly matches the order of the freezing point minus one, which is reminiscent of the “dips” shown in Fig. 3(b). However, for later RG times, the OPEE saturates to a value similar to the first freezing point, irrespective of the ratio of Ω/Ω_* . (ii) The instanton events leave their fingerprints on the OPEE as “steplike” features as a function of λ .

B. Diagonal ensemble average

To study how the memory of an initial state is preserved or lost for the driven Hamiltonian, we compute the diagonal ensemble average, which is intended to capture the expectation value of a local observable in the infinite-time limit. The DE average is defined as

$$\begin{aligned} \mathcal{O}_{\text{DEA}} &= \langle \Psi(NT \rightarrow \infty) | \mathcal{O} | \Psi(NT \rightarrow \infty) \rangle \\ &= \sum_n |c_n|^2 \langle \phi_n | \mathcal{O} | \phi_n \rangle, \end{aligned} \quad (16)$$

where $|\phi_n\rangle$ are the Floquet states and $|\Psi(0)\rangle = \sum_n c_n |\phi_n\rangle$ is an initial state written in the Floquet basis. We are interested in this quantity in the presence of a static magnetic field B_x , which as discussed previously improves the quality of freezing. We evaluate the DE average of S_x starting from a thermal pure state. We first perform this calculation at the first-order freezing point while varying the driving frequency across different system sizes. Figure 9(a) shows the normalized diagonal ensemble average $[S_{x,\text{DEA}}/S_x(t=0)]$ as a function of Ω for several system sizes. At low drive frequencies ($\Omega/J \lesssim 2$), we find that this ratio is consistent with thermalizing behavior, with a value approaching 0 and far below 1, where the latter corresponds to perfect freezing (i.e., complete memory of the initial S_x value). With increasing Ω , the DE average at the first-order freezing point exhibits better memory of the initial state. Importantly, larger systems exhibit a delayed onset associated with this ratio of near-unity values, indicating that stronger driving is needed to stabilize freezing as the system size grows. For $\Omega/J \gtrsim 12$, all of the system sizes we have studied exhibit a value of this ratio close to 1 (> 0.99), consistent with near-perfect freezing. The small dips in the $L = 10$ and $L = 12$ data at intermediate Ω suggest finite-size resonances or residual heating processes that modestly reduce the quality of freezing. All of these trends align with the OPEE results of the previous subsection—small Ω yields RMT-like operator entanglement and loss of memory of initial state at freezing, whereas large Ω suppresses saturation of OPEE and preserves initial state memory. The threshold frequency Ω_* between the two regimes weakly increases with system size L . In Fig. 9(b), we present results for the same ratio at a fixed, but few different values of Ω while increasing the

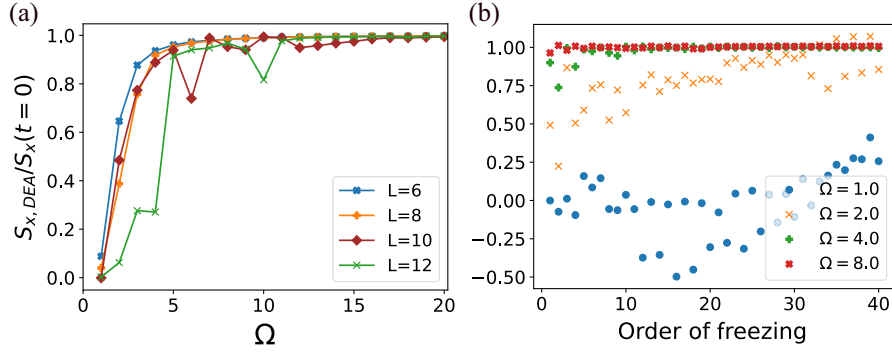


FIG. 9. Diagonal ensemble average of S_x for a thermal pure state as a function of (a) driving frequency for different system sizes and (b) order of freezing for different driving frequencies. Other parameters: $L = 10$, $B_x = 0.1$, and $J_2 = 0.9$.

freezing order. As expected based on the discussion up until now, we do not find the ratio to be close to 1 for small Ω for even high freezing orders (up to the 40th order). Interestingly, for a fixed drive amplitude, larger frequencies yield higher DE average values than smaller ones. Therefore, even with the magnetic field present, freezing is not associated with an “exact” conservation. This behavior stands in sharp contrast to results obtained with a square-wave drive [53], suggesting that these two driving protocols may belong to different universal classes.

VI. ANALYTICAL RESULTS

In this section, we present analytical results related to the solutions of the flow equations (3), especially in the putative dynamically frozen regime and beyond. Our discussion is divided into two parts. First, at early times, which includes the ramp-up regime and the approach to the prethermal fixed point, the flow equation can be analyzed in a perturbative manner that is similar in spirit to the Floquet-Magnus expansion but with important key differences that appear in the beyond-leading-order terms. Second, in the late-time regime (which is the main regime of interest), the flow equation describes the departure from the prethermal fixed point and is analyzed nonperturbatively. This is where we obtain an approximate analytical handle on the instanton events that control the flow toward subsequent thermalization.

A. Early time: Approach to prethermal behavior

The early-time behavior of Eq. (3) with a fRG time $\lambda \sim \Omega^{-1}$ is associated with the emergence of the prethermal behavior in the ramp-up regime, which is more pronounced in the high-frequency $\Omega \rightarrow \infty$ limit. Here, both $H_0(\lambda)$ and $H_1(\lambda)$ are expected to be exponentially local (operator strings of length l have a coefficient that decays exponentially in l). To describe the associated flow dynamics, it is useful to perform a high-frequency expansion in terms of

the initial Hamiltonian $H_0(0)/\Omega$. Formally, we rescale the static component at $\lambda = 0$ by $H_0(0) \rightarrow H_0(0)\zeta$ and then organize the Hamiltonian components for $\lambda > 0$ in powers of ζ :

$$H_0(\lambda) = \zeta H_0^{(1)}(\lambda) + \zeta^2 H_0^{(2)}(\lambda) + \dots, \quad (17a)$$

$$H_1(\lambda) = H_1^{(0)}(\lambda) + \zeta H_1^{(1)}(\lambda) + \dots. \quad (17b)$$

Note that ζ simply serves as a convenient dimensionless (bookkeeping) parameter; the actual expansion is organized in powers of $H_0(0)/\Omega$.

The expansion in Eq. (17) clearly shares some similarity with the standard Floquet-Magnus expansion, which is believed to capture the asymptotic high-frequency behavior of a Floquet system [66]. For instance, the leading-order term can be computed analytically, as discussed in detail in Appendix. D. Under the simplifying condition $H_1(0) = H_{-1}(0)$, we obtain, at leading order in ζ ,

$$H_0^{(1)}(\lambda) = [J_1(\hat{z}e^{-\lambda\Omega})Y_0(\hat{z}) - Y_1(\hat{z}e^{-\Omega\lambda})J_0(\hat{z})] \times \frac{\pi}{2}e^{-\Omega\lambda}\hat{z}H_0(0), \quad (18a)$$

$$H_1^{(0)}(\lambda) = H_1(0)e^{-\Omega\lambda}, \quad (18b)$$

where $\hat{z} = 2\text{Ad}_{H_1(0)}/\Omega$ is an operator that acts on $H_0(0)$ and J_m and Y_m are Bessel functions which are interpreted as a formal series in $\hat{z}$. We use the notation Ad_X as the adjoint operator: $\text{Ad}_X Y = [X, Y]$.

The leading-order solution in Eq. (18) describes the approach to the prethermal fixed point in the ramp-up regime. The typical fRG timescale associated with this approach is *rapid*, of $\mathcal{O}(\Omega^{-1})$. As we approach the fixed point, the oscillatory component $H_1(\lambda)$ decays exponentially with the fRG time λ , and the static component $H_0(\lambda)$ approaches a high-frequency effective Hamiltonian. The

resulting effective Hamiltonian agrees with the (modified) Magnus expansion:

$$H_0^{(1)}(\lambda \rightarrow \infty) = H_F^{(0)} \equiv J_0 \left(\frac{2\text{Ad}_{H_1(0)}}{\Omega} \right) H_0(0), \quad (19)$$

which serves as the leading-order description of the prethermal fixed point. Here, the $\lambda \rightarrow \infty$ limit can be refined to $\lambda \gg (1/\Omega) \ln[\|H_1(0)\|/(\Omega\|\hat{1}\|)]$ based on Eq. (18). Higher-order corrections can be obtained order by order using the expansion in Eq. (17). However, we find that the higher-order terms are *different* from the usual Magnus expansion. In particular, the correction term from the Magnus expansion for the present problem would be $\mathcal{O}(1/\Omega)$, but from the above flow expansion we obtain the correction to be $\mathcal{O}(1/\Omega^2)$.

