

Non-Hermitian Bethe-Salpeter Equation for Open Systems: Emergence of Exceptional Points in Excitonic Spectra from First Principles

Zhenlin Zhang¹, Wei Hu¹, Enrico Perfetto^{2,3} and Gianluca Stefanucci^{2,3}

¹*Hefei National Research Center for Physical Sciences at the Microscale, University of Science and Technology of China, Hefei, Anhui 230026, China*

²*Dipartimento di Fisica, Università di Roma Tor Vergata, Via della Ricerca Scientifica 1, 00133 Rome, Italy*

³*INFN, Sezione di Roma Tor Vergata, Via della Ricerca Scientifica 1, 00133 Rome, Italy*



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In open quantum systems hosting excitons, dissipation mechanisms critically shape the excitonic dynamics, band structure, and topological properties. A microscopic understanding of excitons in such non-Hermitian settings demands a first-principles generalization of the Bethe-Salpeter equation (BSE). Building on a recently introduced nonequilibrium Green's function formalism compatible with Lindbladian dynamics, we derive a non-Hermitian BSE from diagrammatic perturbation theory on the Keldysh contour, and obtain a microscopic excitonic Hamiltonian that incorporates dissipation while preserving causality. We apply the formalism to valley excitons in transition metal dichalcogenides coupled to structured photon baths. We uncover a rich landscape of exceptional points in momentum space, forming either discrete sets or continuous manifolds, depending on bath structure. The exceptional points give rise to nonanalytic valley polarization, unusual polarization pattern in photoluminescence, and nontrivial topological signatures. Our results establish a first-principles framework for predicting and controlling excitonic behavior in open quantum materials, showing how engineered environments can be leveraged to induce and manipulate non-Hermitian and topological properties.

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I. INTRODUCTION

Excitons, bound states of electrons and holes, play a central role in the optical and transport properties of semiconductors and insulators. In recent years, there has been growing interest in studying excitons in *open systems* [1–4], motivated by emerging experimental platforms such as exciton-polariton condensates in microcavities, excitonic devices coupled to lossy environments, and topological photonic structures [5–8]. These systems, characterized by non-Hermitian dynamics, energy loss, and coupling to baths, open the door to novel quantum phenomena that do not exist in closed equilibrium settings [9–18]. Understanding how dissipation affects exciton dynamics, lifetimes, and even topological properties is both fundamentally intriguing and technologically relevant.

In the context of open systems, excitons have often been modeled as effective bosonic quasiparticles—a framework that has proven powerful in capturing collective phenomena

such as condensation [19,20]. These models, typically based on phenomenological approaches or effective Hamiltonians, provide valuable intuition and have been useful in advancing our understanding of many-body exciton physics. However, they generally abstract away the underlying fermionic nature of excitons and the microscopic origins of dissipation. As a result, while highly insightful, the effective bosonic description lacks the accuracy required for material-specific quantitative analysis.

A rigorous, first principles description of excitons is provided by the Bethe-Salpeter equation (BSE) [21], a Green's function formalism built upon many-body perturbation theory [22–27]. The BSE has proven to be a powerful and accurate tool in describing excitonic effects in closed, equilibrium systems, particularly in the context of *ab initio* calculations of solids and nanostructures.

As early as 1966, Sham and Rice established a foundational link between exciton models and many-body theory [28], recognizing the necessity of a field-theoretic treatment of excitonic phenomena. In 1974, Hanke and Sham performed the first nonmodel-based exciton calculations in solids using the BSE [29,30], employing the time-dependent Hartree-Fock approximation and later incorporating static electron-hole screening [31]. A significant milestone was reached in 1995, when the Na₄ cluster

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became the first realistic system treated using a fully *ab initio* BSE scheme [32]. Since then, the application of BSE has flourished, with widespread use in the quantitative prediction of optical properties of both finite and extended systems [33–39].

Despite these successes, the generalization of the BSE to open, dissipative systems remains lacking. In modern experimental platforms—ranging from cavity-coupled materials to detuned or near-resonant driven systems—loss, decoherence, and environmental coupling are intrinsic and unavoidable. These features fundamentally alter the nature of excitations, even in the weak light-matter coupling regime (i.e., without exciton-polariton formation) [40]. They introduce complex eigenvalues, non-Hermitian dynamics, and nonequilibrium steady states—all of which lie beyond the scope of traditional BSE treatments.

For weak system-environment coupling, a suitable formalism to describe the aforementioned physics is the combination of nonequilibrium Green’s functions (NEGF) theory [41–43] and Lindbladian dynamics [44,45]. This has been achieved using field theoretical techniques [16,46–48] and the second-quantization approach [49]. In this work, we build upon the formalism of Ref. [49], which is closer to the *ab initio* community and more suitable to study transient phenomena and relaxation dynamics of carriers, excitons, and phonons [50–55]. By generalizing the BSE within this framework, we derive a first-principles approach to treat excitons in dissipative environments and extract the microscopic non-Hermitian excitonic Hamiltonian. This development provides a powerful tool for quantitative predictions of excitonic phenomena beyond the limits of closed and Hermitian systems.

We also apply the formalism to study the non-Hermitian excitonic spectrum of two-dimensional semiconductors, such as transition metal dichalcogenides (TMDs). These systems feature strong excitonic effects and high environmental sensitivity. Consequently, the optical properties of TMD—whether suspended on or supported on substrates—can undergo significant modifications when embedded in a photonic bath [56]. Here we investigate two engineered baths: one consisting of linearly polarized photons and another of circularly polarized photons carrying orbital angular momentum. Depending on the bath, we find either a finite set or a continuous ring of exceptional points (EPs) in momentum space. These EPs give rise to nontrivial valley polarizations, photoluminescence features, and topological properties. Our microscopic, non-Hermitian treatment highlights the emergence of rich excitonic physics even for weak exciton-photon coupling, and outlines a route toward controlling and tailoring excitonic behavior through bath engineering.

The paper is organized as follows. In Sec. II, we briefly introduce the Lindblad equation that governs the dynamics of open quantum systems and outline the dissipation mechanisms considered in this work—namely, one-body

loss and gain, as well as particle-hole recombination. In Sec. III, we develop the dissipative nonequilibrium Green’s function formalism that arises from the underlying Lindbladian dynamics. In particular, we show that the problem can be reformulated as a generalized diagrammatic perturbation theory on the Keldysh contour, made possible by a dissipative version of Wick’s theorem. Foundational results for the one-particle NEGF are presented in Sec. IV. We introduce the self-energy, address subtleties related to mean-field approximations, and present the framework for conserving approximations. These results are preparatory for deriving the BSE in an open setting. In Sec. V, we establish the exact form of the non-Hermitian BSE for the four-time two-particle NEGF. By gradually introducing physically motivated approximations for the interaction kernel and the quasiparticle Green’s functions, we then derive a stationary BSE for the two-time response function. In Sec. VI, we specialize the stationary BSE formalism to semiconductors and derive the non-Hermitian excitonic Hamiltonian. The microscopic derivation highlights that a Lindbladian treatment of dissipation imposes constraints on the allowed dissipation channels: arbitrary choices can either violate the stationarity assumption or compromise causality. Physically relevant and formally admissible dissipation mechanisms are explicitly discussed. In Sec. VII we apply the formalism to valley excitons in TMDs embedded in different photon baths. We present a 2×2 effective non-Hermitian description of the excitonic subspace and analyze its eigenvalues and eigenvectors, revealing the emergence of EPs. The occurrence of EPs induces a nonanalytic behavior in the valley polarization, and leads to a complex polarization pattern in the photoluminescence emission. Finally, we compute the Berry curvature and report a nontrivial topological character of the non-Hermitian excitons. A summary of the main results and an outlook for future directions are presented in Sec. VIII.

II. THE OPEN SYSTEM

The time evolution for the reduced many-body density matrix $\hat{\rho}(t)$ of an open quantum system governed by a Lindbladian dynamics reads [44,57]:

$$\frac{d}{dt}\hat{\rho}(t) = -i[\hat{H}(t), \hat{\rho}(t)]_- + \hat{\mathcal{D}}[\hat{\rho}](t). \quad (1)$$

The coherent (unitary) evolution is determined by the many-body Hamiltonian of the closed system

$$\hat{H}(t) = \sum_{ij} h_{ij}(t) \hat{d}_i^\dagger \hat{d}_j + \frac{1}{2} \sum_{ijmn} v_{ijmn}^{\text{phys}}(t) \hat{d}_i^\dagger \hat{d}_j^\dagger \hat{d}_m \hat{d}_n, \quad (2)$$

where the annihilation operators $\hat{d}_i$ are either bosonic or fermionic, obeying the commutation or anticommutation

rules $[\hat{d}_i, \hat{d}_j^\dagger]_\mp = \delta_{ij}$ (upper sign for bosons and lower sign for fermions). The global U(1) symmetry, $\hat{d}_j \rightarrow \hat{d}_j e^{i\alpha}$, of $\hat{H}(t)$ guarantees that the total number of particles of the closed system is conserved.

The one-particle Hamiltonian $h(t)$ comprises the kinetic energy as well as static or time-dependent external fields. The time dependence of the physical interaction v^{phys} could describe, e.g., the adiabatic switching of the Coulomb interaction for electronic system or the quenching of the dipolar interaction in ultracold atomic and molecular systems [58].

The dissipation induced by the environment is described by a set of so-called dissipator superoperators

$$\hat{\mathcal{D}}[\hat{\rho}](t) = \sum_{\lambda} \left[\hat{\mathcal{D}}_{\lambda}^{\text{p}}[\hat{\rho}](t) + \hat{\mathcal{D}}_{\lambda}^{\text{h}}[\hat{\rho}](t) + \hat{\mathcal{D}}_{\lambda}^{\text{ph}}[\hat{\rho}](t) \right], \quad (3)$$

which represent the different kinds of dissipation considered in this work—one-body loss (p), one-body gain (h), and particle-hole recombination (ph)—in multiple channels indexed by λ . For a many-body system, there are typically several decay channels, e.g., when the system is coupled to several baths.

The dissipators consist of a jump term and an anti-commutator

$$\hat{\mathcal{D}}_{\lambda}^{\kappa}[\hat{\rho}](t) = 2\hat{L}_{\lambda}^{\kappa}(t)\hat{\rho}(t)\hat{L}_{\lambda}^{\kappa\dagger}(t) - \left[\hat{L}_{\lambda}^{\kappa\dagger}(t)\hat{L}_{\lambda}^{\kappa}(t), \hat{\rho}(t) \right]_+, \quad (4)$$

where the quantum jump operators $\hat{L}_{\lambda}^{\kappa}$ of type $\kappa = \text{p, h, ph}$ describe the interaction between the system and the environment. The jump term guarantees that the density matrix maintains constant trace and remains positive during the evolution. Consequently, the Lindbladian dynamics cannot be reduced to evolution under a non-Hermitian Hamiltonian.

It is worth recalling that deriving the Lindblad equation from the underlying system-environment Hamiltonian generally leads to non-Hermitian jump operators. If the jump operators are Hermitian and conserved by the coherent dynamics ($[\hat{L}_{\lambda}^{\kappa}, \hat{H}]_- = 0$), then they can only induce pure decoherence [16,18]. As for Hermitian jump operators that do not commute with the physical Hamiltonian, they drive the system toward an infinite-temperature state [18,59–61].

The most general form of the jump operators in Eq. (4) is

$$\hat{L}_{\lambda}^{\text{p}}(t) = \sum_n \eta_n^{\lambda}(t) \hat{d}_n, \quad (5a)$$

$$\hat{L}_{\lambda}^{\text{h}}(t) = \sum_n \tilde{\eta}_n^{\lambda*}(t) \hat{d}_n^{\dagger}, \quad (5b)$$

$$\hat{L}_{\lambda}^{\text{ph}}(t) = \sum_{mn} \xi_{mn}^{\lambda}(t) \hat{d}_m^{\dagger} \hat{d}_n, \quad (5c)$$

where the coefficients η , $\tilde{\eta}$, and ξ determine the dissipation rates due to one-body loss, one-body gain, and particle-hole recombination.

The symmetries preserved in Lindbladian dynamics can be classified into two categories: *weak* and *strong*. A symmetry is said to be weak if both the Hamiltonian $\hat{H}$ and the dissipators $\hat{\mathcal{D}}$ remain invariant under the corresponding symmetry transformations, even though the jump operators $\hat{L}$ themselves do not. In contrast, a weak symmetry is considered strong when the jump operators themselves are invariant under the symmetry transformations.

The forms of the one-body jump operators ($\kappa = \text{p, h}$) imply that the corresponding dissipators have *weak* global U(1) symmetry: the total particle number is *not* conserved, yet the many-body density matrix $\hat{\rho}$ does not develop quantum coherence between Hilbert space sectors of different particle number [16,62–68]. The one-body jump operators are under intense investigation in the community of, e.g., non-Hermitian skin effect [65,69,70] and topological superconductor [71–73]. More generally, one-body loss and gain are relevant in systems subject to particle injection (source) and leakage (drain), such as those encountered in quantum transport setups.

The particle-hole jump operator, quadratic in field operators, possess a *strong* global U(1) symmetry—both $\hat{\mathcal{D}}^{\text{ph}}$ and $\hat{L}^{\text{ph}}$ are invariant under global U(1) transformation—and renormalizes both the single particle Hamiltonian and the physical interaction during time evolution. Thus, analytical solutions are in general unavailable even for $v^{\text{phys}} = 0$. Particle-hole losses have drawn special attention in the context of critical phenomena out of equilibrium [64] and novel quantum platform such as exciton-polariton [74,75] and disordered systems [76,77]. As we see later, they are also relevant in the context of materials embedded in bad photonic cavities.

In the following, we develop a general framework that encompasses all types of jump operators simultaneously.

III. DISSIPATIVE NEGF FORMALISM

A. Hamiltonian and n -particle Green's functions on the Keldysh contour

The n -particle nonequilibrium Green's function of the open system governed by the Lindbladian dynamics can be written as [49]

$$G_n(j_1 z_1, \dots, j_n z_n; j'_1 z'_1, \dots, j'_n z'_n) \equiv \frac{1}{i^n} \left\langle \hat{d}_{j_1}(z_1) \dots \hat{d}_{j_n}(z_n) \hat{d}_{j'_n}^{\dagger}(z'_n) \dots \hat{d}_{j'_1}^{\dagger}(z'_1) \right\rangle, \quad (6)$$

where the times of the field operators run along the oriented Keldysh contour $C = C_- \cup C_+ = (t_0, \infty) \cup (\infty, t_0)$, and

we use the notation z^+ to denote a time infinitesimally later than z on C . In Eq. (6) we use the shorthand notation

$$\langle \dots \rangle \equiv \text{Tr} \left[\hat{\rho}(t_0) \mathcal{T} \left\{ e^{-i \int_C d\bar{z} \hat{\mathcal{H}}(\bar{z})} \dots \right\} \right], \quad (7)$$

where $\mathcal{T}$ is the contour ordering operator, and the dissipative Hamiltonian generator $\hat{\mathcal{H}}(z)$ reads

$$\begin{aligned} \hat{\mathcal{H}}(z) = & \hat{H}(z) + \sum_{\kappa, \lambda} \left[-is(z) \hat{L}_{\lambda}^{\kappa\dagger}(z) \hat{L}_{\lambda}^{\kappa}(z) \right. \\ & \left. + i\theta_{-}(z) \hat{L}_{\lambda}^{\kappa\dagger}(z^*) \hat{L}_{\lambda}^{\kappa}(z) - i\theta_{+}(z) \hat{L}_{\lambda}^{\kappa\dagger}(z) \hat{L}_{\lambda}^{\kappa}(z^*) \right]. \end{aligned} \quad (8)$$

Let us discuss in some detail the Hamiltonian in Eq. (8). We write a time $z \in C$ as $z = t_{\pm}$ if $z \in C_{\pm}$. The operators $\hat{H}(z = t_{\pm}) = \hat{H}(t)$ and $\hat{L}_{\lambda}^{\kappa}(z = t_{\pm}) = \hat{L}_{\lambda}^{\kappa}(t)$ are the same on both branches of the contour. The functions $\theta_{\pm}(z) = 1$ if $z \in C_{\pm}$ and zero otherwise, whereas $s(z) = \theta_{-}(z) - \theta_{+}(z)$ is a sign function. Finally, we define $z^* = t_{\mp}$ if $z = t_{\pm}$. Although the dissipative Hamiltonian $\hat{\mathcal{H}}(z)$ depends on both z and z^* , it is easy to verify that $\hat{\mathcal{H}}(t_{-}) = \hat{\mathcal{H}}(t_{+}) = \hat{H}(t)$, and therefore the operator $\hat{K}(t) \equiv \mathcal{T} \{ e^{-i \int_{-}^{+} d\bar{z} \hat{\mathcal{H}}(\bar{z})} \} = 1$ for all t , reflecting the conservation of probability.

The dissipative Hamiltonian can be split into a non-interacting part $\hat{\mathcal{H}}_0(z)$ and interacting part $\hat{\mathcal{H}}_{\text{int}}(z)$

$$\hat{\mathcal{H}}(\bar{z}) = \hat{\mathcal{H}}_0(\bar{z}) + \hat{\mathcal{H}}_{\text{int}}(\bar{z}), \quad (9)$$

which are, respectively, quadratic in the field operators,

$$\hat{\mathcal{H}}_0(\bar{z}) = \sum_{ij} \int_C d\bar{z}' h_{ij}(\bar{z}', \bar{z}) \hat{d}_i^{\dagger}(\bar{z}') \hat{d}_j(\bar{z}), \quad (10)$$

and quartic in the field operators

$$\begin{aligned} \hat{\mathcal{H}}_{\text{int}}(\bar{z}) = & \frac{1}{2} \sum_{ijmn} \int_C d\bar{z}' v_{ijmn}(\bar{z}, \bar{z}') \\ & \times \hat{d}_i^{\dagger}(\bar{z}^+) \hat{d}_j^{\dagger}(\bar{z}^+) \hat{d}_m(\bar{z}) \hat{d}_n(\bar{z}'). \end{aligned} \quad (11)$$

The explicit forms of Eqs. (10) and (11) were derived in Ref. [49]. For completeness, and to ensure the manuscript is self-contained, a more detailed derivation is provided in Appendix A. Below, we summarize the main results.