The difference between these two approaches have a deeper explanation that can be revealed from the point of view of a gauge dependence on the choice of t_0 —the initial time for the drive cycle—in Eq. (2). In general, the Floquet Hamiltonian is gauge dependent, as a different choice of t_0 leads to different H_F that is related via a gauge transformation. The Floquet-Magnus expansion directly approximates H_F and, therefore, inherits the gauge dependence on t_0 . The expansion yields $H_F = H_F^{(0)} + H_F^{(1)} + \mathcal{O}(1/\Omega^2)$. The leading-order term $H_F^{(0)}$ is given by Eq. (19). The first appearance of an explicit gauge dependence in the Floquet-Magnus expansion occurs at the next-to-leading order, which takes the form

$$H_F^{(1)} = -\sum_{m \neq 0} \frac{1}{m\Omega} [h_m, h_0]. \quad (20)$$

Here, h_m is the Fourier coefficient of the comoving frame Hamiltonian (see Appendix D for more details). Note that Eq. (20) is explicitly gauge dependent under a shift of $t_0 \rightarrow t_0 + \delta t$, where $h_m \rightarrow h_m \exp(im\Omega\delta t)$.

On the other hand, the flow equation yields an approximation which is *manifestly gauge invariant* and is, therefore, able to capture many of the *intrinsic* gauge-invariant timescales in a more elegant fashion. This follows from the fact that the flow equation is invariant under $H_1 \rightarrow e^{i\theta} H_1$, $H_0 \rightarrow H_0$. Therefore, the static component $H_0(\lambda)$ represents a gauge-averaged approximation of H_F , and, in hindsight, it is natural to expect a difference between the standard Floquet-Magnus expansion and the flow result.

Let us now apply the solution in Eq. (18) to explicitly address dynamical freezing. The setup is as follows: We choose the bare drive Hamiltonian $H_1(0) = H_{-1}(0) = A\mathcal{Q}$. Here, $\mathcal{Q}$ is the charge of the emergent symmetry, and A is the driving amplitude, as before. We choose the bare static Hamiltonian to be $H_0(0) = H_{0,\mathcal{Q}=0} + H_{0,\mathcal{Q}=1}$. As the subscripts suggest, the two parts have zero and unit

eigenvalues, respectively, under the action of $\mathcal{Q}$: $[\mathcal{Q}, H_{0,\mathcal{Q}=q}] = qH_{0,\mathcal{Q}=q}$. The static Hamiltonian associated with the prethermal fixed point is then

$$H_0(\lambda_{\text{pre}}) \approx H_{0,\mathcal{Q}=0} + J_0 \left(\frac{2A}{\Omega} \right) H_{0,\mathcal{Q}=1} + \mathcal{O} \left(\frac{1}{\Omega^2} \right). \quad (21)$$

Here, we have chosen to flow the system to $\lambda = \lambda_{\text{pre}}$, such that $\Omega\lambda_{\text{pre}} \gg 1$, but still short compared to the thermalization timescales, which is discussed later. Therefore, by choosing $2A/\Omega = \nu_i$, where ν_i are zeros of the Bessel function $J_0(\dots)$, the prethermal Hamiltonian will display an (approximate) emergent symmetry $[H_0(\lambda_{\text{pre}}), \mathcal{Q}] = \mathcal{O}(1/\Omega^2)$ in the high-frequency limit. To leading order, the locations of the freezing points agree with the prediction of the Magnus expansion.

The leading-order solution in Eq. (18) also helps explains several features of the numerical results presented in Sec. IV. First, in the approach to the prethermal Hamiltonian, the commutator $[H_0(\lambda), \mathcal{Q}]$ shows dips whose number is given by $(n-1)$ for a freezing point FP_n , respectively. Substituting $\hat{z} = \nu_i$ into Eq. (18), we see that the dips are associated with zeros of $J_1(\dots)$, which correspond to extreme points of J_0 , and between every zero of J_0 there is one extreme point. Second, to approach the prethermal fixed point, we require $(2A/\Omega) \exp(-\Omega\lambda) \ll 1$. Therefore, for higher freezing points corresponding to larger A/Ω , it takes longer fRG time to reach the prethermal plateau.

However, as is well known, for a generic driven system that thermalizes to an infinite-temperature state, the expansion in Eq. (17) must eventually break down; the higher-order terms in H_0/Ω should become increasingly relevant at a later fRG time. This connection between the breakdown of Eq. (17) and heating can be proven based on our earlier discussion of Eq. (5). Suppose there is no breakdown, such that the exponential decay of $H_1(\lambda)$ implies that $\exp[-itH_0(\lambda)]$ is an increasingly better approximation of the actual time-evolution operator. However, this contradicts heating because $\exp[-itH_0(\lambda)]$ conserves energy. In other words, the prethermal fixed point is unstable in a generic nonintegrable system. In the next subsection, we present a qualitative and semianalytic treatment of the subsequent dynamics even at freezing, beyond the perturbative considerations of the present section.

B. Late time: Thermalization via instantons

To describe the nonlinear effects in Eq. (3), consider running the flow equations up to some initial fRG time λ_{pre} , during which the prethermal flow in Eq. (18) is a good approximation. Recall that λ_{pre} satisfies $\lambda_{\text{pre}} \gg \Omega^{-1}$ but is smaller than the *a priori* unknown thermalization fRG time. We also assume that heating is sufficiently slow so that λ_{pre} satisfies $\|H_1(\lambda_{\text{pre}})\| \ll \|H_0(\lambda_{\text{pre}})\|$. Starting from $\lambda = \lambda_{\text{pre}}$,

we can integrate Eq. (3b) by assuming $H_0(\lambda) \approx H_0(\lambda_{\text{pre}})$, yielding

$$H_1(\lambda) \approx e^{-\Omega(\lambda-\lambda_{\text{pre}})} e^{-(\lambda-\lambda_{\text{pre}})H_0(\lambda_{\text{pre}})} H_1(\lambda_{\text{pre}}) e^{(\lambda-\lambda_{\text{pre}})H_0(\lambda_{\text{pre}})}. \quad (22)$$

Expanding Eq. (22) in the eigenbasis $(E_m, |m\rangle)$ of $H_0(\lambda_{\text{pre}})$, we find

$$\langle m|H_1(\lambda)|n\rangle \approx e^{(-\Omega-E_m+E_n)(\lambda-\lambda_{\text{pre}})} \langle m|H_1(\lambda_{\text{pre}})|n\rangle. \quad (23)$$

According to Eq. (23), $H_1(\lambda)$ will decay if $\Omega > |E_m - E_n|$ for all pairs of $\{m, n\}$ with $\langle m|H_1(\lambda_{\text{pre}})|n\rangle \neq 0$. This case corresponds to the prethermal fixed point being stable, and the associated expansion in Eq. (17) is expected to converge. However, this condition is not always satisfied, even for exponentially local Hamiltonians. Taking a spin chain as an example, an exponentially local drive term $H_1(\lambda_{\text{pre}})$ can contain terms that flip l spins simultaneously for arbitrary l , despite its amplitude decaying exponentially in l . If the typical energy of a single spin flip is J , the matrix elements for $l > \mathcal{O}(\Omega/J)$ spin flips will eventually grow after a sufficiently long fRG time. In Appendix E, we provide an estimate for these timescales associated with the evolution of $\|H_1(\lambda)\|$.

We argue that $\|H_1(\lambda)\|$ is expected to show a minimum at $\lambda = \lambda_{\text{min}} \sim (1/J) \ln(\Omega/J)$, where $\|H_1(\lambda)\| \sim \exp[-(\Omega/J) \ln(\Omega/J)]$; see Fig. 17 in Appendix E for numerical evidence of such scaling. This implies that we can choose $\lambda_{\text{pre}} = \lambda_{\text{min}}$, and $H_0(\lambda_{\text{min}})$ is a good approximation of the time evolution up to exponentially long time in Ω . This is consistent with generic expectations that heating in rapidly driven system is exponentially slow in Ω [30–38]. When $\lambda \gg \lambda_{\text{min}}$, we expect a growth in $\|H_1(\lambda)\|$, where different operators in $H_1(\lambda)$ are expected to have distinct thermalization timescales. For an operator that flips l spins, the thermalization timescale is expected to be

$$\lambda_{\text{th}}^{(l)} \sim \frac{f(l) + l \ln(\Omega/J)}{lJ - \Omega}, \quad (24)$$

where the function $f(l)$ is a dimensionless function of l that depends on the microscopic details of the system. The timescale $\lambda_{\text{th}}^{(l)}$ is such that, at $\lambda = \lambda_{\text{th}}^{(l)}$, the matrix element of $H_1(\lambda)$ corresponding to l spin flips becomes $\mathcal{O}(1)$. As discussed before, only operators that have grown to be longer than $l > \mathcal{O}(\Omega/J)$ lead to that associated thermalization. For $l \gg \mathcal{O}(\Omega/J)$, $\lambda_{\text{th}}^{(l)}$ is parametrically similar to λ_{min} .

We can describe the evolution of $H_1(\lambda)$ approximately, under the assumption that only one of the matrix elements $\langle m|H_1(\lambda)|n\rangle$ grows to be $\mathcal{O}(1)$ —we dub this a single-instanton event. In this scenario, we ignore all of the other matrix elements of $H_1(\lambda)$. Then, the term $[H_1, H_1^\dagger]$ in

Eq. (3) becomes approximately diagonal in the instantaneous eigenbasis of H_0 . The flow equations simplify to

$$\partial_\lambda E_m(\lambda) = 2|\langle m|H_1(\lambda)|n\rangle|^2, \quad (25a)$$

$$\partial_\lambda E_n(\lambda) = -2|\langle m|H_1(\lambda)|n\rangle|^2, \quad (25b)$$

$$\partial_\lambda \langle m|H_1(\lambda)|n\rangle = [-\Omega - E_m(\lambda) + E_n(\lambda)] \langle m|H_1(\lambda)|n\rangle. \quad (25c)$$

Here, $E_m(\lambda)$ and $E_n(\lambda)$ are the eigenvalues of the instantaneous eigenvectors $\{|m\rangle, |n\rangle\}$ of $H_0(\lambda)$, and $|m\rangle$ and $|n\rangle$ are approximately independent of λ . The analytical solutions are given by

$$E_n(\lambda) - E_m(\lambda) = \Omega - \tilde{\Omega} \tanh[\tilde{\Omega}(\lambda - \tilde{\lambda})], \quad (26a)$$

$$|\langle m|H_1(\lambda)|n\rangle| = \frac{\tilde{\Omega}}{2 \cosh[\tilde{\Omega}(\lambda - \tilde{\lambda})]}, \quad (26b)$$

where $\tilde{\Omega}$ and $\tilde{\lambda}$ are integration constants determined from initial conditions at $\lambda = \lambda_{\text{pre}}$. Several remarks are now in order.

For a generic nonintegrable Hamiltonian, we expect the above single-instanton scenario to be a good approximation for the thermalization dynamics, in the absence of any special degeneracies in $E_m(\lambda) - E_n(\lambda)$, and when only one pair of levels are coupled strongly by $H_1(\lambda)$ via $\langle m|H_1(\lambda)|n\rangle$. Under this assumption, different matrix elements of $H_1(\lambda)$ will become $\mathcal{O}(1)$ at different fRG times, and at a given fRG time only one particular matrix element is large. As a corollary, near $\lambda = \tilde{\lambda}$, the norm $\|H_1(\lambda)\|$ will be dominated by a single matrix element. This line of reasoning and the associated analytical form of $\|H_1(\lambda)\|$ in Eq. (26b) is consistent with our numerical results, as shown by the excellent fit in Fig. 7 (additional examples are also shown in Appendix C). We note that, in the presence of additional symmetries that enforce degeneracies in $E_m(\lambda) - E_n(\lambda)$, Eqs. (25) can be extended to include the whole irreducible representation of the symmetry. We leave this generalization for future work.

Because of the explicitly broken time-translation symmetry, the Floquet Hamiltonian is well defined only modulo Ω . However, at $\lambda = 0$ the initial Hamiltonian $H_0(0)$ can have a many-body bandwidth larger than Ω . The instanton solution in Eq. (26) connects these two limits by describing the band-folding dynamics of $H_0(\lambda)$; see, e.g., the schematic in Fig. 1. During the instanton event, $(E_n - E_m)$ decreased from $\Omega + \tilde{\Omega}$ to $\Omega - \tilde{\Omega}$. If $|\tilde{\Omega}| > \tilde{\Omega}$, there will appear to be an accidental degeneracy at $\lambda = \tilde{\lambda}$. We expect that this degeneracy will be lifted after including the other matrix elements neglected in Eq. (25). Additionally, we note that, when $|\Omega - \tilde{\Omega}| > \Omega$, a single-instanton event is not enough to fold (E_n, E_m) into an

energy interval shorter than Ω . In that case, this can trigger subsequent instanton events driven by the matrix element $\langle n|H_1(\lambda)|m\rangle$ (note that m and n are exchanged), where now it satisfies the growth criterion $E_m - E_n > \Omega$. On the contrary, there will be no subsequent instanton events in $\langle n|H_1(\lambda)|m\rangle$ nor $\langle m|H_1(\lambda)|n\rangle$ if $|\Omega - \tilde{\Omega}| < \Omega$. Therefore, the energy levels of $H_0(\lambda)$ are folded step by step in a series of instanton events. When no further instanton events occur, $H_0(\lambda)$ has the property that $|E_m - E_n| < \Omega$ for any pair of eigenstates m, n , and at the same time $H_1(\lambda)$ will decay exponentially to zero.

VII. OUTLOOK

Going beyond conventional Floquet-Magnus high-frequency expansions and numerical exact diagonalization of microscopic Hamiltonians, here we have employed the Floquet flow formalism to analyze the phenomenology of dynamical freezing from a fresh (and universal) perspective. Most interestingly, we find that the bare driven Hamiltonian flows through a complex landscape of (unstable) fixed points and shows a strong tendency toward *slow* thermalization at the dynamically frozen points. Freezing itself is associated with an emergent approximate conservation law, that appears at the first prethermal fixed point along the flow. By combining numerical solutions of the flow equations using exact diagonalization and matrix-product operator-based techniques, as well as analytical approximations that are nonperturbative in the inverse drive frequency (Ω), we have shown that freezing is approximate at any finite Ω . Moreover, the violation of the emergent conservation law, quantified by the commutator $[H_0(\lambda), \mathcal{Q}]$, vanishes as $1/\Omega^2$ in the high-frequency limit.

One of the key contributions of this work is the analysis of thermalization dynamics of the system beyond the prethermal fixed point, where we find that the flow between the intermediate fixed points can be described using *instanton* events. For the sake of concreteness, we have chosen to study the driven spin chain as our main numerical example using complementary methods. However, the demonstrated features for freezing and thermalization are quite generic, as demonstrated by the beautiful agreement between our numerical results and the general model-independent analytical computations. Using a combination of different numerical diagnostics, we have demonstrated that the system flows to a random-matrix-like Hamiltonian when the frequency Ω is below a system-size-dependent threshold $\Omega_*(L)$. Because of present numerical limitations of simulating only small system sizes, we have been able to obtain only the weakly increasing L dependence of $\Omega_*(L)$ in a narrow range. Further numerical and analytical advances will be required to obtain the form of $\Omega_*(L)$ for a generic nonintegrable Hamiltonian. If $\Omega_*(L)$ diverges with L in the thermodynamic limit, that indicates fully thermalizing behavior even at dynamical freezing. However, it is possible that dynamical freezing may slow

down the rate of divergence or keep $\Omega_*(L)$ to be finite in the $L \rightarrow \infty$. This remains an interesting open problem for the future.

From our perspective, the fundamental new ingredient offered by the flow formalism at freezing analyzed here corresponds to the instanton events. We speculate that the instantons correspond to the real-time dynamics in the following sense: In the high-frequency limit, the system starts to exhibit heating after a time $t_{\text{th}} \sim (\Omega/J)^{\Omega/J}$. However, the energy absorption is not a continuous process, and the system can absorb only a quantum of Ω each time. The energy absorption process is associated with a transition between two eigenstates $|m\rangle$ and $|n\rangle$ of the prethermal Hamiltonian whose energy differs by integer multiples of Ω [38], and the transition is alternatively described as the hybridization between $|m\rangle$ and $|n\rangle$ due to the drive. We speculate that (a) the instanton event effectively implements the hybridization process, by mixing the eigenvectors in a step-by-step fashion. In addition, the energy absorption events are expected to be rare in real time, and the typical interval between them is of the order of t_{th} . This implies that, after absorbing one energy quantum, the system should enter a new “prethermal” regime as the next absorption event is t_{th} away. To that end, we conjecture that (b) the intermediate fixed points in between the instantons are the effective Hamiltonians for these new “prethermal” regimes. To verify this picture, it requires simulations of real-time dynamics up to exponentially long times, which is a challenging but exciting future direction.

We note that, despite the simple (yet elegant) solution in Eq. (26) for a single-instanton event, the thermalization dynamics is still quite complex and “chaotic” in nature. There are several aspects that our simple analytical treatment does not capture: First, in the single-instanton scenario, we have focused on only a single large matrix element which grows to $\mathcal{O}(1)$ but ignored the others, which is sufficient to describe the current instanton event. However, the other possibly small matrix elements are likely strongly perturbed by the presence of the instantons, which substantially affects the following instanton events. Second, while we have argued that single-instanton events are more likely, it is possible that two instanton events happen in relatively quick succession [e.g., in Fig. 7(a)]. We leave a detailed analytical study of such two- and multiple-instanton events for future investigation.