The noninteracting one-particle Hamiltonian reads (upper and lower sign for bosons and fermions)

$$\begin{aligned} h_{ij}(z', z) = & \delta(z', z) h_{ij}(t) \\ & - is(z) \delta(z', z) \left(V_{ij}^{\text{loss}}(t) \pm V_{ij}^{\text{gain}}(t) + V_{ij}^{\text{diss}}(t) \right) \\ & + 2i\delta(z', z^*) \left(\theta_{-}(z) V_{ij}^{\text{loss}}(t) \mp \theta_{+}(z) V_{ij}^{\text{gain}}(t) \right), \end{aligned} \quad (12)$$

where we introduce the Dirac delta function on the contour, i.e., $\int_C d\bar{z}' \delta(z, z') f(z') = f(z)$ for any function f . In addition to the physical one-particle Hamiltonian h_{ij} of Eq. (2), the noninteracting one-particle Hamiltonian of Eq. (12) contains complex potentials due to one-body loss and gain mechanisms, i.e.,

$$V_{ij}^{\text{loss}}(t) = \sum_{\lambda} \eta_i^{\lambda*}(t) \eta_j^{\lambda}(t), \quad (13a)$$

$$V_{ij}^{\text{gain}}(t) = \sum_{\lambda} \tilde{\eta}_i^{\lambda*}(t) \tilde{\eta}_j^{\lambda}(t), \quad (13b)$$

as well as from particle-hole recombination mechanisms, i.e.,

$$V_{ij}^{\text{diss}}(t) = \frac{1}{2} \sum_m v_{imjm}^{\text{diss}}(t) \quad (14)$$

with

$$v_{ijmn}^{\text{diss}}(t) = 2 \sum_{\lambda} \xi_{ni}^{\lambda*}(t) \xi_{jm}^{\lambda}(t). \quad (15)$$

The generalized interaction in Eq. (11) is the sum of the physical interaction and the dissipation-induced interaction driven by the particle-hole jump operators

$$v_{ijmn}(z, z') = v_{ijmn}^{\text{phys}}(z, z') + v_{ijmn}^{\text{diss}}(z, z') \quad (16)$$

$$v_{ijmn}^{\text{phys}}(z, z') = \delta(z, z') v_{ijmn}^{\text{phys}}(t) \quad (17)$$

$$\begin{aligned} v_{ijmn}^{\text{diss}}(z, z') = & -is(z) [\delta(z, z') + \delta(z, z'^*)] v_{ijmn}^{\text{s,diss}}(t) \\ & - i\delta(z, z'^*) v_{ijmn}^{\text{a,diss}}(t), \end{aligned} \quad (18)$$

where the symmetric ($v^{\text{s,diss}}$) and antisymmetric ($v^{\text{a,diss}}$) tensors are defined in terms of the tensor introduced in Eq. (15):

$$v_{ijmn}^{\text{s,diss}}(t) = \frac{1}{2} \left[v_{ijmn}^{\text{diss}}(t) + v_{jinm}^{\text{diss}}(t) \right], \quad (19a)$$

$$v_{ijmn}^{\text{a,diss}}(t) = \frac{1}{2} \left[v_{ijmn}^{\text{diss}}(t) - v_{jinm}^{\text{diss}}(t) \right]. \quad (19b)$$

The generalized interaction preserves the symmetry property

$$v_{ijmn}(z, z') = v_{jinm}(z', z) \quad (20)$$

of the physical interaction, an essential feature for deriving the prefactors of the Feynman diagrams [49]. If the particle-hole jump operators are chosen to be either Hermitian, i.e., $\xi_{nm}^{\lambda*}(t) = \xi_{mn}^{\lambda}(t)$, or anti-Hermitian, i.e., $\xi_{nm}^{\lambda*}(t) = -\xi_{mn}^{\lambda}(t)$, then the antisymmetric part vanishes ($v^{\text{a,diss}} = 0$). The

many-body Hamiltonian of the closed system is recovered by setting all the Lindblad coefficients to zero, as it should be.

Henceforth, to avoid ambiguity in expressions involving $\pm$, the upper (lower) sign refers to bosons (fermions). When the sign appears as a subscript on time variables, e.g., $t_{\pm} \in C_{\pm}$, it denotes the corresponding branch of the Keldysh contour.

B. Diagrammatic expansions

Differentiating the n -particle NEGF in Eq. (6) with respect to z or z' gives rise to the so called Martin-Schwinger hierarchy (MSH) [41,78], which couples G_n to the $(n+1)$ and $(n-1)$ -particle NEGFs. In Appendix B we prove that the MSH for our choice of jump operators reads

$$\begin{aligned} & \sum_j \int_C d\bar{z} \left[i \frac{d}{dz_q} \delta_{j_q \bar{j}} \delta(z_q, \bar{z}) - h_{j_q \bar{j}}(z_q, \bar{z}) \right] G_n(j_1 z_1, \dots, \bar{j} \bar{z}, \dots, j_n z_n; j'_1 z'_1, \dots, j'_n z'_n) \\ &= \sum_{l=1}^n (\pm)^{q+l} \delta_{j_q j'_l} \delta(z_q, z'_l) G_{n-1}(j_1 z_1, \dots, \overset{\square}{j_q} z_q, \dots, j_n z_n; j'_1 z'_1, \dots, \overset{\square}{j'_l} z'_l, \dots, j'_n z'_n) \\ & \pm i \sum_{prs} \int_C d\bar{z} v_{j_q p r s}(\bar{z}, z_q) G_{n+1}(j_1 z_1, \dots, r \bar{z}, s z_q, \dots, j_n z_n; j'_1 z'_1, \dots, p \bar{z}^+, j'_q z'_q, \dots, j'_n z'_n), \end{aligned} \quad (21)$$

and

$$\begin{aligned} & \sum_{j'} \int_C d\bar{z}' G_n(j_1 z_1, \dots, j_n z_n; j'_1 z'_1, \dots, \bar{j}' \bar{z}', \dots, j'_n z'_n) \left[\frac{1}{i} \frac{d}{dz'_q} \delta_{\bar{j}' j'_q} \delta(\bar{z}', z'_q) - h_{\bar{j}' j'_q}(\bar{z}', z'_q) \right] \\ &= \sum_{l=1}^n (\pm)^{q+l} \delta_{j_l j'_q} \delta(z_l, z'_q) G_{n-1}(j_1 z_1, \dots, \overset{\square}{j_l} z_l, \dots, j_n z_n; j'_1 z'_1, \dots, \overset{\square}{j'_q} z'_q, \dots, j'_n z'_n) \\ & \pm i \sum_{prs} \int_C d\bar{z}' v_{s r p j'_q}(\bar{z}', z'_q) G_{n+1}(j_1 z_1, \dots, p \bar{z}'^+, j_q z_q, \dots, j_n z_n; j'_1 z'_1, \dots, r \bar{z}', s z'_q, \dots, j'_n z'_n), \end{aligned} \quad (22)$$

where the symbol $\square$ above an index signifies that the index is missing. By setting the general interaction $v(z, z')$ to zero the n -particle NEGF couples exclusively to the $(n-1)$ -particle NEGF, and the solution G_n^0 of the noninteracting MSH is equivalent to Wick's theorem [49].

By construction, the noninteracting NEGF G_n^0 can alternatively be written as in Eq. (6) with the replacement $\hat{\mathcal{H}}(z) \rightarrow \hat{\mathcal{H}}_0(z)$. Therefore, G_n^0 fully accounts for one-body loss and gain mechanisms. In the presence of particle-hole recombination, $\hat{\mathcal{H}}_0(z)$ also contains the complex potential V^{diss} in Eq. (14). Because of this term the contour integral $\int_C dz dz' h(z, z') \neq 0$, and, consequently, the operator $\hat{K}_0(t) \equiv \mathcal{T} \{ e^{-i \int_{t_-}^{t_+} d\bar{z} \hat{\mathcal{H}}_0(\bar{z})} \} \neq 1$. This fact implies that all G_n^0 diverge in the long time limit. As we show below, the physical stability is restored to lowest order in perturbation theory, i.e., in the mean-field Hartree-Fock (HF) approximation [49]. Rigorously speaking, the Wick's theorem can be proven only for the mean-field MSH.

Expanding Eq. (6) in powers of $\hat{\mathcal{H}}_{\text{int}}(z)$ and using Wick's theorem, each term of the expansion can be represented by a Feynman diagram. The terms containing vacuum diagrams cancel with the expansion of the denominator

$1 = \text{Tr}[\hat{\rho}(t_0) \hat{K}(t_0)]$. Furthermore, the number of topological equivalent diagrams of order k is the same as for physical interactions, i.e., $2^k k!$, since the generalized interaction has the same symmetry properties, see Eq. (20). Hence, it is enough to consider only connected and topologically inequivalent diagrams. The final results for the one-particle and two-particle NEGFs are (integrals are over $z_1, z'_1, \dots, z_k, z'_k$ and repeated indices are summed over):

$$\begin{aligned} G_{ab}(z_a, z_b) &= \sum_{k=0}^{\infty} i^k \int_C v_{i_1 j_1 m_1 n_1}(z_1, z'_1) \dots v_{i_k j_k m_k n_k}(z_k, z'_k) \\ & \times \begin{vmatrix} G_{ab}^0(z_a, z_b) & G_{aj_1}^0(z_a, z'_1) & \dots & G_{ai_k}^0(z_a, z'_k) \\ G_{m_1 b}^0(z_1, z_b) & G_{m_1 j_1}^0(z_1, z'_1) & \dots & G_{m_1 i_k}^0(z_1, z'_k) \\ \vdots & \vdots & \ddots & \vdots \\ G_{n_k b}^0(z'_k, z_b) & G_{n_k j_1}^0(z'_k, z'_1) & \dots & G_{n_k i_k}^0(z'_k, z'_k) \end{vmatrix} \Big|_{\substack{\pm \\ t.i.}}, \end{aligned} \quad (23)$$

$$\begin{aligned}
G_2(a z_a, b z_b; c z_c, d z_d) \\
= \sum_{k=0}^{\infty} i^k \int_C v_{i_1 j_1 m_1 n_1}(z_1, z'_1) \dots v_{i_k j_k m_k n_k}(z_k, z'_k) \\
\times \begin{vmatrix} G_{ac}^0(z_a, z_c) & G_{ad}^0(z_a, z_d) & \dots & G_{ai_k}^0(z_a, z'_k) \\ G_{bc}^0(z_b, z_c) & G_{bd}^0(z_b, z_d) & \dots & G_{bi_k}^0(z_b, z'_k) \\ \vdots & \vdots & \ddots & \vdots \\ G_{n_k c}^0(z'_k, z_c) & G_{n_k d}^0(z'_k, z_d) & \dots & G_{n_k i_k}^0(z'_k, z'_k) \end{vmatrix}_{\substack{+ \\ t.i.}}
\end{aligned} \quad (24)$$

and the like for higher order NEGF. In the above equations

$$G_{jj'}^0(z_j, z'_j) \equiv G_1^0(j z_j; j' z'_j) \quad (25)$$

is the noninteracting one-particle NEGF, and the labels “c” and “t.i.” indicate that in the permanent (+) and determinant (−) only connected and topologically inequivalent diagrams are retained.

IV. ONE-PARTICLE GREEN'S FUNCTION

Before delving into the Bethe-Salpeter equation for the two-particle Green's function we need to introduce a few concepts emerging from the analysis of the one-particle Green's function.

A. Kadanoff-Baym equations

The Wick's expansion of the one-particle NEGF provides the Feynman rules for extracting the irreducible self-energy $\Sigma = \Sigma[G^0, v]$ depending on the noninteracting G^0 and the generalized interaction v . The full set of Σ diagrams can alternatively be obtained by dressing G^0 and discarding the nonskeletonic diagrams (a diagram is skeletonic if it does not contain self-energy insertions) [41]. Denoting by Σ^s the sum of only skeletonic Σ diagrams, we have $\Sigma = \Sigma[G^0, v] = \Sigma^s[G, v]$. The equations of motion for G can then be written as (in matrix form)

$$i \frac{d}{dz} G(z, z') - \int_C d\bar{z} [h(z, \bar{z}) + \Sigma^s(z, \bar{z})] G(\bar{z}, z') = \delta(z, z'), \quad (26a)$$

$$\frac{1}{i} \frac{d}{dz'} G(z, z') - \int_C d\bar{z} G(z, \bar{z}) [h(\bar{z}, z') + \Sigma^s(\bar{z}, z')] = \delta(z, z'). \quad (26b)$$

Let us separate out the Hartree-Fock (HF) contribution from the full self-energy, i.e., $\Sigma^s = \Sigma^{\text{HF}} + \Sigma^c$, where $\Sigma^{\text{HF}} = \Sigma^{\text{H}} + \Sigma^{\text{F}}$ is the sum of the tadpole diagram

$$\begin{aligned}
\Sigma_{in}^{\text{H}}(z, z') &= \pm i \delta(z, z') \sum_{mj} \int_C d\bar{z} v_{ijmn}(\bar{z}, z) G_{mj}(\bar{z}, \bar{z}^+) \\
&= \delta(z, z') \sum_{mj} \left(v_{ijmn}^{\text{phys}}(t) - i v_{ijmn}^{\text{a,diss}}(t) \right) \rho_{mj}^<(t), \quad (27)
\end{aligned}$$

where $\rho^<(t) \equiv \pm i G(z, z^+)$ is the one-particle density matrix, and oyster diagrams

$$\Sigma_{in}^{\text{F}}(z, z') = i \sum_{mj} G_{mj}(z, z') v_{ijnm}(z', z). \quad (28)$$

The time local structure of the generalized interaction in Eqs. (16)–(18) implies that the HF self-energy effectively renormalizes the noninteracting one-particle Hamiltonian $h(z, z')$ in Eq. (12), and stabilizes the system dynamics since $\int_C dz dz' [h(z, z') + \Sigma^{\text{HF}}(z, z')] = 0$, see Appendix C. Notice that the quantity

$$V_{in}^{\text{a,diss}}(t) \equiv -i \sum_{mj} v_{ijmn}^{\text{a,diss}}(t) \rho_{mj}^<(t) \quad (29)$$

in the Hartree self-energy is a Hermitian matrix and therefore it provides a dissipation-induced renormalization of the one-particle energy levels.

The dissipative Kadanoff-Baym equations (KBE) are equations of motion for the Keldysh components of the NEGF $G(z, z')$ with times on the contour. Setting $z = t_{\mp}$ and $z' = t'_{\pm}$ we have the lesser and greater component $G^{\lessgtr}(t, t') \equiv G(t_{\mp}, t'_{\pm})$. Using the Langreth rules [41,79] to evaluate the convolutions in Eqs. (26) we find [49,80]

$$\begin{aligned}
\left[i \frac{d}{dt} - h^{\text{HF}}(t) \right] G^<(t, t') \pm 2i \gamma^<(t) G^A(t, t') \\
= \int_{t_0}^{\infty} d\bar{t} [\Sigma^{c,<}(t, \bar{t}) G^A(\bar{t}, t') + \Sigma^{c,R}(t, \bar{t}) G^<(\bar{t}, t')], \quad (30a)
\end{aligned}$$

$$\begin{aligned}
\left[i \frac{d}{dt} - h^{\text{HF}}(t) \right] G^>(t, t') + 2i \gamma^>(t) G^A(t, t') \\
= \int_{t_0}^{\infty} d\bar{t} [\Sigma^{c,>}(t, \bar{t}) G^A(\bar{t}, t') + \Sigma^{c,R}(t, \bar{t}) G^>(\bar{t}, t')], \quad (30b)
\end{aligned}$$

$$\begin{aligned}
G^<(t, t') \left[\frac{1}{i} \frac{d}{dt'} - h^{\text{HF}\dagger}(t') \right] \pm 2i G^R(t, t') \gamma^<(t') \\
= \int_{t_0}^{\infty} d\bar{t} [G^<(t, \bar{t}) \Sigma^{c,A}(\bar{t}, t') + G^R(t, \bar{t}) \Sigma^{c,<}(\bar{t}, t')], \quad (30c)
\end{aligned}$$

$$\begin{aligned}
G^>(t, t') \left[\frac{1}{i} \frac{d}{dt'} - h^{\text{HF}\dagger}(t') \right] + 2i G^R(t, t') \gamma^>(t') \\
= \int_{t_0}^{\infty} d\bar{t} [G^>(t, \bar{t}) \Sigma^{c,A}(\bar{t}, t') + G^R(t, \bar{t}) \Sigma^{c,>}(\bar{t}, t')], \quad (30d)
\end{aligned}$$

where the retarded and advanced components of the Green's function are defined according to $G^R(t, t') = \theta(t - t')[G^>(t, t') - G^<(t, t')]$ and $G^A(t, t') = -\theta(t' - t)[G^>(t, t') - G^<(t, t')]$, and the like for the self-energy. We emphasize that the dissipative KBE are exact when the correlation self-energy Σ^c is known exactly. Otherwise, they provide an approximate method when $\Sigma^c[G, v]$ is treated within a diagrammatic approximation.

The correction of the HF self-energy has been included into a one-particle non-Hermitian Hamiltonian h^{HF} , which is defined as (see Appendix C)

$$h_{in}^{\text{HF}}(t) = h_{in}(t) + V_{in}^{\text{HF}}(t) + V_{in}^{\text{a,diss}}(t) - i\gamma_{in}(t), \quad (31)$$

where

$$V_{in}^{\text{HF}}(t) = \sum_{mj} \left(v_{ijmn}^{\text{phys}}(t) \pm v_{ijnm}^{\text{phys}}(t) \right) \rho_{mj}^<(t), \quad (32)$$

is the physical (nonlocal in space) Hartree-Fock potential, and

$$\gamma_{in}(t) = \gamma_{in}^>(t) \mp \gamma_{in}^<(t), \quad (33a)$$

$$\gamma_{in}^>(t) = V_{in}^{\text{loss}}(t) \pm \frac{1}{2} \sum_{mj} v_{ijnm}^{\text{diss}}(t) \rho_{mj}^>(t), \quad (33b)$$

$$\gamma_{in}^<(t) = V_{in}^{\text{gain}}(t) + \frac{1}{2} \sum_{mj} v_{jimn}^{\text{diss}}(t) \rho_{mj}^<(t), \quad (33c)$$

is the one-particle decay-rate matrix—in Eq. (33b) $\rho^>(t) = \pm iG^>(t, t) = \rho^< \pm 1$. The decay-rate matrices $\gamma^{\lessgtr}$ are both Hermitian and positive semi-definite (PSD). A stable steady-state solution of the Lindblad equation is guaranteed in fermionic systems and needs to be checked case by case in bosonic systems. A simple example of unstable dynamics is provided by a single bosonic mode with a jump operator $\hat{L} = \hat{d}^\dagger$, which continuously pumps particles into the system [81]. Interestingly, the dissipative Hartree self-energy contributes to a purely Hermitian potential, $V^{\text{a,diss}}$, whereas the dissipative Fock self-energy renormalizes the complex potentials in $i\gamma^{\lessgtr}$.