We conclude by discussing several additional future directions. In the current work, we have considered only the cosine-wave drive for simplicity, as there are just two Fourier components in the flow equation. Recently, there has been numerical evidence suggesting that dynamical freezing is more robust for square-wave drive [45,53]. The flow formalism can be generalized to describe the square-wave drive, by introducing an infinite series of Hamiltonian components corresponding to the Fourier decomposition of the drive function [63]. Whether the flow away from the

first prethermal fixed point is similar in character to the example studied here, but with possibly delayed timescales, is an interesting open question.

Time evolution of matrix-product states and operators is, in general, computationally demanding, due to operator spreading and entanglement growth. In our setup, this problem is mitigated by dynamical freezing, where the slower entanglement growth compared to nonfreezing points enabled simulation up to timescales long enough to access the prethermal regime. In the future, it would be useful to combine constraints imposed by the emergent symmetry at freezing with the recently developed MPO compression algorithm [88] to enable numerical simulation up to even longer timescales. Studying the operator spreading using these algorithms, both at and away from freezing, remains an interesting future direction.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [92].

APPENDIX A: CONNECTION BETWEEN FLOW RENORMALIZATION AND SCHRIEFFER-WOLFF TRANSFORMATION

In this section, we provide a brief pedagogical review of the connection between the flow-renormalization approach [56] and the more standard Schrieffer-Wolff (SW) transformation applied to time-independent Hamiltonians [65]. The SW transformation is often used to perturbatively eliminate the “off-diagonal” high-energy terms in a Hamiltonian, resulting in an effective Hamiltonian within a low-energy subspace. The flow-RG equations essentially use a sequence of such infinitesimal SW transformations to gradually achieve this decoupling over a finite energy interval at a time. The SW transformation defines a unitary transformation $U = e^S$, where S is an anti-Hermitian operator (i.e., $S = -S^\dagger$). The transformed Hamiltonian H' is given by

$$H' = e^S H e^{-S} = H + [S, H] + \frac{1}{2}[S, [S, H]] + \dots, \quad (\text{A1})$$

where S is chosen to eliminate the terms that couple different subspaces (e.g., low-energy and high-energy subspaces).

Typically, we write the original Hamiltonian as $H = H_0 + V$, where H_0 is the unperturbed part and V is a perturbative coupling term. Then, S is chosen such that

$$[S, H_0] = -V. \quad (\text{A2})$$

This condition ensures that the transformed Hamiltonian H' is approximately block diagonal to a certain order in V , decoupling the low-energy and high-energy subspaces.

Within the flow equation approach [56,58–60,71], the Hamiltonian H is transformed continuously by defining a family of Hamiltonians $H(l)$, where $H(0) = H$ and $H(l)$ flows toward a diagonal (or block-diagonal) form as $l \rightarrow \infty$. The flow is governed by the differential equation

$$\frac{dH(l)}{dl} = [\eta(l), H(l)], \quad (\text{A3})$$

where $\eta(l)$ is the generator of the transformation, chosen to be

$$\eta(l) = [H_{\text{diag}}(l), H_{\text{off-diag}}(l)]. \quad (\text{A4})$$

Here, $H_{\text{diag}}(l)$ and $H_{\text{off-diag}}(l)$ are the diagonal and off-diagonal parts of $H(l)$, respectively. This choice of $\eta(l)$ ensures that the flow gradually drives $H(l)$ toward a diagonal form as $l \rightarrow \infty$, effectively decoupling the low-energy and high-energy sectors.

The ideas discussed above can be generalized to the Floquet context using the Sambe representation [63,93]. In this representation, the time-dependent Hamiltonian $H(t) = \sum_m e^{im\Omega t} H_m$ is represented in an extended Hilbert space, where the Floquet operator $H(t) - i\partial_t$ is represented as

$$\mathcal{H} = \sum_m \left[m\Omega \otimes |m\rangle\langle m| + \sum_M H_M \otimes |m+M\rangle\langle m| \right]. \quad (\text{A5})$$

The Sambe representation has a gauge redundancy, so that the eigenvalues of $\mathcal{H}$ can be labeled as

$$\mathcal{H}|\phi_{nM}\rangle = (\varepsilon_n + M\Omega)|\phi_{nM}\rangle, \quad (\text{A6})$$

for any $M \in \mathbb{Z}$. In particular, ε_n are the eigenvalues of the Floquet Hamiltonian H_F .

Following Ref. [63], the Floquet flow is constructed as a unitary flow in the Sambe representation, which automatically preserves the eigenspectrum of $\mathcal{H}$ and, hence, the spectrum of H_F . To ensure unitarity, the flow should be

generated by a commutator between the generator $\eta(l)$ and $\mathcal{H}(l)$ as in Eq. (A3). Additionally, the generator should be chosen such that it strives to remove the oscillatory parts of $\mathcal{H}$ and it does not generate higher harmonics not present in $\mathcal{H}$. The generator that satisfies the above requirements is

$$\eta(l) = \sum_m \sum_{M>0} [H_M(l) \otimes |m+M\rangle\langle m| - \text{H.c.}]. \quad (\text{A7})$$

When there are only two Fourier components H_0 and H_1 , the flow reduces to Eq. (3) in the main text.

APPENDIX B: ADDITIONAL NUMERICAL DETAILS

Here, we provide some additional details about our numerical simulations. In our simulations, we did not explicitly fix a maximum bond dimension; instead, we used the default singular value cutoff in ITensors.jl, which is 10^{-12} . For all numerical results involving MPO evolution, the bond dimensions of both $H_0(\lambda)$ and $H_1(\lambda)$ remained below 500. To compute the entanglement entropy, we initialized the matrix-product state (MPS) in a fully polarized product state along the x direction and evolved it using the TDVP method. Since this initial state is unentangled, we employed the global-Krylov method to enrich the MPS with Krylov vectors. For the time evolution of the MPS, we employ the default single-site TDVP algorithm; for further details, see Ref. [94]. During the time evolution of the MPS, the singular value decomposition cutoff was set to 10^{-10} . For the time evolution using the RK4 method, we set the integration step size to 10^{-2} for both the exact and MPS-based calculations. When studying the Ω scaling of the commutator at higher frequencies—where the associated time period becomes shorter—we reduce the step size to 10^{-3} to accurately resolve the faster dynamics.

APPENDIX C: ADDITIONAL NUMERICAL RESULTS

In this appendix, we present additional numerical results that support many of the important conclusions presented in the main text.

1. Effect of static transverse magnetic field

In continuation of the discussion in Sec. IV B, in Fig. 10(a), we plot the expectation value of S_x in the eigenbasis of $H_0(\lambda_c)$, both with and without B_x , at and away from the freezing point. At the freezing point, the effective Floquet Hamiltonian to leading order is proportional to $YY + ZZ$ terms; see Eq. (F3) in Appendix F. This structure gives rise to an emergent $\mathbb{Z}_2$ symmetry, resulting in degenerate Floquet eigenstates [equivalently, eigenstates of $H_0(\lambda_c)$], and, consequently, the expectation values of S_x are nearly zero for all states. Introducing a small nonzero B_x

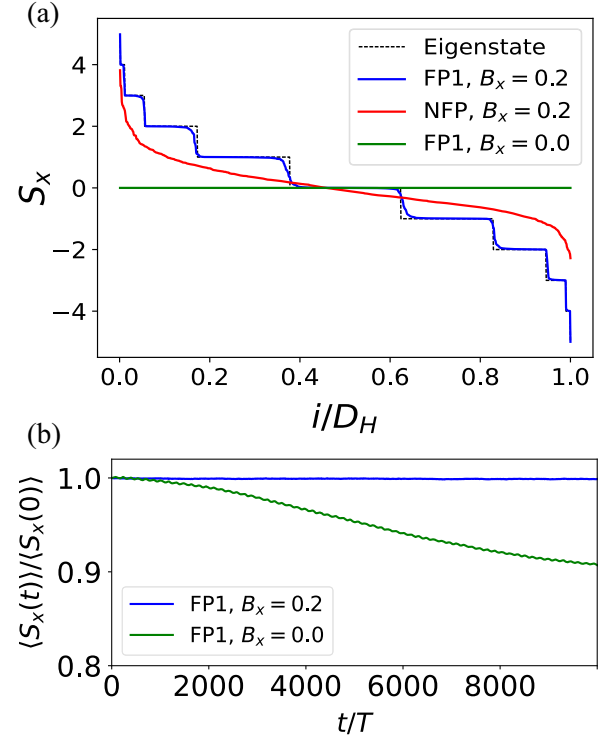


FIG. 10. (a) The values of S_x for the eigenstates of $H_0(\lambda_c)$ are plotted in decreasing order, comparing the freezing point (FP1) and the NFP, both with and without the magnetic field term in the x direction. Without the magnetic field in the x direction, at the freezing point, the leading-order effective Floquet Hamiltonian contains terms proportional to $YY + ZZ$, which implies a global $\mathbb{Z}_2$ symmetry under $X_i \rightarrow -X_i$. As a result, the expectation value of S_x in the eigenstates of $H_0(\lambda_c)$ is vanishingly small. Parameters: $L = 10$, $J = 1$, $J_2 = 0.2$, $\Omega = 5$, and $\lambda_c = 5.0$. (b) Magnetization is plotted as function of real time at the first freezing point with and without the magnetic field term B_x . Parameters: $L = 10$, $J = 1$, $J_2 = 0.5$, and $\lambda_c = 5.0$.

lifts the degeneracy, making the Floquet eigenstates to align with the eigenstates of S_x . In Fig. 10(b), we show the real-time magnetization profile for a thermal pure state evolving under $H_0(\lambda_c)$.