Subtracting Eq. (30a) from Eq. (30b) and using the definition of the retarded Green's function, we have the equation of motion for the retarded component

$$\left[i \frac{d}{dt} - h^{\text{HF}}(t) \right] G^R(t, t') = \delta(t - t') + \int_{t_0}^{\infty} d\bar{t} \Sigma^{\text{c,R}}(t, \bar{t}) G^R(\bar{t}, t'). \quad (34)$$

Likewise, we can derive the equation of motion for the advanced Green's function by subtracting Eq. (30d) from Eq. (30c). The relation $G^A(t, t') = [G^R(t', t)]^\dagger$ is fulfilled in

the dissipative case as well. The PSD property of γ ensures the correct analytical property of $G^{\text{R/A}}$ and, more generally, guarantees the physical stability of the Lindblad dynamics. Indeed, any effective non-Hermitian theory must ensure the PSD condition for the anti-Hermitian (dissipative) part of the Hamiltonian.

The KBE admit a simple solution at the HF level, i.e., for $\Sigma^c = 0$:

$$G^{\text{HF}}(z, z') = \theta(t - t') T \left\{ e^{-i \int_t^{t'} d\bar{t} h^{\text{HF}}(\bar{t})} \right\} G^{\text{HF}}(z'^*, z') + \theta(t' - t) G^{\text{HF}}(z, z^*) \bar{T} \left\{ e^{+i \int_t^{t'} d\bar{t} h^{\text{HF}\dagger}(\bar{t})} \right\}, \quad (35)$$

where $G^{\text{HF}}(z, z^*) = G^{\text{HF},\gtrless}(t, t)$ for $z = t_\pm$. The proof of Eq. (35) is provided in Appendix C.

B. Conserving approximations

The equation of motion for one-particle density matrix $\rho^<(t)$ is derived by subtracting Eq. (30c) from Eq. (30a) and setting $t = t'$

$$\frac{d}{dt} \rho^<(t) = -i h^{\text{HF}}(t) \rho^<(t) + i \rho^<(t) h^{\text{HF}\dagger}(t) + 2\gamma^<(t) + I(t), \quad (36)$$

where $I(t)$ is the collision matrix

$$I(t) = \pm \int_{t_0}^{\infty} d\bar{t} \left[\Sigma^{\text{c},<}(t, \bar{t}) G^A(\bar{t}, t^+) + \Sigma^{\text{c},\text{R}}(t, \bar{t}) G^<(\bar{t}, t^+) \right] + \text{H.c.} \quad (37)$$

Equation (36) reduces to a Lyapunov equation at the HF level, i.e., for $\Sigma^c = 0$.

We inspect the conservation properties entailed by the *weak* and *strong* global U(1) symmetry. The equation of motion for the total particle number $N(t) = \text{Tr}[\rho^<(t)]$ reads

$$\begin{aligned} \frac{d}{dt} N(t) &= \text{Tr}[I(t)] - 2\text{Tr}[(\gamma^>(t) \mp \gamma^<(t)) \rho^<(t) - \gamma^<(t)] \\ &= \text{Tr}[I(t) - 2V^{\text{loss}}(t) \rho^<(t) \pm 2V^{\text{gain}}(t) \rho^>(t)]. \end{aligned} \quad (38)$$

In general, the second and third terms are nonzero, leading to a nonconserved particle number—consistent with the weak U(1) symmetry of one-body loss and gain mechanisms. Conversely, the exact $I(t)$ is traceless since the particle-hole jump operators possess a strong U(1) symmetry. This exact property is preserved by all Φ -derivable approximations to the correlation self-energy [82]:

$$\Sigma_{mn}^c(z, z') = \frac{\delta \Phi}{\delta G_{nm}(z', z^+)}. \quad (39)$$

The functional Φ contains all vacuum diagrams with both physical and dissipative-induced scatterings of order higher than one. It is constructed in the same manner as for the nondissipative case, with the substitution $v^{\text{phys}}(z, z') \rightarrow v(z, z')$.

The proof of the total number of particles conservation for Φ -derivable self-energies is straightforward. Consider the variation of the NEGF induced by the infinitesimal $U(1)$ transformation

$$\delta G_{mn}(z, z') = i[\Lambda(z) - \Lambda(z')]G_{mn}(z, z'). \quad (40)$$

This transformation leaves Φ unchanged. Taking into account Eqs. (39) and (40), as well as that Σ^c is a nonsingular function of its contour times, we have

$$\begin{aligned} 0 &= \sum_{mn} \int_C dz dz' \Sigma_{mn}^c(z, z') [\Lambda(z) - \Lambda(z')] G_{nm}(z', z^+) \\ &= \int_C dz dz' \text{Tr}[\Sigma^c(z, z') G(z', z^+) - G(z, z') \Sigma^c(z', z^+)] \Lambda(z). \end{aligned} \quad (41)$$

Because of the arbitrariness of $\Lambda(z)$, we infer that

$$\begin{aligned} 0 &= \int_C dz' \text{Tr}[\Sigma^c(z, z') G(z', z^+) - G(z, z') \Sigma^c(z', z^+)] \\ &= \int_{t_0}^{\infty} d\bar{t} \text{Tr}[\Sigma^{c,<}(t, \bar{t}) G^A(\bar{t}, t^+) + \Sigma^{c,R}(t, \bar{t}) G^<(\bar{t}, t^+) + \text{H.c.}] \\ &= \pm \text{Tr} I(t). \end{aligned} \quad (42)$$

In conclusion, changes in the total number of particles are entirely governed by one-body loss and gain jump operators. For these operators to conserve $N(t)$, the coefficients η and $\bar{\eta}$ in Eqs. (13a) and (13b) must satisfy the condition $\text{Tr}[V^{\text{loss}}(t)\rho^<(t) \mp V^{\text{gain}}(t)\rho^>(t)] = 0$, which can only be fulfilled in special cases—a relevant example is provided in Sec. VI.

V. DISSIPATIVE BETHE-SALPETER EQUATION

To clarify the logical flow of the presentation, Table I summarizes the most relevant equations and main approximations underlying the different formulations of the BSE discussed in the following sections. Starting from the exact equation on the Keldysh contour, we progressively derive simplified forms, including the HSEX, retarded, steady-state, Bloch basis, and zero-temperature limits.

A. Exact results

The non-Hermitian Bethe-Salpeter equation (BSE) is an equation for the two-particle exchange-correlation (XC) function L , which is obtained from the two-particle Green's function as

TABLE I. Sequence of approximations and limits leading to the different formulations of the BSE. The second column specifies the arguments of the XC function. The third column points to the final equation.

BSE formulation	Arguments	Equations
Exact	Contour times	Eq. (44)
HSEX kernel	Contour times	Eq. (54)
Retarded component	Real times	Eq. (56)
Steady state	Frequency	Eq. (60)
Bloch basis	Frequency	Eq. (64)
Zero temperature	Frequency	Eq. (68)

$$\begin{aligned} \pm L(az_a, bz_b; cz_c, dz_d) &\equiv G_2(az_a, bz_b; cz_c, dz_d) \\ &- G_{ac}(z_a, z_c) G_{bd}(z_b, z_d). \end{aligned} \quad (43)$$

According to the results in Sec. III, dissipation modifies only the noninteracting Green's function G^0 and renormalizes the physical interaction $v^{\text{phys}}(z, z') \rightarrow v(z, z')$, while leaving the rules for diagrammatic expansions otherwise unchanged. Therefore, the dissipative Bethe-Salpeter equation reads [41]

$$\begin{aligned} L(az_a, bz_b; cz_c, dz_d) &= G_{ad}(z_a, z_d) G_{bc}(z_b, z_c) \\ &\pm \sum_{a'b'c'd'} \int_C dz'_a dz'_b dz'_c dz'_d G_{aa'}(z_a, z'_a) G_{c'c}(z'_c, z_c) \\ &\times K(a'z'_a, b'z'_b; c'z'_c, d'z'_d) L(d'z'_d, bz_b; b'z'_b, dz_d), \end{aligned} \quad (44)$$

where the irreducible kernel is related to the self-energy $\Sigma = \Sigma^s[G, v]$ through

$$K(az_a, bz_b; cz_c, dz_d) = \pm \frac{\delta \Sigma_{ac}^s(z_a, z_c)}{\delta G_{db}(z_d, z_b)}. \quad (45)$$

Let us define the *two-time* XC function as

$$L_{ij}^{nm}(z_1, z_2) \equiv L(iz_1, mz_2; jz_1^+, nz_2^+) = L_{ji}^{mn}(z_2, z_1), \quad (46)$$

which has property

$$L_{ij}^{*nm}(z_1, z_2) = L_{ji}^{nm}(z_1^*, z_2^*) = L_{ij}^{nm}(z_2^*, z_1^*). \quad (47)$$

The two-time L also emerges as a response function via the Kubo's formalism. It can be shown that the current and the density variations produced by L , which satisfies the Bethe-Salpeter equation with kernel $K = \pm \delta \Sigma / \delta G$, can equivalently be computed from G , provided that Σ is Φ derivable, as in the case of Hermitian systems [41,83]. Consequently, the poles of the two-time L contain information about absorption and emission energies as well as dissipation-induced lifetimes.

The two-time XC function can also be used to screen the generalized interaction according to

$$W_{ijmn}(z, z') = v_{ijmn}(z, z') + \delta W_{ijmn}(z, z'), \quad (48)$$

where the fluctuation δW reads (repeated indices are summed over and integrals are over $\bar{z}$ and $\bar{z}'$)

$$\delta W_{ijmn}(z, z') = \pm i \int_C v_{qjnp}(z, \bar{z}) L_{pq}(\bar{z}, \bar{z}') v_{isrn}(\bar{z}', z'). \quad (49)$$

The screened interaction has the same symmetry property $W_{ijmn}(z, z') = W_{jinm}(z', z)$ as the bare v . Of particular relevance to the following discussion is the retarded component $W^R \equiv W^T - W^< = W^> - W^{\bar{T}}$, where $W^T(t, t') = W(t_-, t'_-)$ is time ordered and $W^{\bar{T}}(t, t') = W^T(t_+, t'_+)$ is antitime ordered. Using the explicit form of v in Eq. (16) and integrating over the contour times we find

$$\begin{aligned} W_{ijmn}^R(t, t') &= \left[v_{ijmn}^{\text{phys}}(t) + i v_{ijmn}^{\text{a,diss}}(t) \right] \delta(t - t') \pm i \sum_{pqrs} \\ &\times \left[v_{qjnp}^{\text{phys}}(t) + i v_{qjnp}^{\text{a,diss}}(t) \right] L_{pq}^R(t, t') \left[v_{isrn}^{\text{phys}}(t') + i v_{isrn}^{\text{a,diss}}(t') \right]. \end{aligned} \quad (50)$$

The advanced component $W^A \equiv W^T - W^> = W^< - W^{\bar{T}}$ can be extracted similarly. Taking into account that

$$L_{ij}^{R*}(t_1, t_2) = -L_{nm}^A(t_2, t_1), \quad (51)$$

it is straightforward to verify the relation $W_{ijmn}^{R*}(t, t') = W_{mni j}^A(t', t)$.

B. BSE with HSEX kernel

The lowest-order approximation to Σ in a skeletonic expansion in terms of G and W is given by the Hartree term in Eq. (27) and the Fock term in Eq. (28) with $v \rightarrow W$ (screened exchange). We refer to this approximation as the Hartree plus screened exchange (HSEX) approximation. The HSEX kernel follows from Eq. (45). Discarding the functional derivative of W with respect to G , we find

$$\begin{aligned} K^{\text{HSEX}}(a'z'_a, b'z'_b; c'z'_c, d'z'_d) &= i\delta(z'_a, z'_c)\delta(z'_b, z'_d)v_{a'b'd'c'}(z'_b, z'_a) \\ &\pm i\delta(z'_a, z'_d)\delta(z'_b, z'_c)W_{a'b'c'd'}(z'_b, z'_a). \end{aligned} \quad (52)$$

To evaluate the screened interaction we implement the (widely used) static approximation, according to which $W^R(t, t') = \delta(t - t')W^R(\omega = 0)$. Physical dissipation-induced interactions are typically short-ranged and do not necessitate to be screened, see also Sec. VII. Therefore, we can safely discard $v^{\text{a,diss}}$ in the second line of Eq. (48). With these simplifications Eq. (48) becomes

$$W_{ijmn}(z, z') = \delta(z, z')W_{ijmn}^{\text{phys}}(t) + v_{ijmn}^{\text{diss}}(z, z'), \quad (53)$$

where $W^{\text{phys}} = v^{\text{phys}} \pm i v^{\text{phys}} L^R(\omega = 0) v^{\text{phys}}$. This screened interaction fulfills the same symmetry properties as the bare physical interaction, i.e., $W_{ijmn}^{\text{phys}*} = W_{nmji}^{\text{phys}} = W_{mni j}^{\text{phys}}$.

The main advantage of the static HSEX approximation for nondissipative systems is that it reduces the BSE to a closed form for the two-time XC function. As we show below, the presence of dissipation introduces an additional complication, which, however, can be effectively addressed. Inserting the kernel in Eq. (52)—with W from Eq. (53) and v^{diss} from Eq. (18)—into the BSE Eq. (44), and integrating over z'_a, z'_c and z'_d , we obtain

$$\begin{aligned} L(az_a, bz_b; cz_c, dz_d) &= G_{ad}(z_a, z_d)G_{bc}(z_b, z_c) \\ &\pm i \sum_{a'b'c'd'} \int_C dz'_b \left\{ G_{aa'}(z_a, z'_b)G_{c'c}(z'_b, z_c) \left[v_{a'b'd'c'}^{\text{phys}}(t'_b) \pm W_{a'b'c'd'}^{\text{phys}}(t'_b) - i s(z'_b)(v_{a'b'd'c'}^{\text{s,diss}}(t'_b) \pm v_{a'b'c'd'}^{\text{s,diss}}(t'_b)) \right] \right. \\ &+ G_{aa'}(z_a, z_b'^*)G_{c'c}(z_b'^*, z_c) \left[i s(z'_b)v_{a'b'd'c'}^{\text{s,diss}}(t'_b) + i v_{a'b'd'c'}^{\text{a,diss}}(t'_b) \right] \left. \right\} L(d'z'_b, bz_b; b'z'_b, dz_d) \\ &+ i \sum_{a'b'c'd'} \int_C dz'_b G_{aa'}(z_a, z_b'^*)G_{c'c}(z'_b, z_c) \left[i s(z'_b)v_{a'b'd'c'}^{\text{s,diss}}(t'_b) + i v_{a'b'd'c'}^{\text{a,diss}}(t'_b) \right] L(d'z_b'^*, bz_b; b'z'_b, dz_d). \end{aligned} \quad (54)$$

By setting $z_c = z_a$ and $z_d = z_b$, all functions L except the last one reduce to the two-time XC function in Eq. (46). The origin of the last term stems from the anomalous Dirac delta function, $\delta(z, z'^*)$, appearing in Eq. (18). The crucial observation at this point is that

$$L(d'z_b'^*, bz_b; b'z'_b, dz_b) = L(d'z'_b, bz_b; b'z'_b, dz_b) \quad (55)$$

for all physical times $t'_b > t_b$. Therefore, the BSE can be closed for the retarded two-time XC function $L^R(t, t')$, and we find

$$L_{ij}^R(t_1, t_2) = \mathcal{L}_{ij}^R(t_1, t_2) + i \sum_{pqrs} \int_{t_2}^{t_1} d\bar{t} \left\{ \mathcal{L}_{ij}^R(t_1, \bar{t}) \left[W_{psqr}^{\text{phys}}(\bar{t}) \pm v_{psrq}^{\text{phys}}(\bar{t}) \mp i v_{psrq}^{\text{a,diss}}(\bar{t}) \right] \right. \\ \left. + i \left[G_{ip}^<(t_1, \bar{t}) G_{qj}^<(\bar{t}, t_1) - G_{ip}^>(t_1, \bar{t}) G_{qj}^>(\bar{t}, t_1) \right] v_{psqr}^{\text{a,diss}}(\bar{t}) - i G_{ip}^R(t_1, \bar{t}) G_{qj}^A(\bar{t}, t_1) v_{psqr}^{\text{s,diss}}(\bar{t}) \right\} L_{rs}^R(\bar{t}, t_2), \quad (56)$$

where the function $\mathcal{L}$ is the noncorrelated two-time XC function

$$\mathcal{L}_{ij}^R(z, z') \equiv G_{in}(z, z') G_{mj}(z', z). \quad (57)$$

The proof of Eq. (55) and the derivation of Eq. (56) can be found in Appendix D.

C. Steady-state BSE

Let the system Hamiltonian and jump operators be independent of time. The exact numerical solution of Eq. (56) remains an intractable task even for medium-sized systems, due to the many-body nature of the one-particle Green's function G . In practical calculations for nondissipative systems, the so-called quasiparticle approximation $G \simeq G^{\text{qp}}$ is commonly employed to simplify the BSE and analyze the steady-state solution. Under this approximation, the steady-state BSE reduces to the diagonalization of an effective two-particle Hamiltonian, enabling unprecedented accuracy in the description of optical spectra and excitonic properties of semiconductors.

A key feature of G^{qp} is the existence of well defined one-particle quantum numbers, which ensures well defined populations and energies, or equivalently, infinite quasiparticle lifetimes. This condition is fulfilled whenever the quasiparticle Hamiltonian h^{qp} commutes with the one-particle density matrix $\rho^<$. In particular, it is always satisfied within the Hartree-Fock (HF) approximation, where $h^{\text{qp}} = h^{\text{HF}}$.

In dissipative systems the existence of well defined one-particle quantum numbers is generally not guaranteed. Let us inspect the HF approximation. The stationary one-particle density matrix satisfies the Lyapunov equation, see Eq. (36),

$$-i h^{\text{HF}} \rho^< + i \rho^< h^{\text{HF}\dagger} + 2\gamma^< = 0, \quad (58)$$

which differs from the quasiparticle condition $[h^{\text{HF}}, \rho^<]_- = 0$. For both conditions to be satisfied, a sufficient requirement is that the Hermitian matrices $(h^{\text{HF}} + h^{\text{HF}\dagger})$, $\gamma^>$ and $\rho^<$ be simultaneously diagonalizable. In the following we assume that this is the case, and we evaluate G^{qp} as in Eq. (35), replacing $h^{\text{HF}} \rightarrow h^{\text{qp}}$ and $G^{\text{HF}}(t_-, t_+) \rightarrow \mp i \rho^{\text{qp},<}$, where

$$h_{in}^{\text{qp}} = \delta_{in}(\epsilon_i - i\gamma_i), \\ \rho_{in}^{\text{qp},<} = \delta_{in} f_i. \quad (59)$$

The corresponding quasiparticle Green's function is stationary provided that $\rho^{\text{qp},<}$ satisfies the Lyapunov equation with Hamiltonian h^{qp} , or, equivalently, $f_i = \gamma_i^< / \gamma_i$. In Sec. VI we discuss a physical application that fulfills these conditions. In Appendix C we derive all Keldysh components of G^{qp} .

We evaluate Eq. (56) using G^{qp} and then Fourier transform with respect to the time difference. The result is a generalization of the first-principles state-of-the-art Bethe-Salpeter equation

$$[\omega - (\epsilon_i - \epsilon_j) + i(\gamma_i + \gamma_j)] L_{ij}^R(\omega) = \mp i \delta_{in} \delta_{mj} (f_j - f_i) \pm (f_j - f_i) \sum_{rs} \left[W_{isjr}^{\text{phys}} \pm \left(v_{isrj}^{\text{phys}} - i v_{isrj}^{\text{a,diss}} \right) \right] L_{rs}^R(\omega) \\ + i \sum_{rs} \left[v_{isjr}^{\text{s,diss}} \mp (f_j + f_i \pm 1) v_{isjr}^{\text{a,diss}} \right] L_{rs}^R(\omega). \quad (60)$$

In the second line we see that only the antisymmetric part of the dissipation-induced interaction renormalizes the physical Hartree amplitude v_{isrj}^{phys} . As for the physical Fock amplitude W_{isjr}^{phys} , both the symmetric and antisymmetric parts contribute to its renormalization—see terms in the third line. Here, the antisymmetric contribution vanishes only for fermions at zero temperature since $f_j + f_i - 1 = 0$ for all $f_j - f_i \neq 0$. The BSE for the closed system is recovered by setting $v^{\text{diss}} = 0$.

VI. SEMICONDUCTORS

We specialize the general result in Eq. (60) to fermionic systems with discrete translational symmetry and with a finite gap in the one-particle spectrum (semiconductors). If the jump operators preserve the discrete translational symmetry we can rewrite Eq. (60) in a general Bloch basis by replacing [24]

$$\begin{aligned} i &\rightarrow \mathbf{k} + \frac{\mathbf{q}}{2}\mu & j &\rightarrow \mathbf{k} - \frac{\mathbf{q}}{2}\nu & m &\rightarrow \mathbf{k}' - \frac{\mathbf{q}}{2}\rho \\ n &\rightarrow \mathbf{k}' + \frac{\mathbf{q}}{2}\sigma & r &\rightarrow \mathbf{k}'' + \frac{\mathbf{q}}{2}\alpha & s &\rightarrow \mathbf{k}'' - \frac{\mathbf{q}}{2}\beta, \end{aligned}$$

where $\mathbf{k}, \mathbf{k}', \mathbf{k}''$ and $\mathbf{q}$ are quasimomenta and greek indices are collective indices for band and spin. The effects of spin-orbit coupling (SOC) can therefore be incorporated straightforwardly by employing SOC-resolved Bloch states. We introduce the short-hand notation for the two-time XC function

$$\begin{aligned} \left[\omega - \epsilon_{\mu\nu\mathbf{k}}^{\mathbf{q}} + i\gamma_{\mu\nu\mathbf{k}}^{\mathbf{q}} \right] L_{\mu\nu\mathbf{k}}^{\mathbf{q},\text{R}}(\omega) &= i f_{\mu\nu\mathbf{k}}^{\mathbf{q}} \delta_{\mu\sigma} \delta_{\rho\nu} \delta_{\mathbf{k}\mathbf{k}'} - f_{\mu\nu\mathbf{k}}^{\mathbf{q}} \sum_{\alpha\beta\mathbf{k}''} \left(K_{\mu\nu\mathbf{k}}^{\mathbf{q},\text{phys}} + i v_{\mathbf{k}+\frac{\mathbf{q}}{2}\mu\mathbf{k}''-\frac{\mathbf{q}}{2}\beta\mathbf{k}-\frac{\mathbf{q}}{2}\nu\mathbf{k}'+\frac{\mathbf{q}}{2}\alpha}^{\text{a,diss}} \right) L_{\alpha\beta\mathbf{k}''}^{\mathbf{q},\text{R}}(\omega) \\ &+ i \sum_{\alpha\beta\mathbf{k}''} \left[(f_{\mathbf{k}-\frac{\mathbf{q}}{2}\nu} + f_{\mathbf{k}+\frac{\mathbf{q}}{2}\mu} - 1) v_{\mathbf{k}+\frac{\mathbf{q}}{2}\mu\mathbf{k}''-\frac{\mathbf{q}}{2}\beta\mathbf{k}-\frac{\mathbf{q}}{2}\nu\mathbf{k}'+\frac{\mathbf{q}}{2}\alpha}^{\text{a,diss}} + v_{\mathbf{k}+\frac{\mathbf{q}}{2}\mu\mathbf{k}''-\frac{\mathbf{q}}{2}\beta\mathbf{k}-\frac{\mathbf{q}}{2}\nu\mathbf{k}'+\frac{\mathbf{q}}{2}\alpha}^{\text{s,diss}} \right] L_{\alpha\beta\mathbf{k}''}^{\mathbf{q},\text{R}}(\omega). \end{aligned} \quad (64)$$

Taking into account the symmetry property of v^{phys} and v^{diss} we verified that the exact property

$$L_{\mu\nu\mathbf{k}}^{\mathbf{q},\text{R}*}(\omega) = -L_{\sigma\rho\mathbf{k}'}^{\mathbf{q},\text{A}}(\omega) \quad (65)$$

is satisfied.

At zero temperature and in the absence of dissipation the electronic populations are $f_{\mathbf{k}\mu} = 1$ for valence bands ($\mu = v$) and $f_{\mathbf{k}\mu} = 0$ for conduction ($\mu = c$). We consider jump operators that preserve this stationary (ground) state, meaning that the corresponding one-particle density matrix $\rho_{\mathbf{k}\mu\mathbf{k}'\nu} = \delta_{\mathbf{k}\mathbf{k}'} \delta_{\mu\nu} f_{\mathbf{k}\mu}$ satisfies the Lyapunov equation. A physically relevant example of such jump operators is discussed later. The BSE in Eq. (64) can then be closed in the subspace of *resonant* and *antiresonant* band indices. A pair of indices (μ, ν) is called resonant (antiresonant) if μ is a conduction (valence) band and ν is a valence (conduction) band. Accordingly, $f_{\mu\nu\mathbf{k}}^{\mathbf{q}} = 1$ in the resonant sector and $f_{\mu\nu\mathbf{k}}^{\mathbf{q}} = -1$ in the antiresonant sector. Independently of the sector we have $f_{\mathbf{k}-\frac{\mathbf{q}}{2}\nu} + f_{\mathbf{k}+\frac{\mathbf{q}}{2}\mu} = 1$ and therefore the third line of Eq. (64) vanishes. It is then useful to introduce the dissipation kernel

$$\begin{aligned} K_{\mu\nu\mathbf{k}}^{\mathbf{q},\text{diss}} &\equiv f_{\mu\nu\mathbf{k}}^{\mathbf{q}} v_{\mathbf{k}+\frac{\mathbf{q}}{2}\mu\mathbf{k}''-\frac{\mathbf{q}}{2}\beta\mathbf{k}-\frac{\mathbf{q}}{2}\nu\mathbf{k}'+\frac{\mathbf{q}}{2}\alpha}^{\text{s,diss}} \\ &- v_{\mathbf{k}+\frac{\mathbf{q}}{2}\mu\mathbf{k}''-\frac{\mathbf{q}}{2}\beta\mathbf{k}-\frac{\mathbf{q}}{2}\nu\mathbf{k}'+\frac{\mathbf{q}}{2}\alpha}^{\text{a,diss}}, \end{aligned} \quad (66)$$

$$L_{\mu\nu\mathbf{k}}^{\mathbf{q}} \equiv L_{\mathbf{k}+\frac{\mathbf{q}}{2}\mu\mathbf{k}-\frac{\mathbf{q}}{2}\nu}^{\mathbf{q}}, \quad (61)$$

and for the physical BSE kernel

$$\begin{aligned} K_{\mu\nu\mathbf{k}}^{\mathbf{q},\text{phys}} &\equiv W_{\mathbf{k}+\frac{\mathbf{q}}{2}\mu\mathbf{k}''-\frac{\mathbf{q}}{2}\beta\mathbf{k}-\frac{\mathbf{q}}{2}\nu\mathbf{k}'+\frac{\mathbf{q}}{2}\alpha}^{\text{phys}} - v_{\mathbf{k}+\frac{\mathbf{q}}{2}\mu\mathbf{k}''-\frac{\mathbf{q}}{2}\beta\mathbf{k}-\frac{\mathbf{q}}{2}\nu\mathbf{k}'+\frac{\mathbf{q}}{2}\alpha}^{\text{phys}} \\ &= K_{\alpha\beta\mathbf{k}''}^{\mathbf{q},\text{phys}*}{}_{\mu\nu\mathbf{k}}. \end{aligned} \quad (62)$$

We further define

$$\epsilon_{\mu\nu\mathbf{k}}^{\mathbf{q}} \equiv \epsilon_{\mathbf{k}+\frac{\mathbf{q}}{2}\mu} - \epsilon_{\mathbf{k}-\frac{\mathbf{q}}{2}\nu}, \quad (63a)$$

$$\gamma_{\mu\nu\mathbf{k}}^{\mathbf{q}} \equiv \gamma_{\mathbf{k}+\frac{\mathbf{q}}{2}\mu} + \gamma_{\mathbf{k}-\frac{\mathbf{q}}{2}\nu}, \quad (63b)$$

$$f_{\mu\nu\mathbf{k}}^{\mathbf{q}} \equiv f_{\mathbf{k}-\frac{\mathbf{q}}{2}\nu} - f_{\mathbf{k}+\frac{\mathbf{q}}{2}\mu}. \quad (63c)$$

Using these definitions, the BSE in Eq. (60) reads

which shares the same symmetry property as the physical kernel, namely [compare with Eq. (62)]

$$K_{\mu\nu\mathbf{k}}^{\mathbf{q},\text{diss}*}{}_{\alpha\beta\mathbf{k}''} = K_{\alpha\beta\mathbf{k}''}^{\mathbf{q},\text{diss}}{}_{\mu\nu\mathbf{k}}. \quad (67)$$

In deriving the above relation we use Eqs. (15) and (19). We observe that the prefactor $f_{\mu\nu\mathbf{k}}^{\mathbf{q}}$ in K^{diss} flips the sign of the first term when going from the resonant to the antiresonant sector. Time-reversal symmetry is thus broken. It is also worth noticing that the symmetric interaction $v^{\text{s,diss}}$ renormalizes the Fock channel only while the antisymmetric interaction $v^{\text{a,diss}}$ renormalizes the Hartree channel only.

At zero temperature the BSE in Eq. (64) takes the following compact form

$$\begin{aligned} \left[\omega - \epsilon_{\mu\nu\mathbf{k}}^{\mathbf{q}} + i\gamma_{\mu\nu\mathbf{k}}^{\mathbf{q}} \right] L_{\mu\nu\mathbf{k}}^{\mathbf{q},\text{R}}(\omega) &= i f_{\mu\nu\mathbf{k}}^{\mathbf{q}} \delta_{\mu\sigma} \delta_{\rho\nu} \delta_{\mathbf{k}\mathbf{k}'} - f_{\mu\nu\mathbf{k}}^{\mathbf{q}} \sum_{\alpha\beta\mathbf{k}''} \left(K_{\mu\nu\mathbf{k}}^{\mathbf{q},\text{phys}} - i K_{\mu\nu\mathbf{k}}^{\mathbf{q},\text{diss}} \right) L_{\alpha\beta\mathbf{k}''}^{\mathbf{q},\text{R}}(\omega). \end{aligned} \quad (68)$$

The resonant and antiresonant sectors are coupled by the physical kernel $K^{\mathbf{q},\text{phys}}$ and by the dissipation kernel $K^{\mathbf{q},\text{diss}}$. In most first-principles calculation the physical coupling is discarded, a scheme known as the Tamm-Dancoff approximation (TDA) [84]. The possibility of implementing the

TDA for $K^{\text{q,diss}}$ must be carefully assessed on a case-by-case basis. The dissipative resonant-antiresonant coupling reads

$$K_{v'c'k'}^{\text{q,diss}} = v_{\mathbf{k}+\frac{\mathbf{q}}{2}c'\mathbf{k}-\frac{\mathbf{q}}{2}c'\mathbf{k}+\frac{\mathbf{q}}{2}v'}^{\text{s,diss}} - v_{\mathbf{k}+\frac{\mathbf{q}}{2}c'\mathbf{k}-\frac{\mathbf{q}}{2}c'\mathbf{k}+\frac{\mathbf{q}}{2}v'}^{\text{a,diss}}. \quad (69)$$

The antiresonant-resonant coupling is obtained analogously.

A. Stationarity-preserving jump operators

We consider the non-Hermitian jump operators preserving the discrete translational symmetry of the underlying lattice [compare with Eq. (5)]

$$\hat{L}_{\mathbf{k}c}^{\text{p}} = \eta^{\mathbf{k}c} \hat{d}_{\mathbf{k}c}, \quad (70a)$$

$$\hat{L}_{\mathbf{k}v}^{\text{h}} = \bar{\eta}^{\mathbf{k}v*} \hat{d}_{\mathbf{k}v}^{\dagger}, \quad (70b)$$

$$\hat{L}_{\mathbf{q}}^{(1)\text{ph}} = \sum_{\mathbf{k}\mu} \xi_{\mathbf{k}\mu\mu}^{\mathbf{q}} \hat{d}_{\mathbf{k}-\frac{\mathbf{q}}{2}\mu}^{\dagger} \hat{d}_{\mathbf{k}+\frac{\mathbf{q}}{2}\mu}, \quad (70c)$$

$$\hat{L}_{\mathbf{q}}^{(2)\text{ph}} = \sum_{\mathbf{k}vc} \xi_{\mathbf{k}vc}^{\mathbf{q}} \hat{d}_{\mathbf{k}-\frac{\mathbf{q}}{2}v}^{\dagger} \hat{d}_{\mathbf{k}+\frac{\mathbf{q}}{2}c}. \quad (70d)$$

The operators in Eqs. (70a)–(70b) describe one-particle loss from conduction bands and one-particle gain from valence bands. The operator in Eq. (70c) is responsible for scattering particles within the same band whereas the operator in Eq. (70d) is responsible for particle-hole recombination. Using Eq. (15), the dissipation-induced interaction arising from $\hat{L}_{\mathbf{q}}^{(1)\text{ph}}$ and $\hat{L}_{\mathbf{q}}^{(2)\text{ph}}$ reads

$$v_{\mathbf{k}_1\mu_1\mathbf{k}_2\mu_2\mathbf{k}_3\mu_3\mathbf{k}_4\mu_4}^{\text{diss}} = 2\delta_{\mathbf{k}_1-\mathbf{k}_4,\mathbf{k}_3-\mathbf{k}_2} \xi_{\frac{\mathbf{k}_1-\mathbf{k}_4}{2}\mu_4\mu_1}^{\mathbf{k}_1-\mathbf{k}_4*} \xi_{\frac{\mathbf{k}_3-\mathbf{k}_2}{2}\mu_2\mu_3}^{\mathbf{k}_3-\mathbf{k}_2} F_{\mu_1\mu_2\mu_3\mu_4}, \quad (71)$$

where the tensor $F_{cvc'v'} = F_{\mu\nu\mu} = 1$, and vanishes otherwise.

Let us prove that the jump operators in Eqs. (70a)–(70d) preserve the ground state with populations $f_{\mathbf{k}v} = 1 - f_{\mathbf{k}c} = 1$. The strategy is as follows: we calculate h^{HF} in Eq. (31) and $\gamma^{\lessgtr}$ in Eqs. (33) using the ground state density matrix $\rho^<$ of the nondissipative system, and then we verify that the same $\rho^<$ satisfies the stationary Lyapunov equation in Eq. (58). Because of the discrete translational invariance all quantities are diagonal in momentum space. In particular, the ground state density matrix has diagonal elements $\rho_{\mathbf{k}\mu\nu}^< = \delta_{\mu\nu} f_{\mathbf{k}\mu}$. By construction the commutator $[h + V^{\text{HF}}, \rho^<]_- = 0$ (stationarity of the nondissipative system). It is easy to show that the potential $V^{\text{a,diss}}$ and rates $\gamma^{\lessgtr}$ are diagonal in the band indices, and that the diagonal elements read

$$V_{\mathbf{k}\mu}^{\text{a,diss}} = -i \sum_{\mathbf{p}v} \left[\xi_{\mathbf{k}\mu\mu}^{0*} \xi_{\mathbf{p}vv}^0 - \xi_{\mathbf{k}\mu\mu}^0 \xi_{\mathbf{p}vv}^{0*} \right], \quad (72)$$

$$\gamma_{\mathbf{k}\mu}^> = \delta_{\mu\{c\}} \left[|\eta^{\mathbf{k}\mu}|^2 + \sum_{\mathbf{k}'} \left| \xi_{\frac{\mathbf{k}-\mathbf{k}'}{2}\mu\mu}^{\mathbf{k}-\mathbf{k}'} \right|^2 \right], \quad (73)$$

$$\gamma_{\mathbf{k}\mu}^< = \delta_{\mu\{v\}} \left[|\bar{\eta}^{\mathbf{k}\mu}|^2 + \sum_{\mathbf{k}'} \left| \xi_{\frac{\mathbf{k}'+\mathbf{k}}{2}\mu\mu}^{\mathbf{k}'-\mathbf{k}} \right|^2 \right], \quad (74)$$

In Eqs. (73)–(74) the quantity $\delta_{\mu\{c\}} = 1$ ($\delta_{\mu\{v\}} = 1$) if μ is a conduction (valence) band index and zero otherwise. Thus, the stationary Lyapunov equation is solved by the one-particle density matrix

$$\rho_{\mathbf{k}\mu\nu}^< = \delta_{\mu\nu} \frac{\gamma_{\mathbf{k}\mu}^<}{\gamma_{\mathbf{k}\mu}^> + \gamma_{\mathbf{k}\mu}^<} = \delta_{\mu\nu} \frac{\gamma_{\mathbf{k}\mu}^<}{\gamma_{\mathbf{k}\mu}^> + \gamma_{\mathbf{k}\mu}^<} = \delta_{\mu\nu} f_{\mathbf{k}\mu}, \quad (75)$$

which proves our original statement. We emphasize that excitations over the ground state still manifest nontrivial dissipative behavior, as we show in Sec. VII.