We find that the presence of B_x enhances the quality of freezing. In Fig. 11, we show late (real-)time magnetization at the first freezing point for different frequency. Even though at the freezing point the dynamics “slows down,” the magnetization still decays, indicating only an approximate conservation law. In this simulation, Ω is chosen large enough to avoid resonances between the eigenstates of the unperturbed part of the Hamiltonian. For smaller driving frequencies (but $\Omega \gtrsim J$), even at the first freezing point, the magnetization decays substantially. For the same driving frequency, at a higher-order freezing point (blue curve, $A = 15.32$) compared with the lower-order case (yellow curve, $A = 6.01$), the magnetization still decays from its initial value. This indicates that, while the presence of a magnetic field enhances freezing at the level of the

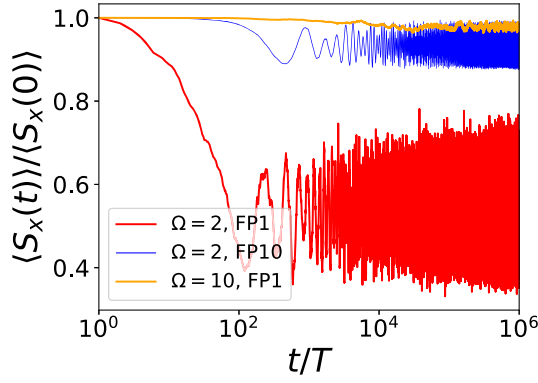


FIG. 11. Magnetization for a thermal pure state [evolving under Hamiltonian $H_0(\lambda_c)$] as a function of real time at the first freezing point for two different driving frequencies. Parameters: $L = 10$, $J = 1$, $J_2 = 0.5$, $B_x = 0.1$, and $\lambda_c = 10$.

expectation value of S_x , it cannot prevent the eventual decay; see also the section on diagonal ensemble average in Supplemental Material [73] for a related discussion. Whether this decay ultimately leads to thermalization to infinite temperature in the $L \rightarrow \infty$ limit remains an open question.

2. Frequency scaling of $P(\lambda_c)$

Recall the definition of $P(\lambda)$ in Eq. (12), which helps identify the extent of freezing and the emergent conservation law. In Fig. 12, we plot $P(\lambda_c)$ at two distinct freezing points with increasing Ω , where λ_c is associated with the initial prethermal plateau. Interestingly, we find that $P(\lambda_c)$ decays as $1/\Omega^2$ at large Ω , which is consistent with our analytical results in Sec. VI. Relatedly, $Q(\lambda)$ [Eq. (13)] falls off exponentially with the drive frequency Ω (Fig. 12 inset). We also point out that $P(\lambda)$, which is normalized to be an intensive quantity, already appears to show convergence to

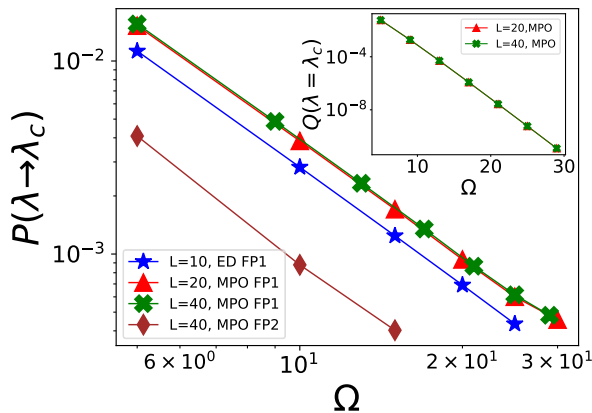


FIG. 12. $P(\lambda \rightarrow \lambda_c)$ for $(L = 10, 20, 40)$ at the first freezing point (FP₁) exhibits a universal $1/\Omega^2$ scaling. Inset: $Q(\lambda = \lambda_c)$ for different system sizes exhibits an exponential decay with increasing driving frequency. The data near FP₂ also follow the same frequency scaling.

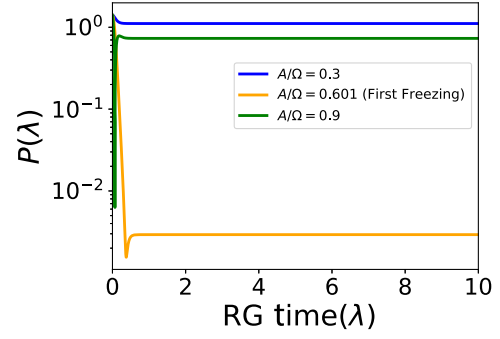


FIG. 13. Normalized commutator $P(\lambda)$ for $L = 10$ as a function of λ at and away from freezing. All other parameters are the same as in Fig. 3.

the thermodynamic limit for $L = 20$, given the lack of any appreciable difference from the $L = 40$ results.

3. Dynamical freezing over longer fRG times

In the main text, we show results for the plateau associated with $P(\lambda)$ up to $\lambda \sim 1$. However, the plateau extends until later fRG times, as shown in Fig. 13, both at and away from freezing point for $\Omega = 10$.

4. Renormalization of freezing points

To examine the possible renormalization of the freezing point away from the leading-order Floquet-Magnus predictions, we compute numerically the location of the minima of $P(\lambda_c)$ for various driving frequencies. In Fig. 14, a fine-grid scan of $P(\lambda_c)$ is performed near FP₁ as a function of A/Ω for two different system sizes ($L = 8, 10$) using flow ED. We observe that, with increasing frequency, the minimum of $P(\lambda_c)$ approaches the value ($= 0.601$) predicted by the zeroth-order Magnus expansion. However, for lower frequencies, there is a slight shift in the location of the freezing point.

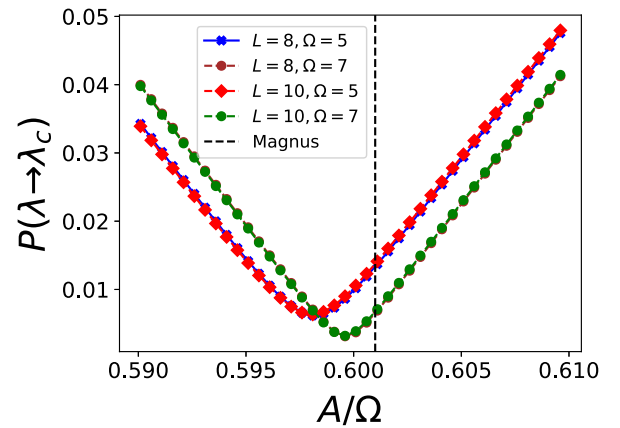


FIG. 14. A fine-grid scan for $P(\lambda \rightarrow \lambda_c)$ around FP₁, for $(L = 8, 10)$ and two different driving frequencies. The dashed line represents the prediction from the zeroth-order Magnus term.

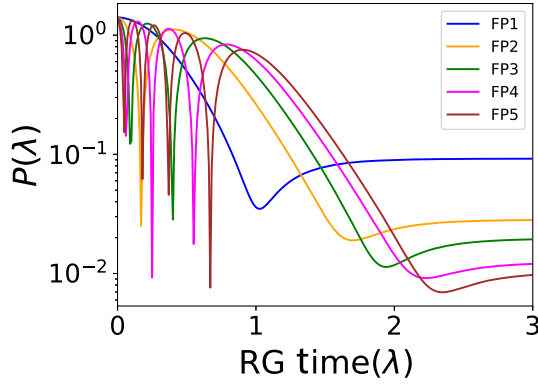


FIG. 15. $P(\lambda)$ is plotted as a function of RG time for different freezing points, where the value of the asymptotic value decreases continuously for higher-order freezing points. Parameters: $L = 10$, $J = 1$, $J_2 = 0.2$, and $\Omega = 2.0$.

5. Higher-order freezing points

In Fig. 15, we illustrate the scenario where the driving frequency is comparable to the single-particle energy scales [in contrast to the choice of Ω in Fig. 3(a)], and the driving amplitude represents the largest energy scale in the system—effectively exploring higher-order freezing points while keeping the frequency fixed. In this case, we did not observe any qualitative differences; as the amplitude increases, the plateau value of $P(\lambda)$ decreases.