The stationarity-preserving jump operators in Eqs. (70c),(70d) have another remarkable feature: the TDA for the dissipation kernel is not an approximation, but an exact property. The resonant-antiresonant coupling is governed by a dissipation-induced interaction v^{diss} with band indices (c, c', v, v') , see Eq. (69). However, for this specific choice of band indices $v^{\text{diss}} = 0$ since $F_{cc'vv'} = 0$, see discussion below Eq. (71). This implies that we can solve the BSE in the resonant and antiresonant sectors separately.

B. Non-Hermitian excitonic Hamiltonian

In the resonant subspace, the BSE in Eq. (68) reduces to the following eigenvalue problem

$$\sum_{c''v''\mathbf{k}''} \left(\delta_{cc''} \delta_{vv''} \delta_{\mathbf{k}\mathbf{k}''} \omega - H_{cv\mathbf{k}c''v''\mathbf{k}''}^{\mathbf{q}} \right) L_{c'v'\mathbf{k}'}^{\mathbf{q},\text{R}}(\omega) = i \delta_{cc'} \delta_{vv'} \delta_{\mathbf{k}\mathbf{k}'}, \quad (76)$$

where the non-Hermitian excitonic Hamiltonian

$$H_{cv\mathbf{k}c'v'\mathbf{k}'}^{\mathbf{q}} = H_{cv\mathbf{k}c'v'\mathbf{k}'}^{\mathbf{q},\text{phys}} - i\Gamma_{cv\mathbf{k}c'v'\mathbf{k}'}^{\mathbf{q}}, \quad (77)$$

comprises the Hermitian excitonic Hamiltonian of the nondissipative system

$$H_{cv\mathbf{k}c'v'\mathbf{k}'}^{\mathbf{q},\text{phys}} = \delta_{cc'} \delta_{vv'} \delta_{\mathbf{k}\mathbf{k}'} \epsilon_{cv\mathbf{k}}^{\mathbf{q}} - K_{c'v'\mathbf{k}'}^{\mathbf{q},\text{phys}}, \quad (78)$$

and the Hermitian dissipation-induced rate

$$\Gamma_{cv\mathbf{k}c'v'\mathbf{k}'}^{\mathbf{q}} = \delta_{cc'} \delta_{vv'} \delta_{\mathbf{k}\mathbf{k}'} \gamma_{cv\mathbf{k}}^{\mathbf{q}} - K_{c'v'\mathbf{k}'}^{\mathbf{q},\text{diss}}, \quad (79)$$

the non-Hermitian excitonic Hamiltonian in the antiresonant sector can be obtained similarly.

It is worth remarking that the microscopic diagrammatic construction of the BSE does not guarantee that Γ^q is PSD for any choice of the coefficients ξ in the particle-hole jump operators. The PSD property is crucial for preserving the causality of the retarded XC function, and it should be satisfied by any sensible approximation. Below we prove that Γ^q in Eq. (79) is PSD only provided that the band-diagonal coefficients satisfy the relation $\xi_{\mathbf{k}\mu\mu}^q = \xi_{\mathbf{k}\mu\mu}^{-q}$ or $\xi_{\mathbf{k}\mu\mu}^q = -\xi_{\mathbf{k}\mu\mu}^{-q}$.

According to Eq. (63b) we have $\gamma_{cv\mathbf{k}}^q = \gamma_{\mathbf{k}+\frac{q}{2}c} + \gamma_{\mathbf{k}-\frac{q}{2}v}$, where we recall that $\gamma_{\mathbf{k}\mu} = \gamma_{\mathbf{k}\mu}^> + \gamma_{\mathbf{k}\mu}^<$, see Eq. (33a), with $\gamma_{\mathbf{k}\mu}^<$ given in Eqs. (73)–(74). To evaluate the dissipation kernel we use Eq. (71), extract the symmetric $v^{s,\text{diss}}$ and antisymmetric $v^{a,\text{diss}}$ interactions, and insert the resulting resonant-resonant dissipative kernel from Eq. (66) into Eq. (79). We find

$$\begin{aligned} \Gamma_{cv\mathbf{k}c'v'\mathbf{k}'}^q &= \delta_{cc'}\delta_{vv'}\delta_{\mathbf{k}\mathbf{k}'} \left(\left| \eta^{\mathbf{k}+\frac{q}{2}c} \right|^2 + \left| \tilde{\eta}^{\mathbf{k}-\frac{q}{2}v} \right|^2 \right. \\ &\quad + \sum_{\mathbf{p}} \left| \xi_{\frac{\mathbf{k}+\mathbf{p}+\mathbf{q}}{2}cc}^{\mathbf{k}-\mathbf{p}} \right|^2 + \sum_{\mathbf{p}} \left| \xi_{\frac{\mathbf{k}+\mathbf{p}-\mathbf{q}}{2}vv}^{\mathbf{p}-\mathbf{k}} \right|^2 \Big) \\ &\quad - \delta_{cc'}\delta_{vv'} \left(\xi_{\frac{\mathbf{k}-\mathbf{k}'+\mathbf{q}}{2}cc}^{\mathbf{k}-\mathbf{k}'} \xi_{\frac{\mathbf{k}+\mathbf{k}'-\mathbf{q}}{2}vv}^{\mathbf{k}-\mathbf{k}'} + \xi_{\frac{\mathbf{k}'+\mathbf{k}+\mathbf{q}}{2}cc}^{\mathbf{k}'-\mathbf{k}} \xi_{\frac{\mathbf{k}+\mathbf{k}'-\mathbf{q}}{2}vv}^{\mathbf{k}'-\mathbf{k}} \right) \\ &\quad + \xi_{\mathbf{k}vc}^q \xi_{\mathbf{k}'v'c'}^q. \end{aligned} \quad (80)$$

The terms in the first two lines arise from γ , the third line arises from $v^{s,\text{diss}}$, and the fourth line arises from $v^{a,\text{diss}}$. The contributions to Γ^q from the first and fourth lines are PSD by inspection. The contributions from the second and third lines are diagonal in the band indices. If we define $C_{\mathbf{k}\mathbf{k}'} \equiv \xi_{\frac{\mathbf{k}+\mathbf{k}'+\mathbf{q}}{2}cc}^{\mathbf{k}-\mathbf{k}'}$ and $V_{\mathbf{k}\mathbf{k}'} \equiv \xi_{\frac{\mathbf{k}+\mathbf{k}'-\mathbf{q}}{2}vv}^{\mathbf{k}-\mathbf{k}'}$, then we are left to prove that the matrix

$$M_{\mathbf{k}\mathbf{k}'} = \delta_{\mathbf{k}\mathbf{k}'} \sum_{\mathbf{p}} (|C_{\mathbf{k}\mathbf{p}}|^2 + |V_{\mathbf{k}\mathbf{p}}|^2) - C_{\mathbf{k}\mathbf{k}'}^* V_{\mathbf{k}'\mathbf{k}} - C_{\mathbf{k}'\mathbf{k}} V_{\mathbf{k}\mathbf{k}'}^* \quad (81)$$

is PSD. At this point of the derivation we have not yet used that $\xi_{\mathbf{k}\mu\mu}^q$ is even or odd in $\mathbf{q}$. If it is, then both C and V are symmetric or antisymmetric matrices, and the diagonal elements of M satisfy the inequality

$$\begin{aligned} M_{\mathbf{k}\mathbf{k}} &= \sum_{\mathbf{p} \neq \mathbf{k}} (|C_{\mathbf{k}\mathbf{p}}|^2 + |V_{\mathbf{k}\mathbf{p}}|^2) + |C_{\mathbf{k}\mathbf{k}} - V_{\mathbf{k}\mathbf{k}}|^2 \\ &\geq \sum_{\mathbf{p} \neq \mathbf{k}} (|C_{\mathbf{k}\mathbf{p}}|^2 + |V_{\mathbf{k}\mathbf{p}}|^2) \\ &\geq \sum_{\mathbf{p} \neq \mathbf{k}} |C_{\mathbf{k}\mathbf{p}}^* V_{\mathbf{p}\mathbf{k}} + C_{\mathbf{p}\mathbf{k}} V_{\mathbf{k}\mathbf{p}}^*| \geq 0. \end{aligned} \quad (82)$$

According to Gershgorin theorem, for every eigenvalue λ of the matrix M , there exists a $\mathbf{k} = \mathbf{k}_\lambda$ such that

$$\begin{aligned} |\lambda - M_{\mathbf{k}_\lambda\mathbf{k}_\lambda}| &\leq \sum_{\mathbf{p} \neq \mathbf{k}_\lambda} |M_{\mathbf{k}_\lambda\mathbf{p}}| = \sum_{\mathbf{p} \neq \mathbf{k}_\lambda} |C_{\mathbf{k}_\lambda\mathbf{p}}^* V_{\mathbf{p}\mathbf{k}_\lambda} + C_{\mathbf{p}\mathbf{k}_\lambda} V_{\mathbf{k}_\lambda\mathbf{p}}^*| \\ &\leq M_{\mathbf{k}_\lambda\mathbf{k}_\lambda}, \end{aligned} \quad (83)$$

where we use Eq. (82) in the last inequality. We conclude that $\lambda \in (0, 2M_{\mathbf{k}_\lambda\mathbf{k}_\lambda})$ and hence that $\lambda \geq 0$.

In a similar manner, one can prove that $\Gamma_{vc\mathbf{k}v'c'\mathbf{k}'}^q$ (antiresonant sector) is PSD, since it has the same form as Eq. (80), with the replacements $\mathbf{q} \rightarrow -\mathbf{q}$ and $\xi \rightarrow \xi^*$. It is worth commenting on the inclusion of the jump operator $\sum_{\mathbf{k}c v} \xi_{cv\mathbf{k}}^q \hat{d}_{\mathbf{k}-\frac{q}{2}c}^\dagger \hat{d}_{\mathbf{k}+\frac{q}{2}v}$. This jump operator promotes an electron from the valence band to the conduction band and is often used to model a pump [40,85]. However, its inclusion would not only destabilize the ground state, but also jeopardize the PSD property of Γ^q , as it modifies Eq. (80) by adding the matrix $-\xi_{\mathbf{k}cv}^{q*} \xi_{\mathbf{k}'c'v'}^q$, which is manifestly negative definite. To avoid issues with the PSD property, and hence with the causal structure of the dissipative NEGF framework, the pump field must be explicitly modeled as a time-dependent perturbation that couples valence and conduction states [80].

VII. VALLEY EXCITONS IN TMD

In this section we apply our theory to a two-dimensional transition metal dichalcogenides (TMD) system, which has band extrema located at the triangular lattice Brillouin zone corners K and K' .

Let us write the physical (Coulomb) kernel in Eq. (62) as $K^{\text{phys}} = K^W + K^v$ where K^W is the direct screened interaction (first term) and K^v is the bare exchange interaction (second term). The two valley excitons are degenerate if we set $K^v = 0$. We define our unperturbed excitonic basis as the eigenvectors of the eigenvalue problem

$$\sum_{c'v'\mathbf{k}'} \left(\delta_{cc'}\delta_{vv'}\delta_{\mathbf{k}\mathbf{k}'} \epsilon_{cv\mathbf{k}}^q - K_{cv\mathbf{k}}^{q,W} \right) A_{c'v'\mathbf{k}'}^{\tau,q} = E_q A_{cv\mathbf{k}}^{\tau,q}, \quad (84)$$

where $E_q \simeq E_0 + [(\hbar^2 q^2)/(2M_0)]$ is the degenerate exciton energy, and $\tau = +$ for the K exciton and $\tau = -$ for the K' exciton. We take $E_0 \simeq 2$ eV as the reference energy, and the excitonic mass M_0 approximately equal to the free electron mass [86].

In the chosen basis the nondissipative excitonic Hamiltonian reduces to the following 2×2 matrix [40,87–90]

$$\begin{aligned}
H_{\tau\tau'}^{\mathbf{q},\text{phys}} &= \sum_{\substack{c\nu\mathbf{k} \\ c'\nu'\mathbf{k}'}} A_{c\nu\mathbf{k}}^{\tau,\mathbf{q}*} H_{c\nu\mathbf{k}c'\nu'\mathbf{k}'}^{\mathbf{q},\text{phys}} A_{c'\nu'\mathbf{k}'}^{\tau',\mathbf{q}} \\
&= \left(\frac{\hbar^2 q^2}{2M_0} + U_q \right) \delta_{\tau\tau'} + \begin{pmatrix} 0 & U_q e^{-2i\varphi} \\ U_q e^{2i\varphi} & 0 \end{pmatrix}_{\tau\tau'},
\end{aligned} \quad (85)$$

where $\mathbf{q} = q(\cos\varphi, \sin\varphi)^\top$ is expressed in polar coordinates, and the bare exchange interaction (stemming from K^ν)

$$U_q = U \frac{q}{q_K} \quad (86)$$

scales linearly with q in the long-wavelength limit [87,90]. In Eq. (86) the quasimomentum $q_K = 4\pi/3a$ is the distance between the Γ point and the K, K' points, $a \approx 3.3 \text{ \AA}$ being a typical lattice constant of TMD monolayers [91]. The value $U \sim 1 \text{ eV}$ has been extracted from first principles calculations [87]. Quasimomenta in the light cone satisfy the approximate inequality $q < q_c = E_0/(\hbar c) \sim 10^{-3} q_K$.

We study the TMD coupled to photon baths. In this context, the relevant jump operators describe band off-diagonal particle-hole recombination processes. Therefore, we set to zero the loss and gain coefficients, $\eta^{\mathbf{k}\mu} = \bar{\eta}^{\mathbf{k}\mu} = 0$, along with the band diagonal coefficients, $\xi_{\mathbf{k}\mu\mu}^{\mathbf{q}} = 0$. Accordingly, see Eq. (80), $\Gamma_{c\nu\mathbf{k}c'\nu'\mathbf{k}'}^{\mathbf{q}} = \xi_{\mathbf{k}\nu c}^{\mathbf{q}*} \xi_{\mathbf{k}'\nu'c'}^{\mathbf{q}}$, where the band off-diagonal ξ coefficients have the form

$$\xi_{\mathbf{k}\nu c}^{\mathbf{q}} = \sqrt{D_q} \mathbf{E}(\mathbf{q}) \cdot \mathbf{d}_{\mathbf{k}\nu c}^{\mathbf{q}}. \quad (87)$$

In this equation $\mathbf{d}_{\mathbf{k}\nu c}^{\mathbf{q}}$ is the unit vector of dipole matrix elements, $\mathbf{E}(\mathbf{q})$ is the unit vector of the electric field generated by photons of momentum $\mathbf{q}$, and $D_q \geq 0$ is a q -dependent proportionality constant.

The 2×2 excitonic Hamiltonian in Eq. (85) is modified by the addition of the 2×2 matrix $-i\Gamma^{\mathbf{q}}$, where

$$\Gamma_{\tau\tau'}^{\mathbf{q}} = \sum_{\substack{c\nu\mathbf{k} \\ c'\nu'\mathbf{k}'}} A_{c\nu\mathbf{k}}^{\tau,\mathbf{q}*} \Gamma_{c\nu\mathbf{k}c'\nu'\mathbf{k}'}^{\mathbf{q}} A_{c'\nu'\mathbf{k}'}^{\tau',\mathbf{q}} = \xi_{\tau}^{\mathbf{q}} \xi_{\tau'}^{\mathbf{q}}. \quad (88)$$

By construction, the ξ coefficients in the excitonic basis are given by

$$\xi_{\tau}^{\mathbf{q}} \equiv \sum_{c\nu\mathbf{k}} A_{c\nu\mathbf{k}}^{\tau,\mathbf{q}} \xi_{\mathbf{k}\nu c}^{\mathbf{q}} = \sqrt{D_q} \mathbf{E}(\mathbf{q}) \cdot \mathbf{d}_{\tau}^{\mathbf{q}} \quad (89)$$

where $\mathbf{d}_{\tau}^{\mathbf{q}} \equiv \sum_{c\nu\mathbf{k}} A_{c\nu\mathbf{k}}^{\tau,\mathbf{q}} \mathbf{d}_{\mathbf{k}\nu c}^{\mathbf{q}}$ is the dipole moment for the recombination of the unperturbed exciton in valley τ . For small momentum transfer $\mathbf{d}_{\tau}^{\mathbf{q}} = (1, \tau i)^\top / \sqrt{2}$, which is independent of $\mathbf{q}$ [92,93].

We observe that the dissipation-induced interaction mixes the valley excitons and introduces a finite exciton lifetime. This originates from the smooth density of states of the bath modes implicitly assumed in Lindbladian dynamics (the so-called bad cavity limit). In the opposite regime, where well-defined modes yield sharp peaks in the density of states (the good cavity limit), photon-mediated interactions also renormalize the exciton energies [94–96].

We consider two distinct baths that are engineered to support structured photons. The first bath (B0) is made of linearly polarized photons with no net orbital angular momentum (OAM); hence $\mathbf{E}(\mathbf{q}) = (\cos\theta_0, \sin\theta_0)^\top$, where the angle θ_0 defines the polarization direction. The second bath (B1) is more exotic as it is made of two types of photons: type “+” photons, which are right-handed circularly polarized and carry an OAM $l = 1$, and type “−” photons, which have opposite handedness and OAM. In B1 the electric field reads $\mathbf{E}(\mathbf{q}) = \mathbf{E}_+(\mathbf{q}) + \mathbf{E}_-(\mathbf{q})$, where $\mathbf{E}_{\pm}(\mathbf{q}) = e^{\pm i(\varphi + \theta_1)} (1, \mp i)^\top / \sqrt{2}$ is the electric field generated by the type $\pm$ photons. The phase θ_1 is a free parameter which, as we see below, controls the symmetry breaking and the existence of EPs.

To our knowledge, experimental realizations of structured optical cavities with smooth spectral densities designed to mimic such baths have not yet been realized. However, significant progress has been made in confining photons with both circular polarization and finite OAM (bath B1), particularly through the use of spiral-phase mirrors and chiral metasurfaces [97–100]. Furthermore, anisotropic cavities that support a single polarization (bath B0) have been recently engineered [101]. We also note that structured photon baths can naturally emerge from a more refined treatment of the system-bath decoupling [102]. In any case, our results depend solely on the form of the ξ coefficients—the photon baths serve to exemplify a possible microscopic realization.

Let us now discuss the function D_q . It encodes all information about the modified mode functions and density of states of the photonic environment. The microscopic form can be determined from the photonic spectral density of the cavity at planar momentum q and energy $\simeq E_0$. Since our results depend only on the general features of D_q , rather than on its detailed form, we adopt here a simple model based on general physical considerations.