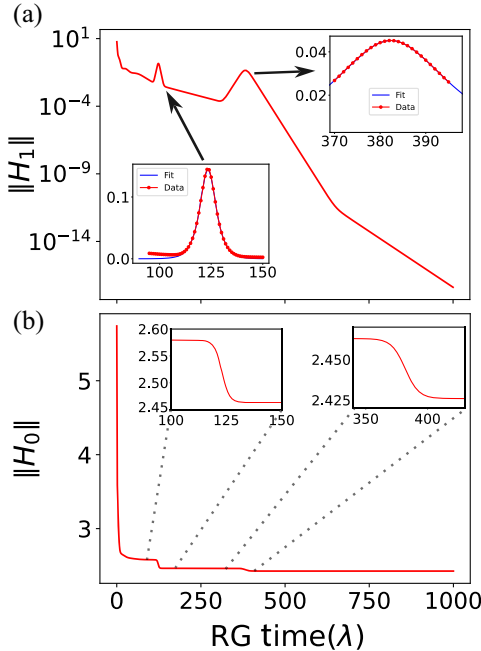


FIG. 16. (a) Frobenius norm of $H_1(\lambda)$ as a function of fRG time at FP_1 for $L = 6$, $\Omega = 1.0J$, $J = 1$, and $J_2 = 0.9J$. (b) Corresponding Frobenius norm of $H_0(\lambda)$. The two instanton events are marked by arrows in (a), along with the corresponding steplike features in (b).

6. Multiple separated instanton events

In Fig. 16(a), we plot the norm of H_1 as a function of fRG time using flow ED for a system of size $L = 6$. The two well-separated instanton events are clearly visible, where the norms of $H_1(\lambda)$ peak at approximately $\lambda \approx 125$ and $\lambda \approx 380$, respectively. In Fig. 16(b), the corresponding norm of $H_0(\lambda)$ is shown, displaying a clear steplike feature precisely at the locations where $H_1(\lambda)$ peaks. Calculations for longer fRG times are constrained by our numerical precision, which in this case is double precision (binary64).

APPENDIX D: EARLY-TIME ANALYSIS

In this appendix, we solve the flow equation in the early-time and large- Ω limit and compare the result with the (modified) Magnus expansion. The Hamiltonian we consider is

$$H(t) = H_0 + 2H_1 \cos(\Omega t), \quad (D1)$$

where we have chosen $H_{-1} = H_1$ and $H_1 = H_1^\dagger$ in Eq. (1) at $\lambda = 0$.

1. Review of Floquet-Magnus expansion

To set the stage for a comparison with our flow analysis, we first perform a (modified) Floquet-Magnus expansion for Eq. (D1), by transforming to the comoving frame Hamiltonian. This is achieved by the unitary

$$W(t) = \exp\left(-\frac{2iH_1}{\Omega} \sin(\Omega t)\right), \quad (D2)$$

and the comoving Hamiltonian is

$$\begin{aligned} H_{\text{mov}}(t) &= W(t)^\dagger (H(t) - i\partial_t) W(t) \\ &= \exp\left(i \frac{2 \sin(\Omega t) \text{Ad}_{H_1}}{\Omega}\right) H_0. \end{aligned} \quad (D3)$$

Here, $\text{Ad}_X Y$ is an adjoint operator acting on Y , i.e., $\text{Ad}_X Y = [X, Y]$.

The Floquet-Magnus expansion connects $H_{\text{mov}}(t)$ to the effective Floquet Hamiltonian H_F . The first two orders of the expansion are given by

$$H_F^{(0)} = \frac{1}{T} \int_0^T dt H_{\text{mov}}(t), \quad (D4a)$$

$$H_F^{(1)} = \frac{1}{2iT} \int_0^T dt_1 \int_0^{t_1} dt_2 [H_{\text{mov}}(t_1), H_{\text{mov}}(t_2)]. \quad (D4b)$$

To evaluate the integrals, we decompose H_{mov} into Fourier components:

$$H_{\text{mov}}(t) = \sum_m h_m e^{im\Omega t}, \quad \text{where} \quad (D5a)$$

$$h_m = J_m \left(\frac{2 \text{Ad}_{H_1}}{\Omega} \right) H_0 \quad (D5b)$$

and J_m is the Bessel function of the first kind. We thus obtain

$$H_F^{(0)} = h_0, \quad (\text{D6a})$$

$$H_F^{(1)} = \sum_{m,n} I_{mn}^{(1)} [h_m, h_n], \quad (\text{D6b})$$

where the coefficient $I_{mn}^{(1)}$ is given by

$$I_{mn}^{(1)} = \frac{1}{2iT} \int_0^T dt_1 \int_0^{t_1} dt_2 \exp[i\omega(mt_1 + nt_2)]. \quad (\text{D7})$$

The only nonzero values for $I_{mn}^{(1)}$ are

$$I_{0m}^{(1)} = -I_{m0}^{(1)} = I_{m,-m}^{(1)} = \frac{1}{2m\Omega}, \quad m \neq 0, \quad (\text{D8})$$

and $I_{00}^{(1)} = -(i\pi)/(2\Omega)$. Since $[h_m, h_{-m}] = 0$, we need only $I_{m0}^{(1)}$ and $I_{0m}^{(1)}$. Therefore, we obtain

$$H_F^{(1)} = 2 \sum_{m \neq 0} \frac{-1}{2m\Omega} [h_m, h_0]. \quad (\text{D9})$$

2. Solution of flow equation

In this subsection, we solve the flow equation in the early-time limit, which we reproduce below:

$$\partial_\lambda H_0(\lambda) = 2[H_1(\lambda), H_1^\dagger(\lambda)], \quad (\text{D10a})$$

$$\partial_\lambda H_1(\lambda) = -\Omega H_1(\lambda) - [H_0(\lambda), H_1(\lambda)], \quad (\text{D10b})$$

where the initial condition is chosen to satisfy $H_1(0) = H_1(0)^\dagger$.

We first integrate Eq. (D10b) to obtain

$$H_1(\lambda) = e^{-\Omega\lambda} U(\lambda) H_1(0) U(\lambda)^{-1}, \quad (\text{D11})$$

where

$$U(\lambda) = \mathcal{T} \exp \left(- \int_0^\lambda d\bar{\lambda} H_0(\bar{\lambda}) \right), \quad (\text{D12})$$

where $\mathcal{T}$ denotes time ordering in λ . Therefore, Eq. (D10) can be cast as

$$\partial_\lambda H_0(\lambda) = 2e^{-2\Omega\lambda} [\bar{H}_1(\lambda), \bar{H}_1(\lambda)^\dagger], \quad (\text{D13})$$

where

$$\bar{H}_1(\lambda) = U(\lambda) H_1(0) U(\lambda)^{-1}, \quad (\text{D14a})$$

$$\bar{H}_1(\lambda)^\dagger = U(\lambda)^{-1} H_1(0)^\dagger U(\lambda). \quad (\text{D14b})$$

Taking an additional derivative in Eq. (D13), we obtain a (nonlinear) second-order differential equation for $H_0(\lambda)$:

$$\begin{aligned} & \partial_\lambda^2 H_0(\lambda) + 2\Omega \partial_\lambda H_0(\lambda) \\ & + 2e^{-2\Omega\lambda} \left\{ \text{Ad}_{\bar{H}_1(\lambda)}, \text{Ad}_{\bar{H}_1(\lambda)^\dagger} \right\} H_0(\lambda) = 0. \end{aligned} \quad (\text{D15})$$

Here, $\bar{H}_1(\lambda)$ and $\bar{H}_1(\lambda)^\dagger$ depend on $H_0(\lambda)$, as described by Eqs. (D14a) and (D14b), and $\{\cdot, \cdot\}$ denotes the usual anticommutator.

We emphasize that all of the manipulations thus far are exact. To make connection with the Floquet-Magnus expansion above, we can expand Eq. (D15) in powers of $H_0(0)$, i.e.,

$$H_0(\lambda) = \zeta H_0^{(1)}(\lambda) + \zeta^2 H_0^{(2)}(\lambda) + \zeta^3 H_0^{(3)}(\lambda) + \dots, \quad (\text{D16})$$

where $H_0^{(k)}(\lambda)$ depends on the k th power of $H_0(0)$ and ζ is a convenient bookkeeping parameter to organize the expansion. We now obtain the forms of these corrections up to the third-order term.

a. First-order term

We first solve for $H_0^{(1)}$, which is obtained by setting $\bar{H}_1(\lambda) = \bar{H}_1(\lambda)^\dagger = H_1(0)$ in Eq. (D15), and the associated linear second-order differential equation:

$$\partial_\lambda^2 H_0^{(1)}(\lambda) + 2\Omega \partial_\lambda H_0^{(1)}(\lambda) + 4e^{-2\Omega\lambda} \text{Ad}_{H_1(0)}^2 H_0^{(1)}(\lambda) = 0. \quad (\text{D17})$$