Electron-hole recombination mainly occurs for momenta within the light cone. Therefore, we let the magnitude D_q vanish exponentially as $q \rightarrow \infty$. The small q behavior of D_q depends on the nature of the bath. For bath B0, the electric field in real space is a plane wave, and therefore we can take $D_q \rightarrow D^{(0)}$ as $q \rightarrow 0$. Electric fields with OAM l , such as those of bath B1, are typically generated using Laguerre-Gaussian beams. In real space they have the form

$E(r, \varphi_r) \propto r^{|l|} e^{-r^2} e^{il\varphi_r}$, where $\mathbf{r} = r(\cos \varphi_r, \sin \varphi_r)^\top$, see Refs. [103–106]. This implies that, in momentum space, $E(q, \varphi) \sim q^{|l|} e^{il\varphi}$ as $q \rightarrow 0$. Therefore, we can take $D_q \rightarrow D^{(1)}(q/q_K)$ in the same limit. To summarize

$$D_q = \begin{cases} D_q^{(0)} = D^{(0)} e^{-\lambda q} & \text{(B0)} \\ D_q^{(1)} = D^{(1)} \frac{q}{q_K} e^{-\lambda q} & \text{(B1)} \end{cases}, \quad (90)$$

where $D^{(0)}, D^{(1)} \geq 0$ and $\lambda \gtrsim 1/q_c$.

From Eq. (89) we have

$$\xi_\tau^{\mathbf{q}} = \sqrt{D_q^{(0)}} e^{i\tau\theta_0} \quad \text{(B0)}; \quad \xi_\tau^{\mathbf{q}} = \sqrt{D_q^{(1)}} e^{i\tau(\varphi+\theta_1)} \quad \text{(B1)}. \quad (91)$$

The two different baths can be treated simultaneously if we introduce the parameter n with value 0 for bath B0 and value 1 for bath B1: $\xi_\tau^{\mathbf{q}} = \sqrt{D_q^{(n)}} e^{i\tau(n\varphi+\theta_n)}$. The explicit form of the 2×2 matrix $\Gamma^{\mathbf{q}}$ reads

$$\Gamma_{\tau\tau'}^{\mathbf{q}} = \begin{pmatrix} D_q^{(n)} & D_q^{(n)} e^{-2i(n\varphi+\theta_n)} \\ D_q^{(n)} e^{2i(n\varphi+\theta_n)} & D_q^{(n)} \end{pmatrix}_{\tau\tau'}. \quad (92)$$

In the 2×2 subspace the non-Hermitian excitonic Hamiltonian is $H_{\tau\tau'}^{\mathbf{q}} = H_{\tau\tau'}^{\mathbf{q},\text{phys}} - i\Gamma_{\tau\tau'}^{\mathbf{q}}$, see Eq. (77), and the left and right eigenvalue problems read

$$\begin{aligned} H^{\mathbf{q}} \vec{u}_{\mathbf{q},\text{R}}^\pm &= E_{\mathbf{q}}^\pm \vec{u}_{\mathbf{q},\text{R}}^\pm, \\ H^{\mathbf{q}\dagger} \vec{u}_{\mathbf{q},\text{L}}^\pm &= E_{\mathbf{q}}^{\pm*} \vec{u}_{\mathbf{q},\text{L}}^\pm. \end{aligned} \quad (93)$$

We find eigenvalues

$$\begin{aligned} E_{\mathbf{q}}^\pm &= \frac{\hbar^2 q^2}{2M_0} + U_q - iD_q^{(n)} \\ &\pm \sqrt{U_q^2 - D_q^{(n)2} - 2iU_q D_q^{(n)} \cos[2\theta_n - 2(1-n)\varphi]} \end{aligned} \quad (94)$$

and unnormalized right or left eigenvectors

$$\vec{u}_{\mathbf{q},\text{R}}^\pm = \begin{pmatrix} \pm e^{-i\varphi} \sqrt{U_q - i e^{-2i[(n-1)\varphi+\theta_n]} D_q^{(n)}} \\ e^{i\varphi} \sqrt{U_q - i e^{2i[(n-1)\varphi+\theta_n]} D_q^{(n)}} \end{pmatrix}, \quad (95a)$$

$$\vec{u}_{\mathbf{q},\text{L}}^\pm = \begin{pmatrix} \pm e^{-i\varphi} \sqrt{U_q + i e^{-2i[(n-1)\varphi+\theta_n]} D_q^{(n)}} \\ e^{i\varphi} \sqrt{U_q + i e^{2i[(n-1)\varphi+\theta_n]} D_q^{(n)}} \end{pmatrix}. \quad (95b)$$

Note that the left and right eigenvectors can be converted into each other by shifting $\theta_n \rightarrow \theta_n + (\pi/2)$.

A. Exceptional points

The angular dependence of the excitonic eigenvalues for the two baths differs solely in the argument of the cosine, see Eq. (94). In B0 we have $\cos[2(\theta_0 - \varphi)]$ whereas in B1 we have $\cos(2\theta_1)$. Therefore, B0 introduces an anisotropy in momentum space, while B1 preserves cylindrical symmetry within each valley. We begin by discussing B1.

In Fig. 1 we plot the real and imaginary parts of the eigenvalues $E_{\mathbf{q}}^\pm$ as functions of the exciton momentum q , for various values of the parameter θ_1 . The dissipation parameters are $D^{(1)} = 1.7U$ and $\lambda = 1/q_c$. The imaginary parts are both negative, in agreement with the general PSD proof we derived in Sec. VI B. The eigenvalues are well separated for all q , except at $\theta_1 = \pm\pi/4$, where both their real and imaginary parts coalesce at the exceptional momentum $q \sim q_e$ (inside the light cone). The momentum q_e follows from Eq. (94) with $\theta_1 = \pm(\pi/4)$. Setting the square root to zero we find $q_e = (1/\lambda) \ln(D^{(1)}/U) \simeq 0.53q_c$. Notice that exceptional momenta are found only for $D^{(1)} > U$. At $q = q_e$ one of the off-diagonal elements of the non-Hermitian excitonic Hamiltonian $H^{\mathbf{q}}$ becomes zero, resembling an effective nonreciprocal coupling as encountered in quantum optics [107]. The degenerate eigenvalues take the form

$$E_{\mathbf{q}_e}^\pm = \frac{\hbar^2 q_e^2}{2M_0} + \frac{q_e}{q_K} (U - iD^{(1)} e^{-\lambda q_e}), \quad (96)$$

and for, e.g., $\theta_1 = \pi/4$ the right and left eigenvectors (normalized to unity)

$$\vec{u}_{\mathbf{q}_e,\text{R}}^\pm = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \vec{u}_{\mathbf{q}_e,\text{L}}^\pm = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (97)$$

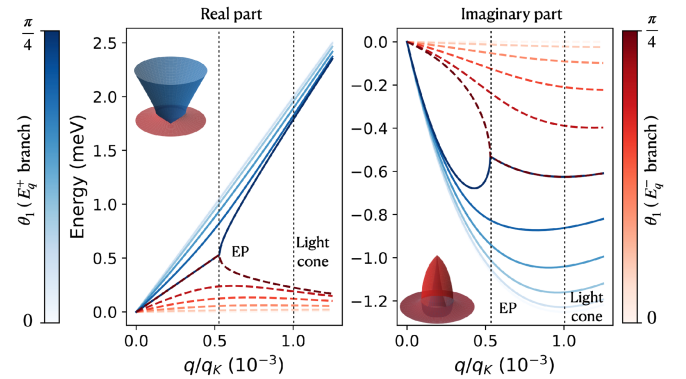


FIG. 1. Excitonic spectrum for bath B1. Real (left) and imaginary (right) part of the eigenvalues $E_{\mathbf{q}}^\pm$ as a function of the exciton momentum q for different symmetry-breaking parameters θ_1 . The upper and lower branches are shown in solid blue and dashed red, respectively. At $\theta_1 = \pi/4$, a ring of EP emerges inside the light cone. The insets show the three-dimensional dispersion at $\theta_1 = \pi/4$, highlighting the cylindrical symmetry.

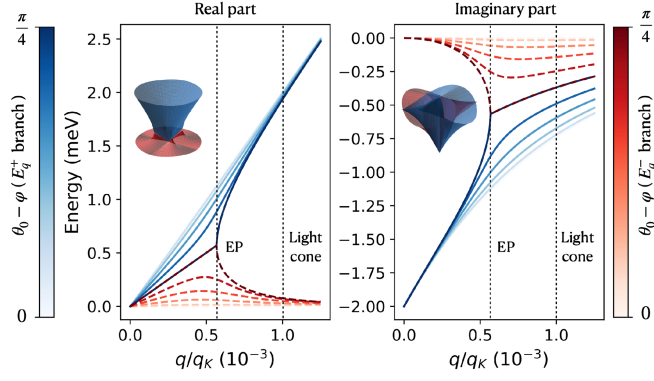


FIG. 2. Excitonic spectrum for bath B0. Real (left) and imaginary (right) part of the eigenvalues $E_{\mathbf{q}}^{\pm}$ as a function of the exciton momentum q for different angles $\theta_0 - \varphi$. The upper and lower branches are shown in solid blue and dashed red, respectively. At $\varphi = \theta_0 \pm \pi/4$, an EP emerges. The insets show the three-dimensional dispersion, highlighting the anisotropic structure in momentum space.

coalesce. As the eigenvalues $E_{\mathbf{q}}^{\pm}$ are independent of the angular coordinate φ of $\mathbf{q}$, a ring of exceptional points (EP) emerges at momentum $q = q_e$.

Figure 2 displays the excitonic eigenvalues in bath B0 for different values of the angle φ . The dissipation parameters are $D^{(0)} = 1$ meV and $\lambda = 1/q_c$. Once again, the imaginary parts are both negative, as it should be. Only excitons with momentum directions tilted by an angle of $\pm(\pi/4)$ with respect to the angle θ_0 of the linearly polarized photons become degenerate. The exceptional momentum satisfies $U(q_e/q_K) = D^{(0)}e^{-\lambda q_e}$, which yields $q_e \simeq 0.56q_c$. The excitonic spectrum exhibits two exceptional points rather than an exceptional ring.

We remark that for more conventional setups hosting two equivalent polarization directions, the coefficient in Eq. (91) becomes $\xi_{\tau}^{\mathbf{q}} = \sqrt{D_q^{(0)}}e^{i\tau\theta_0}(1 + e^{i\tau\pi/2})$. This modification affects the Γ matrix in Eq. (92) only through the shift $\theta_0 \rightarrow \theta_0 + \pi/4$. Consequently, the real and imaginary parts of the eigenvalues remain identical to those shown in Fig. 2. More generally, the choice of the structured baths B0 and B1 is by no means unique. EPs also arise in baths of purely left- or right-handed circularly polarized photons with zero OAM. This diversity enhances the prospects for realizing exceptional points in realistic excitonic platforms.

B. Valley polarization, light polarization, and ellipticity

In this section, we study the valley polarization and characterize the polarization of light in optical emission.

Let us write the normalized right eigenvectors in Eq. (95) as $\vec{u}_{\mathbf{q},R}^{\pm} = (A_{\mathbf{q}}, A'_{\mathbf{q}})^{\top}$. The valley polarization

$$P_{\mathbf{q}} = |A_{\mathbf{q}}|^2 - |A'_{\mathbf{q}}|^2, \quad (98)$$

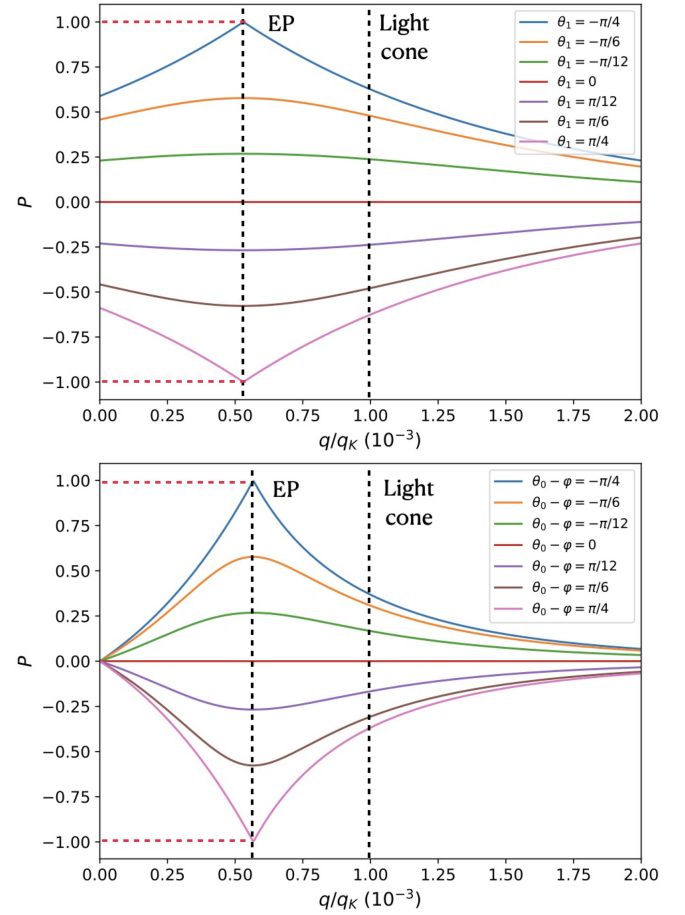


FIG. 3. Valley polarization $P_{\mathbf{q}}$ as a function of momentum q for bath B1 (top) and B0 (bottom). For B1 different values of θ_1 are considered. For $\theta_1 = 0$ the polarization vanishes at all momenta, reflecting perfect balance between the valleys. At finite θ_1 the polarization reaches extremal values at the exceptional momentum. In particular, at $\theta_1 = \pm\pi/4$ the valley polarization saturates, $P_{\mathbf{q}} = \pm 1$, indicating complete localization in either the K or K' valley. For B0 different values of the angle φ are considered. The valley polarization saturates at the exceptional momentum only for $\varphi = \theta_0 \pm \pi/4$.

measures the degree of circular polarization in photoluminescence experiments [108–111]. Its value is independent of the branch, see Eqs. (95). In Fig. 3 we plot $P_{\mathbf{q}}$ as a function of momentum q for bath B1 (top) and B0 (bottom). For B1 several representative values of the parameter θ_1 are displayed. If $\theta_1 = 0$ the valley polarization is zero for all $\mathbf{q}$. Any other choice of θ_1 breaks the balance between the K and K' valleys, with the polarization peaking at $\theta_1 = \pm\pi/4$ near the exceptional momentum q_e (vertical dashed line). At the peak the polarization is unity, indicating that the system is fully polarized. This behavior directly links the appearance of EPs to the emergence of valley-selective excitonic states. A similar discussion can be made for bath B0, where the valley polarization is studied for different angles $\theta_0 - \varphi$. In this case $P_{\mathbf{q}=0} = 0$ for all θ_0 ,

as it should be. The polarization saturates at the exceptional momentum q_e only for momenta $\mathbf{q}$ forming an angle $\varphi = \theta_0 \pm \pi/4$ with the x axis.

Next we explore the momentum resolved light polarization in photoluminescence. Let $\mathbf{J} = (J_x e^{i\phi_x}, J_y e^{i\phi_y})^\top$ be the Jones vector describing a monochromatic electric field propagating along the z direction, which is perpendicular to the TMD plane. The relative phase difference $\phi_x - \phi_y$ can be used to classify the light polarization. Specifically, the light is linearly polarized when $|\phi_x - \phi_y| = 0$, circularly polarized when $|\phi_x - \phi_y| = \pi/2$, and elliptically polarized for a general phase difference. Using the valley contrasting optical selection rules [88,112–114], the Jones vector given by optical emission from the $\vec{u}_{\mathbf{q},R}^\pm$ exciton has components $J_x e^{i\phi_x} = (A_{\mathbf{q}} + A'_{\mathbf{q}})/\sqrt{2}$ and $J_y e^{i\phi_y} = i(A_{\mathbf{q}} - A'_{\mathbf{q}})/\sqrt{2}$.

We first consider the bath B1, and fix $\theta_1 = \pi/4$. In Fig. 4 (top) we show how the electric field vector $\mathbf{E}(t) = \text{Re}[\mathbf{J}e^{i\omega t}]$ moves in the TMD plane over time for different $\mathbf{q}$'s. For both $\pm$ branches, the elliptical polarization rotates as $\mathbf{q}$ varies, and the ellipticity changes continuously except at the EP, where circular polarization emerges. Notably, the orientation angle, defined as the angle between the major axis of the polarization ellipse and the x -axis, undergoes a sudden jump when crossing the EP. In the limit of large q , the two branches converge to orthogonal

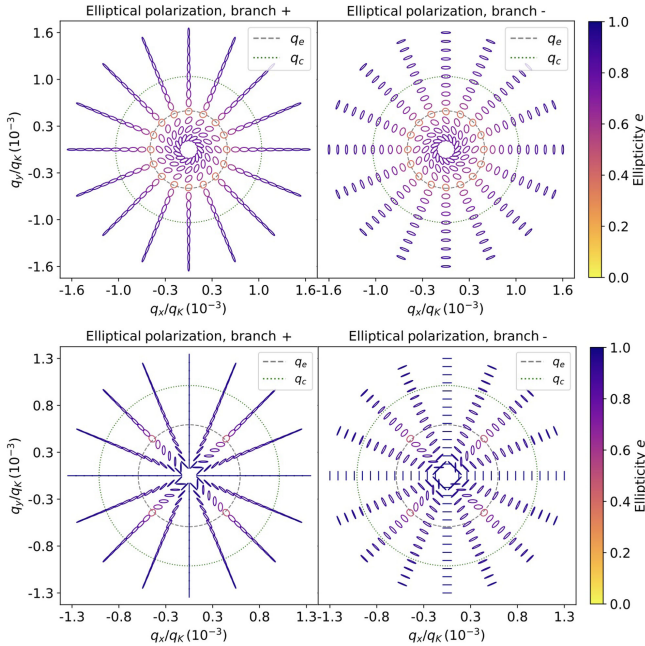


FIG. 4. Elliptical polarization patterns of the electric field $\mathbf{E}(t) = \text{Re}[\mathbf{J}e^{i\omega t}]$ for the $\pm$ branches and for bath B1 (top) and B0 (bottom). The orientation and shape of the ellipses encode the polarization state, while the color scale represents the ellipticity. At the EPs the electric field is circularly polarized, whereas for $q \rightarrow 0$ and $q \rightarrow \infty$ the polarization reduces to linear. The relative orientation of the polarization axes exhibits a $\pi/2$ difference between the two branches at large q .

linear polarizations, consistent with the nondissipative (Hermitian) theory.

The polarization and ellipticity of the electric field $\mathbf{E}(t) = \text{Re}[\mathbf{J}e^{i\omega t}]$ for the bath B0 is displayed in Fig. 4 (bottom). Here, the photons of the bath are polarized along x , i.e., $\theta_0 = 0$ (polarization along y yields the same results). The anisotropy is clearly visible. Circular polarization occurs only at the EPs, that is for $\varphi = \pm(\pi/4)$ and $\varphi = \pm[(3\pi)/4]$.

C. Berry curvature and conductivities

The non-Hermitian excitons in bath B1 have nontrivial topological properties. The Berry connection from the normalized right eigenstates is $\mathcal{C}_{\mathbf{q}}^\pm \equiv i\vec{u}_{\mathbf{q},R}^{\pm\dagger} \nabla_{\mathbf{q}} \vec{u}_{\mathbf{q},R}^\pm$ [40,115], and its real part has vanishing radial component:

$$\text{Re}[\mathcal{C}_{\mathbf{q}}^\pm] = \begin{cases} -\frac{1}{q} \frac{U_q}{D_q^{(1)}} \mathbf{e}_\varphi, & q < q_e \\ -\frac{1}{q} \frac{D_q^{(1)}}{U_q} \mathbf{e}_\varphi, & q > q_e \end{cases}, \quad (99)$$

where $\mathbf{e}_\varphi$ is the angular unit vector (perpendicular to the radial direction). The associated Berry curvature for $\theta_1 = \pi/4$ is independent of the branch and given by

$$\Omega_q \equiv \nabla_{\mathbf{q}} \times \text{Re}[\mathcal{C}_{\mathbf{q}}^\pm] = \begin{cases} -\frac{U}{D^{(1)}} \frac{\lambda e^{\lambda q}}{q}, & q < q_e \\ +\frac{D^{(1)}}{U} \frac{\lambda e^{-\lambda q}}{q}, & q > q_e \end{cases}. \quad (100)$$

Figure 5 (left) highlights the singular nature of the Berry curvature in the vicinity of $q = 0$, where $\Omega_q \sim -1/q$. The discontinuity of magnitude $2\lambda^2/\ln(D^{(1)}/U)$ at the exceptional momentum $q = q_e$ marks a transition between two regimes with opposite analytical forms. For $q > q_e$, the curvature decays exponentially, ensuring that the Berry phase accumulated along a large loop remains finite.

Using the rotational invariance of the eigenvalues in Eq. (94) and the Berry curvature in Eq. (100), the Hall conductivity and Nernst conductivity can be written as [40]

$$\sigma_{xy} \equiv -\sigma_0 \sum_{\tau=\pm} \int_0^\infty dq f_\tau^0(q) (q \Omega_q), \quad (101)$$

$$\alpha_{xy} \equiv \frac{\sigma_0}{T} \sum_{\tau=\pm} \int_0^\infty dq (q \Omega_q) \times \left[\frac{\text{Re}[E_q^\tau] - \mu}{k_B} (f_\tau^0(q) + 1) + T \ln f_\tau^0(q) \right], \quad (102)$$

where $\sigma_0 = 1/h$ is the quantum of conductance and $f_\tau^0(q) = 1/[e^{(\text{Re}[E_q^\tau] - \mu)/k_B T} - 1]$ is the bosonic exciton distribution. Both integrals are convergent and well behaved.

Figure 5 (middle and right) displays the temperature dependence of the conductivities for chemical potential $\mu = -1$ meV, ensuring a dilute population of excitons.

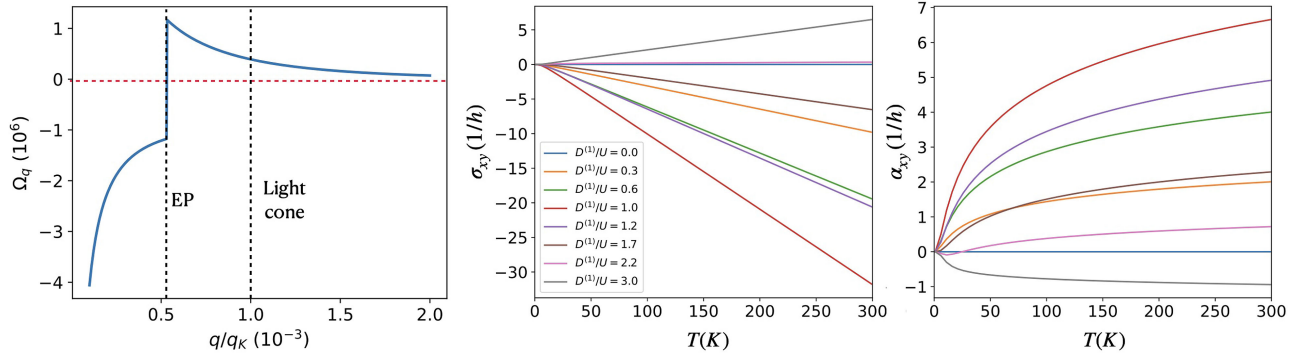


FIG. 5. (Left) Berry curvature Ω_q at $\theta_1 = \pi/4$. The curve shows a divergence for $q \rightarrow 0$ and a discontinuous jump of magnitude $2\lambda^2/\ln(D^{(1)}/U)$ across the exceptional momentum q_e , consistent with the analytic expressions in Eq. (100). Middle-right. Hall conductivity (middle), σ_{xy} , and Nernst conductivity (right), α_{xy} , against temperature for several values of the ratio $D^{(1)}/U$. In both cases, the emergence of the exceptional ring at $D^{(1)}/U = 1$ inverts the trend in how these quantities vary with $D^{(1)}/U$.

The Hall conductivity depends almost linearly on T . Interestingly, for $D^{(1)}/U < 1$ —where there is no exceptional ring—the ratio σ_{xy}/T is negative and decreases as $D^{(1)}/U$ increases. The emergence of the exceptional ring for $D^{(1)}/U > 1$ inverts this trend: σ_{xy}/T begins to increase with increasing $D^{(1)}/U$, eventually turning positive. The exceptional ring has a similar impact on the Nernst conductivity. The quantity α_{xy} is positive and increases with $D^{(1)}/U$ until the formation of the exceptional ring. Upon further increasing $D^{(1)}/U$, the Nernst conductivity decreases and eventually becomes negative, reflecting the entropy-driven nature of the effect. The parameter $D^{(1)}$ acts as a tuning knob, changing the sign of the conductivities.

VIII. CONCLUSIONS

In this work, we have developed a rigorous, microscopic framework for describing excitonic phenomena in open quantum systems. By extending the BSE to incorporate Lindbladian dissipation within the NEGF formalism, we constructed a first-principles approach capable of capturing the interplay between many-body interactions and environmental coupling. This formalism goes beyond effective low-energy or phenomenological models by preserving the underlying fermionic structure of excitons and accounting for *ab initio* band structures and electron-hole interactions. As such, it can be integrated into existing *ab initio* BSE codes [34–38], enabling material-specific predictions. An important outcome of our analysis is that not all forms of dissipation channels are permissible in the stationary limit. Stability and causality provide two key constraints for physically meaningful spectra.

The formalism is applicable to a broad class of dissipative environments. In particular, we investigated baths composed of linearly and circularly polarized photons, and found that dissipation can give rise to exceptional points (EPs) in the excitonic band structure. Depending on the

nature of the bath, these EPs may appear either as a finite set or as a ring-shaped continuum. The presence of EPs also leaves distinctive signatures in observable quantities, including nonanalytic behavior in the valley polarization, complex polarization patterns in photoluminescence, and a nontrivial topological character of the excitonic states. More generally, our results establish a unifying first-principles framework that connects open quantum dynamics, excitonic physics, and non-Hermitian topology.

The methods developed here also provide a foundation for several exciting directions. The framework could be applied to study driven-dissipative phenomena, time-dependent protocols, or excitonic dynamics beyond the steady state. It also opens the possibility of including phonons from first-principles [43]. In fact, exciton-phonon interactions play a crucial role in converting coherent bright excitons into incoherent bright and dark excitons [116–119]. Another active area of application for the non-Hermitian BSE is the study of exciton polaritons in lossy cavities [120–123]. Finally, the appearance of topological features using engineered baths suggests opportunities for designing non-Hermitian quantum devices with controllable optical and valleytronic properties.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: DERIVATION OF THE DISSIPATIVE HAMILTONIAN

The starting point is the Hamiltonian in Eq. (8). Inserting in $\hat{\mathcal{H}}(z)$ the explicit form of the jump operators in Eq. (5) we find (sum over repeated indices is implicit)

$$\begin{aligned}\hat{\mathcal{H}}(z) = & \hat{H}(z) - i s(z) \left[V_{ij}^{\text{loss}}(t) \hat{d}_i^\dagger(z) \hat{d}_j(z) + V_{ij}^{\text{gain}}(t) \hat{d}_j(z) \hat{d}_i^\dagger(z) + \frac{1}{2} v_{ijmn}^{\text{diss}}(t) \hat{d}_i^\dagger(z) \hat{d}_n(z) \hat{d}_j^\dagger(z) \hat{d}_m(z) \right] \\ & + 2i\theta_-(z) \left[V_{ij}^{\text{loss}}(t) \hat{d}_i^\dagger(z^*) \hat{d}_j(z) + \frac{1}{4} v_{ijmn}^{\text{diss}}(t) \hat{d}_i^\dagger(z^*) \hat{d}_n(z^*) \hat{d}_j^\dagger(z) \hat{d}_m(z) \right] \\ & - 2i\theta_+(z) \left[V_{ij}^{\text{gain}}(t) \hat{d}_j(z) \hat{d}_i^\dagger(z^*) + \frac{1}{4} v_{ijmn}^{\text{diss}}(t) \hat{d}_j^\dagger(z) \hat{d}_m(z) \hat{d}_i^\dagger(z^*) \hat{d}_n(z^*) \right],\end{aligned}\quad (\text{A1})$$

where the quantities V_{ij}^{loss} , V_{ij}^{gain} and v_{ijmn}^{diss} are defined in Eqs. (13a)–(13b) and Eq. (15), respectively. Taking into account that under the $\mathcal{T}$ sign the bosonic (fermionic) operators (anti)commute, the normal-ordered form of Eq. (A1) reads

$$\begin{aligned}\hat{\mathcal{H}}(z) = & \hat{H}(z) - i s(z) \left[V_{ij}^{\text{loss}}(t) \pm V_{ij}^{\text{gain}}(t) + V_{ij}^{\text{diss}}(t) \right] \hat{d}_i^\dagger(z) \hat{d}_j(z) + 2i \left[\theta_-(z) V_{ij}^{\text{loss}}(t) \mp \theta_+(z) V_{ij}^{\text{gain}}(t) \right] \hat{d}_i^\dagger(z^*) \hat{d}_j(z) \\ & - \frac{i}{2} s(z) v_{ijmn}^{\text{diss}}(t) \hat{d}_i^\dagger(z) \hat{d}_j^\dagger(z) \hat{d}_m(z) \hat{d}_n(z) + \frac{i}{2} \left[\theta_-(z) v_{ijmn}^{\text{diss}}(t) - \theta_+(z) v_{ijmn}^{\text{diss}}(t) \right] \hat{d}_i^\dagger(z^*) \hat{d}_j^\dagger(z) \hat{d}_m(z) \hat{d}_n(z^*),\end{aligned}\quad (\text{A2})$$

which can be verified to be the same as Eq. (9) once the integral over $\bar{z}'$ in Eqs. (10)–(11) is performed.

APPENDIX B: MARTIN-SCHWINGER HIERARCHY FOR OPEN SYSTEMS

Let $\hat{O}_i(z_i) = f_i(z_i) \hat{C}_i(z_i)$, where $f_i(z_i)$ is a scalar function depending on both z_i and z_i^* , and $\hat{C}_i(z_i)$ is the product of an arbitrary number of field operators, some evaluated at z_i and the remaining at z_i^* . The contour-time derivative of the contour-ordered string of such operators is given by

$$\begin{aligned}\frac{d}{dz_k} \mathcal{T} \{ \hat{O}_1(z_1) \dots \hat{O}_n(z_n) \} = & \sum_{l=1}^{k-1} (\pm)^{k-l} [\delta(z_k, z_l) + \delta(z_k, z_l^*)] \mathcal{T} \left\{ \hat{O}_1(z_1) \dots \hat{O}_l(z_l) \dots [\hat{O}_k(z_k), \hat{O}_l(z_l)]_{\mp} \dots \hat{O}_n(z_n) \right\} \\ & + \sum_{l=k+1}^n (\pm)^{l-k-1} [\delta(z_k, z_l) + \delta(z_k, z_l^*)] \mathcal{T} \left\{ \hat{O}_1(z_1) \dots [\hat{O}_k(z_k), \hat{O}_l(z_l)]_{\mp} \dots \hat{O}_l(z_l) \dots \hat{O}_n(z_n) \right\},\end{aligned}\quad (\text{B1})$$

where the lower sign applies when both $\hat{O}_k$ and $\hat{O}_l$ are fermionic operators. We use this identity to derive the MSH. Taylor expanding the exponential of $\hat{\mathcal{H}}(z)$ in Eq. (6), the G_n reads

$$G_n(j_1 z_1, \dots, j_n z_n; j'_1 z'_1, \dots, j'_n z'_n) = \frac{1}{i^n} \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \int_C d\bar{z}_1 \dots d\bar{z}_k \text{Tr} \left[\hat{\rho}(t_0) \mathcal{T} \left\{ \hat{\mathcal{H}}(\bar{z}_1) \dots \hat{\mathcal{H}}(\bar{z}_k) \hat{d}_{j_1}(z_1) \dots \hat{d}_{j'_n}^\dagger(z'_n) \right\} \right]. \quad (\text{B2})$$

Taking the contour-time derivative of Eq. (B2) with respect to z_q [corresponding to field operator $\hat{d}_{j_q}(z_q)$], using Eq. (B1), and performing the integrals over the delta functions arising from the commutators with $\hat{\mathcal{H}}$, the equation of motion for G_n follows

$$\begin{aligned}i \frac{d}{dz_q} G_n(j_1 z_1, \dots, j_q z_q, \dots, j_n z_n; j'_1 z'_1, \dots, j'_n z'_n) \\ = \sum_{l=1}^n (\pm)^{q+l} \delta(z_q, z'_l) \delta_{j_q j'_l} G_{n-1} \left(j_1 z_1, \dots, j_q z_q, \dots, j_n z_n; j'_1 z'_1, \dots, j'_l z'_l, \dots, j'_n z'_n \right) \\ + \frac{1}{i^n} \left\langle \hat{d}_{j_1}(z_1) \dots \{ [\hat{d}_{j_q}(z_q), \hat{\mathcal{H}}(z_q)]_- - [\hat{d}_{j_q}(z_q), \hat{\mathcal{H}}(z_q^*)]_- \} \dots \hat{d}_{j'_n}^\dagger(z'_n) \right\rangle.\end{aligned}\quad (\text{B3})$$

To prove the equivalence of this result with the MSH we need to evaluate the commutator between $\hat{d}_{j_q}(z_q)$ and Hamiltonians $\hat{\mathcal{H}}(z_q)$ and $\hat{\mathcal{H}}(z_q^*)$. Let us introduce the quantities

$$U_{ij}(z) = h_{ij}(t) - is(z) \left[V_{ij}^{\text{loss}}(t) \pm V_{ij}^{\text{gain}}(t) + V_{ij}^{\text{diss}}(t) \right], \quad (\text{B4a})$$

$$W_{ij}(z) = 2i \left[\theta_-(z) V_{ij}^{\text{loss}}(t) \mp \theta_+(z) V_{ij}^{\text{gain}}(t) \right], \quad (\text{B4b})$$

$$u_{ijmn}(z) = v_{ijmn}^{\text{phys}}(t) - is(z) v_{ijmn}^{\text{s,diss}}(t), \quad (\text{B4c})$$

$$w_{ijmn}(z) = -is(z) v_{ijmn}^{\text{s,diss}}(t) - i v_{ijmn}^{\text{a,diss}}(t), \quad (\text{B4d})$$

and rewrite Eqs. (12) and (18) as

$$h_{ij}(z', z) = \delta(z', z) U_{ij}(z) + \delta(z', z^*) W_{ij}(z), \quad (\text{B5})$$

and

$$v_{ijmn}(z, z') = \delta(z, z') u_{ijmn}(z) + \delta(z, z^*) w_{ijmn}(z). \quad (\text{B6})$$

Inserting these results into Eqs. (10)–(11) and performing the integral over $\bar{z}'$ we obtain (sum over repeated indices is implicit)

$$\begin{aligned} \hat{\mathcal{H}}(z) &= U_{ij}(z) \hat{d}_i^\dagger(z) \hat{d}_j(z) + W_{ij}(z) \hat{d}_i^\dagger(z^*) \hat{d}_j(z) \\ &+ \frac{1}{2} u_{ijmn}(z) \hat{d}_i^\dagger(z) \hat{d}_j^\dagger(z) \hat{d}_m(z) \hat{d}_n(z) \\ &- \frac{1}{2} w_{ijmn}(z) \hat{d}_i^\dagger(z^*) \hat{d}_j^\dagger(z) \hat{d}_m(z) \hat{d}_n(z^*). \end{aligned} \quad (\text{B7})$$

Under the contour-ordering sign we have the following identities [field operators on different branches (anti)commute]

$$\begin{aligned} [\hat{d}_q(z), \hat{\mathcal{H}}(z)]_- &= U_{qn}(z) \hat{d}_n(z) + u_{jqnm}(z) \hat{d}_j^\dagger(z) \hat{d}_m(z) \hat{d}_n(z) \\ &- \frac{1}{2} w_{jqnm}(z) \hat{d}_j^\dagger(z^*) \hat{d}_m(z^*) \hat{d}_n(z), \end{aligned} \quad (\text{B8})$$

and

$$\begin{aligned} [\hat{d}_q(z), \hat{\mathcal{H}}(z^*)]_- &= W_{qn}(z^*) \hat{d}_n(z^*) \\ &- \frac{1}{2} w_{qjmn}(z^*) \hat{d}_j^\dagger(z^*) \hat{d}_m(z^*) \hat{d}_n(z). \end{aligned} \quad (\text{B9})$$

Subtracting Eq. (B9) from Eq. (B8) yields

$$\begin{aligned} &[\hat{d}_{j_q}(z_q), \hat{\mathcal{H}}(z_q)]_- - [\hat{d}_{j_q}(z_q), \hat{\mathcal{H}}(z_q^*)]_- \\ &= \int_C dz' h_{j_q n}(z_q, z') \hat{d}_n(z') \\ &+ \int_C dz' v_{jj_q nm}(z_q, z') \hat{d}_j^\dagger(z') \hat{d}_m(z') \hat{d}_n(z_q). \end{aligned} \quad (\text{B10})$$

Substituting Eq. (B10) into Eq. (B3) we recover the first set of equations of the MSH, see Eq. (21). The second set of equations can be derived in a similar way.