The analytical solution, with an initial condition $H_0^{(1)}(0) = H_0(0)$, $\partial_\lambda H_0^{(1)}(0) = 0$, is given by

$$H_0^{(1)}(\lambda) = \frac{\pi}{2} e^{-\Omega\lambda} \hat{z} [J_1(\hat{z} e^{-\lambda\Omega}) Y_0(\hat{z}) - Y_1(\hat{z} e^{-\lambda\Omega}) J_0(\hat{z})] H_0(0), \quad (\text{D18})$$

where $\hat{z} = 2\text{Ad}_{H_1(0)}/\Omega$ and Y_m denotes Bessel function of the second kind. Taking the $\lambda \rightarrow \infty$ limit, we obtain a result consistent with the leading-order Magnus expansion:

$$H_0^{(1)}(\lambda \rightarrow \infty) = J_0 \left(\frac{2\text{Ad}_{H_1(0)}}{\Omega} \right) H_0(0). \quad (\text{D19})$$

We apply the results above to the driven Harmonic oscillator example discussed in Sec. III. It implies that the $U(1)$ -violating component $B_0(\lambda)$ of $H_0(\lambda)$, in the $\omega_0 \ll \Omega$ limit, will take the form

$$\begin{aligned} B_0(\lambda) = \frac{\pi A \omega_1}{2\Omega} e^{-\Omega\lambda} & \left[J_1 \left(\frac{A}{\Omega} e^{-\Omega\lambda} \right) Y_0 \left(\frac{A}{\Omega} \right) \right. \\ & \left. - J_0 \left(\frac{A}{\Omega} \right) Y_1 \left(\frac{A}{\Omega} e^{-\Omega\lambda} \right) \right]. \end{aligned} \quad (\text{D20})$$

Then, the $\lambda \rightarrow \infty$ limit reads $B_0(\lambda \rightarrow \infty) = \omega_1 J_0(A/\Omega)$. Therefore, to leading order the U(1) symmetry emerges when A/Ω equals zeros of the Bessel function J_0 , which agrees with prediction of the Magnus expansion.

b. Second-order term

The term $H_0^{(2)}(\lambda)$ can be obtained by expanding Eq. (D15) to $\mathcal{O}(\zeta^2)$, with a resulting equation:

$$\partial_\lambda^2 H_0^{(2)}(\lambda) + 2\Omega \partial_\lambda H_0^{(2)}(\lambda) + 4e^{-2\Omega\lambda} \text{Ad}_{H_1(0)}^2 H_0^{(2)}(\lambda) = S^{(2)}(\lambda). \quad (\text{D21})$$

The source term $S^{(2)}(\lambda)$ is obtained by expanding the last term of Eq. (D15) to $\mathcal{O}(\zeta^2)$:

$$S^{(2)}(\lambda) = -2e^{-2\Omega\lambda} \left[\left\{ \text{Ad}_{\bar{H}_1^{(1)}(\lambda)}, \text{Ad}_{\bar{H}_1^{(0)}(\lambda)^\dagger} \right\} + \left\{ \text{Ad}_{\bar{H}_1^{(0)}(\lambda)}, \text{Ad}_{\bar{H}_1^{(1)}(\lambda)^\dagger} \right\} \right] H_0^{(1)}(\lambda). \quad (\text{D22})$$

Here, we have similarly expanded $\bar{H}_1(\lambda)$ in powers of ζ :

$$\bar{H}_1(\lambda) = \bar{H}_1^{(0)}(\lambda) + \zeta \bar{H}_1^{(1)}(\lambda) + \dots, \quad (\text{D23})$$

and the leading-order term is just the bare drive Hamiltonian $\bar{H}_1^{(0)}(\lambda) = H_1(0)$. The first-order term is obtained by expanding Eq. (D14):

$$\bar{H}_1^{(1)}(\lambda) = - \int_0^\lambda d\bar{\lambda} \left[H_0^{(1)}(\bar{\lambda}), H_1(0) \right]. \quad (\text{D24})$$

Interestingly, due to conjugation, $\bar{H}_1^{(1)}(\lambda)^\dagger = -\bar{H}_1^{(1)}(\lambda)$, the two terms in Eq. (D22) cancel, and, hence, the source term, $S^{(2)}(\lambda) = 0$, in Eq. (D22) vanishes. Combining with the initial condition $H_0^{(2)}(0) = \partial_\lambda H_0^{(2)}(0) = 0$, the second-order term vanishes identically.

c. Third-order term

The first nontrivial correction appears at the third order, which satisfies

$$\partial_\lambda^2 H_0^{(3)}(\lambda) + 2\Omega \partial_\lambda H_0^{(3)}(\lambda) + 4e^{-2\Omega\lambda} \text{Ad}_{H_1(0)}^2 H_0^{(3)}(\lambda) = S^{(3)}(\lambda), \quad (\text{D25})$$

with a source term

$$S^{(3)}(\lambda) = -2e^{-2\Omega\lambda} \left[\left\{ \text{Ad}_{\bar{H}_1^{(2)}(\lambda)}, \text{Ad}_{\bar{H}_1^{(0)}(\lambda)^\dagger} \right\} + \left\{ \text{Ad}_{\bar{H}_1^{(0)}(\lambda)}, \text{Ad}_{\bar{H}_1^{(2)}(\lambda)^\dagger} \right\} + 2 \left\{ \text{Ad}_{\bar{H}_1^{(1)}(\lambda)}, \text{Ad}_{\bar{H}_1^{(1)}(\lambda)^\dagger} \right\} \right] H_0^{(1)}(\lambda). \quad (\text{D26})$$

Here, the second-order term $\bar{H}_1^{(2)}(\lambda)$ is

$$\bar{H}_1^{(2)}(\lambda) = \int_0^\lambda d\bar{\lambda}_1 \int_0^{\bar{\lambda}_1} d\bar{\lambda}_2 \left[H_0^{(1)}(\bar{\lambda}_1), \left[H_0^{(1)}(\bar{\lambda}_2), H_1(0) \right] \right]. \quad (\text{D27})$$

The source term $S^{(3)}$ is, in general, nonzero. Combining Eq. (D25) with the initial condition $H_0^{(3)}(0) = \partial_\lambda H_0^{(3)}(0) = 0$, we can, in principle, calculate the correction to $H_0(\lambda \rightarrow \infty)$. Instead of carrying out this computation explicitly, we perform dimensional analysis of the result. Since the expansion here is in powers of H_0 , an additional energy scale must be included, which turns out to be the driving frequency Ω . This can be seen by noting that each integral in λ [Eqs. (D24) and (D27)] introduces a factor of $1/\Omega$, due to the fact that λ appears in Eq. (18) in the form of $\Omega\lambda$. Therefore, we expect $H_0^{(3)}(\lambda)$ to take the form

$$H_0^{(3)}(\lambda) \sim \frac{1}{\Omega^2} \hat{\mathcal{F}} \left(\frac{H_1(0)}{\Omega} \right) [H_0(0) \otimes H_0(0) \otimes H_0(0)], \quad (\text{D28})$$

where $\hat{\mathcal{F}}$ is a linear functional that acts on $H_0(0) \otimes H_0(0) \otimes H_0(0)$. This is the promised $1/\Omega^2$ correction, which we have found numerical evidence for previously in Fig. 4. Furthermore, we find that the flow formalism agrees with the modified Magnus expansion at the leading order, but the subsequent correction terms do not agree, as described previously using arguments relying on gauge invariance.

APPENDIX E: ESTIMATE OF THERMALIZATION TIMESCALES

In this section, we provide an estimate for the thermalization timescale of the Floquet system in the limit of a large drive frequency, Ω . Our discussion mainly involves a spin chain as an illustrative example, where the typical energy cost of a single spin flip is denoted J . We also assume that the amplitude A of the drive $H_1(0)$ is at most of the same order as the frequency Ω , and both $H_0(0)$ and $H_1(0)$ are sum of local terms. As discussed in the main text, thermalization happens in the form of instantons, which corresponds to a matrix element of $H_1(\lambda)$ that flips k spins at the same time, with $k > \mathcal{O}(\Omega/J)$. Our approach relies on first flowing to the prethermal regime and estimating the typical matrix element, $H_1(\lambda_{\text{pre}})$, that flips k spins. Next, we estimate the fRG time λ_{min} where $\|H_1(\lambda)\|$ is minimized. Finally, we translate this to a real timescale using the relation $t_{\text{eff}} \sim \|H_1(\lambda_{\text{min}})\|^{-1}$.