APPENDIX C: HARTREE-FOCK ONE-PARTICLE HAMILTONIAN AND GREEN'S FUNCTION

We first prove that $\int_C dz dz' [h(z, z') + \Sigma^{\text{HF}}(z, z')] = 0$. From Eqs. (12) and (27), it is easy to verify that

$$\int_C dz dz' h_{in}(z, z') = -2i \int_{t_0}^{\infty} d\bar{t} V_{in}^{\text{diss}}(\bar{t}), \quad (\text{C1})$$

$$\int_C dz dz' \Sigma_{in}^{\text{H}}(z, z') = 0. \quad (\text{C2})$$

We evaluate the contour integral of the Fock self-energy using Eqs. (16)–(18) and Eq. (28). We find (sum over repeated indices is implicit)

$$\begin{aligned} &\int_C dz dz' \Sigma_{in}^{\text{F}}(z, z') \\ &= \int_C dz \left\{ s(z) G_{mj}^<(t, t) v_{ijnm}^{\text{s,diss}}(t) \right. \\ &\quad \left. + G_{mj}(z, z^*) \left[s(z^*) v_{ijnm}^{\text{s,diss}}(t) + v_{ijnm}^{\text{a,diss}}(t) \right] \right\} \\ &= \int_{t_0}^{\infty} dt \left[G_{mj}^<(t, t) - G_{mj}^>(t, t) \right] v_{ijnm}^{\text{diss}}(t) \\ &= 2i \int_{t_0}^{\infty} dt V_{in}^{\text{diss}}(t), \end{aligned} \quad (\text{C3})$$

where we took into account that $G(t_{\mp}, t'_{\pm}) = G^{\lessgtr}(t, t')$. We conclude that stability is recovered through the Fock self-energy.

In order to derive the solution of the HF KBE, we split the self-energy as $\Sigma^{\text{s}} = \Sigma^{\text{HF}} + \Sigma^{\text{c}}$. The HF approximation corresponds to set $\Sigma^{\text{c}} = 0$. Thus the equation of motion in Eq. (26) becomes

$$i \frac{d}{dz} G(z, z') - \int_C d\bar{z} [h(z, \bar{z}) + \Sigma^{\text{HF}}(z, \bar{z})] G(\bar{z}, z') = \delta(z, z'). \quad (\text{C4})$$

We rewrite the HF Hamiltonian in the form

$$h(z, \bar{z}) + \Sigma^{\text{HF}}(z, \bar{z}) = \delta(z, \bar{z})U^{\text{HF}}(\bar{z}) + \delta(z, \bar{z}^*)W^{\text{HF}}(\bar{z}). \quad (\text{C5})$$

In terms of the quantities introduced in Eqs. (B5)–(B6) we have

$$U_{in}^{\text{HF}}(\bar{z}) = U_{in}(\bar{z}) + V_{in}^{\text{HF}}(\bar{t}) + V_{in}^{\text{a,diss}}(\bar{t}) \mp \text{is}(\bar{z})v_{ijnm}^{\text{s,diss}}(\bar{t})\rho_{mj}^<(\bar{t}), \quad (\text{C6})$$

$$W_{in}^{\text{HF}}(\bar{z}) = W_{in}(\bar{z}) - \text{i}G_{mj}(\bar{z}^*, \bar{z})w_{ijnm}(\bar{z}). \quad (\text{C7})$$

Substituting Eq. (C5) into Eq. (C4) and taking the integral over $\bar{z}$ we obtain

$$\left[\text{i} \frac{d}{dz} - U^{\text{HF}}(z) \right] G(z, z') + W^{\text{HF}}(z^*)G(z^*, z') = \delta(z, z'). \quad (\text{C8})$$

The HF KBE for the lesser component follows when setting $z = t_-$ and $z' = t'_+$, and is identical to Eq. (30) with $\Sigma^c = 0$. The other HF KBEs can be derived in a similar way.

Let us prove that the HF KBEs admit the following solution:

$$G^{\text{HF}}(z, z') = \theta(t - t')T \left\{ e^{-\text{i} \int_{t'}^t d\bar{t} h^{\text{HF}}(\bar{t})} \right\} G^{\text{HF}}(z^*, z') + \theta(t' - t)G^{\text{HF}}(z, z^*)\bar{T} \left\{ e^{+\text{i} \int_t^{t'} d\bar{t} h^{\text{HF}\dagger}(\bar{t})} \right\}, \quad (\text{C9})$$

according to which the lesser and greater components read

$$G^{\text{HF},\leq}(t, t') = \theta(t - t')T \left\{ e^{-\text{i} \int_{t'}^t d\bar{t} h^{\text{HF}}(\bar{t})} \right\} G^{\text{HF},\leq}(t', t') + \theta(t' - t)G^{\text{HF},\leq}(t, t)\bar{T} \left\{ e^{+\text{i} \int_t^{t'} d\bar{t} h^{\text{HF}\dagger}(\bar{t})} \right\}, \quad (\text{C10})$$

and, consequently, the retarded and advanced components are given by

$$G^{\text{HF},\text{R}}(t, t') = -\text{i}\theta(t - t')T \left\{ e^{-\text{i} \int_{t'}^t d\bar{t} h^{\text{HF}}(\bar{t})} \right\}, \quad (\text{C11})$$

$$G^{\text{HF},\text{A}}(t, t') = +\text{i}\theta(t' - t)T \left\{ e^{+\text{i} \int_t^{t'} d\bar{t} h^{\text{HF}\dagger}(\bar{t})} \right\}. \quad (\text{C12})$$

Taking the derivative of $G^<(t, t')$ in Eq. (C10) with respect to the real time t we find

$$\begin{aligned} \text{i} \frac{d}{dt} G^{\text{HF},<}(t, t') &= \theta(t - t')h^{\text{HF}}(t)T \left\{ e^{-\text{i} \int_{t'}^t d\bar{t} h^{\text{HF}}(\bar{t})} \right\} G^{\text{HF},<}(t', t') \\ &+ \theta(t' - t) \left[\text{i} \frac{d}{dt} G^{\text{HF},<}(t, t) \right] \bar{T} \left\{ e^{+\text{i} \int_t^{t'} d\bar{t} h^{\text{HF}\dagger}(\bar{t})} \right\} \\ &+ \theta(t' - t)G^{\text{HF},<}(t, t)h^{\text{HF}\dagger}(t)\bar{T} \left\{ e^{+\text{i} \int_t^{t'} d\bar{t} h^{\text{HF}\dagger}(\bar{t})} \right\}. \end{aligned} \quad (\text{C13})$$

The equation of motion for equal time $G^{\text{HF},<}(t, t)$ —appearing in third line of the above equation—can be expressed through Eq. (36) [with collision integral $I(t) = 0$]:

$$\text{i} \frac{d}{dt} G^{\text{HF},<}(t, t) = h^{\text{HF}}(t)G^{\text{HF},<}(t, t) - G^{\text{HF},<}(t, t)h^{\text{HF}\dagger}(t) \pm 2\gamma^<(t). \quad (\text{C14})$$

Substituting Eq. (C14) into the third line of Eq. (C13) and identifying the expressions of $G^{\text{HF},<}$ and $G^{\text{HF},\text{A}}$ in Eqs. (C10) and (C12), we find

$$\text{i} \frac{d}{dt} G^{\text{HF},<}(t, t') = h^{\text{HF}}(t)G^{\text{HF},<}(t, t') \mp 2\text{i}\gamma^<(t)G^{\text{HF},\text{A}}(t, t'), \quad (\text{C15})$$

which correctly agrees with the HF KBE in Eq. (30). The other KBEs can be verified in a similar way.

Finally, we consider the stationary case of a one-particle density matrix that satisfies the Lyapunov equation with time-independent $\hat{H}$ and $\hat{D}$, see Eq. (36),

$$-\text{i}h^{\text{HF}}\rho^< + \text{i}\rho^<h^{\text{HF}\dagger} + 2\gamma^< = 0. \quad (\text{C16})$$

We specialize the discussion to Hermitian matrices ($h^{\text{HF}} + h^{\text{HF}\dagger}$), $\gamma^{\geq}$ and $\rho^<$ that can be simultaneously diagonalized. We replace HF with qp, as any quasiparticle approximation with the same properties leads to identical results. In the basis that diagonalizes these matrices we have $h_{in}^{\text{qp}} = \delta_{in}(\epsilon_i - \text{i}\gamma_i)$, and the stationary $\rho_{in}^< = \delta_{in}f_i$ that solves the Lyapunov equation is given by $f_i = \gamma_i^</\gamma_i$. From Eqs. (C10) and (C11)–(C12) we then have the following Keldysh components for the quasiparticle NEGF

$$G_{mn}^{\text{qp},\text{R}}(t_1, t_2) = -\text{i}\delta_{mn}\theta(t_1 - t_2)e^{-\text{i}(\epsilon_m - \text{i}\gamma_m)(t_1 - t_2)}, \quad (\text{C17a})$$

$$G_{mn}^{\text{qp},\text{A}}(t_1, t_2) = +\text{i}\delta_{mn}\theta(t_2 - t_1)e^{+\text{i}(\epsilon_m + \text{i}\gamma_m)(t_2 - t_1)}, \quad (\text{C17b})$$

$$G_{mn}^{\text{qp},<}(t_1, t_2) = \mp \text{i}\delta_{mn}f_m \left[\theta(t_1 - t_2)e^{-\text{i}(\epsilon_m - \text{i}\gamma_m)(t_1 - t_2)} + \theta(t_2 - t_1)e^{+\text{i}(\epsilon_m + \text{i}\gamma_m)(t_2 - t_1)} \right], \quad (\text{C17c})$$

$$G_{mn}^{\text{qp},>}(t_1, t_2) = -\text{i}\delta_{mn}\bar{f}_m \left[\theta(t_1 - t_2)e^{-\text{i}(\epsilon_m - \text{i}\gamma_m)(t_1 - t_2)} + \theta(t_2 - t_1)e^{+\text{i}(\epsilon_m + \text{i}\gamma_m)(t_2 - t_1)} \right], \quad (\text{C17d})$$

where $\bar{f} \equiv 1 \pm f$. Extracting the retarded component in Eq. (57) and using Eqs. (C17) we also have

$$\ell_{ij}^{\text{qp,R}}(t_1, t_2) = \mp \delta_{ip} \delta_{qj} \theta(t_1 - t_2) (f_j - f_i) e^{-i[\epsilon_i - \epsilon_j - i(\gamma_i + \gamma_j)](t_1 - t_2)}. \quad (\text{C18})$$

The dissipation gives rise to a finite quasiparticle lifetime even in the absence of interactions.

APPENDIX D: RETARDED BETHE-SALPETER EQUATION

From Eq. (43), the anomalous and normal XC functions are given by

$$\pm L(d'z_b^*, bz_b; b'z_b', dz_b) = -\langle \hat{d}_{b'}^\dagger(z_b'^+) \hat{d}_{d'}(z_b'^*) \hat{d}_d^\dagger(z_b^+) \hat{d}_b(z_b) \rangle - G_{d'b'}(z_b^*, z_b') G_{bd}^<(t_b, t_b), \quad (\text{D1})$$

and

$$\pm L(d'z_b', bz_b; b'z_b', dz_b) = -\langle \hat{d}_{b'}^\dagger(z_b'^+) \hat{d}_{d'}(z_b') \hat{d}_d^\dagger(z_b^+) \hat{d}_b(z_b) \rangle - G_{d'b'}^<(t_b', t_b') G_{bd}^<(t_b, t_b). \quad (\text{D2})$$

We consider the physical time $t_b' > t_b$, set $z_b' = t_{b-}'$ and $z_b = t_{b-}$, and subtract Eq. (D2) from (D1). We find the identity

$$\begin{aligned} & \pm [L(d't_{b+}', bt_{b-}; b't_{b-}', dt_{b-}) - L(d't_{b-}', bt_{b-}; b't_{b-}', dt_{b-})] \\ &= [\hat{d}_{b'}^\dagger, \hat{d}_{d'}]_{\mp} \langle \hat{d}_d^\dagger(t_{b-}) \hat{d}_b(t_{b-}) \rangle + [G_{d'b'}^<(t_b', t_b') - G_{d'b'}^>(t_b', t_b')] G_{bd}^<(t_b, t_b) = 0, \end{aligned} \quad (\text{D3})$$

where we use that $\langle \hat{d}_d^\dagger(t_{b-}) \hat{d}_b(t_{b-}) \rangle = \pm i G_{bd}^<(t_b, t_b)$ along with $G^<(t, t) - G^>(t, t) = i$. Similarly, we have for $z_b' = t_{b-}'$ and $z_b = t_{b+}$

$$\begin{aligned} & \pm [L(d't_{b+}', bt_{b+}; b't_{b-}', dt_{b+}) - L(d't_{b-}', bt_{b+}; b't_{b-}', dt_{b+})] \\ &= [\hat{d}_{b'}^\dagger, \hat{d}_{d'}]_{\mp} \langle \hat{d}_d^\dagger(t_{b+}) \hat{d}_b(t_{b+}) \rangle + [G_{d'b'}^<(t_b', t_b') - G_{d'b'}^>(t_b', t_b')] G_{bd}^<(t_b, t_b) = 0. \end{aligned} \quad (\text{D4})$$

In the same way we can show that the difference between the normal and anomalous XC functions is zero when $z_b' = t_{b+}'$ and $z_b = t_{b+}$ as well as when $z_b' = t_{b+}'$ and $z_b = t_{b-}$. Thus, the equivalence in Eq. (55) for all the physical times $t_b' > t_b$ is established.

Setting $z_c = z_a = z_1$ and $z_d = z_b = z_2$ in Eq. (54), and renaming the indices $a \rightarrow i, b \rightarrow m, c \rightarrow j$, and $d \rightarrow n$, we obtain

$$L_{ij}^{nm}(z_1, z_2) = \ell_{ij}^{nm}(z_1, z_2) + \mathcal{A}_{ij}^{nm}(z_1, z_2) + \mathcal{B}_{ij}^{nm}(z_1, z_2) + \mathcal{C}_{ij}^{nm}(z_1, z_2), \quad (\text{D5})$$

where (sum over repeated indices is implicit)

$$\mathcal{A}_{ij}^{nm}(z_1, z_2) = i \int_C d\bar{z} \ell_{ij}^{pq}(z_1, \bar{z}) \left[W_{psqr}^{\text{phys}}(\bar{t}) \pm v_{psrq}^{\text{phys}}(\bar{t}) - i s(\bar{z}) (v_{psqr}^{\text{s,diss}}(\bar{t}) \pm v_{psrq}^{\text{s,diss}}(\bar{t})) \right] L_{rs}^{nm}(\bar{z}, z_2), \quad (\text{D6a})$$

$$\mathcal{B}_{ij}^{nm}(z_1, z_2) = i \int_C d\bar{z} G_{ip}(z_1, \bar{z}^*) G_{qj}(\bar{z}, z_1) \left[i s(\bar{z}) v_{psqr}^{\text{s,diss}}(\bar{t}) + i v_{psrq}^{\text{a,diss}}(\bar{t}) \right] L(r\bar{z}^*, m\bar{z}_2; s\bar{z}, n\bar{z}_2^+), \quad (\text{D6b})$$

$$\mathcal{C}_{ij}^{nm}(z_1, z_2) = \pm i \int_C d\bar{z} G_{ip}(z_1, \bar{z}^*) G_{qj}(\bar{z}^*, z_1) \left[i s(\bar{z}) v_{psqr}^{\text{s,diss}}(\bar{t}) + i v_{psrq}^{\text{a,diss}}(\bar{t}) \right] L_{rs}^{nm}(\bar{z}, z_2). \quad (\text{D6c})$$

By definition, to extract the retarded component, we must evaluate $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ for a physical time $t_1 > t_2$. We can replace the domain of integration C by the shrunken contour $\tilde{C} = (t_2, t_1) \cup (t_1, t_2)$ since all integrals over $C/\tilde{C}$ are zeros. This allows us to replace $L(r\bar{z}^*, m\bar{z}_2; s\bar{z}, n\bar{z}_2^+)$ in Eq. (D6b) by $L_{mn}^{rs}(\bar{z}, z_2)$. Consequently, we find

$$\mathcal{A}_{ij}^{\text{R}}(t_1, t_2) = i \int_{t_2}^{t_1} d\bar{t} \left\{ \ell_{ij}^{\text{R}}(t_1, \bar{t}) \left[W_{psqr}^{\text{phys}}(\bar{t}) \pm v_{psrq}^{\text{phys}}(\bar{t}) \right] - i \left[\ell_{ij}^>(t_1, \bar{t}) + \ell_{ij}^<(t_1, \bar{t}) \right] \left[v_{psqr}^{\text{s,diss}}(\bar{t}) \pm v_{psrq}^{\text{s,diss}}(\bar{t}) \right] \right\} L_{rs}^{\text{R}}(\bar{t}, t_2) \quad (\text{D7a})$$

$$\mathcal{B}_{nm}^R(t_1, t_2) = i \int_{t_2}^{t_1} d\bar{t} \left\{ i G_{ip}^<(t_1, \bar{t}) G_{qj}^<(\bar{t}, t_1) v_{psqr}^{\text{diss}}(\bar{t}) + i G_{ip}^>(t_1, \bar{t}) G_{qj}^>(\bar{t}, t_1) v_{sprq}^{\text{diss}}(\bar{t}) \right\} L_{nm}^R(\bar{t}, t_2), \quad (\text{D7b})$$

$$\mathcal{C}_{nm}^R(t_1, t_2) = \mp i \int_{t_2}^{t_1} d\bar{t} \left\{ i \mathcal{C}_{ij}^R(t_1, \bar{t}) v_{psrq}^{\text{a,diss}}(\bar{t}) - i \left[\mathcal{C}_{ij}^>(t_1, \bar{t}) + \mathcal{C}_{ij}^<(t_1, \bar{t}) \right] v_{psrq}^{\text{s,diss}}(\bar{t}) \right\} L_{nm}^R(\bar{t}, t_2). \quad (\text{D7c})$$

Substituting Eqs. (D7) into $L_{nm}^R(t_1, t_2) = \mathcal{C}_{ij}^R(t_1, t_2) + \mathcal{A}_{nm}^R(t_1, t_2) + \mathcal{B}_{nm}^R(t_1, t_2) + \mathcal{C}_{nm}^R(t_1, t_2)$, and using $\mathcal{C}_{ij}^<(t_1, \bar{t}) = G_{ip}^<(t_1, \bar{t}) G_{qj}^>(\bar{t}, t)$ from Eq. (57) we recover Eq. (56).

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