Let us estimate the typical size of the matrix element, $\langle m | H_1(\lambda_{\text{pre}}) | n \rangle$, which flips k spins. From Eq. (D18), we see that $H_0(\lambda)$ will converge to its prethermal limit in fRG

time of the order of a few drive cycles $\lambda \sim \Omega^{-1}$. Therefore, we can choose a λ_{pre} so that $\Omega\lambda_{\text{pre}} = \zeta$. Here, we choose $\zeta \gg 1$ to ensure we are close to the prethermal fixed point, but ζ is still parametrically smaller than the thermalization timescale; i.e., ζ is an $\mathcal{O}(1)$ number that does not diverge in the $\Omega/J \rightarrow \infty$ limit with A/Ω kept fixed. This assumption is verified *a posteriori* by the fact that $H_0(\lambda)$ approaches the zeroth-order Magnus result faster at larger Ω [see Fig. 4(b)]. Assuming now $H_0 \approx H_0(\lambda_{\text{pre}})$, we can approximate $U(\lambda) \approx \exp[-\lambda H_0(\lambda_{\text{pre}})]$ in Eq. (D11). Expanding $H_1(\lambda_{\text{pre}})$ in Eq. (D11) using the Baker-Campbell-Hausdorff (BCH) formula,

$$H_1(\lambda_{\text{pre}}) = e^{-\Omega\lambda_{\text{pre}}} \sum_k \frac{(-1)^k \lambda_{\text{pre}}^k}{k!} \text{Ad}_{H_0(\lambda_{\text{pre}})}^k H_1(0). \quad (\text{E1})$$

The commutator with $H_0(\lambda_{\text{pre}})$ will increase the range of operators by $\mathcal{O}(1)$. Therefore, terms involving k spin flips are from the order of $\mathcal{O}(k)$ of the expansion. We replace λ_{pre}^k by $\zeta^k \Omega^{-k}$. Using $\|\text{Ad}_{H_0(\lambda_{\text{pre}})}^k H_1(0)\| \leq J^k k!$ [30], the matrix element that involves k spin flips will be bounded by the order of $e^{-\zeta} (\zeta J/\Omega)^k$. In the following analysis, we are interested in the asymptotic limit of $\Omega/J \gg 1$, and we, therefore, drop the ζ factor which does not grow with Ω/J .

Now, let us resume the flow and analyze the evolution of $\|H_1(\lambda)\|$. As discussed in the main text, matrix element of $H_1(\lambda)$ with k spin flips will evolve with the factor $\exp[(kJ - \Omega)(\lambda - \lambda_{\text{pre}})]$. Summing contributions of different k to $\|H_1(\lambda)\|$, we obtain

$$\|H_1(\lambda)\| \sim \sum_k a_k \left(\frac{J}{\Omega}\right)^k e^{(\lambda - \lambda_{\text{pre}})(kJ - \Omega)}. \quad (\text{E2})$$

Here, a_k are numerical coefficients that we expect to decay to zero as $k \rightarrow \infty$.

Since $\Omega \gg J$, $\|H_1(\lambda)\|$ will initially show decay for small $(\lambda - \lambda_{\text{pre}})$, followed by a growth for larger $(\lambda - \lambda_{\text{pre}})$. To estimate the minimum time λ_{min} , we rewrite $k = \Omega/J + \delta k$, and we have

$$\|H_1(\lambda)\| \sim \sum_{\delta k} a_k \left(\frac{J}{\Omega}\right)^{\delta k + \Omega/J} e^{(\lambda - \lambda_{\text{pre}})\delta k J}. \quad (\text{E3})$$

Since $\Omega/J \gg 1$, the dependence of a_k on δk will be weak, and we approximate the summation by δk from $-\infty$ to ∞ . The minimum is reached when the terms of exponent δk are balanced by terms of exponent $-\delta k$, meaning that $(J/\Omega) \exp[J(\lambda - \lambda_{\text{pre}})] \sim \mathcal{O}(1)$. Therefore, we obtain

$$\lambda_{\text{min}} - \lambda_{\text{pre}} \sim \frac{1}{J} \ln \frac{\Omega}{J}. \quad (\text{E4})$$

The dependence of a_k on k does not alter the estimate significantly. For example, if $a_k \propto (k!)^{-\alpha}$, it changes only

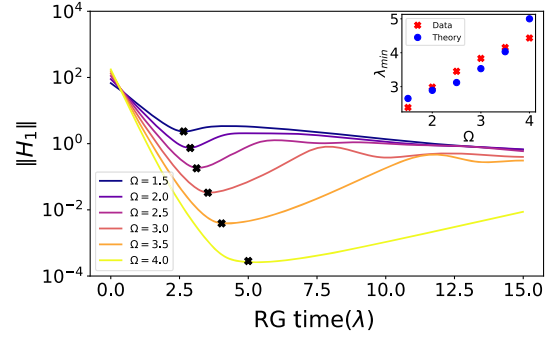


FIG. 17. Driving frequency (Ω) scaling of $H_1(\lambda)$ as a function of fRG time for $J = 1$, $J_2 = 0.9$, $L = 10$, and $A/\Omega = 0.9$. Inset: value of λ corresponding to the minimum norm of H_1 as a function of Ω , demonstrating an order-of-magnitude agreement with the theoretical prediction in Eq. (E4). Here, the theoretical data points are obtained by substituting $J = 0.47$ in Eq. (E4), which is roughly the spin-flip energy associated with the non-integrable coupling J_2 .

the numerical prefactor of the estimate due to $k \sim \Omega/J$. At the same time, $\|H_1\|$ is roughly

$$\|H_1(\lambda_{\text{min}})\| \sim \left(\frac{J}{\Omega}\right)^{\Omega/J} \sim \exp\left(-\frac{\Omega}{J} \ln \frac{\Omega}{J}\right). \quad (\text{E5})$$

Therefore, the heating rate is exponentially small in Ω , and our estimate is consistent with Ref. [30]. We also see that our choice of $\Omega\lambda_{\text{pre}} \gg 1$ is consistent as $\lambda_{\text{min}} - \lambda_{\text{pre}} \gg 1/\Omega$. In Fig. 17, we plot the Ω scaling of the norm of $H_1(\lambda)$ which shows order-of-magnitude agreement with our theoretical prediction.

Another fRG timescale λ_{th} describes the time when $\|H_1(\lambda)\|$ grows to the order of one. Each term of Eq. (E2) will yield a different estimate. For $k = l$, we obtain

$$\lambda_{\text{th}}^{(l)} \sim \frac{f(l) + l \ln(\Omega/J)}{lJ - \Omega}. \quad (\text{E6})$$

Here, $f(l) = \ln(1/a_l)$ and is expected to be a smooth function. For example, $a_l \propto (l!)^{-\alpha}$ will lead to $f(l) \sim \alpha l \ln l$. The estimate above shows that only operators that flip $l > \mathcal{O}(\Omega/J)$ spins will lead to growth of $H_1(\lambda)$ and, hence, an instanton. The minimum of $\lambda_{\text{th}}^{(l)}$ is expected to be parametrically similar to λ_{min} .

APPENDIX F: FLOQUET-MAGNUS ANALYSIS FOR DRIVEN SPIN CHAIN

We compute the Floquet-Magnus expansion for the spin-chain Hamiltonian, utilizing results derived in Appendix D. The unitary transformation to the rotating frame is

$$W(t) = \exp\left(-i \frac{\alpha}{2} \sin(\Omega t) \sum_i S_i^x\right), \quad (\text{F1})$$

where $\alpha = 4A/\Omega$.

The comoving Hamiltonian is

$$H_{\text{mov}}(t) = - \sum_{d=1,2} \frac{J^{(d)}}{2} \sum_i \left\{ S_i^y S_{i+d}^y [1 - \cos(\alpha \sin(\Omega t))] + S_i^z S_{i+d}^z [1 + \cos(\alpha \sin(\Omega t))] + [S_i^y S_{i+d}^z + S_i^z S_{i+d}^y] \sin(\alpha \sin(\Omega t)) \right\}, \quad (\text{F2})$$

where $J^{(1)} = J$ and $J^{(2)} = J_2$. Expanding into Fourier components, $h_m = (1/T) \int_0^T H_{\text{mov}}(t) e^{-im\Omega t} dt$, we obtain

$$h_m = - \sum_{d=1,2} \frac{J^{(d)}}{2} \sum_i \left\{ S_i^y S_{i+d}^y \left[\delta_{m,0} - \frac{1 + (-1)^m}{2} J_m(\alpha) \right] + S_i^z S_{i+d}^z \left[\delta_{m,0} + \frac{1 + (-1)^m}{2} J_m(\alpha) \right] + [S_i^y S_{i+d}^z + S_i^z S_{i+d}^y] \frac{1 - (-1)^m}{2i} J_m(\alpha) \right\}. \quad (\text{F3})$$

The zeroth Floquet Hamiltonian is, therefore,

$$H_F^{(0)} = h_0. \quad (\text{F4})$$

When $J_0(\alpha) = 0$, $H_F^{(0)} \sim \sum_i (S_i^y S_{i+1}^y + S_i^z S_{i+1}^z) + \sum_i (S_i^y S_{i+2}^y + S_i^z S_{i+2}^z)$, which commutes with $\sum_i S_i^x$. The result above is compared with the flow Hamiltonian $H_0(\lambda)$ in Fig. 4, where the difference between the two decays as $1/\Omega^2$ for large Ω , implying that the leading-order result is the same.

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